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Sequential arbitrage measurement in bond markets: Theory and empirical applications in the Euro-zone

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Sequential arbitrage measurement in bond markets: Theory and empirical applications in the Euro-zone

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Abstract

We develop a mathematical programing approach in order to measure the arbitrage size in bond markets. Transaction costs may be incorporated. The obtained arbitrage measures have two interesting interpretations: On the one hand they provide the highest available arbitrage profit with respect to the price of the sold (bought) securities. On the other hand they give the minimum relative (per dollar) bid (ask) price modification leading to an arbitrage free market. Moreover, some primal problems lead to optimal arbitrage strategies (if available), while their dual problems generate proxies for the Term Structure of Interest Rates.

The developed methodology permits us to implement an empirical test in the Euro-zone during the Euro crisis. Classical literature justifies the relevance of empirical analyses verifying the degree of efficiency during market turmoils. Our empirical study of the German, French and Spanish sovereign bonds markets finds that the main arbitrage opportunities come from the price differences between maturity-matched strips or "On-The-Run Premium" for zero-coupon bonds. When we remove the strips and the zero-coupon bonds the arbitrage still exists in the Spanish market.

Key words. Portfolio optimization, Sequential arbitrage measurement, Pricing error, Sovereign debt, Euro crisis.

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1 Introduction

This paper deals with a mathematical programming approach in order to introduce new measures of the level of sequential arbitrage in bond markets. The approach extends former analyses developed in Balbás and López (2008) for friction-free markets, since market imperfections and other transaction costs may be incorporated. This extension is critical in practical applications, since real markets always reflect frictions. In fact, the second part of the paper is devoted to empirically testing the sovereign bond markets efficiency in the Euro-zone, before and during the Euro crisis started in the late 2009.

In the fundamental theory of finance the absence of arbitrage is a key assumption, often extended in such a manner that markets must be also good deal-free, *i.e.*, strategies with a very large return/risk ratio should not be available to traders either (Cochrane and Saa-Requejo, 2000, Balbás *et al.*, 2013, etc.). However, some researches have empirically evidenced the existence of arbitrage opportunities in practice. For instance, Chen and Knez (1995) examined $NYSE$ and $NASDAQ$ market samples, and they found that these markets did not assign the same price to the same common payoff. Similarly, Balbás *et al.* (2000) pointed out the existence of arbitrage between the Spanish index $IBEX$ and its derivatives. With respect to bond markets, Grinblatt and Longstaff (2000) and Halpern and Rumsey (2000) used data of the $U.S$ and Canada respectively, and they found a significant valuation difference between government bonds and the equivalent packages of strips. Also, the analysis of Armitage *et al.* (2012) showed that in the $U.K.$ sovereign bond market the package of strips was overpriced even when accounting for transaction cost. These works motivated us to investigate the European sovereign bond market efficiency, particularly during the debt crisis, where the sovereign bond prices were persistently volatile.[1] We consider the liquidity effect in our examination because the bid-ask spread for most countries in the Euro zone is significantly wider than before.

There are several papers investigating the arbitrage measurement such as Holden (1995), Chen and Knez (1995) and Balbás and López (2008). We mainly follow the approaches of Ronn (1987) and Balbás and López (2008), because their works are based on a linear programming (LP), which is easy to apply in practice and

[1] Market inefficiencies are more obvious in presence of market turmoils (Balbás *et al.,*2000), and mathematical methods are very effective to verify efficiency when facing or anticipating insatiability and/or crisis (Cheng *et al.,* 2006, Balbás *et al.,* 2008, etc.).

also provides the size and degree of arbitrage.[2] Hence, in this paper, we use LP with transaction cost analysis to measure the degree of sequential arbitrage. Primal problems maximize the sequential income of a portfolio composed of available bonds, given that the short (long) position is established at the bid (ask) price. Meanwhile, they give optimal strategies for obtaining maximum arbitrage profits (if arbitrage is available). Moreover, these optimal values provide an important clue for investors to modify the bid (ask) price of the bond with highest pricing error. In contrast, the variables in the dual problems are closely related to the discount factors, and they provide proxies for the Term Structure of Interest Rate ($TSIR$).

We will apply the above methodology in order to empirically test the European sovereign bond market during the period from $2007 - 2012$. We will find that the European sovereign bond market reflects inefficiencies, particularly during the debt crisis. Most arbitrage opportunities come from the price difference between old and new-issued zero-coupon bonds, or strips with identical maturities. The former refers to "On-The-Run Premium", which is a popular liquidity measure in treasury bond markets. The latter is consistent with the findings of Daves *et al.* (1993) who investigated with the *U.S.* treasury strips. Although some previous literature indicates that arbitrage resulted from these price discrepancies are not pure and even very risky in a highly volatile market, rich funds from institutional investors in a fair period definitely can induce high arbitrage returns. For instance, the arbitrage income in the German sovereign bond market in 2007 can be easily obtained by rich investors, because German market is highly liquid in a whole Euro-zone sovereign bond market. In addition, we will also remove all the zero coupon bonds and strips which produce main arbitrage opportunities to examine the sequential arbitrage again, but we will still find that the existence of sequential arbitrage cannot be rejected in Spain during the crisis. Hu *et al.* (2013) indicate that financial market liquidity closely relates the amount of arbitrage capitals available, which is crucial for implementing the arbitrage strategy. Specially during liquidity crises, the arbitrage capitals become scare and big investors are not willing to deploy them to supply liquidity. Then the lack of funds hugely limits arbitrageurs trading and even forces them to abandon high return arbitrage. Nevertheless, if there exist institutional investors who have deep pocket and also willing to invest in the Spanish sovereign market, we cannot deny the existence of arbitrage in Spanish market.

[2] Numerical and computational methods are becoming more and more important in Mathematical and/or Computational Finance (Chiarella *et al.*, 2014, Martín-Vaquero *et al.*, 2014, etc.), but LP may be also a good alternative if it provides us with appropriate investment strategies, pricing rules, risk measure-linked methods etc. (Mansini *et al.*, 2007, among others).

