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RESEARCH ARTICLE

Eigenvectors and minimal bases for some families of Fiedler-like linearizations

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In this paper we obtain formulas for the left and right eigenvectors and minimal bases of some families of Fiedler-like linearizations of square matrix polynomials. In particular, for the families of Fiedler pencils, generalized Fiedler pencils, and Fiedler pencils with repetition. These formulas allow us to relate the eigenvectors and minimal bases of the linearizations with the ones of the polynomial. Since the eigenvectors appear in the standard formula of the condition number of eigenvalues of matrix polynomials, our results may be used to compare the condition numbers of eigenvalues of the linearizations within these families and the corresponding condition number of the polynomial eigenvalue problem.

Keywords: polynomial eigenvalue problem, Fiedler pencils, matrix polynomials, linearizations, eigenvector, minimal bases, symmetric matrix polynomials

AMS Subject Classification: 65F15, 15A18, 15A22. 65F15, 15A18, 15A22.

1. Introduction

In the present paper we are concerned with eigenvectors and minimal bases of linearizations of square matrix polynomials over the complex field $\mathbb{C}$. A square $n \times n$ matrix polynomial over $\mathbb{C}$

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i \,, \quad A_0, \ldots, A_k \in \mathbb{C}^{n\times n}, \quad A_k \neq 0 \,, \tag{1}$$

is said to be *regular* if the determinant of $P(\lambda)$ is not the identically zero polynomial. The matrix polynomial $P(\lambda)$ is *singular* otherwise. The *finite eigenvalues* and associated *eigenvectors* of a regular matrix polynomial (1) are defined as those values $\lambda_0 \in \mathbb{C}$ and nonzero vectors $v \in \mathbb{C}^n$, respectively, such that $P(\lambda_0)v = 0$. They are of relevance in several applied problems where matrix polynomials arise (see, for instance, [23] for a survey on quadratic polynomials, and [20, 21, 25] for recent examples of applications of higher degree polynomials). The problem of the computation of eigenvalues and eigenvectors of regular matrix polynomials, which is know as the Polynomial Eigenvalue Problem (PEP), has attracted the attention of many researchers in numerical linear algebra. When the matrix polynomial is singular, instead of the eigenvectors we are interested in *minimal bases*, which are particular bases of the right and left nullspaces of $P(\lambda)$ and are also relevant in applications [2, 11].

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The standard way to numerically solve the PEP for regular polynomials is through the use of *linearizations*. These are essentially matrix pencils $H(\lambda) = \lambda X + Y$, with $X, Y \in \mathbb{C}^{nk\times nk}$, sharing certain information with the polynomial $P(\lambda)$, in particular, the *invariant polynomials*, which include the eigenvalues and its associated *partial multiplicities* (see [13] for the definition of these notions). However, the eigenvectors of $H(\lambda)$ and $P(\lambda)$ are not the same, and actually they can never be the same because the sizes of $H(\lambda)$ and $P(\lambda)$ are different. Similarly, for singular matrix polynomials, minimal bases are not usually preserved by linearization. Then, the problem of relating the eigenvectors and minimal bases of $P(\lambda)$ with the ones of $H(\lambda)$ becomes essential in numerical computations.

An important issue to determine the errors in the numerical computation of eigenvalues is the *eigenvalue condition number*. The standard formula for the condition number of eigenvalues of a matrix polynomial $P(\lambda)$ involves the associated left and right eigenvectors of $P(\lambda)$ [22]. When using linearizations to compute eigenvalues of $P(\lambda)$, we have to consider the eigenvalue condition numbers corresponding to the linearization $H(\lambda)$, which are, in general, larger than the ones of the polynomial $P(\lambda)$. Actually, these condition numbers involve the eigenvectors of $H(\lambda)$, instead of the eigenvectors of $P(\lambda)$. Hence, in order to compare the condition numbers of the eigenvalues corresponding to $H(\lambda)$ with the condition numbers corresponding to $P(\lambda)$, the knowledge of the left and right eigenvectors of $H(\lambda)$ is relevant. Moreover, it would be desirable to know the relationship between these eigenvectors and the eigenvectors of $P(\lambda)$.

The classical linearizations of matrix polynomials used in practice have been the *first* and *second (Frobenius) companion forms* [13]. However, during the last decade several new families of linearizations have been introduced by different authors [1, 3, 9, 19, 24], some of them extending other known families, like the one introduced back in the 1960's in [17]. The natural subsequent step is to analyze the advantages or disadvantages of these new families and, in particular, to study their numerical features. In connection with the problems mentioned in the previous paragraphs, a natural first step for this would be:

(P1) Find recovery formulas for eigenvectors and minimal bases of $P(\lambda)$ from the ones of the linearizations.
(P2) Obtain explicit formulas for the eigenvectors and minimal bases of the linearizations in terms of the eigenvectors and minimal bases of $P(\lambda)$.

We want to stress that solving (P2) implies solving (P1), but the converse is not true.

For the families of linearizations introduced in [19], Problem (P1) has been solved in [7, 15, 19], but (P2) has been only partially solved. For the family of *Fiedler pencils*, introduced in [3] (and named later in [8]), both (P1) and (P2) have been completely solved in [8] for square matrix polynomials and in [10] for rectangular polynomials. For the family of *generalized Fiedler pencils*, also introduced in [3] (though named in [5]) (P1) has been solved in [5], but (P2) remains open. The present paper deals with problem (P2). Our main goal is to obtain formulas for the eigenvectors and minimal bases of the generalized Fiedler pencils and the *Fiedler pencils with repetition*, which is the family recently introduced in [24]. These formulas will be given in terms of the eigenvectors and minimal bases of the matrix polynomial. We will also provide a simpler expression of the formula obtained in [8] for the eigenvectors of Fiedler pencils.

The paper is organized as follows. In Section 2 we introduce basic notation and definitions, and we recall the families of linearizations that we have mentioned above. In Section 2.4 we recall the notion of eigenvectors and minimal bases of matrix polynomials. In Section 3 we present the main results of the paper, namely, formulas for the left and right eigenvectors and minimal bases of the families of Fiedler pencils, proper generalized Fiedler pencils and Fiedler pencils with repetition. We have also included a subsection where we illustrate how these formulas could be useful in the comparison of condition numbers of eigenvalues of linearizations. Section 4 is devoted to the proofs of the mains results, and in Section 5 we summarize the main contributions of the paper and we pose some open problems that appear as a natural continuation of this work. The case of non-proper generalized Fiedler pencils is addressed in Appendix A, because this is a very particular case which deserves a separate treatment. Finally, in Appendix B we obtain formulas for left and right eigenvectors associated with the infinite eigenvalue of regular polynomials. This case is also addressed in a final appendix because the techniques employed in this case have nothing to do with the main techniques of the paper, and even the formulas for this case are very specific.

2. Basic definitions

Along the paper we use the following notation: I_m will denote the $m \times m$ identity matrix. When no subindex appear in this identity, we will assume it to be n, which is the size of the matrix polynomial in (1). We also deal with block-partitioned matrices with blocks of size $n \times n$. For these matrices, we will use the following operation.

Definition 2.1: If $A = [A_{ij}]$ is a block $r \times s$ matrix consisting of block entries A_{ij} with size $n \times n$, then its *block transpose* is a block-partitioned $s \times r$ matrix $A^{\mathcal{B}}$ whose (i,j) block is $(A^{\mathcal{B}})_{ij} = A_{ji}$.

Two matrix polynomials $P(\lambda)$ and $Q(\lambda)$ are said to be *equivalent* if there are two matrix polynomials with constant nonzero determinant, $U(\lambda)$ and $V(\lambda)$ (such matrix polynomials are known as *unimodular*), such that $Q(\lambda) = U(\lambda)P(\lambda)V(\lambda)$. If $U(\lambda)$ and $V(\lambda)$ are constant matrices, then $P(\lambda)$ and $Q(\lambda)$ are said to be *strictly equivalent*.

The *reversal* of the matrix polynomial $P(\lambda)$ is the matrix polynomial obtained by reversing the order of the coefficient matrices, that is

$$\operatorname{rev} P(\lambda) := \sum_{i=0}^{k} \lambda^i A_{k-i}.$$

We use in this paper the classical notion of linearization for square $n \times n$ polynomials (see [13] and [12] for regular matrix polynomials and [7] for singular ones).

Definition 2.2: A matrix pencil $H(\lambda) = \lambda X + Y$ with $X, Y \in \mathbb{C}^{nk \times nk}$ is a *linearization* of an $n \times n$ matrix polynomial $P(\lambda)$ of degree k if there exist two unimodular $nk \times nk$ matrices $U(\lambda)$ and $V(\lambda)$ such that

$$U(\lambda)H(\lambda)V(\lambda) = \begin{bmatrix} I_{(k-1)n} & 0 \\ 0 & P(\lambda) \end{bmatrix}, \tag{2}$$

or, in other words, if $H(\lambda)$ is equivalent to $\operatorname{diag}(I_{(k-1)n}, P(\lambda))$. A linearization $H(\lambda)$ is called a *strong linearization* if $\operatorname{rev} H(\lambda)$ is also a linearization of $\operatorname{rev} P(\lambda)$.

In Section 2.3 we introduce the families of linearizations which are the subject of the present paper. They are constructed using the following $nk \times nk$ matrices, partitioned into $k \times k$ blocks of size $n \times n$. Here and hereafter, A_i denotes the ith coefficient of the matrix polynomial (1).

$$M_{-k} := \begin{bmatrix} A_k & \\ & I_{(k-1)n} \end{bmatrix}, \quad M_0 := \begin{bmatrix} I_{(k-1)n} & \\ & -A_0 \end{bmatrix}, \tag{3}$$

and

$$M_i := \begin{bmatrix} I_{(k-i-1)n} & & & \\ & -A_i & I & \\ & I & 0 & \\ & & & I_{(i-1)n} \end{bmatrix}, \quad i = 1, \ldots, k-1\,. \tag{4}$$

The M_i matrices in (4) are always invertible, and the inverses are given by

$$M_i^{-1} = \begin{bmatrix} I_{(k-i-1)n} & & & \\ & 0 & I & \\ & I & A_i & \\ & & & I_{(i-1)n} \end{bmatrix}. \tag{5}$$

However, note that M_0 and M_{-k} are invertible if and only if A_0 and A_k, respectively, are.

We will also use the notation

$$M_{-i} := M_i^{-1}, \qquad \text{for } i = 0, 1, ..., k-1, \qquad \text{and} \quad M_k := M_{-k}^{-1}.$$

The notation for M_{-k} differs from the standard one used in [3, 5, 8]. The reason for this change here is that, for all but one of the families of linearizations considered in this paper (and this last one is addressed only in Appendix A), M_{-k} will appear in the leading term of the linearization, and we follow the convention of using negative indices for the matrices in this term. We want to emphasize also that $M_{-0} := M_0^{-1}$. For this reason, we will use along this paper both 0 and -0, with different meanings.

It is easy to check the commutativity relations

$$M_i M_j = M_j M_i \quad \text{for} \quad ||i| - |j|| \neq 1\,. \tag{6}$$

For $0 \leq i \leq k$ we will make use along the paper of the polynomial

$$P_i(\lambda) = A_{k-i} + \lambda A_{k-i+1} + \cdots + \lambda^i A_k.$$

This polynomial is known as the *ith Horner shift of* $P(\lambda)$, with $P(\lambda)$ as in (1). Notice that $P_0(\lambda) = A_k$, $P_k(\lambda) = P(\lambda)$ and $\lambda P_i(\lambda) = P_{i+1}(\lambda) - A_{k-i-1}$, for $0 \leq i \leq k-1$.

2.1. *Index tuples, column standard form, and the SIP property*

In this paper we are concerned with pencils constructed from products of M_i and M_{-i} matrices. In our analysis, the order in which these matrices appear is relevant. For this reason, we will associate an index tuple with each of these products to simplify our developments. We also introduce some additional concepts defined in [24] which are related to this notion. We use boldface letters, namely $\mathbf{t}, \mathbf{q}, \mathbf{z} \ldots$, for ordered tuples of indices (or *index tuples* in the following).

Definition 2.3: Let $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ be an index tuple containing indices from $\{0, 1, \ldots, k, -0, -1, \ldots, -k\}$. We say that $\mathbf{t}$ is *simple* if $i_j \neq i_l$ for all $j, l \in \{1, 2, \ldots, r\}$ with $j \neq l$.

Definition 2.4: Let $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ be an index tuple containing indices from $\{0, 1, \ldots, k, -0, -1, \ldots, -k\}$. Then,

$$M_{\mathbf{t}} := M_{i_1} M_{i_2} \cdots M_{i_r}. \tag{7}$$

We set also $M_\emptyset := I_{nk}$.

We want to insist on the fact that 0 and -0 are different. We include -0 along this section for completeness, though the only case where it is relevant is the one addressed in Appendix A, where matrix M_{-0} appears.

Unless otherwise stated, the matrices M_i, $i = 0, \ldots, k$, and $M_{\mathbf{t}}$ refer to the matrix polynomial $P(\lambda)$ in (1). When necessary, we will explicitly indicate the dependence on a certain polynomial $Q(\lambda)$ by writing $M_i(Q)$ and $M_{\mathbf{t}}(Q)$.

Definition 2.5: Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be two index tuples containing indices from $\{0, 1, \ldots, k, -0, -1, \ldots, -k\}$. We say that $\mathbf{t}_1$ is *equivalent* to $\mathbf{t}_2$, and we will write $\mathbf{t}_1 \sim \mathbf{t}_2$, if $M_{\mathbf{t}_1} = M_{\mathbf{t}_2}$.

Notice that this is an equivalence relation and that if $M_{\mathbf{t}_2}$ can be obtained from $M_{\mathbf{t}_1}$ by the repeated application of the commutativity relations (6), then $\mathbf{t}_1$ is equivalent to $\mathbf{t}_2$.

