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TESIS DOCTORAL

Gromov hyperbolicity in graphs

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Departamento de Matemáticas

Escuela Politécnica Superior

Leganés, 10 de enero de 2014

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Gromov hyperbolicity in graphs

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to my family

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Resumen

Sea X un espacio métrico geodésico y $x_1, x_2, x_3 \in X$. Un *triángulo geodésico* $T = \{x_1, x_2, x_3\}$ es la unión de tres geodésicas $[x_1x_2]$, $[x_2x_3]$ y $[x_3x_1]$ de X. El espacio X es δ-*hiperbólico* (en el sentido de Gromov) si todo lado de T está contenido en la δ-vecindad de la unión de los otros dos lados, para todo triángulo geodésico T de X. Se denota por $\delta(X)$ a la constante de hiperbolicidad óptima de X, es decir, $\delta(X) := \inf\{\delta \geq 0 : \ X \ \text{ es } \delta\text{-hiperbólico}\}$. El estudio de los grafos hiperbólicos es un tema interesante dado que la hiperbolicidad de un espacio métrico geodésico es equivalente a la hiperbolicidad de un grafo más sencillo asociado al espacio.

Uno de los principales objetivos de esta Tesis Doctoral es obtener información cuantitativa sobre la variación de la constante de hiperbolicidad del grafo $G \setminus e$ que se obtiene del grafo G mediante la eliminación de una arista arbitraria e de él. Estas desigualdades permiten caracterizar, de forma cuantitativa, la hiperbolicidad de cualquier grafo en términos de su hiperbolicidad local.

En esta memoria se obtene información acerca de la constante de hiperbolicidad del grafo línea $\mathcal{L}(G)$ en términos de propiedades del grafo G. En particular, se obtienen los siguientes resultados cualitativos: un grafo G es hiperbólico si y sólo si $\mathcal{L}(G)$ es hiperbólico; si $\{G_n\}$ es una T-descomposición de G, el grafo línea $\mathcal{L}(G)$ es hiperbólico si y sólo si $\sup_n \delta(\mathcal{L}(G_n))$ es finito. Además, se obtienen resultados cuantitativos cuando las aristas de G y $\mathcal{L}(G)$ tienen longitud k. Se demuestra que $g(G)/4 \leq \delta(\mathcal{L}(G)) \leq c(G)/4 + 2k$, donde $g(G)$ es el cuello de G y $c(G)$ es su circunferencia. También se prueba que $\delta(\mathcal{L}(G)) \geq \sup\{L(g) : g \text{ es un ciclo isométrico de } G\}/4$. Igualmente, se obtienen cotas para $\delta(G) + \delta(\mathcal{L}(G))$ y se caracterizan los grafos G tales que $\delta(\mathcal{L}(G)) < k$.

Por otra parte, se consideran grafos G con aristas de longitudes arbitrarias, y $\mathcal{L}(G)$ con aristas de longitudes no constante. En particular, se demuestra que los ciclos de G se transforman isométricamente en ciclos de $\mathcal{L}(G)$ con la misma longitud. También se obtiene la relación $\delta(G) \leq \delta(\mathcal{L}(G)) \leq 5\delta(G) + 3l_{max}$, donde $l_{max} := \sup_{e \in E(G)} L(e)$. Este resultado prueba la monotonía de la constante de hiperbolicidad bajo una transformación no trivial (el grafo línea de un grafo).

También, se obtienen criterios que permiten decidir, para una clase amplia de grafos, cuando son estos hiperbólicos o no. Se presta especial atención en los grafos planares que son la "frontera" (el 1-esqueleto) de una teselación del plano euclídeo. Además, se prueba que los grafos que se obtienen como 1-esqueleto de un CW 2-complejo general son hiperbólicos si y sólo si su grafo dual es hiperbólico.

Uno de los principales problemas en este área es relacionar la hiperbolicidad de un grafo con otras propiedades de la teoría de grafos. En este trabajo se extiende de dos maneras (arista-cordalidad y camino-cordalidad) la definición clásica de cordalidad con el fin de relacionar esta propiedad con la hiperbolicidad. De hecho, se demuestra que todo grafo

arista-cordal es hiperbólico y que todo grafo hiperbólico es camino-cordal. También, se demuestra que todo grafo cúbico camino-cordal (con constante pequeña de camino-cordalidad) es hiperbólico.

Algunos trabajos previos caracterizan la hiperbolicidad de productos de grafos (para el producto cartesiano, el producto fuerte y el producto lexicográfico) en términos de propiedades de los grafos factores. En el último capítulo, se caracteriza la hiperbolicidad de dos productos de grafos: el grafo join $G_1 \uplus G_2$ y el corona $G_1 \diamond G_2$. El grafo join $G_1 \uplus G_2$ siempre es hiperbólico, y el corona $G_1 \diamond G_2$ es hiperbólico si y sólo si G_1 es hiperbólico. Además, obtenemos fórmulas sencillas para la constante de hiperbolicidad del grafo join $G_1 \uplus G_2$ y del corona $G_1 \diamond G_2$.

Review

If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a geodesic triangle $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X. The space X is δ-*hyperbolic* (in the Gromov sense) if any side of T is contained in the δ-neighborhood of the union of the two other sides, for every geodesic triangle T in X. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) := \inf\{\delta \geq 0 : X \text{ is } \delta\text{-hyperbolic}\}$. The study of hyperbolic graphs is an interesting topic since the hyperbolicity of a geodesic metric space is equivalent to the hyperbolicity of a graph related to it.

One of the main aims of this PhD Thesis is to obtain quantitative information about the distortion of the hyperbolicity constant of the graph $G \setminus e$ obtained from the graph G by deleting an arbitrary edge e from it. These inequalities allow to obtain other main result, which characterizes in a quantitative way the hyperbolicity of any graph in terms of local hyperbolicity.

In this work we also obtain information about the hyperbolicity constant of the line graph $\mathcal{L}(G)$ in terms of properties of the graph G. In particular, we prove qualitative results as the following: a graph G is hyperbolic if and only if $\mathcal{L}(G)$ is hyperbolic; if $\{G_n\}$ is a T-decomposition of G ($\{G_n\}$ are simple subgraphs of G), the line graph $\mathcal{L}(G)$ is hyperbolic if and only if $\sup_n \delta(\mathcal{L}(G_n))$ is finite. Besides, we obtain quantitative results when k is the length of the edges of G and $\mathcal{L}(G)$. Two of them are quantitative versions of our qualitative results. We also prove that $g(G)/4 \leq \delta(\mathcal{L}(G)) \leq c(G)/4 + 2k$, where $g(G)$ is the girth of G and $c(G)$ is its circumference. We show that $\delta(\mathcal{L}(G)) \geq \sup\{L(g) : g \text{ is an isometric cycle in } G\}/4$. Besides, we obtain bounds for $\delta(G) + \delta(\mathcal{L}(G))$. Also, we characterize the graphs G with $\delta(\mathcal{L}(G)) < k$.

Furthermore, we consider G with edges of arbitrary lengths, and $\mathcal{L}(G)$ with edges of non-constant lengths. In particular, we prove that a cycle of the graph G is transformed isometrically into a cycle of the graph $\mathcal{L}(G)$ with the same length. We also prove that $\delta(G) \leq \delta(\mathcal{L}(G)) \leq 5\delta(G) + 3l_{max}$, where $l_{max} := \sup_{e\in E(G)} L(e)$. This result implies the monotony of the hyperbolicity constant under a non-trivial transformation (the line graph of a graph).

Also, we obtain criteria which allow us to decide, for a large class of graphs, whether they are hyperbolic or not. We are especially interested in the planar graphs which are the "boundary" (the 1-skeleton) of a tessellation of the Euclidean plane. Furthermore, we prove that a graph obtained as the 1-skeleton of a general CW 2-complex is hyperbolic if and only if its dual graph is hyperbolic.

One of the main problems on this subject is to relate the hyperbolicity with other properties on graph theory. We extend in two ways (edge-chordality and path-chordality) the classical definition of chordal graphs in order to relate this property with Gromov hyperbolicity. In fact, we prove that every edge-chordal graph is hyperbolic and that every hyperbolic graph is path-chordal. Furthermore, we prove that every path-chordal cubic graph (with small path-chordality constant) is hyperbolic.

Some previous works characterize the hyperbolic product graphs (for the Cartesian product, strong product and lexicographic product) in terms of properties of the factor graphs. Finally, we characterize the hyperbolic product graphs for two important kinds of products: the graph join $G_1 \uplus G_2$ and the corona $G_1 \diamond G_2$. The graph join $G_1 \uplus G_2$ is always hyperbolic, and $G_1 \diamond G_2$ is hyperbolic if and only if G_1 is hyperbolic. Furthermore, we obtain simple formulae for the hyperbolicity constant of the graph join $G_1 \uplus G_2$ and the corona $G_1 \diamond G_2$.

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Introduction.

Hyperbolic spaces play an important role in geometric group theory and in geometry of negatively curved spaces (see, e.g., [1, 37, 38]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [1, 37, 38]).

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory; see, for instance [5, 7, 8, 9, 13, 18, 19, 24, 34, 49, 50, 51, 52, 53, 55, 58, 59, 63, 64, 65, 66, 72, 73, 75, 78, 81].

The theory of Gromov spaces was used initially for the study of finitely generated groups (see [38, 39] and the references therein), where it was demonstrated to have a practical importance. This theory was applied principally to the study of automatic groups (see [60]), which play an important role in the science of computation. The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. For example, it has been shown empirically in [76] that the internet topology embeds with better accuracy into a hyperbolic space than into an Euclidean space of comparable dimension. A few algorithmic problems in hyperbolic spaces and hyperbolic graphs have been considered in recent papers (see [27, 32, 36, 57]). Another important application of these spaces is secure transmission of information by internet (see [49, 50, 51]). In particular, the hyperbolicity plays an important role in the spread of viruses through the network (see [50, 51]). The hyperbolicity is also useful in the study of DNA data (see [13]).

In recent years several researchers have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Gehring-Osgood j-metric is Gromov hyperbolic; and the Vuorinen j-metric is not Gromov hyperbolic except in the punctured space (see [43]). The study of Gromov hyperbolicity of the quasihyperbolic and the Poincaré metrics is the subject of [3, 10, 44, 45, 66, 67, 68, 73, 75]. In particular, in [66, 73, 75, 78] it is proved the equivalence of the hyperbolicity of many negatively curved surfaces and the hyperbolicity of a simple graph; hence, it is useful to know hyperbolicity criteria for graphs.

There are several definitions of Gromov hyperbolicity (see Chapter 1). These different definitions are equivalent in the sense that if X is δ-hyperbolic with respect to the definition A, then it is δ'-hyperbolic with respect to the definition B for some δ' (see, e.g., [11, 37]). We will work primarily with the definition of Gromov hyperbolicity given by the Rips condition for "geodesic triangles" (see Definition 1.1.1).

Three main problems on Gromov hyperbolic graphs are the following:

I. To characterize the hyperbolicity for some class of graphs.

II. To obtain inequalities relating the hyperbolicity constant and other parameters of graphs.

III. To study the invariance of the hyperbolicity of graphs under appropriate transformations.

In this work, we study:

1. The distortion of the hyperbolicity constant when we remove one edge in a graph, and several consequences (Problem III).

2. Inequalities involving the hyperbolicity constants of a graph and its line graph (Problems II and III).

3. The hyperbolicity of planar graphs that are the 1-skeleton of a tessellation of the Euclidean plane or an abstract CW 2-Complex (Problem I and III).

4. The relationship of hyperbolicity and some natural generalizations of the classical concept of chordality (Problem II).

5. The hyperbolicity of the graph join and corona of two graphs (Problem I).

The outline of this PhD Thesis is as follows.

In Chapter 1 we give a brief introduction to hyperbolic spaces in the Gromov sense. Furthermore, we show some previous results which will be useful.

In Chapter 2 we study the distortion of the hyperbolicity constant of the graph $G \setminus e$ obtained from a graph G by removing an edge e, see Theorems 2.2.7 and 2.2.13. In the context of graphs, to remove an edge of the graph is a very natural transformation. To obtain a quantitative control on this distortion is a very important result. These bounds allow to obtain the characterization, in a quantitative way, of the hyperbolicity of any graph in terms of local hyperbolicity. We call *S-graph* (see Section 2.3) to the graph G obtained by "pasting" the subgraphs $\{G_n\}_{n\geq 1}$ "following the combinatorial design given by a graph G_0"; Theorem 2.3.2 states that G is δ-hyperbolic if and only if G_n is δ'-hyperbolic for every $n \geq 0$, in a simple quantitative way.

In Chapter 3 we study the line graphs with edges of constant length. One of the main aim of this Chapter is to obtain information about the hyperbolicity constant of the line graph $\mathcal{L}(G)$ in terms of properties of the graph G. In particular, we obtain results on the hyperbolicity of $\mathcal{L}(G)$ in terms of the hyperbolicity of G (see Theorems 3.1.1, 3.1.3) and in terms of the hyperbolicity of the line graphs of certain elements of a decomposition in subgraphs of G (we call it T-decomposition, see Theorem 3.3.7). We obtain some relations between the hyperbolicity constant of the line graph $\mathcal{L}(G)$ and some natural properties of G such as its girth and its circumference (see Theorem 3.3.12). Furthermore, we characterize the graphs G with $\delta(\mathcal{L}(G)) < k$ (see Theorem 3.3.6).

In Chapter 4, we deal with graphs with edges with arbitrary lengths. In order to obtain the main result of the chapter we construct a 1-Lipschitz continuous function between the line graph and the original graph (see Proposition 4.1.2). The main result of this Chapter is the inequality

$$\delta(G) \leq \delta(\mathcal{L}(G)) \leq 5\,\delta(G) + 3 \sup_{e \in E(G)} L(e), \tag{1}$$

for the line graph $\mathcal{L}(G)$ of every graph G (see Theorem 4.1.10). The second inequality in (1) can be improved for graphs with edges of length k (see Corollary 4.1.12). Also, we obtain for graphs with edges of length k other inequalities involving the hyperbolicity constant of $\mathcal{L}(G)$ (see Theorem 4.1.13 and Corollary 4.1.14).

Chapter 5 deals with a wide class of planar graphs: the graphs which are the "boundary" (the 1-skeleton) of a tessellation of the Euclidean plane. In fact, in Section 5.1 we provide several criteria in order to conclude that many tessellation graphs of the Euclidean plane $\mathbb{R}^2$ are non-hyperbolic. In Section 5.2 we deal with a wider class of graphs: the graphs which are the 1-skeleton of an abstract CW 2-complex. In fact, we prove that a graph obtained as the 1-skeleton of a CW 2-complex is hyperbolic if and only if its dual graph is hyperbolic (see Theorem 5.2.4). This result improves [65, Theorem 4.1].

One of the main problems on the theory of hyperbolic graphs is to relate the hyperbolicity with other properties on graph theory. In Chapter 6, we extend in two ways (edge-chordality and path-chordality) the classical definition of chordal graphs in order to relate this property with Gromov hyperbolicity. In fact, we prove in Section 6.1 that every edge-chordal graph is hyperbolic (see Theorem 6.1.3) and that every hyperbolic graph is path-chordal (see Theorem 6.1.8). Although the reciprocal of these two Theorems do not hold (see Examples 6.1.6 and 6.1.9), we prove in Section 6.2 that every path-chordal cubic graph (with small path-chordality constant) is hyperbolic. However, Example 6.2.10 shows that a path-chordal graph can be non-hyperbolic.

In [16, 17, 59] the authors study the hyperbolicity of the strong product graphs, the lexicographic product graphs and the Cartesian product graphs, respectively. In Chapter 7 we study the hyperbolicity of two binary operations of graphs, the hyperbolicity of graph join and corona of two graphs. In Section 7.2 we obtain that the graph join of two graphs is always hyperbolic and we compute its hyperbolicity constant in the terms of G_1 and G_2, see Corollary 7.2.1 and Theorem 7.2.14. In Section 7.3 we prove that the corona $G_1 \diamond G_2$ of two graphs G_1, G_2 is hyperbolic if and only if G_1 is hyperbolic, and also, we obtain the precise value of the hyperbolicity constant of the corona $G_1 \diamond G_2$, see Theorem 7.3.2. We want to remark that it is not usual at all to obtain explicit formulae for the hyperbolicity constant of large classes of graphs.

The results in this work appear in [4, 18, 19, 20, 21, 24] (besides, I have worked in the papers [16, 17, 22, 23] related with this topic); these papers have been published or submitted to international mathematical journals which appear in the Journal Citation Reports. These results were presented in the following international and national conferences:

- Workshop into Doc-Course: Triangulations And Extremal Graph Theory, March 2012, Sevilla, Spain.
- VIII Jornadas de Matemática Discreta y Algorítmica, July 2012, Almería, Spain.
- International Conference of Numerical Analysis and Applied Mathematics, September 2012, Kos, Greece.
- XIV Evento Científico Internacional MATECOMPU 2012, November 2012, Matanzas, Cuba.
- Workshop of Young Researchers in Mathematics 2013, September 2013, Universidad Complutense de Madrid, Spain.
- VIII Encuentro Andaluz de Matemática Discreta, October 2013, Universidad de Sevilla, Spain.

They were presented also in the following Seminars:

- Seminario de Matemáticas Avanzadas de la Unidad Académica de Matemática, Universidad Autónoma de Guerrero, México. June 2012.
- Coloquio de la Unidad Académica de Matemática, Universidad Autónoma de Guerrero, México. May 2013.
- Seminario CUICBAS de la Facultad de Ciencias, Universidad de Colima, México. June 2013.
- Seminar on Orthogonality, Approximation Theory and Applications. Group of Applied Mathematical Analysis (GAMA), Universidad Carlos III de Madrid. October 2013.

Chapter 1

A brief introduction on Gromov hyperbolic spaces.

Let (X, d) be a metric space and let $\gamma : [a, b] \longrightarrow X$ be a continuous function. We say that γ is a *geodesic* if $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$, where L denotes the length of a curve. We say that X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y; we denote by $[xy]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected. If the metric space X is a graph, we use the notation $[u, v]$ for the edge joining the vertices u and v.

In order to consider a graph G as a geodesic metric space, we must identify any edge $[u, v] \in E(G)$ with the real interval $[0, l]$ (if $l := L([u, v])$, i.e., if l is the length of $[u, v]$); therefore, any point in the interior of any edge is a point of G and, if we consider the edge $[u, v]$ as a graph with just one edge, then it is isometric to $[0, l]$. A connected graph G is naturally equipped with a distance defined on its points, induced by taking shortest paths in G. Then, we see G as a metric graph.

Throughout the work we just consider connected graphs which are locally finite (i.e., in each ball there are just a finite number of edges) or have all edges with constant length; these properties guarantee that the graphs are geodesic metric spaces (since we consider that every point in any edge of a graph G is a point of G, whether or not it is a vertex of G). Note that the edges can have arbitrary lengths. We want to remark that by [9] the study of the hyperbolicity of graphs with loops and multiple edges can be reduced to the study of the hyperbolicity of simple graphs (see Theorems 1.3.12 and 1.3.13). Usually we just consider simple graphs, but in Chapters 2 and 6 we consider graphs with loops and multiple edges.

1.1 Hyperbolic spaces in the Gromov sense.

The concept of hyperbolicity offers a global approach to spaces like the hyperbolic plane, simply-connected Riemannian manifolds with pinched negative sectional curvature, metric trees and others classical hyperbolic spaces. Several of their properties were introduced by Mikhael Gromov in the context of finitely generated groups but its generality reached new horizons.

If X is a geodesic metric space and $J = \{J_1, J_2, \dots, J_n\}$ is a polygon, with sides $J_j \subseteq X$, we say that J is δ-*thin* if for every $x \in J_i$ we have that $d(x, \cup_{j\neq i} J_j) \leq \delta$. We denote by $\delta(J)$ the sharp thin constant of J, i.e., $\delta(J) := \inf\{\delta \geq 0 : \ J \ \text{ is } \delta\text{-thin}\}$.

Definition 1.1.1. *Given $x_1, x_2, x_3 \in X$. A* geodesic triangle *$T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$. The space X is δ-*hyperbolic *(or satisfies the Rips condition with constant δ) if every geodesic triangle in X is δ-thin.*

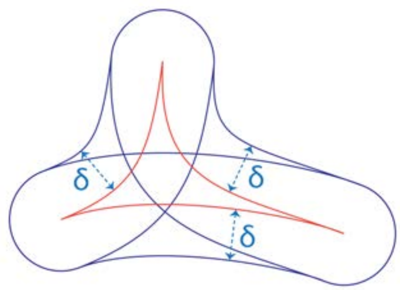


Figure 1.1: δ-thin triangle.

We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) := \sup\{\delta(T) : T \ \text{ is a geodesic triangle in } \ X\}$. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$.

Sometimes we write the geodesic triangle T as $T = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$.

Remark 1.1.2. *If X is hyperbolic, then $\delta(X) = \inf\{\delta \geq 0 : \ X \ \text{ is } \delta\text{-hyperbolic}\}$.*

One can check that every geodesic polygon in X with n sides is $(n-2)\delta(X)$-thin; in particular, any geodesic quadrilateral is $2\delta(X)$-thin. The above result is obtained by dividing the polygon into $n-2$ triangles.

There are several definitions of Gromov hyperbolicity. These different definitions are equivalent in the sense that if X is δ-hyperbolic with respect to the definition A, then it is δ'-hyperbolic with respect to the definition B for some δ' (see, e.g., [11, 37]). We have chosen this definition since it has a deep geometric meaning (see, e.g., [37]).

The following are interesting examples of hyperbolic spaces.

Example 1.1.3. *Any point of a geodesic triangle in the real line belongs to two sides of the triangle simultaneously, and therefore $\mathbb{R}$ is 0-hyperbolic.*

Example 1.1.4. *The Euclidean plane $\mathbb{R}^2$ is not hyperbolic: it is clear that equilateral triangles can be drawn with arbitrarily large diameter.*

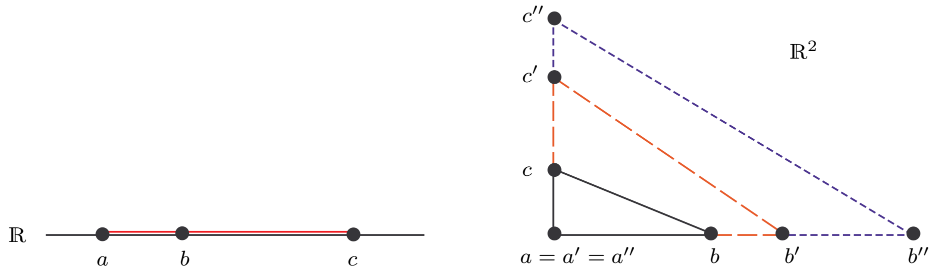


Figure 1.2: $\mathbb{R}$ and $\mathbb{R}^2$ as examples of hyperbolic spaces.

The argument in Example 1.1.4 can be generalized to higher dimensions:

a normed vector space E is hyperbolic if and only if $\dim\ E = 1$.

Example 1.1.5. *Every metric tree is* 0*-hyperbolic: in fact, every point of a geodesic triangle in a tree belongs simultaneously to two sides of the triangle (see Figure 1.3).*

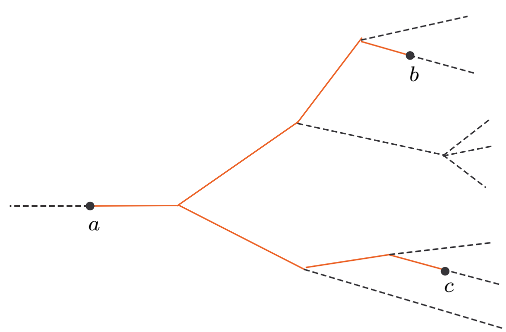


Figure 1.3: Any metric tree T verifies $\delta(T) = 0$.

Example 1.1.6. *Every bounded metric space X is* $(\operatorname{diam} X/2)$*-hyperbolic: in fact, the distance from any point of a geodesic triangle to the endpoints of its geodesic is at most* $\operatorname{diam}(X)/2$.

Example 1.1.7. *Every simply connected complete Riemannian manifold with sectional curvature verifying $K \leq -c^2$, for some positive constant c, is hyperbolic.*

The following example is an exercise in [71, p.191], it is a particular case of Example 1.1.7.

Example 1.1.8. *The open unit disk in the complex plane with its Poincaré metric is* $\log(1+\sqrt{2})$*-hyperbolic.*

We refer to [11, 37] for more background and further results.

We want to remark that the main examples of hyperbolic graphs are the trees. In fact, the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how "tree-like" the space is, since those spaces X with $\delta(X) = 0$ are precisely the metric trees. This is an interesting subject since, in many applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see, e.g., [25]).

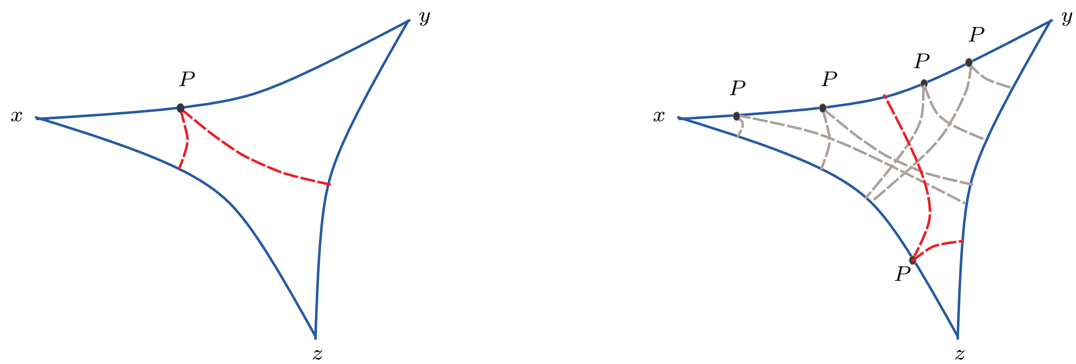


Figure 1.4: First steps in order to compute the hyperbolicity constant of X.

Given a Cayley graph (of a presentation with solvable word problem) there is an algorithm which allows to decide if it is hyperbolic. However, for a general graph or a general geodesic metric space deciding whether or not a space is hyperbolic is usually very difficult. We have to consider an arbitrary geodesic triangle T, and calculate the minimum distance from an arbitrary point P of T to the union of the other two sides of the triangle to which P does not belong to (see Figure 1.4). And then we have to take the supremum over all the possible choices for P and then over all the possible choices for T (see Figures 1.4 and 1.5).

Without disregarding the difficulty of solving this minimax problem, notice that in general the main obstacle is that we do not know the location of geodesics in the space. Therefore, it is interesting to obtain inequalities involving the hyperbolicity constant and other parameters of graphs. Another natural problem is to study the invariance of the hyperbolicity of graphs under appropriate transformations.

Since to obtain a characterization of hyperbolic graphs is a very ambitious goal, it seems reasonable to study this problem for particular classes of graphs (see Chapters 3, 4, 5, 6 and 7). We also study the hyperbolicity of graphs under some transformations (see Chapters 2, 3 and 4).