It seems that the law of one price does not hold in these markets due to the apparent price differentials between maturity-matched strips and zero-coupon bonds. Nonetheless, Armitage *et al.* (2012) pointed out that the low liquidity in strips market may lead to the difficulty in exploiting the arbitrage opportunities and the transaction costs somehow cannot be fully captured by bid-ask spreads. Also, the short-selling position in principle strips is not completely risk-free, because a margin or collateral is required in this case. Since the sequential arbitrage requires the position to be held for a certain period, the more collateral or margin are likely to be required if price diverges (Huij *et al.*, 2012). That is why Liu and Longstaff (2004) presented an insightful model permitting the price difference of principle and coupon bonds with the same maturity in equilibrium if collateral is required for short position. However, if the price difference exists between the coupon bond and the corresponding package of strips, the arbitrage is definitely risk-free because these two securities are perfect substitute in the market.

The paper is organized as follows. Section 2 presents preliminaries and notations. In section 3 we develop a linear programming approach by considering the liquidity effect, and we explain the relations among the optimal arbitrage strategies, arbitrage profits and proxies for the $TSIR$, which is the main contribution of this paper. Section 4 presents the government bonds information for Germany, France and Spain. In section 5 we report the empirical results of the arbitrage examination, and analyze the degree of arbitrage in details. Section 6 summarizes and concludes.

2 Preliminaries and notations

Let us consider n available bonds B_j, $j = 1, 2, ..., n$, and suppose that the bond market is not friction-free (there are transaction costs). As usual in finance (Jouini and Kallal, 1995), in a very general setting one can represent frictions by means of bid-ask spreads. Thus, denote by $P^a = (p_1^a, p_2^a, ..., p_n^a)$ and $P^b = (p_1^b, p_2^b, ..., p_n^b)$ the family of ask and bid prices, and suppose that $p_j^a \geq p_j^b > 0$, $j = 1, 2, ..., n$ holds.

Denote by $t_1 < t_2 < .... < t_m$ the set of future maturities of the cash flows paid by the bonds above. Without loss of generality we will impose the inequality

$$\sum_{i=1}^{m} c_{ij} > p_j^a \geq p_j^b,$$

$j = 1, \ldots, n$, where c_{ij} denotes the cash flow of B_j ($j = 1, \ldots, n$) at t_i ($i = 1, \ldots, m$).

In order to simplify some notations, let us introduce the pay-off matrix below

$$C = (c_{i,j})_{i=1,j=1}^{i=m,j=n}.$$

Following usual conventions (Jouini and Kallal, 1995), portfolios will be represented by a couple of matrices (X,Y), $X = (x_1, x_2, ..., x_n)^T$ being the portfolio of long position (purchases) and $Y = (y_1, y_2, \ldots, y_n)^T$ being the portfolio of short ones (sales), and $x_j \geq 0$, $y_j \geq 0$, $j = 1, 2, ..., n$ must hold. The current price of portfolio (X,Y) can be expressed as

$$P(X,Y) = P^a X - P^b Y = \sum_{j=1}^{n} p_i^a x_j - \sum_{j=1}^{n} p_i^b y_j,$$

and its future cash flows can be represented by means of matrix C. Indeed, consider that $C^a = \binom{-P^a}{C}$ denotes a $(m+1) \times n$ matrix combining C with the ask price $-P^a$ and $C^b = \binom{P^b}{-C}$ is obtained by combining A with P^b. Then, $C^a X$ and $C^b Y$ are the whole sets of cash flows of X and Y respectively, $C^a X + C^b Y$ is the set of cash flows of (X,Y), and the future payoff of portfolio (X,Y) equals $C(X-Y)$.

Let us introduce the concepts of arbitrage and sequential arbitrage.

Definition 1. (X,Y) *is said to be an arbitrage portfolio (AP) if* $C^a X + C^b Y \neq 0$ *and* $C^a X + C^b Y \geq 0$. (X,Y) *is said to be a sequential arbitrage portfolio (SAP) if* $I_{m+1}^*(C^a X + C^b Y) \neq 0$ *and* $I_{m+1}^*(C^a X + C^b Y) \geq 0$.[3] □

We can see that the arbitrage portfolio requires non-negative cash flows for every date t_i and generates at least a positive amount on some date. The conditions of the sequential arbitrage portfolio are not so restrictive, since negative cash flows are allowed as long as they are compensated by the amount of money previously received.

Additionally, it is known that the absence of (sequential) arbitrage in a frictionless market can be characterized by the existence of discount factors or a Term Structure of Interest Rate ($TSIR$). But if bond prices are quoted with spreads we will state that there must exist a bundle of discount factors $\{\mu_i\}$ satisfying $p_j^b \leq \sum_{i=1}^{m} c_{ij}\mu_i \leq p_j^a$ for $j = 1, ..., n$. The proof will be showed later.

[3] Henceforth

$$I_r^* = \begin{pmatrix} 1, 0, 0, ..., 0 \\ 1, 1, 0, ..., 0 \\ .... \\ 1, 1, 1, ..., 1 \end{pmatrix}$$

will be a $r \times r$ square matrix for every $r \in \mathbb{N}$.

To measure the level of sequential arbitrage we adopt the concept of strong sequential arbitrage by extending the Definition 1:

Definition 2. (X, Y) *is said to be a strong sequential arbitrage portfolio (SSA) if* $P(X, Y) < 0$ *and* $I_m^* C(X - Y) \geq 0$. □

Compared with sequential arbitrage, the strong sequential arbitrage is more concerned about current profit, thereby requires a positive initial cash flow (negative price) in the trading strategy which will not be used to compensate negative components in the portfolio payoff.

3 Sequential arbitrage measurement and TSIR proxies

Under the notations above, we will measure the level of SSA by means of the following linear optimization problems with decision variables $x_j, y_j,\ h_j,\ k_j,\ j = 0, 1, ..., n$:

$$\begin{aligned} & Max \quad -(P^a X - P^b Y) \\ s.t. \quad & I_m^* C(X - Y) \geq 0 \\ & x_j \leq k_j,\ j = 1, 2, ..., n \\ & \sum_{j=1}^{n} k_j p_j^a \leq 1 \\ & x_j \geq 0,\ \ y_j \geq 0,\ \ k_j \geq 0,\ \ j = 1, 2, ..., n \end{aligned} \tag{1}$$

and

$$\begin{aligned} & Max \quad -(P^a X - P^b Y) \\ s.t. \quad & I_m^* C(X - Y) \geq 0 \\ & y_j \leq h_j,\ j = 1, 2, ..., n \\ & \sum_{j=1}^{n} h_j p_j^b \leq 1 \\ & x_j \geq 0,\ \ y_j \geq 0,\ \ k_j \geq 0,\ \ j = 1, 2, ..., n \end{aligned} \tag{2}$$

