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A GAME THEORETIC APPROACH TO GROUP CENTRALITY*

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Abstract. This paper is centered in the valuation of the centrality of groups following a problem-specific approach (Friedkin, 1991). Assuming a TU-game that reflects the interests which motivate the interactions among individuals in a network, we extend the game theoretic centrality measure of Gomez et al. (2003) to the case of groups, and define the game theoretic group centrality of a group as the variation of its value or power due to their social relations. We rely on the Shapley group value (Flores et al., 2014) for measuring the value of a group in a game without any restriction, and we introduce the Myerson group value in order to measure the value when the social structure is considered.

Keywords: Centrality, social capital, Shapley group value, Myerson value.

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A game theoretic approach to group centrality*

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Abstract

This paper is centered in the valuation of the centrality of groups following a *problem-specific approach* (Friedkin, 1991). Our aim is to measure the relative position of a group taking also into account the purpose of the network. Assuming a TU game that reflects the interests which motivate the interactions among individuals in a network, we extend the game theoretic centrality measure of Gomez *et al.* (2003) to the case of groups, and define the *game theoretic group centrality* of a group as the variation of its value or power due to their social relations. We rely on the Shapley group value (Flores *et al.*, 2014) for measuring the value of a group in a game without any restriction, and we introduce the *Myerson group value* in order to measure the value of a group in a game that considers the limitations in the cooperation imposed by the social structure.

Keywords: Centrality, social capital, Shapley group value, Myerson value.

1 Introduction

A *social network* consists of a finite set of actors and the relations defined among them. In this paper we are concerned with social networks in which for every pair of actors we have at most a single relation that is dichotomous and non directional. Then, the social network can be represented by a graph (N, Γ) where $N = \{1, 2, \ldots, n\}$ is the finite set of actors (*nodes*) and Γ is a collection of (unordered) pairs $\{i, j\}$ of elements of N (*edges*), which show the possible relations; i.e., individuals i and j are related if, and only if, $\{i, j\} \in \Gamma$.

Wasserman and Faust (2004) point out that one of the primary uses of graph theory in this context is the identification of the "most important" or "prominent" actors in a social network. Definitions of *individual importance* have been proposed by many authors. For example, classical measures of *centrality* –*degree*, *betweenness* or *closeness*– that rely only upon the information given by the structure of the social network, were defined and widely studied (see the review of Freeman, 1979). Other individual measures, such as the *individual game theoretic centrality* index of Gomez *et al.* (2003), which takes also into account the purpose of the graph, have been described.

On the contrary, *group measures* of importance have not deserved such a detailed analysis. Nevertheless, the relevance of defining an appropriate *group centrality* measure has been pointed out by different authors in a variety of contexts, mainly in the framework of information diffusion models (see Kempe *et al.*, 2005), but also in the social networks context (Everett and Borgatti (1999); Borgatti (2006); Latora and Marchiori (2007); Kolaczyk *et al.* (2009)).

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Following the approach given by Gomez *et al.* (2003), we propose in this paper a new *group centrality measure* for *social networks*, which is based on a group valuation. They adopt a game theoretic approximation to the problem of measuring *individual centrality* by considering a cooperative game in characteristic function form to reflect the interests that motivate the interactions among individuals in a network. Then, they define the *game theoretic individual centrality* of an actor as the difference between his value in the game restricted by the graph which captures the social structure -measured by his Myerson value (Myerson, 1977)- and his value in the unrestricted game -measured by his Shapley value (Shapley, 1953)-. If the underlying game is symmetric, and therefore, a priori differences among agents are not taken into account, the resulting variations measure the positional advantage of each actor. On the other hand, when players' differences are modeled by means of a non-symmetric game, it is more appropriate to refer to the resulting variations as the *individual game theoretic social capital* (Salisbury, 1969) of each actor. These variations measure player's relational importance admitting that they possibly have different cooperative abilities (see Gónzalez-Arangüena, Khmelnitskaya, Manuel and Pozo, 2011).

Following this approach, we consider also a cooperative game in characteristic function form to model the purpose of the social network, and then we measure the difference between the value of a given group in the graph-restricted game when their members act together -as one unit- relative to the rest of actors, and their global value in the game without any communication restriction, nor any integration effect. We rely on the *Shapley group value*, and we introduce the *Myerson group value*, for measuring the value of a group depending on whether the social structure is taken into account. In fact, according to the specific meaning of the underlying game the resulting variation measures the *centrality* of the group -if the game describes the *functionality* of the network- or the *capital* of the group -if the game details the resources that are available to every group of actors. In the first case, the game describes the use of the network that anonymous actors do and therefore the underlying game must be symmetric. The second case, in which different actors can have different access to the resources, and thus the underlying game can be non-symmetric, is left for subsequent work.

The remainder of the paper is organized as follows. Section 2 is devoted to a general presentation of the problem we deal with. We describe some illustrative examples for motivating the need for a group measure, and also to show the key points we need to consider when evaluating the performance of a group. In Section 3 we introduce some standard concepts and notation on Game Theory and Graph Theory that will be used throughout this paper, and we give a brief description of the individual centrality index (Gomez *et al.*, 2003) on which our proposal is based. In Section 4 we introduce the *Myerson group value* and the *game theoretic group centrality* and we show how they may answer the relevant questions established from the examples. In Section 5 we study some of their properties. In Section 6 we illustrate its behavior using two datasets which have been previously analyzed in the literature of group centrality measures. Section 7 concludes the paper.

2 Motivation

Let us start by considering four examples that illustrate the need for a group centrality measure that allows to consider the specific "purpose of the network" when we wonder about which is the most central group with certain characteristics (of a given size, for instance). We use these examples to go into detail about the differences between functionality and capital, as well as to show that the aim of obtaining a group valuation can be *endogenous* -when the decision makers are the actors of the social network- or *exogenous* -when there exists an external decision maker who wants to achieve his own specific goal.

First, let us consider the social network of a criminal organization. For instance, the terrorist network of the 11S depicted in Figure 2.

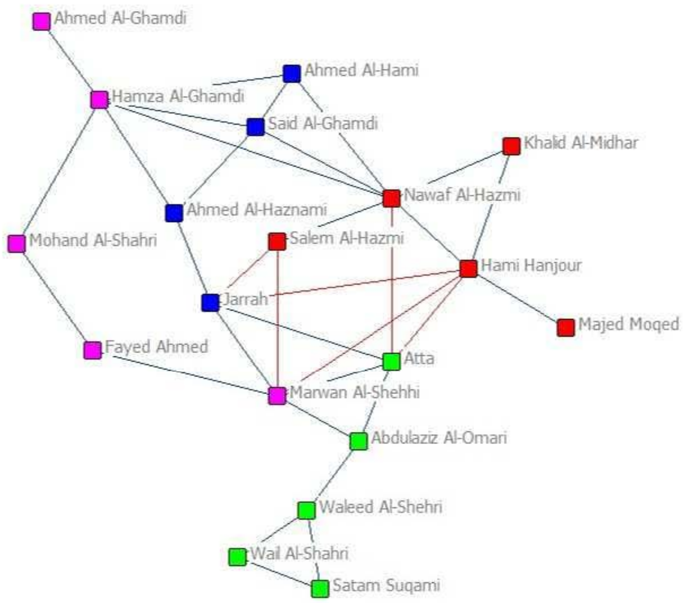


Figure 1: 11S social network. Image taken from Valids E.Krebs (Copyright ©2002, First Monday)

In this event, the purpose of the terrorist group is to mount a coordinated action. In the case of the terrorists, they use the group valuation to identify a small group of agents to act as leaders in order to guarantee the success of the terrorist attack. This is an *endogenous* goal. On the contrary, there exists an external decision agent, the police, who use these information about the valuation of each group to identify a small group of agents to neutralize in order to break up the criminal organization. This is an *exogenous* goal. We can assume here that the *functionality* of the network is to transmit information among the nineteen hijackers which prepared and executed the attack (distributed in four cells).

Many species have a social structure in which individuals form groups and interaction among members within each group is much more frequent than interaction of individuals across groups. This is the case of the second example: the social network of a troop of monkeys based on three months of interactions between them, observed in the wild by Linda Wolfe as they sported by a river in Ocala, Florida. In the Wolfe primate data, which is given as a standard dataset in UCINET (Borgatti, Everett and Freeman, 2002), joint presence at the river once was coded as a interaction. This data set was used by Everett and Borgatti (1999) to illustrate the group centrality measures they proposed. They dichotomized the data, formerly symmetric and valued, by taking the presence of a tie if there were more than six interactions over the time period (see Figure 3 in page 26). In that case, from the point of view of *group selection* in Evolutionary Biology it is worthy to evaluate the relevance of each of these groups in order to promote the fitness of the troop, which will be given mainly by their ability to form groups which improve the defense of their territory, as well as their defense against predators. In this example, the purpose of the primates troop is to promote its fitness. The specific feature relevant to the fitness of a group in which the analyst is interested will determine the *functionality* of the network.

The third example is a network within a consulting company analyzed in Borgatti (2006), which consists of advice-seeking ties among members of a the global company. The data were collected on a one to five strength-of-tie-scale, but for his analysis the author examined only the strongest ties (rated 5). The derived social network is shown in Figure 4 (see page 29). In this framework, the social network of informal relationships between the members of an organization have "become a pervasive feature of organizations" (see Cross and Parker, 2004). Thus, the identification of a small group of actors who are able to lead the formation of optimal working teams, or whose deletion would disrupt most the ability of the social structure to form them, is crucial for the top managers of the organization, which play the role of an external decision maker. In this case, to achieve the goal of the organization, we consider that the purpose of the social network's actors (the company members) is to form working teams. The specific kind of working teams that the organization wants to promote will determine the particular expression of the network's *functionality*.

In the last example, let us consider a voting body, or a compound system of voting bodies. In this situation, the capital of a group is given by its ability to form *winning coalitions*, i.e., coalitions of voters that can force the passing of a bill according to the prearranged voting rules. Let us think in the social network of possible alliances between parties in the Italian Parliament as a result of the elections in 1983 (see Gónzalez-Arangüena *et al.* (2011) for more details).

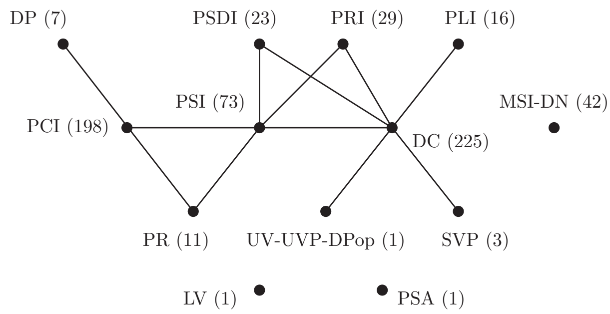


Figure 2: Social affinities between parties in the Italian Parliament after elections 1983

On the one hand, a lobbyist's goal is to choose a small group of constitutional voters in order to force the passing (blocking) of a bill. On the other hand, for this specific example an endogenous goal was to elect the Prime-Minister taking into account the *voting power* of each party, which is in turn a balanced measure between constitutional voting power and social capital.

The questions stated above are related with group centrality rather than individual centrality. Therefore, since the k more central agents (from an individual point of view) do not form in general the most valuable group of k agents, we need a group measure that adjust the sum of centralities so as to account for the possible complementarity and substitutability relations among group's members. That is, we must define a specific group measure to deal with the *ensemble issue* (see Borgatti, 2006). In that sense, Everett and Borgatti (1999) extend the standard network centrality measures of degree, closeness and betweenness to apply to groups and classes as well as individuals. Kolaczyk *et al.* (2009) propose an alternative extension of vertex betweenness to sets of vertices, which they called *group co-betweenness*, and study its relation with the *group betweenness* extension introduced by Everett and Borgatti (1999).

Nevertheless, none of these generalizations takes into account the purpose of the network -its functionality or its capital- which is not the same in each of the four examples described above, and must be incorporated to the problem at hand to measure the importance of a group in relation with the goal that is being pursued. Our intention is to define an appropriate *group* measure that, by means of extending an existing individual centrality measure, takes into account both the ensemble issue and the purpose of the network. We extend the individual game theoretic centrality measure introduced by Grofman and Owen (1982) and Gomez *et al.* (2003), and that of the individual game theoretic social capital index analyzed in Gónzalez-Arangüena *et al.* (2011). Latora and Marchiori (2007) introduce a new class of measures of structural centrality, which they called *delta centralities*, that applies to groups as well as individuals, and also try to capture the functionality of the network. The crucial difference of delta centralities and our proposal that delta centralities only consider the marginal contribution of a group to the whole society, whereas our proposal aggregates the marginal contribution of a group in all their possible cooperation opportunities.

3 Preliminaries

We first recall some standard notation and concepts from graph theory and game theory that will be used throughout the paper.

A *undirected graph* or simply a *graph* (N, Γ) consists of a finite set $N = \{1, \dots, n\}$ of nodes and a set Γ of *edges* whose elements are unordered pairs of distinct nodes. If $\{i, j\} \in \Gamma$, then i and j are called *adjacent* nodes in Γ. The *node adjacency list* $A_\Gamma(i)$ of a node i is the set of nodes that are directly linked to i in Γ, that is, $A_\Gamma(i) = \{j \in N \;/\; \{i, j\} \in \Gamma\}$. The *degree* of a node is the number of its adjacent nodes, i.e. $\delta_\Gamma(i) = |A_\Gamma(i)|$.

A graph (N', Γ') is a *subgraph* of (N, Γ) if $N' \subseteq N$ and $\Gamma' \subseteq \Gamma$, where the edge $\{i, j\}$ can be in Γ' only if i and j are in N'. We say that (N', Γ') is the *subgraph induced by* N' if Γ' contains each edge of Γ with endpoints in N'.

A *path* between two nodes i and j in a graph (N,Γ) is a subgraph of (N,Γ) consisting of a sequence of nodes and edges $P(i,j) = \{i = i_1, i_2, \dots, i_{k-1}, i_k = j\}$, with $k \geq 2$ satisfying the property that for all $1 \leq r \leq k-1$, $\{i_r, i_{r+1}\} \in \Gamma$. A *cycle* is a path $P = \{i = i_1, i_2, \dots, i_{k-1}, i_k = j\}$, together with the edge $\{i,j\}$.

A graph is *connected* if every pair $i,j \in N$ of its nodes is connected, i.e., if there is a path in the graph from node i to node j; otherwise, the graph is *disconnected*. The maximal connected subgraphs of a disconnected graph are called its *connected components*. Let $S \subseteq N$ be a subset of nodes, then $con_\Gamma(S)$ will denote the set of connected components of the subgraph (S, Γ_S) induced by S. We will refer to $con_\Gamma(S)$ as the set of connected components of S in Γ. A *tree* is a connected graph that contains no cycle. Note that a every two nodes of a tree are connected by a unique path. In this case, when (N,Γ) is a tree, then for every $S \subseteq N$ there exists a unique smallest connected subgraph in Γ which contains the subgraph (S,Γ_S), which we will call its *connected hull* and denote by $H_\Gamma(S)$.[1] For a general connected graph (N,Γ), let $\mathcal{M}_\Gamma(S) = \{S_1, \dots, S_r\} \neq \emptyset$ be the set of *minimal connection sets* of S in Γ, and $\mathcal{AM}_\Gamma(S) = \bigcup_{\ell=1}^{r} S_\ell$ be the set of agents in $\mathcal{M}_\Gamma(S)$. That is, every subgraph $(S_\ell, \Gamma_{S_\ell})$, $\ell = 1, \dots, r$, is connected and contains the subgraph (S,Γ_S) and, for every other connected subgraph (T,Γ_T) containing (S,Γ_S), there exists $\ell \in \{1,\dots,r\}$ such that $S_\ell \subseteq T$.