We will refer to an index tuple consisting of consecutive integers as a *string*. We will use the notation $(q : l)$ for the string of integers from q to l, that is

$$(q : l) := \begin{cases} (q, q+1, \ldots, l), & \text{if } q \leq l \\ \emptyset, & \text{if } q > l \end{cases}.$$

Definition 2.6: Given an index tuple $\mathbf{t} = (i_1, \ldots, i_r)$, we define the *reverse* tuple of $\mathbf{t}$, denoted by $\operatorname{rev} \mathbf{t}$, as $\operatorname{rev} \mathbf{t} := (i_r, \ldots, i_1)$.

Given an index tuple $\mathbf{t} = (i_1, \ldots, i_r)$ and an integer h, we will use the following notation:

$$-\mathbf{t} := (-i_1, \ldots, -i_r), \qquad \text{and} \qquad h + \mathbf{t} := (h + i_1, \ldots, h + i_r).$$

The following two notions are basic in our developments.

Definition 2.7: [24] Let $\mathbf{t} = (i_1, i_2, \ldots, i_r)$ be an index tuple. Then $\mathbf{t}$ is said to satisfy the *Successor Infix Property (SIP)* if for every pair of indices $i_a, i_b \in \mathbf{t}$ with $1 \leq a < b \leq r$, satisfying $i_a = i_b$, there exists at least one index $i_c = i_a + 1$ such that $a < c < b$.

Definition 2.8: [24] Let h be a nonnegative integer and $\mathbf{t}$ be an index tuple containing indices from $\{0, 1, \ldots, h\}$. Then $\mathbf{t}$ is said to be in *column standard form* if

$$\mathbf{t} = (a_s : b_s, a_{s-1} : b_{s-1}, \ldots, a_2 : b_2, a_1 : b_1),$$

with $0 \leq b_1 < b_2 < \cdots < b_{s-1} < b_s \leq h$ and $0 \leq a_j \leq b_j$, for all $j = 1, \ldots, s$. Let $\mathbf{t}'$ be an index tuple containing indices from $\{-h, -h+1, \ldots, -1\}$. Then $\mathbf{t}'$ is said to be in *column standard form* if $h + \mathbf{t}'$ is in column standard form.

The connection between the column standard form and the SIP property of an index tuple is shown in the following result and the subsequent definition.

Lemma 2.9: [24] *Let* $\mathbf{t} = (i_1, \ldots, i_r)$ *be an index tuple containing indices from* $\{0, 1, \ldots, h\}$ *or from* $\{-h, -h+1, ..., -1\}$*, for some* $h \geq 1$*. Then* $\mathbf{t}$ *satisfies the SIP if and only if* $\mathbf{t}$ *is equivalent to a (unique) tuple in column standard form.*

Definition 2.10: Let $\mathbf{t} = (i_1, \ldots, i_r)$ be an index tuple containing indices from $\{0, 1, \ldots, h\}$ or from $\{-h, -h+1, ..., -1\}$, for some $h \geq 1$, and satisfying the SIP. The *column standard form of* $\mathbf{t}$ is the unique tuple in column standard form equivalent to $\mathbf{t}$. We denote this tuple by $csf(\mathbf{t})$.

Note that, in particular, if $\mathbf{t}$ is simple, then $\mathbf{t}$ satisfies the SIP and, therefore, is equivalent to a tuple in column standard form. In the more particular case of a permutation we can obtain an expression for $\mathbf{t}$ in column standard form that will be used in further developments.

Lemma 2.11: *Let* $\mathbf{t}$ *be a permutation of* $\{h_0, h_0+1, \ldots, h\}$*, with* $0 \leq h_0 \leq h$*. Then* $\mathbf{t}$ *is in column standard form if and only if*

$$\mathbf{t} = (t_{\alpha-1}+1 : h, t_{\alpha-2}+1 : t_{\alpha-1}, \ldots, t_2+1 : t_3, t_1+1 : t_2, h_0 : t_1)$$

for some positive integers $h_0 \leq t_1 < t_2 < \cdots < t_{\alpha-1} < h$*.*

Denote $t_0 = h_0 - 1$ *and* $t_\alpha = h$*. We call each sequence of consecutive integers* $(t_{i-1}+1 : t_i)$*, for* $i = 1, \ldots, \alpha$*, a* string *in* $\mathbf{t}$*.*

The proof of Lemma 2.11 is straightforward and is left to the reader. Notice that we have an analogue to Lemma 2.11 for tuples of negative integers, because, if $\mathbf{t}'$ is a permutation of $\{-q_0, -q_0+1, \ldots, -q-2, -q\}$, where $1 \leq q \leq q_0$, then $\mathbf{t}'$ is in column standard form if and only if $q_0 + \mathbf{t}'$ is in column standard form.

2.2. *Consecutions and inversions of simple index tuples*

Here we recall some definitions introduced in [8] which are key in the formulas for the eigenvectors and minimal bases.

Definition 2.12: Let $h \geq 1$ be an integer and $\mathbf{q}$ be a simple index tuple with all its elements from $\{0, 1, \ldots, h\}$ or all from $\{-h, -h+1, \ldots, -1\}$.

(a) We say that $\mathbf{q}$ has a *consecution* at j if both $j, j+1 \in \mathbf{q}$ and j is to the left of $j+1$ in $\mathbf{q}$. We say that $\mathbf{q}$ has an *inversion* at j if both $j, j+1 \in \mathbf{q}$ and j is to the right of $j+1$ in $\mathbf{q}$.

(b) We say that $\mathbf{q}$ has $\mathfrak{c}_j$ (resp. $\mathfrak{i}_j$*)* consecutions *(*resp. inversions) at j if $\mathbf{q}$ has consecutions (resp. inversions) at $j, j+1, \ldots, j+\mathfrak{c}_j-1$ (resp. at $j, j+1, \ldots, j+\mathfrak{i}_j-1$) and $\mathbf{q}$ has not a consecution (resp. inversion) at $j+\mathfrak{c}_j$ (resp. $j+\mathfrak{i}_j$).

Example 2.13 Let $\mathbf{q} = (11 : 13, 10, 6 : 9, 5, 4, 0 : 3)$. This tuple has consecutions at $0, 1, 2, 6, 7, 8, 11$ and 12. Moreover, $\mathbf{q}$ has three consecutions at 0, it has two consecutions at 1, and just one consecution at 2.

2.3. *Fiedler pencils, generalized Fiedler pencils, and Fiedler pencils with repetition*

In this section we recall the families of Fiedler pencils, generalized Fiedler (GF) pencils, and Fiedler pencils with repetition (FPR) of a given matrix polynomial, and some of their properties. The Fiedler and GF families were introduced in [3] for regular matrix polynomials (although the authors did not assign any specific name to these pencils). They were also studied, and named, in [8] and [5], respectively, for square singular polynomials. The Fiedler pencils have been addressed recently in [10] for rectangular matrix polynomials. Finally, the FPR have been introduced in [24]. It is worth to mention also that the GF pencils have been used in the construction of structured linearizations, like symmetric [3] and palindromic [9]. Quite recently, also symmetric [4] and palindromic [6] linearizations have been found within the family of FPR.

In the following definitions we make use of the matrices introduced in Definition 2.4 associated with index tuples.

Definition 2.14: (Fiedler pencils) Let $P(\lambda)$ be the matrix polynomial in (1) and let $\mathbf{q}$ be a permutation of $\{0, 1, \ldots, k-1\}$. Then the *Fiedler pencil* of $P(\lambda)$ associated with $\mathbf{q}$ is

$$F_{\mathbf{q}}(\lambda) = \lambda M_{-k} - M_{\mathbf{q}}.$$

Next we introduce GF pencils. In the following, if $\mathcal{E} = \{i_1, \ldots, i_r\}$ is a set of indices, then $-\mathcal{E}$ denotes the set $\{-i_1, \ldots, -i_r\}$.

Definition 2.15: (GF and PGF pencils). Let $P(\lambda)$ be the matrix polynomial in (1). Let $\{C_0, C_1\}$ be a partition of $\{0, 1, \ldots, k\}$ (C_0 and C_1 can be the empty set), and $\mathbf{q}$, $\mathbf{m}$ be permutations of C_0 and $-C_1$, respectively. Then the *generalized Fiedler (GF)* pencil of $P(\lambda)$ associated with $(\mathbf{m}, \mathbf{q})$ is the $nk \times nk$ pencil

$$K(\lambda) := \lambda M_{\mathbf{m}} - M_{\mathbf{q}}.$$

If $0 \in C_0$ and $k \in C_1$, then the pencil $K(\lambda)$ is said to be a *proper generalized Fiedler (PGF)* pencil of $P(\lambda)$.

If, in Definition 2.15 we admit $C_0 = \emptyset$, then $M_{\mathbf{q}} = I_{nk}$ and, if $C_1 = \emptyset$ then $M_{\mathbf{m}} = I_{nk}$.

It is obvious that any Fiedler pencil $F_{\mathbf{q}}(\lambda)$ of $P(\lambda)$ is a particular case of a GF pencil with $C_0 = \{0, 1, \ldots, k-1\}$ and $C_1 = \{k\}$. We stress that GF pencils that are not proper are defined only if A_k and/or A_0 are nonsingular.

The following result is proved in [5, Theorem 2.2]. We include it here for completeness.

Theorem 2.16: *Let $P(\lambda)$ be an $n \times n$ matrix polynomial. Then any GF pencil of $P(\lambda)$ is a strong linearization for $P(\lambda)$.*

Theorem 2.16 is true for both regular and singular polynomials $P(\lambda)$, but in this last case we recall that the only GF pencils that are defined are the PGF pencils.

Now we recall the notion of FPR, recently introduced in [24].

Definition 2.17: (FPR). Let $P(\lambda)$ be the matrix polynomial in (1), where A_0 and A_k are nonsingular matrices. Let $0 \leq h \leq k-1$, and let $\mathbf{q}$ and $\mathbf{m}$ be permutations of $\{0, 1, \ldots, h\}$ and $\{-k, -k+1, \ldots, -h-1\}$, respectively. Assume that $\mathbf{l}_q$ and $\mathbf{r}_q$ are index tuples with elements from $\{0, 1, \ldots, h-1\}$ such that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ satisfies the SIP. Similarly, let $\mathbf{l}_m$ and $\mathbf{r}_m$ be index tuples with elements from $\{-k, -k+1, \ldots, -h-2\}$ such that $(\mathbf{l}_m, \mathbf{m}, \mathbf{r}_m)$ satisfies the SIP. Then, the pencil

$$L(\lambda) = \lambda M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{m}} M_{\mathbf{r}_q} M_{\mathbf{r}_m} - M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{q}} M_{\mathbf{r}_q} M_{\mathbf{r}_m}$$

is a *Fiedler pencil with repetition (FPR)* associated with $P(\lambda)$.

Remark 1: The constraint A_0 and A_k being nonsingular can be relaxed. We need A_0 to be nonsingular only if 0 is an index in $\mathbf{l}_q$, or $\mathbf{r}_q$, or both. Similarly with A_k and the index $-k$ in $\mathbf{l}_m$ and $\mathbf{r}_m$.

Notice that if $\mathbf{l}_q$, $\mathbf{r}_q$, $\mathbf{l}_m$, and $\mathbf{r}_m$ are all the empty index tuple in Definition 2.17, then $L(\lambda)$ is a GF pencil (actually, a PGF pencil). Note also that not all GF pencils are FPR.

We have the analogue of Theorem 2.16 for FPR.

Theorem 2.18: [24] *Let $P(\lambda)$ be an $n \times n$ matrix polynomial. Then every FPR of $P(\lambda)$ is a strong linearization of $P(\lambda)$.*

The requirement that $(\mathbf{l}_q, \mathbf{q}, \mathbf{r}_q)$ and $(\mathbf{l}_m, \mathbf{m}, \mathbf{r}_m)$ satisfy the SIP in Definition 2.17 is introduced in order to keep the product of the M_i matrices defining $L(\lambda)$ *operation free* [24]. As a consequence, the coefficients of $L(\lambda)$ are block-partitioned matrices, whose $n \times n$ blocks are of the form $0, \pm I$, or $\pm A_i$ (that is, no products of A_i blocks appear). This requirement imposes some constraints on the indices of $\mathbf{l}_q$, $\mathbf{r}_q$, $\mathbf{l}_m$ and $\mathbf{r}_m$ that we analyze next. In particular, we focus on $\mathbf{r}_q$ and $\mathbf{r}_m$ because they are the only relevant tuples in the construction of the right eigenvectors and minimal bases (as we will see in Section 4.3).

Lemma 2.19: *Let h be a nonnegative integer and $\mathbf{q}$ be a permutation of $\{0,1,\ldots,h\}$ in column standard form. Let $\mathbf{r}_q = (s_1,...,s_r)$ be such that $(\mathbf{q},\mathbf{r}_q)$ satisfies the SIP, where s_i is the ith index of $\mathbf{r}_q$. Then, for each $i = 1,..,r$, there exists a string $(a : b)$ in $csf(\mathbf{q}, s_1,...,s_{i-1})$ such that $a \le s_i < b$.*

Proof: Let $1 \le i \le r$. Since $(\mathbf{q}, s_1,...,s_{i-1})$ satisfies the SIP, by Lemma 2.9, it is equivalent to a tuple in column standard form. On the other hand, we have $(\mathbf{q}, s_1,....,s_i) \sim (csf(\mathbf{q}, s_1,\ldots,s_{i-1}), s_i)$. Now, notice that $csf(\mathbf{q}, s_1,...,s_{i-1})$ contains all indices in $\{0,1,...,h\}$ and, in particular, s_i. The result follows from the fact that $(\mathbf{q},\mathbf{r}_q)$ satisfies the SIP. □

Lemma 2.19 motivates the following definition.