Both problems attempt to maximize the SSA income $P^b Y - P^a X$, if available

($i.e.$, if the optimal value does not vanish). The unique difference between both optimization problems is given by their constraints, which affect purchases ($x_j \leq k_j$) and sales ($y_j \leq h_j$), respectively. If these constraints are not imposed then their dual problems will easily illustrate that (1) and (2)) will be unbounded unless their optimal value vanish. In order words, our $SSA-$measures could only reach the values 0 or ∞. Finally note that the common constraint $I_m^* C(X-Y) \geq 0$ guarantees that every solution of (1) and (2) will be a SSA portfolio or will replicate the null strategy $(X, Y) = (0, 0)$.[4]

Now we move to the dual problems, which are given by:

$$\begin{array}{rl} & Min \quad \theta \\ s.t. & \mu C - \lambda \leq P^a \\ & \mu C \geq P^b \\ & \lambda_j \leq \theta p_j^a, \; j = 1, 2, ..., n \\ & \lambda_j \geq 0, \; j = 1, 2, ..., n \\ & \mu_1 \geq \mu_2 \geq ... \geq \mu_m \geq 0 \end{array} \tag{3}$$

and

$$\begin{array}{rl} & Min \quad \theta \\ s.t. & \mu C \leq P^a \\ & \mu C + \lambda \geq P^b \\ & \lambda_j \leq \theta p_j^b, \; j = 1, 2, ..., n \\ & \lambda_j \geq 0, \; j = 1, 2, ..., n \\ & \mu_1 \geq \mu_2 \geq ... \geq \mu_m \geq 0 \end{array} \tag{4}$$

where the decision variables are $\theta \in \mathbb{R}$, $\lambda = (\lambda_i)_{i=1}^n$ and $\mu = (\mu_i)_{i=1}^m$.

Both dual Problems (3) and (4) minimize the highest committed error θ for ask and bid prices in percentage. The dual variables μ in (3) and (4) give a proxy for the family of discount factors, but both of them misprice bonds indicated by the respective first four constraints if bond market is not efficient. The first constraint in Problem (3) implies that the difference between market ask price P^a and the

[4]Actually, we could integrate (1) and (2) in a single vector (or multiobjective) optimization problem with two objectives. Then we could apply both the scalarization method or the balance space approach (Galperin and Wiecek, 1999) in order to find Pareto solutions. Nevertheless, we will see that this extension is not interesting in this case because there is a close relationship between the solutions of both (1) and (2) (see Lemma 4 below).

theoretical price μC measured by μ is determined by the value of λ. If λ is significantly greater than 0, discount factor μ will overestimate the bonds ask prices and the estimated price μC will be within the interval $[P^a, P^a + \lambda]$. Using a similar argument, a set of discount factors μ given by Problem (4) will underestimate bonds bid prices in portfolio if $\lambda > 0$.

Next, we start to investigate the properties of the solutions of the primal Problems (1) and (2):

Lemma 1. *Problems (1), (2), (3) and (4) are feasible and bounded. If ℓ^* and ℓ_* are their optimal values, then $0 \leq \ell_* < 1$, $0 \leq \ell^*$ and $\ell_* = 0 \Longleftrightarrow \ell^* = 0 \Longleftrightarrow$ the market is $SSA-$free.*

Proof. $0 \leq \ell_*$ and $0 \leq \ell^*$ are clear since $(0,0,0)$ is feasible for (1) and (2). (3) and (4) will be bounded if (3) and (4) are feasible. Obviously, $\mu = (1,1,...,1)$, $\lambda = \mu C - P^a$, $\theta = Max\left\{\lambda_j/p_j^a;\ j=1,...,n\right\}$ and $\mu = (0,0,...,0)$, $\lambda = P^b$, $\theta = 1$ provide us with feasible elements for (3) and (4) respectively.

To prove that $\ell_* < 1$ suppose that (X,Y,h) is (2)-feasible. Then, $Y \leq h \Rightarrow P^bY \leq P^bh \leq 1 \Rightarrow -P^aX + P^bY \leq P^bY \leq 1$, so $\ell_* < 1$. Moreover, $\ell_* = 1$ holds if and only if

$$P^aX + P^bY = P^bh = 1. \tag{5}$$

Thus, $P^aX = P^bY - P^bh \leq P^b(Y-h) \leq 0$. Since $P^aX \geq 0$, it follows that $P^aX = 0$ and $X = 0$. Combined with (5) we have $P^bY = P^bh$ and $Y = h$, so the first constraint in (2) leads to

$$-h_1\left(\sum_{i=1}^{m} a_{i,1}\right) - ... - h_n\left(\sum_{i=1}^{m} a_{i,n}\right) \geq 0$$

Since $P_j^b \leq \sum_{i=1}^m a_{i,j}$ we have $P^bh \leq 0$, which contradicts (5). Hence $0 \leq \ell_* < 1$.

Finally, if the market is $SSA-$free, the first constraint in (1) or (4) will imply $-(P^aX - P^bY) \leq 0$ as long as (X,Y) is (1) or (4) feasible, so $\ell_* = \ell^* = 0$. Conversely, if $\ell_* = \ell^* = 0$, suppose that (X,Y) satisfies $I_m^*C(X-Y) \geq 0$ and $Y = h$. It is clear that (X,Y,h) is (1)-feasible. Since $\ell_* = 0$ implies that $-(P^aX - P^bY) \leq 0$, it contradicts the definition of SSA, so there are no feasible portfolios generating SSA. With a similar argument, it holds for $\ell^* = 0$. □

Based on Lemma 1, we state that in a SSA free market every bond is priced within the bid-ask spread by a fitted set of discount factors.

Theorem 2. *There are no SSA portfolios if and only if there exists* μ_* *such that* $P^b \leq \mu_* C \leq P^a$ *and* $\mu_{*1} \geq \mu_{*2} \geq ... \geq \mu_{*m} \geq 0$.

Proof. It is clear that the absence of SSA portfolios holds if and only if $\ell_* = \ell^* = 0$ which is equivalent to $\theta_* = \theta^* = 0$. Hence $\lambda_* = \lambda^* = 0$ and $\mu_* C$ will be in the range of $[P^b, P^a]$. □

Lemma 3. *Suppose that* $\ell_* > 0$. *if* (X^*, Y^*, k^*) *solves (1) and* (X_*, Y_*, h_*) *solves (2) then* $Y_* = h_*$, $P^b h_* = 1$, $X^* = k^*$ *and* $P^a k^* = 1$.