An n-person cooperative game in coalitional form with side payments, or with transferable utility (TU game), is an ordered pair (N,v) where N is a finite set of n players and $v : \mathcal{P}(N) \to \mathbb{R}$ is a map assigning a real number $v(S)$, called the *worth* of S, to each coalition $S \subseteq N$, and where $v(\emptyset) = 0$. The real number $v(S)$ represents the reward that coalition S can achieve by itself if all its members act together.

A game (N,v) is *super-additive* when $v(S) + v(T) \leq v(S \cup T)$, for every pair of disjoint coalitions $S,T \subseteq N$. If $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$, for every pair $S,T \subseteq N$, then the game is *convex*. A convex game can also be characterized by $v(S \cup i) - v(S) \leq v(T \cup i) - v(T)$, for every $S \subseteq T \subseteq N \setminus i$.

A game (N,v) is *symmetric* when all the agents play the same role in the cooperative situation and thus the only relevant information is the size of the coalition which is formed. In this case, the characteristic function $v(S)$ is defined by a real function $f(\cdot)$ such that $f(0) = 0$ and $v(S) = f(s)$ for all non-empty $S \subseteq N$ cardinality is s.

A game (N,u) is a *simple game* if $u(S) \in \{0,1\}$, for all $S \subseteq N$, and it is *monotonic* (i.e., $u(S) \leq u(T)$ whenever $S \subseteq T$). Simple games just make a difference between two types of coalitions: *winning and losing* coalitions, with worths 1 and 0, respectively. A *minimal winning coalition* $S \subseteq N$ is a winning coalition such that all its proper subcoalitions lose. The set $\mathcal{MW}(u)$ of minimal winning coalitions of the game (N,u) characterizes it.

Let $\mathcal{G}_n$ be the vector space of all TU games with fixed player set N, where $n = |N|$, and identify $(N,v) \in \mathcal{G}_n$ with its characteristic function v when no ambiguity appears. A *value* φ for TU games is an assignation which associates to each n-person game $(N,v) \in \mathcal{G}_n$ a vector $\varphi(N,v) \in \mathbb{R}^n$, where $\varphi_i(N,v) \in \mathbb{R}$ represents the *value* of player i, $i \in N$. Shapley (1953) defines his value as

[1] Formally, the subgraph which contains (S,Γ_S) is the subgraph induced by $H_\Gamma(S)$.

follows:

$$\phi_i(N,v) = \sum_{\substack{S\subseteq N\\ i\notin S}} \frac{s!(n-s-1)!}{n!}\big(v(S\cup\{i\})-v(S)\big), \quad i=1,\ldots,n. \tag{1}$$

Here $s=|S|$ denotes the cardinality of coalition $S\subseteq N$, and being $(N,v)\in\mathcal{G}_n$.

The *value* $\phi_i(N,v)$ of each player admits different interpretations, such as the *payoff* that player i receives when the Shapley value is used to predict the allocation of resources in multiperson interactions, or as a measure of his *power* when averages are used to aggregate the power of players in their various cooperation opportunities. Note that the difference $v(S\cup i)-v(S)$ measures the ability of player i to change the worth of coalition S in case she joins it.

Following the question originally addressed by Shapley in his seminal paper: we interpret the value as the *expectations* of a player in a game (N,v), so we will refer to $\phi(N,v)\in\mathbb{R}^n$ as a *valuation vector*. Then, $\phi_i(N,v)\in\mathbb{R}$ will measure the (a priori) value that playing the game (N,v) has for player i, and can be used as an objective function for selecting *key players*. Since the Shapley value is symmetric[2] and efficient[3] then $\phi_i(N,v)=\frac{f(n)}{n}$ for every player $i=1,\ldots,n$ and for every symmetric game (N,v).

The Shapley value admits an alternative expression in terms of the *Harsanyi dividends* of every coalition T in (N,v), which are given by $\Delta^N(v,T)=\sum_{\substack{S\subseteq T\\ S\neq\emptyset}}(-1)^{t-s}v(S)$, s and t being the cardinalities of S and T, respectively. The formula is based on the linearity of the Shapley value and the expression of the game (N,v) in terms of the basis of the *unanimity games*[4] (see Shapley (1953)).

$$\phi_i(N,v)=\sum_{\substack{S\subseteq N\\ i\in S}}\frac{\Delta^N(v,S)}{s}, \quad i=1,\ldots,n. \tag{2}$$

Now we are ready to recall the individual centrality measure in which our proposal is based.

A game theoretic approach to individual centrality

Let (N,Γ) be a graph describing the possible interactions among individuals in a social network, and let (N,v) be a symmetric TU-game describing the *functionality* of the network, i.e. the interests that motivate the interactions among the actors of the social network. Then, Gomez *et al.* (2003), following Myerson (1977), consider the *graph-restricted game* (N,v_Γ) to represent the economic possibilities of the agents when the available communication possibilities are taken into account. That is,

$$v_\Gamma(S)=\sum_{T_k\in con_\Gamma(S)} v(T_k), \quad S\subseteq N, \tag{3}$$

[2] $\phi_i(N,v)=\phi_j(N,v)$ for all symmetric players $i,j\in N$ (i.e., $v(S\cup i)=v(S\cup j)$, for all $S\subseteq N\setminus\{i,j\}$).

[3] $\sum_{i=1}^n\phi_i(N,v)=v(N)$, for every n-person TU game (N,v).

[4] Recall that a game (N,v) is a *unanimity game* if there exists a coalition S such that for every $T\subseteq N$, $v(T)=1$ if $S\subseteq T$, and $v(T)=0$ otherwise; in this case, we will denote the game by (N,u^S) and its Shapley value is $\phi_i(N,u^S)=\frac{1}{s}$, for all $i\in S$, and $\phi_i(N,u^S)=0$, otherwise. Unanimity games are a basis of the vector space of the n-person TU-games.

where $con_\Gamma(S)$ is the set of connected components of S in Γ. In that setting, they argue that if we consider the Shapley and the Myerson values as a reasonable outcome for these two games (N, v) and (N, v_Γ), respectively, then the differences between the corresponding outcomes can be considered as a result of the different positions which players have in the graph Γ. Therefore, they propose the difference

$$\gamma_i(N, v, \Gamma) = \phi_i(N, v_\Gamma) - \phi_i(N, v), \quad i = 1, \ldots, n, \tag{4}$$

as a measure of the **centrality** of player $i \in N$ in the graph (N, Γ), given (N, v).

Note that (N, v) describes the functionality of the network, each specific choice of game (N, v) determines a centrality measure, and the final centrality figures and relative ordering depend on this specific choice. Thus, it is worthwhile to propose games that properly model some functionalities of interest to social networks analysis. In that sense, Gomez *et al.* (2003) define some particular relevant cases of centrality measures, based on three games for modeling three scenarios, that allow us to measure group centrality considering three appealing purposes.

i) *Overhead game*, to mount a coordinated action:

$$v^{overh}(S) = -1, \quad \forall \emptyset \neq S \subseteq N,$$

which focuses on the "general cost" that any set of players should pay to perform an action, considering that all players are related.

ii) *Messages game*, to send information through the network via bilateral transmissions:

$$v^{msg}(S) = 2\binom{s}{2} = s^2 - s, \quad \forall S \subseteq N,$$

which focuses on the level communication activity between pairs of individuals, where the benefit of a binary communication is independent of the actors performing and it is the same for each relation. Thus, the payoff $v^{msg}(S)$ of coalition S is proportional to the number of pairs in S, and thus $v(S) = k\binom{s}{2}$. The proportionality constant k is chosen as 2 in order to represent communication in both ways.

iii) *Conferences game*, to send information through the network via multilateral transmissions:

$$v^{conf}(S) = 2^s - s - 1, \quad \forall S \subseteq N,$$

which reflects the level of communication among groups of two or more individual in a coalition. The payoff of coalition S is the number of subsets or *conferences* in S with a cardinality greater than one.

Using these characteristic functions we may describe the functionality of the motivational examples, as we will explain in the sequel.

With respect to the terrorist group, the *functionality* of the network is to transmit information among the nineteen hijackers which prepared and executed the attack. In that case, taking into account that bilateral communications are more safe than multilateral ones, the messages game gives a good functionality model. Moreover, if we extend the social network to a more dense network which included some people which did not take direct part in the attack but support the terrorists, the purpose of the new network remains the same. Then, an asymmetric version of the messages game in which only the messages between pairs of hijackers have a non–zero worth could be used.

In the second example, since one of the purposes of the social structure is to promote the survival of the troop, the overhead game may provide a functionality modeling. This game describes the common vital task of group survival. The workload is always the same regardless of the number of primates in the group. Then, a bigger group is more efficient in a per capita basis than a smaller one. Nevertheless, in Section 6 we analyze in deep this example and we propose an alternative specific game.

In the third example, taking into account that a social network of informal relationships among the members of an organization have a critical influence on the formation of valuable working teams, the conferences game may be a good candidate to describe the functionality of the network. Again, in Section 6 we propose slight variations of the conference game in order to account for the optimal size of a group in the context of advice seeking.

In the last example we deal with *capital*, rather than with functionality. In this kind of situations, the capital of each group can be described by means of a *simple game* (N, u) that is in general non-symmetric, and therefore the above difference (4) is interpreted as an index of *social capital*[5] (Salisbury, 1969). Frequently, the voting procedure can be described by means of a *weighted simple game* (N, w), characterized by a vector of agents' weights $(w_1, \ldots, w_n) \in \mathbb{R}^n$ and a *quota* $q \in \mathbb{R}$ such that a coalition $S \subseteq N$ is winning when the sum of the weights of the players in S meets or exceeds the quota that is requested to pass the proposal. For the specific example of the Italian Parliament the capital is described by means of the weighted simple game with quota $q = 316$ and $w(316; 225, 198, 73, 42, 29, 23, 16, 11, 7, 3, 1, 1, 1)$, being $N = \{DC, PCI, PSI, MSI - DN, PRI, PSDI, PLI, PR, DP, SVP, LV, PSA, UV - UVP - DPop\}$.

In this paper we propose some examples of characteristic functions, although the definition of the characteristic function to be used in a specific situation should be subject to a thorough analysis. We want to remark the versatility of the game theoretic measures of centrality, since each characteristic function v may provide a different centrality measure that takes into account the specific purpose described by v.

4 Myerson group value and centrality

In this section we introduce our main ideas. As the game theoretic centrality measure is based on the variation in each agents's value due to the social structure, then we first need to measure the value of a group in the two games (N, v) and (N, v_Γ). We base our proposal on the extension

[5]The reader is referred to González-Arangüena, Khmelnitskaya, Manuel and Pozo (2011), where the properties of the individual game theoretic centrality measure as a social capital index for individuals are studied.

of the Shapley value from individuals to groups considered in Flores, Molina and Tejada (2013), which we recall below.

Given a TU-game (N, v) and a coalition $C \subseteq N$, Derks and Tijs (2000) define the merging game (N_C, v_C) in which all the agents of $C \subseteq N$ are replaced by a single "*super–player*" $\mathbf{c}$, who can act as a proxy of any player in C. Formally, in the merging game (N_C, v_C), the player set N will become $N_C = (N \setminus C) \cup \{\mathbf{c}\}$ with $\mathbf{c}$ as a single player $\mathbf{c} \equiv C$, and the characteristic function v_C describing the new situation is of the form:

$$v_C(S) = \begin{cases} v(S), & \text{if } \mathbf{c} \notin S, \\ v(S \cup C) & \text{if } \mathbf{c} \in S, \end{cases} \quad \forall\, S \subseteq N_C. \tag{5}$$

Note that when considering the previous merging game[6] for evaluating the expectations of group C in the game, we do not need to suppose necessarily that the agents know each other nor agree to act jointly; instead, we may assume the existence of an *external decision maker* that is able and willing to compute the value of the different groups. For instance, in the case of the criminal organization, there exists a external agent, the police, whose goal is to establish which is the group of terrorists more likely to turn back into double agents, or the more appropriate to to spread misinformation through the criminal organization network.

Now we define the *Shapley group value* of the game $(N, v) \in \mathcal{G}_n$ as the mapping ϕ^g that assigns for every $C \subseteq N$ a real number $\phi^g_C(N, v) \in \mathbb{R}$ which represents the *a priori value* of group $C \subseteq N$ in (N, v), and its is given by

$$\phi^g_C(N, v) := \phi_{\mathbf{c}}(N_C, v_C), \text{ for all } \emptyset \neq C \subseteq N, \text{ and } \phi_\emptyset(N, v) = 0.$$

The Shapley group value represents a priori valuation of the expectations of a group in an unrestricted game (N, v) when they act as a unit. Now, in order to measure the value of each group when the restrictions in the cooperation due to the social relations are considered, we introduce the *Myerson group value* as the Shapley group value of the graph-restricted merging game.

Definition 1. Let (N, Γ) be a social network, and (N, v) a TU game reflecting its functionality. Then for every group $C \subseteq N$ the *graph-restricted merging game* $(N_C, v_{\Gamma,C})$ is defined as:

$$v_{\Gamma,C}(S) = v_\Gamma(S) = \sum_{T_k \in con_\Gamma(S)} v(T_k), \tag{6}$$

$$v_{\Gamma,C}(S \cup \mathbf{c}) = v_\Gamma(S \cup C) = \sum_{T_k \in con_\Gamma(S \cup C)} v(T_k), \tag{7}$$

for every coalition $S \subseteq N \setminus C = N_C \setminus \{\mathbf{c}\}$.

Definition 2. Let (N, Γ) be a social network, and (N, v) a TU game reflecting its functionality.

[6] Which is also called the *quotient game* (Owen, 1977) of (N, v) with respect to the *coalition structure*.

Then the *Myerson value of group* $C \subseteq N$ is defined to be:

$$\phi^g_C(N, v, \Gamma) := \phi^g_C(N, v_\Gamma) = \phi_{\mathbf{c}}(N_C, v_{\Gamma,C}), \text{ for every group } C \subseteq N.$$

The Myerson value of C can be interpreted as a priori valuation of the expectation of group C in the game (N, v) when communications between the players are restricted by the graph (N, Γ). In this context, the Myerson group value is more related to the kind of questions Borgatti's work (2006) deals with, and also with the *Target Set Selection Problem* analyzed by Kempe *et al.* (2005). That is, to choose a set of individuals to target if we want to diffuse widely and rapidly an *announcement* or an *attitude* among the social network; or to maximally disrupt the network's ability to mount coordinated action, etc. However, if we are interested in a group centrality measure, then we must account for the variations in value due to their position in the graph. Thus, following Gomez *et al.* (2003) approach, we propose the formal next definitions.