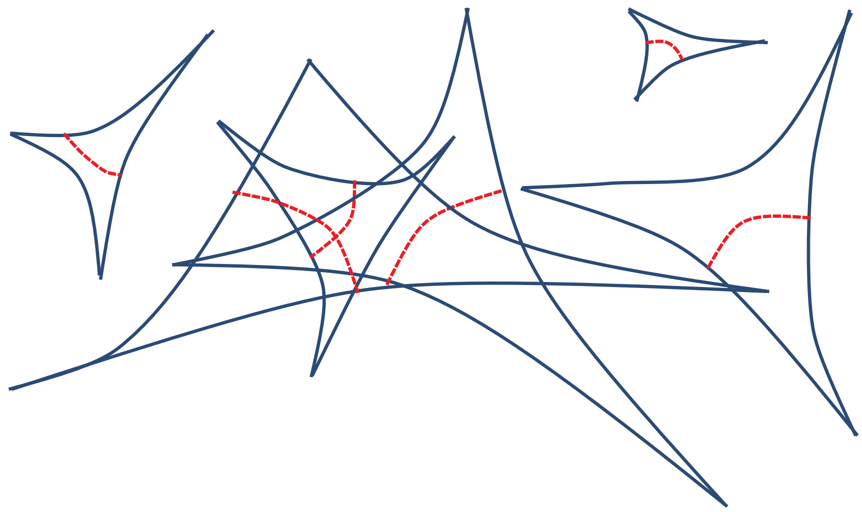


Figure 1.5: Calculating the supremum over all geodesic triangles.

1.2 Others definitions of Gromov hyperbolicity.

Gromov product definition

Definition 1.2.1. *Given a metric space X, we define the* Gromov product *of $x, y \in X$ with base point $w \in X$ by*

$$(x|y)_w := \frac{1}{2}\big(d(x,w) + d(y,w) - d(x,y)\big). \tag{1.1}$$

We say that the Gromov product is δ hyperbolic *if there is a constant $\delta \geq 0$ such that*

$$(x|z)_w \geq \min\big\{(x|y)_w, (y|z)_w\big\} - \delta$$

for every $x, y, z \in X$.

The following result (see [1, Proposition 2.2] and [38, Lemma 1.1A]) gives that the definition is independent of base point.

Proposition 1.2.2. *Let X be a metric space and $w, w' \in X$. If the Gromov product based at w is δ-hyperbolic, then the Gromov product based at w' is 2δ-hyperbolic.*

We say that X is *δ-hyperbolic product* if its Gromov product is δ-hyperbolic for any base point, i.e.,

$$(x|z)_w \geq \min\big\{(x|y)_w, (y|z)_w\big\} - \delta \tag{1.2}$$

for every $x, y, z, w \in X$ (see, e.g., [37]).

It is well known that (1.2) is equivalent to our definition of Gromov hyperbolicity (Definition 1.1.1). Furthermore, we have the following quantitative result about this equivalence.

Theorem 1.2.3. *[37, Proposition 2.21, p.41] Let us consider a geodesic metric space X.*
(1) *If X is δ-hyperbolic, then it is 4δ-hyperbolic product.*
(2) *If X is δ-hyperbolic product, then it is 3δ-hyperbolic.*

Fine definition

First, we recall the definition of fine triangles.

Definition 1.2.4. *Given a geodesic triangle $T = \{x, y, z\}$ in a geodesic metric space X, let T_E be a Euclidean triangle with sides of the same length than T. Since there is no possible confusion, we will use the same notation for the corresponding points in T and T_E. The maximum inscribed circle in T_E meets the side $[xy]$ (respectively $[yz]$, $[zx]$) in a point z' (respectively x', y') such that $d(x, z') = d(x, y')$, $d(y, x') = d(y, z')$ and $d(z, x') = d(z, y')$. We call the points x', y', z', the* internal points *of $\{x, y, z\}$. There is a unique isometry f_{xyz} of $\{x, y, z\}$ onto a tripod (a star graph with one vertex w of degree 3, and three vertices x'', y'', z'' of degree one, such that $d(x'', w) = d(x, z') = d(x, y')$, $d(y'', w) = d(y, x') = d(y, z')$ and $d(z'', w) = d(z, x') = d(z, y')$), see Figure 1.6. The triangle $\{x, y, z\}$ is δ-*fine *if $f_{xyz}(p) = f_{xyz}(q)$ implies that $d(p, q) \leq \delta$. The space X is δ-*fine *if every geodesic triangle in X is δ-fine.*

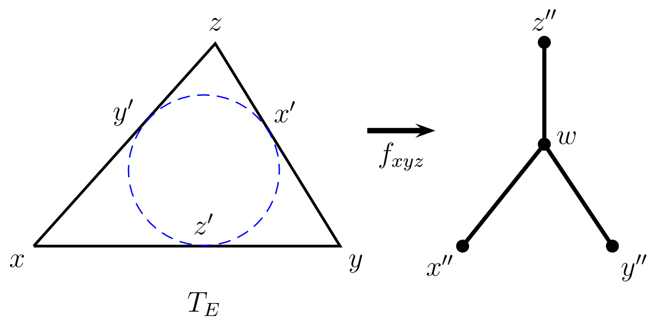


Figure 1.6: Isometry f_{xyz} of the triangle $T_E = \{x, y, z\}$ onto a tripod.

We also allow *degenerated tripods*, i.e., path graphs P_1, P_2 with one or two vertices, respectively. These situations correspond with triangles with several vertices repeated; in these cases the inscribed circle in T_E is a point.

It is known that this definition of fine is also equivalent to our definition of Gromov hyperbolicity. Furthermore, we have the following quantitative result.

Theorem 1.2.5. *[37, Proposition 2.21, p.41] Let us consider a geodesic metric space X.*
(1) *If X is δ-hyperbolic, then it is 4δ-fine.*
(2) *If X is δ-fine, then it is δ-hyperbolic.*

Insize definition

Definition 1.2.6. *Given a geodesic metric space X, let $T = \{x, y, z\}$ be a geodesic triangle in X and let x', y', z' be the internal points on T defined in Definition 1.2.4. We define the* insize *of the geodesic triangle T to be*

$$insize(T) := \operatorname{diam}\{x', y', z'\}. \tag{1.3}$$

*The space X is δ-*insize *if every geodesic triangle in X has insize at most δ.*

This definition of insize is also equivalent to our definition of Gromov hyperbolicity. Besides, we have the following quantitative result.

Theorem 1.2.7. *[37, Proposition 2.21, p.41] Let us consider a geodesic metric space X.*
(1) *If X is δ-hyperbolic, then it is 4δ-insize.*
(2) *If X is δ-insize, then it is 2δ-hyperbolic.*

Minsize definition

Definition 1.2.8. *Given a geodesic metric space X, let $T = \{x, y, z\}$ be a geodesic triangle in X and let $x' \in [yz]$, $y' \in [zx]$, $z' \in [xy]$. We define the* minsize *of the geodesic triangle T to be*

$$minsize(T) := \min_{x', y', z' \in T} \operatorname{diam}\{x', y', z'\}. \tag{1.4}$$

*The space X is δ-*minsize *if every geodesic triangle in X has minsize at most δ.*

It is known that this definition of minsize is also equivalent to our definition of Gromov hyperbolicity, in a quantitative way.

Theorem 1.2.9. *[37, Proposition 2.21, p.41] Let us consider a geodesic metric space X.*
(1) *If X is δ-hyperbolic, then it is 4δ-minsize.*
(2) *If X is δ-minsize, then it is 8δ-hyperbolic.*

Geodesics diverge

As usual, we denote by $B_k(x)$ the open ball in a metric space, i.e.,

$$B_k(x) := \{y \in X : d(x, y) < k\} \quad \text{for any } x \in X \text{ and } k > 0.$$

Definition 1.2.10. *Given a geodesic metric space X, we say that $e : [0, \infty) \to (0, \infty)$ is a* divergence function *for X, if for every point $x \in X$ and all geodesics $\gamma = [xy]$, $\gamma' = [xz]$, the function e satisfies the following condition:*

For every $R, r > 0$ such that $R + r \le \min\{L([xy]), L([xz])\}$, if $d(\gamma(R), \gamma'(R)) \ge e(0)$, and α is a path in $X \setminus B_{R+r}(x)$ from $\gamma(R + r)$ to $\gamma'(R + r)$, then we have $L(\alpha) > e(r)$ (see Figure 1.7).

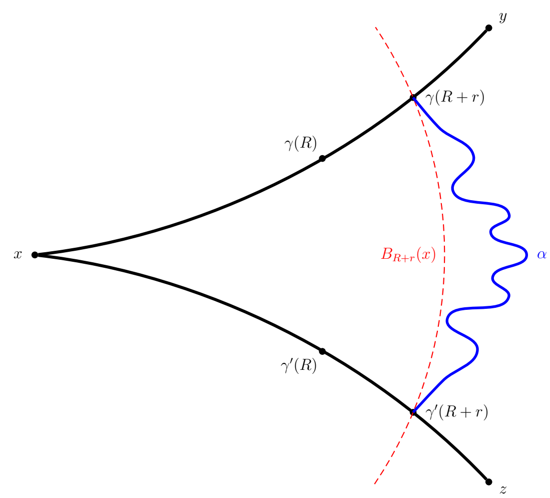


Figure 1.7: Geodesics diverge.

We say that *geodesics diverge* in X if there is a divergence function $e(r)$ such that

$$\lim_{r\to\infty} e(r) = \infty.$$

We say that *geodesics diverge exponentially* in X if there is an exponential divergence function. Theorem 1.1 in [62] shows that in a geodesic metric space X, geodesics diverge in X if and only if geodesics diverge exponentially in X.

It is known that Definition 1.2.10 is also equivalent to our definition of Gromov hyperbolicity (see [1, 62]). However, a quantitative result of this is not possible.

Gromov boundary

Let X be a metric space and we fix a base point $w \in X$. We say that a sequence $\widehat{x} = \{x_i\}_{i=1}^{\infty}$ in X is a *Gromov sequence* if $(x_i|x_j)_w \to \infty$ as $i, j \to \infty$.

Since, we have

$$\big|(x|y)_w - (x|y)_{w'}\big| = \frac{1}{2}\big|d(x,w) - d(x,w') + d(y,w) - d(y,w')\big| \le d(w,w'),$$

this concept is independent of the base point. The Gromov sequences are usually called sequences converging at infinity or tending to infinity (see, e.g., [1]). For the sake of brevity, we shall omit the base point w in the notation.

We say that two sequences $\widehat{x}$ and $\widehat{y}$ in X are *equivalent* and write $\widehat{x} \sim \widehat{y}$ if $(x_i|y_i) \to \infty$ as $i \to \infty$. This relation is always reflexive and symmetric, but it is transitive if X is hyperbolic (it is necessary to use (1.2) in order to prove it).

Definition 1.2.11. *Let X be a hyperbolic product metric space. Let $\overline{x}$ denote the equivalence class containing the Gromov sequence $\widehat{x}$ in X. The set of all equivalence classes*

$$\partial X := \{\overline{x} : \widehat{x} \text{ is a Gromov sequence in } X\}$$

is the Gromov boundary *of X, and the set*

$$\overline{X} := X \cup \partial X$$

is the Gromov closure *of X.*

A *geodesic ray* in a space X is an isometric image of the half line $[0, \infty)$. In the case of geodesic metric spaces one can alternatively define a boundary point as an equivalence class of geodesic rays [37, p.119].

We want to define the Gromov product $(a|b)$ for all $a, b \in \overline{X}$. Assume first that $a, b \in \partial X$ and choose Gromov sequences $\widehat{x} \in a$, $\widehat{y} \in b$. The numbers $(x_i|y_j)$ need not converge to a limit but they converge to a rough limit in the following sense (see [79, Lemma 5.6]):

Lemma 1.2.12. *Let X be a δ-hyperbolic product metric space. Let $a, b \in \partial X$, $a \neq b$, and let $\widehat{x}, \widehat{x'} \in a$, $\widehat{y}, \widehat{y'} \in b$, $z \in X$. Then*

$$\limsup_{i,j\to\infty}(x'_i|y'_j) \le \liminf_{i,j\to\infty}(x_i|y_j) + 2\delta < \infty,$$

$$\limsup_{i\to\infty}(x'_i|z) \le \liminf_{i\to\infty}(x_i|z) + \delta < \infty.$$

Given $a, b \in \partial X$, there exist several definitions for $(a|b)$. We choose the following one, since it is very useful.

Definition 1.2.13. *Let X be a hyperbolic product metric space and $a, b \in \partial X$. We define*

$$(a|b) := \inf\{\liminf_{i,j\to\infty}(x_i|y_j) : \widehat{x} \in a, \widehat{y} \in b\}. \tag{1.5}$$

The same definition is used in [1, 28, 79], but [37] uses sup instead of inf.

In order to provide a topological structure to $\overline{X}$, we consider the set $\mathcal{B}$ consisting of all:

1. open balls $B_r(x)$, for any $x \in X$ and $r > 0$;
2. sets of the form $N_{x,k} := \{y \in \overline{X} : (x|y) > k\}$, for any $x \in \partial X$ and $k > 0$.

Proposition 4.8 in [1] shows that the set $\mathcal{B}$ is a basis for a topology of $\overline{X}$. Furthermore, we have the following result.

Proposition 1.2.14. *[1, Proposition 4.10] Let X be a locally compact hyperbolic product metric space. Then $\overline{X}$ is a Hausdorff compact metric space, and X is open and dense in $\overline{X}$.*

1.3 Previous results on hyperbolic graphs.

Let us return to our framework: graphs as geodesic metric spaces. In this section we present some previous results about hyperbolic graphs. These results will be used throughout the thesis or are benchmark results on the subject.

Definition 1.3.1. *We say that a subgraph* Γ *of* G *is* isometric *if* $d_\Gamma(x, y) = d_G(x, y)$ *for every* $x, y \in \Gamma$.

We will need the following results (see [72, Lemma 5] and [73, Lemma 2.1]).

Lemma 1.3.2. *If* Γ *is an isometric subgraph of* G*, then* $\delta(\Gamma) \le \delta(G)$.

Lemma 1.3.3. *Let us consider a geodesic metric space* X*. If every geodesic triangle in* X *that is a simple closed curve is* δ*-thin, then* X *is* δ*-hyperbolic.*

This lemma has the following direct consequence. As usual, by *cycle* we mean a simple closed curve, i.e., a path with different vertices in a graph, except for the last one, which is equal to the first vertex.

Corollary 1.3.4. *In any graph* G*,*

$$\delta(G) = \sup\{\delta(T) : T \textit{ is a geodesic triangle that is a cycle}\}.$$

Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \longrightarrow Y$ is said to be an (α, β)-*quasi-isometric embedding*, with constants $\alpha \ge 1,\ \beta \ge 0$ if for every $x, y \in X$:

$$\alpha^{-1} d_X(x, y) - \beta \le d_Y(f(x), f(y)) \le \alpha d_X(x, y) + \beta.$$

The function f is ε-*full* if for each $y \in Y$ there exists $x \in X$ with $d_Y(f(x), y) \le \varepsilon$.

A map $f : X \longrightarrow Y$ is said to be a *quasi-isometry*, if there exist constants $\alpha \ge 1,\ \beta, \varepsilon \ge 0$ such that f is an ε-full (α, β)-quasi-isometric embedding.

Note that a quasi-isometric embedding, in general, is not continuous.

A fundamental property of hyperbolic spaces is the following (see, e.g., [37]).

Theorem 1.3.5 (Invariance of hyperbolicity)**.** *Let* $f : X \longrightarrow Y$ *be an* (α, β)*-quasi-isometric embedding between the geodesic metric spaces* X *and* Y*. If* Y *is hyperbolic, then* X *is hyperbolic.*

Besides, if f *is* ε*-full for some* $\varepsilon \ge 0$ *(a quasi-isometry), then* X *is hyperbolic if and only if* Y *is hyperbolic.*

Furthermore, if X *(respectively,* Y*) is* δ*-hyperbolic, then* Y *(respectively,* X*) is* δ'*-hyperbolic, where* δ' *is a constant which just depends on* δ*,* α*,* β *and* ε *(respectively,* δ*,* α *and* β*).*

We denote by $G \setminus \{v\}$ the metric space obtained by removing the point $\{v\}$ from the metric space G. We say that a vertex v of a graph G is a *connection vertex* if $G \setminus \{v\}$ is not connected. Sometimes connection vertices are called tree-vertices; note that any vertex with degree greater than one in a tree is a connection vertex.

Given a graph G, a family of subgraphs $\{G_n\}_{n\in\Lambda}$ of G is a *T-decomposition* of G if $\cup_n G_n = G$ and $G_n \cap G_m$ is either a connection vertex or the empty set for each $n \neq m$.

A T-decomposition of G always exists, as we will show by introducing the canonical T-decomposition of G below.

We denote by $\{G_n\}_n$ the closures in G of the connected components of the set

$$G \setminus \Big\{ v \in V(G) : \ v \ \text{ is a connection vertex of } \ G \Big\} .$$

It is clear that $\{G_n\}_n$ is a T-decomposition of G; we call it the *canonical T-decomposition* of G.

Consider a graph G and a family of subgraphs $\{G_n\}_{n\in\Lambda}$ of G such that $\cup_n G_n = G$ and, for each $n \neq m$, $G_n \cap G_m$ is either the empty set or an edge e_{nm} such that the graph $G_n \cap G_m \setminus \{e_{nm}\}$ is not connected. We define a graph R as follows: for each index $n \in \Lambda$, let us consider a point v_n (v_n is an abstract point which is not contained in G_n) and we define the set of vertices of R as $V(R) = \{v_n\}_{n\in\Lambda}$; two vertices of R are neighbors if and only if $G_n \cap G_m \neq \varnothing$. We say that the family of subgraphs $\{G_n\}_n$ of G is a *T-edge-decomposition* of G if the graph R is a tree. T-decompositions are a useful tool in the study of hyperbolic graphs (see e.g. [9, 24, 58, 72]).

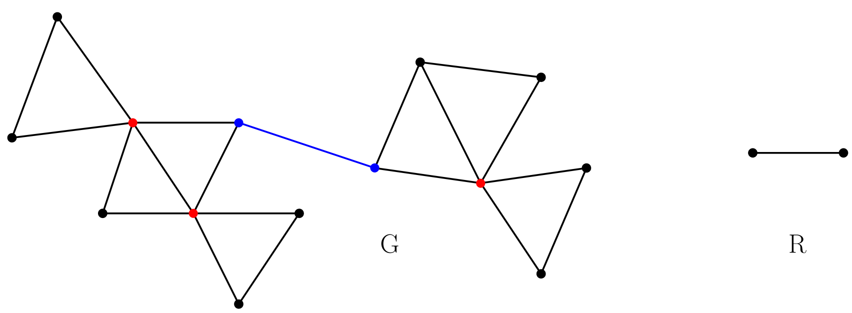


Figure 1.8: Graph G with the tree R corresponding to a T-edge-decomposition.

Remark 1.3.6. *Note that every G_n in any T-decomposition of G is an isometric subgraph of G.*

We will need the following result (see [9, Theorem 5]), which allows to obtain global information about the hyperbolicity of a graph from local information.

Theorem 1.3.7. *Let G be any graph and let $\{G_n\}_n$ be any T-decomposition of G. Then $\delta(G) = \sup_n \delta(G_n)$.*

The following result (see [72, Theorem 8]) will be useful.

Lemma 1.3.8. *In any graph G the inequality $\delta(G) \le (\operatorname{diam} G)/2$ holds, and it is sharp.*

The following results provide inequalities involving the hyperbolicity constant of a graph and its edges.

Theorem 1.3.9. *[72, Theorem 19] Let G be any graph with m edges. Then $\delta(G) \le \sum_{k=1}^{m} l_k/4$, where $l_k = L(e_k)$ for every edge $e_k \in E(G)$. Moreover, $\delta(G) = \sum_{k=1}^{m} l_k/4$ if and only if G is isomorphic to C_m.*

This result have a direct consequence (see [72, Corollary 20]).

Lemma 1.3.10. *Let G be any graph with m edges such that every edge has length k. Then $\delta(G) \le km/4$, and the equality is attained if and only if G is a cycle graph.*

In [58, Theorem 30] we find the following result, which relates the hyperbolic constant of a graph and its order. Note that in [58] multiple edges and loops are allowed.

The following family of graphs allows to characterize the extremal graphs in Theorem 1.3.11 below. Let $\mathcal{F}_n$ be the set of Hamiltonian graphs G of order n with every edge of length k and such that there exists a Hamiltonian cycle G_0 which is the union of two geodesics Γ_1, Γ_2 in G with length $nk/2$ such that the midpoint x_0 of Γ_1 satisfies $d_G(x_0, \Gamma_2) = nk/4$.

Theorem 1.3.11. *Let G be any graph with n vertices. If every edge has length k, then*

$$\delta(G) \le nk/4.$$

Moreover, if $n \ge 3$ we have $\delta(G) = nk/4$ if and only if $G \in \mathcal{F}_n$; if $n = 2$, $\delta(G) = k/2$ if and only if G has a multiple edge; if $n = 1$, $\delta(G) = k/4$ if and only if G has a loop.

The following results appear in [9, Theorems 8 and 10]. They allow to reduce the study of the hyperbolicity of graphs to the study of the hyperbolicity of simple graphs.

Given a graph G, we define $A(G)$ as the graph G without its loops, and $B(G)$ as the graph G without its multiple edges, obtained by replacing each multiple edge by a single edge with the minimum length of the edges corresponding to that multiple edge.

Theorem 1.3.12. *If G is a graph with some loop, then G is hyperbolic if and only if $A(G)$ is hyperbolic and $\sup\{L(g) : g$ is a loop of $G\} < \infty$. Besides,*

$$\delta(G) = \max\left\{ \delta(A(G)) \ , \ \frac{1}{4} \sup\{L(g) : g \text{ is a loop of } G\} \right\}.$$

In particular, if every edge has length k, then $\delta(G) = \max\{\delta(A(G)), k/4\}$.

Theorem 1.3.13. *If G is a graph with some multiple edge, then G is hyperbolic if and only if $B(G)$ is hyperbolic and $J := \sup\{L(\beta) : \beta$ is a multiple edge of $G\} < \infty$. Besides, if $j := \inf\{L(\beta) : \beta$ is a multiple edge of $G\}$,*

$$\max\left\{\delta(B(G)), \frac{J+j}{4}\right\} \le \delta(G) \le \delta(B(G)) + J.$$

Furthermore, if every edge of G has length k, then

$$\delta(G) = \max\left\{\delta(B(G)), \frac{k}{2}\right\} = \max\left\{\delta(A(B(G))), \frac{k}{2}\right\}.$$

If every edge of G has length k and $B(G)$ is not a tree, then $\delta(G) = \delta(B(G)) = \delta(A(B(G)))$.

We will need the following results in [58].

Proposition 1.3.14. *[58, Proposition 5] Let G be any graph with edges of length k. If there exists a cycle g in G with length $L(g) = 3k$, then $\delta(G) \ge 3k/4$.*

Theorem 1.3.15. *[58, Theorem 7] Let G be any graph with edges of length k. If there exists a cycle g in G with length $L(g) \ge 4k$, then*

$$\delta(G) \ge \frac{1}{4} \min\left\{L(\sigma) : \sigma \text{ is a cycle in } G \text{ with } L(\sigma) \ge 4k\right\}.$$

Note that Theorem 1.3.15 improves Proposition 1.3.14: for instance, if there are cycles of lengths $3k$ and $7k$ in a graph G and there are not cycles of lengths $4k$, $5k$ and $6k$, then Proposition 1.3.14 gives $\delta(G) \ge 3k/4$ and Theorem 1.3.15 gives $\delta(G) \ge 7k/4$.

Theorem 1.3.16. *[58, Theorem11] Let G be any graph with edges of length k.*

- *$\delta(G) < k/4$ if and only if G is a tree.*
- *$\delta(G) < k/2$ if and only if $A(G)$ is a tree.*
- *$\delta(G) < 3k/4$ if and only if $B(A(G))$ is a tree.*
- *$\delta(G) < k$ if and only if every cycle g in G has length $L(g) \le 3k$.*

Furthermore, if $\delta(G) < k$, then $\delta(G) \in \{0, k/4, k/2, 3k/4\}$.

The following results characterize the graphs with hyperbolicity constant k and greater than k, respectively.

Theorem 1.3.17. *[6, Theorem 3.10] Let G be any graph with edges of length k. Then $\delta(G) = k$ if and only if the following conditions hold:*

(1) There exists a cycle isomorphic to C_4.

(2) For every cycle σ with $L(\sigma) \geq 5k$ and for every vertex $w \in \sigma$, we have $\deg_\sigma(w) \geq 3$.

Theorem 1.3.18. *[6, Theorem 3.2] Let G be any graph with edges of length k. Then $\delta(G) \geq 5k/4$ if and only if there exist a simple cycle σ in G with length $L(\sigma) \geq 5k$ and a vertex $w \in V(\sigma)$ such that $\deg_\sigma(w) = 2$.*

We will also need the following result (see [72, Theorem 11]).

Theorem 1.3.19. *The following graphs with edges of length k have the following hyperbolicity constants:*

- *The path graphs verify $\delta(P_n) = 0$ for every $n \geq 1$.*
- *The cycle graphs verify $\delta(C_n) = nk/4$ for every $n \geq 3$.*
- *The complete graphs verify $\delta(K_1) = \delta(K_2) = 0$, $\delta(K_3) = 3k/4$, $\delta(K_n) = k$ for every $n \geq 4$.*
- *The complete bipartite graphs verify $\delta(K_{1,1}) = \delta(K_{1,2}) = \delta(K_{2,1}) = 0$, $\delta(K_{m,n}) = k$ for every $m, n \geq 2$.*
- *The Petersen graph P verifies $\delta(P) = 3k/2$.*
- *The wheel graph with n vertices W_n verifies $\delta(W_4) = \delta(W_5) = k$, $\delta(W_n) = 3k/2$ for every $7 \leq n \leq 10$, and $\delta(W_n) = 5k/4$ for $n = 6$ and for every $n \geq 11$.*

Cubic graphs (graphs with all of their vertices of degree 3) are very interesting in many situations (see, e.g., [12, 14, 29]). Theorem 1.3.5 and the following results show that they are also very important in the study of Gromov hyperbolicity.

Theorem 1.3.20. *[9, Theorem 21] Given any graph G and any $\varepsilon > 0$ there exist a cubic graph G' and an ε-full $(1+\varepsilon, \varepsilon)$ quasi-isometry $f : G \longrightarrow G'$.*

Theorem 1.3.21. *[63, Theorem 3.4] For any graph G with maximum degree Δ and edges of length k, there exists a b-full $(2b, b)$-quasi-isometry $f : G \to G^*$, where G^* is a cubic graph with every edge of length k and b is a constant depending just on k and Δ.*

These results allow to reduce the study of the hyperbolicity of graphs to the study of the hyperbolicity of cubic simple graphs.

