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Robust bootstrap forecast densities for GARCH models: returns, volatilities and value-at-risk

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Keywords: BM estimator, Outliers, smooth bootstrap, variance targeting, winsorized bootstrap

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1 Introduction

In the context of financial time series, density forecast of future returns is important for researchers and practitioners as it is fundamental to obtain, for example, risk measures as the Value-at-Risk (VaR). Furthermore, measuring the uncertainty around future volatilities is also of interest for trading/pricing volatility derivatives and for designing volatility hedges for generic portfolios; see, for example, Avellaneda et al. (1995), Corradi et al. (2009, 2011) and Vorbrink (2014). It is well known that conditional variances of financial returns evolve over time and, in order to construct forecast densities for future returns and volatilities, one should take into account this evolution. One of the most popular models

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to represent the evolution of daily conditional variances of financial returns is the generalized autoregressive conditional heteroscedasticity (GARCH) model of Engle (1982) and Bollerslev (1986). After estimating the GARCH parameters and assuming that the standardized returns have a conditional Gaussian distribution, the forecast densities of future returns are usually approximated using this distribution; see, for example, Linsmeier and Pearson (2000) and Kuester et al. (2006) among many others. However, approximating the forecast densities of future returns in this way has two main drawbacks. First, the corresponding densities do not take into account the parameter uncertainty and, second, the distribution of standardized returns may have heavy tails and can even be asymmetric; see, for instance, Bollerslev (1987), Pagan (1996) and Cont (2001) among many others. Furthermore, the construction of forecast intervals for future GARCH volatilities is still a difficult task; see, for instance, Jacquier et al. (2002). To solve these problems, Pascual et al. (2006) propose a bootstrap procedure for GARCH models that not only allows for the construction of forecast densities of future returns but also of future volatilities incorporating the parameter uncertainty without assuming any particular error distribution; see Christoffersen and Gonçalves (2005), Grigoletto and Lisi (2011), Huang and Wang (2012), Wang et al. (2012) and Trucíos and Hotta (2016) for implementations.

In practice, when dealing with long time series of financial returns, it is not unusual to find extreme observations that cannot be explained by a GARCH model even when assuming standardized returns with a heavy tailed distribution; see, for example, Franses and Ghijssels (1999), Charles and Darné (2005) and Hotta and Zavallos (2013). Catalán and Trávez (2007) and Carnero et al. (2012) show that these outliers may cause biases on the usual maximum likelihood (ML) estimator of the parameters of GARCH models and on the estimated volatilities. As a consequence, it is expected that bootstrap forecast densities for returns and volatilities based on ML parameter estimators and standard filters for volatilities could be distorted giving a misleading picture of what can be expected in the future.

To overcome the problems caused by outliers when fitting GARCH models to forecast returns and volatilities, one can use robust estimators of the parameters and volatilities; see Muler and Yohai (2008), Carnero et al. (2012) and Boudt et al. (2013). In the context of time series of financial returns, Mancini and Trojani (2011) propose a semi-parametric bootstrap procedure to estimate the forecast distribution of returns using the robust optimal bounded influence M-estimator of Mancini et al. (2005) to estimate the GARCH parameters and a robustified estimator of the tails of the residuals. However, this bootstrap procedure assumes fixed parameters and, consequently, does not allow for the construction of bootstrap forecast densities for future volatilities. Furthermore, it is computationally very expensive and will not be considered further in this paper.

Our first objective is to analyze the effect of outliers on the forecast densities of returns and volatilities constructed using the bootstrap procedure proposed by Pascual et al. (2006), denoted as PRR, when the parameters and volatilities are estimated using the non-robust ML estimator and alternative robust estimators available in the literature. In particular, we consider the following robust parameter estimators: the M-estimator proposed by Muler and Yohai (2008); the bounded quasi maximum likelihood (QML) estimator based on maximizing the Student likelihood proposed by Carnero et al. (2012); and, finally, the variance targeting (VT) estimator proposed by Boudt et al. (2013). With respect to the robust filters to estimate the underlying volatilities, we consider the filter proposed by Muler and Yohai (2008) in which when a squared standardized return is larger than a given threshold it is substituted by the threshold. Carnero et al. (2012) propose the same robust filter but substituting large squared standardized returns by their conditional expectation. We implement the PRR algorithm using these estimators

and filters and an improvement is obtained in its performance compared with the original PRR based on ML estimator and standard volatility filter. However, the performance of the algorithm is still poor, specially, for one-step-ahead volatility forecasts. Consequently, we propose a further modification of the filter in which squared standardized returns larger than a given threshold are substituted by a bootstrap extraction. The performance of the proposed PRR bootstrap modified procedure, based on an appropriate combination of robust estimates of parameters and the latter filter, showed to be adequate when constructing forecast densities for future GARCH returns and volatilities.

In the PRR procedure, and also in the robust implementation proposed in this paper, the bootstrap residuals are random extractions without replacement from the empirical distribution of standardized residuals. To evaluate if alternative bootstrap extractions could improve the results obtained in the robust implementation, we also consider bootstrapping from the smoothing distribution as in Silverman and Young (1987) and the winsorized bootstrap of Singh (1998) which is based on trimming a fraction of the bootstrapped standardized residuals. We carry out Monte Carlo experiments to compare the performance of these alternatives and show that they have similar performance to the robust implementation, but the classic non parametric bootstrap extraction is simpler.

Finally, we implement the proposed robust bootstrap PRR procedure to forecasting the density of returns and volatilities of a real time series of daily returns of the Euro/Dollar (EUR/USD) exchange rates and show that the differences between the ML procedure and that proposed in this paper can be important when obtaining forecast densities for future volatilities.

The rest of the paper is organized as follows. Section 2 establishes notation by describing the GARCH model contaminated by outliers and the alternative estimators and filters considered. The PRR bootstrap procedure to construct forecast densities for returns and volatilities is described in section 3 which also reports the results of Monte Carlo experiments to analyze its finite sample performance in the presence of outliers. We also analyze the performance of two robust bootstrap extraction procedures, namely, the smoothing bootstrap and the winsorized bootstrap. Section 4 presents an empirical application of the robust PRR procedure to forecast returns, volatilities and VaRs of a daily series of EUR/USD exchange rates. Finally, section 5 concludes.

2 Contaminated GARCH models: estimation and filtering

In this section, we describe the GARCH model and the contamination scheme considered in this paper. We also describe alternative estimators of the parameters and volatilities and illustrate their small sample performance.

2.1 Additive outliers in GARCH models

GARCH models were proposed by Engle (1982) and Bollerslev (1986) to represent the dynamic dependence often observed in the second order moments of economic and financial time series. Given its popularity in empirical applications, in this paper, we focus on the GARCH(1,1) model. The GARCH(1,1) model contaminated by observational or

additive outliers is defined by Hotta and Tsay (2012) as follows

$$y_t = z_t + \text{sign}(z_t)w_t I_t(t \in A) \quad (1a)$$

$$z_t = \sigma_t \epsilon_t \quad (1b)$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 z_{t-1}^2 + \beta \sigma_{t-1}^2, \quad (1c)$$

for $t = 2, \dots, T$, where y_t is the return observed at time t and z_t is the uncontaminated GARCH(1,1) process with σ_t being the volatility that depends on past uncontaminated returns. The disturbances, ϵ_t , are an independent white noise sequence with variance one and w_t is the size of the outlier at time t . $I_t(\cdot)$ is the indicator function and A is the set of contaminated observations. Finally, the parameters are assumed to satisfy the usual positivity and stationary conditions, namely $\alpha_0 > 0$, $\alpha_1, \beta \geq 0$ and $\alpha_1 + \beta < 1$. The unconditional variance of the uncontaminated returns is given by $\sigma_z^2 = E(z_t^2) = \frac{\alpha_0}{1 - \alpha_1 - \beta}$.

In the context of uncontaminated GARCH models, h -step-ahead point forecasts of future volatilities are obtained as follows

$$\sigma_{T+h|T}^2 = \frac{\alpha_0 - (\alpha_1 + \beta)^h}{1 - \alpha_1 - \beta} + (\alpha_1 + \beta)^{h-1}(\alpha_1 y_T^2 + \beta \sigma_T^2). \quad (2)$$

Note that, when $h \rightarrow \infty$, the volatility forecasts tend to the marginal variance, σ_z^2 . Furthermore, given that in empirical applications, β is rather close to 1, for short forecast horizons, the value of the conditional variance at time T , σ_T^2 , has an important weight on the volatility forecast.

2.2 Parameter and volatility estimation

Several alternative estimators have been proposed in the literature to estimate the parameters of GARCH models, with some of them meant to be robust to additive outliers. Next, we describe the most popular ones.

Consider first the Gaussian QML estimator obtained by maximizing the following log-likelihood

$$l_G(\theta) \propto -\frac{1}{2} \left(\sum_{t=2}^T \frac{y_t^2}{s_t^2} + \sum_{t=2}^T \log(s_t^2) \right), \quad (3)$$

where

$$s_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + \beta s_{t-1}^2, \quad (4)$$

with $s_1^2 = \sigma_z^2$ and θ being the vector of unknown parameters. Alternatively, one can obtain s_1^2 using the sample variance of returns, which is a consistent estimator of σ_z^2 if there are not outliers. Horváth et al. (2006) show that the QML estimators implemented with both initial values are asymptotically equivalent with their difference being of order $T^{-1/2}$ if both ϵ_t and σ_t have finite fourth order moments. Furthermore, if there are no outliers, the QML estimator, denoted as $\hat{\theta}^{(N)}$, is consistent and asymptotically normal under mild regularity conditions; see Francq and Zakoian (2011) for a summary of the asymptotic properties of $\hat{\theta}^{(N)}$. However, several authors show that the QML estimator is badly affected by the presence of outliers; see, for example, Carnero et al. (2007) and Muler and Yohai (2008).

Once the parameters in θ are estimated, in-sample one-step-ahead volatility estimates are obtained by expression (4) with the parameters substituted by their corresponding estimates and \hat{s}_1^2 being given by the sample variance of returns¹. Denote these volatility estimates by \hat{s}_t .

¹The results are very similar when the initial value for the conditional variance is the plug-in estimator. These results are available upon request.

To illustrate the effect of outliers on parameter and volatility estimation, we carry out Monte Carlo experiments based on 500 replicates of sample size 1500 generated by an uncontaminated GARCH(1,1) model defined as in equations (1) with $\alpha_0 = 0.05$, $\alpha_1 = 0.1$ and $\beta = 0.85$ and Gaussian errors². After discarding the first 500 observations to avoid the effect of initial conditions, the sample size used for parameter estimation is $T = 1000$. The returns are then contaminated with $w = 5$ and 10 marginal standard deviations of the uncontaminated returns. We consider four different patterns of contamination: (1) each series is contaminated by an isolated outlier at $t = 500$; (2) an isolated outlier near the end of the sample at $t = 999$; (3) two consecutive outliers in the middle of the series at $t = 500$ and 501; and (4) two consecutive outliers at the end of the series at $t = 998$ and 999. For each simulated series, the GARCH parameters are estimated implementing the QML estimator³. Given that the main interest in this paper is not parameter estimation but volatility forecasting, we also obtain the plug-in estimator of the unconditional variance, $\hat{\sigma}_z^2 = \frac{\hat{\alpha}_0}{1 - \hat{\alpha}_1 - \hat{\beta}}$, which is related with long-run forecasts of volatility. Table 1 reports the Monte Carlo averages and root mean square errors (RMSE) of the parameters and of the plug-in estimator of the unconditional variance. When looking at the results corresponding to the individual parameters, we can observe that the QML estimator has important biases which are larger when the outliers are consecutive and appear in the middle of the estimation period. However, the QML plug-in estimator of σ_z^2 has very large positive biases which can be huge when the outliers appear at the end of the estimation period. Also we can observe that the variability of the estimated parameters and plug-in unconditional variance can be very large when compared with that obtained in the uncontaminated series. The results are in concordance with those obtained by Carnero et al. (2012) and are included for completeness.