Definition 3. Let (N, Γ) be a social network, and (N, v) a TU symmetric game reflecting its functionality. Then, the *centrality of the group* $C \subseteq N$, is defined to be:

$$\gamma^g_C(N, v, \Gamma) := \phi^g_C(N, v_\Gamma) - \sum_{i \in C} \phi_i(N, v), \quad \forall\, C \subseteq N. \tag{8}$$

In general, if the a priori differences among agents must be taken into account, then a non–symmetric game that properly reflects those differences must be considered. Therefore, the resulting variations measure the social capital of a group, rather than its centrality. Formally:

Definition 4. Let (N, Γ) be a social network, and (N, v) a TU game reflecting the amount of resources which are available to each group, i.e. its capital. Then, the *social capital of the group* $C \subseteq N$, is defined to be:

$$\kappa^g_C(N, v, \Gamma) := \phi^g_C(N, v_\Gamma) - \sum_{i \in C} \phi_i(N, v), \quad \forall\, C \subseteq N. \tag{9}$$

The above differences (8) and (9) represent the increase (or decrease) in the value of group C due to two factors: its position in the graph, i.e. its *positional value*, and the synergies among their agents, i.e. their *integration effect* -which depend only on the functionality of the network. In fact, the centrality (social capital) of group C can be reformulated as

$$\gamma^g_C(N, v, \Gamma) = \underbrace{\Big(\phi^g_C(N, v_\Gamma) - \phi^g_C(N, v)\Big)}_{\textit{positional value}} + \underbrace{\Big(\phi^g_C(N, v) - \sum_{i \in C} \phi_i(N, v)\Big)}_{\textit{integration effect}}.$$

The first difference $\phi^g_C(N, v_\Gamma) - \phi^g_C(N, v)$ measures the variation in the value of group C due to their position in the social network, whereas the second difference accounts in turn for the benefits derived from their agreement to act jointly taking into account the purpose of the network.

We will focus on measuring the centrality of a group and thus we will restrict ourselves to the symmetric case (the general asymmetric case is left for subsequent work). Note then that $\sum_{i \in C} \phi_i(N, v) = \frac{c}{n} v(N)$ if the game (N, v) is symmetric. Thus, if we are interested in comparing centralities of groups of the same cardinality c, it is possible to avoid the last term in (8) and use $\phi_C^g(N, v_\Gamma)$ as a measure for the centrality of all groups of size c.

Definitions 2 and 3 provides us with two measurements of value and centrality which take into account the *ensemble issue*. Next, we will show that the proposed Myerson group value simultaneously captures the *positive* and *negative* value of every group, which measure respectively its value to achieve the specific goal that is being pursued, and its value to force its underachievement. First we recall the definition of the *dual game*, as well as the *self-duality* property of the Shapley value.

Definition 5. Let $(N, v) \in \mathcal{G}_n$ be a TU game. Then its *dual game*, denoted by (N, v^D), is defined for all $S \subseteq N$ by $v^D(S) = v(N) - v(N \setminus S)$.

The worth $v^D(S)$ of $S \subseteq N$ in the dual game measures the losses of the grand coalition if coalition S leaves the game (N, v). That is, $v^D(S)$ is the benefit that their members can block in the situation without any coalition structure. Now, in order to properly measure the underachievement of the goal pursued by the network when coalition S leaves the society, then the *dual graph-restricted game*, which we will denote by (N, v_Γ^D), must be defined as follows:

$$v_\Gamma^D(S) := (v_\Gamma)^D(S) = v_\Gamma(N) - v_\Gamma(N \setminus S), \quad \forall\, S \subseteq N. \tag{10}$$

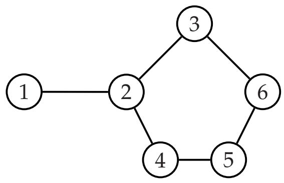
The worth of coalition S in the dual game is then $v(N) - \sum_{T_k \in con_\Gamma(N \setminus S)} v(T_k)$, whenever the social network (N, Γ) is connected. Otherwise, the graph-restricted game is an additive game over its connected components, and the dual operator applies to each of these components.

The Shapley value is *self-dual*, that is, the Shapley value of the game (N, v) equals the Shapley value of its dual, and thus it simultaneously captures the positive and the negative value of every player. Next proposition shows that the same remains true at a group level.

Proposition 1. *Let (N, Γ) be a social network, and (N, v) be a general TU game reflecting its functionality. Then, the Myerson group value and the game theoretic group centrality are self-dual, i.e., $\phi_C^g(N, v^D, \Gamma) = \phi_C^g(N, v, \Gamma)$ and $\gamma_C^g(N, v^D, \Gamma) = \gamma_C^g(N, v, \Gamma)$, for every group $C \subseteq N$.*

All the proofs here and below are postponed to the Appendix, not to interrupt the natural flow of the arguments. Next, we show an example to illustrate the previous definitions.

Example 1. Let us consider the same *kite* with 6 nodes as in Gomez *et al.* (2003), and let (N, v^{msg}) be the messages game:



Let us calculate the Myerson value and centrality of the group $C = \{1,2,5,6\}$, whose a reasonable size which allows to describe the corresponding merging game; observe, in particular, that the merged player set is $N_C = \{c,3,4\}$, of cardinality three. The characteristic function $v^{msg}_{\Gamma,C}$ is given by:

$$v^{msg}_{\Gamma,C}(c) = v^{msg}_{\Gamma}(C) = 2+2 = 4, \quad v^{msg}_{\Gamma,C}(3) = v^{msg}_{\Gamma}(3) = 0, \quad v^{msg}_{\Gamma,C}(4) = v^{msg}_{\Gamma}(4) = 0,$$

$$v^{msg}_{\Gamma,C}(\{\mathbf{c},3\}) = v^{msg}_{\Gamma}(\{1,2,3,5,6\}) = 5\cdot 4 = 20, \quad v^{msg}_{\Gamma,C}(\{\mathbf{c},4\}) = v^{msg}_{\Gamma}(\{1,2,4,5,6\}) = 20,$$

$$v^{msg}_{\Gamma,C}(\{3,4\}) = v^{msg}_{\Gamma}(\{3,4\}) = 0, \quad v^{msg}_{\Gamma,C}(N_C) = v^{msg}_{\Gamma}(N) = 6\cdot 5 = 30.$$

The Myerson group value of C is:

$$\phi^g_C(N, v^{msg}_{\Gamma}, \Gamma) = \phi_{\mathbf{c}}(N_C, v^{msg}_{\Gamma,C}) =$$

$$= \sum_{S\subseteq\{3,4\}} \frac{s!(2-s)!}{3!}\left(v^{msg}_{\Gamma,C}(S\cup\{\mathbf{c}\}) - v^{msg}_{\Gamma,C}(S)\right) = \left(\frac{4}{3}\right) + \left(2\cdot\frac{20}{6}\right) + \left(\frac{30}{3}\right) = 18.$$

Then, since $\phi_i(N, v^{msg}) = \frac{30}{6} = 5$, its centrality is $\gamma^g_C(N, v^{msg}, \Gamma) = 18 - 4\cdot 5 = -2$.

The corresponding dual graph-restricted game $(N_C, (v^{msg}_{\Gamma,C})^D)$, which measures the blocking worth of each coalition, will be given by:

$$(v^{msg}_{\Gamma,C})^D(\mathbf{c}) = 30 - v^{msg}_{\Gamma}(\{3,4\}) = 30,$$

$$(v^{msg}_{\Gamma,C})^D(3) = 30 - v^{msg}_{\Gamma}(\{1,2,4,5,6\}) = 10, \quad (v^{msg}_{\Gamma,C})^D(4) = 30 - v^{msg}_{\Gamma}(\{1,2,3,5,6\}) = 10,$$

$$(v^{msg}_{\Gamma,C})^D(\{\mathbf{c},3\}) = 30 - v^{msg}_{\Gamma}(\{4\}) = 30, \quad (v^{msg}_{\Gamma,C})^D(\{\mathbf{c},4\}) = 30 - v^{msg}_{\Gamma}(\{3\}) = 30,$$

$$(v^{msg}_{\Gamma,C})^D(\{3,4\}) = 30 - v^{msg}_{\Gamma}(\{1,2,5,6\}) = 16, \quad (v^{msg}_{\Gamma,C}(N_C))^D = 30 - v^{msg}_{\Gamma}(\emptyset) = 30.$$

It can be checked that

$$\phi^g_C(N, v^D, \Gamma) = \phi_{\mathbf{c}}(N_C, (v^{msg}_{\Gamma,C})^D) = \phi_{\mathbf{c}}(N_C, v^{msg}_{\Gamma,C}) = \phi^g_C(N, v, \Gamma),$$

and therefore $\phi^g_C(N, v, \Gamma)$ measures the negative and positive value of group C.

With respect to the smaller groups $C_1 = \{1,2\}$ and $C_2 = \{5,6\}$, we omit the calculations which are more tedious since the merging games $(N_{C_1}, v^{msg}_{\Gamma,C_1})$ and $(N_{C_2}, v^{msg}_{\Gamma,C_2})$ are 5-person TU games.

We show the final results:

$$\phi^g_{C_1}(N, v^{msg}_\Gamma, \Gamma) = \phi_{\mathbf{c}_1}(N_{C_1}, v^{msg}_{\Gamma, C_1}) = \frac{286}{30},$$
$$\phi^g_{C_2}(N, v^{msg}_\Gamma, \Gamma) = \phi_{\mathbf{c}_2}(N_{C_2}, v^{msg}_{\Gamma, C_2}) = \frac{271}{30},$$

and thus group C_1 is more valuable and central than it is group C_2:

$$\gamma^g_{C_1}(N, v^{msg}_\Gamma, \Gamma) = \phi^g_{C_1}(N, v^{msg}_\Gamma) - 2 \cdot 5 = -\frac{14}{30},$$
$$\gamma^g_{C_2}(N, v^{msg}_\Gamma, \Gamma) = \phi^g_{C_2}(N, v^{msg}_\Gamma) - 2 \cdot 5 = -\frac{29}{30}.$$

Now, if we compare the centrality and Myerson value of groups C_1 and C_2 separately with the centrality and Myerson value that both groups have when they act as a unit and form group C we obtain:

$$\gamma^g_{C_1}(N, v, \Gamma) + \gamma^g_{C_2}(N, v, \Gamma) - \gamma^g_C(N, v, \Gamma) = \phi^g_{C_1}(N, v, \Gamma) + \phi^g_{C_2}(N, v, \Gamma) - \phi^g_C(N, v, \Gamma) = \frac{17}{30}.$$

This difference, which represents the 3.148% of their joint value, can be interpreted as a measure of redundancy between groups C_1 and C_2.

$\diamondsuit$

Now, in connection with the previous example, let us formalize the concept of *redundancy* between groups of agents, which we will analyze in more detail in next section.

Definition 6. Let (N, Γ) be a social network, and (N, v) a TU game reflecting its functionality and the a priori differences among the agents. Then, the *redundancy* between groups C_1 and C_2, is defined to be:

$$Red(C_1, C_2, N, v, \Gamma) := \gamma^g_{C_1}(N, v, \Gamma) + \gamma^g_{C_2}(N, v, \Gamma) - \gamma^g_{C_1 \cup C_2}(N, v, \Gamma) =$$
$$= \phi^g_{C_1}(N, v, \Gamma) + \phi^g_{C_2}(N, v, \Gamma) - \phi^g_{C_1 \cup C_2}(N, v, \Gamma),$$

for every pair of groups $C_1, C_2 \subseteq N$.

Their *relative redundancy* is defined as the quotient

$$RRed(C_1, C_2, N, v, \Gamma) := \frac{Red(C_1, C_2, N, v, \Gamma)}{\phi^g_{C_1 \cup C_2}(N, v, \Gamma)}.$$

Note that negative redundancies must be interpreted as a positive feature. If $Red(C_1, C_2, N, v, \Gamma) < 0$, then groups C_1 and C_2 strengthen each other, i.e., they are *complementaries*.

5 Properties

In this section we show some interesting properties of the proposed group measures. It is organized in four subsections.

First, we give a a general decomposition of the Myerson group value in two kinds of value: *communication* and *betweenness*. As a result, we obtain that the Myerson value of a group counts for its contribution to enable the formation of coalitions, either through mediation or connection, taking into account the benefit of each coalition, which is problem-specific and depends on the selected game (N, v). Thus, game theoretic point of view allows to adopt the *problem-specific approach* to centrality advocated by Friedkin (1991) and followed by Borgatti (2006), since the Myerson value and centrality emerge as a result of a combination of two different features. Not only the network topology determines the centrality of a group, but also the purpose of the network does. We also use this decomposition to elaborate on redundancy in Example 2.

Then, in subsection 5.2, we study the marginal effect that the incorporation of a new member has over an already integrated group. This marginal effect is related with a measure of average complementarity between the entrant player and the incumbent group.

In the third subsection we identify, under some restrictions, the groups of lowest and highest centrality of a given size k.

Subsection 5.4 concludes by showing that, for a particular type of connected graphs, the game theoretic group measures we propose are closely related to certain standard measures of group centrality and *cohesiveness*.

5.1 Myerson group value decomposition: communication and betweenness

The main goal of this paragraph is to give a a general decomposition of the Myerson group value (see Propositions 2 and 3 below) in two kinds of value – *communication* and *betweenness*– along the lines of Gomez *et al.* (2003). Next lemmas, that allow to express $\phi_C(N, v_\Gamma)$ in terms of the Harsanyi dividends of the original game (N, v), are needed for obtaining such decomposition. Recall that the connected convex hull $H_\Gamma(S)$ of a subset S in Γ and the set $\mathcal{M}_\Gamma(S)$ of its minimal connection sets is defined in Section 3.

Lemma 1. *Let (N, Γ) be a social network, and let (N, v) a TU game. If (N, Γ) is a tree, then*

$$\phi^g_C(N, v_\Gamma) := \phi_{\mathbf{c}}(N_C, v_{\Gamma,C}) = \sum_{H_\Gamma(S)\cap C\neq\emptyset} \frac{\Delta^N(v, S)}{|N_C/H_\Gamma(S)|}, \tag{11}$$

where $N_C/S \subseteq N_C$ is the coalition

$$N_C/S = \begin{cases} (S \setminus C) \cup \{c\}, & \text{if } S \cap C \neq \emptyset, \\ S, & \text{if } S \cap C = \emptyset. \end{cases}$$

Lemma 2. *Let (N, Γ) be a social network, and let (N, v) a TU game. If (N, Γ) is a general connected* *graph (N, Γ), let $\mathcal{M}_\Gamma(S) = \{S_1, \ldots, S_r\} \neq \emptyset$ be the set of minimal connection sets of S in Γ. Then, we have*

$$\phi^g_C(N, v_\Gamma) := \phi_{\mathbf{c}}(N_C, v_{\Gamma,C}) = \sum_{\substack{S \subseteq N \\ C \cap \mathcal{AM}_\Gamma(S) \neq \emptyset}} \Delta^N(v, S)\phi_{\mathbf{c}}(N_C, u^S_{\Gamma,C}), \tag{12}$$

where $\mathcal{AM}_\Gamma(S) = \bigcup_{\ell=1}^r S_\ell$ is the set of agents in $\mathcal{M}_\Gamma(S)$ and $(N_C, u^S_{\Gamma,C})$ is the simple game with minimal winning coalitions set:

$$\mathcal{MW}(N_C, u^S_{\Gamma,C}) = \{N_C/S_1, \ldots, N_C/S_r\},$$

for all $S \subseteq N$.

From the previous lemmas the decomposition results established in Propositions 2 and 3 follow straightforward. First, let us formalize the notion of being an *intermediary*. If (N, Γ) is a graph abd $S \subseteq N$ a coalition, then the set of intermediaries of S agents in Γ is determined by

$$Bet_\Gamma(S) = \mathcal{AM}_\Gamma(S) \setminus S = \{j \notin S \,/\, \exists\, S_\ell \in \mathcal{M}_\Gamma(S) \text{ with } j \in S_\ell\}.$$

Now according to the group C we can distinguish two kind of coalitions $S \subseteq N$ with $C \cap \mathcal{AM}_\Gamma(S) \neq \emptyset$: the coalitions $S \subseteq N$ that incorporate C members, and the coalitions that do not incorporate any member of C but in which some members of C may be needed to be connected. Therefore, according to (12), the Myerson value of group C can be decomposed in two kinds of value:

- *communication value*, which is the portion of value corresponding to those payoffs received as members of different coalitions S with $S \cap C \neq \emptyset$, and
- *betweenness value*, which is the portion of value corresponding to those payoffs received as intermediaries between members of different coalitions S such that $S \cap C = \emptyset$.