Definition 2.20: (Type 1 indices relative to a simple index tuple). Let h be a nonnegative integer and $\mathbf{q}$ be a permutation of $\{0,1,\ldots,h\}$. Let s be an index in $\{0,1,...,h-1\}$. Then s is said to be a *right index of type 1 relative to* $\mathbf{q}$ if there is a string $(t_{d-1}+1 : t_d)$ in $csf(\mathbf{q})$ such that $s = t_{d-1}+1 < t_d$.

We have the analogues of Lemma 2.19 and Definition 2.20 for tuples of negative integers. They follow directly from the fact that, if $\mathbf{q}'$ is a permutation of $\{-h,-h+1,\ldots,-1\}$, then $\mathbf{q}'$ is in column standard form if and only if $h+\mathbf{q}'$ is in column standard form.

The following definition allows us to associate a simple tuple to the tuple obtained by adding a type 1 index to a given permutation.

Definition 2.21: (Associated simple tuple) Let h be a nonnegative integer and $\mathbf{q}$ be a permutation of $\{0,1,\ldots,h\}$. Let $csf(\mathbf{q}) = (\mathbf{b}_{\alpha+1}, \mathbf{b}_\alpha, \ldots, \mathbf{b}_1)$, where $\mathbf{b}_i = (t_{i-1}+1 : t_i)$, $i = 1,\ldots,\alpha+1$, are the strings of $csf(\mathbf{q})$. We say that the *simple tuple associated with* $\mathbf{q}$ is $csf(\mathbf{q})$ and denote it by $\mathfrak{s}(\mathbf{q})$. If s is an index of type 1 with respect to $\mathbf{q}$, say $s = t_{d-1}+1 < t_d$, then the *simple tuple associated with* $(\mathbf{q}, s)$ is the simple tuple:

- $\mathfrak{s}(\mathbf{q}, s) := \left(\mathbf{b}_{\alpha+1}, \mathbf{b}_\alpha, \ldots, \mathbf{b}_{d+1}, \widetilde{\mathbf{b}}_d, \widetilde{\mathbf{b}}_{d-1}, \mathbf{b}_{d-2}, \ldots, \mathbf{b}_1\right)$, where

$$\widetilde{\mathbf{b}}_d = (t_{d-1}+2 : t_d) \qquad \text{and} \qquad \widetilde{\mathbf{b}}_{d-1} = (t_{d-2}+1 : t_{d-1}+1)$$

  if $s \neq 0$.
- $\mathfrak{s}(\mathbf{q}, 0) := \left(\mathbf{b}_{\alpha+1}, \mathbf{b}_\alpha, \ldots, \widetilde{\mathbf{b}}_1, \widetilde{\mathbf{b}}_0\right)$, where

$$\widetilde{\mathbf{b}}_1 = (1 : t_1) \qquad \text{and} \qquad \widetilde{\mathbf{b}}_0 = (0).$$

Definition 2.21 can be extended to the case where we adjoin tuples containing more than one index. This is done in Definition 2.22, which is key in Theorem 3.6.

Definition 2.22: (Index tuple of type 1) Let h be a nonnegative integer, $\mathbf{q}$ be a permutation of $\{0,1,\ldots,h\}$, and $\mathbf{r}_q$ and $\mathbf{l}_q$ be tuples with indices from $\{0,1,...,h-1\}$, possibly with repetitions. We say that $\mathbf{r}_q = (s_1,...,s_r)$, where s_i is the ith index of $\mathbf{r}_q$, is an *index tuple of type 1* relative to $\mathbf{q}$ if, for $i = 1,...,r$, s_i is a right index of type 1 with respect to $\mathfrak{s}(\mathbf{q}, (s_1,...,s_{i-1}))$, where $\mathfrak{s}(\mathbf{q}, (s_1,...,s_{i-1})) := \mathfrak{s}(\mathfrak{s}(\mathbf{q}, (s_1,...,s_{i-2})), s_{i-1})$ for $i > 2$.

2.4. *Eigenvalues and eigenvectors, minimal indices and minimal bases.*

The right and left *eigenspaces* of an $n \times n$ regular matrix polynomial $P(\lambda)$ at $\lambda_0 \in \mathbb{C}$ are the right and left null spaces of $P(\lambda_0)$, i.e.,

$$\mathcal{N}_r(P(\lambda_0)) := \{x \in \mathbb{C}^n \,:\, P(\lambda_0)x = 0\} \,,$$
$$\mathcal{N}_\ell(P(\lambda_0)) := \left\{y \in \mathbb{C}^n \,:\, P(\lambda_0)^T y = 0\right\} \,.$$

If $P(\lambda)$ is a regular matrix polynomial and $\mathcal{N}_r(P(\lambda_0))$ (or, equivalently, $\mathcal{N}_\ell(P(\lambda_0))$) is nontrivial, then λ_0 is said to be a (*finite*) *eigenvalue*, and a vector $x \neq 0$ (respectively, $y \neq 0$) in $\mathcal{N}_r(P(\lambda_0))$ (resp. $\mathcal{N}_\ell(P(\lambda_0))$) is a *right* (resp. *left*) *eigenvector of P associated with* λ_0. Matrix polynomials may also have infinite eigenvalues. In this work we will focus on finite eigenvalues. Infinite eigenvalues are considered only in Appendix B, because the techniques used for this case are completely different (though simpler) than the ones employed for finite eigenvalues.

In the case of $P(\lambda)$ being a square singular $n\times n$ matrix polynomial, the previous notion of eigenvalue (and eigenvector) makes no sense, because with this definition all complex values would be eigenvalues of $P(\lambda)$. In this case we are interested in minimal bases of $P(\lambda)$ instead of eigenvectors. This notion is related to the *right* and *left nullspaces* of $P(\lambda)$, which are, respectively, the following subspaces of $\mathbb{C}(\lambda)^n$,

$$\mathcal{N}_r(P) := \{x(\lambda) \in \mathbb{C}(\lambda)^n \,:\, P(\lambda)x(\lambda) \equiv 0\} \,,$$
$$\mathcal{N}_\ell(P) := \left\{y(\lambda) \in \mathbb{C}(\lambda)^n \,:\, P(\lambda)^T y(\lambda) \equiv 0\right\} \,,$$

where $\mathbb{C}(\lambda)^n$ is the vector space of dimension n with coordinates in the the field $\mathbb{C}(\lambda)$ of rational functions in λ with complex coefficients. A *polynomial basis* of a vector space over $\mathbb{C}(\lambda)$ is a basis consisting of polynomial vectors (that is, vectors whose coordinates are polynomials in λ). The *order* of a polynomial basis is the sum of the degrees of its vectors. Here the *degree* of a polynomial vector is the maximum degree of its components. A *right* (respectively, *left*) minimal basis of $P(\lambda)$ is a polynomial basis of $\mathcal{N}_r(P)$ (resp. $\mathcal{N}_\ell(P)$) such that the order is minimal among all polynomial bases of $\mathcal{N}_r(P)$ (resp. $\mathcal{N}_\ell(P)$) [11].

Eigenvectors and minimal bases are the central object of this paper, as we see in Section 3.

In the following, when referring to eigenvectors of matrix polynomials (or their linearizations), we will assume that the polynomial is regular, and when referring to minimal bases, we assume it to be singular.

3. Main results

By theorems 2.16 and 2.18, all pencils within the families considered in Section 2.3 are (strong) linearizations. Our goal is to derive formulas for the left and right eigenvectors and the left and right minimal bases of these linearizations. In particular, we want to relate the left and right eigenvectors and the left and right minimal bases of these linearizations with the ones of the polynomial $P(\lambda)$. Lemma 5.3 in [8] shows how to do this for Fiedler pencils. By using suitable strict equivalence relations between GF, FPR and appropriate Fiedler pencils, we obtain formulas for GF pencils and FPR associated with type 1 tuples as well. These formulas are given in sections 3.1, 3.2 and 3.3. As we will see, the presence of an identity block within these formulas allow us to reverse the process and recover the eigenvectors and minimal bases of $P(\lambda)$ from the eigenvectors and minimal bases of the linearizations, as it was done in [5] for the GF pencils, and in [8] for Fiedler pencils. The proofs of all these formulas are addressed in Section 4.

From now on, when considering an ordered tuple $\mathbf{z}$ with ℓ entries, we will follow the convention of assigning the position 0 to the first entry in the tuple. Also, for each $0 \leq i \leq \ell$, $\mathbf{z}(i)$ will denote the number occupying the ith position in $\mathbf{z}$ and, for each $j \in \mathbf{z}$, $\mathbf{z}^{-1}(j)$ denotes the position of j in $\mathbf{z}$ (starting with 0). In other words, we see an index tuple $\mathbf{z}$ with ℓ elements, $j_1, \ldots, j_\ell$, as a bijection $\mathbf{z} : \{0, 1, \ldots, \ell - 1\} \to \{j_1, \ldots, j_\ell\}$. We will also associate tuples of blocks to tuples of numbers. Then, according to the previous convention, when referring to “the position of a block" we understand that we start counting in 0 (the 0th position)

3.1. *Eigenvectors and minimal bases of Fiedler pencils*

The following theorem is a restatement of Lemma 5.3 in [8].

Theorem 3.1 : *Let $P(\lambda)$ be an $n \times n$ matrix polynomial of degree k, P_i be its ith Horner shift, for $i = 0, \ldots, k$, and $\mathbf{q}$ be a permutation of $\{0, 1, \ldots, k-1\}$ with $csf(\mathbf{q}) = (\mathbf{b}_\alpha, \mathbf{b}_{\alpha-1}, \ldots, \mathbf{b}_1)$, where $\mathbf{b}_j = (t_{j-1} + 1 : t_j)$, for $j = 1, \ldots, \alpha$. Let $F_\mathbf{q}(\lambda) = \lambda M_{-k} - M_\mathbf{q}$ be the Fiedler pencil of $P(\lambda)$ associated with $\mathbf{q}$. Let*

$$\mathcal{R}_\mathbf{q}(P, \lambda) := \begin{bmatrix} B_0 \; B_1 \; \ldots \; B_{k-1} \end{bmatrix}^\mathcal{B}, \tag{8}$$

where, if $\mathbf{q}(i) \in \mathbf{b}_j$, for some $j = 1, \ldots, \alpha$, then

$$B_i = \begin{cases} \lambda^{j-1} I \,, & \text{if } \; i = k - t_j - 1, \\ \lambda^{j-1} P_i \,, & \text{otherwise.} \end{cases} \tag{9}$$

Let $\mathcal{L}_\mathbf{q}(P, \lambda) := \mathcal{R}_{\mathrm{rev}\,\mathbf{q}}(P^T, \lambda)$. Then

(a) $\{v_1(\lambda), \ldots, v_p(\lambda)\}$ is a right minimal basis of $P(\lambda)$ if and only if $\{\mathcal{R}_\mathbf{q}(P, \lambda) v_1(\lambda), \ldots, \mathcal{R}_\mathbf{q}(P, \lambda) v_p(\lambda)\}$ is a right minimal basis of $F_\mathbf{q}(\lambda)$.
(b) v is a right eigenvector of $F_\mathbf{q}(\lambda)$ associated with the eigenvalue λ_0 if and only if $v = \mathcal{R}_\mathbf{q}(P, \lambda_0)x$, where x is a right eigenvector of $P(\lambda)$ associated with λ_0.
(c) $\{w_1(\lambda), \ldots, w_p(\lambda)\}$ is a left minimal basis of $P(\lambda)$ if and only if $\{\mathcal{L}_\mathbf{q}(P, \lambda) w_1(\lambda), \ldots, \mathcal{L}_\mathbf{q}(P, \lambda) w_p(\lambda)\}$ is a left minimal basis of $F_\mathbf{q}(\lambda)$.
(d) w is a left eigenvector of $F_\mathbf{q}(\lambda)$ associated with the eigenvalue λ_0 if and only if $w = \mathcal{L}_\mathbf{q}(P, \lambda_0)y$, where y is a left eigenvector of $P(\lambda)$ associated with λ_0.

Moreover, if $\mathbf{q}$ has $\mathfrak{c}_0$ consecutions at 0, then the $(k - \mathfrak{c}_0)$th block of $\mathcal{R}_\mathbf{q}(P, \lambda)$ is equal to I_n, and if $\mathbf{q}$ has $\mathfrak{i}_0$ inversions at 0, then the $(k - \mathfrak{i}_0)$th block of $\mathcal{L}_\mathbf{q}(P, \lambda)$ is equal to I_n.

Remark 1 : We want to stress that $k - t_j - 1$ in (9) is the position in $csf(\mathbf{q})$, (counting from left to right and starting with 0) of the smallest index in $\mathbf{b}_j$ (that is, $\mathbf{q}^{-1}(t_{j-1} + 1) = k - t_j - 1$). Thus, we may see $\mathcal{R}_\mathbf{q}(P, \lambda)$ as partitioned into α strings of blocks, each one corresponding to a string $\mathbf{b}_j$ in $csf(\mathbf{q})$. More precisely, the string in $\mathcal{R}_\mathbf{q}(P, \lambda)$ associated with $\mathbf{b}_j$ is of the form $\lambda^{j-1} \begin{bmatrix} I \; P_{\mathbf{q}^{-1}(t_{j-1}+2)} \; \ldots \; P_{\mathbf{q}^{-1}(t_j)} \end{bmatrix}^\mathcal{B}$. Hence, $\mathcal{R}_\mathbf{q}(P, \lambda)$ can be easily obtained from $csf(\mathbf{q})$.

Remark 2 : There is a duality between the formulas for $\mathcal{R}_\mathbf{q}$ and $\mathcal{L}_\mathbf{q}$ given in Theorem 3.1. More precisely, if the ith block, B_i, of $\mathcal{R}_\mathbf{q}$ in (8), with $i \neq 0$, is of the form $\lambda^{j-1} P_i$, then the ith block, B'_i, of $\mathcal{L}_\mathbf{q}$ is $\lambda^{k-(j+i)} I$ and, similarly, if the ith block of $\mathcal{L}_\mathbf{q}$ is $\lambda^{j-1} P_i^T$, with $i \neq 0$, then the ith block of $\mathcal{R}_\mathbf{q}$ is $\lambda^{k-(j+i)} I$. Notice, finally, that $B_0 = \lambda^{\alpha-1} I$ and $B'_0 = \lambda^{\beta-1} I$, with $\alpha + \beta = k + 1$.