Proof. . Since $Y_* \leq h_*, P^b Y_* \leq P^b h_* \leq 1$ and $P^b Y_*$ is the only strictly positive components in ℓ_*, it is obvious to see that $Y_* = h_*$ and $P^b Y_* = P^b h_* = 1$. Clearly, $X^* \leq k^*, P^a X^* \leq P^a k^* \leq 1$. Suppose that $P^a X^* < 1$ and set $X = X^*/P^a X^*, Y = Y^*/P^a X^*, k = X^*/P^a X^*$, it is obvious that portfolio (X,Y) is feasible, so it gives that

$$-(P^a X - P^b Y) = -\frac{-(P^a X^* - P^b Y^*)}{P^a X^*} = \frac{\ell^*}{P^a X^*} > \ell^*$$

which obviously has a contradiction. so we have $X^* = k^*$ and $P^a k^* = 1$. □

Lemma 4. (a) $\ell^* = \dfrac{\ell_*}{1 - \ell_*}$, $\ell_* = \dfrac{\ell^*}{1 + \ell^*}$ *and* $\ell_* \leq \ell^*$.
(b) $X_* = (1 - \ell_*) k^*$ *and* $Y^* = (1 + \ell^*) h_*$

Proof. $a)$ Consider the functions $\phi_i : \mathcal{A} \longrightarrow \mathbb{R}$, $i = 1, 2$, given by

$$\phi_1 (X, Y) = \frac{-P^a X + P^b Y}{P^b Y}, \qquad \phi_2 (X, Y) = \frac{-P^a X + P^b Y}{P^a X}$$

where $\mathcal{A} = \{(X, Y) \in \mathbb{R}^n \times \mathbb{R}^n;\ P(X, Y) < 0,\ I_m^* C (X - Y) \geq 0\}$ is non void due to $\ell_* > 0$. Notice that the denominator will never vanish in the definitions above, because $P^b Y = 0$ would imply $P(X, Y) = P^a X \geq 0$, contadicting $(X, Y) \in \mathcal{A}$, and $P^a X = 0$ would imply $\ell_* = 1$, contradicting Lemma 1.

Expression $0 < \phi_1 (X, Y) < 1$

$$\phi_2 (X, Y) = \frac{\phi_1 (X, Y)}{1 - \phi_1 (X, Y)} \tag{6}$$

are obvious. Since $[0, 1) \ni t \longrightarrow t/(1 - t) \in [0, \infty)$ is a one to one increasing function, Problems

$$Max\ \{\phi_i (X, Y);\ (X, Y) \in \mathcal{A}\}, \tag{7}$$

$i = 1, 2$, attain the optimal value at the same solutions. It is clear that if $(X, Y) \in \mathcal{A}$ then $\left(X/P^b Y, Y/P^b Y\right)$ is (2)-feasible, and therefore $\left(-P^a X + P^b Y\right) / \left(P^b Y\right) \leq \ell_*$.

Hence Lemma 3 implies that $Y_* = h_*$ and $P^b h_* = 1$. Therefore

$$\phi_1(X_*, Y_*) = \left(-P^a X_* + P^b Y_*\right) / \left(P^b Y_*\right) = -P^a X_* + P^b Y_* = \ell_*,$$

and (X_*, Y_*) solves (7). Similarly, (X^*, Y^*) solves (7) and $\phi_2(X^*, Y^*) = \ell^*$. Therefore (see (6))

$$\ell^* = \phi_2(X^*, Y^*) = \phi_2(X_*, Y_*) = \frac{\phi_1(X_*, Y_*)}{1 - \phi_1(X_*, Y_*)} = \frac{\ell_*}{1 - \ell_*},$$

and the inequality $\ell_* \leq \ell^*$ obviously holds form equation above.

b) Consider a (2)-feasible strategy $(\gamma X_*, \gamma Y_*)$ with $\gamma > 0$ such that $\gamma P^a X_* = 1$. Then,

$$1 = \gamma P^a X_* = \gamma\left(-\ell_* + P^b Y_*\right) = \gamma(-\ell_* + 1)$$

and therefore

$$\gamma = \frac{1}{1 - \ell_*} = 1 + \ell^*.$$

Proceeding as in a very parallel proof of Balbás and López (2008), the function $f(X, h)$ equaling the optimal value of (2) for every fixed h is increases with h, and the function $f(k, , Y)$ equaling the optimal value of (1) for every fixed k is increases with k. Since $P^a k^* = 1$ and $P^a(\gamma X_*) = 1$, it gives $k^* = \gamma X_* = (1/(1 - \ell_*) X_*)$Analogously, $h_* = (1/(1 + \ell^*) X_*)$. □

Now we transfer our attention to the solutions of the dual problems. Assume that $(\ell^*, \lambda^*, \mu^*)$ and $(\ell_*, \lambda_*, \mu_*)$ are the solutions of (3) and (4). If SSA does not exist in the market, which is indicated by $\ell_* = \ell^* = \lambda_* = 0$, the theoretical prices $P_* = \mu_* C$ and $P^* = \mu^* C$ will be within the interval of $[P^b, P^a]$. However, in a non-efficient market they will satisfy the following relations:

Theorem 5. *(a)if $k_j^* > 0$ then $p_j^a = \frac{p_j^*}{1+\ell^*}$. If $h_{*j} > 0$ then $p_j^b = \frac{p_{*j}}{1-\ell_*}$.*

(b) $p_{*j} \leq p_j^b \leq p_j^a \leq p_j^*$, $j = 1, 2, ..., n$.

Proof. The dual optimal values will satisfy $\theta_* = \ell_*, \theta^* = \ell^*$. If $h_* > 0$ previous lemmas ensure that $Y_* > 0$, so the complementary slackness conditions lead to $\lambda_* = \theta_* P^b = \ell_* P^b$ and $\mu_* A + \lambda_* = P^b$. It gives that $\lambda_* = P^b - \mu_* A = P^b - P_*$, and then $P^b - P_* = \ell_* P^b$. Hence $P^b = \frac{p_*}{1-\ell_*}$. With a similar argument, we can derive that $p_j^a = \frac{p_j^*}{1+\ell^*}$ if $k_{*j} > 0$.