Next propositions formalize this decomposition.

Proposition 2. *Let (N, Γ) be a social network and let (N, v) a TU game. If (N, Γ) is a tree, then*

$$\phi^g_C(N, v_\Gamma) = \phi^{Com}_C(N, v_\Gamma) + \phi^{Bet}_C(N, v_\Gamma) = \overbrace{\sum_{\substack{S \subseteq N \\ S \cap C \neq \emptyset}} \frac{\Delta^N(v, S)}{|H_\Gamma(S) \setminus C| + 1}}^{\textit{communication value}} + \overbrace{\sum_{\substack{S \subseteq N \setminus C \\ H_\Gamma(S) \cap C \neq \emptyset}} \frac{\Delta^N(v, S)}{|H_\Gamma(S) \setminus C| + 1}}^{\textit{betweenness value}} \tag{13}$$

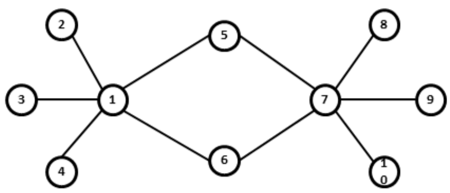
Proposition 3. *Let (N, Γ) be a social network and let (N, v) a TU game. If (N, Γ) is a connected graph,* *then*

$$\phi^g_C(N, v_\Gamma) = \phi^{Com}_C(N, v_\Gamma) + \phi^{Bet}_C(N, v_\Gamma) =$$

$$= \overbrace{\sum_{\substack{S\subseteq N \\ S\cap C\neq\emptyset}} \Delta^N(v,S)\phi_{\mathbf{c}}(N_C, u^S_{\Gamma,C})}^{\textit{communication value}} + \overbrace{\sum_{\substack{S\subseteq N\setminus C \\ Bet_\Gamma(S)\cap C\neq\emptyset}} \Delta^N(v,S)\phi_{\mathbf{c}}(N_C, u^S_{\Gamma,C})}^{\textit{betweenness value}}. \quad (14)$$

In the next example we use the above decomposition to analyze how the proposed group measures account for the two kinds of *redundancy* considered by Borgatti (2006): redundancy with respect to adjacency and distance, and redundancy with respect to bridging. We omit here some of the algebraic manipulations because they are tedious and technical. They are available upon request.

Example 2. Let us consider the following social network, and let (N, v) be a symmetric TU game:



Now let us consider the group $C = \{5, 6\}$; then the redundancy of agents 5 and 6 (see Definition 6) is given by:

$$Red(\{5\}, \{6\}, N, v, \Gamma) = \phi_5(N, v, \Gamma) + \phi_6(N, v, \Gamma) - \phi^g_{\{5,6\}}(N, v, \Gamma).$$

If we apply the formula (14) to the computation of $\phi_5(N, v, \Gamma)$, $\phi_6(N, v, \Gamma)$ and $\phi_{\{5,6\}}(N, v, \Gamma)$, we can make the difference between two kinds of redundancy of 5 and 6 : with respect to *communication* and with respect to *betweenness*. The variation of communication value when agents 5 and 6 form a group accounts for their redundancy with respect to adjacency and distance, which we will call *communication-redundancy*, whereas the variation of betweenness value accounts for their redundancy with respect to bridging, which we will call *betweenness-redundancy*, i.e.:

$$Red(\{5\}, \{6\}, N, v, \Gamma) = ComRed(\{5\}, \{6\}, N, v, \Gamma) + BetRed(\{5\}, \{6\}, N, v, \Gamma).$$

The *communication-redundancy* of agents 5 and 6, using the first summand of expression (14) and after some calculations, is given by:

$$ComRed(\{5\},\{6\},N,v,\Gamma) = \frac{1}{6}\Delta^N(v,\{5,6\}) + \sum_{\substack{S\subseteq N\\ \{5,6\}\subset S}} \underbrace{\Big(\frac{|H_\Gamma(S)|-2}{|H_\Gamma(S)|(|H_\Gamma(S)|-1)}\Big)}_{\geq 0}\Delta^N(v,S) \qquad (15)$$

Analogously, the *betweenness-redundancy* of agents 5 and 6, using now the second summand of expression (14), is given by:

$$BetRed(\{5\},\{6\},N,v,\Gamma) =$$
$$\sum_{\substack{\emptyset\neq S_1\subseteq\{1,2,3,4\}\\ \emptyset\neq S_2\subseteq\{7,8,9,10\}}} \underbrace{\Big(\frac{1}{|H_\Gamma(S_1)|+|H_\Gamma(S_2)|+1} - \frac{2}{|H_\Gamma(S_1)|+|H_\Gamma(S_2)|+2}\Big)}_{\leq 0}\Delta^N(v,S_1\cup S_2). \qquad (16)$$

Thus, being the coefficients in (15) positive and the coefficients in (16) negative, we may state that agents 5 and 6 are redundant for spreading purposes in this case, although both are strictly necessary when the goal is to break the communications. The above coefficients only deal with the structure of the network, but not with the interest on forming a coalitions. Such interests, measured through the Harsanyi dividends of coalition S in (15), and those of $S_1\cup S_2$ in (16), also determine the amount of positive and negative redundancy, respectively.

Now, if the functionality of the network is to transmit information through bilateral communications then, applying the above calculations to the messages game, we obtain the following results:[7]

$$ComRed(\{5\},\{6\},N,v^{msg},\Gamma) = \frac{1}{6}\Delta^N(v^{msg},\{5,6\}) + 0 = \frac{1}{3},$$
$$BetRed(\{5\},\{6\},N,v^{msg},\Gamma) = \sum_{\substack{i\in\{1,2,3,4\}\\ j\in\{7,8,9,10\}}} \Big(\frac{1}{3}-\frac{2}{4}\Big)\Delta^N(v^{msg},\{i,j\}) = -\frac{8}{3}.$$

If we go back to Example 1, then we can easily compute the communication and betweenness values of each of the involved groups C_1, C_2 and $C = C_1\cup C_2$. We obtain:

$$\phi_C^{Com}(N,v^{msg},\Gamma) = 2\sum_{\{i,j\}\subseteq C}\phi_{\mathbf{c}}(N_C,u_{\Gamma,C}^{\{i,j\}}) + 2\sum_{i\in C}(\phi_{\mathbf{c}}(N_C,u_{\Gamma,C}^{\{i,5\}}) + \phi_{\mathbf{c}}(N_C,u_{\Gamma,C}^{\{i,6\}}) = \frac{52}{3},$$

$$\phi_C^{Bet}(N,v^{msg},\Gamma) = 2\phi_{\mathbf{c}}(N_C,u_{\Gamma,C}^{\{5,6\}}) = \frac{2}{3}.$$

For the subgroups C_1 and C_2 we obtain $\phi_{C_1}^{Com}(N,v^{msg},\Gamma) = \frac{136}{15}$, $\phi_{C_1}^{Bet}(N,v^{msg},\Gamma) = \frac{7}{15}$, $\phi_{C_2}^{Com}(N,v,\Gamma) =$

[7]Note that the Harsanyi dividends of the messages game verify $\Delta^N(v^{msg},S) = 2$, for every coalition $S\subseteq N$ with $|S| = 2$, and $\Delta^N(v^{msg},S) = 0$, otherwise.

$\frac{266}{30}$, and $\phi^{Bet}_{C_2}(N, v, \Gamma) = \frac{1}{6}$. Then

$$ComRed(C_1, C_2, N, v^{msg}, \Gamma) = \phi^{Com}_{C_1}(N, v^{msg}, \Gamma) + \phi^{Com}_{C_2}(N, v, \Gamma) - \phi^{Com}_{C}(N, v^{msg}, \Gamma) = \frac{3}{5} > 0,$$

$$BetRed(C_1, C_2, N, v^{msg}, \Gamma) = \phi^{Bet}_{C_1}(N, v^{msg}, \Gamma) + \phi^{Bet}_{C_2}(N, v, \Gamma) - \phi^{Bet}_{C}(N, v^{msg}, \Gamma) = -\frac{1}{30} < 0,$$

and therefore groups C_1 and C_2 are redundant for spreading purposes, but they are complementaries with respect to bridging. $\diamondsuit$

Previous decomposition results only deal with connected social networks. Nevertheless, they are general enough since the Myerson value of a group in a disconnected graph can be obtained as the sum of the Myerson value of its subgroups, when considered as a group in the different connected subgraphs to which each subgroup belongs.

Proposition 4. *Let (N, Γ) be a social network, and let (N, v) a TU game. If (N, Γ) is a disconnected graph, let (N^k, Γ^k), $k = 1, \dots, r$, be its connected components. Then,*

$$\phi^g_C(N, v, \Gamma) = \sum_{k=1}^{r} \phi^g_{C^k}(N^k, v^k, \Gamma^k),$$

where v^k is the restriction of v to N^k and $C^k = C \cap N^k$.

Note that the above lemma does not generalize straightforward when dealing with centralities. For instance, let us consider the case of two components. Then it follows that

$$\gamma^g_C(N, v, \Gamma) = \gamma^g_{C^1}(N^1, v^1, \Gamma^1) + \gamma^g_{C^2}(N^2, v^2, \Gamma^2) - \sum_{i \in C^1} \Big(\phi_i(N, v) - \phi_i(N^1, v^1)\Big) -$$

$$- \sum_{i \in C^2} \Big(\phi_i(N, v) - \phi_i(N^2, v^2)\Big).$$

Observe that some externalities across disconnected components are present. The contribution of N^1's players to the members of C^2 and the contribution of N^2's members to the agents in C^1 must be subtracted.

5.2 Looking for the best partner

Now we analyze the elements that determine the contributions of each player $i \notin C$ to the Myerson value and centrality of a group $C \cup \{i\}$, which are given by

$$\partial_i \phi^g(C, N, v, \Gamma) := \phi^g_{C \cup i}(N, v, \Gamma) - \phi^g_C(N, v, \Gamma), \tag{17}$$

$$\partial_i \gamma^g(C, N, v, \Gamma) := \gamma^g_{C \cup i}(N, v, \Gamma) - \gamma^g_C(N, v, \Gamma) = \partial_i \phi^g(C, N, v, \Gamma) - \phi_i(N, v), \tag{18}$$

respectively. Next proposition shows that the contribution of player i to the Myerson value of group C depends on his *average complementarity* with the group C, and on the part of his value that is independent of C. These features are frequently conflicting.

Before stating that result, we recall the concept of being *strategic complements* and *substitutes*, which in turn rely on the *second-order difference operator* for a pair of agents $i, j \in N$.

The *second-order difference operator* for a pair of players $i, j \in N$ (Segal, 2003) is defined as a composition of marginal contribution operators (i.e., first-order difference operators) as follows:

$$\partial^2_{ij} v(S, N, v) = v(S \cup \{i, j\}) - v(S \cup j) - v(S \cup i) + v(S) = \partial^2_{ji} v(S, N, v), \quad \forall\, S \subseteq N \setminus \{i, j\}.$$

Here $\partial^2_{ij}(S, N, v)$ expresses player i's effect over the marginal contribution of player j (or vice versa). Note that $v(S \cup \{i, j\}) - v(S) = \partial^2_{ij}(S, N, v) + \partial_i(S, N, v) + \partial_j(S, N, v)$, and thus $\partial^2_{ij}(S, N, v) > 0$ implies that the marginal contribution of player i, j as a group exceeds the sum of the individual marginal contributions of each player. Therefore, $\partial^2_{ij}(S, N, v)$ can be interpreted as a measure of players i and j *complementarity* with respect to the players in S.

In fact, following Bulow, Geanakoplos, and Klemperer (1985), players i and j are said to be *strategic complements* whenever $\partial^2_{ij}(S, N, v) \geq 0$, for all $S \subseteq N \setminus \{i, j\}$. They are said to be *strategic substitutes* whenever $\partial^2_{ij}(S, N, v) \leq 0$, for all $S \subseteq N \setminus \{i, j\}$.

In this framework we define the *average complementarity* of players $i, j \in N$ as the following average of second-order differences:

$$\psi_{ij}(N, v) := \sum_{S \subseteq N \setminus \{i,j\}} \frac{s!(n-s-1)!}{n!} \partial^2_{ij}(S, N, v), \quad \text{for all } i \neq j \in N. \tag{19}$$

Next proposition is a particular case of Proposition 4 in Flores *et al.* (2013), and shows that the marginal contribution of a new player i to the incumbent group C is the sum of the a priori value $\phi_i(N \setminus C, v_\Gamma|_{N \setminus C})$ in the social network of player i which does not emerge from his relation with players in C, and the average complementarity between i and C. Here $v_\Gamma|_{N \setminus C}$ stands for the restriction of the characteristic function v to the set of players $N \setminus C$.

Proposition 5. *Let (N, Γ) be a social network, and (N, v) a TU game that reflects its functionality. Then for every group $C \subseteq N$, and every player $i \notin C$, the* marginal contribution *of player $i \in N \setminus C$ to the Myerson value of group C equals:*

$$\partial_i \phi^g(C, N, v, \Gamma) = \phi_i(N \setminus C, v_\Gamma|_{N \setminus C}) + \psi_{\mathbf{c}i}(N_C, v_{\Gamma, C}), \quad \textit{for all } i \notin C. \tag{20}$$

We need to recall now a couple of definitions. Two nodes i and j are said to be *automorphically equivalent* if there exists a bijection $\zeta : N \to N$ with $\zeta(i) = j$, $\zeta(j) = i$ such that the set of edges is the same.[8] Two groups $C_1 \subseteq N$ and $C_2 \subseteq N$ with $|C_1| = |C_2|$ are *automorphically equivalent groups* in (N, Γ) if there exists a bijection $\eta : C_1 \to C_2$ between the agents of both groups verifying that

[8] Formally, the induced map over the edge set is also a bijection.

$i \in C_1$ is automorphically equivalent to $\eta(i) \in C_2$, for every $i \in C_1$. Symmetric groups are defined accordingly.

Example 3. Let us consider again Example 1. Then, the individual Myerson value of each player is (see Gomez *et al.*, 2003):

$$\phi(N, v^{msg}, \Gamma) = (\frac{104}{30}, \frac{224}{30}, \frac{147}{30}, \frac{147}{30}, \frac{139}{30}, \frac{139}{30})$$

The two most valuable players are 2, and either 3 or 4 (who are in fact automorphically equivalent to 3, see the remark below for a definition). Now we show that the best partners for player 2 are either 5 or 6 (who again are automorphically equivalent), rather than players 3 or 4.