Example 3.2 Let $k = 13$ and $\mathbf{q} = (10 : 12, 9, 8, 6 : 7, 5, 2 : 4, 0 : 1)$. Note that $\mathbf{q}$ contains seven strings. Each string induces a string of blocks in $\mathcal{R}_\mathbf{q}$ corresponding to

$F_{\mathbf{q}}(\lambda) = \lambda M_{-k} - M_{\mathbf{q}}$. The first entries of these strings correspond to the positions 0, 3, 4, 5, 7, 8 and 11, respectively. Then $\mathcal{R}_{\mathbf{q}}$ is

$$\mathcal{R}_{\mathbf{z}} = \left[\lambda^6 I \ \lambda^6 P_1 \ \lambda^6 P_2 | \lambda^5 I | \lambda^4 I | \lambda^3 I \ \lambda^3 P_6 | \lambda^2 I | \lambda I \ \lambda P_9 \ \lambda P_{10} | I \ P_{12}\right]^{\mathcal{B}}.$$

For the left eigenvectors and minimal bases, we have $csf(\operatorname{rev}\mathbf{q}) = (12, 11, 7 : 10, 4 : 6, 3, 1 : 2, 0)$, so

$$\mathcal{L}_{\mathbf{q}} = \left[\lambda^6 I | \lambda^5 I | \lambda^4 I \ \lambda^4 P_3^T \ \lambda^4 P_4^T \ \lambda^4 P_5^T | \lambda^3 I \ \lambda^3 P_7^T \ \lambda^3 P_8^T | \lambda^2 I | \lambda I \ \lambda P_{11}^T | I\right]^{\mathcal{B}}.$$

3.2. *Eigenvectors and minimal bases of GF pencils*

In this section we present an explicit relationship between left and right eigenvectors and minimal bases of GF pencils and left and right eigenvectors and minimal bases of $P(\lambda)$. Here we only address the case of PGF pencils and we postpone to Appendix A the case of non-proper GF pencils since these pencils do not seem to be relevant in applications (except in the particular case of the symmetric linearizations of even-degree regular matrix polynomials in [3]) and the study of eigenvectors and minimal bases in this case requires techniques other than those used in the PGF case. It should be remarked that index tuples $\mathbf{q}$ and $\mathbf{m}$ in Definition 2.15 are both permutations and, so, they are equivalent to tuples in column standard form.

Theorem 3.3 : *Let $P(\lambda)$ be an $n \times n$ matrix polynomial with degree k and let $K(\lambda) = \lambda M_{\mathbf{m}} - M_{\mathbf{q}}$ be a PGF pencil of $P(\lambda)$. Let P_i, for $i = 0, 1, \ldots, k$, be the ith Horner shift of P. Assume that $\mathbf{m}$ has $\mathfrak{c}_{-k}$ consecutions at $-k$, and $csf(\mathbf{m}) = (\mathbf{m}_1, -k : -k + \mathfrak{c}_{-k})$. Set $\mathbf{z} := csf(-\operatorname{rev}\mathbf{m}_1, \mathbf{q}) = (\mathbf{b}_\alpha, \mathbf{b}_{\alpha-1}, \ldots, \mathbf{b}_1)$, and let $\mathcal{R}_K(P, \lambda)$ be the following $nk \times n$ matrix polynomial:*

(i) If $\mathfrak{c}_{-k} = 0$, then $\mathcal{R}_K(P, \lambda) := \mathcal{R}_{\mathbf{z}}(P, \lambda)$, with $\mathcal{R}_{\mathbf{z}}(P, \lambda)$ as in (8).

(ii) If $\mathfrak{c}_{-k} > 0$, then

$$\mathcal{R}_K(P, \lambda) := \left[\lambda^\alpha [P_0 \ P_1 \ \ldots \ P_{\mathfrak{c}_{-k}-1}] | B_{\mathfrak{c}_{-k}} \ B_{\mathfrak{c}_{-k}+1} \ \ldots \ B_{k-1}\right]^{\mathcal{B}}, \qquad (10)$$

where, if $\mathbf{z}(i) \in \mathbf{b}_j$, for some $j = 1, 2, \ldots, \alpha$, then the block $B_{i+\mathfrak{c}_{-k}}$ is as in (9).

Finally, set $\mathcal{L}_K(P, \lambda) := \mathcal{R}_{K^\sharp}(P^T, \lambda)$, where $K^\sharp(\lambda) = \lambda M_{\operatorname{rev}\mathbf{m}}(P^T) - M_{\operatorname{rev}\mathbf{q}}(P^T)$. Then:

(a) $\{v_1(\lambda), \ldots, v_p(\lambda)\}$ is a right minimal basis of $P(\lambda)$ if and only if $\{\mathcal{R}_K(P, \lambda)v_1(\lambda), \ldots, \mathcal{R}_K(P, \lambda)v_p(\lambda)\}$ is a right minimal basis of $K(\lambda)$.

(b) v is a right eigenvector of $K(\lambda)$ associated with the eigenvalue λ_0 if and only if $v = \mathcal{R}_K(P, \lambda_0)x$, where x is a right eigenvector of $P(\lambda)$ associated with λ_0.

(c) $\{w_1(\lambda), \ldots, w_p(\lambda)\}$ is a left minimal basis of $P(\lambda)$ if and only if $\{\mathcal{L}_K(P, \lambda)w_1(\lambda), \ldots, \mathcal{L}_K(P, \lambda)w_p(\lambda)\}$ is a left minimal basis of $K(\lambda)$.

(d) w is a left eigenvector of $K(\lambda)$ associated with the eigenvalue λ_0 if and only if $w = \mathcal{L}_K(P, \lambda_0)y$, where y is a left eigenvector of $P(\lambda)$ associated with λ_0.

Moreover, if $\mathbf{q}$ has $\mathfrak{c}_0$ consecutions at 0, then the $(k - \mathfrak{c}_0)$th block of $\mathcal{R}_K(P, \lambda)$ is equal to I_n, and if $\mathbf{q}$ has $\mathfrak{i}_0$ inversions at 0, then the $(k - \mathfrak{i}_0)$th block of $\mathcal{L}_K(P, \lambda)$ is equal to I_n.

Remark 3 : Notice that the B_i blocks in (10) follow the same rule as in (9). More precisely, the ith block B_i is of the form $\lambda^{j-1}I$ if $\mathbf{z}(i - \mathfrak{c}_{-k})$ is the first element in $\mathbf{b}_j$, and it is of the form $\lambda^{j-1}P_i$ if $\mathbf{z}(i - \mathfrak{c}_{-k}) \in \mathbf{b}_j$ but is not the first element of $\mathbf{b}_j$.

In the following, for simplicity and when there is no risk of confusion, we will drop the dependence on P and λ in $\mathcal{R}_K(P,\lambda)$ and $\mathcal{L}_K(P,\lambda)$.

Example 3.4 Let $k = 12$, $\mathbf{m} = (-4 : -3, -6, -12 : -10)$ and $\mathbf{q} = (7 : 9, 5, 0 : 2)$. Then, $\mathfrak{c}_{-k} = 2$. Note that $\mathbf{z} = csf(-\mathrm{rev}\,\mathbf{m}_1, \mathbf{q}) = (6 : 9, 3 : 5, 0 : 2)$, so $\alpha = 3$. Also, $csf(\mathrm{rev}\,\mathbf{m}) = (-3, -4, -6, -10, -11, -12) = (\mathbf{m}_1', -12)$, and $csf(\mathrm{rev}\,\mathbf{q}) = (9, 8, 7, 5, 2, 1, 0)$. Then, $\mathbf{z}' = csf(-\mathrm{rev}\,\mathbf{m}_1', \mathrm{rev}\,\mathbf{q}) = (11, 10, 9, 8, 6 : 7, 4 : 5, 3, 2, 1, 0)$, so $\alpha = 10$ in this case. If $K(\lambda) = \lambda M_{\mathbf{m}} - M_{\mathbf{q}}$, Theorem 3.3 gives

$$\mathcal{R}_K = \left[\, \lambda^3 P_0 \ \lambda^3 P_1 \middle| \lambda^2 I \ \lambda^2 P_3 \ \lambda^2 P_4 \ \lambda^2 P_5 \middle| \lambda I \ \lambda P_7 \ \lambda P_8 \middle| I \ P_{10} \ P_{11} \,\right]^{\mathcal{B}},$$

and

$$\mathcal{L}_K = \left[\, \lambda^9 I \middle| \lambda^8 I \middle| \lambda^7 I \middle| \lambda^6 I \middle| \lambda^5 I \ \lambda^5 P_5^T \middle| \lambda^4 I \ \lambda^4 P_7^T \middle| \lambda^3 I \middle| \lambda^2 I \middle| \lambda I \middle| I \,\right]^{\mathcal{B}}.$$

Example 3.5 Let $k = 12$, $\mathbf{m} = (-12 : -8)$, and $\mathbf{q} = (6 : 7, 5, 4, 0 : 3)$. In this case, $\mathfrak{c}_{-k} = 4$, $-\mathbf{m}_1$ is the empty tuple, and $\mathbf{z} = \mathbf{q}$. Therefore, $\alpha = 4$. Similarly, $\mathrm{rev}\,\mathbf{m} = (-8, -9, -10, -11, -12) = (\mathbf{m}_1', -12)$, which is already in column standard form, $\mathrm{rev}\,\mathbf{q} = (3, 2, 1, 0, 4 : 5, 7, 6)$, so $\mathbf{z}' = csf(-\mathrm{rev}\,\mathbf{m}_1', \mathrm{rev}\,\mathbf{q}) = (11, 10, 9, 8, 7, 3 : 6, 2, 1, 0)$, and $\alpha = 9$ in this case. Then, if $K(\lambda) = \lambda M_{\mathbf{m}} - M_{\mathbf{q}}$, Theorem 3.3 gives

$$\mathcal{R}_K = \left[\, \lambda^4 P_0 \ \lambda^4 P_1 \ \lambda^4 P_2 \ \lambda^4 P_3 \middle| \lambda^3 I \ \lambda^3 P_5 \middle| \lambda^2 I \middle| \lambda I \middle| I \ P_9 \ P_{10} \ P_{11} \,\right]^{\mathcal{B}},$$

and

$$\mathcal{L}_K = \left[\, \lambda^8 I \middle| \lambda^7 I \middle| \lambda^6 I \middle| \lambda^5 I \middle| \lambda^4 I \middle| \lambda^3 I \ \lambda^3 P_6^T \ \lambda^3 P_7^T \ \lambda^3 P_8^T \middle| \lambda^2 I \middle| \lambda I \middle| I \,\right]^{\mathcal{B}}.$$

3.3. *Eigenvectors and minimal bases of FPR*

We provide in this section formulas for the right (respectively, left) eigenvectors and minimal bases of FPR with $\mathbf{r}_m$ and $\mathbf{r}_q$ (resp. $\mathrm{rev}\,\mathbf{l}_m$ and $\mathrm{rev}\,\mathbf{l}_q$) in Definition 2.17 being type 1 tuples relative to $\mathbf{m}$ and $\mathbf{q}$ (resp. $\mathrm{rev}\,\mathbf{m}$ and $\mathrm{rev}\,\mathbf{q}$). This case seems to be the most relevant for applications. For example, all symmetric and palindromic families of linearizations considered in [6, 24] correspond to this case. However, there are examples of symmetric FPR linearizations in which the previous tuples are not of type 1 [4].

The families of symmetric linearizations in [24] are addressed in Section 4.3.1. To derive appropriate formulas for the eigenvectors and minimal bases of FPR when the tuples are not of type 1 seems to be quite involved and remains an open problem.

Theorem 3.6: *Let $P(\lambda)$ be a matrix polynomial of degree k and let $L(\lambda) = \lambda M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{m}} M_{\mathbf{r}_q} M_{\mathbf{r}_m} - M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{q}} M_{\mathbf{r}_q} M_{\mathbf{r}_m}$ be a FPR.*

(a) Assume that $\mathbf{r}_m$ and $\mathbf{r}_q$ are type 1 tuples relative to $\mathbf{m}$ and $\mathbf{q}$, respectively. Let $\mathfrak{s}(\mathbf{q}, \mathbf{r}_q)$ and $\mathfrak{s}(\mathbf{m}, \mathbf{r}_m)$ be the simple tuple associated with $(\mathbf{q}, \mathbf{r}_q)$ and $(\mathbf{m}, \mathbf{r}_m)$, respectively. Set $\mathcal{R}_L(P,\lambda) := \mathcal{R}_{\widetilde{K}}(P,\lambda)$, where $\widetilde{K}(\lambda) = \lambda M_{\mathfrak{s}(\mathbf{m},\mathbf{r}_m)} - M_{\mathfrak{s}(\mathbf{q},\mathbf{r}_q)}$ is a GF pencil. Then

(a1) $\{v_1(\lambda), \ldots, v_p(\lambda)\}$ is a right minimal basis of $P(\lambda)$ if and only if $\{\mathcal{R}_L(P,\lambda)v_1(\lambda), \ldots, \mathcal{R}_L(P,\lambda)v_p(\lambda)\}$ is a right minimal basis of $L(\lambda)$.

(a2) v is a right eigenvector of $L(\lambda)$ associated with the eigenvalue λ_0 if and only if $v = \mathcal{R}_L(P,\lambda_0)x$, where x is a right eigenvector of $P(\lambda)$ associated with λ_0.