We will use the following results which allow to reduce the study of the hyperbolicity of graphs to a countable set of geodesic triangles.

Given a graph G, we define $PMV(G)$ as the set of points of the graph G which are either vertices or midpoints of the edges.

Theorem 1.3.22. *[5, Theorem 2.6] For every hyperbolic graph G with edges of lengths k, $\delta(G)$ is a multiple of $k/4$.*

Theorem 1.3.23. *[5, Theorem 2.7] For any hyperbolic graph G with edges of length k, there exists a geodesic triangle $T = \{x, y, z\}$ that is a cycle with $\delta(T) = \delta(G)$ and $x, y, z \in PMV(G)$.*

Chapter 2

Distortion of the hyperbolicity constant of a graph.

One of the important problems in the study of any mathematical property is to determine its stability under appropriate deformations, in other words, to determine what type of perturbations preserve this property (with a quantitative control of the distortion). In the context of graphs, to delete an edge of the graph is a very natural transformation. One of the main aims of this Chapter is to obtain quantitative information about the distortion of the hyperbolicity constant of the graph $G \setminus e$ obtained from the graph G by deleting an arbitrary edge e from it (see Section 1.2). Note that this is a difficult task, since deleting an edge can change dramatically (or not) the hyperbolicity constant: on the one hand, if C is a cycle graph and $e \in E(C)$, then $\delta(C) = L(C)/4$ and $C \setminus e$ is a path graph (a tree) with $\delta(C \setminus e) = 0$; on the other hand, if G is any graph with a vertex v of degree one and $e \in E(G)$ is the edge starting in v, then $\delta(G \setminus e) = \delta(G)$. However, Theorems 2.2.7 and 2.2.13 give precise upper bounds, respectively, for $\delta(G \setminus e)$ in terms of $\delta(G)$, and for $\delta(G)$ in terms of $\delta(G \setminus e)$.

These bounds allow to obtain the other main result of this Chapter, Theorem 2.3.2, which characterizes in a quantitative way the hyperbolicity of any graph in terms of local hyperbolicity. That was the idea that lead us to think of a graph G as the union of some subgraphs $\{G_n\}_{n\geq 1}$. In order to obtain that, we call S-graph (see Section 1.3) to the graph G obtained by "pasting" the subgraphs $\{G_n\}_{n\geq 1}$ "following the combinatorial design given by a graph G_0"; Theorem 2.3.2 states that G is δ-hyperbolic if and only if G_n is δ'-hyperbolic for every $n \geq 0$, in a simple quantitative way. Note that any graph can be viewed as a S-graph (see Section 1.3).

In order to prove Theorem 2.3.2 we need to introduce a new definition of hyperbolicity (equivalent to the previous definition) which we think that it is interesting by itself: quadrilaterals δ-fine (see Section 1.1).

We want to remark that in the context of hyperbolic graphs it is usually not possible to obtain precise inequalities with explicit constants like the ones appearing in Theorems 2.2.7, 2.2.13 and 2.3.2.

2.1 A new definition of hyperbolicity in geodesic metric spaces

There are several definitions of Gromov hyperbolicity. These different definitions are equivalent in the sense that if X is δ-hyperbolic with respect to the definition A, then it is δ'-hyperbolic with respect to the definition B for some δ' (see Section 1.2).

A basic result is that hyperbolicity (thin) is equivalent to be fine (see Theorem 1.2.5). In this Chapter we need a new definition of fine hyperbolicity for geodesic quadrilaterals which will play an important role in the proof of Theorem 2.3.2.

Definition 2.1.1. *A* quatripod *is a double star graph, i.e, a tree with two vertices v_1, v_2 of degree 3 which are connected by an edge and four vertices of degree 1 two of them connected to v_1 and the other two connected to v_2. We also allow* degenerated quatripods*, i.e., star graphs $K_{1,4}$ (complete bipartite graph).*

Remark 2.1.2. *We also allow more degenerated quatripods, as star graphs $K_{1,3}$ (respectively, $K_{1,2}$). These situations correspond with quadrilaterals with several vertices repeated.*

Definition 2.1.3. *A geodesic metric space X es τ-*fine for quadrilaterals *if given any geodesic quadrilateral $Q = \{x, y, z, w\}$ in X there exists a quatripod $\mathcal{Q}$ with vertices of degree one, x_0, y_0, z_0, w_0, and a map $F : Q \longrightarrow \mathcal{Q}$ such that:*

i) $F(x) = x_0$, $F(y) = y_0$, $F(z) = z_0$, *and* $F(w) = w_0$.

ii) F *is an isometry between* $[xy]$ *and* $[x_0y_0]$, $[yz]$ *and* $[y_0z_0]$, $[zw]$ *and* $[z_0w_0]$, *and between* $[wx]$ *and* $[w_0x_0]$.

iii) *If* $F(p) = F(q)$ *then* $d(p, q) \le \tau$.

This new concept of fine quadrilaterals is an equivalent definition of hyperbolicity, as Theorems 1.2.5 and 2.1.4 show.

Theorem 2.1.4. *Let us consider a geodesic metric space X.*

- *If X is δ-fine for quadrilaterals, then it is δ-fine (for triangles).*
- *If X is δ-fine (for triangles), then it is 2δ-fine for quadrilaterals.*

Proof. The first statement is direct, since a triangle is a degenerated quadrilateral with two vertices repeated. We prove now the second statement.

Given a geodesic quadrilateral $Q = \{x, y, z, w\}$, we are going to find an Euclidean quadrilateral Q_E with sides of the same length than the sides of Q. Let us choose, for example, a geodesic $[xz]$ joining the vertex x with the vertex z. We have divided in this way the quadrilateral Q into two geodesic triangles $T_1 = \{x, y, z\}$ and $T_2 = \{x, z, w\}$. Let us consider two Euclidean triangles $T_{1,E}$, $T_{2,E}$ with sides of the same length than the sides of T_1 and T_2; without loss of generality we can assume that the sides of $T_{1,E}$ and $T_{2,E}$ corresponding to $[xz]$ are the real interval $[0, d(x, z)]$ in the complex plane, $T_{1,E}$ is contained in the upper halfplane and $T_{2,E}$ is contained in the lower halfplane. Since there is no possible confusion, we will use the same notation for the corresponding points in T_j and $T_{j,E}$, $j = 1, 2$. Then Q_E is the Euclidean quadrilateral $Q_E = \{x, y, z, w\}$.

Now, the maximum inscribed circle in $T_{1,E}$ meets the side $[xy]$ (respectively $[yz]$, $[zx]$) in the internal point z' (respectively x', y') such that $d(x, z') = d(x, y')$, $d(y, x') = d(y, z')$ and $d(z, x') = d(z, y')$. Similarly, the maximum inscribed circle in $T_{2,E}$ meets the side $[xz]$ (respectively $[zw]$, $[wx]$) in the internal point w'' (respectively x'', z'') such that $d(x, z'') = d(x, w'')$, $d(z, w'') = d(z, x'')$ and $d(w, x'') = d(w, z'')$.

There is a unique isometry f_1 of the triangle $T_1 = \{x, y, z\}$ onto a *tripod* $\mathcal{T}_1$, with one vertex v_1 of degree 3, and three vertices x_1, y_1, z_1 of degree 1, such that $d(x_1, v_1) = d(x, z') = d(x, y')$, $d(y_1, v_1) = d(y, x') = d(y, z')$ and $d(z_1, v_1) = d(z, x') = d(z, y')$. As X is δ-fine for triangles, if $f_1(p) = f_1(q)$ then we have that $d(p, q) \le \delta$. Similarly, there is also a unique isometry f_2 of the triangle $T_2 = \{x, z, w\}$ onto a tripod $\mathcal{T}_2$ with one vertex v_2 of degree 3, and three vertices x_2, z_2, w_2 of degree 1, such that $d(x_2, v_2) = d(x, z'') = d(x, w'')$, $d(w_2, v_2) = d(w, x'') = d(w, z'')$ and $d(z_2, v_2) = d(z, w'') = d(z, x'')$. Again as X is δ-fine for triangles, if $f_2(p) = f_2(q)$ then we have that $d(p, q) \le \delta$.

Let us consider the quatripod $\mathcal{Q}$ obtained from $\mathcal{T}_1$ and $\mathcal{T}_2$ by identifying $[x_1z_1] \subset \mathcal{T}_1$ with $[x_2z_2] \subset \mathcal{T}_2$: i.e., $\mathcal{Q}$ is a tree with two vertices v_1, v_2 of degree 3 which are connected by an edge with length equal to $d(y', w'')$ and four vertices of degree one $x_1 = x_2, y_1, z_1 = z_2, w_2$. Assume that $d(x_1, v_1) < d(x_2, v_2)$ (the case $d(x_1, v_1) > d(x_2, v_2)$ is similar). Then the vertices x_1, y_1 are connected to v_1 as in the tripod $\mathcal{T}_1$ and the other two z_2, w_2 are connected to v_2 as in the tripod $\mathcal{T}_2$. If $d(x_1, v_1) = d(x_2, v_2)$, then $\mathcal{Q}$ is a degenerated quatripod which is a limit case: $y' = w''$, $v_1 = v_2$ and $\mathcal{Q}$ is a tree with a vertex $v_1 = v_2$ with degree 4.

Then there is a unique map F of the quadrilateral $Q = \{x, y, z, w\}$ onto the quatripod $\mathcal{Q}$ satisfying properties *i)* and *ii)* in Definition 2.1.3.

Assume now that $p, q \in Q$ satisfy $F(p) = F(q)$. We have the following cases:

i) If $F(p) = F(q)$ belongs to $[x_1z_1] = [x_2z_2]$, then, a fortiori, it must exists a point $u \in [xz]$ such that $f_1(p) = f_1(u)$ and $f_2(q) = f_2(u)$. Therefore, $d(p, u) \le \delta$ and $d(q, u) \le \delta$ and it follows that $d(p, q) \le 2\delta$ in this case.

ii) If $F(p) = F(q)$ belongs to the edge $[v_1, y_1]$, then $f_1(p) = f_1(q)$ and so $d(p, q) \le \delta$.

iii) If $F(p) = F(q)$ belongs to the edge $[v_2, w_2]$, then $f_2(p) = f_2(q)$ and so $d(p, q) \le \delta$.

□

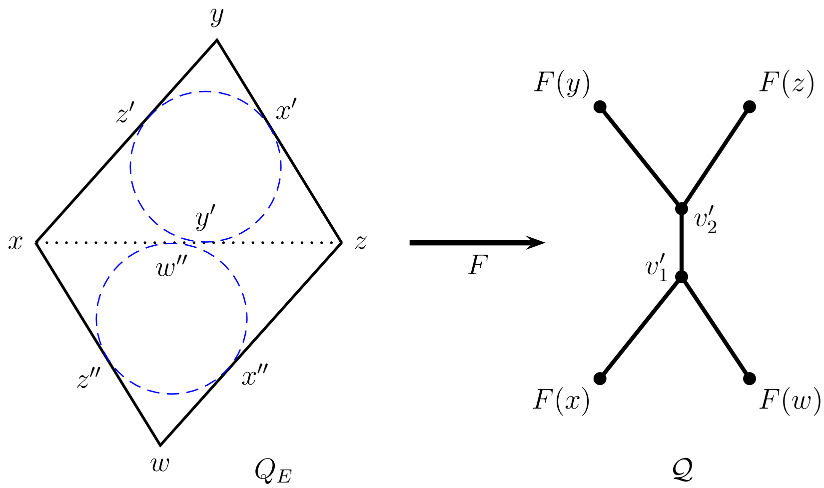


Figure 2.1: Map F of the quadrilateral $Q_E = \{x, y, z, w\}$ onto the quatripod $\mathcal{Q}$.

2.2 Deleting an edge

In this section we deal with one of the main problems in the Chapter: to obtain quantitative relations between $\delta(G \setminus e)$ and $\delta(G)$, where e is any edge of G. As usual, we define the graph $G \setminus e$ as the graph with $V(G \setminus e) = V(G)$ and $E(G \setminus e) = E(G) \setminus \{e\}$.

Since the proofs of these inequalities are long and technical, in order to make the arguments more transparent, we collect some results we need along the proof in technical lemmas.

Lemma 2.2.1. *Let G be any graph, $e \in E(G)$ with $G \setminus e$ connected and $x, y \in G \setminus e$. If a geodesic $\Gamma_G = [xy]_G \subset G$ contains e, then there exists a point $z \in \Gamma_{G\setminus e} = [xy]_{G\setminus e} \subset G\setminus e$ such that the subcurve γ_{xz} (respectively, γ_{zy}) contained in $\Gamma_{G\setminus e}$ and joining x and z (respectively, z and y) is a geodesic in G.*

Proof. Consider the points $A, B \in \Gamma_{G\setminus e}$ such that $d_{G\setminus e}(x, A) = d_G(x, e)$ and $d_{G\setminus e}(y, B) = d_G(y, e)$, and choose z as the midpoint of $[A, B] \subset \Gamma_{G\setminus e}$. (The points A and B always exist since $L(\Gamma_G) \le L(\Gamma_{G\setminus e})$.) From the fact that $\gamma_{xz} \subset \Gamma_{G\setminus e}$ and $\gamma_{zy} \subset \Gamma_{G\setminus e}$ are geodesics in $G \setminus e$, we obtain $d_G(z, e) \ge L([A, B])/2$; hence, γ_{xz} and γ_{zy} are geodesics in G. ☐

Lemma 2.2.2. *Let G be any graph and $e \in E(G)$ with $G \setminus e$ connected. For all $x, y \in G \setminus e$, if $\Gamma_G = [xy]_G$ is a geodesic in G containing e and $\Gamma_{G\setminus e} = [xy]_{G\setminus e}$ is a geodesic in $G \setminus e$, then*

$$\forall\ u \in \Gamma_{G\setminus e}\ ,\ \exists\ u' \in \Gamma_G \setminus e\ :\ d_{G\setminus e}(u, u') \le 2\delta(G). \tag{2.1}$$

Remark 2.2.3. *In $\Gamma_G \setminus e$ we include the vertices connected by e.*

Proof. Without loss of generality we can assume that G is hyperbolic, since otherwise the inequality is direct. By Lemma 2.2.1 we have a point $z \in \Gamma_{G\setminus e}$ such that $T = \{\Gamma_G, [yz]_{G\setminus e}, [zx]_{G\setminus e}\}$ is a geodesic triangle in G. Without loss of generality we can assume that $u \in [yz]_{G\setminus e}$. If $L([yz]_{G\setminus e}) \le \delta(G)$, then there exists $u' = y \in \Gamma_G$ such that $d_{G\setminus e}(u,u') \le \delta(G)$. If $L([yz]_{G\setminus e}) > \delta(G)$, then we can take a point $C \in [yz]_{G\setminus e}$ such that $d_{G\setminus e}(C,z) = \delta(G)$; therefore, if $u \in [Cy] \setminus \{C\}$, then the hyperbolicity of G implies $d_G(u, \Gamma_G \cup [zx]_{G\setminus e}) \le \delta(G)$; note that if $d_G(u, [zx]_{G\setminus e}) \le \delta(G)$ then the geodesic γ joining u and $[zx]_{G\setminus e}$ is not contained in $G \setminus e$; in fact, $e \subset \gamma$, and since $e \subset \Gamma_G$ we have $d_G(u,\Gamma_G) \le L(\gamma) \le \delta(G)$; otherwise, $d_G(u,\Gamma_G) \le \delta(G)$; in both cases, since $e \subset \Gamma_G$, we deduce $d_{G\setminus e}(u,\Gamma_G) = d_G(u,\Gamma_G) \le \delta(G)$. Assume now that $u \in [Cz]_{G\setminus e}$ (i.e., $u \in [yz]_{G\setminus e}$ with $d_{G\setminus e}(u,z) \le \delta(G)$); for every $\varepsilon > 0$ there exists $u_\varepsilon \in [yz]_{G\setminus e}$ such that $d_{G\setminus e}(u,u_\varepsilon) \le \delta(G) + \varepsilon$ and $d_{G\setminus e}(u_\varepsilon, z) > \delta(G)$. Then there exists $u'_\varepsilon \in \Gamma_G$ with $d_{G\setminus e}(u'_\varepsilon, u_\varepsilon) \le \delta(G)$ and $d_{G\setminus e}(u,u'_\varepsilon) \le 2\delta(G) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, by compactness of Γ_G there exists $u' \in \Gamma_G$ with $d_{G\setminus e}(u',u) \le 2\delta(G)$.

In order to finish the proof it suffices to note that if u' belongs to the interior of e, we can replace u' by one of the vertices joined by e. □

We also obtain this similar result.

Lemma 2.2.4. *Let G be any graph and $e \in E(G)$ with $G \setminus e$ connected. For all $x, y \in G \setminus e$, if $\Gamma_G = [xy]_G$ is a geodesic in G containing e and $\Gamma_{G\setminus e} = [xy]_{G\setminus e}$ is a geodesic in $G \setminus e$, then*

$$\forall\ u' \in \Gamma_G\ ,\ \exists\ u \in \Gamma_{G\setminus e}\ :\ d_G(u',u) \le \delta(G). \tag{2.2}$$

Furthermore,

$$\forall\ u' \in \Gamma_G \setminus e\ ,\ \exists\ u \in \Gamma_{G\setminus e}\ :\ d_{G\setminus e}(u',u) \le 2\delta(G). \tag{2.3}$$

Proof. Without loss of generality we can assume that G is hyperbolic, since otherwise the inequalities are direct. By Lemma 2.2.1 we have a point $z \in \Gamma_{G\setminus e}$ such that $T = \{\Gamma_G, [yz]_{G\setminus e}, [zx]_{G\setminus e}\}$ is a geodesic triangle in G; so (2.2) follows directly since G is hyperbolic. We prove now (2.3).

Let A and B be the vertices of e, such that $[xA]_G \subset \Gamma_G$ and $[By]_G \subset \Gamma_G$ are geodesics in G with $[xA]_G \cap [By]_G = \varnothing$. Without loss of generality we can assume that $u' \in [xA]_G$. If $L([xA]_G) \le \delta(G)$, then there exists $u \in \Gamma_{G\setminus e}$ such that $d_{G\setminus e}(u',u) \le \delta(G)$. If $L([Ax]_G) > \delta(G)$, then let us consider the point $A' \in [Ax]_G$ such that $d_{G\setminus e}(A',A) = \delta(G)$; if $u' \in [A'x]_G$, then we have $d_G(u',e) \ge \delta(G)$ and therefore $d_{G\setminus e}(u',u) \le \delta(G)$. Finally, if $u' \in [AA']_G$, then there exits $u'' \in [A'x]_G$ such that $d_{G\setminus e}(u',u'') \le \delta(G)$; hence, there exits $u \in \Gamma_{G\setminus e}$ such that $d_{G\setminus e}(u',u) \le d_{G\setminus e}(u',u'') + d_{G\setminus e}(u'',u) \le 2\delta(G)$. □

The argument in the proof of Lemma 2.2.4 also gives the following result.

Corollary 2.2.5. *Let G be any graph, $e \in E(G)$ with $G \setminus e$ connected, and $x, y, z \in G \setminus e$; let $T = \{[xy], [yz], [zx]\}$ be a geodesic triangle in G such that $[xy]$ contains e and $[yz]$, $[zx]$ do not contain e. Then*

$$\forall\ u' \in [xy] \setminus e,\ \exists\ u \in [yz] \cup [zx]\ :\ d_{G\setminus e}(u', u) \le 2\delta(G). \tag{2.4}$$

Lemma 2.2.6. *Let G be any graph and $e \in E(G)$ with $G \setminus e$ connected. Let $T_G = \{[xy]_G, [yz]_G, [zx]_G\}$ be a geodesic triangle in G with $x, y, z \in G \setminus e$. Then e is contained at most in two of the three sides of T_G.*

Proof. Without loss of generality we can assume that $e = [A, B]$ is contained in $[xy]_G$ and $[xz]_G$. Since $[xy]_G = [xA]_{G\setminus e} \cup [A, B] \cup [By]_{G\setminus e}$, we have $L([xB]_{G\setminus e}) \ge L([xA]_{G\setminus e}) + L(e)$ and $L([Ay]_{G\setminus e}) \ge L(e) + L([By]_{G\setminus e})$; since $[xz]_G = [xA]_{G\setminus e} \cup [A, B] \cup [Bz]_{G\setminus e}$, we have $L([Az]_{G\setminus e}) \ge L(e) + L([zB]_{G\setminus e})$. Hence, $\min\{L(\gamma) : \gamma$ is a path in G between y and z with $e \subset \gamma\} \ge L(e) + d_{G\setminus e}(y, B) + d_{G\setminus e}(B, z)$; since $d_G(y, z) \le d_{G\setminus e}(y, B) + d_{G\setminus e}(B, z)$, then e is not contained in $[yz]_G$. ☐

We can prove now the following Theorem.

Theorem 2.2.7. *Let G be any graph and $e \in E(G)$ with $G \setminus e$ connected. The following inequality holds*

$$\delta(G \setminus e) \le 5\delta(G). \tag{2.5}$$

Proof. Without loss of generality we can assume that G is hyperbolic, since otherwise the inequality is direct. If $e = [A, B]$ and $L(e) \ge d_{G\setminus e}(A, B)$, then $G \setminus e$ is an isometric subgraph of G and Lemma 1.3.2 gives $\delta(G \setminus e) \le \delta(G)$. Assume now that $L(e) < d_{G\setminus e}(A, B)$.

Let us consider an arbitrary geodesic triangle $T_{G\setminus e} = \{[xy]_{G\setminus e}, [yz]_{G\setminus e}, [zx]_{G\setminus e}\}$ in $G \setminus e$. Let T_G be a geodesic triangle of G with the same vertices of $T_{G\setminus e}$, i.e., $T_G = \{[xy]_G, [yz]_G, [zx]_G\}$, satisfying the following property: if a and b are vertices of $T_{G\setminus e}$ with $d_{G\setminus e}(a, b) = d_G(a, b)$, then we choose $[ab]_G$ as $[ab]_{G\setminus e}$. If n is the number of the geodesic sides of T_G containing e, then by Lemma 2.2.6 n is either 0, 1 or 2.

Case $n = 0$ In this case we have $T_G = T_{G\setminus e}$. Let us consider any $\alpha \in T_{G\setminus e}$; without loss of generality we can assume that $\alpha \in [xy]_{G\setminus e}$.

Since G is hyperbolic, there exists $\beta \in [xz]_{G\setminus e} \cup [yz]_{G\setminus e}$ such that $d_G(\alpha, \beta) \le \delta(G)$. If $d_{G\setminus e}(\alpha, \beta) = d_G(\alpha, \beta) \le \delta(G)$, then $d_{G\setminus e}(\alpha, [xz]_{G\setminus e} \cup [yz]_{G\setminus e}) \le \delta(G)$. Hence, we can assume that $d_{G\setminus e}(\alpha, \beta) > d_G(\alpha, \beta)$; then the geodesic in G joining α and β contains e. Let γ_1 be the geodesic contained in $[xy]_{G\setminus e}$ joining α and x, and let γ_2 be the geodesic contained in $[xy]_{G\setminus e}$ joining α and y; then $\gamma_1 \cup \gamma_2 = [xy]_{G\setminus e}$.

If $L(\gamma_1) \le 2\delta(G)$ or $L(\gamma_2) \le 2\delta(G)$, then there exists $\beta \in \{x, y\} \subset [xz]_{G\setminus e} \cup [yz]_{G\setminus e}$ such that $d_{G\setminus e}(\alpha, \beta) \le 2\delta(G)$.

If $L(\gamma_1) > 2\delta(G)$, then consider the point $\alpha' \in \gamma_1$ such that $d_{G\setminus e}(\alpha, \alpha') = 2\delta(G)$. Since G is hyperbolic, there exists $\beta' \in [xz]_{G\setminus e} \cup [yz]_{G\setminus e}$ such that $d_G(\alpha', \beta') \leq \delta(G)$. If $d_{G\setminus e}(\alpha', \beta') = d_G(\alpha', \beta') \leq \delta(G)$, then we conclude $d_{G\setminus e}(\alpha, [xz]_{G\setminus e} \cup [yz]_{G\setminus e}) \leq 3\delta(G)$. If $d_{G\setminus e}(\alpha', \beta') > d_G(\alpha', \beta')$, then the geodesic in G joining α' and β' contains e; recall that the geodesic in G joining α and β contains e; hence, there exists a path in G joining α and α' with length less than $2\delta(G)$, and therefore $[xy]_{G\setminus e}$ is not a geodesic in G. This is a contradiction, and we conclude $d_{G\setminus e}(\alpha, [xz]_{G\setminus e} \cup [yz]_{G\setminus e}) \leq 3\delta(G)$.

Therefore, $\delta(T_{G\setminus e}) \leq 3\delta(G)$ in the case $n = 0$.

Case $n = 1$ In this case, without loss of generality we can assume that $[xz]_G = [xz]_{G\setminus e}$ and $[yz]_G = [yz]_{G\setminus e}$.

By Lemma 2.2.2, for any $\alpha_1 \in [xy]_{G\setminus e}$ there exists $\alpha' \in [xy]_G \setminus e$ such that $d_{G\setminus e}(\alpha_1, \alpha') \leq 2\delta(G)$. Furthermore, by Corollary 2.2.5 there exists $\beta_1 \in [xz]_G \cup [yz]_G$ such that $d_{G\setminus e}(\alpha', \beta_1) \leq 2\delta(G)$. Hence, we have $d_{G\setminus e}(\alpha_1, \beta_1) \leq 4\delta(G)$.

Let us consider now any $\alpha_2 \in [xz]_{G\setminus e} \cup [yz]_{G\setminus e}$; without loss of generality we can assume that $\alpha_2 \in [yz]_{G\setminus e}$. Since $T' = \{[xy]_G, [yz]_{G\setminus e}, [zx]_{G\setminus e}\}$ is a geodesic triangle in G, there exists $\alpha' \in [xy]_G \cup [xz]_{G\setminus e}$ such that $d_G(\alpha_2, \alpha') \leq \delta(G)$. Hence, there exists $\alpha'' \in ([xy]_G \setminus e) \cup [xz]_G$ such that $d_{G\setminus e}(\alpha_2, \alpha'') \leq \delta(G)$ (if the geodesic joining α_2 and α' contains $e = [A, B]$, then $A, B \in [xy]_G$). If $\alpha'' \in [xz]_{G\setminus e}$, then we obtain $d_{G\setminus e}(\alpha_2, \alpha'') \leq \delta(G)$. Assume now that $\alpha'' \in [xy]_G \setminus e$. By Lemma 2.2.4 there exists $\beta_2 \in [xy]_{G\setminus e}$ such that $d_{G\setminus e}(\alpha'', \beta_2) \leq 2\delta(G)$, and we have $d_{G\setminus e}(\alpha_2, \beta_2) \leq 3\ d(G)$.