The Myerson value $\phi_i(\{1,3,4,5,6\}, v_\Gamma^{msg})$ of players $i = 3,4,5$ and 6 that is independent of 2 equals the value of these players in the 4-nodes chain in which players 3 and 4 are the two extremes:

$$\phi_3(N \setminus 2, v_\Gamma^{msg}|_{N\setminus 2}) = \phi_4(N \setminus 2, v_\Gamma^{msg}|_{N\setminus 2}) = \frac{65}{30},$$

$$\phi_5(N \setminus 2, v_\Gamma^{msg}|_{N\setminus 2}) = \phi_6(N \setminus 2, v_\Gamma^{msg}|_{N\setminus 2}) = \frac{115}{30}.$$

The average complementarities of players 3 and 5 with player 2 equal:

$$\psi_{23}(N, v_\Gamma^{msg}) = \sum_{S \subseteq \{1,4,5,6\}} \frac{s!(5-s)!}{6!} (v^{msg}(S \cup \{2,3\}) - v^{msg}(S \cup \{2\}) - v(S^{msg} \cup \{3\}) + v^{msg}(S)) = \frac{64}{30}$$

$$\psi_{25}(N, v_\Gamma^{msg}) = \sum_{S \subseteq \{1,3,4,6\}} \frac{s!(5-s)!}{6!} (v^{msg}(S \cup \{2,5\}) - v^{msg}(S \cup \{2\}) - v^{msg}(S \cup \{5\}) + v^{msg}(S)) = \frac{17}{30}.$$

Then, despite the fact that 3 is more complementary on average to player 2 than it is 5, and since the value of player 3 depends more on the presence of player 2 than it does the value of 5, 5 is a better partner for player 2 than it is 3:

$$\phi_{23}^g(N, v^{msg}, \Gamma) = \phi_2(N, v^{msg}, \Gamma) + \phi_3(N \setminus 2, v_\Gamma^{msg}|_{N\setminus 2}) + \psi_{23}(N, v_\Gamma^{msg}) = \frac{353}{30} <$$

$$\frac{416}{30} = \phi_2(N, v^{msg}, \Gamma) + \phi_5(N \setminus 2, v_\Gamma^{msg}|_{N\setminus 2}) + \psi_{25}(N, v_\Gamma^{msg}) = \phi_{25}^g(N, v^{msg}, \Gamma).$$

The messages game is symmetric,so $\phi_3(N, v^{msg}) = \phi_3(N, v^{msg}) = \frac{1}{5}$, and therefore 5 is also a better partner for player 2 than it is 3 when we measure centrality rather than the Myerson value. $\diamondsuit$

Remark 1. Note that if two agents $i, j \in N$ are *automorphically equivalent* in the social network (N, Γ) and they are also symmetric in the game (N, v), then they are symmetric players in the graph-restricted game (N, v_Γ). Thus, their individual Myerson values and their individual centralities coincide. Likewise, if $C_1 \subseteq N$ and $C_2 \subseteq N$ with $|C_1| = |C_2|$ are *automorphically equivalent groups* in (N, Γ) and *symmetric groups* in (N, v), then their group Myerson values and their group centralities coincide (since the Shapley group value verifies *G-anonymity*,[9] see Flores *et al.*, 2013)

5.3 The most and the least central group

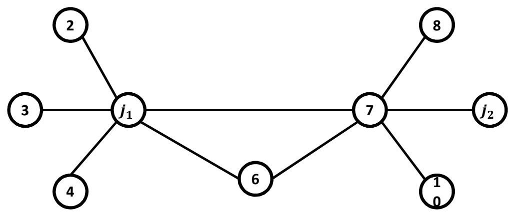
In the sequel we identify, under some restrictions, the groups of lowest and highest centrality of a given size k. Proposition 6 states that if the game (N, v) is symmetric and super-additive the group of size k of highest centrality is such that their nodes shrink to the hub of a star in the *shortened* graph, which is formally defined below. Under the same conditions, in Proposition 7, we show that a group of k isolated nodes have lowest centrality among all the groups of size k. This result can be strengthen when we restrict to symmetric convex games. In that case, Proposition 8 states that the k extreme nodes of a n–node chain have lowest centrality among all connected groups of size k. Partial results about the order among groups for the particular case of a chain are given in Proposition 9.

Definition 7. Let (N, Γ) be a social network, and let $C \subseteq N$. Then, the *C-shortened graph* $(N /_s C, \Gamma / C)$ is given by $N /_s C = (N \setminus C) \cup \{j_1, \ldots, j_p\}$, where there is a new node j_ℓ for every connected component T_ℓ of C in (N, Γ), and

$$\Gamma /_s C = \{\{i, j\} \ / \ i, j \notin C\} \cup \Big(\bigcup_{\ell=1}^{p} \{(i, j_\ell) \ / \ \text{ for all } i \notin C \text{ s.t. } \exists j \in T_\ell \text{ with } \{i, j\} \in \Gamma \} \Big)$$

Let us illustrate this concept by means of the following example.

Example 4. Let us consider the social network (N, Γ) of Example 2 and the group $C = \{1, 5, 9\}$. Then, the *C-shortened graph* $(N /_s C, \Gamma /_s C)$ is depicted below:



Note that $con_\Gamma(C) = \{\{1, 5\}, \{9\}\}$ and thus $N /_s C = \{2, 3, 4, 6, 7, 8, 10\} \cup \{j_1, j_2\}$, where the new nodes $j_1, j_2 \in N /_s C$ are the representatives of group C in the shortened graph. $\diamondsuit$

Now, using this concept we can state the following result about the most central group of a given size.

Proposition 6. *Let (N, Γ) be a social network, and let (N, v) a TU game. If (N, v) is symmetric and super-additive, and C_0 is a group of size k and (N, Γ^S) such that C_0 nodes shrink to the hub of a star in the*

[9] That is, for every $C \subseteq N$, and for all permutations $\pi \in \Pi(N)$, being $\Pi(N)$ the set of permutations of n elements, $\phi^g_{\pi(C)}(\pi(N), \pi v) = \xi^g_C(N, v)$, where $\pi v(S) := v(\pi(S))$, and being $\pi(S) = \{\pi(i) \mid i \in S\}$.

C_0-shortened *graph, then*

$$\phi^g_{C_0}(N, v, \Gamma^S) \geq \phi^g_C(N, v, \Gamma), \text{ for all } C \subseteq N \text{ with } |C| = k,$$

and for every social network (N, Γ)*.*

Propositions 7 and 8 deal with the inverse problem. Previously, in order to prove Proposition 7, we need to analyze the behavior of the Myerson value and game theoretic centrality of group C when they lose some of their *connections*, and also to obtain the centrality of a group of isolated nodes C. These are relevant questions by themselves.

Lemma 3. *Let* (N, Γ) *be a social network, and let* $C \subseteq N$*. If* $C \subseteq N$ *is a group of isolated nodes in* (N, Γ)*, then the Myerson value of a group* C *is* $\phi^g_C(N, v, \Gamma) = \sum_{i \in N} v(\{i\})$*, and its centrality equals* $\gamma^g_C(N, v, \Gamma) = c(f(1) - \frac{f(n)}{n})$*.*

In this context, we will give the following formal definition of a connection of a group C.

Definition 8. Let (N, Γ) be a social network, and let (N, v) a TU game. Then, edge $\{i, j\} \in \Gamma$ is a *connection* of group $C \subseteq N$ in (N, Γ) if $C \cap \{i, j\} \neq \emptyset$. If besides does not exist an alternative path $P(i, j)$ between nodes i, j in Γ with $(P(i, j) \setminus \{i, j\}) \subseteq C$, then $\{i, j\}$ is said to be a *vital* connection for C.

Lemma 4. *Let* (N, Γ) *be a social network, and let* $C \subseteq N$*. If* (N, v) *is a super–additive game, then removing a connection* $\{i, j\}$ *of* C *in* (N, Γ)*, the Myerson value of group* C *will not increase. Moreover, if* $\{i, j\}$ *is not a vital connection for* C *then* $\phi^g_C(N, v, \Gamma) = \phi^g_C(N, v, \Gamma_{-ij})$*, where* $\Gamma_{-ij} = \Gamma \setminus \{i, j\}$*.*

If $\{i, j\}$ *is not a connection for* C *and there exists an alternative path* $P(i, j)$ *between nodes* i, j *in* Γ *with* $(P(i, j) \setminus \{i, j\}) \subseteq C$*, then the Myerson value of group* C *will not decrease.*

Proposition 7. *Let* (N, Γ) *be a social network, and let* (N, v) *a TU game. If* (N, v) *is symmetric and super-additive, and* C_0 *is a group of* k *isolated nodes in* (N, Γ_0)*, then*

$$\phi^g_{C_0}(N, v, \Gamma_0) \leq \phi^g_C(N, v, \Gamma), \text{ for all } C \subseteq N \text{ with } |C| = k,$$

and for every social network (N, Γ)*.*

If the game in convex and the group is connected we may provide a more precise statement.

Proposition 8. *Let* $(N, \Gamma^{1,n})$ *be the chain with* n *nodes numbered in the natural way*[10]*, and let* (N, v) *be a TU game. If* (N, v) *is symmetric and convex, then*

$$\phi^g_{\{1,2,\ldots,k\}}(N, v, \Gamma^{1,n}) \leq \phi^g_C(N, v, \Gamma),$$

for every connected group $C \subseteq N$ *of cardinality* $1 \leq k \leq n$*, and for all connected graphs* (N, Γ)*.*

Now, applying Proposition 5 to compare the Myerson value of consecutive connected groups in a chain, we deduce the following result.

[10] That is, $(N, \Gamma^{1,n})$ is the chain $1 - 2 - 3 - \cdots - (n-1) - n$.

Proposition 9. *Let $(N, \Gamma^{1,n})$ be the chain with n nodes numbered in the natural way, and let $C_i^k = \{i, i+1, \ldots, k+i-1\}$, $1 \leq k \leq n$, $1 \leq i \leq n+1-k$. If (N, v) is a symmetric and convex game then for all $k = 1, \ldots, n-2$ the following holds:*

$$\phi^g_{C_i^k}(N, v, \Gamma^{1,n}) \leq \phi^g_{C_{i+1}^k}(N, v, \Gamma^{1,n}), \text{ for all } 1 \leq i \leq \frac{n-k}{2},$$

and

$$\phi^g_{C_i^k}(N, v, \Gamma^{1,n}) = \phi^g_{C^k_{n-i+2-k}}(N, v, \Gamma^{1,n}), \text{ for all } 1 + \left(\frac{n-k}{2}\right) < i \leq n-k+1 \,.$$

In general, if we consider also disconnected groups the most central connected groups are not in general the most valuable groups. Let us consider for instance a 5-nodes chain; then $\phi^g_{24}(N, v^{msg}, \Gamma^{1,5}) > \phi^g_{23}(N, v^{msg}, \Gamma^{1,5}) = \phi_{34}(N, v^{msg}, \Gamma^{1,5})$.

If the social network is a chain and we think that the above properties of the proposed group measures must be satisfied, then we must define a convex TU game to describe the functionality of the network. Observe that the three games defined in Section 3 are convex.

5.4 Particular cases and relation with classical measures

In this paragraph we obtain a general expression for the Myerson group value and game theoretic group centrality measures for the games introduced in Gomez *et al.* (2003) in some particular cases. These formulas show that, for a particular type of connected graphs, the game theoretic group measures we propose are very related to certain standard measures of group centrality and *cohesiveness*. In the sequel c denoted the cardinal of the distinguished coalition C.

Proposition 10. *Let (N, Γ) be a tree. Then,*

(i) *The* overhead-game *Myerson group value can be obtained as*

$$\phi^g_C(N, v, \Gamma) = \sum_{i \in C} \phi_i(N, v_\Gamma) = \frac{1}{2} \sum_{i \in C} \delta_i(\Gamma) - c, \tag{21}$$

where $\delta_i(\Gamma)$ is the degree centrality *of player $i \in C$.*

(ii) *The* messages-game *Myerson group value can be obtained as the sum of:*

- *A* group closeness *centrality, measured as the sum of the closeness of all members of C to every player outside C (when distances are shortened if agents of C act as intermediaries):*

$$\sum_{\substack{j \notin C \\ i \in C}} \frac{2}{d(i,j) - \sum_{k \in C} \delta_{ij}(k) + 2}, \tag{22}$$

where $\delta_{ij}(k) = 1$, if k is in the geodesic path from i and j in Γ, and 0 otherwise.

- *A* cohesiveness *index, measured by means of the number of agents in which group C has to rely on for communicate:*

$$\sum_{\substack{i,j \in C \\ i<j}} \frac{2}{1 + \sum\limits_{k \notin C} \delta_{ij}(k)}, \tag{23}$$

which reaches its maximum $v(C)$ when C is connected.

- *A* group betweenness *centrality, measured by a weighted sum of the geodesics that connect pairs of non-group members and, at the same time contain at least one member of the group. The weight depends on the number of agents of C in the geodesic and its length.*

$$\sum_{\substack{i,j \notin C \\ i<j}} \frac{2}{d(i,j) - \sum\limits_{k \in C} \delta_{ij}(k) + 2} \delta_{ij}(C), \tag{24}$$

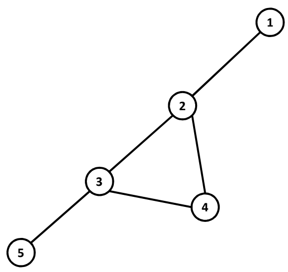
where $\delta_{ij}(C) = 1$, if $\sum_{k \in C} \delta_{ij}(k) \geq 1$, and 0, otherwise.

If the social network is a tree, the group centrality measure according to the overhead is additive and equal to:

$$\gamma_C^g(N, v^{overh}, \Gamma) = \sum_{i \in C} \gamma_i(N, v^{overh}, \Gamma) = \Big(\frac{1}{2} \sum_{i \in C} \delta_\Gamma(i)\Big) - c\frac{(n+1)}{n}. \tag{25}$$

In the next example we show that the above additivity result does not generalize to graphs with cycles.

Example 5. Let us consider following social network, and let (N, v^{overh}) be the overhead game:



Now, let us consider the group $C = \{2, 3\}$, then:

$$\phi_C^g(N, v^{overh}, \Gamma) = \frac{1}{2} > \phi_2(N, v^{overh}, \Gamma) + \phi_3(N, v^{overh}, \Gamma) = 2 \cdot \frac{1}{6} = \frac{1}{3}$$

$\diamondsuit$

With respect to the game theoretic group centrality measure based on the messages game, if the social network is a tree then the size of group C is penalized by a factor of $n-1$:

$$\gamma_C^g(N, v^{msg}, \Gamma) = \phi_C^g(N, v_G^{msg}) - \sum_{i \in C} \phi_i(N, v^{msg}) = \phi_C^g(N, v_G^{msg}) - c(n-1). \tag{26}$$

6 Applications

We will show now the application of the Myerson group value and game theoretic group centrality to two social networks which have been considered by Everett and Borgatti (1999) and Borgatti (2006) to illustrate the centrality group measures they proposed.

Following Castro, Gomez and Tejada (2009) and Castro *et al.* (2012) we have estimated in polynomial time, via Monte Carlo simulation, the Myerson group value of the examples previously described. We give the point estimation and the confidence intervals at the 99% of confidence level for each of the estimations we made.

First, we analyze the Wolfe primate data, which appears as a standard dataset in UCINET (Borgatti, Everett and Freeman, 2002). To be specific, we use the dichotomized data analyzed by Everett and Borgatti (1999) and depicted in Figure 3. In the original data joint presence at the river was coded as a interaction. They dichotomized the data by assuming the presence of a tie if there were more than six interactions over the considered time period.