Table 1 also reports the Monte Carlo average bias and RMSE of the in-sample volatility estimates of σ_t . When $\omega \neq 0$, the filter in (4) with QML parameter estimates overestimates the in-sample volatility with the overestimation being larger when the outliers are consecutive and appear in the middle of the sample. For example, the bias in estimating σ_t , when the series are contaminated by two consecutive outliers of size $\omega = 10$ at times $t = 998$ and 999 is 6.9%. Also note that the RMSE could be rather large in the presence of outliers. Finally, given that the short-run forecasts of volatility depend on the estimated volatility at time $t = T$, Table 1 also reports the Monte Carlo biases and RMSE of \hat{s}_{1000} . Obviously, in this case, the biases of the QML volatility estimates are larger when the outliers appear at the end of the sample period. Note that these biases can be huge even if the size of the outlier is moderate ($\omega = 5$). Figure 1 plots kernel densities of the true and estimated volatilities at $t = T$ obtained through the Monte Carlo simulations. We can observe that, when there are not outlier (first columns) or when they appear at the middle of the sample (second and fourth columns), the kernel densities of the true σ_T and estimated \hat{s}_T are rather close. However, when the outliers appear at the end of the sample period (third and fifth columns), there is a large displacement to the right of the density of the estimated volatilities at $t = T$ which will clearly affect the short-run forecasts of future volatilities.

To deal with the lack of robustness of the QML estimator and filter, Muler and Yohai (2008) propose an M-estimator of the GARCH parameters whereby the propagation of the impact of outliers on volatility is bounded. The bounded M-estimator (BM) is based

²Results for alternative GARCH models and sample sizes are available upon request. They are not reported to save space but the main conclusions are the same.

³In fact, in the Gaussian simulated case, the QML estimator is the ML estimator, but we still call it QML estimator because in real application we never have Gaussianity.

on minimizing the following function

$$M(\theta) = \frac{1}{T-1} \sum_{t=2}^T \rho \left(\log \left(\frac{y_t^2}{h_t} \right) \right), \quad (5)$$

where h_t is a filter for the volatility whose specification is given later in this paper and $\rho(\cdot)$ is a nondecreasing and bounded function given by

$$\rho(x) = \begin{cases} -\log(g(x)), & \text{if } -\log(g(x)) \leq 4 \\ P(-\log(g(x))), & \text{if } 4 < -\log(g(x)) \leq 4.3 \\ 4.15, & \text{if } -\log(g(x)) > 4.3, \end{cases} \quad (6)$$

with $g(x) = \phi(e^{\frac{x}{2}}) e^{\frac{x}{2}}$, $\phi(\cdot)$ being the standardized Gaussian density and

$$P(x) = \frac{2}{0.3^3} \left(\frac{1}{4}(x^4 - 4^4) - \frac{12.3}{3}(x^3 - 4^3) + \frac{28.3}{2}(x^2 - 4^2) \right) - \frac{137.6}{0.3^3}(x - 4) - \frac{1}{0.27}(x - 4)^3 + x.$$

Denote by $M_1(\cdot)$ and $M_2(\cdot)$ the function in (5) defined with $h_t = s_t^2$ and $h_t = (s_t^R)^2$ respectively, where $(s_t^R)^2$ is a bounded robust specification of the conditional variance given by

$$(s_t^R)^2 = \alpha_0 + \alpha_1 (s_{t-1}^R)^2 r_c \left(\frac{y_{t-1}^2}{(s_{t-1}^R)^2} \right) + \beta (s_{t-1}^R)^2, \quad (7)$$

with $(s_1^R)^2 = \frac{\alpha_0}{1-\beta}$ and $r_c(\cdot)$ given by

$$r_c(x) = \begin{cases} x, & \text{if } x \leq c \\ c, & \text{if } x > c, \end{cases} \quad (8)$$

with $c = 5.02$ for a convenient trade off between efficiency and robustness in the context of GARCH models with Gaussian errors. Finally, the BM estimator, denoted by $\hat{\theta}^{(BM)}$, is given by:

$$\hat{\theta}^{(BM)} = \begin{cases} \hat{\theta}_1, & M_1(\hat{\theta}_1) \leq M_2(\hat{\theta}_2) \\ \hat{\theta}_2, & M_1(\hat{\theta}_1) > M_2(\hat{\theta}_2), \end{cases} \quad (9)$$

where $\hat{\theta}_1 = \operatorname{argmin}_{\theta} M_1$ and $\hat{\theta}_2 = \operatorname{argmin}_{\theta} M_2$. Once the parameters are estimated, one-step-ahead estimates of the volatility are obtained by substituting them into expression (7).

The Monte Carlo results reported in Table 1 show that the averages and RMSE of the BM estimates of the parameters are similar regardless of the position, number and size of the outliers with both being clearly reduced with respect to those of the QML estimator. The same is true with respect to the plug-in estimator of the unconditional variance. When looking at the results for the in-sample estimates of the volatility, we can observe that they are negatively biased. The magnitudes and RMSE of the in-sample one-step-ahead estimates of the volatility are also reduced with respect to those obtained when they are estimated using QML. However, we can observe that the volatility estimates at $t = T$ may still have important positive biases and large dispersions when the outliers appear close to the end of the in-sample period. Figure 1, which plots the corresponding kernel densities, show that the BM filter still generates estimates of σ_T which are located to the right of the true volatilities when the outliers appear close to T .

Alternatively, Carnero et al. (2012) propose estimating the GARCH parameters using a Bounded QML estimator, denoted as BS, based on combining the maximization of the Student-t log-likelihood with the bounding mechanism proposed by Muler and Yohai (2008). The BS estimator, denoted as $\hat{\theta}^{(BS)}$, is given by

$$\hat{\theta}^{(BS)} = \begin{cases} \hat{\theta}_1, & l_{S_1}(\hat{\theta}_1) \leq l_{S_2}(\hat{\theta}_2) \\ \hat{\theta}_2, & l_{S_1}(\hat{\theta}_1) > l_{S_2}(\hat{\theta}_2), \end{cases} \quad (10)$$

where $l_{S_1}(\theta)$ is defined as the following Student-t log-likelihood

$$l_S(\theta) \propto - \sum_{t=2}^T \left(\log \left(\sqrt{\frac{\nu}{\nu-2}} t_\nu \left(\frac{y_t}{\sqrt{h_t}} \right) \right) - \frac{1}{2} \log(h_t) \right), \quad (11)$$

where $t_\nu(\cdot)$ is the density function of a Student-t variable with ν degrees of freedom and h_t is given as for the BM estimator above with $r_c(\cdot)$ given by

$$r_c(x) = \begin{cases} x, & \text{if } x \leq c \\ 1, & \text{if } x > c, \end{cases} \quad (12)$$

with $c = 9$ in the context of Gaussian errors. Note that the vector of parameters, θ , in expression (11) has an additional parameter to be estimated, in particular, the degrees of freedom, ν .

Once more, one-step-ahead estimates of the volatility are obtained substituting the estimated parameters in the corresponding expression of the volatility. Note that, the difference between the specification of the trimming function $r_c(x)$ in expressions (8) and (12) is important. In (8) the outlying squared standardized returns are equal to the threshold constant, c , when entering the volatility equation. Therefore, the corresponding squared returns, enter as $c(s_t^R)^2$. However, in expression (12), they enter the volatility equation as $(s_t^R)^2$ which is its estimated conditional standard deviation. According to standard time series results, unknown observations should be replaced by their conditional expectations so that $r_c(x)$ as defined in (12) is expected to have better properties than (8) when estimating the underlying volatilities.

The results in Table 1 show that when $\omega \neq 0$, the BS estimator of the parameters still has problems if the outliers are consecutive and appear in the middle of the sample period; these results are in concordance with those in Carnero et al. (2012). Furthermore, the averages and RMSE of the parameters are different depending on the outlier size. As a consequence, the plug-in estimator of the unconditional variance is rather different depending on the size of the outlier. Moreover, observe that the RMSE of the estimator of the unconditional variance are larger than those of the BM estimator. Note that, for example, when the series is contaminated by two consecutive outliers at time $t = 500$ and 501, the RMSE is huge. We can also observe that the magnitude of the biases of BS in-sample volatility estimates increase with the outlier size. These biases are positive and the RMSE are similar to those of the estimates obtained with the BM estimator. Finally, the results when estimating the volatility at the end of the in-sample period, are better than when the BM filter is implemented. When the outliers appear at the end of the sample period, the biases and RMSE of s_T are clearly reduced. Consequently, even if the BS parameter estimates are not truly robust and have worse robust properties than those of the BM estimator, using the filter for volatilities as in equation (7) with $r_c(\cdot)$ defined in (12) improves the performance of the estimator of σ_T which is crucial to obtain short-run forecasts of future volatilities with good properties. The same conclusion can be obtained

from Figure 1, where we can observe that the kernel densities of s_T obtained using BS are much closer to those of σ_T than when the BM filter is implemented.

Recently, Boudt et al. (2013) propose a further robust VT estimator; see Francq et al. (2011) for its asymptotic properties. The VT estimator is based on the following reparametrization of the volatility equation

$$s_t^2 = s_{t-1}^2 + k(\sigma_z^2 - s_{t-1}^2) + \alpha_1(y_{t-1}^2 - s_{t-1}^2), \quad (13)$$

where $k = 1 - \alpha_1 - \beta$. The robust VT estimator is a two-step estimator. In the first step, the marginal variance is estimated as

$$\hat{\sigma}_z^{2(BVT)} = 1.318 \frac{\sum_{t=1}^T (y_t - \hat{\mu})^2 J_t}{\sum_{t=1}^T J_t}, \quad (14)$$

where $J_t = I\left(\frac{(y_t - \hat{\mu})^2}{(1.486 \times MAD_{t,K}(y_t))^2} \leq \chi_1^2(95\%)\right)$ with $I(\cdot)$ being the indicator function, $\chi_N^2(\delta)$ the δ quantile of the χ^2 distribution with N degrees of freedom, $\hat{\mu} = \frac{\sum_{t=1}^T y_t I_t}{\sum_{t=1}^T I_t}$ with $I_t = I\left(\frac{(y_t - Median_{t,K}(y_t))^2}{(1.486 \times MAD_{t,K}(y_t))^2} \leq \chi_1^2(95\%)\right)$, $MAD_{t,K}(y_t) = Median_{t,K}(|y_t - Median_{t,K}(y_t)|)$ and $Median_{t,K}(y_t)$ is the local median estimated in a window of size K around y_t . At or near the borders, the window is given by $[1, K + 1]$ when $t < K/2$, or by $[T - K, T]$ when $t > T - K/2$. We use $K = 30$ as in Boudt et al. (2013). Conditioning on the robust estimate of the marginal variance, $\hat{\sigma}_z^{2(BVT)}$, the remaining parameters, α_1 and β , are estimated by BM in the second step, minimizing the function in (5) with $\rho(\cdot)$ defined as

$$\rho(x) = -x + 4.13 \times \log\left(1 + \frac{e^x}{2}\right). \quad (15)$$

The conditional variance is defined as

$$(s_t^{Rm})^2 = \alpha_0 + \alpha_1 (s_{t-1}^{Rm})^2 \times c_\gamma \times r_c\left(\frac{y_{t-1}^2}{(s_{t-1}^{Rm})^2}\right) + \beta (s_{t-1}^{Rm})^2, \quad (16)$$

where $c_\gamma = \frac{1}{F_{\chi_3^2}(\chi_1^2(\gamma)) + (1-\gamma)\chi_1^2(\gamma)}$, with $F_{\chi_N^2}$ being the distribution function of a χ^2 variable with N degrees of freedom and $(s_1^{Rm})^2 = \hat{\sigma}_z^{2(BVT)}$. Boudt et al. (2013) show that, when $r_c(\cdot)$ is defined as in (8), the robust VT estimator, denoted by $\hat{\theta}^{(BVT)}$, has smaller biases than $\hat{\theta}^{(BM)}$. Furthermore, the computer time involved in the BVT estimator is smaller than those of the BM and BS estimators. In this paper, we implement the BVT estimator defining $r_c(\cdot)$ as in (12) because of the better results when estimating σ_T as described previously⁴. There are two issues about the BVT estimator that should be clarified. First of all, regardless of whether one choose the trimming function $r_c(\cdot)$ as defined by Muler and Yohai (2008) or as defined by Carnero et al. (2012), one should determine the threshold parameter, c . The former authors choose $c = 5.02$ while the latter choose $c = 9$. After carrying out a small Monte Carlo experiment in the context of the GARCH model described above, we conclude that the best compromise between the properties of the BVT estimator of the parameters, the unconditional variance and the one-step-ahead estimator of σ_T^2 , is obtained when $c = 9$. The second issue one should take into account when implementing the BVT estimator is that it has been designed under the assumption of Gaussian errors. Relaxing the assumption of Gaussianity affects the way in which the unconditional variance in (14) is estimated as well as the scaling constant, c_γ , in equation (16). Furthermore, the appropriate threshold constant, c , to be used in the trimming

⁴Results using $r_c(\cdot)$ as in (8) are available upon request.

function $r_c(\cdot)$ can also be different depending on the distribution of ϵ_t . In applications in which other distributions are considered, the constant c should be adequately calibrated.