Moreover, if $\mathfrak{s}(\mathbf{q}, \mathbf{r}_q)$ has $\widetilde{\mathfrak{c}}_0$ consecutions at 0, then the$(k - \widetilde{\mathfrak{c}}_0)$th block of $\mathcal{R}_L$ is equal to I_n.

(b) Assume that $\mathrm{rev}\,\mathbf{l}_m$ and $\mathrm{rev}\,\mathbf{l}_q$ are type 1 tuples relative to $\mathrm{rev}\,\mathbf{m}$ and $\mathrm{rev}\,\mathbf{q}$, respectively. Let $\mathfrak{s}(\mathrm{rev}\,\mathbf{q}, \mathrm{rev}\,\mathbf{l}_q)$ and $\mathfrak{s}(\mathrm{rev}\,\mathbf{m}, \mathrm{rev}\,\mathbf{l}_m)$ be the simple tuple associated with

$(\text{rev}\,\mathbf{q}, \text{rev}\,\mathbf{l}_q)$ *and* $(\text{rev}\,\mathbf{m}, \text{rev}\,\mathbf{l}_m)$*, respectively. Set* $\mathcal{L}_L(P,\lambda) := \mathcal{R}_{\widehat{K}}(P,\lambda)$*, where* $\widehat{K}(\lambda) = \lambda M_{\mathfrak{s}(\text{rev}\,\mathbf{m},\text{rev}\,\mathbf{l}_m)}(P^T) - M_{\mathfrak{s}(\text{rev}\,\mathbf{q},\text{rev}\,\mathbf{l}_q)}(P^T)$ *is a GF pencil. Then*

(b1) $\{w_1(\lambda), \dots, w_p(\lambda)\}$ *is a left minimal basis of* $P(\lambda)$ *if and only if* $\{\mathcal{L}_L(P,\lambda)w_1(\lambda), \dots, \mathcal{L}_L(P,\lambda)w_p(\lambda)\}$ *is a left minimal basis of* $L(\lambda)$*.*

(b2) w *is a left eigenvector of* $L(\lambda)$ *associated with the eigenvalue* λ_0 *if and only if* $w = \mathcal{L}_L(P,\lambda_0)y$*, where* y *is a left eigenvector of* $P(\lambda)$ *associated with* λ_0*.*

Moreover, if $\mathfrak{s}(\text{rev}\,\mathbf{q}, \text{rev}\,\mathbf{r}_q)$ *has* $\widehat{\mathfrak{c}}_0$ *consecutions at* 0*, then the*$(k-\widehat{\mathfrak{c}}_0)$*th block of* $\mathcal{L}_L$ *is equal to* I_n*.*

Example 3.7 Let $L(\lambda) = \lambda M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{m}} M_{\mathbf{r}_q} M_{\mathbf{r}_m} - M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{q}} M_{\mathbf{r}_q} M_{\mathbf{r}_m}$ be the FPR associated with a matrix polynomial of degree $k = 12$, with $\mathbf{q} = (6, 1:5, 0)$, $\mathbf{r}_q = (1:4)$, $\mathbf{m} = (-7, -8, -12:-9)$, $\mathbf{r}_m = (-12:-10, -12:-11)$, $\mathbf{l}_q = (0)$, $\mathbf{l}_m = (-8, -9)$. Then, $(\mathbf{q}, \mathbf{r}_q) = (6, 1:5, 0:4)$ and $\mathfrak{s}(\mathbf{q}, \mathbf{r}_q) = (6, 5, 0:4)$. Similarly, $(\mathbf{m}, \mathbf{r}_m) = (-7, -8, -12:-9, -12:-10, -12:-11)$ and $\mathfrak{s}(\mathbf{m}, \mathbf{r}_m) = (-7, -8, -9, -10, -12:-11)$, so $\widetilde{\mathfrak{c}}_{-k} = 1$. Also, $(\text{rev}\,\mathbf{q}, \text{rev}\,\mathbf{l}_q) \sim (5:6, 4, 3, 2, 0:1, 0)$, $\mathfrak{s}(\text{rev}\,\mathbf{q}, \text{rev}\,\mathbf{l}_q) = (5:6, 4, 3, 2, 1, 0)$, $(\text{rev}\,\mathbf{m}, \text{rev}\,\mathbf{l}_m) \sim (-9:-7, -10, -11, -12, -9:-8)$, and $\mathfrak{s}(\text{rev}\,\mathbf{m}, \text{rev}\,\mathbf{l}_m) = (-7, -10:-8, -11, -12)$, so $\widehat{\mathfrak{c}}_{-k} = 0$. Let $\widetilde{K}(\lambda) = \lambda M_{\mathfrak{s}(\mathbf{m},\mathbf{r}_m)} - M_{\mathfrak{s}(\mathbf{q},\mathbf{r}_q)}$, and $\widehat{K}(\lambda) = \lambda M_{\mathfrak{s}(\text{rev}\,\mathbf{m},\text{rev}\,\mathbf{l}_m)} - M_{\mathfrak{s}(\text{rev}\,\mathbf{q},\text{rev}\,\mathbf{l}_q)}$. Following the notation in the statement of Theorem 3.3, we have $\widetilde{\mathbf{m}}_1 = (-7, -8, -9, -10)$ and then $\widetilde{\mathbf{z}} = (10, 9, 8, 7, 6, 5, 0:4)$. Similarly, $\widehat{\mathbf{m}}_1 = (-7, 10:-8, -11)$ and $\widehat{\mathbf{z}} = (11, 8:10, 7, 5:6, 4, 3, 2, 1, 0)$. Hence

$$\mathcal{R}_L = \left[\, \lambda^7 P_0 \middle| \lambda^6 I \middle| \lambda^5 I \middle| \lambda^4 I \middle| \lambda^3 I \middle| \lambda^2 I \middle| \lambda I \middle| I \;\; P_8 \;\; P_9 \;\; P_{10} \;\; P_{11} \,\right]$$

and

$$\mathcal{L}_L = \left[\, \lambda^8 I \middle| \lambda^7 I \;\; \lambda^7 P_2^T \;\; \lambda^7 P_3^T \middle| \lambda^6 I \middle| \lambda^5 I \;\; \lambda^5 P_6^T \middle| \lambda^4 I \middle| \lambda^3 I \middle| \lambda^2 I \middle| \lambda I \middle| I \,\right].$$

3.4. *Application: conditioning of eigenvalues*

Although all linearizations of a given matrix polynomial $P(\lambda)$ have the same eigenvalues as $P(\lambda)$, the presence of rounding errors may produce quite different results when the eigenvalues are computed using different linearizations and when computing the eigenvalues directly from the polynomial. The notions of *conditioning* and *backward error* [22] measure the effect of rounding errors in the final (computed) quantities. In particular, condition numbers measure how perturbations in the data affect the final result. In the 2-norm, the (normwise) condition number of the simple eigenvalue λ_0 of the matrix polynomial (1) is given by

$$\kappa_P(\lambda_0) = \frac{\left(\sum_{j=0}^{k} |\lambda_0|^j \|A_j\|_2\right) \|y\|_2 \|x\|_2}{|\lambda_0| |y^* P'(\lambda_0) x|},$$

where y and x are, respectively, a left and a right eigenvector associated with λ_0 and P' denotes the derivative of P with respect to the variable λ [22]. Similarly, when considering a linearization $H(\lambda) = \lambda X + Y$ of $P(\lambda)$, we have

$$\kappa_H(\lambda_0) = \frac{(|\lambda_0| \|X\|_2 + \|Y\|_2) \|w\|_2 \|v\|_2}{|\lambda_0| |w^* H'(\lambda_0) v|},$$

where now w and v denote a left and a right eigenvector of H associated with λ_0. It can be seen that $y^* P'(\lambda_0) x = w^* H'(\lambda_0) v$ [14, Lemma 3.2]. Hence, the ratio between the condition number of λ_0 as an eigenvalue of the linearization and the condition number of λ_0 as an eigenvalue of the matrix polynomial is equal to

$$\frac{\kappa_H(\lambda_0)}{\kappa_P(\lambda_0)} = \frac{(|\lambda_0|\|X\|_2 + \|Y\|_2)}{\left(\sum_{j=0}^{k} |\lambda_0|^j \|A_j\|_2\right)} \cdot \frac{\|w\|_2\|v\|_2}{\|y\|_2\|x\|_2}.$$

As a consequence, the ratio between the norm of eigenvectors $(\|w\|_2\|v\|_2) / (\|y\|_2\|x\|_2)$ plays a relevant role in comparing the conditioning of λ_0 in H with the conditioning of λ_0 in P. To measure this ratio, our formulas relating the eigenvectors of linearizations with the eigenvectors of the matrix polynomial may be useful.

4. Proof of the main results

In the following subsections we will prove Theorems 3.1, 3.3 and 3.6. We will only prove the part regarding the right eigenvectors and minimal bases. The statements about the left eigenvectors can be obtained from the right ones by using the following observation. Given a index tuple $\mathbf{t}$, let $M_{\mathbf{t}}(P)$ be the matrix in (7). Let $H(\lambda) = \lambda M_{\mathbf{a}}(P) - M_{\mathbf{b}}(P)$, where $\mathbf{a}$ and $\mathbf{b}$ are index tuples satisfying the SIP with indices (maybe with repetitions) from $\{0, 1, \ldots, k, -0, -1, -2, \ldots, -k\}$ (notice that this includes all three families of Fiedler pencils, GF pencils and FPR). Then $H(\lambda)^T = \lambda M_{\mathrm{rev}\,\mathbf{a}}(P^T) - M_{\mathrm{rev}\,\mathbf{b}}(P^T)$. Since the left eigenvectors and left minimal bases of $H(\lambda)$ are the right eigenvectors and right minimal bases of $H(\lambda)^T$, we can get formulas for the left eigenvectors and minimal bases by reversing the tuples of the coefficient matrices of $H(\lambda)$ and replacing the coefficients A_i by A_i^T in the formulas for the right eigenvectors and right minimal bases.

4.1. *The case of Fiedler pencils*

Theorem 3.1 follows almost immediately from Lemma 5.3 in [8], where the authors derive formulas for the last block-column of $V(\lambda)$ and the last block-row of $U(\lambda)$ in (2) with $H(\lambda)$ being a Fiedler pencil. Our proof of Theorem 3.1 consists of relating our formulas (8) and (9) with the ones obtained in [8].

Proof of Theorem 3.1. First, let us recall the notion of *Consecution Inversion Structure Sequence (CISS)* of a permutation $\mathbf{q}$ of $\{0, 1, ..., k-1\}$, introduced in [8, Def. 3.3]. Assume that $\mathbf{q}$ has c_1 consecutions at 0, i_1 inversions at c_1, c_2 consecutions at $c_1 + i_1$, i_2 inversions at $c_1 + i_1 + c_2$, and so on. Then,

$$CISS(\mathbf{q}) := (c_1, i_1, c_2, i_2, \ldots, c_\ell, i_\ell).$$

Notice that c_1 and i_ℓ in this list may be zero, but the remaining numbers are nonzero. Using this notation, and following Remark **1**, we may write

$$\mathcal{R}_{\mathbf{q}} = \left[\mathcal{I}_\ell\, \mathcal{C}_\ell \ldots \mathcal{I}_1\, \mathcal{C}_1\right]^{\mathcal{B}},$$

where, for $j = 1, \ldots, \ell$,

$$\mathcal{I}_j = \lambda^{i_1+\cdots+i_{j-1}+j} \begin{bmatrix} \lambda^{i_j-1} I \\ \vdots \\ \lambda I \\ I \end{bmatrix}^{\mathcal{B}} \quad \text{and} \quad \mathcal{C}_j = \lambda^{i_1+\cdots+i_{j-1}+j-1} \begin{bmatrix} I \\ P_{\alpha_1^j} \\ \vdots \\ P_{\alpha_{c_j}^j} \end{bmatrix}^{\mathcal{B}},$$

(we set $i_0 := 0$) and

$$\alpha_i^j = k - (c_1 + i_1 + \cdots + c_{j-1} + i_{j-1} + c_j) + i - 1\,, \qquad \text{for} \quad i = 1, \ldots, c_j.$$

These are precisely the formulas (5.3) in [8], which are the building blocks of formula (5.4) (also in [8]), which generates the right eigenvectors and minimal bases of the Fiedler pencil $F_{\mathbf{q}}$. The fact that $\mathcal{R}_{\mathbf{q}}$ contains an identity block follows immediately from this formula. □

4.2. *The case of PGF pencils*

To prove Theorem 3.3 we use the following elementary observation. Let $\mathbf{B}$ be a block-column matrix consisting of k square blocks of size n. When $\mathbf{B}$ is multiplied on the left by M_{k-1}, only the first and second blocks of $\mathbf{B}$ are modified. When multiplied by $M_{k-2}M_{k-1}$ only the first, second, and third blocks of $\mathbf{B}$ are modified. Thus, when multiplying $M_{(k-j:k-1)}\mathbf{B}$ the only blocks of $\mathbf{B}$ that can be altered are the blocks with indices from 1 to $j+1$.

Proof of Theorem 3.3. Let $K(\lambda) = \lambda M_{\mathbf{m}} - M_{\mathbf{q}}$ be a PGF pencil associated with a matrix polynomial $P(\lambda)$ and such that $\mathbf{m}$ and $\mathbf{q}$ are index tuples in column standard form. We only prove (b), since (a) can be obtained using similar arguments. We construct a right eigenvector of $K(\lambda)$ from strict equivalence with a Fiedler pencil and show that this strict equivalence preserves an identity block in the formulas that lead to the eigenvectors of the Fiedler pencil, proving the last part of the statement.