(b) is obvious from the results of (a). □

Measures ℓ_* and ℓ^* appropriately give the level of SSA since they reflect a relative (per dollar) arbitrage gain value. Moreover, according to Theorem 5, the difference between bid (ask) prices and estimated prices p_{*j} (p_j^*) is closely related to the value of ℓ^* and ℓ_*. In fact, based on the value of ℓ^* and ℓ_*,we can modify mispriced prices of some bonds that have large percentage in producing arbitrage opportunities. In addition, let us investigate a new property of ℓ_* and ℓ^* stating that they minimize the maximum relative variation of prices to prevent the existence of SSA, and they also provide a new explanation for the risk premium.

Theorem 6. *Let* $Q^a = (q_1^a, q_2^a, ..., q_n^a)$ *and* $Q^b = (q_1^b, q_2^b, ..., q_n^b)$ *be vectors of ask and bid prices for bonds* $B_1, B_2, ..., B_n$*. Suppose that* Q^a *and* Q^b *do not generate SSA opportunities. Suppose also that* $0 < q_j^b \leq p_j^b$, $p_j^a \leq q_j^a$ *for* $j = 1, 2, \ldots, n$*. Then,*

$$\ell_* = Max\{\frac{p_j^b - p_{*j}}{p_j^b} : j = 1, 2, ..., n\} \leq Max\{\frac{p_j^b - q_j^b}{p_j^b} : j = 1, 2, ..., n\}$$

$$\ell^* = Max\{\frac{p^{*j} - p_j^a}{p_j^a} : j = 1, 2, ..., n\} \leq Max\{\frac{q_j^a - p_j^a}{p_j^a} : j = 1, 2, ..., n\}$$

Proof. Assume that $\ell_* > 0$. The dual constraints lead to $\theta_* = \ell_* \geq \frac{\lambda_{*j}}{p_j^b}, j = 1, 2, \ldots, n$. Theorem 5($a$) shows that $\ell_* = \frac{\lambda_{*j}}{p_j^b}$ if $h_* > 0$. Hence, the arbitrage profit ℓ_* satisfies that

$$\ell_* = Max\{\frac{p_j^b - p_{*j}}{p_j^b} : j = 1, 2, ..., n\}$$

Theorem 2 guarantees the existence of a set of discount factors μ such that $Q^b \leq \mu C \leq Q^a$. Take $\theta = Max\{\frac{p_j^b - q_j^b}{p_j^b} : j = 0, 1, ..., N\}$ and $\lambda = P^b - \mu C = P^b - Q^b \geq 0$. (μ, λ, θ) is dual-feasible, so $\theta_* \leq \theta$. The remaining statement can be derived with a similar argument. □

The above theorem indicates that the "authentic" bid (ask) price p_{*j} (p^{*j}) provided by our optimization model can minimize the maximum modification of bond quotes leading to a $SSA-$free market. ℓ_* and ℓ^* play important roles in measuring this minimum difference in percentage. Additionally, they can be understood as a lower bound of the risk premium of a risky bond. To clarify this idea, we assume a portfolio consisted in default free bonds and a risky bond j with bid price p_j^b, and suppose that the SSA disappears ($\ell_* = 0$) when only dealing with default free bonds. But if we include risky bond j, ℓ_* will be greater than zero because bond j should be involved in the buying position h. Although discount factors cannot reflect all the information provided by bond j, we can see that the arbitrage profit ℓ_* will imply a minimum risk premium in percentage to compensate risk for investors. If more riskier bonds are considered, ℓ_* will provide the largest required premium of bonds in portfolio.

The next sections provide an empirical analysis in European sovereign bond markets. We adopt the methodology above to examine wether there exists sequential arbitrage before and during the Euro crisis.

4 Data

4.1 Data Source

The existence of SSA is tested in the Euro-zone. We will deal with government bond markets due to the European sovereign debt crisis beginning in the late 2009. We will choose Germany, France and Spain, which, along with Italy own the largest government bonds and strips markets in the Euro-zone, and also because they differ significantly in credit ratings.

The main source of our data on daily bond price quotation is *Datastream*. Quoted bid (ask) prices are composite prices calculated by *Datastream* from the average of all the available contributors bid (ask) quotes, excluding the highest and the lowest values. Since bid-ask quotes information are limited, we also take prices without spread measured by "Market Default Prices" (MDP). MPD are reference prices estimated by retrieving the composite bid prices provided from *Datastream′s* or *Thomson Reuters′s* valuation bid prices, if the prices are liquid.[5] Although there is still a fraction of electronic transactions such as MTS for European bond markets, these data are not easily available. Another data source is the *Bank of Spain*, which is the biggest dealer for Spanish Treasury Bonds, and provides daily information of all traded securities in over the counter market. Here, this data source is mainly used to examine the data reliability provided by *Datastream* for the Spanish market. This is because we only find little data about liquidity information as measured by turnover. Our sample ranges from January 2007 to December 2012. This period is particularly suitable for analyzing the Euro-zone Sovereign bonds market efficiency as it covers the stable period before 2008 as well as the chaotic period following a Greek debt crisis.

[5] A price is liquid if it changed in the previous five days.

Our model requires data containing bonds with perfectly predictable cash flows, so only default-free and option-free government bonds are included in the sample. In order to examine the market efficiency for each country, the bid and ask quotation are analyzed during the period from 2010 to 2012, because they are only available in *Datastream* from late 2009. For the remaining years from 2007 to 2009, we use MDP. In fact, coupon bearing instruments are traded at their "Gross Prices", which involves calculating accrued interest. So we add it to the quoted bid/ask prices and MPD for coupon bonds in the model.

4.2 Data Concerns

Although *Datastream* has the largest data information for financial markets, we find some quoted prices in our data keeping constant for more than five trading days in the three-country data-set. To exclude any possibilities of no liquidity problems, we remove these bonds on the day where their quotations are exactly the same as the preceding day when doing daily arbitrage test. After cleaning the data, the daily traded bonds and strips information in the Spanish market during 2007 to 2009 is consistent with the ones provided by *Bank of Spain*. In addition, we also delete some outliers as they appear to be due to obvious data-entry errors.

4.3 Descriptive Statistics

Table 1 provides descriptive statistics about the mean (median) of bid-ask spreads form 2010 to 2012 for the three countries. In each year, the mean for all the traded securities are presented in the first column. Then we segment the entire sample into coupon bonds and zero coupon bonds consisting of strips and Treasury Bills, shown in the 'Fixed-Income Security' and 'Treasury Bills and Strips' columns.