Table 1 reports the Monte Carlo results for the BVT estimator which has similar averages and RMSE of the parameters and unconditional variance to those of the BM estimator. Comparing the BM and BVT in-sample volatility estimates, we can also observe that the magnitudes of the biases and RMSE are similar. However, the estimation of σ_T obtained using the BVT filter with the modification mentioned previously, has smaller biases when consecutive outliers appear at the end of the sample period. We can also observe a large reduction in the RMSE. In Figure 1, which plots the densities of \hat{s}_{1000} and σ_{1000} , we can observe that the kernel densities of \hat{s}_T obtained using the BVT estimator and filter are very close to the densities of σ_T and similar to those obtained when using the BS procedure. By looking jointly at the properties of the estimator of σ_z^2 , related with long-run forecasts, and the properties of the estimator of σ_T^2 , related with short-run forecasts, it seems that the BVT procedure is the best compromise.

In order to illustrate the practical implications when the alternative estimators/filters are implemented in real time series, Figure 2 plots, for a particular simulated series, the in-sample estimates of the volatility obtained using the QML, BM, BS and BVT filters when the series is contaminated by one isolated outlier of size $\omega = 5$ generated in the middle (top panel) and at the end (bottom panel) of the estimation sample. The results when the series are contaminated by consecutive outliers are similar. Figure 2 shows that, in this particular example, the QML filter overestimates the volatility while it is difficult to distinguish the estimates obtained using any of the three robust filters which closely follow the dynamics of the true volatilities. However, when the outliers appear at the end of the in-sample period, BM estimates a volatility which is larger than the true one while BS and BVT estimate volatilities close to the true one.

3 Bootstrap Forecast

3.1 The bootstrap algorithm

As mentioned in the introduction, Pascual et al. (2006) propose a bootstrap procedure to construct forecast densities of returns and volatilities that incorporates the parameter uncertainty without relying on particular assumptions about the error distribution. The PRR algorithm is described next for the sake of clarity.

- **Step 1:** Estimate the parameters θ by QML, $\hat{\theta}^{(N)} = (\hat{\alpha}_0^{(N)}, \hat{\alpha}_1^{(N)}, \hat{\beta}^{(N)})$ and obtain the corresponding standardized residuals $\hat{\epsilon}_t = \frac{y_t}{\hat{s}_t}$, $t = 1, \dots, T$ where \hat{s}_t , $t = 2, \dots, T$, is defined as in (4) with the parameters substituted by the corresponding estimates and \hat{s}_1^2 is the sample variance of y_t . Denote by \hat{F}_ϵ the empirical distribution of the centered standardized residuals.
- **Step 2:** Generate a bootstrap series y_t^* , $t = 1, \dots, T$, as

$$\begin{aligned} y_t^* &= s_t^* \epsilon_t^*, \\ s_{t+1}^{*2} &= \hat{\alpha}_0^{(N)} + \hat{\alpha}_1^{(N)} y_t^{*2} + \hat{\beta}^{(N)} s_t^{*2}, \end{aligned} \tag{17}$$

where ϵ_t^* are random draws with replacement from \hat{F}_ϵ and $s_1^{*2} = \hat{s}_1^2$. Compute the bootstrap estimates $\hat{\theta}^{*(N)} = (\hat{\alpha}_0^{*(N)}, \hat{\alpha}_1^{*(N)}, \hat{\beta}^{*(N)})$.

- **Step 3:** For $h = 1, \dots, H$, construct $\hat{y}_{T+h|T}^*$ and $\hat{s}_{T+h|T}^*$ using the bootstrap estimates $\hat{\theta}^{*(N)}$ as

$$\begin{aligned}\hat{s}_{T+h|T}^{*2} &= \hat{\alpha}_0^{*(N)} + \hat{\alpha}_1^{*(N)} \hat{y}_{T+h-1|T}^{*2} + \hat{\beta}^{*(N)} \hat{s}_{T+h-1|T}^{*2} \\ \hat{y}_{T+h|T}^* &= \epsilon_{T+h}^* \hat{s}_{T+h|T}^*,\end{aligned}\quad (18)$$

where $\hat{y}_{T|T}^* = y_T$, ϵ_{T+h}^* are random draws with replacement from \hat{F}_ϵ and

$$\hat{s}_{T|T}^{*2} = \frac{\hat{\alpha}_0^{*(N)}}{1 - \hat{\alpha}_1^{*(N)} - \hat{\beta}^{*(N)}} + \hat{\alpha}_1^{*(N)} \sum_{j=0}^{T-2} \hat{\beta}^{*j(N)} \left(y_{T-j-1}^2 - \frac{\hat{\alpha}_0^{*(N)}}{1 - \hat{\alpha}_1^{*(N)} - \hat{\beta}^{*(N)}} \right). \quad (19)$$

- **Step 4:** Repeat steps 2 and 3 B times obtaining B bootstrap replicates $(\hat{y}_{T+h|T}^{*(1)}, \dots, \hat{y}_{T+h|T}^{*(B)})$ of y_{T+h} and $(\hat{s}_{T+h|T}^{*(1)}, \dots, \hat{s}_{T+h|T}^{*(B)})$ of s_{T+h} .

Observe that all bootstrap replicates of $\hat{s}_{T|T}^{*2}$ are based on the observed time series y_1, \dots, y_{T-1} so that, we obtain conditional estimates of the volatility at time T that incorporate the parameter uncertainty. Consequently, the bootstrap forecasts of y_{T+h} and σ_{T+h} are conditional on the available data set and incorporate the parameter uncertainty.

Denote by $\hat{G}_y^*(x) = \#(\hat{y}_{T+h|T}^* \leq x) / B$, the empirical bootstrap distribution function of the B bootstrap replicates $(\hat{y}_{T+h|T}^{*(1)}, \dots, \hat{y}_{T+h|T}^{*(B)})$. Using $\hat{G}_y^*(x)$, one can compute $100(1 - \delta)\%$ forecast intervals for returns as

$$[L_y^*(h), U_y^*(h)] = [G_y^{*-1}(\frac{\delta}{2}), G_y^{*-1}(1 - \frac{\delta}{2})]. \quad (20)$$

Estimation of the one-step-ahead density of returns is important for the estimation of VaR which is crucial for risk management. Using bootstrap procedures is very popular when computing the VaR; see, for example, Ruiz and Pascual (2002), Charles and Darné (2005), Hartz et al. (2006) and Grigoletto and Lisi (2011). Note that from $\hat{G}_y^*(x)$ one can easily compute the VaR of returns.

Finally, bootstrap forecast intervals of future volatilities can be obtained in a similar way, as that described for returns in equation (20).

The PRR algorithm can be easily modified using robust parameter estimators and robust filters of the volatility. Next, we describe the modification when the BVT procedure is implemented.

- **Step 1:** Estimate the parameters θ by BVT, $\hat{\theta}^{(BVT)} = (\hat{\alpha}_0^{(BVT)}, \hat{\alpha}_1^{(BVT)}, \hat{\beta}^{(BVT)})$ and obtain the corresponding standardized residuals $\hat{\epsilon}_t = \frac{y_t}{\hat{s}_t^{Rm}}$, $t = 1, \dots, T$ where \hat{s}_t^{Rm} , $t = 2, \dots, T$, is defined as in (16) with the parameters substituted by the corresponding estimates and $(\hat{s}_1^{Rm})^2 = \hat{\sigma}_z^2$. Denote by \hat{F}_ϵ the empirical distribution of the centered standardized residuals.
- **Step 2:** Generate a bootstrap series y_t^* , $t = 1, \dots, T$, as

$$\begin{aligned}y_t^* &= s_t^* \epsilon_t^*, \\ s_{t+1}^{*2} &= \hat{\alpha}_0^{(BVT)} + \hat{\alpha}_1^{(BVT)} s_t^{*2} c_\gamma r_c \left(\frac{y_t^{*2}}{s_t^{*2}} \right) + \hat{\beta}^{(BVT)} s_t^{*2},\end{aligned}\quad (21)$$

where ϵ_t^* are random draws with replacement from \hat{F}_ϵ , $s_1^{*2} = (\hat{s}_1^{Rm})^2$ and the function $r_c(\cdot)$ is defined as in equation (12). Compute the bootstrap estimates $\hat{\theta}^{*(BVT)} = (\hat{\alpha}_0^{*(BVT)}, \hat{\alpha}_1^{*(BVT)}, \hat{\beta}^{*(BVT)})$.

- **Step 3:** For $h = 1, \dots, H$, construct $\hat{y}_{T+h|T}^*$ and $\hat{s}_{T+h|T}^*$ using the bootstrap estimates $\hat{\theta}^{*(BVT)}$ and the original series y_t , $t = 1, \dots, T$ as

$$\hat{s}_{T+h|T}^{*2} = \hat{\alpha}_0^{*(BVT)} + \hat{\alpha}_1^{*(BVT)} \hat{s}_{T+h-1|T}^{*2} c_\gamma r_c \left(\frac{y_{T+h-1|T}^{*2}}{\hat{s}_{T+h-1|T}^{*2}} \right) + \hat{\beta}^{*(BVT)} \hat{s}_{T+h-1|T}^{*2} \quad (22)$$

$$\hat{y}_{T+h|T}^* = \epsilon_{T+h}^* \hat{s}_{T+h|T}^*,$$

where $\hat{y}_{T|T}^* = y_T$, ϵ_{T+h}^* are random draws with replacement from \hat{F}_ϵ and $\hat{s}_{T|T}^{*2}$ is obtained using the recursion

$$\hat{s}_{t|T}^{*2} = \hat{\alpha}_0^{*(BVT)} + \hat{\alpha}_1^{*(BVT)} \hat{s}_{t-1|T}^{*2} c_\gamma r_c \left(\frac{y_{t-1}^2}{\hat{s}_{t-1|T}^{*2}} \right) + \hat{\beta}^{*(BVT)} \hat{s}_{t-1|T}^{*2} \quad (23)$$

for $t = 2, \dots, T$ and $\hat{s}_{1|T}^{*2} = (\hat{s}_1^{Rm})^2$.

- **Step 4:** Repeat steps 2 and 3 B times obtaining B bootstrap replicates $(\hat{y}_{T+h|T}^{*(1)}, \dots, \hat{y}_{T+h|T}^{*(B)})$ of y_{T+h} and $(\hat{s}_{T+h|T}^{*(1)}, \dots, \hat{s}_{T+h|T}^{*(B)})$ of s_{T+h} .

In the robust bootstrap procedure described above, it is important to point out that, in steps 2 and 3, the robust filter for volatilities is implemented both to generate the series used to obtain bootstrap replicates of the parameters (step 2) and to obtain bootstrap replicates of future returns and volatilities. In this way, the effect of outliers is mitigated. However, using the bootstrap methodology together with a robust filter for the volatility allows to think about a better proposal of the trimming function $r_c(\cdot)$. Note that when using (12), if a squared observation is detected as an outlier, it is substituted by its conditional expectation. However, we can think about substituting it by a bootstrap estimate by defining $r_c(\cdot)$ as

$$r_c(x) = \begin{cases} x, & \text{if } x \leq c \\ \varepsilon_t^{*2}, & \text{if } x > c. \end{cases} \quad (24)$$

with $c = 9$ in the context of Gaussian errors. In the following, we explore the finite sample performance of the PRR procedure when implemented using the QML estimator and filter as in equations (17) to (19). We also explore the properties of the PRR procedure when implemented with the robust estimators and filters described above.