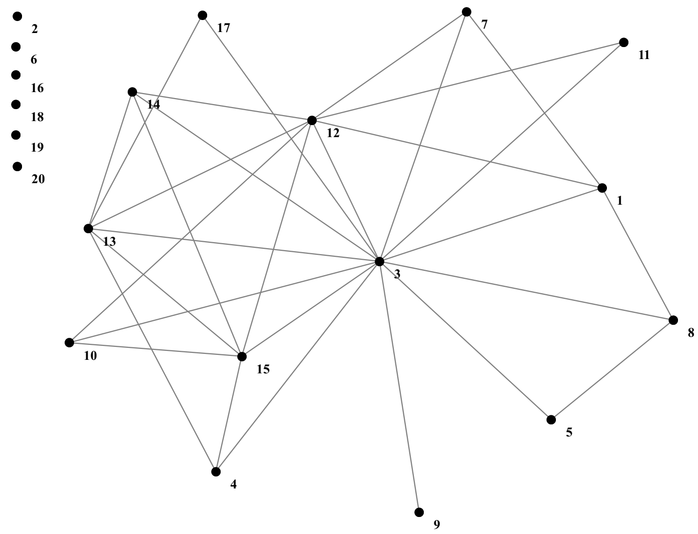


Figure 3: Wolfe Primate binary social network

In that case, from the point of view of *group selection* in Evolutionary Biology, it is worthy to evaluate the relevance of a group in order to promote the fitness of the troop. This fitness will be given mainly by their ability to form groups which improve for instance their ability to feed themselves, the defense of their territory, or their defense against predators. Taking this purpose into account we propose two TU games to model the functionality of the social network:

- On the one hand, we propose the overhead game (N, v^{overh}) for describing the common vital task of group survival. The workload is always the same regardless of the number of primates in the group. Then, a bigger group is more efficient in a per capita basis than a smaller one.

- On the other hand, we propose an alternative symmetric game which evaluates the expected size of the group, considering that the probability of survival of each primate increases when he/she is integrated in a bigger group. To be specific, let $p \in (0, 1)$ be the probability of mortality of a primate when he stays alone in a given period of time. We consider that each primate's mortality probability decreases to p^s when he takes part of a group S os size s. Then, assuming independence, the expected number of survivors of a given group S will be given by $s(1 - p^s) = f(s)$. For illustrative purposes, we choose $p = 0.5$ and denote the game as (N, v^{sur}).

The results obtained for individuals are listed in Table 1. The figures correspond to the individual Myerson value of each primate. Note that the value of primates 2, 6, 16, 18, 19 and 20 are calculated with no error. These primates are isolated nodes in the social network, so they are dummy[11] players in the graph-restricted game, and their individual values are therefore $\phi_i(N, v_\Gamma) = v(\{i\})$, for every $i = 2, 6, 16, 18, 19, 20$.

Monkey	Age	Sex	$\phi_i(N, v^{overh}, \Gamma)$	$\phi_i(N, v^{sur}, \Gamma)$	Deg.	Close.	Betw.
1	14-16	M	$-0,0833 \pm 1,3 \cdot 10^{-4}$	$0,9922 \pm 2,3 \cdot 10^{-4}$	4	142	1
2	10-13	M	-1	$0,5$	0	380	0
3	10-13	M	$1,3000 \pm 3,0 \cdot 10^{-4}$	$1,7621 \pm 5,4 \cdot 10^{-4}$	13	133	$44,5$
4	7-9	M	$-0,2500 \pm 1,0 \cdot 10^{-4}$	$0,9012 \pm 1,7 \cdot 10^{-4}$	3	143	0
5	7-9	M	$-0,3333 \pm 1,0 \cdot 10^{-4}$	$0,8494 \pm 1,8 \cdot 10^{-4}$	2	144	0
6	14-16	F	-1	$0,5$	0	380	0
7	4-5	F	$-0,2500 \pm 1,0 \cdot 10^{-4}$	$0,9403 \pm 2,2 \cdot 10^{-4}$	3	143	0
8	10-13	F	$-0,1667 \pm 1,3 \cdot 10^{-4}$	$0,7609 \pm 1,9 \cdot 10^{-4}$	3	143	$0,5$
9	7-9	F	$-0,5000 \pm 1,2 \cdot 10^{-4}$	$0,9000 \pm 1,7 \cdot 10^{-4}$	1	145	0
10	7-9	F	$-0,2500 \pm 1,0 \cdot 10^{-4}$	$0,9021 \pm 1,7 \cdot 10^{-4}$	3	143	0
11	14-16	F	$-0,3333 \pm 1,1 \cdot 10^{-4}$	$0,8537 \pm 1,9 \cdot 10^{-4}$	2	144	0
12	10-13	F	$0,3832 \pm 2,2 \cdot 10^{-4}$	$1,2617 \pm 4,1 \cdot 10^{-4}$	9	137	$10,33$
13	14-16	F	$-0,0333 \pm 1,2 \cdot 10^{-4}$	$1,0222 \pm 2,2 \cdot 10^{-4}$	6	140	$1,83$
14	4-5	F	$-0,2000 \pm 0,9 \cdot 10^{-4}$	$0,9302 \pm 1,6 \cdot 10^{-4}$	4	142	0
15	7-9	F	$-0,0334 \pm 1,2 \cdot 10^{-4}$	$1,0220 \pm 2,2 \cdot 10^{-4}$	6	140	$1,83$
16	10-13	F	-1	$0,5$	0	380	0
17	7-9	F	$-0,2500 \pm 1,0 \cdot 10^{-4}$	$0,9020 \pm 1,7 \cdot 10^{-4}$	3	143	0
18	4-5	F	-1	$0,5$	0	380	0
19	14-16	F	-1	$0,5$	0	380	0
20	4-5	F	-1	$0,5$	0	380	0

Table 1: Linda Wolfe primates network individual rankings

The relative order among the most central primates remains the same for the five rankings.

[11] A player $i \in N$ is dummy in the game (N, v) whenever $v(S \cup \{i\}) = v(S) + v(\{i\})$, for all $S \subseteq N \setminus \{i\}$.

However, it is remarkable that the centralities of primates 13 and 15 on the one hand, and the centralities of 4, 7, 10 and 17 on the other, are the same for all the classical measures. The reason for the centralities of 13 and 15 to coincide is that they are automorphically equivalent. The same remains true when comparing the centralities of monkeys 10 and 17. However, this is not the case of monkeys 4 and 7, as they do not take up the same position in the social network, and also they have a different position from 10 and 17. In this example, the unique measure which distinguishes among them is the game theoretic centrality obtained through the survival game (N, v^{sur}), that gives the relative order:

$$\phi_4(N, v^{sur}, \Gamma) < \phi_{10}(N, v^{sur}, \Gamma) = \phi_{17}(N, v^{sur}, \Gamma) < \phi_7(N, v^{sur}, \Gamma).$$

Here, when we compare two estimates we say that they are equal when their difference is not statistically significant at a 1%.

Everett and Borgatti consider six different groups to illustrate the group measures they proposed. Four of the groups were formed by age: (*i*) Age 14-16 (1,6,11,13,19); (*ii*) Age 10-13 (2,3,8,12,16); (*iii*) Age 7-9 (4,5,9,10,15,17); (*iv*) Age 4-5 (7,14,18,20); and two formed by sex (*v*) Male (1 to 5); and (*vi*) Female (6 to 20).

We compare the importance of each group according to Everett and Borgatti group measures with our two game theoretic proposals. Following their approach, we first compute the Myerson value of each group $\phi_C^g(N, v, \Gamma)$. Then, in order to avoid the influence of the different sizes of each group, we compute the game theoretic centrality of each group $\gamma_C^g(N, v, \Gamma)$; that is, we adjust the Myerson value of each group by means of substracting the individual value of each member derived form the functionality of the network:

$$\gamma_C^g(N, v, \Gamma) = \phi_C^g(N, v, \Gamma) - \sum_{i \in C} \phi_i(N, v).$$

Then, we compare our results with the absolute and relative group centralities obtained by Everett and Borgatti. These authors obtained their results deleting the isolated nodes $D = \{2, 6, 16, 18, 19, 20\}$, so we do the same for comparative purposes.[12]

Table 2 shows the results for the six groups. Group closeness depends on the definition of distance from the group to an outside node. Everett and Borgatti considered the minimum distance to each of the group's nodes, the average of the distances, and the maximum distance.

Note that the group closeness centralities and the Myerson group value based on the survival game are highly dependent on the size of the group. In fact, the Myerson group value $\phi_C^g(N, v^{sur}, \Gamma)$ preserves the relative order given by the sizes of the groups, even for the female group. However, not only the size matters: the group (*ii*) (age 10-13) takes more advantage of its position than the group (*i*) (age 14-16) does, which has its same size when isolated monkeys are removed. For the remaining measures, degree, betweenness and Myerson group value $\phi_C^g(N, v^{overh}, \Gamma)$, the presence of monkey 3, who is related to all other monkeys, is crucial.

[12] Since the Shapley group value verifies the *G-dummy player* property, then the value of each group C including the isolated monkeys can be calculated as $\phi_C^g(N, v^{overh}, \Gamma) = \phi_{C \setminus D}^g(N \setminus D, v^{overh}|(N \setminus D), \Gamma_{N \setminus D}) - |C \cap D|$, and $\phi_C^g(N, v^{sur}, \Gamma) = \phi_{C \setminus D}^g(N \setminus D, v^{sur}|(N \setminus D), \Gamma_{N \setminus D}) + 0.9999 \cdot |C \cap D|$.

Group	Myerson group value		Degree	Closeness			Between.
	$\phi_C^g(N, v^{overh}, \Gamma)$	$\phi_C^g(N, v^{sur}, \Gamma)$		Min.	Average	Max.	
Age 14-16	$-0,45 \pm 2,8 \cdot 10^{-4}$	$2,87 \pm 1,5 \cdot 10^{-4}$	8	14	18	20	$2,84$
Age 10-13	$2,33 \pm 4,1 \cdot 10^{-4}$	$4,44 \pm 2,1 \cdot 10^{-4}$	11	11	15	21	$43,5$
Age 7-9	$-1,28 \pm 3,6 \cdot 10^{-4}$	$5,54 \pm 1,8 \cdot 10^{-4}$	5	11	$13,7$	15	0
Age 4-5	$-0,45 \pm 1,8 \cdot 10^{-4}$	$1,83 \pm 1,0 \cdot 10^{-4}$	5	19	$20,5$	22	0
Male	$1,30 \pm 3,0 \cdot 10^{-4}$	$4,90 \pm 1,6 \cdot 10^{-4}$	10	10	16	20	$24,34$
Female	$-0,83 \pm 3,0 \cdot 10^{-4}$	$10,0 \pm 1,4 \cdot 10^{-4}$	4	4	$6,4$	7	$0,5$

Table 2: Myerson group value for the Primate data. Classical absolute values.

Group	Game theoretic group centrality		Degree	Closeness			Between.
	$\gamma_C^g(N, v^{overh}, \Gamma)$	$\gamma_C^g(N, v^{sur}, \Gamma)$		Min.	Average	Max.	
Age 14-16	$-0,236398$	$-0,129248$	$0,73$	0.7	0,61	0,55	0,03
Age 10-13	$2,548456$	$1,445639$	1	1	0,73	0,52	0,41
Age 7-9	$-0,854471$	$-0,454262$	$0,625$	0,73	0,58	0,53	0
Age 4-5	$-0,306954$	$-0,168622$	$0,42$	0,63	0,59	0,55	0
Male	$1,585754$	$0,899111$	1	1	0,63	0,5	0,23
Female	$-0,118949$	$0,004103$	1	1	0,63	0,57	0,05

Table 3: Group centrality for the Primate data. Classical normalized values.

Next we do the same comparisons avoiding the size effect. According to Everett and Borgatti we normalize the classical group measures dividing the centrality of each group of size c by the maximum centrality that a group of size c can achieve. When considering the game theoretic group measures derived from the survival (N, v^{sur}) and the overhead (N, v^{overh}) games, we adjust the Myerson value of a group by substracting the amount $c\frac{f(n)}{n}$, being $f(n) = v(N)$ (-1 for the overhead game and $13,999$ for the survival game). Again, the results depicted in Table 3 are obtained deleting the isolated nodes $D = \{2, 6, 16, 18, 19, 20\}$.

All the group measures except maximum group closeness rank groups of Male, Female and Age 10-13 primates as the three best ones. However, there are some interesting aspects supporting game theoretic group centralities that should be remarked. With respect to its relation with closeness, first note that minimum and average closeness do not provide much sensitivity. All the three aforementioned groups attain the maximal value with the minimum method, whereas the average method does not make difference among Male and Female groups. On the contrary, as they use more information, game theoretic centralities appear to be more sensitive. The same occurs when comparing with betweenness, which do not discriminate among the groups with a minimum value of zero. Note that the female group is now less central than the male group when considering the survival game.

The second example we analyze here is the network within a consulting company analyzed in Borgatti (2006), which consists of advice-seeking ties among members of a the global company. The data were collected on a one to five strength-of-tie-scale, but for his analysis the author examined only the strongest ties (rated 5). The derived social network is shown in Figure 4 below.

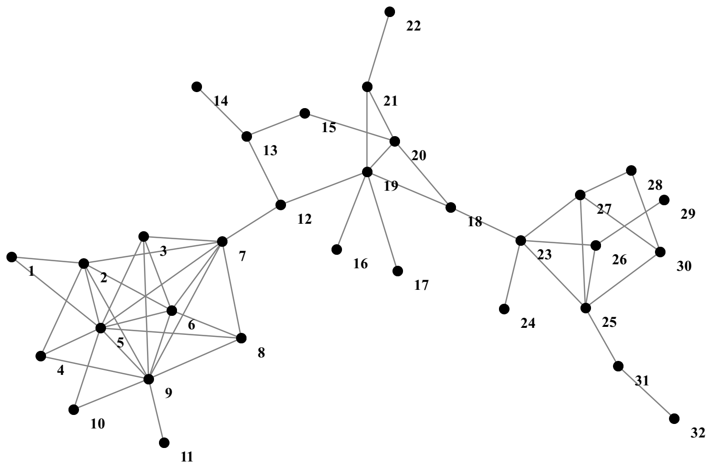


Figure 4: Strong advice-seeking ties in a global consulting company (Borgatti, 2006)

As commented, the social network of informal relationships between the members of an organization have "become a pervasive feature of organizations" (see Cross and Parker, 2004). Thus, the identification of a small group C of agents who are able to lead the formation of efficient working teams, or whose deletion would disrupt most the ability of the social structure to form them, is crucial for the top managers of the organization.

If the purpose of the organization is the identification of a small group C who is capable to integrate all the agents in the development of a common project, the overhead game describes appropriately the functionality of the social network. If the social network is intended to *collaborative learning*, the model game must take into account what is the ideal group size for that purpose. The main idea in the literature about this facet is "to keep groups midsized: small groups of 3 or less lack enough diversity and may not allow divergent thinking to occur. Groups that are too large create "freeloading" where not all members participate. A moderate size group of 4-5 is ideal".[13] We propose a *trimmed* version of the conferences game in which the worth of coalition $S \subseteq N$ counts the subsets of midsized (4 or 5) conferences S. Formally, the trimmed conference game $(N, v^{conf-mid})$ is defined as

$$v^{conf-mid}(S) = \sum_{k=4}^{\min\{5,s\}} \binom{s}{k}, \quad \forall\, S \subseteq N \text{ with } |S| \geq 4, \text{ and } v^{conf-mid}(S) = 0, \text{ otherwise.}$$

We analyze the most valuable groups up to size 4 when the purpose of the organization is to promote small collaborative learning groups, and we compare our results with those ones obtained by Borgatti in his analysis of the network of advice seeking. As a result we obtain that the trimmed conference game gives results that are related to the Key Player Problem Negative analysis (KPP-Neg in the sequel).