Therefore, we obtain $\delta(T_{G\setminus e}) \leq 4\delta(G)$ in the case $n = 1$.

Case $n = 2$ Without loss of generality we can assume that $[yz]_{G\setminus e} = [yz]_G$.

Let us consider $\alpha \in [xy]_{G\setminus e} \cup [xz]_{G\setminus e}$; without loss of generality we can assume that $\alpha \in [xy]_{G\setminus e}$ and that $d_G(x, A) < d_G(x, B)$. By Lemma 2.2.2, for any $\alpha \in [xy]_{G\setminus e}$ there exists $\alpha' \in [xy]_G \setminus e$ such that $d_{G\setminus e}(\alpha, \alpha') \leq 2\delta(G)$. If $\alpha' \in [yB]_G$, then since $T' = \{[yB]_G, [Bz]_G, [zy]_{G\setminus e}\}$ is $\delta(G)$-thin in G there exists $\beta'_0 \in [yz]_G \cup [Bz]_G$ such that $d_G(\alpha', \beta'_0) \leq \delta(G)$; hence, there exists $\beta' \in [yz]_G \cup ([xz]_G \setminus e)$ such that $d_{G\setminus e}(\alpha', \beta') \leq \delta(G)$, since if the geodesic joining α' and β'_0 contains e, then we can take $\alpha' \in \{A, B\}$. Moreover, if $\beta' \in [yz]_G$, then $d_{G\setminus e}(\alpha, \beta') \leq 3\delta(G)$. If $\beta' \in [xz]_G \setminus e$, then by Lemma 2.2.4, there exists $\beta \in [xz]_{G\setminus e}$ such that $d_{G\setminus e}(\beta', \beta) \leq 2\delta(G)$. Hence, we have $d_{G\setminus e}(\alpha, \beta) \leq 5\delta(G)$. If $\alpha' \in [xA]_G$, then we also obtain $d_{G\setminus e}(\alpha, \beta) \leq 5\delta(G)$ with a similar argument.

Consider now $\alpha \in [yz]_{G\setminus e}$; since $T' = \{[yB]_G, [Bz]_G, [zy]_{G\setminus e}\}$ is $\delta(G)$-thin in G, there exists $\alpha'_0 \in [yB]_G \cup [Bz]_G$ such that $d_G(\alpha, \alpha'_0) \leq \delta(G)$. Thus, there exists $\alpha' \in ([xy]_G \cup [xz]_G) \setminus e$ such that $d_{G\setminus e}(\alpha, \alpha') \leq \delta(G)$, since if the geodesic joining α and α'_0 contains e, then we can take $\alpha' \in \{A, B\}$. Hence, without loss of generality we can suppose that $\alpha' \in [xy]_G \setminus e$; then by Lemma 2.2.4 there exists $\beta \in [xy]_{G\setminus e}$ such that $d_{G\setminus e}(\alpha', \beta) \leq 2\delta(G)$. Therefore, we have $d_{G\setminus e}(\alpha, \beta) \leq 3\delta(G)$.

Finally, we obtain $\delta(T_{G\setminus e}) \leq 5\delta(G)$ in this case.

□

We will prove now a kind of converse of Theorem 2.2.7. First of all, note that it is not possible to have the inequality $\delta(G) \leq c\,\delta(G \setminus e)$ for some fixed constant c, since if G is the cycle graph with n vertices and edges with length 1, and e is any edge of G, then $\delta(G) = n/4$ and $\delta(G \setminus e) = 0$.

We prove first some previous results.

Lemma 2.2.8. *Let G be any graph and $e \in E(G)$ with $G \setminus e$ connected. Let T_G be a geodesic triangle in G contained in $G \setminus e$. Then T_G is $\delta(G \setminus e)$-thin in G, i.e.,*

$$\delta(T_G) \leq \delta(G \setminus e). \tag{2.6}$$

Proof. This result is straightforward since T_G is a geodesic triangle in $G \setminus e$ also, and $d_G(x, y) \leq d_{G\setminus e}(x, y)$ for every $x, y \in G \setminus e$. □

Lemma 2.2.9. *Let G be any graph and $e = [A, B] \in E(G)$ with $G \setminus e$ connected. For all $x, y \in G \setminus e$, if $\Gamma_G = [xy]_G$ is a geodesic in G containing e and $\Gamma_{G\setminus e} = [xy]_{G\setminus e}$ is a geodesic in $G \setminus e$, then*

$$\forall\ u \in \Gamma_{G\setminus e}\ ,\ \exists\ u' \in \Gamma_G \setminus e\ :\ d_G(u, u') \leq 2\delta(G \setminus e) + \frac{1}{2} d_{G\setminus e}(A, B). \tag{2.7}$$

Proof. Without loss of generality we can assume that $G \setminus e$ is hyperbolic, since otherwise the inequality is direct. We can assume also that $\Gamma_G = [xy]_G = [xA] \cup e \cup [By]$. Let us consider the geodesic quadrilateral $P_4 = \{[xy]_{G\setminus e}, [xA], [AB]_{G\setminus e}, [By]\}$ in $G \setminus e$. Since P_4 is $2\delta(G \setminus e)$ -thin in $G \setminus e$, then

$$\forall\ u \in \Gamma_{G\setminus e}\ ,\quad d_{G\setminus e}(u, [xA] \cup [AB]_{G\setminus e} \cup [By]) \leq 2\delta(G \setminus e),$$

and inequality (2.7) follows. □

Lemma 2.2.10. *Let G be any graph and $e = [A, B] \in E(G)$ with $G \setminus e$ connected. For all $x, y \in G \setminus e$, if $\Gamma_G = [xy]_G$ is a geodesic in G containing e and $\Gamma_{G\setminus e} = [xy]_{G\setminus e}$ is a geodesic in $G \setminus e$, then*

$$\forall\ u' \in \Gamma_G\ ,\ \exists\ u \in \Gamma_{G\setminus e}\ :\ d_G(u', u) \leq 5\delta(G \setminus e) + d_{G\setminus e}(A, B). \tag{2.8}$$

Proof. Without loss of generality we can assume that $G \setminus e$ is hyperbolic, since otherwise the inequality is direct. We can assume also that $\Gamma_G = [xy]_G = [xA] \cup e \cup [By]$. Denoted by P the middle point of $[AB]_{G\setminus e}$. Note that the condition $e \subseteq \Gamma_G = [xy]_G$, implies $d_{G\setminus e}(A, B) \geq L(e)$.

Note also that

$$\forall\, u' \in \Gamma_G \ , \ \exists\, u^* \in [xA] \cup [By] \quad : \quad d_G(u,u^*) \le \frac{1}{2}L(e).$$

Without loss of generality we can assume that $u^* \in [xA]$. Since $T = \{[xA], [AP]_{G\setminus e}, [xP]_{G\setminus e}\}$ is a geodesic triangle in $G \setminus e$, there exists $\alpha \in [AP]_{G\setminus e} \cup [xP]_{G\setminus e}$ such that $d_G(u^*,\alpha) \le d_{G\setminus e}(u^*,\alpha) \le \delta(G\setminus e)$, and so

$$\forall\, u^* \in [xA] \ , \ \exists\, \beta \in [xP]_{G\setminus e} \quad : \quad d_G(u^*,\beta) \le \delta(G\setminus e) + \frac{1}{2}d_{G\setminus e}(A,B).$$

Now, $T = \{[xy]_{G\setminus e}, [xP]_{G\setminus e}, [Py]_{G\setminus e}\}$ is a geodesic triangle in $G \setminus e$ and T is $4\delta(G\setminus e)$-fine by Theorem 1.2.5. Let us denote by r, s and t the internal points in the geodesics $[xy]_{G\setminus e}$, $[xP]_{G\setminus e}$ and $[Py]_{G\setminus e}$, respectively. Since $L([sP]_{G\setminus e}) = L([Pt]_{G\setminus e}) = \frac{1}{2}(L([xP]_{G\setminus e}) + L([Py]_{G\setminus e}) - L([xy]_{G\setminus e}))$, we have

$$\forall\, \beta \in [xP]_{G\setminus e} \cup [Py]_{G\setminus e}, \exists\, u \in [xy]_{G\setminus e} :$$

$$d_G(\beta,u) \le 4\delta(G\setminus e) + \frac{1}{2}\big(L([xP]_{G\setminus e}) + L([Py]_{G\setminus e}) - L([xy]_{G\setminus e})\big).$$

Triangle inequality gives $L([xP]_{G\setminus e}) + L([Py]_{G\setminus e}) \le L([xA]_{G\setminus e}) + L([AP]_{G\setminus e}) + L([PB]_{G\setminus e}) + L([By]_{G\setminus e}) = L([xy]_G) + d_{G\setminus e}(A,B) - L(e)$; since $L([xy]_{G\setminus e}) \ge L([xy]_G)$, we deduce

$$\frac{1}{2}\big(L([xP]_{G\setminus e}) + L([Py]_{G\setminus e}) - L([xy]_{G\setminus e})\big) \le \frac{1}{2}\big(d_{G\setminus e}(A,B) - L(e)\big).$$

Finally, if we consider the path $[u'u^*] \cup [u^*\beta] \cup [\beta u]$, then we obtain

$$\begin{aligned} d_G(u',u) &\le \frac{1}{2}L(e) + \delta(G\setminus e) + \frac{1}{2}d_{G\setminus e}(A,B) + 4\delta(G\setminus e) + \frac{1}{2}\big(d_{G\setminus e}(A,B) - L(e)\big) \\ &= 5\delta(G\setminus e) + d_{G\setminus e}(A,B). \end{aligned}$$

$\square$

Lemma 2.2.11. *Let G be any graph and $e = [A,B] \in E(G)$ with $G \setminus e$ connected. Let $T_G = \{[xy]_G, [yz]_G, [zx]_G\}$ be a geodesic triangle in G, such that $e \subseteq [xy]_G$ and $[yz]_G, [zx]_G \subset G \setminus e$. Then*

$$\delta(T_G) \le 6\delta(G\setminus e) + d_{G\setminus e}(A,B). \tag{2.9}$$

Proof. Without loss of generality we can assume that $G \setminus e$ is hyperbolic, since otherwise the inequality is direct. Let $[xy]_{G\setminus e}$ be a geodesic in $G \setminus e$; then $T = \{[xy]_{G\setminus e}, [yz]_G, [zx]_G\}$ is a geodesic triangle in $G \setminus e$. Hence, for any $\alpha \in [yz]_G$ we have

$$d_G(\alpha, [zx]_G \cup [xy]_{G\setminus e}) \le d_{G\setminus e}(\alpha, [zx]_G \cup [xy]_{G\setminus e}) \le \delta(G\setminus e).$$

By Lemma 2.2.9, for any $\beta \in [xy]_{G\setminus e}$, we have $d_G(\beta, [xy]_G) \le 2\delta(G \setminus e) + \frac{1}{2} d_{G\setminus e}(A, B)$. Then we obtain

$$d_G(\alpha, [zx]_G \cup [xy]_G) \le 3\delta(G \setminus e) + \frac{1}{2} d_{G\setminus e}(A, B).$$

If $\alpha \in [zx]_G$, then the same argument gives the last inequality.

By Lemma 2.2.10, for any $\alpha \in [xy]_G$, there exists $\beta \in [xy]_{G\setminus e}$ such that $d_G(\alpha, \beta) \le 5\delta(G \setminus e) + d_{G\setminus e}(A, B)$. If we consider again the geodesic triangle T in $G \setminus e$, then we have

$$d_G(\beta, [yz]_G \cup [zx]_G) \le d_{G\setminus e}(\beta, [yz]_G \cup [zx]_G) \le \delta(G \setminus e),$$

Therefore, for any $\alpha \in [xy]_G$, we obtain

$$d_G(\alpha, [yz]_G \cup [zx]_G) \le 6\delta(G \setminus e) + d_{G\setminus e}(A, B).$$

□

Lemma 2.2.12. *Let G be any graph and $e = [A, B] \in E(G)$ with $G \setminus e$ connected. Let $T_G = \{[xy]_G, [yz]_G, [zx]_G\}$ be a geodesic triangle in G, such that $\{x, y, z\} \cap e \ne \varnothing$. Then*

$$\delta(T_G) \le \max\big\{2\delta(G \setminus e) + d_{G\setminus e}(A, B),\ L(e)\big\}. \tag{2.10}$$

Proof. Without loss of generality we can assume that $G \setminus e$ is hyperbolic, since otherwise the inequality is direct. If every vertex of T_G belongs to e, then we have $T_G \subseteq e \cup [AB]_{G\setminus e}$; hence, $\delta(T_G) \le \frac{1}{4} L(T_G) = \frac{1}{4}(L(e) + d_{G\setminus e}(A, B))$.

Assume now that there are exactly two vertices of T_G in e; without loss of generality we can assume that $x, y \in e$, $z \notin e$, $A \in [xz]_G$ and $B \in [yz]_G$. In order to bound $\delta(T_G)$, let us choose any $\alpha \in T_G$. If $\alpha \in [xy]_G$, then we have $d_G(\alpha, [xz]_G \cup [yz]_G) = d_G(\alpha, \{x, y\}) \le L(e)$. If $\alpha \in [xz]_G \cup [yz]_G$, then without loss of generality we can assume that $\alpha \in [xz]_G$. If $\alpha \in [xA]_G \subset [xz]_G$, then we have $d_G(\alpha, [xy]_G \cup [yz]_G) \le d_G(\alpha, x) \le L(e)$. If $\alpha \in [Az]_G \subset [xz]_G$, then let us consider the geodesic triangle $T^* = \{[Az]_G, [zB]_G, [AB]_{G\setminus e}\}$ in $G \setminus e$; then there exists $\beta \in [Bz]_G \cup [AB]_{G\setminus e}$ such that $d_G(\alpha, \beta) \le d_{G\setminus e}(\alpha, \beta) \le \delta(G \setminus e)$, and we obtain $d_G(\alpha, [yz]_G \cup [xy]_G) \le \delta(G \setminus e) + d_{G\setminus e}(A, B)$. Hence,

$$\delta(T_G) \le \max\big\{\delta(G \setminus e) + d_{\setminus e}(A, B), L(e)\big\}.$$

Finally, assume that there is exactly one vertex of T_G in e; without loss of generality we can assume that $x \in e$, $z, y \notin e$, $A \in [xy]_G$ and $B \in [xz]_G$. In order to bound $\delta(T_G)$, let us choose any $\alpha \in T_G$. If $\alpha \in [yz]_G$, then $T_4^* = \{[Ay]_G, [yz]_G, [zB]_G, [AB]_{G\setminus e}\}$ is a geodesic quadrilateral in $G \setminus e$ and there exists $\beta \in [yA]_G \cup [AB]_{G\setminus e} \cup [Bz]_G$ such that $d_G(\alpha, \beta) \le d_{G\setminus e}(\alpha, \beta) \le 2\delta(G \setminus e)$; hence, we obtain $d_G(\alpha, [yx]_G \cup [xz]_G) \le 2\delta(G \setminus e) + \frac{1}{2} d_{G\setminus e}(A, B)$. If $\alpha \in [xy]_G \cup [xz]_G$, then without loss of generality we can assume that $\alpha \in [xy]_G$. If $\alpha \in [xA]_G \subset [xy]_G$, then we have $d_G(\alpha, [xz]_G \cup [yz]_G) \le d_G(\alpha, x) \le L(e)$. If $\alpha \in [Ay]_G \subset [xy]_G$, then let us consider again the geodesic quadrilateral T_4^*; hence, there exists $\beta \in [AB]_{G\setminus e} \cup [Bz]_G \cup [zy]_G$ such that $d_G(\alpha,\beta) \le d_{G\setminus e}(\alpha,\beta) \le 2\delta(G\setminus e)$, and we obtain $d_G(\alpha,[yz]_G \cup [zx]_G) \le 2\delta(G\setminus e) + d_{G\setminus e}(A,B)$. Hence,

$$\delta(T_G) \le \max\big\{2\delta(G\setminus e) + d_{G\setminus e}(A,B), L(e)\big\}.$$

$\square$

Finally, we can prove a kind of converse of Theorem 2.2.7.

Theorem 2.2.13. *Let G be any graph and $e = [A,B] \in E(G)$ with $G\setminus e$ connected. Then*

$$\max\Big\{\frac{1}{5}\delta(G\setminus e), \frac{1}{4}d_{G\setminus e}(A,B), \frac{1}{4}L(e)\Big\} \le \delta(G) \le \max\big\{6\delta(G\setminus e) + d_{G\setminus e}(A,B), L(e)\big\}. \quad (2.11)$$

Proof. Theorem 2.2.7 gives $\delta(G\setminus e)/5 \le \delta(G)$. If $d_{G\setminus e}(A,B) \le L(e)$, then let C_1 be the midpoint of e and w_1 the midpoint of $[AC_1]_G$; since $T_1 = \{A,B,C_1\}$ is a geodesic triangle in G, we have $\delta(G) \ge \delta(T_1) \ge d_G(w_1,[AB]_G \cup [BC_1]_G) = L(e)/4$. If $d_{G\setminus e}(A,B) \ge L(e)$, then let C_2 be the midpoint of a geodesic $[AB]_{G\setminus e}$ and w_2 the midpoint of $[AC_2]_{G\setminus e} \subset [AB]_{G\setminus e}$ (note that $[AC_2]_{G\setminus e}$ is a geodesic in G also); since $T_2 = \{A,B,C_2\}$ is a geodesic triangle in G, we have $\delta(G) \ge \delta(T_2) \ge d_G(w_2, e \cup [BC_2]_G) = d_{G\setminus e}(A,B)/4$. These facts prove the lower bound for $\delta(G)$.

In order to prove the second inequality, let us consider a geodesic triangle T_G in G. By Lemma 2.2.12 we can assume that every vertex of T_G is contained in $G\setminus e$. By Lemma 2.2.6 at most two geodesics sides of T_G contain e. If $T_G \subseteq G\setminus e$, then Lemma 2.2.8 gives the result. If just one geodesic side of T_G contains e, then it suffices to apply Lemma 2.2.11. If two geodesics sides of T_G contain e, then we can split T_G in the union of e, a geodesic bigon in $G\setminus e$ and a geodesic triangle in $G\setminus e$, and Lemma 2.2.8 finishes the proof. $\square$

We have the following direct consequences.

Corollary 2.2.14. *Let G be any graph and $e = [A,B] \in E(G)$ with $G\setminus e$ connected. Then*

$$\frac{1}{5}\max\big\{\delta(G\setminus e), d_{G\setminus e}(A,B), L(e)\big\} \le \delta(G) \le 12\max\big\{\delta(G\setminus e), d_{G\setminus e}(A,B), L(e)\big\}.$$

Corollary 2.2.15. *Let G be any graph and $e = [A,B] \in E(G)$ such that $G\setminus e$ is connected and $L(e) \le d_{G\setminus e}(A,B)$. Then*

$$\delta(G) \le 6\delta(G\setminus e) + d_{G\setminus e}(A,B). \quad (2.12)$$

2.3 Hyperbolic S-graphs

Using the previous results, we prove in this section that local hyperbolicity guarantees the hyperbolicity of any graph, in a quantitative way. In order to do that we need to introduce the concept of S-graph.

Definition 2.3.1. *Let us consider a graph G_0 with $E(G_0) = \{[a_n, b_n]\}_{n\geq 1}$, and a family of graphs $\{G_n\}_{n\geq 1}$ such that for all $n \geq 1$ there exist $a'_n, b'_n \in V(G_n)$ such that $d_{G_n}(a'_n, b'_n) = L_{G_0}([a_n, b_n])$. We define the* S-graph *G associated to $\{G_n\}_{n\geq 0}$ as follows; we replace each edge $[a_n, b_n] \in E(G_0)$ by the whole graph G_n in the following way: a_n and b_n are substituted, respectively, by a'_n and b'_n, for each $n \geq 1$.*

A very simple example of S-graph is the following: Let G be any graph with at least two connection vertices v, w (recall that a connection vertex is a vertex whose removal renders G disconnected). We denote by G_1, G_2, G_3, the closures in G of the connected components of the metric graph G minus the points $\{v, w\}$. Without loss of generality we can assume that $v \in G_1$, $v, w \in G_2$ and $w \in G_3$. If $\alpha \neq v$ is a vertex of G_1 and $\beta \neq w$ is a vertex of G_3, we define G_0 as the graph with $V(G_0) = \{\alpha, v, w, \beta\}$, $E(G_0) = \{[\alpha, v], [v, w], [w, \beta]\}$ and $L([\alpha, v]) = d_G(\alpha, v), L([v, w]) = d_G(v, w), L([w, \beta]) = d_G(w, \beta)$. Then G is the S-graph associated to $\{G_0, G_1, G_2, G_3\}$.

The previous example shows that we can see graphs as S-graphs. Besides, Figure 2.3 show another example of a particular S-graph G associated to $G_0 := K_4 \setminus e$ with edges of length 2 and $\{G_n\}_{n=1}^5 = \{C_6, K_{3,3}, K_4 \setminus e, \overline{C_6}, C_5 \cup e\}$ all with edges of length 1.

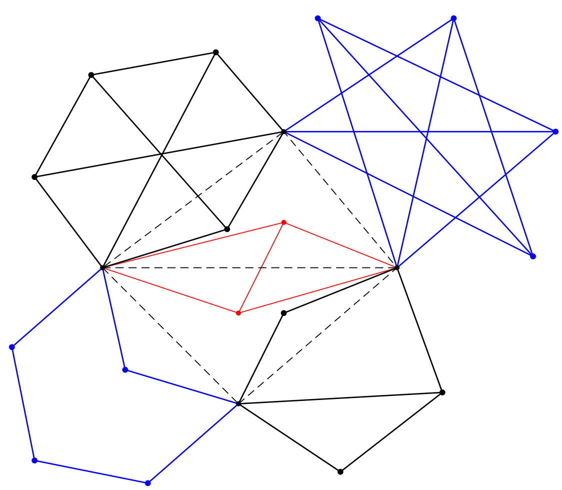


Figure 2.2: S-graph associated to $\{K_4 \setminus e, C_6, K_4 \setminus e, K_{3,3}, C_5 \cup e, \overline{C_6}\}$.

Theorem 2.3.2. *Let G be the S-graph associated to $\{G_n\}_{n\geq 0}$. Then, G is hyperbolic if and only if $\{G_n\}_{n\geq 0}$ are hyperbolic with the same hyperbolicity constant. Furthermore,*

$$\frac{1}{5}\sup_{n\geq 0}\delta(G_n) \leq \delta(G) \leq 11\sup_{n\geq 0}\delta(G_n).$$

Proof. Assume first that G is hyperbolic. For each $n \geq 1$, let us denote by $[a'_n b'_n]_{G_n}$ a geodesic in G_n, and define G^* as the subgraph of G given by $G^* = \cup_{n\geq 1}[a'_n b'_n]_{G_n}$. Note that G^* and G_0 are isometric. We have that G^* is an isometric subgraph of G and Lemma 1.3.2 gives $\delta(G_0) = \delta(G^*) \leq \delta(G)$. In what follows we identify G^* and G_0. For each $n \geq 1$, if $G \setminus G_n$ is connected, let us consider a geodesic α_n in $G \setminus G_n$ joining a'_n and b'_n; if $G \setminus G_n$ is not connected, we define $\alpha_n = \varnothing$; then $G_n \cup \alpha_n$ is an isometric subgraph of G. Therefore, by Theorem 2.2.7 and Lemma 1.3.2, we have that $\delta(G_n) \leq 5\delta(G_n \cup \alpha_n) \leq 5\delta(G)$. Hence, G_n is $5\delta(G)$-hyperbolic for every $n \geq 0$.

Assume now that G_n is δ-hyperbolic for every $n \geq 0$. Let us consider any fixed geodesic triangle $T = \{x, y, z\}$ in G; by Lemma 1.3.4 we can assume that T is a cycle.

If x, y, z belong to different subgraphs G_s, G_r, G_t, respectively, then let us consider the three geodesic triangles $T_s = \{x, a'_s, b'_s\}$, $T_r = \{y, a'_r, b'_r\}$ and $T_t = \{z, a'_t, b'_t\}$ in G_s, G_r and G_t, respectively, and their tripods (see Definition 1.2.4). Let P_x (respectively, P_y, P_z) be the internal point of T_s in $[a'_s b'_s]$ (respectively, T_r in $[a'_r b'_r]$, T_t in $[a'_t b'_t]$).

Since T is a cycle and we are identifying G^* and G_0, without loss of generality we can assume that

$$[xy] = [xb_s]_{G_s} \cup [b_s a_r]_{G_0} \cup [a_r y]_{G_r},$$
$$[yz] = [yb_r]_{G_r} \cup [b_r a_t]_{G_0} \cup [a_t z]_{G_t}$$

and

$$[zx] = [zb_t]_{G_t} \cup [b_t a_s]_{G_0} \cup [a_s x]_{G_s}.$$

We are going to prove that $[P_x b_s]_{G_0} \cup [b_s a_r]_{G_0} \cup [a_r P_y]_{G_0}$ is a geodesic in G_0. Let $[P_x P_y]_{G_0} = [P_x c_s]_{G_0} \cup [c_s c_r]_{G_0} \cup [c_r P_y]_{G_0}$ be a geodesic in G_0 joining P_x and P_y, where $c_s \in \{a_s, b_s\}$ and $c_r \in \{a_r, b_r\}$. Seeking for a contradiction, assume that $L([P_x P_y]_{G_0}) < L([P_x b_s]_{G_0}) + L([b_s a_r]_{G_0}) + L([a_r P_y]_{G_0})$. Denote by P_{a_s} and P_{b_s} the internal points of T_s in $[xa'_s]_{G_s}$ and $[xb'_s]_{G_s}$, respectively; denote by P_{a_r} and P_{b_r} the internal points of T_r in $[ya'_r]_{G_s}$ and $[yb'_r]_{G_s}$, respectively. Then

$$\begin{aligned}
L([P_x c_s]_{G_s}) + L([c_s c_r]_{G_0}) + L([c_r P_y]_{G_r}) &< L([P_x b_s]_{G_s}) + L([b_s a_r]_{G_0}) + L([a_r P_y]_{G_r}),\\
L([P_{c_s} c_s]_{G_s}) + L([c_s c_r]_{G_0}) + L([c_r P_{c_r}]_{G_r}) &< L([P_{b_s} b_s]_{G_s}) + L([b_s a_r]_{G_0}) + L([a_r P_{a_r}]_{G_r}),\\
d_G(x, y) \leq L([x P_{c_s}]_{G_s}) + L([P_{c_s} c_s]_{G_s}) &+ L([c_s c_r]_{G_0}) + L([c_r P_{c_r}]_{G_r}) + L([P_{c_r} y]_{G_r}) <\\
< L([x P_{b_s}]_{G_s}) + L([P_{b_s} b_s]_{G_s}) &+ L([b_s a_r]_{G_0}) + L([a_r P_{a_r}]_{G_r}) + L([P_{a_r} y]_{G_r}) =\\
&= L([xy]) = d_G(x, y),
\end{aligned}$$

which is a contradiction. Then, we have that $[P_x b_s]_{G_0} \cup [b_s a_r]_{G_0} \cup [a_r P_y]_{G_0}$ is a geodesic in G_0 joining P_x and P_y. A similar argument proves that $[P_y b_r]_{G_0} \cup [b_r a_t]_{G_0} \cup [a_t P_z]_{G_0}$ and $[P_z b_t]_{G_0} \cup [b_t a_s]_{G_0} \cup [a_s P_x]_{G_0}$ are also geodesics in G_0. Now, let us consider the geodesic triangle $T_0 = \{P_x, P_y, P_z\}$ in G_0 with these geodesics.