3.2 Monte Carlo experiments

In order to analyze how outliers affect the performance of the PRR procedure when implemented with the alternative robust estimators and filters, we carry out Monte Carlo experiments. We use the same design described above. Bootstrap forecast densities are obtained implementing the PRR procedure using the QML estimates and the standard volatility filter as defined in (4). We also implement the PRR bootstrap with the BM and BS estimators and the bounded filter in (7) with $r_c(\cdot)$ defined as in (8) and (12), respectively. Finally, we implement the PRR procedure using the estimator proposed by Boudt et al. (2013) with its corresponding robust filter in which $r_c(\cdot)$ is defined either as in (12), denoted by BVT1, or as in (24), denoted by BVT2. For each replicate and procedure, we construct out-of-sample bootstrap densities for future returns and volatilities for $h = 1, \dots, 20$. Based on the bootstrap densities of returns we obtain 1% VaR forecasts for $h = 1$ and 95% forecast intervals. The bootstrap densities of volatilities are also used to obtain 95% forecast intervals.

Consider first the results for the VaR. The one-step-ahead 1% bootstrap VaR forecasts obtained using the PRR procedure implemented with the alternative estimators and filters considered are compared with the corresponding empirical quantile. Figure 3 plots empirical densities of the differences between the one-step-ahead empirical and bootstrap 1% VaR when the series are contaminated by outliers of size $\omega = 5$. We can observe that the differences between the bootstrap and empirical VaRs are similar regardless the outliers position when the BS, BVT1 and BVT2 procedures are implemented to estimate parameters and volatilities. When there are not outliers or they are in the middle of the sample period, these differences are centered around zero in all cases. However, when the outliers appear at the end of the sample period, the VaR is clearly overestimated mainly when QML but also when BM is used, so that it seems that the estimated risk is larger than it truly is. The VaR forecasts obtained when implementing the BVT2 procedure are close to the true VaR regardless of the position of the outliers.

Table 2, which reports the proportion in the Monte Carlo replicates of failures of the one-step-ahead 1% VaR when $\omega = 5$ and 10, illustrates the danger involved in using bootstrap to obtain VaR forecasts based on GARCH models when a small number of outliers appear close to the end of the sample period. We can observe that, if the filter is not adequately robustified as in QML and BM estimators, then the percentage of failures is well below the nominal level of the VaR even if $\omega = 5$. This result may explain the empirical results in Dupuis et al. (2015) who show that the VaR is overestimated in the presence of outliers. Furthermore, if the parameter estimator is not truly robust as in the case of BS, the percentage of failures can be larger than the nominal when the outliers are large ($\omega = 10$). Note that, in this case, the number of failures is larger when the outliers are consecutive than when they are isolated. This results can explain why Chiu et al. (2005) find that the number of failures is larger than the nominal.

Consider now the results for 95% forecast intervals for returns. For each series we also generate 1000 future values of y_{T+h} , for $h = 1, \dots, 20$, and count the number of observations lying within the 95% bootstrap forecast intervals and in the left and right intervals to compute the empirical coverage and coverages above and below, respectively. Figure 4 plots, for outliers of size $\omega = 5$, the Monte Carlo averages of the coverages and of the coverages above and below of each interval for returns when $h = 1, 5$ and 20. We can observe that, when there are not outliers or they are in the middle of the sample period, the coverage of the 95% forecast intervals for returns is close to the nominal regardless of the procedure implemented to estimate parameters and volatilities. However, when the outliers appear at the end of the sample period, the coverages of the bootstrap intervals are too large when QML and BM are used. Better results are obtained when BS, BVT1 and BVT2 procedures are implemented. In this case, the average coverage is rather close to the nominal. Therefore, constructing bootstrap forecast intervals for returns using the appropriate filter for the volatility has an adequate performance in presence of outliers.

To illustrate the effect in practice of outliers on the PRR algorithm, we consider again the particular simulated series considered in Figure 2 contaminated by consecutive outliers at times $t = 500$ and 501 and $t = 998$ and 999 and implement the QML and robust procedures to obtain 95% bootstrap forecast intervals for returns for $h = 1, \dots, 20$. We also generate 1000 future returns y_{T+h} and construct the corresponding 95% empirical forecast intervals. Figure 5 plots the empirical and bootstrap intervals for returns. We can observe that, in concordance with the Monte Carlo results above, when there are not outliers or they appear in the middle of the sample, the bootstrap intervals are adequate regardless of the parameter estimator and filter used. Only the BM intervals are slightly wider than the others. However, when the outliers appear at the end of the sample, the bootstrap intervals based on the QML estimator and filter are badly affected by the outliers. When the PRR

algorithm is implemented with QML, the forecast intervals for returns are much wider than the true empirical intervals. Furthermore, if the PRR procedure is implemented using the BM procedure, we can observe that the forecast intervals are slightly larger than the empirical ones. Finally, the intervals constructed using BS, BVT1 and BVT2 are indistinguishable and very similar to the empirical intervals. It is also important to note that the bootstrap intervals based on QML are closer to the empirical intervals as the forecast horizon, h , increases.

Consider now the results when the PRR bootstrap is implemented to obtain forecast densities for volatilities. Figure 6, which plots the same quantities as in Figure 4 for volatilities, shows that when there are not outliers the estimators QML, BVT1 and BVT2 produce the best results. When outliers appear in the middle of the sample period the worst results are observed when the BM estimator and filter are implemented to obtain bootstrap intervals for volatilities. The main problem could be the inadequacy of the BM filter for volatilities. It is important to point out that, in this case, using robust estimators is not a big advantage with respect to using the more popular QML estimator. However, when the outliers appear at the end of the sample period, the bootstrap forecast intervals for volatilities using QML are useless, with the coverages in the left tail of the bootstrap density being close to 100%. This fact could be expected due to the overestimation of the marginal variance and of the volatility in the last moment of the estimation period; see Table 1. Note that the increase of the coverage in the left tail of the forecast densities of future volatilities can also be too large when the outliers appear at the end of the sample period and the BM filter is implemented. On the other hand, the situation is less dramatic when the robust filters BS and BVT1 are implemented. The average coverages for $h = 5$ and 20 are rather close to the nominal. However, when $h = 1$, we can observe that the coverages of the BS and BVT1 bootstrap intervals are still smaller than the nominal. The undercoverage, also observed by Pascual et al. (2006), when the series are uncontaminated, or the outliers appear in the middle of the sample period, could be attributed to the fact that the GARCH volatility is observed one-step-ahead and, consequently, the only uncertainty associated with s_{T+1}^* is due to the parameter estimation. However, the undercoverage observed when the outliers appear at the end of the sample period cannot be only explained by the parameter uncertainty. Looking back at the results on the estimates of \hat{s}_T obtained using the BS and BVT procedures and plotted in Figure 1, we can observe that, when the outliers appear at the end of the sample, there is a number of series in which the true volatilities are larger than the estimated ones. In these cases, the right coverage of the intervals should be larger than expected. However, when BVT is implemented with the filter defined with the trimming function as in (24), BVT2, then the coverages of the bootstrap intervals are close to the nominal regardless of the position of the outliers. Therefore, an appropriate specification of the trimming function, as that proposed in this paper, yields bootstrap forecast intervals for volatilities with the desired properties.

Figure 7 plots the forecast intervals for volatilities obtained for an specific simulated series together with the corresponding empirical intervals. We can observe that, when there are outliers in the middle of the sample period, the bootstrap intervals are wider than they should be if the algorithm is implemented using QML and BM procedures. The discrepancies between the bootstrap and empirical intervals increase with the forecast horizon. The situation is even worse when the outliers appear at the end of the sample as, in this case, the bootstrap intervals are completely misplaced with respect to the empirical intervals. An improvement is achieved using BS, BVT1 and BVT2 procedures. When using these estimators and filters implemented to the particular series generated for the illustration, we can observe that the intervals are rather similar and very close to

the empirical intervals for future volatilities.

3.3 Robust bootstrap extraction procedures

Several authors argue that bootstrap procedures may be distorted when the data contains outliers even if they are based on robust estimators and filters. Part of these distortions could be attributed to the important negative effect of bootstrap replicates in which the proportion of outliers could be even higher than in the original series. The PRR bootstrap procedure is based on random extractions from the empirical distribution of the standardized residuals and, therefore, could be affected by this problem. Alternatively, several authors propose to reject those extractions that correspond to residuals that can be considered as outliers. In this subsection, we explore whether the results above can be improved when using alternative robust bootstrap extraction procedures. We consider two alternatives. We first consider bootstrapping from the smoothing distribution and, second, the winsorized bootstrap which is based on trimming a fraction of the bootstrapped standardized residuals.

The smooth bootstrap consists in replacing the empirical distribution of the residuals by a smoothed version. As described in Silverman (1986), the smooth bootstrap residuals are given by

$$\varepsilon_t^* = \epsilon_t^* + \lambda \times \eta_t, \quad (25)$$

where ϵ_t^* are random extractions from \hat{F}_ε , λ is a positive bandwidth and η_t are random extractions from a symmetric density function. Following Silverman and Young (1987), Wang (1995) and Polansky (2001), in this paper, the latter density function is the standard Gaussian density and we use $\lambda = T^{-1/4}$ as in Neumeyer (2009).

Alternatively, in the context of *iid* data, Singh (1998) proposes using a winsorized bootstrap to reduce the effect of outliers and poor bootstrap samples. The winsorization of the ξ -fraction of the residuals from each end of the sample is based on obtaining random draws with replacement from the original residuals, ϵ_t^* , and define

$$\varepsilon_t^* = \begin{cases} \hat{\epsilon}_{(\rho+1)}, & \text{if } \epsilon_t^* \leq \hat{\epsilon}_{(\rho)} \\ \hat{\epsilon}_{(T-\rho)}, & \text{if } \epsilon_t^* \geq \hat{\epsilon}_{(T-\rho+1)} \\ \epsilon_t^*, & \text{otherwise,} \end{cases} \quad (26)$$

where $\hat{\epsilon}_{(r)}$ is the r th order statistic of the residuals and $\rho = [T\xi]$ is the largest integer smaller or equal to $T\xi$. The choice of ξ depends on the potential number of outliers present in the data. When this number is small, ξ could be small, for example, $\xi = 0.01$, that represents 2% of atypical observations. If we suspect that the number of outliers is large, ξ may be larger. The correct choice of ξ is still a topic of research. In this paper, because the number of outliers in the contaminated series is very small, we choose $\xi = 0.01$.

When running the PRR algorithm with the BVT2 procedure and using the alternative smooth and winsorized bootstrap extraction procedures instead of the random draws with replacement, we do not observe any improvement. The results are available upon request.

4 Application

In this section, we implement the bootstrap procedures described above to construct forecast intervals for returns and volatilities of the buying rates at noon time in New York for cable transfers payable in currencies EUR/USD available at www.federalreserve.gov.

The rates are observed daily from January 3, 2000 to December 31, 2013, with a total of $T = 3520$ observations. Returns are computed as usual by $r_t = 100 \times \log(P_t/P_{t-1})$, where P_t denotes the closing price at day t . Figure 8 plots the centered daily returns together with the sample autocorrelations of returns and squared returns. Descriptive statistics are reported in Table 3. There is no statistically significant serial correlation in returns while the autocorrelations of squared returns are significant. Consequently, the GARCH model is fitted with its parameters estimated by QML and BVT. The parameter estimates were $\hat{\alpha}_0^{(N)} = 0.0012$, $\hat{\alpha}_1^{(N)} = 0.0292$, $\hat{\beta}^{(N)} = 0.9679$ and $\hat{\alpha}_0^{(BVT)} = 0.0024$, $\hat{\alpha}_1^{(BVT)} = 0.0520$, $\hat{\beta}^{(BVT)} = 0.9415$ using QML and BVT estimators respectively. Note that both parameter estimates imply large persistence of the volatility shocks, which is slightly larger when implementing the QML estimator (0.9971) than when using BVT estimator (0.9935). Also, the plug-in estimate of the marginal variance is larger when obtained using the QML estimates (0.414) than when it is obtained using the BVT estimates (0.369). This result could be expected from Monte Carlo results reported in Table 1. The sample autocorrelations of the standardized returns and of their squares are not significant. Finally, when returns are standardized using QML in-sample volatility estimates, we can observe that there are 19 squared standardized returns larger than 9 and, consequently, that could be considered as outliers. However, it is important to note that only two out of the 19 outliers are larger than 4 conditional standard deviations. The potential residuals are isolated and appear at different positions of the in-sample period with none of them very close to the end. The largest standardized return appears at $t = 2317$ with a size of 4.7 conditional standard deviations while the latest one appears at $t = 3146$ with a size of 3.2 conditional standard deviations.