Let us assume that $\mathbf{m}$ has $\mathfrak{c}_{-k}$ consecutions at $-k$. Then, there exists an index tuple $\mathbf{m}_1$ such that

$$K(\lambda) = \lambda\, M_{\mathbf{m}_1} M_{(-k:-k+\mathfrak{c}_{-k})} - M_{\mathbf{q}}. \tag{11}$$

Notice that the index tuple $(-\mathrm{rev}\,\mathbf{m}_1, \mathbf{q})$ is a permutation of $\{0, 1, \ldots, k - \mathfrak{c}_{-k} - 1\}$. Let $\mathbf{z} = csf(-\mathrm{rev}\,\mathbf{m}_1, \mathbf{q}) = (\mathbf{b}_\alpha, \mathbf{b}_{\alpha-1}, \ldots, \mathbf{b}_1)$ and $\widetilde{\mathbf{z}} = csf(-\mathrm{rev}\,\mathbf{m}_1, \mathbf{q}, k - \mathfrak{c}_{-k} : k-1)$. We construct the following Fiedler pencil associated with $P(\lambda)$:

$$F_{\widetilde{\mathbf{z}}}(\lambda) = M_{-\mathrm{rev}\,\mathbf{m}_1} K(\lambda) M_{(k-\mathfrak{c}_{-k}:k-1)} = \lambda M_{-k} - M_{(-\mathrm{rev}\,\mathbf{m}_1, \mathbf{q}, k-\mathfrak{c}_{-k}:k-1)}\,, \tag{12}$$

where $M_{(k-\mathfrak{c}_{-k}:k-1)} = I$ if $\mathfrak{c}_{-k} = 0$. We know that there exist unimodular matrices $U(\lambda)$ and $V(\lambda)$ such that

$$U(\lambda) F_{\widetilde{\mathbf{z}}}(\lambda) V(\lambda) = \begin{bmatrix} I & 0 \\ 0 & P(\lambda) \end{bmatrix},$$

which can be rewritten as

$$(U(\lambda) M_{-\mathrm{rev}\,\mathbf{m}_1}) K(\lambda) (M_{(k-\mathfrak{c}_{-k}:k-1)} V(\lambda)) = \begin{bmatrix} I & 0 \\ 0 & P(\lambda) \end{bmatrix}.$$

Note that $K(\lambda)v(\lambda) = 0$ if and only if $v(\lambda) = M_{(k-\mathfrak{c}_{-k}:k-1)} \mathcal{R}_{\widetilde{\mathbf{z}}} x(\lambda)$, for some $x(\lambda)$ with $P(\lambda)x(\lambda) = 0$, where $M_{(k-\mathfrak{c}_{-k}:k-1)}\mathcal{R}_{\widetilde{\mathbf{z}}}$ is the last block-column of $M_{(k-\mathfrak{c}_{-k}:k-1)} V(\lambda)$. Recall that the explicit expression for $\mathcal{R}_{\widetilde{\mathbf{z}}}$ is given in Theorem 3.1. Thus, if $\mathfrak{c}_{-k} = 0$, then $\mathcal{R}_K = \mathcal{R}_{\widetilde{\mathbf{z}}} = \mathcal{R}_{\mathbf{z}}$, and this proves part *(i)* in the statement.

Now assume that $\mathfrak{c}_{-k} \neq 0$. Let $\mathbf{b}_\alpha = (w : k - \mathfrak{c}_{-k} - 1)$, for some $w > 0$. Then $\widetilde{\mathbf{z}}$ is equivalent to $(w : k-1, \mathbf{b}_{\alpha-1}, \ldots, \mathbf{b}_1)$. By Theorem 3.1,

$$\mathcal{R}_{\widetilde{\mathbf{z}}} = \left[\, \lambda^{\alpha-1} [I\ P_1\ \ldots\ P_{k-1-w}] \middle| B_{k-w}\ B_{k-w+1}\ \ldots\ B_{k-1} \right]^{\mathcal{B}}, \tag{13}$$

where B_i, for $i = k-w, \dots, k-1$, are as in the statement. Now, multiplying $\mathcal{R}_{\widetilde{\mathbf{z}}}$ on the left by $M_{(k-\mathfrak{c}_{-k}:k-1)}$ only affects the first $\mathfrak{c}_{-k}+1$ blocks of $\mathcal{R}_{\widetilde{\mathbf{z}}}$. Since $(w : k-1)$ contains at least $\mathfrak{c}_{-k}+1$ elements, only some of the first $k-w$ blocks in (13) will be modified.

It is easy to check by direct multiplication that $M_{(k-\mathfrak{c}_{-k}:k-1)}\mathcal{R}_{\widetilde{\mathbf{z}}}$ is equal to

$$\left[\lambda^{\alpha}[P_0\ P_1 \dots P_{\mathfrak{c}_{-k}-1}]\ \lambda^{\alpha-1}[I\ P_{\mathfrak{c}_{-k}+1} \dots P_{k-1-w}]\ B_{k-w} \dots B_{k-1}\right]^{\mathcal{B}},$$

and this proves *(ii)*.

Finally, for the claim on the identity block, we first assume that $k-\mathfrak{c}_{-k} \neq \mathfrak{c}_0+1$, and then $\mathfrak{c}_0+1 \in \mathbf{m}_1$ or $\mathfrak{c}_0+1 \in \mathbf{q}$. This implies that $s \geq 2$. From Theorem 3.1, the $(k-\mathfrak{c}_0)$th block of $\mathcal{R}_{\widetilde{\mathbf{z}}}$ (given by (13)) is equal to I_n and, since multiplying on the left by $M_{(k-\mathfrak{c}_{-k}:k-1)}$ does not affect this block, the identity block remains in $\mathcal{R}_K$. If $k-\mathfrak{c}_{-k} = \mathfrak{c}_0+1$, then $s=1$ and, by the previous arguments, $\mathcal{R}_K = [\mathbf{B}_1\ \mathbf{B}_2]^{\mathcal{B}}$, where the first block of of $\mathbf{B}_2$ is equal to I_n. This is, precisely, the $(k-\mathfrak{c}_0)$th block of $\mathcal{R}_K$. □

4.3. *The case of FPR*

Proof of Theorem 3.6. We first notice that, from the conditions in the statement of the theorem, we get

$$(\mathbf{q}, \mathbf{r}_q) \sim (\mathbf{r}_q, \mathfrak{s}(\mathbf{q}, \mathbf{r}_q)) \qquad \text{and} \qquad (\mathbf{m}, \mathbf{r}_m) \sim (\mathbf{r}_m, \mathfrak{s}(\mathbf{m}, \mathbf{r}_m)). \tag{14}$$

We may prove (14) inductively on the number of indices of $\mathbf{r}_q$ and $\mathbf{r}_m$. Let us focus, for instance, on the first identity (for the second one we can proceed in a similar way). Let us assume that $\mathbf{r}_q = (s_1, ..., s_r)$, where s_i denotes the ith index in $\mathbf{r}_q$, and set $\mathbf{q} = (\mathbf{b}_\alpha, \mathbf{b}_{\alpha-1}, \dots, \mathbf{b}_1)$, with $\mathbf{b}_i = (t_{i-1}+1, t_i)$, for $i = 1, \dots, \alpha$. Since $\mathbf{r}_q$ is of type 1 relative to $\mathbf{q}$, we have $s_1 = t_{d-1}+1 < t_d$, for some $1 \leq d \leq \alpha$. Hence $(\mathbf{q}, s_1) \sim (t_{d-1}+1, \mathbf{b}_\alpha, \dots, \mathbf{b}_{d+1}, t_{d-1}+2 : t_d, \mathbf{b}_{d-1}, t_{d-1}+1, \dots, \mathbf{b}_1) = (\mathbf{r}_q, \mathfrak{s}(\mathbf{q}, s_1))$, if $d > 1$, and $(\mathbf{q}, s_1) \sim (0, \mathbf{b}_\alpha, \dots, \mathbf{b}_2, 1 : t_1, 0) = (\mathbf{r}_q, \mathfrak{s}(\mathbf{q}, s_1))$, if $d = 1$. We can proceed recursively to prove the claim.

Now, let $L(\lambda) = \lambda M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{m}} M_{\mathbf{r}_q} M_{\mathbf{r}_m} - M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{q}} M_{\mathbf{r}_q} M_{\mathbf{r}_m}$, as in the statement. Here we assume that A_0 (resp. A_k) is nonsingular if 0 (resp. $-k$) is an index in $\mathbf{l}_q$, $\mathbf{r}_q$, or both (resp. in $\mathbf{l}_m$, $\mathbf{r}_m$, or both). Notice that, by definition of FPR, $M_{\mathbf{r}_m}$ commutes with $M_{\mathbf{q}}$ and $M_{\mathbf{r}_q}$, and $M_{\mathbf{m}}$ commutes with $M_{\mathbf{r}_q}$. This fact, together with (14) gives

$$\begin{aligned} L(\lambda) &= \lambda M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{r}_m} M_{\mathbf{r}_q} M_{\mathfrak{s}(\mathbf{m},\mathbf{r}_m)} - M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{r}_m} M_{\mathbf{r}_q} M_{\mathfrak{s}(\mathbf{q},\mathbf{r}_q)} \\ &= M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{r}_m} M_{\mathbf{r}_q} (\lambda M_{\mathfrak{s}(\mathbf{m},\mathbf{r}_m)} - M_{\mathfrak{s}(\mathbf{q},\mathbf{r}_q)}) = M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{r}_m} M_{\mathbf{r}_q} \widetilde{K}(\lambda). \end{aligned}$$

Now the result follows, since multiplication on the left by nonsingular matrices do not change the eigenvectors and the minimal bases. □

Example 4.1 Let $L(\lambda) = \lambda M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{m}} M_{\mathbf{r}_q} - M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{q}} M_{\mathbf{r}_q}$ be the FPR of a matrix polynomial $P(\lambda)$ of degree $k = 15$ with $\mathbf{q} = (8, 4 : 7, 0 : 3)$, $\mathbf{m} = (-11 : -9, -12, -15 : -13)$, and $\mathbf{r}_q = (4 : 6)$, $\mathbf{r}_m = \emptyset$. Then, the simple tuple associated with $(\mathbf{q}, \mathbf{r}_q)$ is $\widetilde{\mathbf{q}} = (8, 7, 0 : 6)$. Following the notation in Theorem 3.6, we have $\widetilde{K}(\lambda) = \lambda M_{\mathbf{m}} - M_{\widetilde{\mathbf{q}}}$. In this case, $(-\operatorname{rev} \mathbf{m}_1, \mathbf{q}) = (12, 9 : 11, 8, 7, 0 : 6)$, thus

$$\mathcal{R}_{\widetilde{K}} = \left[\lambda^5 P_0\ \lambda^5 P_1 \middle| \lambda^4 I \middle| \lambda^3 I\ \lambda^3 P_4\ \lambda^3 P_5 \middle| \lambda^2 I \middle| \lambda I \middle| I\ P_9\ P_{10}\ P_{11}\ P_{12}\ P_{13}\ P_{14}\right]^{\mathcal{B}}.$$

Now set $\mathbf{r}_q = \emptyset$, $\mathbf{r}_m = (-15 : -14)$. Then, the simple tuple associated with $(\mathbf{m}, \mathbf{r}_m)$ is $\widetilde{\mathbf{m}} = (-11 : -9, -12, -13, -15 : -14)$. We now have $\widetilde{K}(\lambda) = \lambda M_{\widetilde{\mathbf{m}}} - M_{\mathbf{q}}$. In this case, $(-\operatorname{rev}\mathbf{m}_1,\mathbf{q}) = (13, 12, 9:11, 8, 4:7, 0:3)$, thus

$$\mathcal{R}_{\widetilde{K}} = \left[\lambda^6 P_0 | \lambda^5 I | \lambda^4 I | \lambda^3 I \ \lambda^3 P_4 \ \lambda^3 P_5 | \lambda^2 I | \lambda I \ \lambda P_8 \ \lambda P_9 \ \lambda P_{10} | I \ P_{12} \ P_{13} \ P_{14}\right]^{\mathcal{B}}.$$

Example 4.2 Let $K(\lambda) = \lambda M_{-5}M_{-4}M_{-3}M_{-8}M_{-7}M_{-6} - M_2M_0M_1$ be the PGF pencil associated with a matrix polynomial $P(\lambda)$ with degree $k = 8$. We have $\mathbf{m} = (-5 : -3, -8 : -6)$ and $\mathbf{q} = (2, 0 : 1)$ in column standard form. By direct computation we get

$$K(\lambda) = \begin{bmatrix} -I & 0 & \lambda A_8 & 0 & 0 & 0 & 0 & 0 \\ \lambda I & -I & \lambda A_7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & \lambda I & 0 & 0 \\ 0 & \lambda I & \lambda A_6 & -I & 0 & \lambda A_5 & 0 & 0 \\ 0 & 0 & 0 & \lambda I & -I & \lambda A_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda I & \lambda A_3 + A_2 & A_1 & -I \\ 0 & 0 & 0 & 0 & 0 & -I & \lambda I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_0 & \lambda I \end{bmatrix}$$

and, from Theorem 3.3,

$$\mathcal{R}_K = \left[\lambda^3 A_8 \ \lambda^3 P_1 | \lambda^2 I \ \lambda^2 P_3 \ \lambda^2 P_4 | \lambda I | I \ P_7\right]^{\mathcal{B}}.$$

It is straightforward to see that $K(\lambda)\mathcal{R}_K = \left[0\,0\,0\,0\,0\,0\,0\,P(\lambda)\right]^{\mathcal{B}}$, so $K(\lambda)\mathcal{R}_K(\lambda)x = 0$ if and only if $P(\lambda)x = 0$. Now, set $\mathbf{r}_m = (-5 : -4)$ and $\mathbf{r}_q = (0)$. We have that both $(\mathbf{m}, \mathbf{r}_m)$ and $(\mathbf{q}, \mathbf{r}_q)$ satisfy the SIP and also that both $\mathbf{r}_m$ and $\mathbf{r}_q$ are of type 1 relative to $\mathbf{m}$ and $\mathbf{q}$, respectively. Moreover, a simple computation gives

$$\mathcal{R}_L := M_{-\operatorname{rev}\mathbf{r}_m} M_{-\operatorname{rev}\mathbf{r}_q} \mathcal{R}_K = \left[\lambda^3 A_8 \ \lambda^3 P_1 \ \lambda^3 P_2 \ \lambda^3 P_3 | \lambda^2 I | \lambda I | I \ -A_0^{-1} P_7\right]^{\mathcal{B}}.$$

