Games and Economic Behavior 79 (2013) May, pp. 56-66

DOI:10.1016/j.geb.2013.01.004

Heterogeneous network games: Conflicting preferences

Penélope Hernández [a,*], Manuel Muñoz-Herrera [b], Ángel Sánchez [c,d]

[a] *ERI-CES, Departamento de Análisis Económico, Universitat de Valencia, Spain*
[b] *ICS, Faculty of Behavioral Social Sciences, University of Groningen, Grote Rozenstraat 15, 9712 TG, Groningen, The Netherlands*
[c] *GISC, Departamento de Matemáticas, Universidad Carlos III de Madrid, Madrid, Spain*
[d] *BIFI, Universidad de Zaragoza, Spain*

Abstract: In many economic situations, a player pursues coordination or anti-coordination with her neighbors on a network, but she also has intrinsic preferences among the available options. We here introduce a model which allows to analyze this issue by means of a simple framework in which players endowed with an idiosyncratic identity interact on a social network through strategic complements or substitutes. We classify the possible types of Nash equilibria under complete information, finding two thresholds for switching action that relate to the two-player setup of the games. This structure of equilibria is considerably reduced when turning to incomplete information, in a setup in which players only know the distribution of the number of neighbors of the network. For high degrees of heterogeneity in the population the equilibria is such that every player can choose her preferred action, whereas if one of the identities is in the minority frustration ensues.

Keywords: Heterogeneity, Social networks, Incomplete information, Bayesian equilibrium.

1. Introduction

In multiple social and economic interactions agents aim to coordinate their choices to improve their wellbeing. Thus, for instance, individuals purchase products, choose schools or attend to social events influenced by the decisions of those around them, be they their friends, family or colleagues. As this environment generally takes the form of a social network, it is clear that the study of coordination problems in networks is key to understand social outcomes and welfare through strategic behavior. In this regard, the current literature in economics assumes these interactions to be anonymous, so there is no intrinsic difference between agents involved in them: What mainly determines a player's choices is her position in the network. Nonetheless, individual preferences are fundamental to place in context the strength of social influence over such decisions. For example, when choosing between two products, a potential purchaser requires less influence from her neighbors to acquire the one she likes than the other option.

We here propose a model that considers how agents decide on what behavior to adopt when they have intrinsic differences between their preferences. Within this framework, we develop a complete characterization of the network configurations in equilibrium for two classes of games: strategic complements and strategic substitutes. The equilibrium characterization is made for two conditions: complete and incomplete information. With such a simple model, we are able to pinpoint the effect of heterogeneity on preferences and to identify the impact on refinement as compared to the frame-work introduce by Galeotti et al. (2010). These authors analyzed two well-known and important classes of games, namely strategic complements and strategic substitutes, finding that on a network, local information arising from the network

* Corresponding author.
E-mail address: penelope.hernandez@uv.es (P. Hernández).

structure leads to a set of (symmetric) equilibria that is very much related to the pairwise case. When information is incomplete, equilibria are refined and only certain types of them are allowed, in particular those with specific monotonicity properties of the actions with respect to the number of connections of the player. We here show that introducing heterogeneity in this framework leads, first, to a very large set of Nash equilibria under complete information, which can be classified in different types according to the satisfaction of the players and the diversity in their actions; and, second, the refinement attained under incomplete information is much more stringent, in so far as only a restricted class of equilibria survives. It is important to stress that we are able to obtain this results in a very simple model, which by virtue of its simplicity allows to make clear all these differences and provides an intuition of what could take place in more complicated situations.

1.1. Framework

Our model studies a broad set of coordination games with strategic complements (substitutes). Agents interact in a fixed network by choosing an action from a binary choice set. Each individual has an order for the available choices so that one is preferred over the other. This order represents a player's identity. Coordinating in the liked option gives greater payoffs than in the disliked one. Independently of the chosen action, the more neighbors a player coordinates (anti-coordinates) with, the greater her utility in games with strategic complements (substitutes). This allows an analysis of heterogeneity in which players are endowed with different identities so that their incentives to choose one action or another vary, even with the same number of neighbors. In addition, we consider heterogeneity in the level of connectivity each player has, so that two players with the same identity need not have equal number of neighbors. The linear payoff structure of our model allows to differentiate the behavior in the two classes of games for different distributions of identities and connectivities in the network. We specifically model networks as random graphs of the Erdös–Renyi type. In this scenario, the probability that a pair of connected players share a common neighbor is very low. Thus, the behavior of a player's neighbors can be safely assumed to be independent of each other.

Our work makes a contribution to the research program on strategic behavior in social and economic networks by considering a rich set of coordination games with heterogeneous players, both in complete and incomplete information. The equilibrium characterization of our network games is carried in two directions. First we provide a complete characterization when players are informed about the size and shape of the network and the distribution of identities in it. This leads to the Nash equilibria of the network games. The natural relaxation of the informational assumptions allows for asymmetries where a player knows her identity but not that of others in the network. Individuals know the number of their contacts and the distribution of connections in the population, but are informed only of the probability of the identity of their contacts at the moment of making a choice (note that if there is no information at all on the neighbors' identity, we would be in a setup of homogeneous networks, similar to that studied by Galeotti et al., 2010). Subsequently, we characterize Bayesian–Nash equilibria for such games.

1.2. Our contributions

We distinguish two threshold functions, which determine the tipping point where players switch from their liked to their disliked option. The threshold value is the minimum proportion of neighbors necessary to coordinate with to guarantee that choosing the preferred option gives greater payoffs. For example, in the case of choosing between two social events, say two parties, someone who likes party A over B, needs less of her friends to go to the first than to the second one for her to choose to go with them. Because of this, identity heterogeneity affects the structure and conditions of the game through the identity of the players interacting. There are conflicting preferences on the desired outcome when two players of different (the same) identity interact in games with strategic complements (substitutes). Given their best responses we observe multiple Nash equilibria expressed by the distribution of choices in the network. We denote as *specialized* the cases where the entire set of players coordinates in one same action. Depending on the distribution of identities in the network, in this action profile there are players who choose what they like (*satisfactory*) and others who choose the disliked option (*frustrated*). When both actions coexist in the network, the Nash equilibrium portrays a *hybrid* case which can be satisfactory or frustrated as well.

The integration of heterogeneity in a simple manner into the analysis of social networks allows to disentangle the system of incentives of the players involved. Even when two action profiles in networks with the same configuration of links are identical, the distribution of identities in the population reveals key differences in terms of payoffs in their choice. The simplest example is a society where all individuals coordinate in the same choice. By the inclusion of an analysis on identities in terms of preferences over the available options, we observe that if one or more players are *frustrated* because their choice is based on the influence of their neighbors but not on their preference, they have stronger incentives to deviate.