In the German and French market, bid-ask spread changes of all traded sovereign bonds are small from 2010 to 2012. The average spreads were always around 48 basic points (*bp*), but the mean of the bid-ask spread in the Spanish market kept increasing every year and increased by more than one half in 2012, which potentially reflects investor's lack of confidence in government recovery in debt crisis. For fixed income bonds, then German market showed higher vitality than the other two countries. The average spreads over the three years are less than 13 *bp*. In contrast, the mean spread for less liquid French and Spanish fixed-income securities were approximately 24 *bp* and 50 *bp*, respectively. Surprisingly, overall liquidity performance in strips and zero-coupon bonds for Germany was dismal, the average spread in 2012 is up

Table 1:

	All bonds		Fixed-income Securities		zero-coupon bonds and Strips	
Year: 2010	Mean	Median	Mean	Median	Mean	Median
Germany	0.4928	0.4100	0.1362	0.0800	0.7471	0.7000
France	0.4354	0.3000	0.2334	0.1900	0.6158	0.4800
Spain	0.4082	0.3800	0.4102	0.3900	0.4105	0.3800
Year: 2011						
Germany	0.4618	0.3500	0.1255	0.0700	0.7702	0.7500
France	0.3847	0.3000	0.2627	0.2100	0.4828	0.4500
Spain	0.5178	0.3100	0.4925	0.4400	0.5234	0.2700
Year: 2012						
Germany	0.5296	0.2900	0.1003	0.0400	0.9709	1.0300
France	0.4811	0.2900	0.2269	0.1800	0.6891	0.5000
Spain	0.6252	0.4800	0.5127	0.4900	0.6859	0.4700

to 100 *bp*, even worse than in Spain and France which were less than 70 *bp*. In general, Spanish government bond market appears to face higher liquidity risk than Germany and France, based on their high bid-ask spread. However, Germany, who owns the most liquid strips market, shows an apparent liquidity problem.

5 Empirical Results

Tables 2, 3 and 4 summarize the days with SSA opportunities from 2007 to 2012 on both aggregate and percentage basis for Germany, France and Spain, respectively. The value of the arbitrage income is divided into eight intervals whose length equals 0.0005, as shown at every row. In each column, "Upper" indicates the maximum profit generated by Problem (1). In contrast, "Lower" represents the maximum profit of Problem (2). The tables show a pronounced difference in the days with arbitrage for the three countries. In the stable period from 2007 to 2008, Germany who owned one of the largest and most liquid market for sovereign debt, surprisingly, showed quite high frequency of daily SSA. Particularly in 2008, bond pricing errors were more than 1% in approximately 67% working days and they were even above 5% in 30 days at the end of the year. However, sovereign debt markets in France and Spain performed regularly during this period, since more than 97% of the days exhibited low margin close to zero. Although Juji *et al* (2011) showed that price difference between principals and coupon strips with the same maturity from 2002 to 2007 for these three countries may lead to an arbitrage by switching two strips, we could not find any significant riskless profits given by the model in 2007 and 2008, except for Germany. In the following turbulent period from 2009 to 2010 SSA opportunities began to appear in the latter half of 2009 for the French and Spanish sovereign markets. The results show that investors can obtain at most 1% to 5% price differences with the ones provided by arbitrage free market in 36% of trading days. From 2010 to 2012, Germany and Spain showed obvious mispricing problems that persistently existed. German market had maintained a large percentage of arbitrage opportunities over the three years, particularly in 2010. For Spain, the arbitrage profits $\ell^*(\ell_*)$ were lying within the spread $1\% - 5\%$ more than 200 days in 2012, but the days of arbitrage slowly decreased. French market, in contrast, seems to be much more efficient. Arbitrage tended to decrease gradually and almost disappeared in 2011. However, in 2012 it became wrong again.

Table 2: Arbitrage days in German sovereign bond market

Germany	All default-free and option-free bonds											
Year	2007				2008				2009			
Days of Examination	260				260				260			
Arbitrage Profits	Upper		Lower		Upper		Lower		Upper		Lower	
	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%
$\ell_*, \ell^* \leq 0.0005$	15	5.8%	15	5.8%	15	5.8%	15	5.8%	9	3.5%	9	3.5%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	14	5.4%	14	5.4%	5	1.9%	5	1.9%	1	0.4%	1	0.4%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	122	46.9%	125	48.1%	32	12.3%	33	12.7%	3	1.2%	3	1.2%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	102	39.2%	99	38.1%	175	67.4%	177	68.1%	144	55.3%	157	60.3%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	6	2.3%	7	2.7%	28	10.7%	25	9.6%	102	39.2%	89	34.2%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	1	0.4%	0	0.0%	5	1.9%	5	1.9%	1	0.4%	1	0.4%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%
Year	2010				2011				2012			
Days of Examination	256				252				244			
$\ell_*, \ell^* \leq 0.0005$	42	16.4%	42	16.4%	87	34.5%	87	34.5%	147	60.3%	147	60.3%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	13	5.1%	13	5.1%	5	2.0%	5	2.0%	22	9.0%	22	9.0%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	57	22.3%	58	22.7%	73	29.0%	74	29.4%	63	25.8%	63	25.8%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	62	24.2%	62	24.2%	50	19.8%	50	19.8%	10	4.1%	10	4.1%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	81	31.6%	80	31.2%	37	14.7%	36	14.3%	2	0.8%	2	0.8%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	1	0.4%	1	0.4%	0	0.00%	0	0.00%	0	0.00%	0	0.0%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%

Table 3: Arbitrage days in French sovereign bond market

*

France	All default-free and option-free bonds											
Year	2007				2008				2009			
Days of Examination	260				260				260			
Arbitrage Profits	Upper		Lower		Upper		Lower		Upper		Lower	
	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%
$\ell_*, \ell^* \leq 0.0005$	260	100.0%	260	100.0%	0	0.0%	0	0.0%	148	56.9%	148	56.9%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	1	0.4%	1	0.4%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	5	1.9%	5	1.9%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	3	1.2%	3	1.2%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	94	36.2%	96	36.9%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	9	3.4%	7	2.7%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
Year	2010				2011				2012			
Days of Examination	258				255				255			
$\ell_*, \ell^* \leq 0.0005$	47	18.2%	47	18.2%	226	88.6%	226	88.6%	29	11.4%	29	11.4%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	47	18.2%	47	18.2%	9	3.5%	9	3.5%	2	0.8%	0	0.8%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	159	61.6%	159	61.6%	5	2.0%	5	2.0%	2	0.8%	2	0.8%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	2	0.8%	2	0.8%	2	0.8%	2	0.8%	9	3.5%	8	3.1%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	3	1.2%	3	1.2%	11	4.3%	12	4.7%	212	83.1%	212	83.1%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	1	0.4%	1	0.4%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	2	0.8%	1	0.4%	0	0.00%	0	0.0%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%