Let us consider any $\alpha \in T$. Without loss of generality we can assume that $\alpha \in [xy]$. If $\alpha \in [xb_s]_{G_s}$, then since T_s is δ-thin there exists $\alpha' \in [xa_s]_{G_s} \cup [a_s b_s]_{G_s}$ such that $d_{G_s}(\alpha, \alpha') \le \delta$. If $\alpha' \in [xa_s]_{G_s}$, then $\alpha' \in [xz]$. Assume now that $\alpha' \in [a_s b_s]_{G_s}$. If $\alpha' \in [a_s P_x]_{G_s}$, then there exists $\beta \in [xa_s]_{G_s} \subset [xz]$ such that $d_{G_s}(\alpha', \beta) \le 4\delta$ and $d_G(\alpha, \beta) \le 5\delta$. If $\alpha' \in [P_x b_s]_{G_s} \subset [P_x P_y]_{G_0}$ since T_0 is δ-thin, there exists $\beta' \in [P_y P_z]_{G_0} \cup [P_z P_x]_{G_0}$ such that $d_{G_0}(\alpha', \beta') \le \delta$. Then, β' belongs to $[b_r a_t]_{G_0} \cup [b_t a_s]_{G_0} \subset [yz]_G \cup [zx]_G$ or to one of the subgraphs T_s, T_r or T_t (if β' belongs to $[P_y b_r]_{G_r}, [a_t P_z]_{G_t}, [P_z b_t]_{G_t}, [a_s P_x]_{G_s}$) and there exists $\beta \in [yz]_G \cup [zx]_G$ such that $d_G(\beta', \beta) \le 4\delta$. Then, we obtain $d_G(\alpha, \beta) \le d_{G_s}(\alpha, \alpha') + d_{G_0}(\alpha', \beta') + d_G(\beta', \beta) \le 6\delta$. Note that, by symmetry, if $\alpha \in [a_r y]_{G_r}$ we have the same result. If $\alpha \in [b_s a_r]_{G_0}$, then since T_0 is δ-thin there exists $\beta' \in [P_y P_z]_{G_0} \cup [P_z P_x]_{G_0}$ such that $d_{G_0}(\alpha, \beta') \le \delta$. Then, β' belongs to $[b_r a_t]_{G_0} \cup [b_t a_s]_{G_0} \subset [yz]_G \cup [zx]_G$ or to one of the subgraphs T_s, T_r or T_t (if β' belongs to $[P_y b_r]_{G_r}, [a_t P_z]_{G_t}, [P_z b_t]_{G_t}, [a_s P_x]_{G_s}$) and there exists $\beta \in [yz]_G \cup [zx]_G$ such that $d_G(\beta', \beta) \le 4\delta$. Then, we obtain $d_G(\alpha, \beta) \le d_{G_0}(\alpha, \beta') + d_G(\beta', \beta) \le 5\delta$. Consequently, if x, y, z belong to different subgraphs, then

$$\delta(T) \le 6\delta.$$

If x, y belong to the same subgraph G_s and $z \in G_r$ with $s \neq r$, then consider two geodesic polygons $F_s = \{x, y, a_s, b_s\}$ and $T_r = \{z, a_r, b_r\}$ in G_s and G_r, respectively. Consider the tripod of T_r and a quatripod of F_s respectively, into the definition of *fine*. Let P'_x, P'_y, P'_z be the vertices with degree 3 in the quatripod and the tripod, respectively; let P_z be the point in $[a_r b_r]$ related with P'_z (the internal point), and $P_x, P_y \in [a_s b_s]$ related with P'_x, P'_y (note that it is possible to have $P_x = P_y$, in particular, if P'_x or P'_y is neighbor of the two vertices corresponding to a_s and b_s).

Without loss of generality we can assume that

$$[yz] = [yb_s]_{G_s} \cup [b_s a_r]_{G_0} \cup [a_r z]_{G_r},$$

$$[xz] = [xa_s]_{G_s} \cup [a_s b_r]_{G_0} \cup [b_r z]_{G_r}$$

and

$$[a_s b_s]_{G_s} = [a_s P_x]_{G_s} \cup [P_x P_y]_{G_s} \cup [P_y b_s]_{G_s}.$$

As in the previous case, it is possible to check that $[P_x a_s]_{G_0} \cup [a_s b_r]_{G_0} \cup [b_r P_z]_{G_0}$, $[P_z a_r]_{G_0} \cup [a_r b_s]_{G_0} \cup [b_s P_y]_{G_0}$ and $[P_x P_y]_{G_0}$ are geodesics in G_0. Let us consider the geodesic triangle $T_0 = \{P_x, P_y, P_z\}$ in G_0 with these geodesics.

Let us fix any $\alpha \in [xy]$; there exists $\alpha' \in [xa_s]_{G_s} \cup [a_s b_s]_{G_s} \cup [b_s y]_{G_s}$ such that $d_{G_s}(\alpha, \alpha') \le 2\delta$. If $\alpha' \in [xa_s]_{G_s} \cup [b_s y]_{G_s}$, then $\alpha' \in [xz] \cup [zy]$. If $\alpha' \in [a_s P_x]_{G_s} \cup [P_y b_s]_{G_s}$, then by definition of fine quatripod there exists $\beta' \in [xa_s]_{G_s} \cup [b_s y]_{G_s} \subset [xz] \cup [zy]$ such that $d_{G_s}(\alpha', \beta') \le 8\delta$.

If $P_x \neq P_y$ and $\alpha' \in [P_xP_y]_{G_s}$, then since T_0 is δ-thin there exists $\beta' \in [P_yP_z]_{G_0} \cup [P_zP_x]_{G_0}$ such that $d_{G_0}(\alpha', \beta') \leq \delta$; then, $\beta' \in [b_sa_r]_{G_0} \cup [b_ra_s]_{G_0} \subset [yz]_G \cup [zx]_G$ or since F_s is 8δ-fine and T_r is 4δ-fine there exists $\beta \in [yz]_G \cup [zx]_G$ such that $d_{G_i}(\beta', \beta) \leq 8\delta$ with $i \in \{r, s\}$. Therefore, we conclude $d_G(\alpha, \beta) \leq 11\delta$.

Let us fix now any $\alpha \in [xz] \cup [yz]$; without loss of generality we can assume that $\alpha \in [yz]$.

Assume first that $\alpha \in [yb_s]_{G_s}$; then since F_s is 2δ-thin there exists $\alpha' \in [xy]_G \cup [xa_s]_{G_s} \cup [a_sb_s]_{G_s}$ such that $d_{G_s}(\alpha, \alpha') \leq 2\delta$. If $\alpha' \in [xy] \cup [xa_s]_{G_s}$, then $\alpha' \in [xy] \cup [xz]$. Assume now that $\alpha' \in [a_sb_s]_{G_s}$. If $\alpha' \in [a_sP_x]_{G_s}$, then since F_s is 8δ-fine there exists $\beta' \in [xa_s]_{G_s} \subset [xz]$ such that $d_{G_s}(\alpha', \beta') \leq 8\delta$ and $d_G(\alpha, \beta') \leq 10\delta$. If $P_x \neq P_y$ and $\alpha' \in [P_xP_y]_{G_s}$, then there exists $\beta' \in [xy]_G$ such that $d_{G_s}(\alpha', \beta') \leq 8\delta$ and $d_G(\alpha, \beta') \leq 10\delta$. If $\alpha' \in [P_yb_s]_{G_s} \subset [P_yP_z]_{G_0}$, then since T_0 is δ-thin, there exists $\beta' \in [P_zP_x]_{G_0} \cup [P_xP_y]_{G_0}$ such that $d_{G_0}(\alpha', \beta') \leq \delta$. If $\beta' \in [P_xP_y]_{G_s}$, then since F_s is 8δ-fine there exists $\beta \in [xy]_G$ such that $d_{G_s}(\beta', \beta) \leq 8\delta$ and $d_G(\alpha, \beta) \leq 11\delta$. Assume that $\beta' \in [P_zP_x]_{G_0}$; if $\beta' \in [P_zb_r]_{G_r} \cup [P_xa_s]_{G_s}$, then since F_s is 8δ-fine and T_r is 4δ-fine there exists $\beta \in [zx]$ such that $d_{G_i}(\beta', \beta) \leq 8\delta$ with $i \in \{r, s\}$ and therefore $d_G(\alpha, \beta) \leq 11\delta$; otherwise, $\beta' \in [b_ra_s]_{G_0} \subset [zx]$ and $d_G(\alpha, \beta') \leq 3\delta$.

Assume that $\alpha \in [b_sa_r]_{G_0} \subset [P_yP_z]_{G_0}$. Since T_0 is δ-thin there exists $\alpha' \in [P_zP_x]_{G_0} \cup [P_xP_y]_{G_0}$ such that $d_{G_0}(\alpha, \alpha') \leq \delta$; using the previous arguments for $\alpha' \in [P_yb_s]_{G_s}$, we obtain that there exists $\beta \in [xy] \cup [xz]$ such that $d_G(\alpha, \beta) \leq 9\delta$.

Assume that $\alpha \in [a_rz]_{G_r}$; then since T_r is δ-thin there exists $\alpha' \in [zb_r]_{G_r} \cup [b_ra_r]_{G_r}$ such that $d_{G_r}(\alpha, \alpha') \leq \delta$. If $\alpha' \in [zb_r]_{G_r}$, then $\alpha' \in [zx]$. If $\alpha' \in [b_rP_z]_{G_r}$, then since T_r is 4δ-fine there exists $\beta' \in [zb_r]_{G_r} \subset [zx]$ such that $d_{G_r}(\alpha', \beta') \leq 4\delta$ and $d_G(\alpha, \beta') \leq 5\delta$. If $\alpha' \in [P_za_r]_{G_r}$, then since T_0 is δ-thin there exists $\beta' \in [P_zP_x]_{G_0} \cup [P_xP_y]_{G_0}$ such that $d_{G_0}(\alpha', \beta) \leq \delta$; using the previous arguments for $\alpha' \in [P_yb_s]_{G_s}$, we obtain that there exists $\beta \in [xy] \cup [xz]$ such that $d_G(\alpha, \beta) \leq 10\delta$.

Consequently, if x, y belong to the same subgraph G_s and $z \in G_r$ with $s \neq r$, then

$$\delta(T) \leq 11\delta.$$

Finally, assume that x, y, z belong to the same subgraph G_s. If T is contained in G_s, then $\delta(T) \leq \delta(G_s) \leq \delta$. Assume that T is not contained in G_s; then $e = [a_s, b_s] \in E(G_0)$, $G_0 \setminus e$ is connected and $L(e) \leq d_{G_s}(a'_s, b'_s)$. Hence, T is contained in $G_s \cup \alpha_s$, where α_s is a geodesic in $G_0 \setminus e$ joining a_s and b_s. Corollary 2.2.15 gives

$$\delta(T) \leq \delta(G_s \cup \alpha_s) \leq 6\delta(G_s) + d_{G_s}(a'_s, b'_s) \leq 6\delta + d_{G_s}(a'_s, b'_s).$$

Note that $[a_s, b_s] \cup \alpha_s$ is an isometric cycle in G_0; therefore,

$$\frac{1}{4} d_{G_s}(a'_s, b'_s) = \frac{1}{4} L([a_s, b_s]) \leq \frac{1}{4} L([a_s, b_s] \cup \alpha_s) \leq \delta([a_s, b_s] \cup \alpha_s) \leq \delta(G_0) \leq \delta.$$

Consequently, if x, y, z belong to the same subgraph, then $\delta(T) \leq 10\delta$.

Finally, we obtain that G is hyperbolic with $\delta(G) \leq 11\delta$. $\square$

Chapter 3

Hyperbolicity of line graph with edges of length k.

In this Chapter we study the line graphs with constant length of edges. Line graphs were initially introduced in the papers [80] and [56], although the terminology of line graph was used in [42] for the first time. They are an active topic of research at this moment.

Definition 3.0.3. *Let G be a graph with edges $E(G) = \{e_i\}_{i\in\mathcal{I}}$. The* line graph $\mathcal{L}(G)$ *of G is a graph which has a vertex $V_{e_i} \in V(\mathcal{L}(G))$ for each edge e_i of G, and an edge joining V_{e_i} and V_{e_j} when $e_i \cap e_j \neq \varnothing$.*

Some authors define the edges of line graph with length 1 or another fixed constant, but we define the length of the edge $[V_{e_i}, V_{e_j}] \in E(\mathcal{L}(G))$ as $(L(e_i)+L(e_j))/2$. Obviously, if every edge of G has length k for some constant k, then every edge of $\mathcal{L}(G)$ has length k.

One of the main aim of this Chapter is to obtain information about the hyperbolicity constant of the line graph $\mathcal{L}(G)$ in terms of properties of the graph G.

In particular, we prove qualitative results as the following: a graph G is hyperbolic if and only if $\mathcal{L}(G)$ is hyperbolic (see Theorem 3.1.1); if $\{G_n\}$ is a T-decomposition of G, then the line graph $\mathcal{L}(G)$ is hyperbolic if and only if $\sup_n \delta(\mathcal{L}(G_n))$ is finite (see Theorem 3.3.7).

Besides, we obtain quantitative results. Some of them are quantitative versions of our qualitative results:

$$\frac{1}{12}\,\delta(G) - \frac{3k}{4} \leq \delta(\mathcal{L}(G)) \leq 12\,\delta(G) + 18k, \tag{3.1}$$

for graphs G with edges of length k (see Theorem 3.1.3); and, if $\{G_n\}_n$ is any T-decomposition of any graph G, then

$$\sup_n \delta(\mathcal{L}(G_n)) \leq \delta(\mathcal{L}(G)) \leq \sup_n \delta(\mathcal{L}(G_n)) + k$$

(see Theorem 3.3.7).

We also prove (see Theorem 3.3.12) that

$$\frac{g(G)}{4} \leq \delta(\mathcal{L}(G)) \leq \frac{c(G)}{4} + 2k,$$

where $g(G)$ is the girth of G (the infimum of the lengths of the cycles in G) and $c(G)$ its circumference (the supremum of the lengths of its cycles). We show that

$$\delta(\mathcal{L}(G)) \geq \frac{1}{4} \sup\{L(g) : \, g \, \text{ is an isometric cycle in } \, G\}$$

(see Theorem 3.2.2).

Furthermore, we characterize the graphs G with $\delta(\mathcal{L}(G)) < k$ (see Theorem 3.3.6).

If G is any graph with $\delta(\mathcal{L}(G)) < k$, then there are just two possibilities: $\delta(\mathcal{L}(G)) = 0$ or $\delta(\mathcal{L}(G)) = 3k/4$. Furthermore,

- $\delta(\mathcal{L}(G)) = 0$ if and only if G is a tree with maximum degree $\Delta \leq 2$,
- $\delta(\mathcal{L}(G)) = 3k/4$ if and only if G is either a tree with maximum degree $\Delta = 3$ or isomorphic to C_3.

3.1 Hyperbolicity of Line Graphs.

Using the *Invariance of hyperbolicity* (see Theorem 1.3.5), we can obtain the main qualitative aim in this section.

Theorem 3.1.1. *There exists a $(k/2)$-full $(1,k)$-quasi-isometry from G on its line graph $\mathcal{L}(G)$ and, consequently, G is hyperbolic if and only if $\mathcal{L}(G)$ is hyperbolic.*

Furthermore, if G (respectively, $\mathcal{L}(G)$) is δ-hyperbolic, then $\mathcal{L}(G)$ (respectively, G) is δ'-hyperbolic, where δ' is a constant which just depends on δ and k.

Proof. By Theorem 1.3.5, it suffices to find a $(k/2)$-full $(1,k)$-quasi-isometry $f : G \to \mathcal{L}(G)$. If $e \in E(G)$, we denote by $p(e)$ its corresponding vertex in $V(\mathcal{L}(G))$. We define a function $f : G \to \mathcal{L}(G)$ in the following way: if x belongs to the interior of some $e \in E(G)$, then $f(x) := p(e)$; if $x \in V(G)$, let us choose some edge $e \in E(G)$ starting in x and then $f(x) := p(e)$. Since $f(G) = V(\mathcal{L}(G))$, we deduce that f is $(k/2)$-full.

Fix $x, y \in G$. If $f(x) = f(y)$, then $d_G(x,y) \leq k = d_{\mathcal{L}(G)}(f(x), f(y)) + k$. Let us assume that $d_{\mathcal{L}(G)}(f(x), f(y)) = km$, with $m = 1$. Then there exist vertices $w_0 = f(x), w_1, \dots, w_{m-1}$, $w_m = f(y) \in V(\mathcal{L}(G))$ and a geodesic $\gamma := [f(x), w_1] \cup [w_1, w_2] \cup \cdots \cup [w_{m-1}, f(y)]$ in $\mathcal{L}(G)$ joining $f(x)$ and $f(y)$. Therefore, there exist vertices $v_0, v_1, \dots, v_{m+1} \in V(G)$, such that $x \in [v_0, v_1]$, $y \in [v_m, v_{m+1}]$ and $p([v_j, v_{j+1}]) = w_j$ for $j = 0, 1, \dots, m$. Then $d_G(x,y) \leq k(m+1) = d_{\mathcal{L}(G)}(f(x), f(y)) + k$.

Let us consider now $x, y \in G \setminus V(G)$ and a geodesic $\eta := [xu_0] \cup [u_0, u_1] \cup \cdots \cup [u_{r-1}, u_r] \cup [u_r y]$ joining them in G. Then $d_G(x, y) \geq kr$ and the path in $\mathcal{L}(G)$ given by the $r+2$ vertices $f(x), p([u_0, u_1]), \ldots, p([u_{r-1}, u_r]), f(y)$ joins $f(x)$ and $f(y)$ in $\mathcal{L}(G)$. Consequently, $d_{\mathcal{L}(G)}(f(x), f(y)) \leq k(r+1) \leq d_G(x, y) + k$. If we consider now the cases $x \in V(G)$ or $y \in V(G)$, a similar argument also gives $d_{\mathcal{L}(G)}(f(x), f(y)) \leq d_G(x, y) + k$. □

We have obtained also a quantitative version (with explicit constants) for the hyperbolicity constants on Theorem 3.1.1. First recall the Gromov product (see Definition 1.2.1). Let us denote by $\delta^*(G)$ the sharp constant for this inequality, i.e.

$$\delta^*(G) := \sup \big\{ \min \big\{(x, y)_w, (y, z)_w\big\} - (x, z)_w : \, x, y, z, w \in G \big\}.$$

Theorem 3.1.2. *For any graph G we have*

$$\delta^*(G) - 3k \leq \delta^*(\mathcal{L}(G)) \leq \delta^*(G) + 6k.$$

Proof. Recall that we have seen in the proof of Theorem 3.1.1 that there exists a $(k/2)$-full $(1, k)$-quasi-isometry $f : G \to \mathcal{L}(G)$.

Consequently, if $\delta^*(G) = \infty$, then $\delta^*(\mathcal{L}(G)) = \infty$ and the inequalities trivially hold.

Furthermore, if $\delta^*(G) < \infty$, then $\delta^*(\mathcal{L}(G)) < \infty$.

It is not difficult to check that

$$(x, y)_w - \frac{3k}{2} \leq (f(x), f(y))_{f(w)} \leq (x, y)_w + \frac{3k}{2}$$

for every $x, y, w \in G$. Then, we deduce for every $x, y, z, w \in G$,

$$\begin{aligned} (x, z)_w &\geq (f(x), f(z))_{f(w)} - \frac{3k}{2} \geq \min \big\{(f(x), f(y))_{f(w)}, (f(y), f(z))_{f(w)}\big\} - \delta^*(\mathcal{L}(G)) - \frac{3k}{2} \\ &\geq \min \big\{(x, y)_w, (y, z)_w\big\} - \delta^*(\mathcal{L}(G)) - 3k. \end{aligned}$$

Then $\delta^*(G) \leq \delta^*(\mathcal{L}(G)) + 3k$.

Furthermore, given $x', y', z', w' \in \mathcal{L}(G)$ there exist $x, y, z, w \in G$ with $d_{\mathcal{L}(G)}(x', f(x)) \leq k/2$, $d_{\mathcal{L}(G)}(y', f(y)) \leq k/2$, $d_{\mathcal{L}(G)}(z', f(z)) \leq k/2$ and $d_{\mathcal{L}(G)}(w', f(w)) \leq k/2$. It is not difficult to check that

$$(x, y)_w - 3k \leq (x', y')_{w'} \leq (x, y)_w + 3k.$$

Then

$$\begin{aligned} (x', z')_{w'} &\geq (x, z)_w - 3k \geq \min \big\{(x, y)_w, (y, z)_w\big\} - \delta^*(G) - 3k \\ &\geq \min \big\{(x', y')_{w'}, (y', z')_{w'}\big\} - \delta^*(G) - 6k. \end{aligned}$$

Then $\delta^*(\mathcal{L}(G)) \leq \delta^*(G) + 6k$. □

We deduce directly the following result.

Theorem 3.1.3. *For any graph G we have*

$$\frac{1}{12}\,\delta(G) - \frac{3k}{4} \leq \delta(\mathcal{L}(G)) \leq 12\,\delta(G) + 18k.$$

Proof. Using the inequalities in Theorem 1.2.3 relating $\delta^*(G)$ and $\delta(G)$, and Theorem 3.1.2, we conclude

$$\delta(G) \leq 3\,\delta^*(G) \leq 3\,(\delta^*(\mathcal{L}(G)) + 3k) \leq 12\,\delta(\mathcal{L}(G)) + 9k,$$
$$\delta(\mathcal{L}(G)) \leq 3\,\delta^*(\mathcal{L}(G)) \leq 3\,(\delta^*(G) + 6k) \leq 12\,\delta(G) + 18k.$$

□

3.2 Inequalities involving the hyperbolicity constant of line graphs.

The following result is consequence of Lemma 1.3.8.

Theorem 3.2.1. *For any graph G we have $\delta(\mathcal{L}(G)) \leq \frac{1}{2}\operatorname{diam} V(G) + k$, and this inequality is sharp.*

Proof. By Lemma 1.3.8, we have $\delta(\mathcal{L}(G)) \leq \frac{1}{2}\operatorname{diam}\mathcal{L}(G)$. Since $\operatorname{diam}\mathcal{L}(G) \leq \operatorname{diam} V(\mathcal{L}(G))+$ k and $\operatorname{diam} V(\mathcal{L}(G)) \leq \operatorname{diam} V(G) + k$, we conclude that $\delta(\mathcal{L}(G)) \leq \frac{1}{2}\operatorname{diam} V(G) + k$. Note that the equality is attained in the complete graph $G = K_6$. □

Theorem 3.2.2. *We have for any graph G*

$$\delta(\mathcal{L}(G)) \geq \frac{1}{4}\sup\{L(g) : g \textit{ is an isometric cycle in } G\}.$$

Proof. First of all, we prove that if C is any isometric cycle of a graph G, then $p(C)$ is an isometric cycle of the line graph $\mathcal{L}(G)$.

Seeking for a contradiction, assume that $p(C)$ is not an isometric cycle of the line graph $\mathcal{L}(G)$. Then there exist two edges $e_1, e_2 \in E(G)$ of C such that $d_{p(C)}(p(e_1), p(e_2)) = r$ and $d_{\mathcal{L}(G)}(p(e_1), p(e_2)) = k \leq r - 1$.

Since $d_{p(C)}(p(e_1), p(e_2)) = r$, we deduce that $d_C(e_1, e_2) = d_G(e_1, e_2) = r - 1$.

Since $d_{\mathcal{L}(G)}(p(e_1), p(e_2)) = k \leq r - 1$, there exist edges $a_1, a_2, \dots, a_{k-1} \in E(G)$ such that $[p(e_1), p(a_1)] \cup [p(a_1), p(a_2)] \cup \cdots \cup [p(a_{k-1}), p(e_2)]$ is a geodesic joining $p(e_1)$ and $p(e_2)$ in $\mathcal{L}(G)$. Therefore, $a_1 \cup a_2 \cup \cdots \cup a_{k-1}$ is a path joining e_1 and e_2 in G; this implies that $d_G(e_1, e_2) \leq k - 1 \leq r - 2 < r - 1 = d_G(e_1, e_2)$, which is the contradiction we were looking for.

Now, Corollary 1.3.4 gives the result. □

Given any graph G we define, as usual, its *girth* $g(G)$ as the infimum of the lengths of the cycles in G. The following result (see [58, Theorem 17]) relates the girth of a graph and its hyperbolicity constant.

Lemma 3.2.3. *For any graph G*

$$\delta(G) \geq \frac{g(G)}{4},$$

and the inequality is sharp.

Remark 3.2.4. *One can think that the equality $\delta(G) = g(G)/4$ holds if and only if every cycle g in G verifies $L(g) = g(G)$. However, this is false, as shows the following example.*

Let us consider a graph G obtained from a cycle graph C_6 (with edges of length 1) by attaching three edges joining antipodal vertices. It is not difficult to check that $\operatorname{diam} V(G) = 2$, $\operatorname{diam} G = 2$, $\delta(G) = 1$, $g(G) = 4$, *and there exists a cycle with length* 6.

Proposition 3.2.5 below gives a similar upper bound for $\delta(G)$. Let us define the *circumference* $c(G)$ of a graph G as the supremum of the lengths of its cycles.

The following result is a consequence of Corollary 1.3.4.

Proposition 3.2.5. *For any graph G*

$$\delta(G) \leq \frac{1}{4}\, c(G),$$

and this inequality is sharp.

Proof. Let us consider any fixed geodesic triangle $T = \{\gamma_1, \gamma_2, \gamma_3\}$ in G and $p \in T$. Without loss of generality we can assume that $p \in \gamma_1 = [xy]$. By Corollary 1.3.4 we can assume that T is a cycle. Since $L(T) \leq c(G)$, then $L(\gamma_1) \leq c(G)/2$ and $d(p, \gamma_2 \cup \gamma_3) \leq d(p, \{x, y\}) \leq L(\gamma_1)/2 \leq c(G)/4$. Consequently, $\delta(G) \leq c(G)/4$.

The equality is attained in the cycle graph with $n \geq 3$ vertices. □

Proposition 3.2.6. *For any graph G which is not a tree, we have*

$$\delta(G) \geq \frac{g(\mathcal{L}(G))}{4}.$$

Proof. Since G is not a tree, we know that there is at least a cycle in G. By Lemma 3.2.3, we have $\delta(G) \geq g(G)/4$. Then it suffices to note that $g(\mathcal{L}(G)) \leq g(G)$, since for every cycle in G we have a cycle in $\mathcal{L}(G)$ with the same length. □

The following result, which is a consequence of Theorem 3.2.2, is a dual version of Proposition 3.2.6.