Finally, we conclude our analysis with a relaxation of the assumption of complete information. Players are informed about the probability to be connected to different neighbors with different identities but do not know the preferences of these neighbors at the moment of choosing. As shown by Galeotti et al. (2010), this is a natural way to introduce incomplete information in network games, so that a player knows her preferences and has a good forecast of the number of her connections but has incomplete information about the degrees of others and their individual preferences. We characterize

the existence of pure symmetric Bayesian equilibria for different distributions of identities. This means that depending on how the population is distributed, the resulting configurations can either be specialized or frustrated: indeed, if there are enough players of both types, we show that they can play their action of choice, whereas when one large majority dominates over a small minority, we are able to establish monotonicity results which, for instance, lead to (almost) all players choosing the same action in the case of strategic complements. Interestingly, we are able to prove our main result weakening an assumption in Galeotti et al. (2010) while keeping their formation rule, thus obtaining a dramatic reduction of the complete information equilibria.

1.3. Relation to the literature on network games

The literature studying strategic interactions in networks and its applications to economics has grown increasingly for the past years. Our work contributes to this research program and in particular to the study of coordination network games. For detailed surveys of the literature see Goyal (2007), Jackson (2008), and Vega-Redondo (2007); see also Roca et al. (2009) for a review of the literature from the evolutionary point of view. Our main contribution is to model the strategic interaction of agents with different identities or preferences when either complete or incomplete information is available.

The main objective of the paper is to provide a tractable framework where heterogeneity in individual characteristics of the players as well as specifications of informational asymmetries can be analyzed in a single model. Other works that have considered identities of players in terms of individual productivity (Rogers, 2005) or linking costs (Galeotti et al., 2006) are also part of the research on heterogeneity in networks. Our model differs from these for they are focused on strategic link formation and our analysis considers players are located in a fixed network. Several authors have also modeled games with strategic complements or substitutes on networks, see for example Angeletos and Pavan (2007), Ballester et al.(2006), Ballester and Calvó-Armengol (2010), Bergemann and Morris (2009), Calvó-Armengol et al. (2009), Bénabou (2008), Bramoullé and Kranton (2007), Ilkilic (2008), Glaeser and Scheinkman (2003), Goyal and Moraga-Gonzalez (2001), and Vives (1999). Our work belongs in this set of literature and complements it with the analysis of incomplete information and the characterization of the Bayesian–Nash equilibria. In particular, our work feeds and is closely related to Galeotti et al. (2010) for they consider games with strategic complements and substitutes and model equilibria for complete and incomplete information. The main difference in our analysis is the considerations of heterogeneity, by which players can differ in their preference for one action or another, leading to conflicting preferences between them.

Our work also relates and contributes to the literature on threshold[1] models. The pioneering work by Granovetter (1978) referred to thresholds as the proportion of others (neighbors) who must adopt a certain behavior for a given player to do so. In this direction, works such as Morris (2000) and López-Pintado (2006) model coordination games in network structures where there is a contagion threshold that determines how an action spreads. Our work mainly differs from their analysis for we consider static games while they study best-response dynamics. However, we complement these works by assuming heterogeneity in the identities of players. The introduction of players with different preferences over the binary choice set allows for asymmetry in the payoffs between players, which in consequence causes the risk dominant action not to be the same for all players. As a consequence, although in a simplified static model, we study a richer setting where there is not one default action for all individuals, but each of them has preferences for one over the other. Note that this is different from homophily in the sense of Golub and Jackson (2010), where types control the probability of being linked, but do not make any difference in the preferences of individual players. In this sense, part of our contribution is that we obtain two different threshold functions, one for each identity of players for them to adopt the disliked action.

The paper is organized in five sections including this introduction. Section 2 describes the model. Section 3 contains the characterization of equilibria in complete information. Section 4 presents the analysis with incomplete information. Section 5 concludes.

2. The model

Consider the social network (N, g). The set $N = \{1, \ldots, n\}$, where $|N| \geqslant 2$, contains the players interacting in a game. This set is fixed throughout the analysis, so we represent the network by the set of links, g. Prior to the start of the game, players are informed about the size of the network and the identity of all players. The set of potential connections is the complete network, g^N, and any network configuration is part of the set $G = \{g\colon\ g \subset g^N\}$. In the network, if a pair of players i and j are connected by a link, it is denoted as $ij \in g$, and if there is no link between them, we say $ij \notin g$. The set of neighbors a player i has is $k_i(g) = \{j\colon\ ij \in g\},\ \forall j \neq i$. For simplicity we assume that $ii \notin g$, so that all neighbors in $k_i(g)$ are different from i. The cardinality of $k_i(g)$ is k_i, the degree of node i in the network, and is exogenously determined prior to the interactions.

[1] Equilibria in our model depend on a threshold of the chosen actions by a player's neighbors which also relates to the literature on conformism and social norms (i.e., Kandel and Lazear, 1992), where there is a trade off between a player's choice for her preferred action and increasing the distance from the average behavior chosen by her neighbors. This is a relation between peer pressures and partnerships, where peer pressure arises when individuals deviate from a well-established group norm, i.e., individuals are penalized for working less than the group norm. Out threshold functions model when players are willing to *imitate* the behavior of others even if it is not their preferred one and conform to the average behavior or not.

Players from the set N interact in a network game denoted by Γ. Every player $i \in N$ is *ex-ante* and exogenously endowed with an identity $\theta_i \in \{0, 1\}$. Players choose an action from the binary set $X = \{0, 1\}$ which is the same for all players. A player i who has identity 1 (0) prefers action 1 over 0 (0 over 1). We denote $x_{k_i}(g)$ as the vector of actions taken by i's neighbors. The game is expressed through a linear payoff function, $u_i(\Gamma)$, that strategically depends on the choices made by connected players and their identities, as follows:

$$u_i\big(\theta_i, x_i, x_{N_i}(g)\big) = \lambda_{x_i}^{\theta_i}\Bigg[1 + \delta \sum_{j \in k_i(g)} I_{\{x_j = x_i\}} + (1 - \delta) \sum_{j \in k_i(g)} I_{\{x_j \neq x_i\}}\Bigg], \tag{1}$$

where $I_{\{x_j = x_i\}}$ is the indicator function of those neighbors choosing the same action as player i, and $I_{\{x_j \neq x_i\}}$ indicates neighbors choosing the opposite. The parameter λ is defined by $\lambda_{x_i}^{\theta_i} = \alpha$ when a player chooses what she likes, and $\lambda_{x_i}^{\theta_i} = \beta$ otherwise, where $0 < \beta < \alpha$. The class of game played is specified through the multiplier δ, that takes value 1 if the game is of strategic complements (SC) and 0 if it is of strategic substitutes (SS).