The PRR algorithm implemented with QML and BVT2 estimates and filters are then implemented to obtain forecast densities for future returns and volatilities using a rolling window scheme with in-sample size $T = 3275$ and the out-of-sample size $H = 225$. For each window we construct h -step-ahead, $h = 1, \dots, 20$, bootstrap forecast densities for returns and the corresponding VaR and 95% and 99% intervals. We also obtain forecast densities for volatilities and the corresponding intervals.

We first analyze the performance of bootstrap forecast densities for returns when implemented to compute the one-step-ahead 1% VaR. We can observe that the levels of the VaR are very similar in both cases. Figure 9 plots the observed returns and the 1% VaR computed using the PRR algorithm with QML and BVT2 implementations. The percentage of failures are reported in Table 4 together with the p-values of the unconditional coverage (UC) test of Kupiec (1995) and the conditional coverage (CC) and independence (I) tests of Christoffersen (1998). Note that the proportion of failures when the PRR is implemented with QML is larger which may have implications with respect to capital requirements; see, for example, the survey on VaR forecasting by Nieto and Ruiz (2015). When the PRR algorithm is implemented with BVT2, the proportion of failures is closer to the nominal. However, none of the tests reject the null hypothesis of correct proportion of failures regardless of whether the QML or BVT2 estimators are implemented.

Consider now the results on forecast intervals for returns. Table 5, which reports the coverage over the out-of-sample period of the h -step-ahead forecast intervals, $h = 1, 2, 5, 10$ and 20 , shows that the coverages of the bootstrap intervals obtained by both procedures are rather similar and slightly below the nominal. Therefore, it seems that in terms of forecast densities for returns, there are not big differences between obtaining them using PRR procedure with QML or BVT2 estimators and filters. Note that this result is in concordance with the Monte Carlo results given that, as mentioned above, the potential outliers are not very large and appear isolated in the middle of the in-sample

period.

Finally, consider the results for the forecast intervals for volatilities. Given that volatilities are not observable, in order to evaluate the bootstrap forecast intervals, we consider realized volatility as a proxy for actual volatility; see, for instance, Koopman et al. (2005), Pascual et al. (2006) and Boudt et al. (2013), among many others. We estimate realized volatility using 1 minute transaction prices of EUR/USD exchange rates provided by *Disktrading* and implementing the robust MinRV estimator of Andersen et al. (2012) who also provide asymptotic confidence intervals for the corresponding measures⁵. The left panel of Figure 10 plots the 95% one-step-ahead bootstrap forecast intervals for the last 40 windows based on QML and BVT2 estimators and filters together with the corresponding intervals of the MinRV measures while the right panel plots the same quantities for 99% forecast intervals. Note that, regardless of whether we look at 95% or 99% intervals, when the PRR procedure is implemented with QML, the realized volatilities are usually below the interval. As shown in the Monte Carlo experiments above, QML bootstrap intervals for volatilities are displaced upwards. However, when the forecast intervals for volatilities are constructed using BVT2, they contain the corresponding intervals for realized volatilities most of the time. Furthermore, we can observe that the upper limit of both intervals is rather similar while the lower limit of the bootstrap intervals obtained using BVT2 estimator and filter are well below the lower limit of the intervals obtained using QML. Consequently, to obtain density forecasts of future volatilities is crucial to use the adequate robust bootstrap procedure.

5 Conclusions

In this paper, we analyze the effect of outliers on bootstrap forecast densities of returns and volatilities obtained using the PRR procedure proposed by Pascual et al. (2006) in the context of GARCH models. We show that the performance of bootstrap forecast densities is badly affected by outliers when they are based on the QML estimator of the parameters and filter for the volatility. Using robust estimators and filters improves the finite sample performance. We also show that it is important to combine an estimator of the parameters with appropriate robust properties with an adequate filter for the volatility. By using the bootstrap extraction of the residuals, we propose a modification of the volatility filter that has the best properties when combined with the BVT estimator of the parameters. We also consider alternative robust bootstrap extraction procedures, as the smoothing and the winsorized procedures and show that, the performance of the PRR procedure is not improved when using these alternative bootstrap extraction mechanisms.

The proposed procedure is implemented to a series of daily Euro/Dollar exchange rates and compared with the available QML procedure. We show that, although the bootstrap densities for returns are similar in both cases, the forecast intervals obtained for volatilities using QML and the proposed procedure are very different with the latter being more appropriate to explain the out-of-sample dynamics of realized volatilities.

The bootstrap procedure proposed in this paper is a robust tool for practitioners to compute forecast densities for returns and volatilities based on GARCH models in the presence of outliers. Note that the definition of outliers considered in this paper, is related with models with jumps that are at present very popular in the literature; see, for example Chan and Maheu (2002) and Vlaar and Palm (1993). However, as discussed by Boudt

⁵Alternative realized measures, such as realized volatility (RV), bipower variation (BV) and MedRV, were also considered; see Barndorff-Nielsen and Shepard (2004) and Zhang et al. (2005). In all cases, the PRR algorithm with BVT2 is shown to be superior to the PRR algorithm with QML.

et al. (2013), modelling jumps usually requires the specification and estimation of a model which governs both the time-varying jump intensity and the jump size. Moreover, the estimated jumps often have wide confidence bands due to the low frequency of extremes in the sample.

Several issues deserve further research. First of all, it would be important to propose data-driven procedures to choose the threshold constant c in the trimming function. As mentioned above, this constant could be different when the uncontaminated standardized returns, ϵ_t , are non-Gaussian. Also, in the context of non-Gaussian errors, the BVT estimator of the parameters should be modified or alternatively other robust estimators of the parameters not based on Gaussianity as that recently proposed by Hill (2015) could be implemented to estimate the parameters.

The extension of the bootstrap procedure proposed in this paper to models with an asymmetric response of volatility to positive and negative returns is also of interest; see Maheu and McCurdy (2004) and Chiu et al. (2005) for models with asymmetries and jumps.

Furthermore, in the presence of non-Gaussian errors, it is also of interest to explore the alternative specification of the volatility equation proposed by Harvey (2013) and Creal et al. (2013) based on the score. Note that the robust EWMA estimator of the volatility proposed by Guermat and Harris (2002) and based on an underlying Laplace distribution is a particular case of the models proposed by Creal et al. (2013); see Dupuis et al. (2015) for an empirical application. Using the bootstrap techniques developed in this paper to construct forecast densities for returns and volatilities in the context of score models and compare them to those obtained in this paper is also in our research agenda.

Finally, in the spirit of Boudt et al. (2013) for multivariate robust DCC models and Fresoli and Ruiz (2015) for bootstrap procedures in the same models, we project to extend the results in this paper to construct robust bootstrap forecast of volatilities and correlations in multivariate systems.

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Table 1: Monte Carlo results based on 500 replicates of size $T = 1000$ of the bivariate cDCC(1,1) model with Gaussian errors and parameters $\omega_1 = 0.05$, $\alpha_1 = 0.10$, $\beta_1 = 0.85$. The unconditional variance is 1. $\omega = 0, 5$ and 10 . P. stands for outlier position: (1) at observation $t = 500$; (2) at observation $t = 999$; (3) at observations $t = 500$ and 501 ; (4) at observations $t = 998$ and 999 . In parenthesis RMSE. Bottom panel bias and RMSE for volatilities.

		Estimator											
ω	P.	QML			BM			BS			BVT		
		0	5	10	0	5	10	0	5	10	0	5	10
		0.060 (0.029)			0.066 (0.036)			0.062 (0.047)			0.065 (0.040)		
α_0	1		0.076 (0.059)	0.159 (0.198)		0.067 (0.037)	0.067 (0.037)		0.070 (0.059)	0.076 (0.070)		0.066 (0.041)	0.066 (0.041)
	2		0.067 (0.043)	0.098 (0.121)		0.066 (0.036)	0.066 (0.036)		0.065 (0.051)	0.066 (0.053)		0.066 (0.042)	0.067 (0.042)
	3		0.107 (0.101)	0.254 (0.319)		0.073 (0.049)	0.069 (0.039)		0.096 (0.096)	0.148 (0.169)		0.067 (0.043)	0.068 (0.043)
	4		0.078 (0.056)	0.142 (0.182)		0.070 (0.042)	0.067 (0.038)		0.071 (0.055)	0.074 (0.059)		0.067 (0.042)	0.067 (0.042)
		0.101 (0.024)			0.103 (0.028)			0.102 (0.025)			0.105 (0.031)		
α_1	1		0.100 (0.031)	0.122 (0.076)		0.102 (0.028)	0.102 (0.028)		0.108 (0.032)	0.116 (0.037)		0.105 (0.031)	0.105 (0.031)
	2		0.101 (0.030)	0.116 (0.065)		0.103 (0.028)	0.103 (0.028)		0.105 (0.028)	0.109 (0.031)		0.105 (0.031)	0.105 (0.031)
	3		0.122 (0.038)	0.159 (0.087)		0.102 (0.031)	0.099 (0.028)		0.120 (0.035)	0.136 (0.050)		0.104 (0.032)	0.104 (0.032)
	4		0.118 (0.033)	0.146 (0.078)		0.110 (0.034)	0.106 (0.033)		0.117 (0.030)	0.122 (0.034)		0.104 (0.031)	0.105 (0.031)
		0.838 (0.045)			0.830 (0.055)			0.836 (0.059)			0.830 (0.063)		
β	1		0.826 (0.075)	0.742 (0.211)		0.829 (0.056)	0.829 (0.056)		0.828 (0.072)	0.821 (0.082)		0.830 (0.064)	0.829 (0.064)
	2		0.837 (0.059)	0.807 (0.141)		0.831 (0.055)	0.830 (0.055)		0.835 (0.064)	0.834 (0.065)		0.829 (0.065)	0.829 (0.065)
	3		0.775 (0.127)	0.621 (0.334)		0.822 (0.072)	0.829 (0.058)		0.788 (0.119)	0.725 (0.202)		0.829 (0.066)	0.829 (0.066)
	4		0.810 (0.077)	0.743 (0.203)		0.820 (0.067)	0.826 (0.062)		0.818 (0.071)	0.815 (0.073)		0.829 (0.064)	0.829 (0.064)
							75.674						
ν	1							20.792	11.178				
	2							21.770	11.404				
	3							19.998	10.024				
	4							21.534	11.223				
		1.009 (0.157)			1.021 (0.186)			1.025 (0.179)			1.038 (0.179)		
σ_z^2	1		1.061 (0.178)	1.324 (0.517)		1.005 (0.172)	1.008 (0.173)		1.146 (0.306)	1.291 (0.455)		1.039 (0.180)	1.039 (0.180)
	2		1.116 (0.224)	2.339 (3.623)		1.020 (0.185)	1.022 (0.185)		1.132 (0.253)	1.269 (0.633)		1.039 (0.180)	1.039 (0.180)
	3		1.084 (0.190)	1.359 (0.546)		0.983 (0.163)	0.979 (0.160)		1.114 (0.256)	1.319 (1.143)		1.040 (0.180)	1.040 (0.180)
	4		1.165 (0.269)	2.633 (3.568)		1.040 (0.203)	1.030 (0.196)		1.150 (0.259)	1.272 (0.403)		1.040 (0.181)	1.040 (0.180)
		0.000 (0.043)			-.014 (0.064)			-.006 (0.063)			-.002 (0.067)		
σ_t	1		0.019 (0.088)	0.067 (0.202)		-.014 (0.066)	-.014 (0.065)		0.012 (0.063)	0.029 (0.071)		-0.001 (0.067)	-0.001 (0.067)
	2		0.017 (0.063)	0.059 (0.136)		-.013 (0.064)	-.013 (0.064)		0.010 (0.064)	0.024 (0.067)		-0.001 (0.067)	-0.001 (0.067)
	3		0.028 (0.132)	0.082 (0.294)		-.016 (0.072)	-.016 (0.071)		0.009 (0.071)	0.028 (0.087)		-0.001 (0.068)	0.000 (0.068)
	4		0.024 (0.090)	0.069 (0.206)		-.011 (0.067)	-.012 (0.067)		0.012 (0.064)	0.026 (0.068)		-0.001 (0.068)	0.000 (0.068)
		-.003 (0.044)			-.017 (0.066)			-.011 (0.072)			-.007 (0.076)		
σ_T	1		0.008 (0.055)	0.041 (0.108)		-.018 (0.067)	-.018 (0.066)		0.006 (0.073)	0.023 (0.076)		-0.007 (0.077)	-0.007 (0.077)
	2		1.095 (1.124)	2.743 (2.899)		0.167 (0.204)	0.167 (0.204)		0.027 (0.119)	0.039 (0.106)		0.003 (0.105)	0.002 (0.100)
	3		0.012 (0.057)	0.049 (0.114)		-.022 (0.072)	-.022 (0.070)		0.002 (0.075)	0.017 (0.090)		-0.005 (0.076)	-0.006 (0.076)
	4		1.841 (1.864)	4.406 (4.507)		0.416 (0.472)	0.399 (0.458)		0.047 (0.205)	0.051 (0.131)		0.023 (0.161)	0.017 (0.116)