Table 4: Arbitrage days in Spanish sovereign bond market

*

Spain	All default-free and option-free bonds											
Year	2007				2008				2009			
Days of Examination	260				260				260			
Arbitrage Profits	Upper		Lower		Upper		Lower		Upper		Lower	
	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%
$\ell_*, \ell^* \leq 0.0005$	260	100.0%	260	100.0%	260	100.0%	260	100.0%	149	57.2%	149	57.2%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	1	0.4%	1	0.4%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	3	1.2%	3	1.2%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	5	1.9%	5	1.9%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	94	36.2%	94	36.2%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	8	3.1%	8	3.1%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
Year	2010				2011				2012			
Days of Examination	258				255				256			
$\ell_*, \ell^* \leq 0.0005$	12	4.7%	12	4.7%	18	7.1%	18	7.1%	4	1.6%	4	1.6%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	0	0.0%	0	0.0%	4	1.6%	4	1.6%	0	0.0%	0	0.0%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	7	2.7%	7	2.7%	11	4.3%	11	4.3%	7	2.7%	7	2.7%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	4	1.6%	4	1.6%	0	0.00%	0	0.00%	8	3.1%	8	3.1%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	166	64.3%	175	67.8%	193	75.6%	196	76.8%	215	84.0%	223	87.1%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	69	26.7%	60	23.3%	29	11.4%	26	10.2%	22	8.6%	14	5.5%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%

These results are very striking because most of arbitrage profits $\ell_*(\ell^*)$ exceeded 5% , even taking into account a tax effect. This implies that in this three sovereign markets every investor could obtain an arbitrage income in 50% working days on average from 2010 to 2012. Therefore, we examine the corresponding arbitrage strategies to find out the main reason resulting in such a strange result. As presented in Table 5, we summarize arbitrage days in terms of different strategies, which are pricing errors from maturity-matched strips, from On-The-Run Premium and from portfolios composed of fixed-income bonds, strips and zero-coupon bonds. Last two columns show the minimum and maximum arbitrage profits per year. Clearly, in the safe year 2007 for the German sovereign bond market, more than 75% of arbitrage incomes come from complicated strategies by selling or buying a pool of fixed-income bonds, strips and zero-coupon bonds. Although the maximum arbitrage profits are the smallest compared to other years, the arbitrage indeed exist without liquidity risk or capital problem. However, in 2008, the German zero-coupon bond market shows increasing pricing errors due to a strike increase of the arbitrage days of "On-The-Run Premium",[6] which closely relates a liquidity problem in sovereign bond market. In other words, more and more the zero-coupon bonds with shorter maturity are traded at lower prices, compared to recent-issued bonds but with longer maturity. Moreover, a threefold increase in the maximum arbitrage profits also directly implies a lower liquidity in German zero-coupon bond markets than before. By contrast, Spanish and French sovereign bond markets seem quiet and efficient during 2007 and 2008, where the arbitrage profits are close to zero.

Since 2009 three sovereign markets enter into a turbulent period due to the Greece crisis. More than one third of working days shows arbitrage opportunities for three countries. Particularly in the French and Spanish government bond markets all the maximum profits come from price discrepancies between old and recent-issued zero-coupon, showed in "On-The-Run Premium", which strongly suggests a huge liquidity problem in both markets. Moreover, the maximum income for Spain attains 0.20, which means that the riskless return rate is up to 20% if a trader invests 1 Euro by implementing the optimal arbitrage strategy provided by our model. From 2010 to 2012, in order to reduce the liquidity effect on arbitrage, we use market bid-ask prices instead of market trading price in the experiment. However, the results presented in Table 6 still reflect significant arbitrage opportunities for three countries. The corresponding strategies mainly focus on the price differ-

[6] "On-The-Run Premium" is a popular liquidity measure used in Treasury bond markets. The just-issued or called on-the-run Treasury bonds are generally more liquid and traded at a premium compared to other old bonds with similar maturity.

ence from maturity-matched strips and portfolio strategy, other than "On-The-Run Premium". More importantly, we observe that 82% of principle strips were sold at higher price than the coupon strips with identical cash flows in our sample period for the three countries, which highlights a strong violation of the law of one price. This phenomenon is consistent with the findings of previous empirical work on the Treasury strips in the U.S. by Jordan *et al.* (2000), who found that bid quoted price of principle strips in U.S. strip market was on average 10.8 basic points higher than matched-maturity coupon strips. Daves and Ehrhardt (1993) claimed that principle strips were more valuable because of a unique role played in reconstitution, which always guarantees market demand.

In general, we cannot deny the existence of SSA opportunities in sovereign bond markets. Although current works indicate that arbitrage is not pure or riskless when arbitrageurs lack capitals to satisfy margin maintenance, or arbitrage is difficult to implement in low liquid market, it is true that the arbitrage opportunities still exist in a safe and high liquid German market in 2007 and 2008 as long as there exist investors or traders with deep pocket.

Finally, we exclude the maturity-matched strips and "On-The-Run Premium" zero-coupon bonds that might lead to risky arbitrage, and re-examine the arbitrage from 2007 to 2012. The results are shown in Tables 7 to 9. Clearly, there are little SSA opportunities from 2007 to 2011 for the three countries. However, in 2012 we cannot reject the existence of SSA in the Spanish sovereign bond market. Investors can obtain price difference $\ell_*(\ell^*)$ greater than 1% in 28 working days without considering the capital requirement in a high liquidity risk period.