The main feature of our utility specification is that it captures several strategic scenarios in a simple way, allowing for games of strategic complementarities or substitutes. As a result, we can observe the way players' payoffs are affected by the choices of others given their individual preferences. This is motivated by our desire to develop an understanding of how the conflict of preferences interacts with the network structure. In addition, as discussed in the Introduction, by introducing players' types we are extending the applicability of network game models to situations in which the preferences of different players may not be aligned.

We now consider a partial order $\succcurlyeq_i$ on the action profiles of the neighbors for a given player i. Fix a player i with k_i neighbors and identity θ_i, where $x_{k_i}(g)$ and $x'_{k_i}(g)$ are two action profiles of her neighbors. We say that $x_{k_i}(g) \succcurlyeq_i x'_{k_i}(g)$ if $\sum_{j=1}^{k_i} I_{\{x_j=1\}} \geqslant \sum_{j=1}^{k_i} I_{\{x'_j=1\}}$. For player i, the actions of her partners can be ordered depending on the number of neighbors playing the action 1. When more individuals in i's neighborhood play the action 1, the corresponding action profile is ordered in a higher position. The payoff function for an SC ($\delta = 1$) game verifies the following condition when the actions of player i verify $1 = x_i > x'_i = 0$ and the action profiles of her neighbors verify $x_{k_i}(g) \succcurlyeq_i x'_{k_i}(g)$:

$$u_i\big(\theta_i, x_i, x_{k_i}(g)\big) - u_i\big(\theta_i, x'_i, x_{k_i}(g)\big) \geqslant u_i\big(\theta_i, x_i, x'_{k_i}(g)\big) - u_i\big(\theta_i, x'_i, x'_{k_i}(g)\big). \tag{2}$$

Notice that when $x_{k_i}(g) \succcurlyeq_i x'_{k_i}(g)$ then the number of zero actions in $x'_{k_i}(g)$ is larger than in $x_{k_i}(g)$. By multiplying by (-1) we get $u_i(\theta_i, 0, x'_{k_i}(g)) - u_i(\theta_i, 1, x'_{k_i}(g)) \geqslant u_i(\theta_i, 0, x_{k_i}(g)) - u_i(\theta_i, 1, x_{k_i}(g))$. From both equations above, we conclude that in SC it is more profitable to play the action that your neighbors play more.

Analogously, when the game is SS ($\delta = 0$), the payoff function u_i verifies the following condition when $1 = x_i > x'_i = 0$ and $x_{k_i}(g) \succcurlyeq_i x'_{k_i}(g)$:

$$u_i\big(\theta_i, x_i, x_{k_i}(g)\big) - u_i\big(\theta_i, x'_i, x_{k_i}(g)\big) \leqslant u_i\big(\theta_i, x_i, x'_{k_i}(g)\big) - u_i\big(\theta_i, x'_i, x'_{k_i}(g)\big). \tag{3}$$

Analogously than in SC case, we obtain that in SS, it is more profitable to play the action that your neighbors play less.

Players in our game, represented by $\Gamma = \{N,\ \{g\}_{i,j \in N},\ X,\ \{\theta_i\}_{i \in N},\ \{u_i\}_{i \in N}\}$, decide on an action from the binary choice set X. A unilateral deviation by player i changes her choice x_i to choice x'_i, where $x_i \neq x'_i$. When no player has incentives to deviate from an action profile $(x_1^*, \ldots, x_n^*)$, it is a Nash equilibrium. Formally:

$$u_i\big(\theta_i, x_1^*, \ldots, x_i^*, \ldots x_n^*\big) \geqslant u_i\big(\theta_i, x_1^*, \ldots, x'_i, \ldots, x_n^*\big) \quad \forall x_i^* \neq x'_i,\ \forall i \in N.$$

Note that $u_i(\theta_i, x_1^*, \ldots x_n^*) = u_i(\theta_i, x_i, x_{k_i}(g))$, i.e., the actions of players that are not i's neighbors do not change her payoff.

2.1. The 2-person game: strategic complements and substitutes

We model games with strategic complements (SC) or strategic substitutes (SS) in a 2-person setting. Specifically:

Definition 1 *(Strategic complements (coordination)).* Let SC be a 2-person game where every player has an identity $\theta_i \in \{0, 1\}$ and the finite set of actions X. The payoff matrix,[2] where $2\beta > \alpha > \beta > 0$, depends on each player's choices and identity as follows (Table 1):

Table 1
Payoff matrices for SC games.

		0	
		1	0
1	1	$2\alpha, 2\beta$	α, α
	0	β, β	$2\beta, 2\alpha$

$\theta_1 = 1;\ \theta_2 = 0$

		1	
		1	0
1	1	$2\alpha, 2\alpha$	α, β
	0	β, α	$2\beta, 2\beta$

$\theta_1 = 1;\ \theta_2 = 1$

		0	
		1	0
0	1	$2\beta, 2\beta$	β, α
	0	α, β	$2\alpha, 2\alpha$

$\theta_1 = 0;\ \theta_2 = 0$

[2] We consider a payoff structure such that a player prefers to coordinate in the disliked option than staying alone. This payoff structure is observed in the game of the Battle of Sexes, for an example of the n-person game see Szidarovszky et al. (2008).

Definition 2 *(Strategic substitutes (anti-coordination)).* Let SS be a 2-person game where every player has an identity $\theta_i = \{0, 1\}$ and the finite set of actions $x_i = \{0, 1\}$. The payoff matrix, where $2\beta > \alpha > \beta > 0$, depends on each player's choices and identity as follows (Table 2):

Table 2
Payoff matrices for SS games.