It is also immediate to see that the FPR defined as $L(\lambda) := K(\lambda)M_{\mathbf{r}_m}M_{\mathbf{r}_q}$ is

$$L(\lambda) = \begin{bmatrix} -I & 0 & 0 & 0 & \lambda A_8 & 0 & 0 & 0 \\ \lambda I & -I & 0 & 0 & \lambda A_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & \lambda I & 0 & 0 \\ 0 & \lambda I & -I & 0 & \lambda A_6 - A_5 & \lambda A_5 & 0 & 0 \\ 0 & 0 & \lambda I & -I & \lambda A_5 - A_4 & \lambda A_4 & 0 & 0 \\ 0 & 0 & 0 & \lambda I & \lambda A_4 & \lambda A_3 + A_2 & A_1 & A_0 \\ 0 & 0 & 0 & 0 & 0 & -I & \lambda I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_0 & -\lambda A_0 \end{bmatrix},$$

and that $L(\lambda)\mathcal{R}_L = \left[0\,0\,0\,0\,0\,0\,0\,P(\lambda)\right]^{\mathcal{B}}$, so $L(\lambda)\mathcal{R}_L x = 0$ if and only if $P(\lambda)x = 0$. However, Theorem 3.6 gives the following:

$$\mathcal{R}_{\widetilde{K}} := \left[\lambda^4 A_8 \ \lambda^4 P_1 \ \lambda^4 P_2 \ \lambda^4 P_3 | \lambda^3 I | \lambda^2 I | \lambda I | I\right]^{\mathcal{B}},$$

which corresponds to the PGF pencil $\widetilde{K}(\lambda) = \lambda M_{\mathfrak{s}(\mathbf{m},\mathbf{r}_m)} - M_{\mathfrak{s}(\mathbf{q},\mathbf{r}_q)}$, where $\mathfrak{s}(\mathbf{m}, \mathbf{r}_m) = (-3, -8 : -4)$ and $\mathfrak{s}(\mathbf{q}, \mathbf{r}_q) = (2, 1, 0)$ are the simple tuples associated with $(\mathbf{m}, \mathbf{r}_m)$ and $(\mathbf{q}, \mathbf{r}_q)$, in column standard form. It is straightforward to check that $L(\lambda)\mathcal{R}_{\widetilde{K}} = \left[0\,0\,0\,0\,0\,P(\lambda)\,0\,0\right]^{\mathcal{B}}$, so $L(\lambda)\mathcal{R}_{\widetilde{K}}x = 0$ if and only if $P(\lambda)x = 0$.

The case of indices which are not of type 1 will not be addressed in this work. When the column standard form of both $\mathbf{r}_m$ and $\mathbf{r}_q$ contains at most one index not being of type 1, we may determine the blocks in $M_{-\mathrm{rev}\,\mathbf{r}_q}M_{-\mathrm{rev}\,\mathbf{r}_m}\mathcal{R}_K$ by direct multiplication. However, if there is more than one index in $\mathbf{r}_m$ or $\mathbf{r}_q$ not being of type 1, then the problem of keeping track of the blocks which are moved after successive multiplications by the corresponding M_j matrices becomes an involved task, and remains as an open problem.

4.3.1. Symmetric pencils with repetition

Although a full characterization of all symmetric FPR has been recently presented in [4], here we focus on two subfamilies introduced in earlier references because they involve type 1 tuples and allows us to exemplify our results in this paper.

Let us begin with the symmetric linearizations considered in [17] and [18], and recently analyzed in [24] in the context of Fiedler pencils. These linearizations are FPR. In particular, for a given $0 \le h \le k-1$, we set $L^S_{k,h}(\lambda) := \lambda M_{\mathbf{m}}M_{\mathbf{r}_q}M_{\mathbf{r}_m} - M_{\mathbf{q}}M_{\mathbf{r}_q}M_{\mathbf{r}_m}$, with $\mathbf{q} = (0:h)$, $\mathbf{m} = (-k:-h-1)$, $\mathbf{r}_q = (0:h-1, 0:h-2, \ldots, 0:1, 0)$, and $\mathbf{r}_m = (-k:-h-2, -k:-h-3, \ldots, -k:-k+1, -k)$ (see [24, Cor. 2]). Notice that, with the notation introduced in Section 2.3, we have $\mathbf{l}_q = \mathbf{l}_m = \emptyset$ for all these pencils.

Notice that both $\mathbf{r}_q$ and $\mathbf{r}_m$ are of type 1 relative to $\mathbf{q}$ and $\mathbf{m}$, respectively. Moreover, with the notation of Theorem 3.6, we have $\mathfrak{s}(\mathbf{q}, \mathbf{r}_q) = (h, h-1, h-2, \ldots, 1, 0)$ and $\mathfrak{s}(\mathbf{m}, \mathbf{r}_m) = (-h-1, -h-2, \ldots, -k)$. Therefore,

$$\mathcal{R}_{L^S_{k,h}} = \left[\lambda^{k-1}I\ \lambda^{k-2}I\ \lambda^{k-3}I\ \ldots\ \lambda I\ I\right]^{\mathcal{B}}.$$

Note that this expression does not depend on h. By the symmetry of the construction, this is also equal to $\mathcal{L}_{L^S_{k,h}}$. As an example of these pencils, let us consider the case $k = 4$ and $h = 2$. We have

$$L^S_{4,2}(\lambda) = \lambda M_{(-4:-3)}M_{(0:1,0)}M_{(-4)} - M_{(0:2)}M_{(0:1,0)}M_{(-4)}$$

$$= \begin{bmatrix} -A_4 & \lambda A_4 & 0 & 0 \\ \lambda A_4 & \lambda A_3 + A_2 & A_1 & A_0 \\ 0 & A_1 & -\lambda A_1 + A_0 & -\lambda A_0 \\ 0 & A_0 & -\lambda A_0 & 0 \end{bmatrix}.$$

Notice that $L^S_{4,2}\mathcal{R}_{L^S_{4,2}} = \left[0\ P(\lambda)\ 0\ 0\right]^{\mathcal{B}}$, and that $(L^S_{4,2})^T\mathcal{R}_{L^S_{4,2}} = \left[0\ P(\lambda)^T\ 0\ 0\right]^{\mathcal{B}}$, so $\mathcal{R}_{L^S_{4,2}}x = 0$ if and only iof $P(\lambda)x = 0$.

We want to emphasize that, as mentioned in [24, p. 336], the pencils $L^S_{k,h}(\lambda)$ are a basis for the vector space $\mathbb{DL}(P)$ introduced in [19]. This is an immediate consequence of the following three facts:

(i) Every $L^S_{k,h}(\lambda)$ belongs to $\mathbb{DL}(P)$ [18, p. 225].
(ii) The dimension of the vector space spanned by $L^S_{k,0}(\lambda), \ldots, L^S_{k,k-1}(\lambda)$ is k (provided that $A_k \neq 0$) [18, Lemma 10].
(iii) The dimension of the vector space $\mathbb{DL}(P)$ is k [19, Cor. 5.4].

Next we consider a recent construction of symmetric linearizations introduced by Vologiannidis and Antoniou in [24, p. 338]. Let $0 \le h \le k-1$ and consider the cases:

(a) h is odd: Set $\mathbf{q} = (\mathbf{q}_{odd}, \mathbf{q}_{even})$ and $\mathbf{m} = (\mathbf{m}_{odd}, \mathbf{m}_{even})$, where $\mathbf{q}_{odd} = (1, 3, \ldots, h)$, $\mathbf{q}_{even} = (0, 2, \ldots, h-1)$, $\mathbf{m}_{odd} = (-h-2, -h-4, \ldots)$, and $\mathbf{m}_{even} = (-h-1, -h-3, \ldots)$. Also, $\mathbf{l}_q = \mathbf{q}_{even}$, $\mathbf{r}_q = \emptyset$, $\mathbf{l}_m = \emptyset$, $\mathbf{r}_m = \mathbf{m}_{odd}$.
Notice that the column standard form of $\mathbf{q}$ and $\mathbf{m}$ is $(h, h-2:h-1, h-4:h-3, \ldots, 1:2, 0)$ and $(-h-2:-h-1, -h-4:-h-3, \ldots)$, respectively. Thus,

$\mathbf{r}_m$ is of type 1 relative to $\mathbf{m}$. Moreover, with the notation of Theorem 3.6, we have $\mathfrak{s}(\mathbf{m}, \mathbf{r}_m) = (-h-1, -h-3 : -h-2, -h-5 : -h-4, \ldots, -k)$ if k is odd, and $\mathfrak{s}(\mathbf{m}, \mathbf{r}_m) = (-h-1, -h-3 : -h-2, -h-5 : -h-4, \ldots, -k : -k+1)$ if k is even. However, $\operatorname{rev} \mathbf{l}_q$ is not of type 1 relative to $\operatorname{rev} \mathbf{q}$. Nonetheless, by the symmetry of the construction, the same formulas allow us to recover both left and right eigenvectors and minimal bases (replacing A_i by A_i^T).

(b) h is even: Set $\mathbf{q} = (\mathbf{q}_{odd}, \mathbf{q}_{even})$ and $\mathbf{m} = (\mathbf{m}_{odd}, \mathbf{m}_{even})$, where now $\mathbf{q}_{odd} = (1, 3, \ldots, h-1)$, $\mathbf{q}_{even} = (0, 2, \ldots, h)$, $\mathbf{m}_{odd} = (-h-1, -h-3, ...)$, $\mathbf{m}_{even} = (-h-2, -h-4, \ldots)$. Also, $\mathbf{l}_q = \emptyset$, $\mathbf{r}_q = \mathbf{q}_{odd}$, $\mathbf{l}_m = \mathbf{m}_{even}$, $\mathbf{r}_m = \emptyset$.

As in the previous case, $\mathbf{r}_q$ is of type 1 relative to $\mathbf{q}$.

Example 4.3 Let $k = 6$ and $h = 3$. Then $\mathbf{q} = (\mathbf{q}_{even}, \mathbf{q}_{odd}) = ((1,3), (0,2))$ and $\mathbf{m} = (\mathbf{m}_{even}, \mathbf{m}_{odd}) = ((-5), (-4, -6))$, $\mathbf{r}_m = (-5)$, $\mathbf{l}_q = (0 : 2)$ and $\mathbf{r}_q = \emptyset = \mathbf{l}_m$. Then

$$L(\lambda) = \lambda M_{(0,2)} M_{(-5:-4,-6)} M_{-5} - M_{(0,2)} M_{(3,1:2,0)} M_{-5}$$
$$= \begin{bmatrix} 0 & -I & \lambda I & 0 & 0 & 0 \\ -I & \lambda A_6 - A_5 & \lambda A_5 & 0 & 0 & 0 \\ \lambda I & \lambda A_5 & \lambda A_4 + A_3 & A_2 & -I & 0 \\ 0 & 0 & A_2 & -\lambda A_2 + A_1 & \lambda I & A_0 \\ 0 & 0 & -I & \lambda I & 0 & 0 \\ 0 & 0 & 0 & A_0 & 0 & -\lambda A_0 \end{bmatrix}.$$

Notice that $L(\lambda)$ is, indeed, block-symmetric.

The simple tuple associated with $(\mathbf{m}, \mathbf{r}_m)$ in column standard form is $\mathfrak{s}(\mathbf{m}, \mathbf{r}_m) = (-4, -6 : -5)$, and the simple tuple associated with $(\mathbf{q}, \mathbf{r}_q)$ in column standard form is $\mathfrak{s}(\mathbf{q}, \mathbf{r}_q) = (3, 1 : 2, 0)$. Then, following the notation of Theorem 3.6, $\widetilde{\mathbf{m}}_1 = (-4)$ and $\widetilde{\mathbf{z}} = (4, 3, 1 : 2, 0)$ is the tuple in column standard form similar to $(-\widetilde{\mathbf{m}}_1, \mathfrak{s}(\mathbf{q}, \mathbf{r}_q))$. Hence, by Theorem 3.6, we have

$$\mathcal{R}_L = \left[\lambda^4 A_6 \middle| \lambda^3 I \middle| \lambda^2 I \middle| \lambda I \ \lambda P_4 \middle| I\right]^{\mathcal{B}}.$$

It is straightforward to check that $L(\lambda)\mathcal{R}_L = \left[0\ 0\ 0\ 0\ P(\lambda)\ 0\right]^{\mathcal{B}}$, so $L(\lambda)\mathcal{R}_L x = 0$ if and only if $P(\lambda)x = 0$. Since $L(\lambda)$ is block-symmetric, we have that

$$\mathcal{R}_L(P^T) = \left[\lambda^4 A_6^T \middle| \lambda^3 I \middle| \lambda^2 I \middle| \lambda I \ \lambda P_4^T \middle| I\right]^{\mathcal{B}}.$$

5. Conclusions and future work

We have obtained explicit formulas for the left and right eigenvectors and minimal bases of the following families of linearizations of square matrix polynomials: (a) the Fiedler pencils; (b) the GF pencils; and (c) the FPR with type 1 tuples. We have also analyzed two particular families of symmetric linearizations that belong to the last family. It remains, as an open problem, to obtain formulas for eigenvectors and minimal bases of FPR containing tuples which are not of type 1. Our formulas relate the eigenvectors and minimal bases of these linearizations with the eigenvectors and minimal bases of the polynomial. The formulas for the left and right eigenvectors may be useful in the comparison of the conditioning of eigenvalues of matrix polynomials through linearizations. We think that this is now one of the most challenging questions regarding the PEP solved by linearizations. There are several previous pioneer works where the conditioning of eigenvalues of linearizations and the conditioning of eigenvalues of the polynomial have been compared

[15, 16]. The present paper may be useful for the continuation of these works. In particular, to compare the conditioning of eigenvalues in the Fiedler families (including the Fiedler pencils, the GF pencils and the FPR) with the conditioning of eigenvalues in the matrix polynomial.