Table 5:

		Number of Arbitrage days			Arbitrage Profits	
Germany	Total	Maturity-matched strips	On-The-Run premium	Others	Min.	Max.
2007	245	32	28	185	0.0022	0.0818
2008	245	15	97	133	0.0011	0.2393
2009	251	63	141	47	0.0025	0.1092
France						
2007	0	0	0	0	0	0
2008	0	0	0	0	0	0
2009	112	2	110	0	0.0046	0.088
Spain						
2007	0	0	0	0	0	0
2008	0	0	0	0	0	0
2009	111	2	109	0	0.0176	0.1931

* In this table, we separate the arbitrage days into three groups based on different types of pricing errors. Since bid-ask prices are not available in 2007,2008 and 2009 for above three countries, we use 'MDP' in the arbitrage examination.

Table 6:

		Number of Arbitrage days			Arbitrage Profits	
Germany	total	Maturity-matched strips	On-The-Run premium	Others	Min.	Max.
2010	213	181	0	32	$5.34e^{-4}$	0.0714
2011	165	132	0	33	$5.18e^{-4}$	0.0273
2012	97	85	0	12	$5.84e^{-4}$	0.0353
France						
2010	86	85	1	0	$5.33e^{-4}$	0.0397
2011	29	29	0	0	0.0005	0.1006
2012	224	224	0	0	$3.6e^{-4}$	0.0505
Spain						
2010	246	231	1	12	0.0016	0.1351
2011	237	112	1	124	$5.34e^{-4}$	0.0822
2012	252	224	0	28	0.0036	0.0733

* In this table, we summarize the arbitrage days in 2010, 2011 and 2012. We assume that traders can buy bonds or strips at ask prices, and sell at bid prices. The bid-ask prices used in our arbitrage examination are daily average bid-ask prices obtained from Datastream.

Table 7: Arbitrage days in German Fixed-Income sovereign bond market

Germany	All default-free and option-free bonds											
Year	2007				2008				2009			
Days of Examination	260				260				260			
Arbitrage Profits	Upper		Lower		Upper		Lower		Upper		Lower	
	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%
$\ell_*, \ell^* \leq 0.0005260$	100.0%	260	100.0%	260	100.0%	260	100.0%	260	100.0%	260	100.0%	
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
Year	2010				2011				2012			
Days of Examination	256				252				244			
$\ell_*, \ell^* \leq 0.0005$	256	100.0%	256	100.0%	252	0.0%	252	0.0%	244	0.0%	244	0.0%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%	0	0.0%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%

Table 8: Arbitrage days in French Fixed-Income sovereign bond market

*

France	All default-free and option-free bonds											
Year	2007				2008				2009			
Days of Examination	260				260				260			
Arbitrage Profits	Upper		Lower		Upper		Lower		Upper		Lower	
	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%
$\ell_*, \ell^* \leq 0.0005$	260	100.0%	260	100.0%	260	100.0%	260	100.0%	260	100.0%	260	100.0%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	0	0.00%	0	0.00%	0	0.00%	0	0.0%	0	0.0%	0	0.0%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	0	0.00%	0	0.00%	0	0.00%	0	0.0%	0	0.0%	0	0.0%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	0	0.00%	0	0.00%	0	0.00%	0	0.0%	0	0.0%	0	0.0%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%
Year	2010				2011				2012			
Days of Examination	258				255				255			
$\ell_*, \ell^* \leq 0.0005$	260	100.0%	260	100.0%	255	100.0%	255	100.0%	255	100.0%	255	100.0%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.0%

Table 9: Arbitrage days in Spanish Fixed-Income sovereign bond market

Spain	All default-free and option-free bonds											
Year	2007				2008				2009			
Days of Examination	260				260				260			
Arbitrage Profits	Upper		Lower		Upper		Lower		Upper		Lower	
	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%	Num. of days	%
$\ell_*, \ell^* \leq 0.0005$	260	100.0%	260	100.0%	260	100.0%	260	100.0%	260	100.0%	260	100.0%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
Year	2010				2011				2012			
Days of Examination	258				255				256			
$\ell_*, \ell^* \leq 0.0005$	258	100.0%	258	100.0%	255	0.00%	255	0.00%	198	77.2%	198	77.4%
$0.0005 \leq \ell_*, \ell^* \leq 0.001$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.001 \leq \ell_*, \ell^* \leq 0.005$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.005 \leq \ell_*, \ell^* \leq 0.01$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	30	11.8%	31	12.8%
$0.01 \leq \ell_*, \ell^* \leq 0.05$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	28	11.0%	27	10.5%
$0.05 \leq \ell_*, \ell^* \leq 0.1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.1 \leq \ell_*, \ell^* \leq 0.5$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%
$0.5 \leq \ell_*, \ell^* \leq 1$	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%	0	0.00%

*

6 Conclusion

We have presented a mathematical programing approach in order to measure the size of the strong sequential arbitrage of a bond market. Transaction costs may be incorporated. The obtained arbitrage measures ℓ_* and ℓ^* reflect two interesting quantities: On the one hand ℓ_* (ℓ^*) yields the highest available arbitrage profit with respect to the price of the sold (bought) securities. On the other hand ℓ_* (ℓ^*) gives the minimum relative (per dollar) bid (ask) price modification leading to a strong sequential arbitrage free market. The provided primal problems lead to the optimal strong sequential arbitrage strategies (if available), while their dual problems generate proxies for the Term Structure of Interest Rates. Several results have shown the significant analogies between the two provided primal problems and their optimal strategies (X_*, Y_*) and (X^*, Y^*). Similarly, the one to one and increasing relationship $\ell^* = \ell_*/(1 - \ell_*)$ indicates that both arbitrage measures provide analogous information.

The developed theory easily applies in practice. In fact we have empirically studied the existence of strong sequential arbitrage in the European sovereign debt market from 2007 to 2012. The focus has been on sovereign bonds issued by Germany, France and Spain, respectively. During the safe period, from 2007 to 2008, the Spanish and French sovereign bond markets performed efficiently, but the German market reflected strong sequential arbitrage due to the existence of price differences between maturity-matched strips and zero-coupon bonds. In contrast, during the crisis period, from 2009 to 2012, the three bond markets showed market inefficiencies which particularly focused on "On-The-Run Premium" and strips rather than the fix-income bonds. These results are consistent with the findings of Daves and Ehrhardt (1993) and Jordan *et al.* (2000), who claimed that the principle strip price is usually higher than the strip or zero-coupon bonds with the same maturity because of its uniqueness. However, after removing all the zero coupon bonds and strips, we still found a fraction of arbitrage opportunities existing in the Spanish fixed-income bond market, where arbitrage profits were higher than 1%.

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The usual caveat applies.

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