		0	
		1	0
1	1	α, β	$2\alpha, 2\alpha$
	0	$2\beta, 2\beta$	β, α

$\theta_1 = 1;\ \theta_2 = 0$

		1	
		1	0
1	1	α, α	$2\alpha, 2\beta$
	0	$2\beta, 2\alpha$	β, β

$\theta_1 = 1;\ \theta_2 = 1$

		0	
		1	0
0	1	β, β	$2\beta, 2\alpha$
	0	$2\alpha, 2\beta$	α, α

$\theta_1 = 0;\ \theta_2 = 0$

Each 2×2 coordination (anti-coordination) game can be played between two players of equal or opposite identities. There are two Nash equilibria in pure strategies and one in mixed strategies. Let us first discuss the pure strategy equilibria. For the SC case the Nash equilibria in pure strategies $psNE = \{(0, 0), (1, 1)\}$ present conflicting preferences when the two players have opposite identities given that each likes a different action and both want to coordinate. Thus, it is not possible to Pareto rank them. However, in games between players with equal identity there is no conflict in preferences because each one likes the same action, and the equilibrium when both choose the action corresponding to their identity is Pareto dominant in payoffs: $(1, 1)$ Pareto dominates $(0, 0)$ if two players with identity 1 are playing, and the opposite for two players with identity 0. For the case of SS the $psNE = \{(0, 1), (1, 0)\}$ shows no conflicting preferences when the two players have opposite identities because both are better-off when choosing the action corresponding to each of their identities, which is the Pareto dominant Nash equilibrium of the game. For example: $(1, 0)$ Pareto dominates $(0, 1)$ if the first player is of identity 1 and the second is of identity 0. Conflicting preferences arise when two players with the same identity interact, because both of them like the same action and want to anti-coordinate.

Let us now consider the mixed strategy equilibrium. For the SC game, the probability to choose your favorite action when playing against a player of your same identity is obtained from the corresponding payoff matrix and is given by $\underline{q} = (2\beta - \alpha)/(\alpha + \beta)$. When playing against a player of different identity, the result is $q = (2\bar{\alpha} - \beta)/(\alpha + \beta)$. Following Morris (2000) and López-Pintado (2006), these probabilities can be understood as the adoption threshold function, i.e., the proportion of neighbors making a given choice required for a player to adopt that same action. Heterogeneity in preferences gives a new insight to this by showing that the q needed varies depending on the identity of the player choosing, but not on the identity of the player(s) she is interacting with. That is, there exist $\underline{q} < q$, where $\underline{q}$ is the probability of choosing the liked action and q the disliked action. The intuition of this result relates directly to the Nash equilibrium configurations of the network games, and it is associated to many social scenarios where the utility of affiliation is based on choices of others and not on a player's preferences, but the utility of the individual is based both on her choice and her preference. A similar result holds for the SS case exchanging the probabilities.

3. Equilibrium: complete information

In this section of complete information we characterize the set of Nash equilibria for our network game Γ, NE(Γ). We develop in detail the analysis for games with SC, which symmetrically hold for SS unless the opposite is specified.

3.1. Strategies

A player in the network game Γ chooses an action in the set $X = \{0, 1\}$, the same for all her connections. The action profiles in the network are such that either all players coordinate on one action (*specialized*) or both actions are chosen by different players (*hybrid*). Having in mind the identity of the players, there are two possible categories, depending on whether all players coordinate in choosing the action for which $x_i = \theta_i$ (*satisfactory*) or at least one player chooses $x_i \neq \theta_i$ (*frustrated*). Thus, we have four possible configurations: (i) satisfactory specialized (S_S) where all players coordinate on the same action, which is their preferred choice, $x_i = \theta_i$; (ii) frustrated specialized (F_S), where all players coordinate on the same action, but at least one of them is choosing her disliked option, $x_i \neq \theta_i$; (iii) satisfactory hybrid (S_H), where all players choose the action they prefer but there is at least one player with an identity different from the rest, so that both actions are present; and (iv) frustrated hybrid (F_H) which portray both actions and at least one player chooses her disliked option. Fig. 1 illustrates these categories for games with SC.

3.2. Nash equilibrium

We characterize the Nash network games, NE(Γ), in relation to players' unilateral deviations. To that end, we will call the number of player i's neighbors choosing action 1 χ_i; correspondingly the number of her neighbors choosing action 0 is $k_i - \chi_i$. We will denote by $\lceil \ldots \rceil$ and $\lfloor \ldots \rfloor$ respectively the maximum lower integer or the minimum higher integer of the real number considered. Proposition 1 presents the best responses for games with SC, characterizing the thresholds to change actions. Note that for SS the relation of χ_i with the thresholds is inverted.

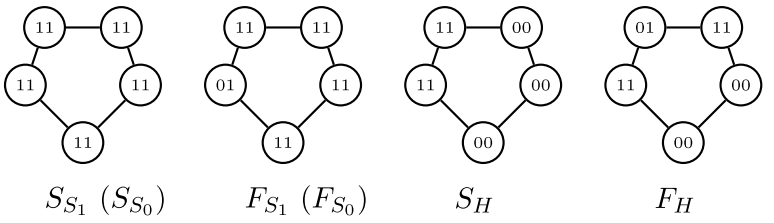

Fig. 1. Types of configurations. The first digit refers to the identity and the second to the action.

Proposition 1. *For an SC game, let*

$$\underline{\tau}(k_i) = \left\lceil \frac{\beta}{\alpha+\beta}k_i - \frac{\alpha-\beta}{\alpha+\beta} \right\rceil, \tag{4}$$

$$\overline{\tau}(k_i) = \left\lfloor \frac{\alpha}{\alpha+\beta}k_i + \frac{\alpha-\beta}{\alpha+\beta} \right\rfloor, \tag{5}$$

defined for any degree $k_i \in \{1, \ldots, N-1\}$. The best response of player i with identity $\theta_i = 1$ and degree k_i, x_i^*, is

$$x_i^* = \begin{array}{ll} 1, & \textit{iff } \chi_i \geqslant \underline{\tau}(k_i), \\ 0, & \textit{otherwise}. \end{array} \tag{6}$$

The best response of player i with identity $\theta_i = 0$ and degree k_i, x_i^*, is

$$x_i^* = \begin{array}{ll} 0, & \textit{iff } \chi_i \leqslant \overline{\tau}(k_i), \\ 1, & \textit{otherwise}. \end{array} \tag{7}$$

□

Proof. For simplicity we develop the proof in terms of action 1 for the case of SC, however it extends naturally for action 0 and also for the case of SS.