[13] Cited from: http://www.opencolleges.edu.au/informed/features/facilitating-collaborative-learning-20-things-you-need-to-know-from-the-pros/

Ranking	1 member	2 members	3 members	4 members
1	$\{12\}$	$\{12, 18\}$	$\{7, 12, 18\}$	$\{7, 12, 18, 23\}$
2	$\{7\}$	$\{12, 23\}$ (*)	$\{7, 12, 23\}$ (*)	$\{7, 12, 18, 19\}$
3	$\{18\}$	$\{7, 18\}$ (*)	$\{12, 18, 23\}$ (*)	$\{7, 12, 19, 23\}$ (*)
4	$\{23\}$	$\{7, 23\}$	$\{7, 18, 23\}$	$\{12, 18, 19, 23\}$ (*)
5	$\{19\}$	$\{12, 19\}$	$\{7, 19, 23\}$ (*)	$\{7, 18, 19, 23\}$
6	$\{25\}$	$\{7, 12\}$	$\{12, 18, 19\}$ (*)	$\{7, 12, 18, 25\}$
7	$\{13\}$ (*)	$\{7, 19\}$	$\{7, 18, 19\}$ (*)	$\{7, 12, 13, 18\}$
8	$\{9\}$ (*)	$\{18, 19\}$	$\{12, 19, 23\}$ (*)	$\{12, 18, 23, 25\}$
9	$\{21\}$ (**)	$\{12, 25\}$	$\{7, 12, 19\}$ (*)	$\{7, 12, 23, 25\}$
10	$\{26\}$ (**)	$\{19, 23\}$	$\{12, 18, 25\}$	$\{7, 9, 12, 18\}$

Table 4: Game theoretic group centrality rankings for the advice-seeking data based on $(N, v^{conf-mid})$.

Clearly, this social network has three distinguished subsocieties that have not too many links connecting them: the first one, $\mathcal{S}_1$ made up by agents 1 to 11, the central one $\mathcal{S}_2$ made up by agents 12 to 22, and the third one $\mathcal{S}_3$ that comprises the remaining agents 23 to 32. Agents 7 in $\mathcal{S}_1$, 12 and 18 in $\mathcal{S}_2$ and 23 in $\mathcal{S}_3$ represent the unique connections among two consecutive groups. Note also that agents 7, 12, 18 and 23, together with player 19 $\in \mathcal{S}_2$, form a chain that pass through the three subsocieties. Considering the three subsocieties in an isolated way, the two most central agents of $\mathcal{S}_1$ are 9 and 9; agents 19 and 13 are the two most central of $\mathcal{S}_2$, and agents 25 and 23 are the two most central of $\mathcal{S}_3$. All of them, according to both game theoretic centralities, with the overhead and the trimmed-conference games. Apart from player 23, who is highly central inside his own subsociety, the remaining connection agents are more peripheral: agents 7 and 12 are in the fourth position inside their own subsocieties, and player 18 occupies the seventh position.

In a natural way these nine agents $\{5, 7, 9, 12, 13, 18, 19, 23, 25\}$ are always in the top positions for all the considered criteria. When the underlying game is the trimmed-conferences game, the ten best groups up to size 4 -arranged in decreasing order of importance- are listed in Table 4. Non-statistically significative differences (at a 99% of confidence level) between two groups centralities are labeled with an asterisk.

In this case, from an individual point of view, the four most central agents are 12, 7, 18 and 23, which are respectively the unique agents connecting consecutive subsocieties. With respect to groups of two agents, the most central is $\{12, 18\}$, which selects the two agents from $\mathcal{S}_2$ who connect this central subsociety with the other two extreme subsocieties. For a KPP-Neg analysis the algorithm proposed by Borgatti (2006) selects the group $\{7, 23\}$, which is similar to $\{12, 18\}$ but selecting the intermediaries in $\mathcal{S}_1$ and $\mathcal{S}_3$. Note that the most central group of two agents consists on the respectively the first and the third more central agents from and individual point of view, rather than the first and the second ones 12 and 7. In this case, player 18 is a better partner for 12 than it is 7; note that the group $\{12, 7\}$ is in the sixth position. Also remark that the two groups $\{12, 23\}$ and $\{7, 18\}$ that are tied in the second place select, in one hand, one of the agents in $\mathcal{S}_2$ who connects with one of the other two subsocieties, and in the other, the player in the remaining subsociety that serves as a bridge.

With respect to the most central groups of three and four agents, the groups with highest centrality consist on the three and four more individually central agents: $\{7, 12, 18\}$ and $\{7, 12, 18, 23\}$. For a KPP-Neg analysis the algorithm proposed selects $\{7, 19, 23\}$, in which the intermediary player $12 \in \mathcal{S}_2$ is replaced by $19 \in \mathcal{S}_2$ who is the most central player inside his subsociety. This group $\{7, 19, 23\}$ is in the fifth position according to the trimmed-conferences game theoretic group centrality.

7 Conclusions

Our interest has been centered in the valuation of the centrality of groups. We have undertaken this problem by means of a twofold approach, whose first ingredient is the game-theoretic centrality measure defined by Gomez *et al.* (2003). Assuming a social network where the agents are connected by a graph, and in which the functionality of the network is modeled by means of a cooperative game where the agents act as players, these authors defined an individual centrality measure by means of the (classical) Myerson value of the graph-restricted game.

The goal is to generalize this individual measure to a group valuation concept. For that, we need our second basic ingredient, which is the Shapley group value defined in a previous work (Flores, Molina and Tejada, 2013), and that is constructed by computing the usual Shapley value of a certain super-player that represent the group of interest in an appropriate quotient game. In this way, we have defined the Myerson group value, which is our key concept to describe a centrality measure when the game is symmetric (and then only positioning is important) and a measure of the capital of the group, when there is no symmetry in the game. So, given a TU-game defined in the set of agents, a graph that relates them and a group of agents, these measures will be defined as the difference between the Myerson group value and the individual Shapley values of the agents in the group. This difference expresses both the value of the agents own to their position and own to the external information codified by the game.

In this paper we have dealt mainly with the first instance, while the non-symmetric case has been left for subsequent work. After describing some motivating examples, we have investigated different properties of the Myerson group value, as its relationship with classical centrality measures, possible redundancy, relevance in the search of the best partner, or identification of the most and least central groups. We end up our analysis by identifying influential groups in two well-known networks of a very different type, as Wolfe primate binary social network and Borgatti advice-seeking network.

It is remarkable that one of the main features of our proposal is that it offers a wide framework to model many kinds of centrality measures. For example, in our study of the advice-seeking network, we have showed that our centrality measure referred to the trimmed conferences game gives rise to similar results than Borgatti's algorithm for solving the key player problem in the negative form (KPP-Neg). At this point it is worth to mention that our definition of group centrality is very flexible, in the sense that it can be combined with any game –which in turn should be selected according to the nature of the concrete problem- and also that other similar measures can be defined starting from other individual values, and not only the Shapley value. In this paper we have analyzed the Myerson group value mainly from the point of view of the optimal group selection problem in social networks. In forthcoming work we plan to describe this value more from the point of view of Cooperative Game Theory, including if possible an axiomatic approach. Moreover, the problem of computing this measure for big networks is very interesting in itself; the computations in this paper have been undertaken using Castro-Gomez and Tejada algorithm, but more sophisticated methods will be needed when the number of nodes is bigger and/or the characteristic function of the game cannot be computed in polynomial time. Finally, another interesting extension of this work may include the identification of central groups in directed graphs, or else in networks where the influence between agents is pondered with weights. This approach may be specially interesting for networks that model diffusion processes.

Appendix

Proof of Proposition 1. It follows straightforward from the Shapley value self-duality ($\phi_i(N,v) = \phi_i(N,v^D)$), for every $i \in N$, and for every n-person TU game (N,v)) because $(v^D)_C \equiv (v_C)^D$. Then, $\phi_C^g(N,v^D,\Gamma) = \phi_C^g(N,v_\Gamma^D) = \phi_C^g(N,v_\Gamma) = \phi_C^g(N,v,D)$, and $\gamma_C^g(N,v^D,\Gamma) = \gamma_C^g(N,v_\Gamma^D) = \gamma_C^g(N,v_\Gamma) = \gamma_C^g(N,v,D)$. □

Proof of Lemma 1. Given a fixed player set N and a fixed social network (N,Γ) it is evident that the Γ-graph-restriction of games on this player set is a linear function. Hence, we have

$$v_\Gamma = \sum_{S \subseteq N} \Delta^N(v,S) u_\Gamma^S, \tag{27}$$

where (N,u_Γ^S) is the Γ-graph-restriction of the unanimity game (N,u^S), which is given by (see Lemma 2.1 in Gomez *et al.* (2003) for a proof)

$$u_\Gamma^S = \mathbf{1} - \prod_{i=1}^{r} (\mathbf{1} - u^{S_i}). \tag{28}$$

Here $\mathbf{1}$ is the game defined by $\mathbf{1}(S) = 1$, for all $\emptyset \neq S \subseteq N$, and $\mathcal{M}_\Gamma(S) = \{S_1,\ldots,S_r\} \neq \emptyset$ is the set of minimal connection sets of S in Γ. Note that this expression equals $u_\Gamma^S = u^{H_\Gamma(S)}$ when $\mathcal{M}_\Gamma(S)$ has only one set[14]. $H_\Gamma(S) = S$ for the special case in which S is connected in Γ.

Now, given a fixed set of agents $C \subseteq N$ acting as a group, the transformation of games on the fixed player set N into quotient games with respect to the partition $\mathcal{P}_C = \{C, \{j\}, j \notin C\}$ is a linear function, and the quotient game derived from the unanimity game with respect to coalition $S \subseteq N$, (N,u^S), is given by the unanimity game $(N_C, u^{N_C/S})$. See Derks and Tijs (2000) for a proof.

Then, if (N,Γ) is a tree and $C \subseteq N$,

$$v_{\Gamma,C} = \sum_{S \subseteq N} \Delta^N(v,S) u_{\Gamma,C}^S = \sum_{S \subseteq N} \Delta^N(v,S) u_C^{H_\Gamma(S)} = \sum_{S \subseteq N} \Delta^N(v,S) u^{N_C/H_\Gamma(S)}. \tag{29}$$

[14]If there are no cycles in Γ connecting two agents in S, then there is only one smallest connected set $H_\Gamma(S) \subseteq N$ which contains $S \subseteq N$.

Since the Shapley value is also linear, and $\phi_i(N, u^S) = \frac{1}{s}$, for all $i \in S$, $\phi_i(N, u^S) = 0$, for all $i \notin S$. Therefore, taking into account that the set of all coalitions S such that $c \in N_C/H_\Gamma(S)$ equals the set of coalitions S with $H_\Gamma(S) \cap C \neq \emptyset$, we have

$$\phi_C^g(N, v_\Gamma) := \phi_{\mathbf{c}}(N_C, v_{\Gamma,C}) = \sum_{H_\Gamma(S) \cap C \neq \emptyset} \frac{\Delta^N(v, S)}{|N_C/H_\Gamma(S)|}$$

□

Proof of Lemma 2. The proof follows the same argument than that of previous lemma. That is,

$$\phi_C^g(N, v_\Gamma) := \phi_{\mathbf{c}}(N_C, v_{\Gamma,C}) = \sum_{S \subseteq N} \Delta^N(v, S) \phi_{\mathbf{c}}(N_C, u_{\Gamma,C}^S).$$

Now, we have to consider that

$$u_{\Gamma,C}^S = \mathbf{1} - \prod_{i=1}^{r} (\mathbf{1} - u^{N_C/S_i}), \tag{30}$$

for every coalition $S \subseteq N$, and therefore $\phi_{\mathbf{c}}(N_C, u_{\Gamma,C}) = 0$, for every S with $C \cap S_i = \emptyset$, and for every $S_i \in \mathcal{M}_\Gamma(S)$. □

The reader interested in a explicit formula for $\phi_{\mathbf{c}}(N_C, u_{\Gamma,C}^S)$ is referred to Gomez *et al.* (2003) (formula (11) in page 36); there, using expression (28) above, $\phi_i(N, u_\Gamma^S)$ is evaluated in the case in which there are alternative paths in Γ connecting the agents of S.

Proof of Proposition 4. It follows straightforward from (12). □

Proof of Proposition 6. The proof resembles much that of Proposition 3.4 in Gomez *et al.* (2003), so we refer to them for the details. We must take into account that if (N, Γ^S) is a social network such that $C_0 \subseteq N$ shrinks to the hub of a star in the C_0–shortened graph, then every coalition $S \subseteq N \setminus C_0$ is a group of $|S| = s$ isolated nodes, whereas $S \cup C_0$ is a connected group in Γ^S. Thus,

$$v_{\Gamma^S, C_0}(S \cup \{c_0\}) := v_{\Gamma^S}(S \cup C_0) = v(S \cup C_0) = f(s + k)$$

and $v_{\Gamma^S, C_0}(S) := v_{\Gamma^S}(S) = \sum_{i \in S} v(i) = sf(1)$, for all symmetric (N, Γ), where $f(\cdot)$ is the real valued function which defines the game. Recall from the preliminaries that $f(\cdot)$ only depends on the cardinal of the coalition.

Now let $C \subseteq N$ and $T \subseteq N$ be any pair of coalitions with $|C| = |C_0| = k$, and $T \subseteq N$ with $|T| = |S| = s$. Then, super-additivity implies:

$$v_{\Gamma,C}(T \cup \{c\}) := v_\Gamma(T \cup C) \leq v(T \cup C) = f(s + k) \text{ and } v_{\Gamma,C}(T) := v_\Gamma(T) \geq sf(1).$$

Therefore, the marginal contributions of $\mathbf{c}_0$ in $(N_{C_0}, v_{\Gamma^S,C_0})$ are always greater or equal than those of $\mathbf{c}$ in $(N_C, v_{\Gamma,C})$, and the result holds. $\square$

Proof of Lemma 3. In order to prove this lemma we recall the notion of *dummy* player. A player $i \in N$ is dummy in the game (N, v) whenever $v(S \cup \{i\}) = v(S) + v(\{i\})$, for all $S \subseteq N \setminus \{i\}$. The Shapley value verifies the *dummy player* property, which states that $\phi_i(N, v) = v(\{i\})$, for every dummy player $i \in N$. Analogously, the Shapley group value (see Flores *et al.*, 2013) verifies the *G-dummy player* property, which implies that $\phi_C^g(N, v) = \sum_{i\in N} v(\{i\})$, for every group C of dummy players. Then, the result follows trivially from the fact that every isolated node in (N, Γ) is a dummy player in the graph-restricted game (N, v_Γ). Since the game (N, v) is symmetric then $\gamma_C^g(N, v, \Gamma) = \sum_{i\in C}(v(\{i\}) - \phi_i(N, v)) = c(f(1) - \frac{f(n)}{n})$. $\square$

Proof of Lemma 4. Let $\{i, j\} \in \Gamma$ a connection of C, then $con_\Gamma(S) = con_{\Gamma_{-ij}}(S)$, for all $S \subseteq N \setminus C$, since $\{i, j\} \notin \Gamma_S$. If $\{i, j\}$ is not vital for C then the equality $con_\Gamma(S \cup C) = con_{\Gamma_{-ij}}(S \cup C)$ also holds for all $S \subseteq N \setminus C$. Thus,

$$\phi_C^g(N, v, \Gamma) = \sum_{S\subseteq N\setminus C} \frac{s!(n-c-s)!}{(n-c+1)!}\big(v_\Gamma(S\cup C) - v_\Gamma(S)\big) =$$

$$= \sum_{S\subseteq N\setminus C} \frac{s!(n-c-s)!}{(n-c+1)!}\big(v_{\Gamma_{-ij}}(S\cup C) - v_{\Gamma_{-ij}}(S)\big) = \phi_C^g(N, v, \Gamma_{-ij})$$

Next we show that $v_\Gamma(S \cup C) \geq v_{\Gamma_{-ij}}(S \cup C)$, for all $S \subseteq N \setminus C$ in case $\{i, j\}$ was vital for C. Let $S \subseteq N \setminus C$ such that $\{i, j\} \subseteq S \cup C$, and let $T^{ij}(S)$ be the connected component of $S \cup C$ in (N, Γ) that contains agents i, j. Two cases are possible:

(i) If there exists an alternative path $P(i, j)$ between nodes i, j in Γ with $(P(i, j) \setminus \{i, j\}) \subseteq C \cup S$, then $con_\Gamma(S \cup C) = con_{\Gamma_{-ij}}(S \cup C)$.