Corollary 3.2.7. *For any graph G, we have*

$$\delta(\mathcal{L}(G)) \geq \frac{g(G)}{4}.$$

The inequality in Corollary 3.2.7 is attained in the cycle graphs with $n \geq 3$ vertices.

A *matching* in a finite graph G is a set of edges pairwise non adjacent. An *independent set* in a finite graph G is a set of vertices pairwise non adjacent. We denote by $M(G)$ (respectively, $I(G)$) the maximum of the cardinal of matching (respectively, independent) sets in G.

Theorem 3.2.8. *For any finite graph G, we have $\delta(\mathcal{L}(G)) \leq M(G)$.*

Proof. Note that $M(G) = I(\mathcal{L}(G))$. It is not difficult to check that $2I(\mathcal{L}(G)) \geq \operatorname{diam} V(\mathcal{L}(G)) + 1 \geq \operatorname{diam} \mathcal{L}(G)$. Then, Lemma 1.3.8 gives $2\delta(\mathcal{L}(G)) \leq 2M(G)$. $\square$

Let G be any graph. We define

$$\sigma_2(G) := \min\{\deg_G(x) + \deg_G(y) : \ x, y \in V(G),\ d_G(x, y) \geq 2k\}.$$

In [61] we find the following result.

Lemma 3.2.9. *Let G be any graph with $\sigma_2(\mathcal{L}(G)) \geq 7$. Suppose that, for some $r \geq 3$, $\mathcal{L}(G)$ has an r-cycle C but no $(r-1)$-cycle. Then C is an isometric subgraph of $\mathcal{L}(G)$.*

Proposition 3.2.10. *Let G be any graph with $\deg_G(x) + \deg_G(y) \geq 6$ for every $[x, y] \in E(G)$. Suppose that, for some $r \geq 3$, $\mathcal{L}(G)$ has an r-cycle C but no $(r-1)$-cycle. Then $\delta(\mathcal{L}(G)) \geq rk/4$.*

Proof. Note that if $[x, y] \in E(G)$ then $\deg_{\mathcal{L}(G)}(p([x, y])) = \deg_G(x) + \deg_G(y) - 2 \geq 4$. Hence, $\sigma_2(\mathcal{L}(G)) \geq 8$. Therefore Lemma 3.2.9 gives that C is an isometric subgraph of $\mathcal{L}(G)$, and then Lemma 1.3.2 and Theorem 1.3.19 give that $\delta(\mathcal{L}(G)) \geq \delta(C) = L(C)/4 = rk/4$. $\square$

We deduce the following direct consequence.

Corollary 3.2.11. *Let G be any graph with $\deg_G(x) \geq 3$ for every $x \in V(G)$. Suppose that, for some $r \geq 3$, $\mathcal{L}(G)$ has an r-cycle but no $(r-1)$-cycle. Then $\delta(\mathcal{L}(G)) \geq rk/4$.*

3.3 T-decompositions and T-edge-decompositions.

We have a similar result to Theorem 1.3.7 for $\{\mathcal{L}(G_n)\}_n$ if $\{G_n\}_n$ is a T-edge-decomposition of G.

Theorem 3.3.1. *If $\{G_n\}_n$ is any T-edge-decomposition of any graph G, then $\delta(\mathcal{L}(G)) = \sup_n \delta(\mathcal{L}(G_n))$.*

Proof. Note that if $\{G_n\}_n$ is a T-edge-decomposition of G, then $\{\mathcal{L}(G_n)\}_n$ is a T-decomposition of $\mathcal{L}(G)$. Then, Lemma 1.3.7 gives the result. $\square$

Proposition 3.3.2. *Let T be any tree with maximum degree Δ. Then*

$$\delta(\mathcal{L}(T)) = \begin{cases} k, & \text{if } \Delta \geq 4\,, \\ 3k/4, & \text{if } \Delta = 3\,, \\ 0, & \text{if } \Delta \leq 2\,. \end{cases}$$

Proof. The canonical T-decomposition $\{G_n\}_n$ of $\mathcal{L}(T)$ has an edge for each vertex $v \in V(T)$ with $\deg_T(v) = 2$ and a graph isomorphic to K_m for each vertex $v \in V(T)$ with $\deg_T(v) = m \geq 3$. Lemma 1.3.7 gives $\delta(\mathcal{L}(T)) = \sup_n \delta(G_n)$. Besides, [72, Theorem 11] gives

$$\delta(K_m) = \begin{cases} k, & \text{if } m \geq 4\,, \\ 3k/4, & \text{if } m = 3\,. \end{cases}$$

These facts give the result. □

From Proposition 1.3.14 and Theorem 1.3.15 we deduce the following results.

Lemma 3.3.3. *If G is any graph with a cycle g with length $L(g) \geq 3k$, then $\delta(G) \geq 3k/4$. If there exists a cycle g in G with length $L(g) \geq 4k$, then $\delta(G) \geq k$.*

Corollary 3.3.4. *Let G be any graph with maximum degree Δ. If $\Delta \geq 3$, then $\delta(\mathcal{L}(G)) \geq 3k/4$. If $\Delta \geq 4$, then $\delta(\mathcal{L}(G)) \geq k$.*

Corollary 3.3.5. *If G is any graph with a cycle g with length $L(g) \geq 3k$, then $\delta(\mathcal{L}(G)) \geq 3k/4$. If there exists a cycle g in G with length $L(g) \geq 4k$, then $\delta(\mathcal{L}(G)) \geq k$.*

In [55], the authors characterize the bridged graphs with edges of length 1 which have hyperbolicity constant 1, for a different definition of hyperbolicity constant.

An interesting question is how to characterize the graphs G with edges of length k and $\delta(\mathcal{L}(G)) = k$, but it seems very difficult to give a description of such graphs in a simple way. However, the following theorem allows to characterize the graphs with $\delta(\mathcal{L}(G)) < k$.

Theorem 3.3.6. *If G is any graph with $\delta(\mathcal{L}(G)) < k$, then there are just two possibilities: $\delta(\mathcal{L}(G)) = 0$ or $\delta(\mathcal{L}(G)) = 3k/4$. Furthermore,*

- *$\delta(\mathcal{L}(G)) = 0$ if and only if G is a tree with maximum degree $\Delta \leq 2$,*
- *$\delta(\mathcal{L}(G)) = 3k/4$ if and only if G is either a tree with maximum degree $\Delta = 3$ or isomorphic to C_3.*

Proof. First of all, Theorem 1.3.16 gives that if $\delta(\mathcal{L}(G)) < k$, then we have either $\delta(\mathcal{L}(G)) = 0$ or $\delta(\mathcal{L}(G)) = 3k/4$.

Proposition 3.3.2 gives that if G is a tree with maximum degree $\Delta \le 2$, then $\delta(\mathcal{L}(G)) = 0$.

It is well known that if $\delta(\mathcal{L}(G)) = 0$, then $\mathcal{L}(G)$ is a tree. Since every cycle in G corresponds with a cycle in $\mathcal{L}(G)$ with the same length, G is a tree. If a vertex of G has degree greater or equal than 3, then there is a cycle g in $\mathcal{L}(G)$ with length $L(g) \ge 3k$, and Lemma 3.3.3 gives that $\delta(G) \ge 3k/4$; then the maximum degree of G verifies $\Delta \le 2$.

If G is a tree with maximum degree $\Delta = 3$, then Proposition 3.3.2 gives that $\delta(\mathcal{L}(G)) = 3k/4$. If G is isomorphic to C_3, then $\mathcal{L}(G)$ is also isomorphic to C_3 and Theorem 1.3.19 gives $\delta(\mathcal{L}(G)) = 3k/4$.

If $\delta(\mathcal{L}(G)) = 3k/4$, then Lemma 3.3.3 gives that every cycle in $\mathcal{L}(G)$ has length $3k$. If a vertex of G has degree greater or equal than 4, then Corollary 3.3.4 gives that $\delta(\mathcal{L}(G)) \ge k$, which is a contradiction; then the maximum degree of G verifies $\Delta \le 3$. If G is a tree, then Proposition 3.3.2 gives $\Delta = 3$ and we have the result. If G has a cycle, then it has length $3k$ by Corollary 3.3.5; assume that G is not isomorphic to C_3; therefore, G contains a cycle isomorphic to C_3 with a vertex of degree at least 3; then $\mathcal{L}(G)$ contains a cycle with length at least $4k$, and Lemma 3.3.3 gives that $\delta(\mathcal{L}(G)) \ge k$, which is a contradiction; hence, G is isomorphic to C_3. □

If $\{G_n\}_n$ is a T-decomposition of G, $\{\mathcal{L}(G_n)\}_n$ is not (in general) a T-decomposition of $\mathcal{L}(G)$; however, it is possible to obtain information about $\delta(\mathcal{L}(G))$ from $\delta(\mathcal{L}(G_n))$.

Theorem 3.3.7. *If $\{G_n\}_n$ is any T-decomposition of any graph G, then*

$$\sup_n \delta(\mathcal{L}(G_n)) \le \delta(\mathcal{L}(G)) \le \sup_n \delta(\mathcal{L}(G_n)) + k\,.$$

Proof. First of all note that if a connection vertex v belongs to $G_{n_1}, G_{n_2}, \dots, G_{n_r}$, $\deg_{G_{n_j}}(v) = d_j$ for $j = 1, \dots, r$, and $\deg_G(v) = d = \sum_{j=1}^r d_j$, then the set of edges starting in v corresponds to a subgraph Γ of $\mathcal{L}(G)$ isomorphic to the complete graph K_d; furthermore, the subgraph $\Gamma \cap \mathcal{L}(G_{n_j})$ is isomorphic to K_{d_j} for $j = 1, \dots, r$. Hence, for each n, $\mathcal{L}(G_n)$ is an isometric subgraph of $\mathcal{L}(G)$; then Lemma 1.3.2 gives $\sup_n \delta(\mathcal{L}(G_n)) \le \delta(\mathcal{L}(G))$.

In order to prove the upper bound of $\delta(\mathcal{L}(G))$, let us consider any geodesic triangle $T = \{\gamma_1, \gamma_2, \gamma_3\}$ in $\mathcal{L}(G)$ and $p \in T$. By Corollary 1.3.4 we can assume that T is a cycle and that each vertex of T is either a vertex in $V(\mathcal{L}(G))$ or a midpoint of some edge in $E(\mathcal{L}(G))$.

Without loss of generality we can assume that $p \in \gamma_1$. Assume first that $p \in \mathcal{L}(G_m)$ for some fixed m.

Since $\mathcal{L}(G_m)$ is an isometric subgraph of $\mathcal{L}(G)$, $\gamma_j^m := \gamma_j \cap \mathcal{L}(G_m)$ is a (connected) geodesic in $\mathcal{L}(G_m)$ for $j = 1, 2, 3$. We are going to construct a geodesic triangle T_m in $\mathcal{L}(G_m)$ containing $\gamma_1^m, \gamma_2^m, \gamma_3^m$. Note that $p \in \gamma_1^m$. We also have that $(\gamma_2^m \cup \gamma_3^m) \cap \mathcal{L}(G_m) \neq \varnothing$ since $\{G_n\}_n$ is a T-decomposition in G. Since T is a cycle, if some endpoint of γ_j^m is not an endpoint of γ_j, then there exists an edge $e_{ji} \in E(\mathcal{L}(G_m))$ connecting $x_j \in \gamma_j^m$ with some $x_i \in \gamma_i^m$ $(i \neq j)$.

The vertices x_j, x_i of the edge $e_{ji} \in E(\mathcal{L}(G_m))$ correspond to two edges $e_1, e_2 \in E(G_m)$ starting in a connection vertex $v \in V(G_m)$. Therefore, v belongs to $G_{m_1}, \dots, G_{m_r}$; if $\deg_G(v) = d$, then the set of edges in G starting in v corresponds to a subgraph Γ of $\mathcal{L}(G)$ isomorphic to the complete graph K_d, and $e_{ji} \in E(\mathcal{L}(G_m)) \cap E(\Gamma)$.

Let us denote by U the closure of the connected component of $T \setminus \{\mathcal{L}(G_m)\}$ which joins x_j and x_i (the endpoints of the edge e_{ji}). Assume first that $L(U) \geq 3k$.

If there is just a vertex of T in U, then there exist vertices $y_j, y_i \in V(\Gamma)$ with $y_j \neq y_i$, $[x_j, y_j] \subset \gamma_j \cap \Gamma$ and $[x_i, y_i] \subset \gamma_i \cap \Gamma$. Let us define w' as the midpoint of the edge e_{ji}, $g_j^m := \gamma_j^m \cup [x_j, w']$ and $g_i^m := \gamma_i^m \cup [x_i, w']$. We will show that g_j^m and g_i^m are geodesics in $\mathcal{L}(G_m)$.

In fact, we prove that if $\gamma_j^m = [x_j z_j]$, then $d_{\mathcal{L}(G)}(x_j, z_j) \leq d_{\mathcal{L}(G)}(x_i, z_j)$. Seeking for a contradiction, assume that $d_{\mathcal{L}(G)}(x_j, z_j) > d_{\mathcal{L}(G)}(x_i, z_j)$. Then

$$d_{\mathcal{L}(G)}(y_j, z_j) \leq d_{\mathcal{L}(G)}(y_j, x_i) + d_{\mathcal{L}(G)}(x_i, z_j) < k + d_{\mathcal{L}(G)}(x_j, z_j) = L([y_j, x_j]) + L(\gamma_j^m),$$

and this implies that γ_j is not a geodesic. This is the contradiction we were looking for. Therefore, $d_{\mathcal{L}(G)}(x_j, z_j) \leq d_{\mathcal{L}(G)}(x_i, z_j)$.

Hence, g_j^m is a geodesic in $\mathcal{L}(G_m)$. With a similar argument we obtain that g_i^m is a geodesic in $\mathcal{L}(G_m)$.

If there are two vertices of T in U, let us define $g_j^m := \gamma_j^m$ and $g_i^m := \gamma_i^m$; in this case we consider as third side of T_m the edge $e_{ji} = [x_i, x_j]$.

Assume now that $L(U) = 2k$. Then $U \subset \Gamma$.

If there is just a vertex of T in U, let us denote by w this vertex of T. If $w \in V(\mathcal{L}(G))$, then let us define w' as the midpoint of the edge e_{ji}, $g_j^m := \gamma_j^m \cup [x_j, w']$ and $g_i^m := \gamma_i^m \cup [x_i, w']$; we have that g_j^m and g_i^m are geodesics in $\mathcal{L}(G_m)$. If w is a midpoint of some edge in $E(\Gamma)$, without loss of generality we can assume that it is the midpoint of $[x_j, a]$, with $a \in V(\Gamma)$; let us define $w' = x_j$, $g_j^m := \gamma_j^m$ and $g_i^m := \gamma_i^m \cup [x_i, x_j]$; we have that g_j^m and g_i^m are geodesics in $\mathcal{L}(G_m)$.

If there are two vertices of T in U, let us define $g_j^m := \gamma_j^m$ and $g_i^m := \gamma_i^m$; in this case we consider as third side of T_m the edge $e_{ji} = [x_i, x_j]$.

Iterating this process at most three times, we obtain a geodesic triangle T_m in $\mathcal{L}(G_m)$ with sides $\gamma_1^{m*}, \gamma_2^{m*}, \gamma_3^{m*}$, containing $\gamma_1^m, \gamma_2^m, \gamma_3^m$, respectively.

Furthermore, $d_{\mathcal{L}(G)}(p, \gamma_2 \cup \gamma_3) \leq d_{\mathcal{L}(G_m)}(p, \gamma_2^{m*} \cup \gamma_3^{m*}) + k \leq \delta(\mathcal{L}(G_m)) + k$.

Assume now that $p \notin \cup_n \mathcal{L}(G_n)$. Then p belongs to a subgraph Γ of $\mathcal{L}(G)$ isomorphic to the complete graph K_d. Since the distance from any vertex in K_d to any point in K_d is less or equal than $3k/2$, then $d_{\mathcal{L}(G)}(p, \gamma_2 \cup \gamma_3) \leq 3k/2$.

Consequently,

$$\delta(T) \leq \max\left\{ \sup_n \delta(\mathcal{L}(G_n)) + k, \frac{3k}{2} \right\}.$$

Since T is arbitrary, we conclude

$$\delta(G) \leq \max\left\{ \sup_n \delta(\mathcal{L}(G_n)) + k, \frac{3k}{2} \right\}.$$

In order to finish the proof, assume first that G is a tree; Proposition 3.3.2 gives that $\delta(\mathcal{L}(G)) \le k$, and then $\delta(\mathcal{L}(G)) \le \sup_n \delta(\mathcal{L}(G_n)) + k$.

Assume now that G is not a tree; then there exists a cycle g in G with $L_G(g) \ge 3k$. Note that g is in G_{n_0} for some n_0, since $\{G_n\}$ is a T-decomposition of G. The corresponding cycle g' to g in $\mathcal{L}(G_{n_0})$ verifies $L_{\mathcal{L}(G)}(g') = L_G(g) \ge 3k$, and Lemma 3.3.3 gives that $\delta(\mathcal{L}(G_{n_0})) \ge 3k/4$. Consequently, $\delta(\mathcal{L}(G_{n_0})) + k > 3k/2$ and $\delta(\mathcal{L}(G)) \le \sup_n \delta(\mathcal{L}(G_n)) + k$. □

The lower bound in Theorem 3.3.7 is attained in any cycle graph C_n with $n \ge 3$, and the upper bound is attained in any star graph S_n with $n \ge 5$.

Theorem 3.3.8. *If G is any graph such that each graph G_n in its canonical T-decomposition is either a cycle or an edge, then*

$$\delta(G) \le \delta(\mathcal{L}(G)) \le \delta(G) + k,$$

and $\delta(G) = \frac{1}{4} \sup\{L(g) : g \text{ is a cycle in } G\}$.

Proof. If G is a tree, then we just need to check that $0 \le \delta(\mathcal{L}(G)) \le k$, and this is a consequence of Proposition 3.3.2.

Assume now that G has at least a cycle.

We prove now the formula for $\delta(G)$. Lemma 1.3.7 and Theorem 1.3.19 give

$$\delta(G) = \sup_n \delta(G_n) = \frac{1}{4} \sup\{L(g) : g \text{ is a cycle in } G\}.$$

By hypothesis each graph G_n is either a cycle (and then $\delta(G_n) = \delta(\mathcal{L}(G_n)) = L(G_n)/4$) or an edge (and then $\delta(G_n) = \delta(\mathcal{L}(G_n)) = 0$). Since G has at least a cycle, there exists n such that G_n (and consequently $\mathcal{L}(G_n)$) is not a tree. These facts and Theorem 3.3.7 give the result. □

The lower bound in Theorem 3.3.8 is attained in any cycle graph C_n with $n \ge 3$, and the upper bound is attained in any star graph S_n with $n \ge 5$.

In particular, we can bound directly the hyperbolicity constant of the line of an unicycle graph.

Corollary 3.3.9. *If G is any unicycle graph and we denote by g its cycle, then*

$$\frac{1}{4} L(g) \le \delta(\mathcal{L}(G)) \le \frac{1}{4} L(g) + k.$$

We can improve the upper bound of $\delta(\mathcal{L}(G))$ in Corollary 3.3.9.

Theorem 3.3.10. *If G is any unicycle graph and we denote by g its cycle, then*

$$\frac{1}{4} L(g) \le \delta(\mathcal{L}(G)) \le \frac{1}{4} L(g) + \frac{k}{2}.$$

Proof. We know that the first inequality holds from Corollary 3.3.9.

We prove now the second inequality. The graph G is the union of g and the trees $T_1, \dots, T_r$. If we denote by G_0 the subgraph of G defined as $G_0 := \{x \in G : d_G(x, g) \leq k\}$, then $\{G_0, T_1, \dots, T_r\}_n$ is a T-edge-decomposition of G, and Theorem 3.3.1 gives $\delta(\mathcal{L}(G)) = \max\{\delta(\mathcal{L}(G_0)), \delta(\mathcal{L}(T_1)), \dots, \delta(\mathcal{L}(T_1))\}$. Since Proposition 3.3.2 gives $\delta(\mathcal{L}(T_j)) \leq k$, we have $\delta(\mathcal{L}(G)) \leq \max\{\delta(\mathcal{L}(G_0)), k\}$. Since $L(g)/4 + k/2 \geq 3k/4 + k/2 > k$, it suffices to prove that $\delta(\mathcal{L}(G_0)) \leq L(g)/4 + k/2$.

In order to do that, we just need to construct the geodesic triangle T_0 in $\mathcal{L}(g)$ with sides $\gamma_1^{0*}, \gamma_2^{0*}, \gamma_3^{0*}$, following the proof of Theorem 3.3.7 (replacing $\mathcal{L}(G_m)$ by $\mathcal{L}(g)$). In this case, if $p \in \mathcal{L}(g)$, then $d_{\mathcal{L}(G_0)}(p, \gamma_2 \cup \gamma_3) \leq \delta(\mathcal{L}(g)) + k/2$. Furthermore, if $p \notin \mathcal{L}(g)$, then $d_{\mathcal{L}(G_0)}(p, \gamma_2 \cup \gamma_3) \leq 5k/4$. Hence, we conclude

$$\delta(\mathcal{L}(G_0)) \leq \max\left\{\delta(\mathcal{L}(g)) + \frac{k}{2}, \frac{5k}{4}\right\},$$

and, since $\delta(\mathcal{L}(g)) = \delta(g) = L(g)/4$ by Theorem 1.3.19, we deduce $\delta(\mathcal{L}(g)) + k/2 \geq 3k/4 + k/2 = 5k/4$ and

$$\delta(\mathcal{L}(G_0)) \leq \delta(\mathcal{L}(g)) + \frac{k}{2} = \frac{1}{4}\,L(g) + \frac{k}{2}.$$

□

Both inequalities in Theorem 3.3.10 are sharp: the first one is attained in the cycle graphs C_n; the second one is attained in the cycle graphs C_{2n} with two edges attached in antipodal vertices.

We also have the following result.

Theorem 3.3.11. *If G is any graph with $\delta(G) < k$, then $\delta(\mathcal{L}(G)) \leq 7k/4$.*

Proof. If G is a tree, then Proposition 3.3.2 gives that $\delta(\mathcal{L}(G)) \leq k < 7k/4$. Assume now that G has a cycle. Since $\delta(G) < k$, Lemma 3.3.3 gives that every cycle g in G has length $L(g) = 3k$. Then each graph G_n in the canonical T-decomposition of G is either a cycle with length $3k$ or an edge, and Theorem 3.3.8 gives $\delta(G) = 3k/4$ and $\mathcal{L}(G) \leq 7k/4$. □

The following theorem is a similar result to Proposition 3.2.5 for line graphs.

Theorem 3.3.12. *For any graph G, we have*

$$\frac{g(G)}{4} \leq \delta(\mathcal{L}(G)) \leq \frac{c(G)}{4} + 2k.$$

Proof. The first inequality is just Corollary 3.2.7.

In order to prove the second inequality, let us consider the canonical T-decomposition $\{G_n\}_n$ of G. Fix any geodesic triangle $T = \{\gamma_1, \gamma_2, \gamma_3\}$ in $\mathcal{L}(G)$ and $p \in T$. By Corollary 1.3.4 we can assume that T is a cycle. Without loss of generality we can assume that $p \in \gamma_1$.

If $p \notin \cup_n \mathcal{L}(G_n)$, we have seen in the proof of Theorem 3.3.7 that $d_{\mathcal{L}(G)}(p, \gamma_2 \cup \gamma_3) \le 3k/2$.

Assume now that $p \in \mathcal{L}(G_m)$ for some fixed m. Since $\mathcal{L}(G_m)$ is an isometric subgraph of $\mathcal{L}(G)$, $\gamma_j^m := \gamma_j \cap \mathcal{L}(G_m)$ is a (connected) geodesic in $\mathcal{L}(G_m)$ for $j = 1, 2, 3$. Then $p \in \gamma_1^m = [xy]$. Note that, since $\{G_n\}_n$ is a T-decomposition in G, $2 \operatorname{diam} G_m \le c(G_m)$. Furthermore, $\operatorname{diam} V(\mathcal{L}(G_m)) \le \operatorname{diam} V(G_m) + k$ and thus $\operatorname{diam} \mathcal{L}(G_m) \le \operatorname{diam} G_m + 2k$. Hence,

$$\begin{aligned} d_{\mathcal{L}(G)}(p, \gamma_2 \cup \gamma_3) &\le d_{\mathcal{L}(G_m)}(p, \gamma_2^m \cup \gamma_3^m) + k \le d_{\mathcal{L}(G)}(p, \{x, y\}) + k \\ &\le \frac{1}{2} L(\gamma_1^m) + k \le \frac{1}{2} \operatorname{diam} \mathcal{L}(G_m) + k \le \frac{1}{2} \operatorname{diam} G_m + 2k \\ &\le \frac{1}{4} c(G_m) + 2k \le \frac{1}{4} c(G) + 2k. \end{aligned}$$

Therefore, in any case we have

$$\delta(\mathcal{L}(G)) \le \frac{1}{4} c(G) + 2k.$$

□

The lower bound in Theorem 3.3.12 is attained in the cycle graphs with $n \ge 3$ vertices.

We show now by two examples that the upper bound in Theorem 3.3.12 is very precise:

If G is the star graph $G = K_{1,4}$, then $\mathcal{L}(G) = K_4$, $\delta(G) = 0$ and $\delta(\mathcal{L}(G)) = k = c(G)/4 + k$.

If G is the cycle C_{2n} with two edges attached in antipodal vertices, then $\mathcal{L}(G)$ is the cycle C_{2n} with two graphs isomorphic to C_3 attached in antipodal edges. It is not difficult to check that $\delta(G) = nk/2$ and $\delta(\mathcal{L}(G)) = nk/2 + k/2 = c(G)/4 + k/2$.

Chapter 4

Hyperbolicity of line graph with edges of arbitrary length.

In Chapter 4, we deal with graphs with edges with arbitrary lengths. It is a remarkable fact that the constants appearing in many results in the theory of hyperbolic spaces depend just on a small number of parameters (also, this is a common place in the theory of negatively curved surfaces and manyfolds). Usually, there is no explicit expression for these constants (see, e.g., Theorem 1.3.5). Even though sometimes it is possible to estimate the constants, those explicit values obtained, in general, are far from being sharp (see, e.g., (3.1)). Although there are important results stating that the hyperbolicity constant of a graph $T(G)$ (obtained from an original graph G via some transformation T), is bounded in terms of the hyperbolicity constant of G, there is still no known example of non-trivial transformation that is monotonous for the hyperbolicity constants, i.e., such that $\delta(G) \le \delta(T(G))$ (or $\delta(G) \ge \delta(T(G))$) for every graph G.