Suppose that $\chi_i \geqslant \underline{\tau}(k_i)$ for a player $i \in N$ with identity $\theta_i = 1$. She gets a payoff $u_i(1, 1, ((x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n))$. Following the payoff functions in Eq. (1), we have: $u_i(1, 1, ((x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)) = u_i(1, 1, x_{k_i(g)})$. Therefore,

$$\begin{aligned} u_i(1,1,x_{k_i(g)}) &= \alpha(1+\chi_i) \geqslant \alpha\big(1+\underline{\tau}(k_i)\big) = \alpha\left(1+\left\lceil \frac{\beta}{\alpha+\beta}k_i - \frac{\alpha-\beta}{\alpha+\beta}\right\rceil\right) \geqslant \alpha\left(1+\frac{\beta}{\alpha+\beta}k_i - \frac{\alpha-\beta}{\alpha+\beta}\right) \\ &= \alpha\left(1+k_i-k_i+\frac{\beta}{\alpha+\beta}k_i - \frac{\alpha-\beta}{\alpha+\beta}\right) = \alpha\left(1+k_i-\left[k_i\left(\frac{-\beta}{\alpha+\beta}+1\right)+\frac{\alpha-\beta}{\alpha+\beta}\right]\right) \\ &= \alpha\left(1+k_i-\left[k_i\frac{\alpha}{\alpha+\beta}+\frac{\alpha-\beta}{\alpha+\beta}\right]\right) > \beta\big(1+k_i-\underline{\tau}(k_i)\big) > \beta(1+k_i-\chi_i) = u_i(1,0,x_{k_i(g)}). \end{aligned}$$

The remaining cases can be proven straightforwardly in the same manner. □

In SC a player i wants to coordinate with the highest number of neighbors making the same choice, and prefers coordination on the action corresponding to her identity. Players with identity $\theta_i = 1$ have incentives to choose the action they like when $\chi_i \geqslant \underline{\tau}(k_i)$. Thus, players with identity $\theta_i = 0$ choose $x_i = 0$ if $\chi_i \leqslant \overline{\tau}(k_i)$. We illustrate this in Fig. 2.

In games with SS a player i wants to anti-coordinate with the highest number of neighbors making the opposite choice. This means that no specialized configuration is Nash in pure strategies. The threshold functions in SS are inverted compared to SC. A players has incentives to choose the action she likes when $\chi_i \leqslant \overline{\tau}$ for $\theta_i = 1$, and $\chi_i \geqslant \underline{\tau}$ for $\theta_i = 0$.

With Proposition 1, we have characterized the best response of every player in terms of her identity and the actions of her neighbors. It is important to stress that this best response does not depend on the identities of her neighbors, but only on their actions. This result allows us to analyze the Nash equilibria and how do they depend on the distribution of identities, actions and network characteristics. From the best responses, it is clear that there will be very many different equilibria. Examples of these equilibria are illustrated in Fig. 3.

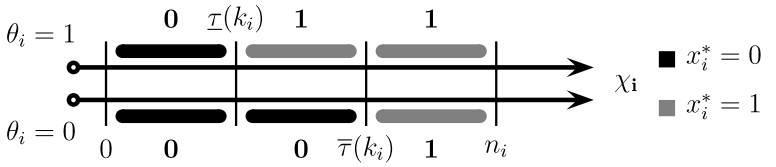

Fig. 2. SC thresholds.

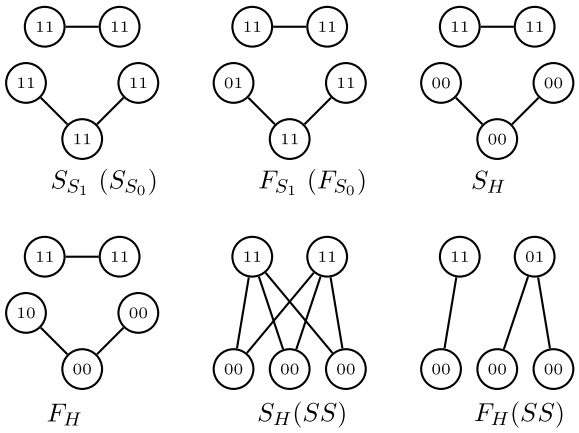

Fig. 3. Examples of Nash equilibria. The first digit refers to the identity and the second to the action. The two bottom-right configurations are examples for strategic substitutes.

As a specific example, consider the following case: Satisfactory specialized equilibria are very restrictive. Indeed, let us assume that all players have identity 1. Then, in an S_S equilibrium, all players choose action 1. However, if for any reason a player or group of players play action 0 and for them the condition $\chi_i \geqslant \underline{\tau}(k_i)$ is not verified, we would again have a Nash equilibrium, but it would not be satisfactory, i.e., it would be frustrated hybrid. In general, if all players have the same identity, if an equilibrium is satisfactory it has to be specialized with players choosing the action they like.

There is another manner in which specialized equilibria emerge, namely when the distribution of identities is not homogeneous and condition either $\chi_i \leqslant \underline{\tau}(k_i)$ or $\chi_i \geqslant \overline{\tau}(k_i)$ holds for all players. As a consequence, for the same distributions of links and identities, two players with opposite identities can best respond with the same action and vice versa.

4. Incomplete information

The above section is devoted to characterize the Nash equilibrium set when players know the realized neighbors' actions, i.e. the complete information framework. We study now how the structure of the network and the configuration on identities affect the set of equilibria. In this setup we introduce an incomplete information framework where players know their own identity and degree, but have no information about the whole network. Players beliefs about the rest of the network are given by a probability distribution over the connections (the formation rule) and the distribution of identities. The consideration of a framework of incomplete information in our setup is motivated by our aim to obtain a more realistic approach of the interaction of conflicting preferences in coordination problems. Based on the assumption of a network structure *a l a* Erdös–Renyi, where the size of the network is very big, incomplete information characterizes the reduction of information into a local level. Moreover, it has been shown by results found in Galeotti et al. (2010) that an incomplete information framework in network games reduces the multiplicity of Nash equilibria obtained in complete information. As we will show below, in our case and in spite of the fact that we use weaker assumptions, the set of equilibria is much more drastically reduced, allowing for configurations in which every player chooses her preferred action when there is enough heterogeneity in the network.

We assume that both processes, formation rule and allocation of identities in the network, are independent. The probability that player i has k neighbors depends on the formation rule, which in our setup accounts for an Erdös–Renyi network structure. In addition, a player i knows that with probability $0 \leqslant \rho \leqslant 1$, each agent in her neighborhood is of identity 1 and with probability $1-\rho$ is of identity 0. Then, the incomplete information set up could be understood as players having local knowledge of the network.

Given a formation rule for a network Γ, we denote by $P(k_i)$ the probability that player i has k_i neighbors. Given that probability, a player i can compute the conditional probability of the degree of her neighbors given that she has k_i neighbors. The distribution of these conditional events with dimension k_i is represented by $P(\mathbf{k}_{N(i)}|k_i)$. For instance, if player i only has one neighbor, $k_i = 1$, player i should consider that her neighbor may have $1, 2, \ldots, N-1$ neighbors. Therefore, player i will compute $P(1|1), P(2|1), \ldots, P(N-1|1)$. If player i has two neighbors, $k_i = 2$, it is necessary to consider all possible degree combination for both neighbors of player i. For example, $P(1,1|2)$, $P(2,1|2)$, $P(2,2|2)$, $P(3,1|2)$ and so on. Notice that each player with degree k has the same information of any other agent with the same degree. It implies that we can assume anonymity among the agents.