(ii) Otherwise, $con_{\Gamma_{-ij}}(S \cup C) = (con_\Gamma(S \cup C) \setminus T^{ij}(S)) \cup T^i(S) \cup T^j(S)$, where $T^i(S) = \{\ell \in T^{ij}(S) \,/\, \exists\, P(\ell, i) \subseteq (S \cup C) \setminus \{j\}\}$ and $T^j(S) = \{\ell \in T^{ij}(S) \,/\, \exists\, P(\ell, j) \subseteq (S \cup C) \setminus \{i\}\}$. Thus, super-additivity of (N, v) implies:

$$\big(v_\Gamma(S\cup C) - v_\Gamma(S)\big) - \big(v_{\Gamma_{-ij}}(S\cup C) - v_{\Gamma_{-ij}}(S)\big) = v(T^{ij}(S)) - v(T^i(S)) - v(T^j(S)) \geq 0.$$

Thus, the statements about removing a connection for C hold.

Let $\{i, j\} \in \Gamma$ with $i, j \in N \setminus C$. Then, $con_\Gamma(S) = con_{\Gamma_{-ij}}(S)$ and $con_\Gamma(S \cup C) = con_{\Gamma_{-ij}}(S \cup C)$, for all $S \subseteq N \setminus C$ with $\{i, j\} \nsubseteq S$. Now, let us consider the case $S \cup \{i, j\}$, then $con_\Gamma(S \cup \{i, j\} \cup C) = con_{\Gamma_{-ij}}(S \cup \{i, j\} \cup C)$ since there exists an alternative path $P(i, j)$ which uses agents of C as intermediaries. However, coincidence between $con_\Gamma(S \cup \{i, j\})$ and $con_{\Gamma_{-ij}}(S \cup \{i, j\})$ can not be assured and $v_\Gamma(S \cup \{i, j\}) \geq v_{\Gamma_{-ij}}(S \cup \{i, j\})$. Thus, the marginal contributions of group C not decrease when edge $\{i, j\}$ is removed. Note that $\phi_C^g(N, v, \Gamma_{-ij}) > \phi_C^g(N, v, \Gamma)$ whenever

$v(\{i,j\}) > v(\{i\}) + v(\{j\})$. $\square$

Proof of Proposition 7. The proof resembles much that of Proposition 3.3 in Gomez *et al.* (2003), taking into account Lemma 4 to analyze the effect of stepwise elimination of the k edges incident in a node j of degree k, so we refer to them. $\square$

Proof of Proposition 8. Again, the proof follows the lines of that of Proposition 3.5 in Gomez *et al.* (2003), and we refer to them. In this case, the nodes in (N, Γ) must be relabeled in such a way that $C = \{1, \ldots, k\}$, which it is possible since C is connected in Γ. We must take into account also that convexity implies $v(S \cup C) - v(S) \leq v(T \cup C) - v(T)$, for all $S \subseteq T \subseteq N \setminus C$. $\square$

Proof of Proposition 9. In this case, the reasoning followed in the proof of the analogous result for connected groups of size 1, Proposition 3.6 in Gomez *et al.* (2003), cannot be generalized to larger groups.

We apply Proposition 5 to compare the Myerson value of two consecutive connected groups in a chain to prove the result. For every pair k, i such that $1 \leq k \leq n-2$ and $1 \leq i \leq \frac{n-k}{2}$, let us consider the set $D_i^k = \{i+1, \ldots, i+k-1\}$. Then, we will show that $i+k$ is a better partner for D_i^k than it is i and thus $\phi^g_{C_i^k}(N, v, \Gamma^{1,n}) \leq \phi^g_{C_{i+1}^k}(N, v, \Gamma^{1,n})$.

In the rest of the proof we will denote the chain $\Gamma^{1,n}$ simply as Γ .

First, we will prove that

$$\phi_i(N \setminus D_i^k, v_\Gamma | N \setminus D_i^k) \leq \phi_{i+k}(N \setminus D_i^k, v_\Gamma | N \setminus D_i^k), \tag{31}$$

for every $k = 1, \ldots, n-2$, and for every $1 \leq i \leq \frac{n-k}{2}$.

Taking into account Proposition 4 about the Myerson value of a group in a disconnected graph, it follows that the Myerson value of agents i and $i+k$ in $(N \setminus D_i^k, v_\Gamma | N \setminus D_i^k)$ equals the Myerson value of a extreme[15] in the two subchains $\Gamma^{1,i}$, $1 - 2 - \cdots - i$ and $\Gamma^{i+k,n}$, $(i+k) - (i+k+1) - \cdots - n$, respectively. Moreover, since $i \leq \frac{n-k}{2} < \frac{n-k+1}{2}$, the subchain $\Gamma^{1,i}$ is shorter than the subchain $\Gamma^{i+k,n}$. Therefore, Lemma 3.4 in Gomez *et al.* (2003) implies inequality (31).

Next we prove that also the interaction between D_i^k and $i+k$ is greater or equal that the interaction between D_i^k and i. The interaction indices $\psi_{\mathbf{d}_i^k, i}(N_{D_i^k}, v_{\Gamma, D_i^k})$ and $\psi_{\mathbf{d}_i^k, i+k}(N_{D_i^k}, v_{\Gamma, D_i^k})$

[15] Note also that the two extremes of a given chain are automorphically equivalent.

can be rewritten. The index $\psi_{\mathbf{d}_i^k,i}(N_{D_i^k}, v_{\Gamma,D_i^k})$ equals the sum:

$$\sum_{\substack{S\subseteq N\setminus D_i^k\\ i,i+k\notin S}} \left\{ \frac{s!(n-k-s+1)!}{(n-k+2)!}\Big(v_\Gamma(S\cup D_i^k\cup\{i\}) - v_\Gamma(S\cup D_i^k) - v_\Gamma(S\cup\{i\}) + v_\Gamma(S)\Big) + \right.$$

$$+\frac{(s+1)!(n-k-s)!}{(n-k+2)!}\Big(v_\Gamma(S\cup D_i^k\cup\{i,i+k\}) - v_\Gamma(S\cup D_i^k\cup\{i+k\}) -$$

$$\left. - v_\Gamma(S\cup\{i,i+k\}) + v_\Gamma(S\cup\{i+k\})\Big)\right\}, \quad (32)$$

while $\psi_{\mathbf{d}_i^k,i+k}(N_{D_i^k}, v_{\Gamma,D_i^k})$ can be expressed as:

$$\sum_{\substack{S\subseteq N\setminus D_i^k\\ i,i+k\notin S}} \left\{ \frac{s!(n-k-s+1)!}{(n-k+2)!}\Big(v_\Gamma(S\cup D_i^k\cup\{i+k\}) - v_\Gamma(S\cup D_i^k) - v_\Gamma(S\cup\{i+k\}) + v_\Gamma(S)\Big) + \right.$$

$$\left. +\frac{(s+1)!(n-k-s)!}{(n-k+2)!}\Big(v_\Gamma(S\cup D_i^k\cup\{i,i+k\}) - v_\Gamma(S\cup D_i^k\cup\{i\}) - v_\Gamma(S\cup\{i,i+k\}) + v_\Gamma(S\cup\{i\})\Big)\right\}. \quad (33)$$

Let us denote the marginal contributions of group D_i^k to a given coalition $T\subseteq N\setminus D_i^k$ as $MC(D_i^k,T) = v_\Gamma(D_i^k\cup T) - v_\Gamma(T)$. Then, the difference $\psi_{\mathbf{d}_i^k,i+k}(N_{D_i^k}, v_{\Gamma,D_i^k}) - \psi_{\mathbf{d}_i^k,i}(N_{D_i^k}, v_{\Gamma,D_i^k})$ is given by:

$$\sum_{\substack{S\subseteq N\setminus D_i^k\\ i,i+k\notin S}} \frac{s!(n-k-s)!}{(n-k+1)!}\Big(MC(D_i^k, S\cup\{i+k\}) - MC(D_i^k, S\cup\{i\})\Big). \quad (34)$$

Next, we show that (34) is non–negative. There are two types of coalitions $S\subseteq N\setminus D_i^k$ with $i, i+k\notin S$, taking into account whether $\{i+k+1,\ldots,2i+k-1\}\subseteq S$ or not. Such a difference emerges from the fact that the subchain $\Gamma^{i+k,2i+k-1}$ has the same length than the subchain $\Gamma^{1,i}$.

Let us denote by $T_{i+k}(S)$ and $T_i(S)$ the connected components of $con_\Gamma(D_i^k\cup S\cup\{i+k\})$ and $con_\Gamma(D_i^k\cup S\cup\{i\})$, respectively, that contain D_i^k. Note that since $i\notin S$ then $T_{i+k}(S) = D_i^k\cup\{i+k,\ldots,\ell_{i+k}(S)\}$, for some $i+k\le \ell_{i+k}(S)\le n$. Analogously, since $i+k\notin S$ then $T_i(S) = D_i^k\cup\{\ell_i(S),\ldots,i\}$, for some $1\le \ell_i(S)\le i$. Let us also denote $S_i = S\cap\{1,\ldots,i-1\}$ and $S_{i+k} = S\cap\{i+k+1,\ldots,n\}$. Thus:

(*i*) If $\{i+k+1,\ldots,2i+k-1\}\nsubseteq S$, then $T_{i+k}(S) = D_i^k\cup\{i+k,\ldots,\ell_{i+k}(S)\}$ with $i+k\le \ell_{i+k}(S)\le 2i+k-1$, and thus it holds:

$$MC(D_i^k, S\cup\{i+k\}) = v(T_{i+k}(S)) - v(\{i+k,\ldots,\ell_{i+k}(S)\}).$$

Now, let us define the coalition $\widetilde{S}\subseteq N\setminus D_i^k$ with $i, i+k\notin\widetilde{S}$ as follows:

- For every $j \in S_i$ consider $2i+k-j \in \widetilde{S}_{i+k}$.
- For every $i+k+j \in S_{i+k}$ being $j \leq i-1$ consider $i-j \in \widetilde{S}_i$.
- For every $i+k+j \in S_{i+k}$ being $j > i-1$ consider $i+k+j \in \widetilde{S}_{i+k}$.

Thus, $T_i(\widetilde{S}) = D_i^k \cup \{\ell_i(\widetilde{S}), \ldots, i\}$ and the cardinality of $\{\ell_i(\widetilde{S}), \ldots, i\}$ is the same as that of $\{i+k, \ldots, \ell_{i+k}(S)\}$. Therefore,

$$MC(D_i^k, \widetilde{S} \cup \{i\}) = v(T_i(\widetilde{S})) - v(\{\ell_i(\widetilde{S}), \ldots, i\}) = v(T_{i+k}(S)) - v(\{i+k, \ldots, \ell_{i+k}(S)\}) = \\ = MC(D_i^k, S \cup \{i+k\}).$$

Analogously, $MC(D_i^k, \widetilde{S} \cup \{i+k\}) = MC(D_i^k, S \cup \{i\})$. Therefore, since $|\widetilde{S}| = |S|$, the sum of the values corresponding to S and $\widetilde{S}$ in (34) is zero.

(ii) Otherwise, if $\{i+k+1, \ldots, 2i+k-1\} \subseteq S$, then $T_{i+k}(S) = D_i^k \cup \{i+k, \ldots, \ell_{i+k}(S)\}$ with $2i+k-1 \leq \ell_{i+k}(S) \leq n$. Thus, $|\{i+k, \ldots, \ell_{i+k}(S)\}| = i+t$, for some $0 \leq t \leq n-2i-k+1$, and

$$MC(D_i^k, S \cup \{i+k\}) = v(T_{i+k}(S)) - v(\{i+k, \ldots, \ell_{i+k}(S)\}) = f(i+k-1+t) - f(i+t), \tag{35}$$

where $f(\cdot)$ is the real function which defines the symmetric and convex game (N, v). Convexity implies

$$f(i+k-1+t) - f(i+t) \geq f(i+k-1) - f(i) = \\ = v(D_i^k \cup \{1, \ldots, i\}) - v(\{1, \ldots, i\}) \geq MC(D_i^k, S \cup \{i\}). \tag{36}$$

Thus the statement about the positiveness of (34) follows from (35) and (36).

The equalities

$$\phi^g_{C_i^k}(N, v, \Gamma^{1,n}) = \phi^g_{C^k_{n-i+2-k}}(N, v, \Gamma^{1,n}), \text{ for all } 1 + \left(\frac{n-k}{2}\right) < i \leq n-k+1\,,$$

follow from the symmetry of the Shapley group value taking into account that nodes i and $n-i+1$ are symmetric players in the graph-restricted game (N, v_Γ). □

Proof of Proposition 10. Since the Harsanyi dividends of the overhead game (N, v^{overh}) are given by $\Delta^N(v^{overh}, S) = (-1)^s$, for all $\emptyset \neq S \subseteq N$, being $s = |S|$, it follows trivially that

$$\sum_{\substack{H_\Gamma(S) \cap C \neq \emptyset \\ |H_\Gamma(S)| > 2}} \frac{\Delta^N(v, S)}{|N_C / H_\Gamma(S)|} = 0.$$

Thus, taking into account expression (11):

$$\phi_C^g(N, v_\Gamma^{overh}) = \sum_{i \in C} (-1)^1 + \sum_{\substack{i,j \in C \\ \{i,j\} \in \Gamma}} (-1)^2 + \sum_{\substack{i \in C, j \notin C \\ \{i,j\} \in \Gamma}} \frac{1}{2}(-1)^2 =$$

$$= -c + \sum_{i \in C} \Big(\sum_{\substack{\{i,j\} \in \Gamma \\ j \in C}} \frac{1}{2} + \sum_{\substack{\{i,j\} \in \Gamma \\ j \notin C}} \frac{1}{2} \Big) = \sum_{i \in C} \left(\frac{1}{2} \delta_\Gamma(i) - 1 \right) = \sum_{i \in C} \phi_i(N, v_\Gamma^{overh}).$$

The last equality, which establishes the additivity of the overhead Myerson group value measure, follows from Gomez *et al.* (2003). There, for the special case in which (N, Γ) is a tree, formulas of the individual value measures for the overhead and messages games are given.

A similar reasoning gives us the result when the functionality of the network is modeled by means of the messages game. In that case, we must take into account that the Harsanyi dividends of the messages game (N, v^{msg}) are given by $\Delta^N(v^{msg}, S) = 2$, for every two person coalition $\{i, j\} \subseteq N$, and $\Delta^N(v^{msg}, S) = 0$, otherwise. Then, taking into account

$$|N_C / H_\Gamma(\{i, j\})| = d(i, j) - \sum_{k \in C} \delta_{ij}(k) + 2 \quad \text{for all } i \neq j \in N,$$

which equals $1 + \sum_{k \notin C} \delta_{ij}(k)$ for each pair $i, j \in C$. The result follows. □

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