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Appendix A. Eigenvectors and minimal bases of GF pencils that are not proper

Theorem A.1 : *Let $K(\lambda) = \lambda M_{\mathbf{m}} - M_{\mathbf{q}}$ be a GF pencil of a regular matrix polynomial $P(\lambda)$ of degree k. Let $\mathcal{R}_K(P,\lambda)$ be the following $nk \times n$ matrix polynomial.*

(a) *Assume $0, k \in \mathbf{q}$. Let $\mathbf{q}' = \mathbf{q} \setminus \{k\}$ and $\mathbf{z} = csf(-\operatorname{rev}\mathbf{m}, \mathbf{q}')$. We distinguish two cases:*

(a1) *If $k-1$ is to the left of k in $(-\operatorname{rev}\mathbf{m}, \mathbf{q})$, then*

$$\mathcal{R}_K(P,\lambda) := \begin{bmatrix} A_k \\ \mathcal{R}_{\mathbf{z}}(2:k) \end{bmatrix},$$

with $\mathcal{R}_{\mathbf{z}}$ as in (3.1).

(a2) *If $k-1$ is to the right of k in $(-\operatorname{rev}\mathbf{m}, \mathbf{q})$, then*

$$\mathcal{R}_K(P,\lambda) := \mathcal{R}_{\mathbf{z}}.$$

(b) *Assume $-0, -k \in \mathbf{m}$. Set $csf(k+\mathbf{m}) = (k - \mathfrak{c}_{-0} : k, k+\mathbf{m}')$.*

(b1) *If $\mathfrak{c}_{-0} = k$, then*

$$\mathcal{R}_K(P,\lambda) := \begin{bmatrix} \lambda I \ \lambda P_1 \ \dots \ \lambda P_{k-2} \ A_0 \end{bmatrix}^{\mathcal{B}}.$$

(b2) *If $\mathfrak{c}_{-0} < k$, then*

$$\mathcal{R}_K(P,\lambda) := \mathcal{R}_{\widetilde{K}},$$

where $\widetilde{K}(\lambda) = \lambda M_{\mathbf{m}'} - M_{(0:\mathfrak{c}_{-0})} M_{\mathbf{q}}$ is a PGF pencil.

(c) *Assume $-0 \in \mathbf{m}$ and $k \in \mathbf{q}$. Set $csf(k+\mathbf{m}) = (k - \mathfrak{c}_{-0} : k, k+\mathbf{m}')$ and $csf(\mathbf{q}) = (t : k, \mathbf{q}')$. We distinguish the following two cases:*

(c1) *If $t > \mathfrak{c}_{-0} + 1$, then*

$$\mathcal{R}_K(P,\lambda) := \mathcal{R}_{\widetilde{K}},$$

where $\widetilde{K}(\lambda) = \lambda M_{(-k:-t)} M_{\mathbf{m}'} - M_{(0:\mathfrak{c}_{-0})} M_{\mathbf{q}'}$ is a PGF pencil.

(c2) *If* $t = \mathfrak{c}_{-0} + 1$*, then*

$$\mathcal{R}_K(P, \lambda) := \left[A_k \; P_1 \; \dots \; P_{k-1} \right]^{\mathcal{B}}.$$

Then

(a) $\{v_1(\lambda), \dots, v_p(\lambda)\}$ *is a right minimal basis of* $P(\lambda)$ *if and only if* $\{\mathcal{R}_K(P, \lambda)v_1(\lambda), \dots, \mathcal{R}_K(P, \lambda)v_p(\lambda)\}$ *is a right minimal basis of* $F_{\mathbf{z}}(\lambda)$*.*

(b) v *is a right eigenvector of* $K(\lambda)$ *associated with the eigenvalue* λ_0 *if and only if* $v = \mathcal{R}_K(P, \lambda_0)x$*, where* x *is a right eigenvector of* $P(\lambda)$ *associated with* λ_0*.*

Proof : (a1) In the conditions of the statement, we have that $(-\mathrm{rev}\,\mathbf{m}, \mathbf{q})$ is equivalent to $(-\mathrm{rev}\,\mathbf{m}, \mathbf{q}', k)$, so $K(\lambda) = \lambda M_{\mathbf{m}} - M_{\mathbf{q}'}M_k$, and then $F_\sigma(\lambda) := M_{-\mathrm{rev}\,\mathbf{m}}K(\lambda)M_{-k} = \lambda M_{-k} - M_{-\mathrm{rev}\,\mathbf{m}}M_{\mathbf{q}'}$ is a Fiedler pencil. Now the claim is a consequence of Theorem 3.1 applied to $F_\sigma(\lambda)$.

(a2) In this case we have that $(-\mathrm{rev}\,\mathbf{m}, \mathbf{q})$ is equivalent to $(k, -\mathrm{rev}\,\mathbf{m}, \mathbf{q}')$, so $F_\sigma(\lambda) := M_{-k}M_{-\mathrm{rev}\,\mathbf{m}}K(\lambda) = \lambda M_{-k} - M_{-\mathrm{rev}\,\mathbf{m}}M_{\mathbf{q}'}$ is also a Fiedler pencil, and the result is again a consequence of Theorem 3.1 applied to $F_\sigma(\lambda)$.

(b1) In this case we have

$$K(\lambda) = \lambda M_{-k}M_{-k+1} \cdots M_{-1}M_{-0} - I,$$

so $K(\lambda)M_0 = \lambda M_{-k}M_{-k+1} \cdots M_1 - M_0$ is a PGF pencil, and the result is an immediate consequence of Theorem 3.3 applied to this pencil.

(b2) Notice that, in this case, $K(\lambda) = \lambda M_{(-\mathfrak{c}_{-0}:-0)}M_{\mathbf{m}'} - M_{\mathbf{q}}$, so $\widetilde{K}(\lambda) = M_{(0:\mathfrak{c}_{-0})}K(\lambda)$ is a PGF pencil, and the result follows.

(c1) Now we have $K(\lambda) = \lambda M_{(-\mathfrak{c}_{-0}:-0)}M_{\mathbf{m}'} - M_{(t:k)}M_{\mathbf{q}'}$, so $\widetilde{K}(\lambda) = M_{(0:\mathfrak{c}_{-0})}M_{(-k:-t)}K(\lambda)$ is a PGF pencil, and the result follows.

(c2) In this case, we have $K(\lambda) = \lambda M_{(-\mathfrak{c}_{-0}:-0)} - M_{(\mathfrak{c}_{-0}+1:k)}$, so $M_{(0:\mathfrak{c}_{-0})}K(\lambda)M_{-k} = C_1(\lambda)$ is the first companion form. Hence, the claim is a consequence of Theorem 3.1. □

For the left eigenvectors and minimal bases, similar results can be stated using the matrix polynomial P^T and reversal of all tuples appearing in Theorem A.1 and.

Appendix B. The infinite eigenvalue

A matrix polynomial $P(\lambda)$ is said to have an *infinite eigenvalue* if zero is an eigenvalue of $\mathrm{rev}\,P(\lambda)$. Moreover, the left and right eigenspaces of the infinite eigenvalue of $P(\lambda)$ are the left and right eigenspaces of the zero eigenvalue of $\mathrm{rev}\,P(\lambda)$, respectively.

In this appendix we provide formulas for the left and right eigenvectors associated with the infinite eigenvalue in the following cases: (a) Fiedler pencils; (b) PGF pencils; and (c) FRP with type 1 tuples. Hence, the results we will state here are complementary to the ones in Theorems 3.1, 3.3 and 3.6, respectively, for finite eigenvalues.

The key to derive formulas for the left and right eigenvectors associated with the infinite eigenvalue relies in the following fact: Given a matrix polynomial $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$, with $A_k \neq 0$, the vector v (respectively w) is a right (resp. left) eigenvector of $P(\lambda)$ associated with the infinite eigenvalue if and only if $A_k v = 0$ (resp. $A_k^T w = 0$), that is, left and right eigenvectors of a matrix polynomial associated with the infinite eigenvalue are vectors belonging to the left and right nullspace, respectively, of its leading coefficient. In all three statements below, $P(\lambda)$ is assumed to be a regular matrix polynomial as in (1), and the eigenvectors of linearizations are partitioned into k blocks with length n.

Theorem B.1 : *Let* $F_\sigma(\lambda)$ *be a Fiedler pencil of* $P(\lambda)$*. Then:*

(a) A right eigenvector associated with the infinite eigenvalue of $P(\lambda)$ *is of the form* $\begin{bmatrix} v \; 0 \ldots 0 \end{bmatrix}^{\mathcal{B}} \in \mathbb{C}^{nk\times n}$*, where* $v \neq 0$ *is such that* $A_k v = 0$*.*

(b) A left eigenvector associated with the infinite eigenvalue of $P(\lambda)$ *is of the form* $\begin{bmatrix} w \; 0 \ldots 0 \end{bmatrix}^{\mathcal{B}} \in \mathbb{C}^{nk\times n}$*, where* $w \neq 0$ *is such that* $A_k^T w = 0$*.*

Proof: The result is an immediate consequence of the observation in the paragraph just before the statement and the fact that the leading coefficient of every Fiedler pencil is $M_{-k} = \operatorname{diag}(A_k, I_{n(k-1)})$. □

Theorem B.2: *Let* $K(\lambda) = \lambda M_{\mathbf{m}} - M_{\mathbf{q}}$ *be a PGF pencil associated with* $P(\lambda)$*, and* $\mathfrak{c}_{-k}$*,* $\mathfrak{i}_{-k}$ *be, respectively, the number of consecutions and inversions of* $\mathbf{m}$ *at* $-k$*.*

(i) Let $v \neq 0$ *be such that* $A_k v = 0$*. Then* $\begin{bmatrix} v_1 \ldots v_{\mathfrak{c}_{-k}} \; v \; 0 \ldots 0 \end{bmatrix}^{\mathcal{B}}$*, where* $v_i = -A_{k-i}v$*, for* $i = 1, \ldots, \mathfrak{c}_{-k}$*, is a right eigenvector of* $K(\lambda)$ *associated with the infinite eigenvalue.*

(ii) Let $w \neq 0$ *be such that* $A_k^T w = 0$*. Then* $\begin{bmatrix} w_1 \ldots w_{\mathfrak{i}_k} \; w \; 0 \ldots 0 \end{bmatrix}^{\mathcal{B}}$*, where* $w_i = -A_{k-i}^T w$*, for* $i = 1, \ldots, \mathfrak{i}_{-k}$*, is a left eigenvector of* $K(\lambda)$ *associated with the infinite eigenvalue.*

Proof: The result for the right eigenvectors is an immediate consequence of the fact that, if we write $\mathbf{m} = (-\operatorname{rev}\mathbf{m}_1, -k : -k + \mathfrak{c}_{-k})$, then $M_{\mathbf{m}} x = 0$ if and only if $M_{(-k:-k+\mathfrak{c}_{-k})} x = 0$, and

$$M_{(-k:-k+\mathfrak{c}_{-k})} = \left[\begin{array}{cc|c} 0 & A_k & \\ I & A_{k-1} & \\ \ddots & \vdots & \\ & I \; A_{k-\mathfrak{c}_{-k}} & \\ \hline & & I_{n(k-\mathfrak{c}_{-k}-1)} \end{array}\right].$$

The result for the left eigenvectors is a consequence of *(i)* applied to $K(\lambda)^T$. □

Theorem B.3: *Let* $L(\lambda) = \lambda M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{m}} M_{\mathbf{r}_q} M_{\mathbf{r}_m} - M_{\mathbf{l}_m} M_{\mathbf{l}_q} M_{\mathbf{q}} M_{\mathbf{r}_q} M_{\mathbf{r}_m}$ *be a FPR of a matrix polynomial* $P(\lambda)$*. Assume* $\mathbf{r}_m$*,* $\mathbf{r}_q$*,* $\operatorname{rev}\mathbf{l}_m$ *and* $\operatorname{rev}\mathbf{l}_q$ *are of type 1 relative to* $\mathbf{m}$*,* $\mathbf{q}$*,* $\operatorname{rev}\mathbf{m}$ *and* $\operatorname{rev}\mathbf{q}$*, respectively. Let* $\mathfrak{c}_{-k}$ *be the number of consecutions of* $-k$ *in the simple tuple associated with* $(\mathbf{m}, \mathbf{r}_m)$ *and* $\mathfrak{i}_{-k}$ *be the number of inversions of* $-k$ *in the simple tuple associated with* $(\mathbf{l}_m, \mathbf{m})$*.*

(i) Let $v \neq 0$ *be such that* $A_k v = 0$*. Then* $\begin{bmatrix} v_1 \ldots v_{\mathfrak{c}_{-k}} \; v \; 0 \ldots 0 \end{bmatrix}^{\mathcal{B}}$*, where* $v_i = -A_{k-i}v$*, for* $i = 1, \ldots, \mathfrak{c}_{-k}$*, is a right eigenvector of* $L(\lambda)$ *associated with the infinite eigenvalue.*

(ii) Let $w \neq 0$ *be such that* $A_k^T w = 0$*. Then* $\begin{bmatrix} w_1 \ldots w_{\mathfrak{i}_{-k}} \; w \; 0 \ldots 0 \end{bmatrix}^{\mathcal{B}}$*, where* $w_i = -A_{k-i}^T w$*, for* $i = 1, \ldots, \mathfrak{i}_{-k}$*, is a left eigenvector of* $L(\lambda)$ *associated with the infinite eigenvalue.*

Proof: The proof can be carried out in a similar way as the proof of Theorem B.2. □

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