Henceforth, the information structure given a rule of formation and ρ is denoted by the family of anonymous conditional probabilities $\mathbf{P} = \{P(\mathbf{k}, \boldsymbol{\theta}|k, \theta)_{\mathbf{k}\in\mathbb{N}^k, \boldsymbol{\theta}\in\{0,1\}^k}\}$.

Given the above ingredients, the network game with incomplete information is represented by $(\Gamma, \mathbf{P}, \rho)$. In our benchmark individuals have two identities 0 or 1 and this identity establishes a unique pure-best response. Under incomplete information the type of each agent depends on her degree and the identity θ_i, since it is her private information. Notice

that the available information for each agent includes her own identity θ_i but not the realization of her neighbors' identities. Therefore, the Bayesian game has the following features:

- The set of players $N = \{1, \ldots, n\}$.
- The binary set of actions $X = \{0, 1\}$ for $i \in N$. Denote by $\mathcal{X} = \Delta(X)$ the set of mixed strategies on the support of X.
- The type set is $T = \{0, 1, \ldots, N-1\} \times \Theta$ and denote by t_i the type of player i.
- The beliefs $\mathbf{P} = \{P(\mathbf{k}, \boldsymbol{\theta}|k, \theta)_{\mathbf{k}\in\mathbb{N}^k, \boldsymbol{\Theta}\in\{0,1\}^k}\}$.
- Payoffs come from the expected utility criterium.

A strategy of player i is a function from his private information, i.e., her type to her action set: $\sigma_i : \{0, 1, \ldots, N-1\} \times \Theta \to \mathcal{X}$. A Bayesian–Nash equilibrium is a strategy profile σ such that each player plays her best response given the strategy profile of the other agents. Notice that it is crucial on the valuation on the best response of player i, the computation of her expected payoff which depends on her beliefs. The beliefs rely on the network structure and the distribution of θ, or in other words, on the formation rule which generates the network and the distribution of identities ρ.

Given that player i's type is $t_i = (k_i, \theta_i)$, her beliefs[3] are $\mathbf{P}(\cdot|k_i, \theta_i)$. The strategy profile of the other players is σ_{-i} generating an action profile of length k_i. Each of these sequences is included in the expected payoff of player i with the corresponding probability denoted by $P_{-i}(\sigma, t_{-i}, t_i)$ induced by the beliefs $\mathbf{P}(\cdot|k_i, \theta_i)$ and σ. Hence, the expected payoff from choosing the action $x_i \in \mathcal{X}$ is:

$$U_i(\theta_i, x_i, k_i, \sigma_{-i}) = \int_{t_{-i}\in T_{-i}} u_i\big(\theta_i, x_i, \sigma_{-i}(t_{-i})\big)\, dP_{-i}(\sigma, t_{-i}, t_i).$$

The rule of formation taken into account in this paper consists of creating a link independently with equal probability between two nodes. This is the well-known Erdös–Rényi graph formation. This rule of formation has a significant property. If the formation rule is *i.i.d.* then the conditional probability when a player has k_i neighbors represented by $P(\mathbf{k}_{N(i)}|k_i)$ exhibits increasing beliefs with respect to the partial order on T_{-i}. In other words, if one player has more neighbors, she may think that the network is more connected. Moreover as the distribution ρ is independent of the formation rule, the two conditions together may be interpreted as if player i has k_i neighbors, then any node has the same degree and ρ of them have identity 1 and $1-\rho$ have identity 0.

The next proposition characterizes pure symmetric bayesian equilibria fixing the distribution of identities 0 and 1. It does not give information on the existence of other non-symmetric equilibria. We justify this class of equilibria since the belief formation is anonymous, therefore it is natural to assume that any player with the same private information will react with the same behavior. Namely, for a range of ρ and the above formation rule states that the different class of equilibria: non-decreasing, non-increasing and both. We here focus only on the SC case for the sake of brevity, but it is apparent from what follows that a similar result with reversed inequalities holds for SS for the cases of hybrid action profiles.[4]

Proposition 2. *Consider the game with incomplete information $(\Gamma, \mathbf{P}, \rho)$ with SC characteristics, independent formation rule, and independent distribution on degrees. Then,*

- *There exists a pure symmetric equilibrium $(\sigma_i^*)_{i\in N}$.*
- *If $\frac{\alpha}{\alpha+\beta} > \rho > \frac{\beta}{\alpha+\beta}$ then every symmetric equilibrium $(\sigma_i^*)_{i\in N}$ is $\sigma_i^*(k, \theta_i = 1) = 1$ and $\sigma_i^*(k, \theta_i = 0) = 0$ for all k.*
- *If $\rho > \frac{\alpha}{\alpha+\beta}$ then $\sigma_i(k, \theta)$ is non-decreasing in k for all θ. Moreover $\sigma_i^*(k, \theta_i = 1) = 1$ for all k.*
- *If $\rho < \frac{\beta}{\alpha+\beta}$ then $\sigma_i(k, \theta)$ is non-increasing in k for all θ. Moreover $\sigma_i^*(k, \theta_i = 0) = 0$ for all k.* □

Proof. The game Γ is a game with a finite number of players and a binary set of actions for each player. Moreover, the belief space depends on the Erdös–Rényi formation rule, which implies that the neighbors' degrees are stochastically independent, therefore the player's interim beliefs about the degree of her neighbors are weakly increasing in her own degree (see Galeotti et al., 2010).

Given the above conditions, applying Milgrom and Shannon (1994) and van Zandt and Vives (2007) the existence of at least one pure symmetric equilibrium holds. Let $(\sigma_i^*)_{i\in N}$ be a symmetric pure equilibrium.

Let us now characterize a class of symmetric pure equilibria.

Denote by $(\rho k(1), (1-\rho)k(0))$ an action profile with ρk ones and $(1-\rho)k$ zeros. In view of Eqs. (6) and (7), given $\rho > \frac{\alpha}{\alpha+\beta}$, the best response under complete information, we can check that $\rho k > \underline{\tau}(k)$ and $\rho k \leqslant \overline{\tau}(k)$ (note that ρk plays

[3] We assume that N is large enough so the conditional probability that $\theta_k = 1$ given that $\theta_i = 1$ can be approximated by ρ.