The main result of this Chapter is the inequality $\delta(G) \le \delta(\mathcal{L}(G))$ for the line graph $\mathcal{L}(G)$ of every graph G (see Theorem 4.1.10).

Theorem 4.1.10 allows to obtain the main qualitative result: the line graph of G is hyperbolic if and only if G is hyperbolic. Although the multiplicative and additive constants appearing in (3.1) allow to prove this main result, it is a natural problem to improve the inequalities in (3.1). In this chapter we also improve the second inequality; in fact, Theorem 4.1.10 states

$$\delta(G) \le \delta(\mathcal{L}(G)) \le 5\,\delta(G) + 3 \sup_{e \in E(G)} L(e),$$

where here the edges of G can have arbitrary lengths. The second inequality in (1) can be improved for graphs with edges of length k (see Corollary 4.1.12) in the following way:

$$\delta(G) \le \delta(\mathcal{L}(G)) \le 5\delta(G) + 5k/2.$$

We obtain some relations involving $\delta(G)$ and $\delta(\mathcal{L}(G))$ for graphs with edges of length k, as the following: if G is a graph with n vertices $v_1, \dots, v_n$, then

$$\delta(\mathcal{L}(G)) + \delta(G) \le \frac{k}{8} \sum_{i=1}^{n} (\deg_G(v_i))^2.$$

(see Theorem 4.1.14).

4.1 Inequalities involving the hyperbolicity constant of line graphs.

We obtain in this section the results on the hyperbolicity constant of a line graph with edges of arbitrary lengths. The main result in this Chapter is Theorem 4.1.10, which states

$$\delta(G) \le \delta(\mathcal{L}(G)) \le 5\delta(G) + 3l_{max},$$

with $l_{max} = \sup_{e \in E(G)} L(e)$.

For the sake of clarity and readability, we have opted to state and prove several preliminary lemmas. This makes the proof of Theorem 4.1.10 much more understandable.

Let G be a graph such that its edges $E(G) = \{e_i\}_{i \in \mathcal{I}}$ have arbitrary lengths. The *line graph* $\mathcal{L}(G)$ of G is a graph which has a vertex $V_{e_i} \in V(\mathcal{L}(G))$ for each edge e_i of G, and an edge joining V_{e_i} and V_{e_j} when $e_i \cap e_j \neq \varnothing$. Note that we have a complete subgraph K_n in $\mathcal{L}(G)$ corresponding to one vertex v of G with degree $\deg_G(v) = n$.

Let us consider $Pm(e)$ the midpoint of $e \in E(G)$; also, we denote by $PM(G)$ the set of the midpoints of the edges of G, i.e., $PM(G) := \{Pm(e)/e \in E(G)\}$. Besides, let us consider $Pm_{\mathcal{L}}([V_{e_i}, V_{e_j}])$ the point in $[V_{e_i}, V_{e_j}] \in E(\mathcal{L}(G))$ with $L([V_{e_i} Pm_{\mathcal{L}}([V_{e_i}, V_{e_j}])]) = L(e_i)/2$ $\big($and then $L([Pm_{\mathcal{L}}([V_{e_i}, V_{e_j}]) V_{e_j}]) = L(e_j)/2\big)$. Analogously, we denote $PM_{\mathcal{L}}(\mathcal{L}(G))$ the set of these points in each edge of $\mathcal{L}(G)$, i.e., $PM_{\mathcal{L}}(\mathcal{L}(G)) := \{Pm_{\mathcal{L}}(e)/e \in E(\mathcal{L}(G))\}$. Note that $Pm_{\mathcal{L}}([V_{e_i}, V_{e_j}])$ is the midpoint of $[V_{e_i}, V_{e_j}]$ when $L(e_i) = L(e_j)$; thus, if every edge of G has the same length then $PM_{\mathcal{L}}(\mathcal{L}(G))$ is the set of midpoints of the edges of $\mathcal{L}(G)$.

Let us consider the sets $PMV(G) := PM(G) \cup V(G)$ (defined previously in Section 1.3) and $PM_{\mathcal{L}}V(\mathcal{L}(G)) := PM_{\mathcal{L}}(\mathcal{L}(G)) \cup V(\mathcal{L}(G))$.

We define a function $h : PM_{\mathcal{L}}V(\mathcal{L}(G)) \longrightarrow PMV(G)$ as follows: for every vertex V_e of $V(\mathcal{L}(G))$, the image via h of V_e is $Pm(e)$, and for every $Pm_{\mathcal{L}}([V_{e_i}, V_{e_j}])$ in $PM_{\mathcal{L}}(\mathcal{L}(G))$, the image via h of $Pm_{\mathcal{L}}([V_{e_i}, V_{e_j}])$ is the vertex $e_i \cap e_j$ in $V(G)$, i.e.,

$$h(x) := \begin{cases} Pm(e), & \text{if } x = V_e \in V(\mathcal{L}(G)), \\ e_i \cap e_j, & \text{if } x = Pm_{\mathcal{L}}([V_{e_i}, V_{e_j}]) \in PM_{\mathcal{L}}(\mathcal{L}(G)). \end{cases} \tag{4.1}$$

Remark 4.1.1. *If $x \in PM(G)$, then $h^{-1}(x)$ is a single point, but otherwise, $h^{-1}(x)$ can have more than one point (see Figure 4.1).*

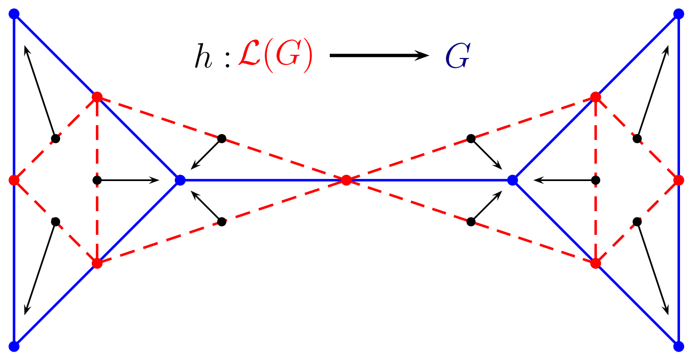


Figure 4.1: Graphical view of h.

The function h defined in (4.1) can be extended to $\mathcal{L}(G)$. Note that every point $x_0 \in \mathcal{L}(G) \setminus PM_{\mathcal{L}}V(\mathcal{L}(G))$ is located in $\mathcal{L}(G)$ between one vertex V_e and one point $Pm_{\mathcal{L}}(V_eV_{e_0})$. For each $x_0 \in \operatorname{int}([V_e Pm_{\mathcal{L}}([V_eV_{e_0}])])$ we define $h(x_0)$ as the point $x \in \operatorname{int}[Pm(e)h(Pm_{\mathcal{L}}([V_eV_{e_0}]))]$ such that $L([xPm(e)]) = L([x_0V_e])$; hence, $L([x_0V_e]) = L([h(x_0)h(V_e)])$ and $L([x_0Pm_{\mathcal{L}}(V_eV_{e_0})]) = L([h(x_0)h(Pm_{\mathcal{L}}(V_eV_{e_0}))])$.

In what follows we denote by h this extension.

We call *half-edge* in G a geodesic contained in an edge with an endpoint in $V(G)$ and an endpoint in $PM(G)$; similarly, a *half-edge* in $\mathcal{L}(G)$ is a geodesic contained in an edge with an endpoint in $V(\mathcal{L}(G))$ and an endpoint in $PM_{\mathcal{L}}(\mathcal{L}(G))$.

Proposition 4.1.2. *h is an 1-Lipschitz continuous function, i.e.,*

$$d_G(h(x), h(y)) \le d_{\mathcal{L}(G)}(x, y)\,, \quad \forall\, x, y \in \mathcal{L}(G). \tag{4.2}$$

Proof. First of all note that, by definition of $\mathcal{L}(G)$, we have for every $x' \in h(\mathcal{L}(G)) \cap PMV(G)$,

$$|h^{-1}(x')| = \begin{cases} 1, & \text{if } x' \in PM(G), \\ \deg_G(x')(\deg_G(x') - 1)/2, & \text{if } x' \in V(G). \end{cases}$$

In order to prove (4.2), we verify that

$$d_G(x', y') = d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y'))\,, \quad \forall\, x', y' \in h(\mathcal{L}(G)) \cap PMV(G). \tag{4.3}$$

We study separately the different cases of $x', y' \in h(\mathcal{L}(G)) \cap PMV(G)$.

Case 1 $x', y' \in PM(G)$.

Let us consider $x' := Pm(e_i)$ and $y' := Pm(e_j)$ with $e_i, e_j \in E(G)$, and define $d := d_G(Pm(e_i),\, Pm(e_j)) \ge 0$.

If $d = 0$, then $e_i = e_j$, so, $h^{-1}(Pm(e_i)) = h^{-1}(Pm(e_j))$ and $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) = 0$.

If $d > 0$, then $e_i \neq e_j$ and $d_G(Pm(e_i), Pm(e_j)) = (L(e_i) + L(e_j))/2 + d_G(e_i, e_j)$. Note that, if $d_G(e_i, e_j) = 0$ and $e_i \neq e_j$, then $d_G(x', y') = (L(e_i)+L(e_j))/2 = d_{\mathcal{L}(G)}(V_{e_i}, V_{e_j}) = d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y'))$. If $d_G(e_i, e_j) > 0$, then a geodesic γ joining e_i and e_j in G contains the edges $e_{i_1}, \dots, e_{i_r}$ in this order, with $r \geq 1$. Now, we have that $d_G(e_i, e_j) = \sum_{k=1}^{r} L(e_{i_k})$; hence, $V_{e_i} V_{e_{i_1}} \dots V_{e_{i_r}} V_{e_j}$ is a path joining V_{e_i} and V_{e_j} with length d. So, $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) \leq d$.

We prove now that $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) = d$. Seeking for a contradiction, assume that $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) = d_{\mathcal{L}(G)}(V_{e_i}, V_{e_j}) < d$. Hence, there exists $V_{e_{j_1}}, \dots, V_{e_{j_m}}$ such that $V_{e_i} V_{e_{j_1}} \dots V_{e_{j_m}} V_{e_j}$ is a geodesic in $\mathcal{L}(G)$ joining V_{e_i} and V_{e_j} with length $(L(e_i) + L(e_j))/2 + \sum_{k=1}^{m} L(e_{j_k}) < d$. Since $d = (L(e_i) + L(e_j))/2 + d_G(e_i, e_j)$, we have $\sum_{k=1}^{m} L(e_{j_k}) < d_G(e_i, e_j)$. By definition of $\mathcal{L}(G)$ we have that $\gamma^* := e_{j_1} \cup \dots \cup e_{j_m}$ is a path in G joining e_i and e_j with length $\sum_{k=1}^{m} L(e_{j_k}) < d_G(e_i, e_j)$. This is the contradiction we were looking for; so we have $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) = d_G(x', y')$.

Case 2 $x' \in PM(G)$ and $y' \in V(G)$.

Let us consider $x' := Pm(e)$ with $e \in E(G)$ and $y' \in V(G) \backslash \{w \in V(G)/ \deg_G(w) = 1\}$, and define $d := d_G(e, y')$; then $d_G(Pm(e), v) = d + L(e)/2$. Note that if $y' \in V(G)$ and $\deg_G(y') = 1$, then $y' \notin h(\mathcal{L}(G))$.

If $d = 0$, then y is an endpoint of e and $d_{\mathcal{L}(G)}(V_e, h^{-1}(y')) = L(e)/2$; note that $|h^{-1}(y')| = \deg_G(y')[\deg_G(y') - 1]/2$, where $|A|$ denotes the cardinality of the set A.

If $d_G(e, y') = d > 0$, then there exist $e_{i_1}, \dots, e_{i_r} \in E(G)$ such that $\gamma := e_{i_1} \cup \dots \cup e_{i_r}$ is a geodesic joining e and y' in G with length $d = \sum_{k=1}^{r} L(e_{i_k})$. Note that e, e_{i_1} are different and adjacent edges. So, we have that $V_e V_{e_{i_1}} \dots V_{e_{i_r}}$ is a path in $\mathcal{L}(G)$ joining V_e and $V_{e_{i_r}}$ with length $L(e)/2 + \sum_{k=1}^{r} L(e_{i_k}) - L(e_{i_r})/2$. Since y' is an endpoint of e_{i_r}, we have $d_{\mathcal{L}(G)}(h^{-1}(y'), V_{e_{i_r}}) = L(e_{i_r})/2$ and $d_{\mathcal{L}(G)}(h^{-1}(y'), V_e) \leq d + L(e)/2$.

We prove now that $d_{\mathcal{L}(G)}(h^{-1}(y'), V_e) = d + L(e)/2$. Seeking for a contradiction, assume that $d_{\mathcal{L}(G)}(h^{-1}(y'), V_e) < d + L(e)/2$. Hence, there exists $V_{e_{j_1}}, \dots, V_{e_{j_m}}$ such that $V_e V_{e_{j_1}} \dots V_{e_{j_m}} \cup [V_{e_{j_m}} z]$ is a geodesic of $\mathcal{L}(G)$ joining V_e and $z \in h^{-1}(y')$ with length $L(e)/2 + \sum_{k=1}^{m} L(e_{j_k}) < d + L(e)/2$. We have $z = Pm_{\mathcal{L}}([V_{e_{j_m}}, V_{e_s}])$ with e_{j_m}, e_s edges in G starting in y'. By definition of $\mathcal{L}(G)$ we have that $\gamma^* := e_{j_1} \cup \dots \cup e_{j_m}$ contains a path in G joining e and y' with length at most $\sum_{k=1}^{m} L(e_{j_k}) < d$. This is the contradiction we were looking for; so we have $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) = d_G(x', y')$.

Case 3 $x', y' \in V(G)$.

Let us consider $x', y' \in V(G) \backslash \{v \in V(G)/ \deg_G(v) = 1\}$, and define $d := d_G(x', y') \geq 0$.

If $d = 0$, then $x' = y'$, so $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) = 0$.

If $d_G(x', y') = d > 0$, then there exists $e_{i_1}, \dots, e_{i_r} \in E(G)$ such that $\gamma := e_{i_1} \cup \dots \cup e_{i_r}$ is a geodesic joining x' and y' in G with length $d = \sum_{k=1}^{r} L(e_{i_k})$. So, we have that there exist $a \in h^{-1}(x')$ and $b \in h^{-1}(y')$ such that $[aV_{e_{i_1}}] \cup V_{e_{i_1}} \dots V_{e_{i_r}} \cup [V_{e_{i_r}} b]$ is a path in $\mathcal{L}(G)$ joining a and b with length $\sum_{k=1}^{r} L(e_{i_k}) = d$. Then, we have that $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) \leq d$.

We prove now that $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) = d$. Seeking for a contradiction, assume that $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) < d$. Hence, there exist $\alpha \in h^{-1}(x')$, $\beta \in h^{-1}(y')$ and $V_{e_{j_1}}, \dots, V_{e_{j_m}}$ vertices in $\mathcal{L}(G)$ such that $[\alpha V_{e_{j_1}}] \cup V_{e_{j_1}} \dots V_{e_{j_m}} \cup [V_{e_{j_m}} \beta]$ is a geodesic joining α and β in $\mathcal{L}(G)$ with length $\sum_{k=1}^{m} L(e_{j_k}) < d$. We have $\alpha = Pm_{\mathcal{L}}([V_{e_s^1}, V_{e_{j_1}}])$ with e_{j_1}, e_s^1 edges in G starting in x', and $\beta = Pm_{\mathcal{L}}([V_{e_{j_m}}, V_{e_s^2}])$ with e_{j_m}, e_s^2 edges in G starting in y'. By definition of $\mathcal{L}(G)$ we have that $\gamma^* := e_{j_1} \cup \dots \cup e_{j_m}$ contains a path in G joining x' and y' with length at most $\sum_{k=1}^{m} L(e_{j_k}) < d$. This is the contradiction we were looking for; so we have $d_{\mathcal{L}(G)}(h^{-1}(x'), h^{-1}(y')) = d_G(x', y')$.

This prove (4.3) and guarantees (4.2) for $x, y \in PM_{\mathcal{L}}V(\mathcal{L}(G))$ when we take $x := h(x')$ and $y := h(y')$. We know that there exist $X_1, X_2, Y_1, Y_2 \in PM_{\mathcal{L}}V(\mathcal{L}(G))$ with $x \in [X_1, X_2]$ and $y \in [Y_1, Y_2]$ such that $d_{\mathcal{L}(G)}(x, X_1) = \varepsilon_x$, $d_{\mathcal{L}(G)}(x, X_2) = \delta_x$, $d_{\mathcal{L}(G)}(y, Y_1) = \varepsilon_y$, $d_{\mathcal{L}(G)}(y, Y_2) = \delta_y$ and $[X_1X_2]$, $[Y_1Y_2]$ are two half-edges in $\mathcal{L}(G)$. Hence, we have $h(x) \in [h(X_1)h(X_2)]$, $h(y) \in [h(Y_1)h(Y_2)]$ with $d_G(h(x), h(X_1)) = \varepsilon_x$, $d_G(h(x), h(X_2)) = \delta_x$, $d_G(h(y), h(Y_1)) = \varepsilon_y$ and $d_G(h(y), h(Y_2)) = \delta_y$; besides $[h(X_1)h(X_2)]$ and $[h(Y_1)h(Y_2)]$ are two half-edges in G.

Note that if $[X_1X_2] = [Y_1Y_2]$, then $d_G(h(x), h(y)) = d_{\mathcal{L}(G)}(x, y)$. Otherwise, we have

$$d_{\mathcal{L}(G)}(x, y) = \min \left\{ \begin{array}{l} d_{\mathcal{L}(G)}(X_1, Y_1) + \varepsilon_x + \varepsilon_y, \\ d_{\mathcal{L}(G)}(X_1, Y_2) + \varepsilon_x + \delta_y, \\ d_{\mathcal{L}(G)}(X_2, Y_1) + \delta_x + \varepsilon_y, \\ d_{\mathcal{L}(G)}(X_2, Y_2) + \delta_x + \delta_y \end{array} \right\} \tag{4.4}$$

and

$$d_G(h(x), h(y)) = \min \left\{ \begin{array}{l} d_G(h(X_1), h(Y_1)) + \varepsilon_x + \varepsilon_y, \\ d_G(h(X_1), h(Y_2)) + \varepsilon_x + \delta_y, \\ d_G(h(X_2), h(Y_1)) + \delta_x + \varepsilon_y, \\ d_G(h(X_2), h(Y_2)) + \delta_x + \delta_y \end{array} \right\}. \tag{4.5}$$

Let us consider X_i, Y_j with $i, j \in \{1, 2\}$, $\alpha \in \{\varepsilon_x, \delta_x\}$ and $\beta \in \{\varepsilon_y, \delta_y\}$ such that $d_{\mathcal{L}(G)}(x, y) = d_{\mathcal{L}(G)}(X_i, Y_j) + \alpha + \beta$. Hence, by (4.3) we have

$$\begin{aligned} d_{\mathcal{L}(G)}(x, y) &= d_{\mathcal{L}(G)}(X_i, Y_j) + \alpha + \beta, \\ &\geq d_G(h(X_i), h(Y_j)) + \alpha + \beta, \\ &\geq d_G(h(x), h(y)). \end{aligned}$$

$\square$

The following result is a consequence of Proposition 4.1.2.

Remark 4.1.3. *Let x and y be in $V(\mathcal{L}(G))$, then we have that*

$$d_{\mathcal{L}(G)}(x, y) = d_G(h(x), h(y)).$$

We also have a kind of reciprocal of Proposition 4.1.2.

Lemma 4.1.4. *For every $x, y \in \mathcal{L}(G)$ we have*

$$d_{\mathcal{L}(G)}(x, y) \le d_G(h(x), h(y)) + 2l_{max}, \tag{4.6}$$

where $l_{max} := \sup_{e \in E(G)} L(e)$.

Proof. First of all, we prove (4.6) for $x, y \in PM_{\mathcal{L}}V(\mathcal{L}(G))$. In order to prove it, we can assume that $\operatorname{diam}_{\mathcal{L}(G)} h^{-1}(h(x)), \operatorname{diam}_{\mathcal{L}(G)} h^{-1}(h(y)) > 0$ (i.e., $h(x), h(y) \in V(G)$ and $\deg_G(h(x)), \deg_G(h(x)) > 2$), since otherwise the argument is easier. Thus, by definition of $\mathcal{L}(G)$ we have a complete subgraph $K_{\deg(v)}$ associated to $v \in V(G)$ and $h^{-1}(v) = PM_{\mathcal{L}}(\mathcal{L}(G)) \cap K_{\deg(v)}$. Let us choose $x'' \in h^{-1}(h(x))$, $y'' \in h^{-1}(h(y))$ with $d_{\mathcal{L}(G)}(x'', y'') = d_{\mathcal{L}(G)}(h^{-1}(h(x)), h^{-1}(h(y)))$. Consider a geodesic γ joining x'' and y'' in $\mathcal{L}(G)$. Let V_1 (respectively, V_2) be the closest vertex to x'' (respectively, y'') in γ. It is easy to check that, since $h^{-1}(v) = PM_{\mathcal{L}}(\mathcal{L}(G)) \cap K_{\deg(v)}$ and $L([V_{e_i}, V_{e_j}]) = (L(e_i) + L(e_j))/2$, we have

$$d_{\mathcal{L}(G)}(V_1, x) \le d_{\mathcal{L}(G)}(V_1, x'') + \sup_{e \in E(G)} L(e),$$

$$d_{\mathcal{L}(G)}(V_2, y) \le d_{\mathcal{L}(G)}(V_2, y'') + \sup_{e \in E(G)} L(e),$$

and since

$$d_{\mathcal{L}(G)}(x'', V_1) + d_{\mathcal{L}(G)}(V_1, V_2) + d_{\mathcal{L}(G)}(V_2, y'') = d_{\mathcal{L}(G)}(x'', y''),$$

we deduce (4.6) for $x, y \in PM_{\mathcal{L}}V(\mathcal{L}(G))$.

Now, let us consider $X_{i'}, Y_{j'}$ with $i', j' \in \{1, 2\}$, $\alpha' \in \{\varepsilon_x, \delta_x\}$ and $\beta' \in \{\varepsilon_y, \delta_y\}$ such that $d_G(h(x), h(y)) = d_G(h(X_{i'}), h(Y_{j'})) + \alpha' + \beta'$. Hence, we have

$$\begin{aligned} d_G(h(x), h(y)) &= d_G(h(X_{i'}), h(Y_{j'})) + \alpha' + \beta', \\ &\ge d_{\mathcal{L}(G)}(X_{i'}, Y_{j'}) - 2l_{max} + \alpha' + \beta', \end{aligned}$$

finally, (4.4) gives the condition. $\square$

It is easy to see that $G \setminus h(\mathcal{L}(G))$ is the union of the half-edges of G such that one of its vertices has degree 1; thus the following fact holds.

Remark 4.1.5. *h is a $(l_{max}/2)$-full $(1, 2l_{max})$-quasi-isometry with $l_{max} = \sup_{e \in E(G)} L(e)$.*

Now, let us consider a cycle C in G. We define $g_C : C \longrightarrow \mathcal{L}(G)$ in the following way; $g_C(Pm(e)) := V_e$ for $e \in E(G)\cap C$; if C^* is the cycle in $\mathcal{L}(G)$ with vertices $\cup_{e\in E(G)} g_C(Pm(e))$, then one can check that $h|_{C^*} : C^* \longrightarrow C$ is a bijection; we define

$$g_C := (h|_{C^*})^{-1} : C \longrightarrow C^*. \tag{4.7}$$

Corollary 4.1.6. *Let C be a geodesic polygon in a graph G that is a cycle and let g_C be the function defined by (4.7). Then, $C^* := g_C(C)$ is a geodesic polygon in $\mathcal{L}(G)$ with the same number of edges than C.*

Furthermore, if γ is a geodesic in C, then $g_C(\gamma)$ is a geodesic in $\mathcal{L}(G)$ with $L(g_C(\gamma)) = L(\gamma)$.

Proof. First of all, note that $L(C) = L(C^*)$ since if $E(C) = \{e_1, \dots, e_n\}$ with $e_1 \cap e_n \neq \varnothing$ and $e_i \cap e_{i+1} \neq \varnothing$ for $1 \leq i < n$, then $L(C) = \sum_{i=1}^{n} L(e_i)$ and $L(C^*) = L(e_1)/2 + \sum_{i=1}^{n-1}(L(e_i) + L(e_{i+1}))/2 + L(e_n)/2$.

Now, let us consider a geodesic γ in C joining x and y. Since $g_C(\gamma)$ is a path joining $g_C(x)$ and $g_C(y)$, we have that $d_{\mathcal{L}(G)}(g_C(x), g_C(y)) \leq d_{C^*}(g_C(x), g_C(y)) = d_G(x, y)$; Proposition 4.1.2 gives $d_{\mathcal{L}(G)}(g_C(x), g_C(y)) \geq d_G(h(g_C(x)), h(g_C(y))) = d_G(x, y)$. Then we obtain that

$$d_{\mathcal{L}(G)}(g_C(x), g_C(y)) = d_G(x, y).$$

Since γ is an arbitrary geodesic in C we obtain that g_C maps geodesics in G (contained in C) in geodesics in $\mathcal{L}(G)$ (contained in C^*). ☐

Now, we deal with the geodesics in $\mathcal{L}(G)$.

Lemma 4.1.7. *Let γ^* be a geodesic joining x and y in $\mathcal{L}(G)$. Then $h(\gamma^*)$ is a path in G joining $h(x)$ and $h(y)$, which is the union of three geodesics γ_1, γ_2, γ_3 in G, with $h(x) \in \gamma_1$, $h(y) \in \gamma_3$ and $0 \leq L(\gamma_1), L(\gamma_3) < \sup_{e\in E(G)} L(e)$.*

Proof. Note that if x, y are contained in one edge $[V_1, V_2]$ of $\mathcal{L}(G)$, then $\gamma^* \subset [V_1, V_2]$ and $h(\gamma^*)$ is a geodesic in G joining $h(x)$ and $h(y)$, since $h(\gamma^*) \subset \gamma := [h(V_1)h(Pm_{\mathcal{L}}([V_1, V_2]))] \cup [h(Pm_{\mathcal{L}}([V_1, V_2]))h(V_2)]$ and γ is a geodesic in G by Remark 4.1.3.

If x, y are not contained in the same edge of $\mathcal{L}(G)$, then let us consider V_α as the closest vertex in γ^* to α, for $\alpha \in \{x, y\}$ (it is possible to have $V_x = x$ or $V_y = y$). By Remark 4.1.3, we have that $h([V_x V_y]) = [h(V_x)h(V_y)]$ is a geodesic joining $h(V_x)$ and $h(V_y)$ in G; moreover, $h(\gamma^*) = [h(x)h(V_x)] \cup [h(V_x)h(V_y)] \cup [h(V_y)h(y)]$ where $[h(x)h(V_x)]$ and $[h(V_y)h(y)]$ are geodesics in G since x, V_x (respectively y, V_y) are contained in the same edge of $\mathcal{L}(G)$. This finishes the proof, since $L(e^*) \leq \sup_{e\in E(G)} L(e)$ for every $e^* \in E(\mathcal{L}(G))$. ☐

The arguments in the proof of Lemma 4.1.7 give the following result.