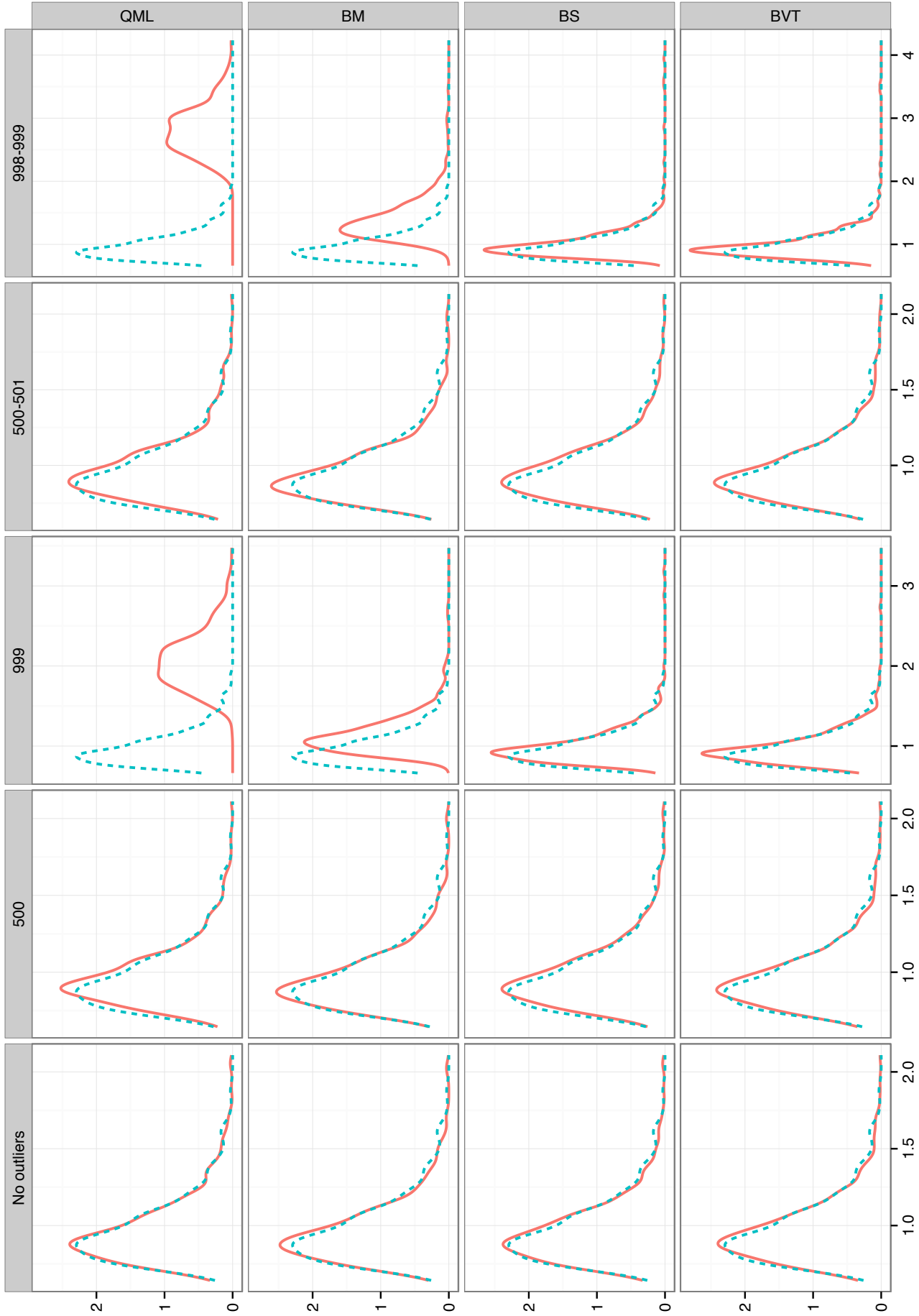


Figure 1: Kernel densities of $\hat{\sigma}_{1000}$ (solid line) and σ_{1000} (dashed line) using QML (first row), BM (second row), BS (third row) and BVT (fourth row) when the series is contaminated with one outlier of size $w_t = 5$ at $t = 500$ (second column) and $t = 999$ (third column) and with two consecutive outliers at $t = 500, 501$ (fourth column) and $t = 998, 999$ (fifth column).

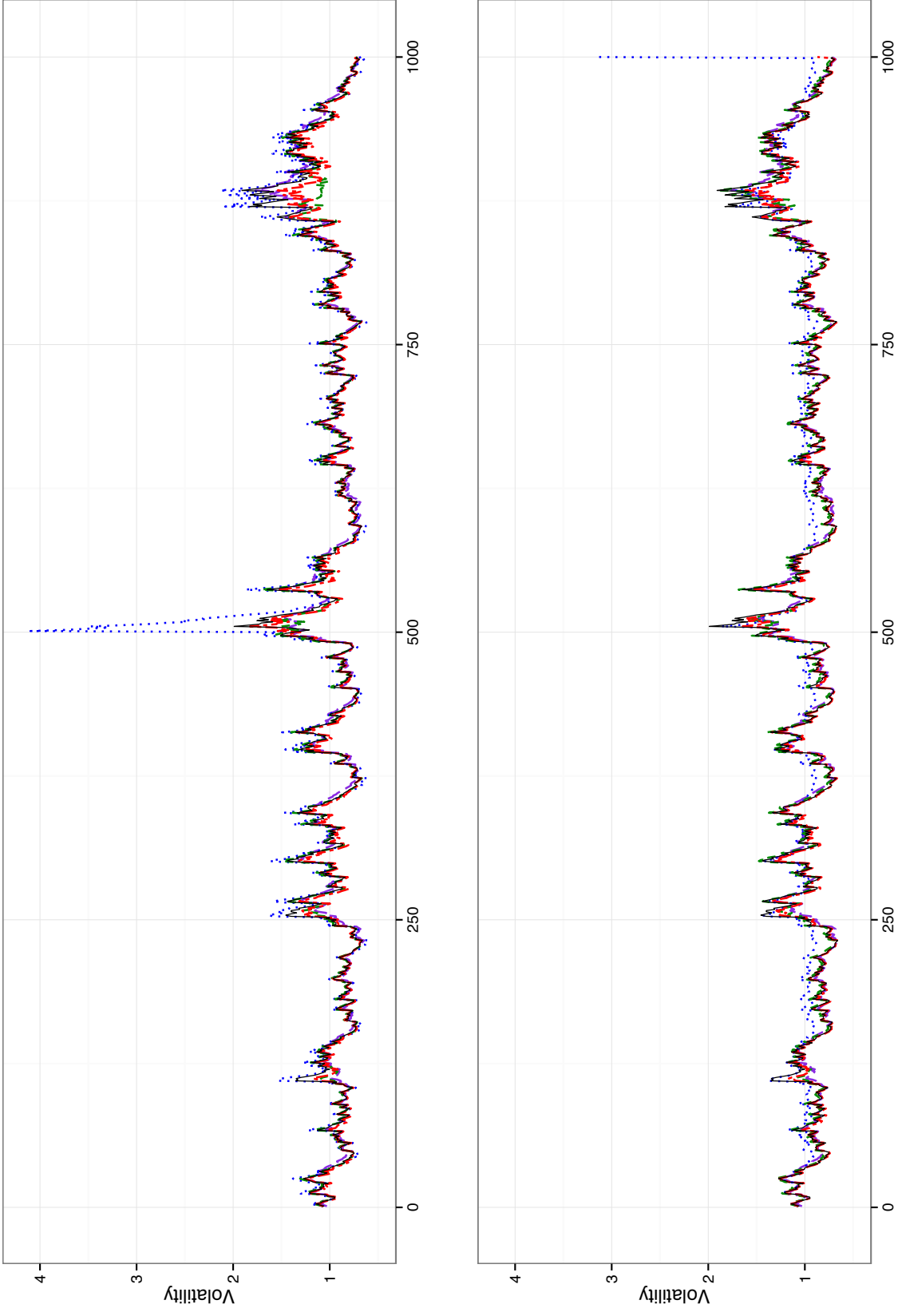


Figure 2: Volatility of a series generated by a GARCH(1,1) model (solid line) and estimated volatilities using QML (dotted line), BM (two dashed line), BS (dashed line) and BVT (long dashed line) when the series is contaminated by one outlier of size $w_t = 5$ at $t = 500$ (top panel) and $t = 999$ (bottom panel).

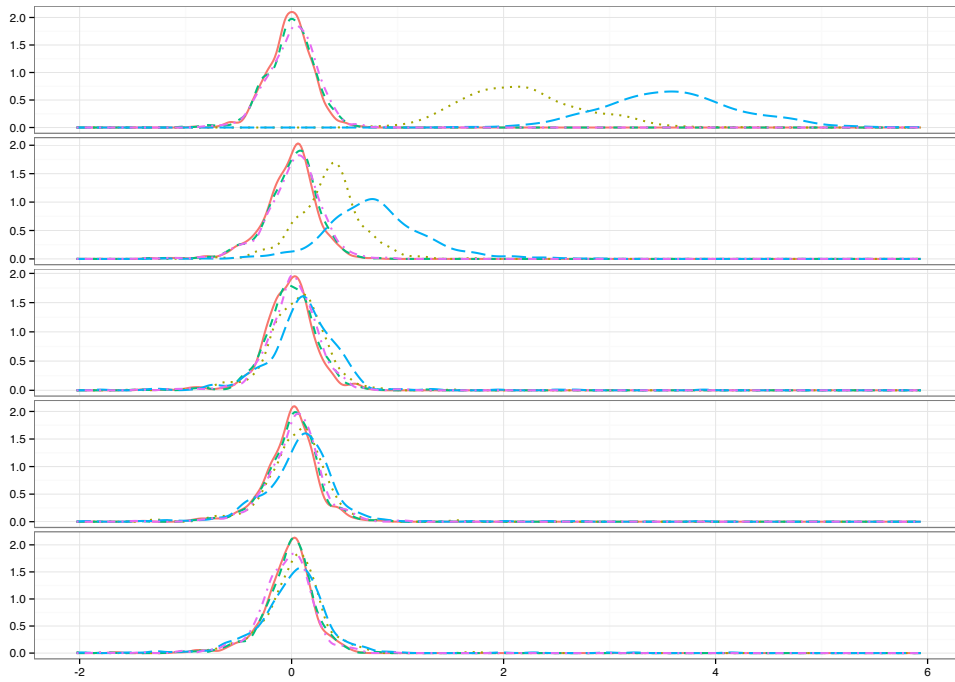


Figure 3: Monte Carlo density of the differences between one-step-ahead empirical and bootstrap 1% VaR forecasts obtained using the PRR algorithm with QML (top panel), BM (second panel), BS (third panel), BVT1 (fourth panel) and BVT2 (bottom panel) procedures in uncontaminated process (solid line), contaminated at $t = 500$ (dot dashed line), at $t = 500$ and 501 (dashed line), at $t = 999$ (dotted line) and finally at $t = 998$ and 999 (long dashed line).