[4] Recall that the case of specialized action profiles where all players choose one same action is not an equilibrium in pure strategies for games with SS. Therefore, only conditions 1 and 2 in Proposition 2 are verified for SS.

the same role as χ_i in complete information). Therefore, for player i with identity $\theta_i = 1$ ($\theta_i = 0$), her best response is $x_i^* = 1$ ($x_i^* = 0$):

$$u_i\big(1, 1, \big(\rho k(1), (1-\rho)k(0)\big)\big) - u_i\big(1, 0, \big(\rho k(1), (1-\rho)k(0)\big)\big) \geqslant 0.$$

As a consequence, under incomplete information we get the same order since the beliefs do not change the increasing difference obtained under complete information:

$$U_i\big(1, k_i, 1; \sigma_{-i}^*\big) \geqslant U_i\big(0, k_i, 1; \sigma_{-i}^*\big).$$

In other words, when the distribution on identities is quite heterogeneous: $\frac{\alpha}{\alpha+\beta} > \rho > \frac{\beta}{\alpha+\beta}$ then, the best response of any agent i with k will be her own realization of identity. Consequently, the equilibrium strategy $(\sigma_i^*)_{i\in N}$ coincides with her identity for all k.

To prove the third point, let us assume that $(\sigma_i^*)_{i\in N}$ is not trivial, i.e., there exists a k' and an action such that $x' \in$ **supp**$(\sigma_{k'}^*)$

Case $\theta = 1$**.** We study now the case of players which identity realization is 1. For that we will compute the difference between the expected payoff for player i when she has $k+1$ nodes with respect to the case in which she has k nodes. Recall that Γ is a game with SC and $\rho > \frac{\alpha}{\alpha+\beta}$. We then have

$$\begin{aligned} &u_i\big(1, 1, \big(\rho(k+1)(1), (1-\rho)(k+1)(0)\big)\big) - u_i\big(1, 0, \big(\rho(k+1)(1), (1-\rho)(k+1)(0)\big)\big) \\ &\quad \geqslant u_i\big(1, 1, \big(\rho k(1), (1-\rho)k(0)\big)\big) - u_i\big(1, 0, \big(\rho k(1), (1-\rho)k(0)\big)\big). \end{aligned}$$

Actually, the above condition holds even for a smaller ρ, $\rho < \frac{\beta}{\alpha+\beta}$. Moreover, $u_i(1, 1, (\rho k(1), (1-\rho)k(0))) - u_i(1, 0, (\rho k(1), (1-\rho)k(0))) \geqslant 0$ then 1 is best response for k since $\theta = 1$. Then 1 is best response for $k+1$.

Given the condition for complete information, we can now express the expected payoff with the same order since the beliefs do not change the direction of the inequality in expected terms because the player's interim beliefs about the degree of her neighbors are weakly increasing in her own degree:

$$U_i\big(1, k+1, 1, \sigma_{-i}^*\big) - U_i\big(1, k+1, 0, \sigma_{-i}^*\big) \geqslant U_i\big(1, k, 1, \sigma_{-i}^*\big) - U_i\big(1, k, 0, \sigma_{-i}^*\big) \geqslant 0. \tag{8}$$

Therefore, $U_i(1, k+1, 1, \sigma_{-i}^*) - U_i(1, k+1, 0, \sigma_{-i}^*) \geqslant 0$ and consequently 1 is as well the best response when the type of player i is $(k+1, 1)$ and players play the symmetric equilibrium (σ_{-i}^*).

Case $\theta = 0$**.** Similarly to the previous case, given that $\rho > \frac{\alpha}{\alpha+\beta}$ for all k, 1 is the best response under complete information. Then we get for incomplete information the fixed strategy $\sigma_i^*(k, \theta) = 1$.

Note that the above results have been obtained for a continuous k. In actual realizations of networks, the degree k is a positive integer and $\overline{\tau}(k) \geqslant 1$. Therefore, isolated players with $\theta = 0$ will choose action 0, and for this reason we cannot conclude that the action is fixed for all players.

The last part of the proposition holds in a manner completely similar to the third point. □

It is natural to compare this result to the homogeneous case studied by Galeotti et al. (2010). To begin with, our proposition relies on an assumption that is weaker than the one used in that paper. Indeed, in Galeotti et al. (2010), adding neighbors who choose one of the actions, e.g., 0 for the SC case, does not change the payoff for any player. In contrast, in our setup, this condition does not apply since depending of the player's identity and always using SC as a specific example, action 0 may be the preferred action for player i and then her payoff improves. On the other hand, if the player's identity
is 1, the payoff may still change if the player has few connected nodes and ρ is at most $\frac{\beta}{\alpha+\beta}$. In spite of the fact that we do not have an analogue of what Galeotti et al. (2010) call property A, our result holds because when links are added, a fraction ρ of them will be of identity 1 and the remaining $1-\rho$ will be of identity 0, which suffices to guarantee the proper order on differences of payoffs. Note also that our proposition does use the same independent formation rule as Galeotti et al. (2010), so we can rightfully compare both results.

Importantly, the above proposition has crucial implications on the nature of the equilibria allowed under incomplete information. In particular we note the following: First, when the distribution on identities is very heterogeneous, $\frac{\alpha}{\alpha+\beta} > \rho > \frac{\beta}{\alpha+\beta}$ then satisfactory hybrid configurations appear as a consequence of symmetric equilibrium. Moreover, an immediate implication of the second statement of the proposition is that the trivial equilibrium is never a symmetric Bayesian equilibrium. Second, when the distribution on identities is extreme, for instance, when there is a large majority of players of identity 1 (a small minority of players of identity 1) $\rho > \frac{\alpha}{\alpha+\beta}$ ($\rho < \frac{\beta}{\alpha+\beta}$) and k is large enough, then frustrated specialized configurations are the result of symmetric equilibria. Our result is quite strong since it holds for $k \geqslant 1$, i.e., unless there are isolated nodes, the frustrated specialized configuration are equilibria, or in other words, the trivial strategy, in which all players choose the same action, arises as the symmetric Bayesian equilibrium. We thus see that the introduction of identities we are proposing in our framework leads to a drastic reduction of the set of equilibria possible under com-plete information. In contrast with the case of Galeotti et al. (2010), our set up, in particular the existence of large enough heterogeneity, adds the satisfactory hybrid configurations as equilibria where everyone plays her favorite action.

5. Discussion

Networks of economic, technological or social interaction are nowadays recognized as a key structure to understand how agents behave and contribute to the general economic activity. However, for all their ubiquity, they have not been considered in the body of economic literature until the beginning of this century. Work carried out so far on this subject has focused on modeling and understanding the effects of having a (possibly complex) network of interactions among anonymous actors where the only source of difference is the pattern of connections a given agent has. The main novelty of this paper is the introduction of intrinsic diversity in this scenario by analyzing the case in which agents have an identity. While, admittedly, this is still a very simplified model, our results show that allowing for heterogeneity in the economic interactions on the network leads to a wealth of interesting results even when sufficiently detailed local information is available.