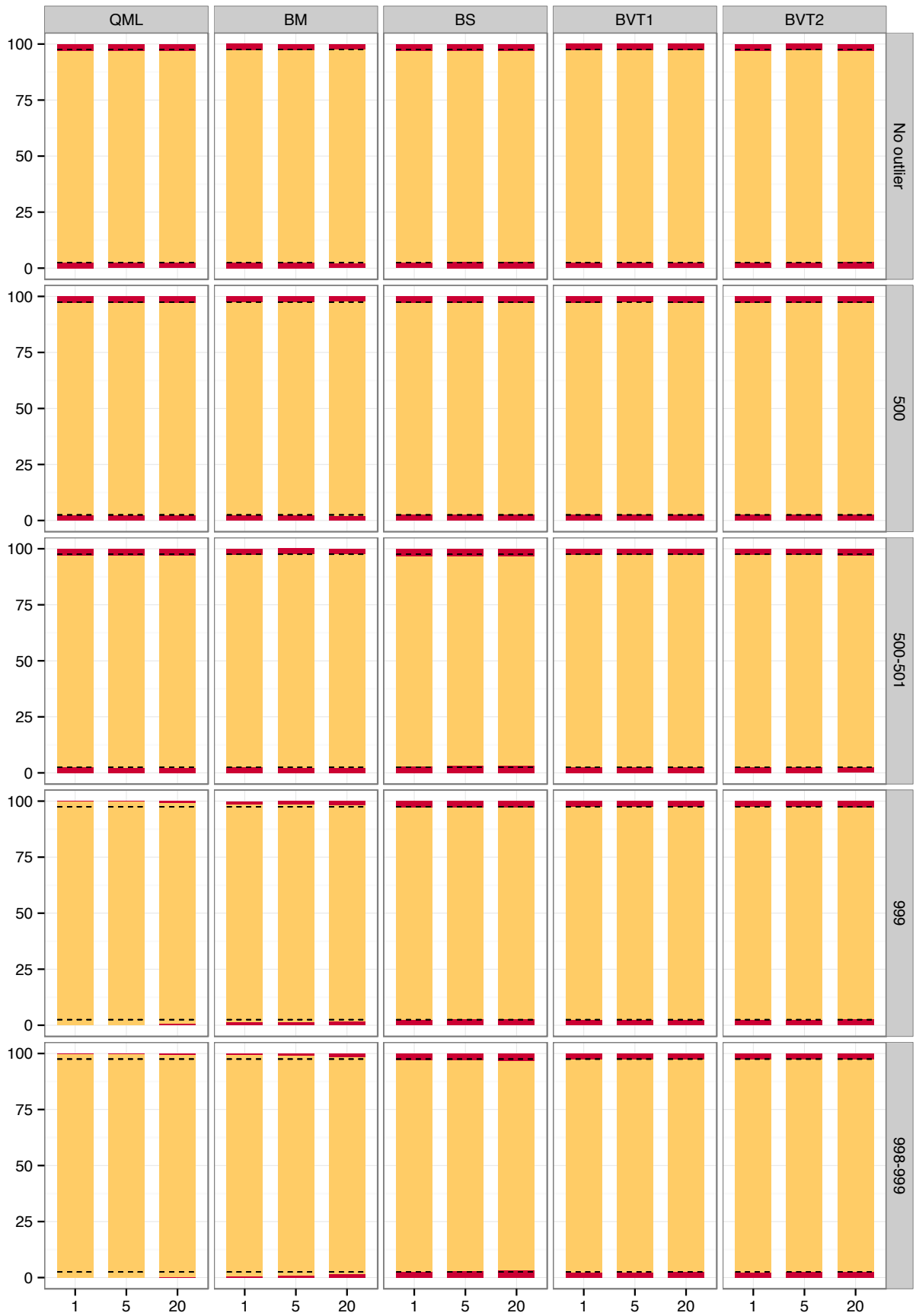


Figure 4: Estimated coverage and coverages above and below of the 95% bootstrap forecast interval for returns using the PRR algorithm with QML (first column), BM (second column), BS (third column) and BVT1 (fourth column) and BVT2 (fifth column) procedures. $T = 1000$

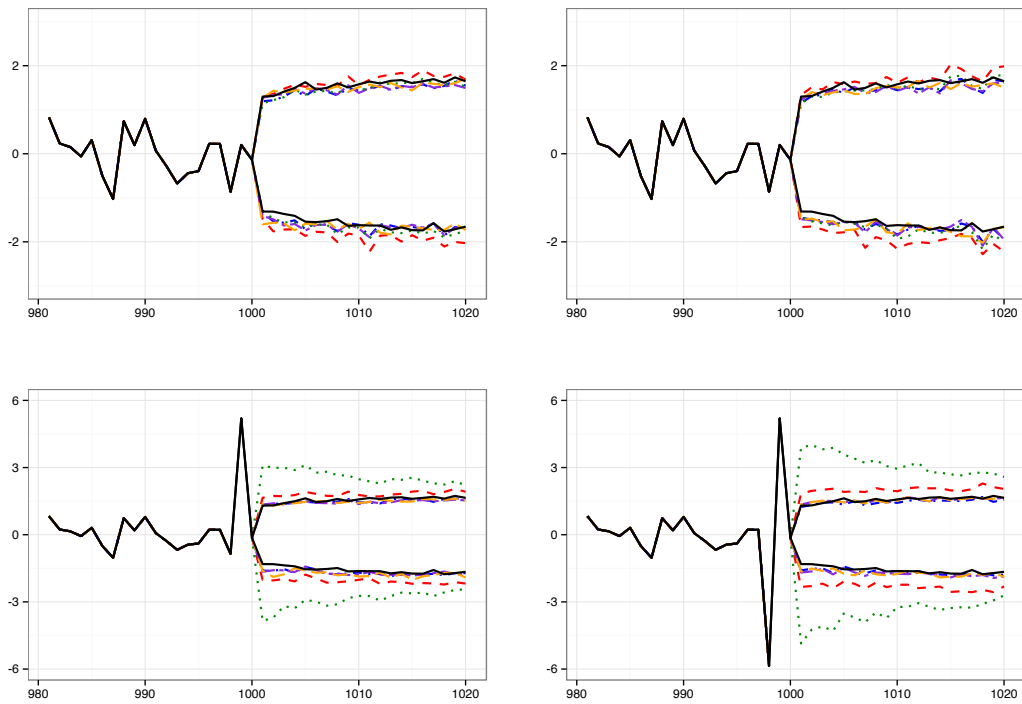


Figure 5: 95% bootstrap forecast intervals for a series generated by a GARCH model (top left panel) and contaminated by two consecutive outliers at $t = 500, 501$ (top right panel), one isolated outlier at $t = 999$ (bottom left panel) and two consecutive outliers at $t = 998, 999$ (bottom right panel) of size $w_t = 5$. The intervals are constructed using the PRR algorithm with QML (dotted line), BM (dashed line), BS (dot dashed line), BVT1 (two dashed line) and BVT2 (long dashed line) procedures. The true interval (solid line) is constructed as the empirical 0.025 and 0.975 quantiles obtained from 1000 futures values of y_{T+h} .

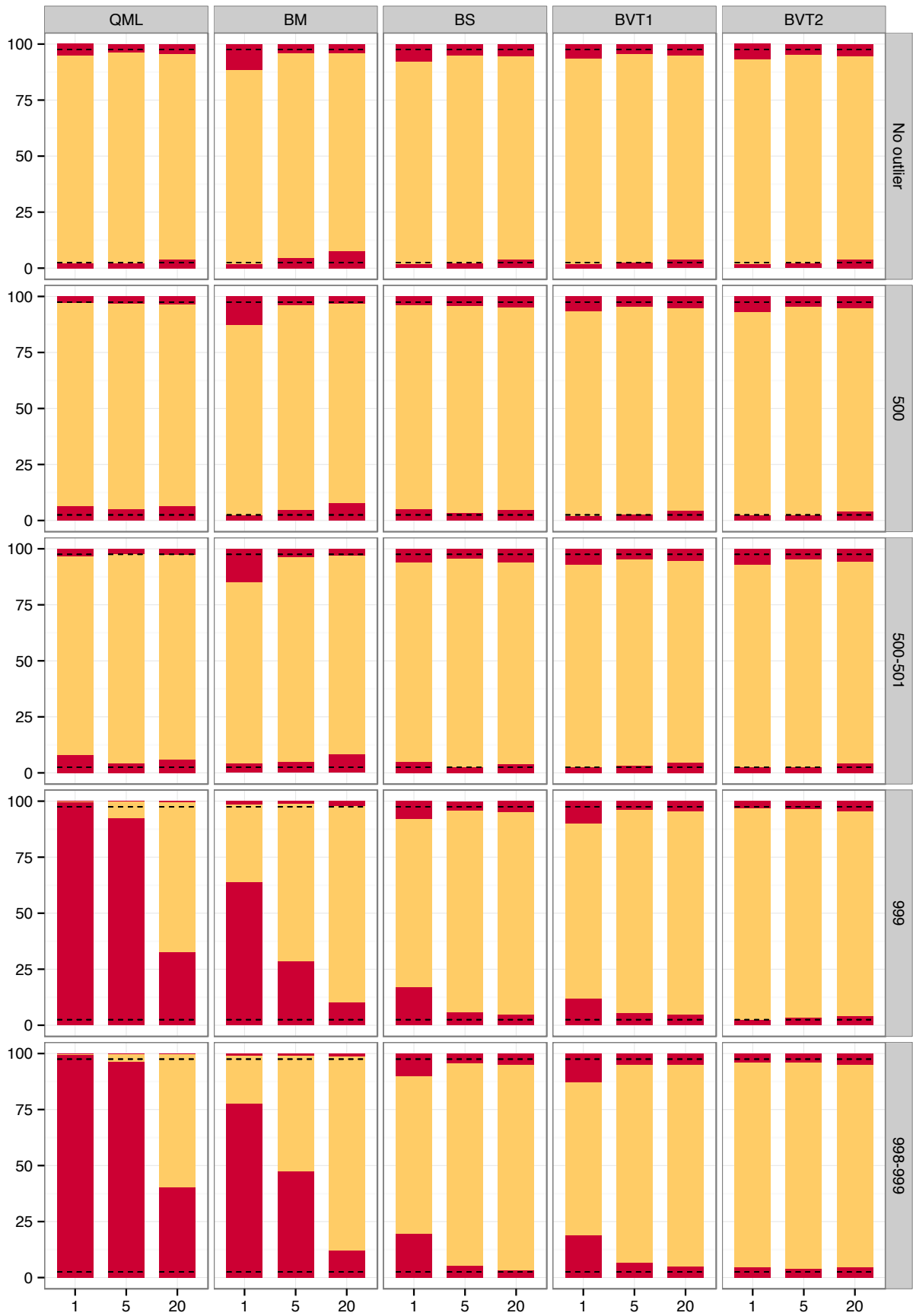


Figure 6: Estimated coverage and coverages above and below of the 95% bootstrap forecast interval for volatilities using the PRR algorithm with QML (first column), BM (second column), BS (third column), BVT1 (fourth column) and BVT2 (fifth column) procedures. $T = 1000$