The results we have obtained by studying a heterogeneous model are a noteworthy contribution to the research on games on networks. Thus, we have shown that the knowledge of the neighbors an agent has as well as their actions does not prevent us from classifying the possible equilibria, finding a rich scenario of Nash equilibria that can be either satisfied or frustrated, specialized or hybrid. While all these cases are possible, we have discussed that in general, under complete information, frustrated hybrid configurations will obtain. For SC games, this implies that even if the desirable outcome is that every player contributes, it will not be possible to reach such a situation in general. For both SC and SS games, the consequence of this result is that most of the times there will be frustrated players playing the action they do not like. Looking in detail to the structure of the network, it can be seen that those frustrated individuals will be those with the smallest degrees, in particular the leaves of the network. It is also interesting to note that the equilibria we have found when there are agents with different identities on the network, such that their identities indicate their preference over one action from the binary choice set, shows a monotonicity property on the number of neighbors an agent has choosing one of the two actions.

We have also considered a relaxation of the informational setup in which players have only limited knowledge of the network. In particular, they know the formation rule for the network (Erdös–Renyi random graph) and the probability that their neighbors are of a given identity. In this scenario, we have shown that specific predictions can be obtained about the equilibria of the games. In particular, for SC games, in case there are no large majorities of any identity, we have proven that every player can be choosing the action she likes in equilibrium. On the contrary, if one of the identities is a relative minority, the threshold depending on the game parameters, then equilibria are mostly specialized, except possibly for isolated players, i.e., equilibria are specialized on connected networks. This implies that in this situation the players in the minority will be frustrated, and the goal of achieving coordination in the network will be possible only at the prize of some agents not choosing what they prefer. On the other hand, when the identities occur in similar numbers, coordination on one same action (specialized) is not likely in equilibrium. All players will choose their preferred actions and both will coexist in the network. For the case of SS, we have not explicitly proven the result, but it is easy to see that the situation also holds. Nonetheless, for the cases of majority/minority only result 2 in Proposition 2 holds, but not results 3 and 4 because there is no specialized equilibrium in SS with pure strategies.

Acknowledgments

The authors wish to thank participants of the 2nd UECE Lisbon Meeting on Game Theory and Applications. P.H. thanks financial support from MEC, Spain (ECO-2010-20584) and Generalitat Valenciana (PROMETEO/2009/068) is gratefully acknowledged. A.S. thanks grants MOSAICO, PRODIEVO and RESINEE from MEC, Spain and MODELICO from Comunidad de Madrid, Spain.

References

Angeletos, George M., Pavan, Alessandro, 2007. Efficient use of information and social value of information. Econometrica 75, 1103–1142.

Ballester, Coralio, Calvó-Armengol, Antoni, Zenou, Yves, 2006. Who is who in networks. Wanted: The key player. Econometrica 74, 1403–1417.

Ballester, Coralio, Calvó-Armengol, Antoni, 2010. Interaction with hidden complementarities. Reg. Sci. Urban Econ. 40, 397–406.

Bénabou, Roland J., 2008. Groupthink: Collective delusions in organizations and markets. National Bureau of Economic Research, Inc. Unpublished manuscript.

Bergemann, Dirk, Morris, Stephen, 2009. Robust implementations in direct mechanisms. Rev. Econ. Stud. 76, 1175–1204.

Bramoullé, Yann, Kranton, Rachel E., 2007. Public goods in networks. J. Econ. Theory 135, 478–494.

Calvó-Armengol, Antoni, Patacchini, Eleonora, Zenou, Yves, 2009. Peers effects and social networks in education. Rev. Econ. Stud. 76, 1239–1267.

Galeotti, Andrea, Goyal, Sanyeev, Kamphorst, Jurjen, 2006. Network formation with heterogeneous players. Games Econ. Behav. 54, 353–372.

Galeotti, Andrea, Goyal, Sanyeev, Jackson, Matthew O., Yariv, Leaat, Vega-Redondo, Fernando, 2010. Network games. Rev. Econ. Stud. 77, 218–244.

Glaeser, Edward L., Scheinkman, Jose A., 2003. Non-market interactions. In: Advances in Economics and Econometrics: Theory and Applications, vol. 1. Cambridge Univ. Press, pp. 339–370.

Golub, B., Jackson, M.O., 2010. How homophily affects the speed of learning and best-response dynamics. Mimeo.

Goyal, Sanyeev, 2007. Connections: An Introduction to the Economics of Networks. Princeton Univ. Press, Princeton.

Goyal, Sanyeev, Moraga-Gonzalez, Jose L., 2001. R&D networks. RAND J. Econ. 32, 686–707.

Granovetter, Mark, 1978. Threshold models of collective behavior. Amer. J. Econ. Sociology 83, 1420–1443.

Ilkilic, Rahmin, 2008. Network of commons. Fondazione Eni Enrico Mattei. Unpublished manuscript.

Jackson, Matthew O., 2008. Social and Economic Networks. Princeton Univ. Press, Princeton.

Kandel, Eugene, Lazear, Edward P., 1992. Peer pressure and partnerships. J. Polit. Economy 100, 801–817.

López-Pintado, Dunia, 2006. Contagion and coordination in random networks. Int. J. Game Theory 34, 371–381.

Milgrom, Paul, Shannon, Chris, 1994. Monotone comparative statics. Econometrica 62, 157–180.
Morris, Stephen, 2000. Contagion. Rev. Econ. Stud. 67, 57–78.
Roca, Carlos P., Cuesta, José A., Sánchez, A., 2009. Evolutionary game theory: Temporal and spatial effects beyond replicator dynamics. Phys. Life Rev. 6, 208–249.
Rogers, Brian W., 2005. A strategic theory of network status. Northwestern Univ. Unpublished manuscript.
Szidarovszky, Ferenc, Szilagyi, Miklos N., Zhao, Jijun, 2008. An n-person battle of sexes game. Physica A 387, 3669–3677.
Van Zandt, Timothy, Vives, Xavier, 2007. Monotone equilibria in Bayesian games of strategic complementarities. J. Econ. Theory 134, 339–360.
Vega-Redondo, Fernando, 2007. Complex Social Networks. Econometric Society Monograph Series. Cambridge Univ. Press, Cambridge.
Vives, Xavier, 1999. Oligopoly Pricing: Old Ideas and New Tools. MIT Press, Cambridge.