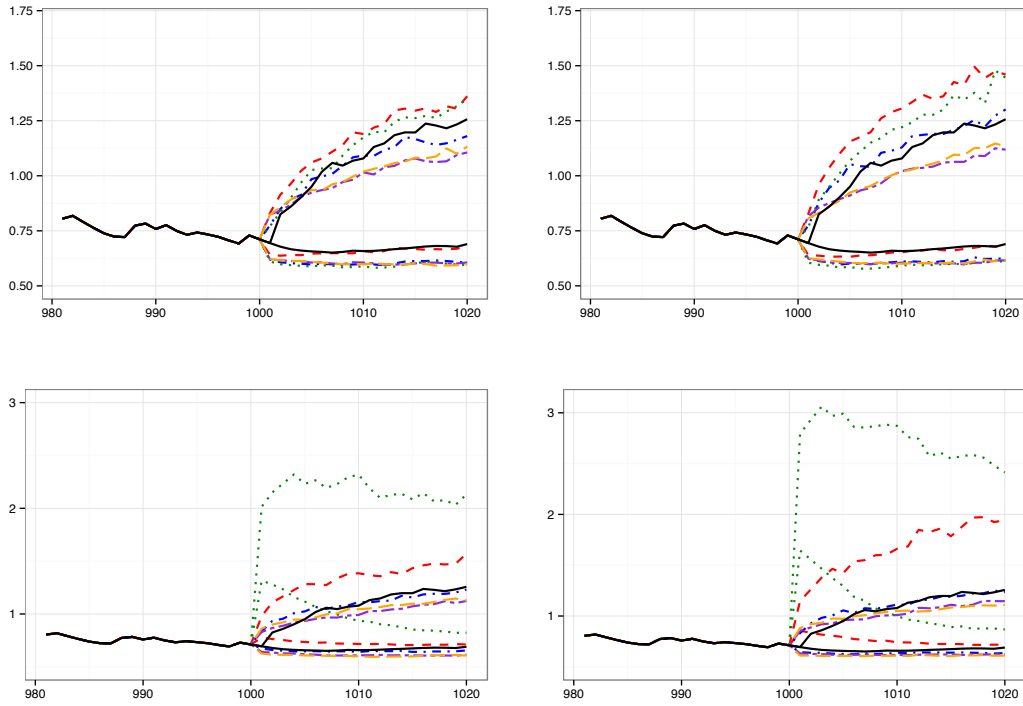


Figure 7: 95% bootstrap forecast intervals for volatilities of a GARCH model (top left panel) and contaminated by two consecutive outliers at $t = 500, 501$ (top right panel), one isolated outlier at $t = 999$ (bottom left panel) and two consecutive outliers at $t = 998, 999$ (bottom right panel) of size $w_t = 5$. The intervals are constructed using the PRR algorithm with QML (dotted line), BM (dashed line), BS (dot dashed line), BVT1 (two dashed line) and BVT2 (long dashed line) procedures. The true interval (solid line) is constructed as the empirical 0.025 and 0.975 quantiles obtained from 1000 futures values of σ_{T+h} .

Table 2: Monte Carlo proportion of failures of one-step-ahead out-of-sample 1% VaR forecasts estimated using the PRR algorithm with alternatives robust procedures.

PRR	No outlier	500	500-501	999	998-999
$\omega = 5$					
QML	0.012	0.011	0.011	0.000	0.000
BM	0.012	0.011	0.011	0.005	0.002
BS	0.012	0.012	0.013	0.012	0.012
BVT1	0.012	0.011	0.011	0.011	0.010
BVT2	0.012	0.012	0.011	0.011	0.010
$\omega = 10$					
QML	-	0.012	0.011	0.000	0.000
BM	-	0.011	0.011	0.005	0.002
BS	-	0.017	0.024	0.017	0.025
BVT1	-	0.011	0.010	0.011	0.010
BVT2	-	0.011	0.011	0.010	0.009

Table 3: Summary statistics for the daily returns exchange rate EUR/USD from January 4, 2000 to December 31, 2013.

Mean	S.D	Min.	Q_1	Med.	Q_3	Max.	Asym.	Kurt.
0.000	0.648	-3.012	-0.358	-0.001	0.374	4.612	0.076	5.118

Table 4: 1% VaR proportion of failures and p-values of backtesting tests.

	Proportion of failures	Unconditional coverage	Conditional coverage	Independence
QML	0.018	0.29	0.53	0.70
BVT2	0.013	0.63	0.85	0.78

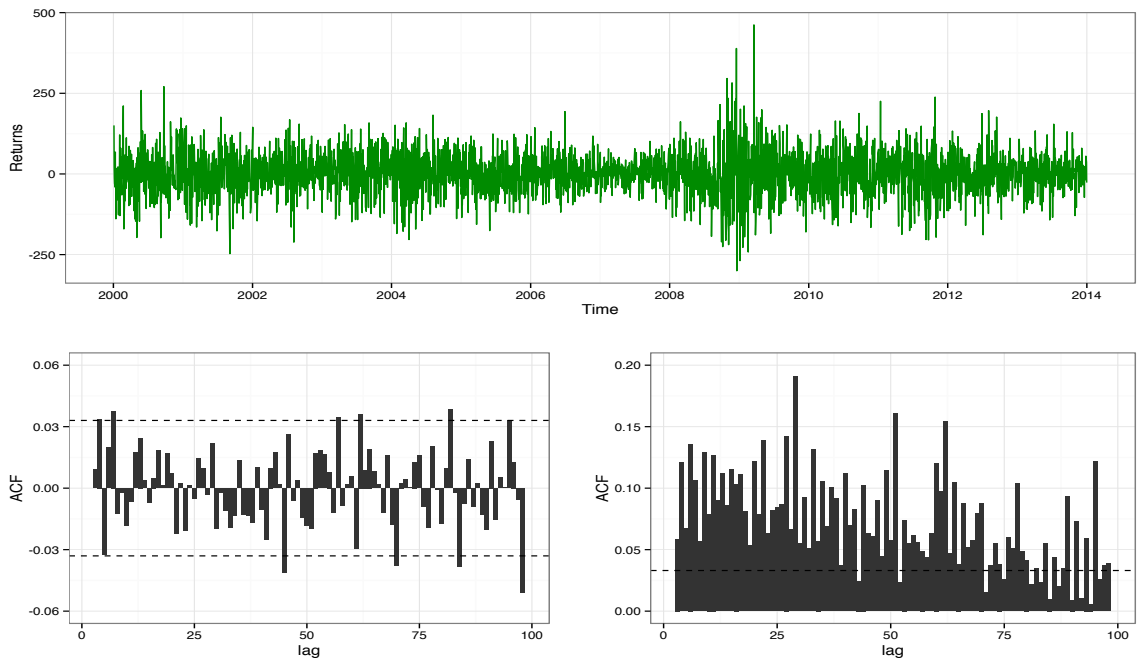


Figure 8: Daily EUR/USD exchange rates returns observed (top panel), sample autocorrelation of returns and squared returns (bottom panel) from January 4, 2000 to December 31, 2013.

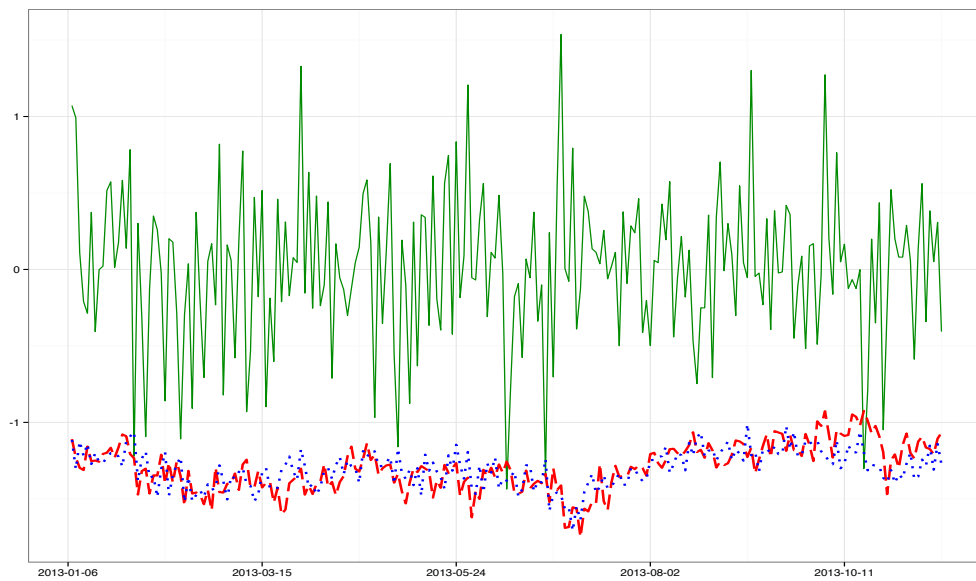


Figure 9: Observed returns and 1% Value-at-Risk obtained using the PRR with QML (dotted line) and BVT2 (dashed line) implementations.

Table 5: Coverage of the 95% and 99% h -step-ahead bootstrap forecast intervals for returns using the QML and BVT2 implementations of the PRR algorithms using rolling window. $h = 1, 2, 5, 10$ and 20 steps-ahead.

Method	Steps ahead	95%			99%		
		Coverage	Below	Above	Coverage	Below	Above
QML	1	93.78	3.11	3.11	98.22	0.89	0.89
	2	94.22	3.11	2.67	99.11	0.00	0.89
	5	95.11	2.67	2.22	98.67	0.44	0.89
	10	94.67	3.11	2.22	99.11	0.00	0.89
	20	94.67	3.11	2.22	100.0	0.00	0.00
BVT2	1	92.89	3.56	3.55	98.22	0.44	1.34
	2	94.22	3.11	2.67	98.22	0.44	1.34
	5	94.22	3.56	2.22	97.78	0.89	1.33
	10	94.67	3.11	2.22	99.11	0.00	0.89
	20	95.11	2.67	2.22	99.56	0.00	0.44

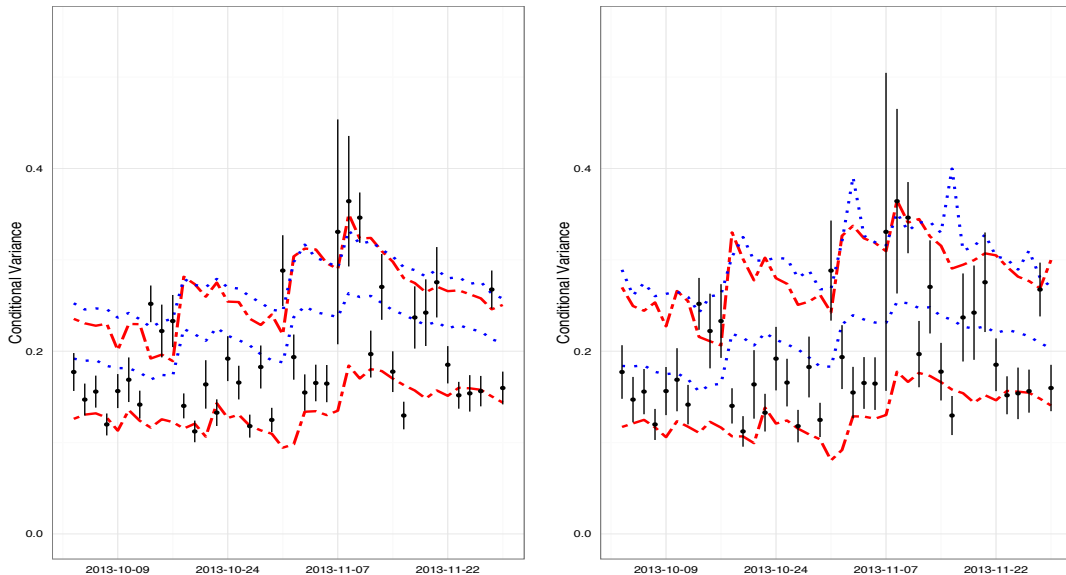


Figure 10: One-step-ahead bootstrap forecast intervals for volatilities based on QML (dotted lines) and BVT2 (dashed lines). Interval with 95% coverages are represented left panel while the right panel plots intervals with 99% coverages. The vertical lines represent confidence intervals for MinRV with the bullet representing the point estimate.