Lemma 4.1.8. *Let G be a graph with edges of length k and γ^* be a geodesic of $\mathcal{L}(G)$ joining x and y with $x, y \in PM_{\mathcal{L}}V(\mathcal{L}(G))$. Then $h(\gamma^*)$ is the union of three geodesics γ_1^*, γ_2^*, γ_3^* in G with $h(x) \in \gamma_1^*$, $h(y) \in \gamma_3^*$ and $0 \le L(\gamma_1^*), L(\gamma_3^*) \le k/2$.*

Also, we shall need a property of geodesic quadrilaterals in G.

Lemma 4.1.9. *For every x, y, u, v in the graph G, let us define $\Gamma := [xu] \cup [uv] \cup [vy]$. If $L([xu]), L([vy]) \le \sup_{e \in E(G)} L(e)$, then*

$$\forall\ \alpha \in \Gamma\ ,\ \exists\ \beta \in [xy]\ :\ d_G(\alpha, \beta) \le 2\delta(G) + \sup_{e \in E(G)} L(e). \tag{4.8}$$

Proof. Let us consider the geodesic quadrilateral $Q = \{[xy], [xu], [uv], [vy]\}$ and $\alpha \in \Gamma$. If $\alpha \in [xu] \cup [vy]$, then there exists $\beta \in \{x, y\} \subset [xy]$ such that $d_G(\alpha, \beta) \le \max\{L([xu]), L([vy])\} \le \sup_{e \in E(G)} L(e)$. If $\alpha \in [uv]$, then there exists $\alpha' \in [xy] \cup [xu] \cup [vy]$ such that $d_G(\alpha, \alpha') \le 2\delta(G)$ since Q is a geodesic quadrilateral in G. So, there exists $\beta \in [xy]$ such that $d_G(\alpha', \beta) \le \sup_{e \in E(G)} L(e)$. Then, we obtain that $d_G(\alpha, \beta) \le d_G(\alpha, \alpha') + d_G(\alpha', \beta) \le 2\delta(G) + \sup_{e \in E(G)} L(e)$. ☐

Theorem 4.1.10. *Let G be a graph and consider $\mathcal{L}(G)$ the line graph of G. Then*

$$\delta(G) \le \delta(\mathcal{L}(G)) \le 5\delta(G) + 3l_{max}, \tag{4.9}$$

with $l_{max} = \sup_{e \in E(G)} L(e)$. Furthermore, the first inequality is sharp: the equality is attained by every cycle graph.

Proof. First, let us consider a geodesic triangle $T = [xy] \cup [yz] \cup [zx]$ in G that is a cycle. Hence, if g_T is defined by (4.7), then Corollary 4.1.6 gives that $T^* = [g_T(x)g_T(y)] \cup [g_T(y)g_T(z)] \cup [g_T(z)g_T(x)]$ is a geodesic triangle in $\mathcal{L}(G)$; besides, by Proposition 4.1.2 we have that $d_G(u, v) \le d_{\mathcal{L}(G)}(g_T(u), g_T(v))$ for every $u, v \in T$.

Let $\Gamma = (\gamma_1, \gamma_2, \gamma_3)$ be a permutation of $([xy], [yz], [zx])$. So, by Proposition 4.1.2 we have

$$\begin{aligned}\sup_{a \in \gamma_1} \inf_{b \in \gamma_2 \cup \gamma_3} d_G(a, b) &\le \sup_{a \in \gamma_1} \inf_{b \in \gamma_2 \cup \gamma_3} d_{\mathcal{L}(G)}(g_T(a), g_T(b)) \\ &\le \sup_{a^* \in g_T(\gamma_1)} \inf_{b^* \in g_T(\gamma_2) \cup g_T(\gamma_3)} d_{\mathcal{L}(G)}(a^*, b^*).\end{aligned}$$

Since Γ is an arbitrary permutation, we obtain

$$\delta(T) \le \delta(T^*) \le \delta(\mathcal{L}(G)).$$

This finishes the proof of the first inequality by Corollary 1.3.4.

Now, let us consider a geodesic triangle $T^* = \{[x^*y^*], [y^*z^*], [z^*x^*]\}$ in $\mathcal{L}(G)$ that is a cycle, and a permutation $\Gamma = (\gamma_1^*, \gamma_2^*, \gamma_3^*)$ of $([x^*y^*], [y^*z^*], [z^*x^*])$. So, by Lemma 4.1.4 we have

$$\begin{aligned}\sup_{a^*\in\gamma_1^*}\inf_{b^*\in\gamma_2^*\cup\gamma_3^*} d_{\mathcal{L}(G)}(a^*,b^*) &\le \sup_{a^*\in\gamma_1^*}\inf_{b^*\in\gamma_2^*\cup\gamma_3^*} d_G(h(a^*),h(b^*))+2l_{max}\\ &\le \sup_{a\in h(\gamma_1^*)}\inf_{b\in h(\gamma_2^*)\cup h(\gamma_3^*)} d_G(a,b)+2l_{max}, \\ \sup_{a^*\in\gamma_1^*} d_{\mathcal{L}(G)}(a^*,\gamma_2^*\cup\gamma_3^*) &\le \sup_{a\in h(\gamma_1^*)} d_G(a,h(\gamma_2^*)\cup h(\gamma_3^*))+2l_{max}.\end{aligned} \tag{4.10}$$

By Lemma 4.1.7 we know that $h([x^*y^*])$ is the union of three geodesics $[\alpha_z^1 P_{\alpha_z^1}]$, $[P_{\alpha_z^1}P_{\alpha_z^2}]$ and $[P_{\alpha_z^2}\alpha_z^2]$ in G:

$$h([x^*y^*])=[\alpha_z^1 P_{\alpha_z^1}]\cup[P_{\alpha_z^1}P_{\alpha_z^2}]\cup[P_{\alpha_z^2}\alpha_z^2].$$

Analogously, $h([y^*z^*])$ and $h([z^*x^*])$ are the union of three geodesics in G:

$$h([y^*z^*])=[\alpha_x^1 P_{\alpha_x^1}]\cup[P_{\alpha_x^1}P_{\alpha_x^2}]\cup[P_{\alpha_x^2}\alpha_x^2],$$

$$h([z^*x^*])=[\alpha_y^1 P_{\alpha_y^1}]\cup[P_{\alpha_y^1}P_{\alpha_y^2}]\cup[P_{\alpha_y^2}\alpha_y^2].$$

Now, let us consider a geodesic triangle $T:=\{[h(x^*)h(y^*)],[h(y^*)h(z^*)],[h(z^*)h(x^*)]\}$ in G. Without loss of generality we can assume that $\gamma_1^*=[x^*y^*]$, $\gamma_2^*=[y^*z^*]$ and $\gamma_3^*=[z^*x^*]$. Hence, by Lemma 4.1.9 we have that, if $\alpha\in h(\gamma_1^*)$ then there exists $\beta\in[h(x^*)h(y^*)]$ such that

$$d_G(\alpha,\beta)\le 2\delta(G)+l_{max}.$$

Since $\beta\in[h(x^*)h(y^*)]$, there exists $\beta'\in[h(y^*)h(z^*)]\cup[h(z^*)h(x^*)]$ such that

$$d_G(\beta,\beta')\le\delta(G).$$

Without loss of generality we can assume that $\beta'\in[h(y^*)h(z^*)]$. If we consider the geodesic quadrilateral $\{[\alpha_x^1\alpha_x^2],[\alpha_x^1 P_{\alpha_x^1}],[P_{\alpha_x^1}P_{\alpha_x^2}],[P_{\alpha_x^2}\alpha_x^2]\}$, then there exists $\alpha'\in h([y^*z^*])$ such that

$$d_G(\beta',\alpha')\le 2\delta(G).$$

Thus, since $d_G(\alpha,h(\gamma_2^*)\cup h(\gamma_3^*))\le d_G(\alpha,\beta)+d_G(\beta,\beta')+d_G(\beta',\alpha')$ we obtain that

$$d_G(\alpha,h(\gamma_2^*)\cup h(\gamma_3^*))\le 5\delta(G)+l_{max}. \tag{4.11}$$

Then, by (4.10) and (4.11) we obtain

$$\sup_{a^*\in\gamma_1^*} d_{\mathcal{L}(G)}(a^*,\gamma_2^*\cup\gamma_3^*)\le 5\delta(G)+3l_{max}.$$

Finally, since Γ is an arbitrary permutation of any triangle that is a cycle, Corollary 1.3.4 gives

$$\delta(\mathcal{L}(G)) \le 5\delta(G) + 3l_{max}.$$

Theorem 1.3.19 gives that $\delta(G) = \delta(\mathcal{L}(G)) = L(G)/4$ for every cycle graph G. □

Remark 4.1.11. *The cycle graphs are not the only graphs G with $\delta(\mathcal{L}(G)) = \delta(G)$, as the following example shows. Let C_n be the cycle graph with n vertices and every edge with length k, and $u, v \in V(C_n)$ with $d_{C_n}(u, v) = 2k$; if G is the graph obtained by adding the edge $[u, v]$ (also with length k) to C_n, one can check that $\delta(\mathcal{L}(G)) = \delta(G) = kn/4$.*

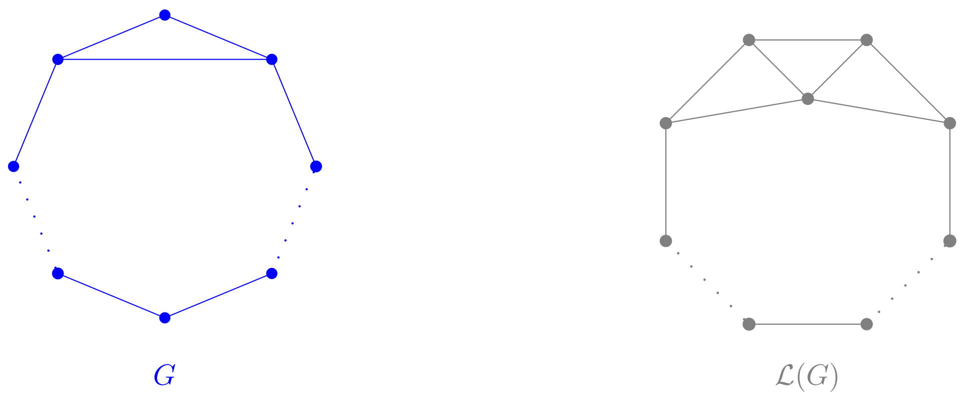


Figure 4.2: Family of graphs such that $\delta(\mathcal{L}(G)) = \delta(G)$.

Let us consider now graphs with edges of length k. We will improve Theorem 4.1.10 in this case.

Corollary 4.1.12. *Let G be any graph such that every edge has length k and consider $\mathcal{L}(G)$ the line graph of G. Then*

$$\delta(G) \le \delta(\mathcal{L}(G)) \le 5\delta(G) + \frac{5k}{2}.$$

Proof. We just need to prove the second inequality. By Theorem 1.3.23 it suffices to consider geodesic triangles in $\mathcal{L}(G)$ with vertices in $PM_{\mathcal{L}}V(\mathcal{L}(G)) = PMV(\mathcal{L}(G))$. Then the arguments in the proof of Theorem 4.1.10, replacing Lemma 4.1.7 by Lemma 4.1.8, give the result. □

Theorem 4.1.13. *Let G be any graph such that every edge has length k, with n vertices and maximum degree Δ. Then*

$$\delta(\mathcal{L}(G)) \le nk\Delta(\Delta - 1)/8,$$

and the equality is attained if and only if G is a cycle graph.

Proof. It is well known that $2(m(\mathcal{L}(G)) + m(G)) = \sum_{i=1}^{n}(\deg_G(v_i))^2$, where $\deg_G(v_i)$ are the degrees of the vertices of G. Since $2m(G) = \sum_{i=1}^{n}\deg_G(v_i)$, Lemma 1.3.10 gives the inequality, and the equality is attained if and only if G is a cycle graph. □

Using the argument in the proof of Theorem 4.1.13 we also obtain the following inequality.

Corollary 4.1.14. *If G is any graph such that every edge has length k, with n vertices $v_1, \dots, v_n$, then*

$$\delta(\mathcal{L}(G)) + \delta(G) \leq \frac{k}{8}\sum_{i=1}^{n}(\deg_G(v_i))^2,$$

and the equality is attained if and only if G is a cycle graph.

Chapter 5

Hyperbolicity of planar graphs and CW complexes.

In this Chapter we obtain information about either the hyperbolicity or the non-hyperbolicity of a wide class of planar graphs: the graphs which are the "boundary" (the 1-skeleton) of a tessellation of the Euclidean plane. The edges of such a tessellation graph are just rectifiable paths in $\mathbb{R}^2$ and have the length induced by the metric in $\mathbb{R}^2$ (they may or may not be geodesics in $\mathbb{R}^2$).

In fact, in Section 5.1 we provide several criteria in order to conclude that many tessellation graphs of the Euclidean plane $\mathbb{R}^2$ are non-hyperbolic. One can think that the tessellation graphs of the Euclidean plane $\mathbb{R}^2$ are always non-hyperbolic, since the plane is non-hyperbolic (and then the theory would be trivial); however, there exists a hyperbolic tessellation graph of $\mathbb{R}^2$ (see [65]).

These tessellation graphs are the 1-skeleton (i.e., the set of 1-cells and 0-cells) of a CW 2-complex contained in the Euclidean plane. In Section 5.2 we deal with a wider class of graphs: the graphs which are the 1-skeleton of an abstract CW 2-complex (not necessarily contained in $\mathbb{R}^2$ or in some Riemannian surface). In fact, we prove that a graph obtained as the 1-skeleton of a CW 2-complex is hyperbolic if and only if its dual graph is hyperbolic, under some reasonable hypotheses (see Theorem 5.2.4). This result improves [65, Theorem 4.1] about a particular kind of CW 2-complexes: the ones that are a tessellation of some complete Riemannian surface without boundary (in this special case every edge belongs exactly to two faces). Furthermore, Section 5.2 contains Theorem 5.2.9, which is a stronger version of Theorem 5.2.4.

5.1 Hyperbolicity of planar graphs.

We obtain in this section results on the hyperbolicity of the 1 skeleton of tessellation graphs of $\mathbb{R}^2$. The main results in this section are Theorems 5.1.2 and 5.1.8, since they are the key tools in order to prove the other results.

We will need the following result (see [65, Theorem 3.1]):

Lemma 5.1.1. *Let $\mathcal{S}$ be a Riemannian surface with curvature satisfying $K \geq -k^2$ for some constant k. Suppose that a graph G is the 1-skeleton of a tessellation of $\mathcal{S}$ with tiles $\{F_n\}$ such that there exist sets $\mathcal{L}_1, \mathcal{L}_2$ which are a partition of the sets of indices n, and positive constants c_1, c_2, verifying the following properties:* $\operatorname{diam}_G \partial F_n \leq c_1$, $\operatorname{diam}_{\mathcal{S}} F_n \leq c_1$ *and* $A_{\mathcal{S}}(F_n) \geq c_2$ *for every $n \in \mathcal{L}_1$, and $d_{\partial F_n}(x,y) \leq c_1 d_{\mathcal{S}}(x,y)$ for every $x,y \in \partial F_n$ and for every $n \in \mathcal{L}_2$. If $\mathcal{S}$ is hyperbolic, then G is hyperbolic, quantitatively.*

Furthermore, if $\operatorname{diam}_{\mathcal{S}} F_n \leq c_1$ *for every $n \in \mathcal{L}_2$, then $\mathcal{S}$ is hyperbolic if and only if G is hyperbolic, quantitatively.*

We denote by $\operatorname{int}(F)$ the topological interior of the set F.

Theorem 5.1.2. *Suppose that a graph G is the 1-skeleton of a tessellation of $\mathbb{R}^2$ with tiles $\{F_n\}$. Denote by c_n the shortest cycle in G homotopic to ∂F_n in $\mathbb{R}^2 \setminus \operatorname{int}(F_n)$. If* $\sup_n L(c_n) = \infty$, *then G is not hyperbolic.*

Proof. First, we prove that c_n is an isometric subgraph of G for every n. Seeking for a contradiction assume that there exists n such that c_n is not an isometric subgraph of G. Then there exist $x, y \in c_n$ and a curve γ in G joining them with $L(\gamma) < d_{c_n}(x,y)$; therefore, if g_1, g_2 are the two curves joining x and y with $g_1 \cup g_2 = c_n$, we have $L(\gamma) < \min\{L(g_1), L(g_2)\}$. Since c_n is homotopic to ∂F_n in $\mathbb{R}^2 \setminus \operatorname{int}(F_n)$, we have that either $\gamma \cup g_1$ or $\gamma \cup g_2$ is homotopic to ∂F_n in $\mathbb{R}^2 \setminus \operatorname{int}(F_n)$. Since $\max\{L(\gamma \cup g_1), L(\gamma \cup g_2)\} < L(c_n)$, we have the contradiction we were aiming for, and we conclude that c_n is an isometric subgraph of G for every n.

Let us fix n. If γ_1, γ_2 are two curves with $\gamma_1 \cup \gamma_2 = c_n$ and $L(\gamma_1) = L(\gamma_2) = L(c_n)/2$, then $B = \{\gamma_1, \gamma_2\}$ is a geodesic bigon (a geodesic triangle such that two of its vertices are the same point) in c_n. If we denote by z the midpoint of γ_1, then $\delta(B) \geq d_{c_n}(z, \gamma_2) = L(c_n)/4$. Lemma 1.3.2 gives $\delta(G) \geq \delta(c_n) \geq \delta(B) \geq L(c_n)/4$ for every n, and we deduce that G is not hyperbolic. □

Corollary 5.1.3. *Suppose that a graph G is the 1-skeleton of a tessellation of $\mathbb{R}^2$ with tiles $\{F_n\}$. Let us assume that there exists a subsequence of tiles $\{F_{n_k}\}_k$ such that they are all convex tiles and, besides,* $\sup_k L(\partial F_{n_k}) = \infty$. *Then G is not hyperbolic.*

Proof. Since each F_{n_k} is a convex polygon, we have $c_{n_k} = \partial F_{n_k}$, where c_{n_k} are the shortest cycles mentioned in Theorem 5.1.2. Applying that theorem, the conclusion is straightforward. □

Corollary 5.1.4. *Suppose that a graph G is the 1-skeleton of a tessellation of $\mathbb{R}^2$ with tiles $\{F_n\}$. If there exists a sequence of balls $\{B_n\}$ with radius r_n such that $B_n \subseteq F_n$ for every n and* $\sup_n r_n = \infty$, *then G is not hyperbolic.*

Proof. Let us consider the cycles c_n as in Theorem 5.1.2. For each n it is obvious that $L(c_n) \geq L(\partial B_n) = 2\pi r_n$. Therefore, $\sup_n L(c_n) = \infty$ and Theorem 5.1.2 gives the conclusion. $\square$

Theorem 5.1.5. *Suppose that a graph G is the* 1*-skeleton of a tessellation of* $\mathbb{R}^2$ *with tiles* $\{F_n\}$*. Assume that every tile F_n can be obtained from a finite set of tiles $\hat{F}_1, \hat{F}_2, \dots, \hat{F}_m$ by means of translations, rotations and dilations. Then, G is not hyperbolic.*

Proof. Since every tile F_n can be obtained from a finite set of tiles $\hat{F}_1, \hat{F}_2, \dots, \hat{F}_m$ by means of translations, rotations and dilations, there exists a positive constant k_1 such that $d_{\partial F_n}(x,y) \leq k_1\ d_{\mathbb{R}^2}(x,y)$ for every $x,y \in \partial F_n$ and for every n.

Let us consider the cycles c_n as in Theorem 5.1.2. If $\sup_n L(\partial F_n) = \infty$, then $\sup_n L(c_n) = \infty$ and Theorem 5.1.2 gives the result. Assume now that $\sup_n L(\partial F_n) < \infty$. Since

$$\sup_n \operatorname{diam}_{\mathbb{R}^2} F_n = \sup_n \operatorname{diam}_{\mathbb{R}^2} \partial F_n \leq \sup_n \operatorname{diam}_G \partial F_n \leq \frac{1}{2} \sup_n L(\partial F_n) < \infty,$$

and $\mathbb{R}^2$ is not hyperbolic, Lemma 5.1.1 (with $\Lambda_1 = \varnothing$) gives that G is not hyperbolic. $\square$

Theorem 5.1.6. *Suppose that a graph G is the* 1*-skeleton of a tessellation of* $\mathbb{R}^2$ *with convex tiles* $\{F_n\}$*. If* $\inf_n A(F_n) > 0$*, then G is not hyperbolic.*

Proof. If $\sup_n L(\partial F_n) = \infty$, then Theorem 5.1.2 gives the result. Assume now that $\sup_n L(\partial F_n) = c_1 < \infty$. Since

$$\operatorname{diam}_{\mathbb{R}^2} F_n = \operatorname{diam}_{\mathbb{R}^2} \partial F_n \leq \operatorname{diam}_G \partial F_n \leq \frac{1}{2} L(\partial F_n) \leq \frac{c_1}{2},$$

$\inf_n A(F_n) > 0$ and $\mathbb{R}^2$ is not hyperbolic, Lemma 5.1.1 (with $\Lambda_2 = \varnothing$) gives that G is not hyperbolic. $\square$

In order to prove our next theorem, we will need the following well-known (and non-trivial) result.

Lemma 5.1.7. *Given any open convex set $C \subset \mathbb{R}^2$ and any curve $g \subset \mathbb{R}^2 \setminus C$ joining two points $x,y \in \partial C$, there exists a curve $\gamma \subset \partial C$ joining x and y with $L(\gamma) \leq L(g)$.*

Theorem 5.1.8. *Suppose that a graph G is the* 1*-skeleton of a tessellation of* $\mathbb{R}^2$ *with convex tiles* $\{F_n\}$*. Let us assume that there exist balls $B_n \subset F_n$ with radius r_n such that $L(\partial F_n) \leq c_1 r_n$ for some positive constant c_1 and for every n. Then G is not hyperbolic.*

Proof. By Corollary 5.1.3, we can assume that there exists a constant k_1 with $L(\partial F_n) \leq 2k_1$ for every n. Note that $\operatorname{diam}_{\mathbb{R}^2} F_n \leq L(\partial F_n)/2 =: R_n \leq k_1$ for every n. If we denote by B_n^* the closed ball with the same center as B_n and radius R_n, then we have $B_n \subset F_n \subset B_n^*$ and

$$\frac{R_n}{r_n} \leq \frac{L(\partial F_n)/2}{c_1^{-1} L(\partial F_n)} = \frac{c_1}{2}\,. \tag{5.1}$$

Let us consider $z, w \in \partial F_n$. We want to show that there exists a constant c, which just depends on c_1, such that $d_{\partial F_n}(z,w) \le c\, d_{\mathbb{R}^2}(z,w)$. If z and w belong to the same edge, then $d_{\partial F_n}(z,w) = d_{\mathbb{R}^2}(z,w)$. Assume now that z and w belong to different edges. On the one hand, if $d_{\mathbb{R}^2}(z,w) \ge r_n$, then

$$d_{\partial F_n}(z,w) \le \frac{1}{2}\, L(\partial F_n) \le \frac{1}{2}\, L(\partial F_n)\, \frac{d_{\mathbb{R}^2}(z,w)}{r_n} \le \frac{c_1}{2}\, d_{\mathbb{R}^2}(z,w).$$

On the other hand, let us consider the case $d_{\mathbb{R}^2}(z,w) < r_n$. Without loss of generality we can assume that the origin O is the center of B_n and B_n^*. Since $d_{\mathbb{R}^2}(z,w) < r_n$, we have that $|\arg z - \arg w| < \pi/3$. Let us consider straight lines S_z and S_w containing the edges in ∂F_n which contain z and w, respectively; let ζ be the point $\zeta := S_z \cap S_w$ (note that if the edges in ∂F_n which contain z and w are not adjacent, then may be have $[z\zeta] \not\subseteq G$ and $[\zeta w] \not\subseteq G$).

Since F_n is convex, Lemma 5.1.7 gives that

$$\frac{d_{\partial F_n}(z,w)}{d_{\mathbb{R}^2}(z,w)} \le \frac{L([z\zeta] \cup [\zeta w])}{d_{\mathbb{R}^2}(z,w)}\,.$$

Let us denote by α the angle at ζ of $[z\zeta]$ and $[\zeta w]$ $(0 < \alpha < \pi)$. Note that

$$\frac{d_{\partial F_n}(z,w)}{d_{\mathbb{R}^2}(z,w)} \le \frac{L([z\zeta] \cup [\zeta w])}{d_{\mathbb{R}^2}(z,w)} \le \max\left\{ \frac{L([u\zeta] \cup [\zeta v])}{d_{\mathbb{R}^2}(u,v)} \,:\, u \in [z\zeta], v \in [\zeta w] \right\} = \frac{1}{\sin(\alpha/2)}\,. \tag{5.2}$$

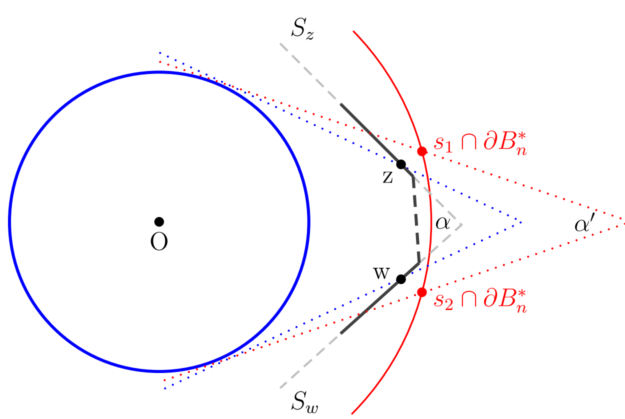


Figure 5.1: Auxiliary graphic of Theorem 5.1.8.

In order to obtain a lower bound for α, consider two straight lines s_1 and s_2 tangent to ∂B_n such that $d_{\mathbb{R}^2}(s_1 \cap \partial B_n^*, s_2 \cap \partial B_n^*) = r_n$ and $s_1 \cap s_2 \notin B_n^*$. One can check that if α' is the angle between s_1 and s_2, then $\alpha \ge \alpha'$, since F_n is a convex tile. Applying now a dilation (or a contraction) we obtain two straight lines s_1' and s_2' tangent to the unit circle such that $d_{\mathbb{R}^2}(s_1' \cap \partial(B_n^*)', s_2' \cap \partial(B_n^*)') = 1$, where $(B_n^*)'$ is the ball centered at O with radius

$R_n/r_n \le c_1/2$. Then α' is the angle between s_1' and s_2', and it is clear that $\alpha' = f(R_n/r_n)$ for some positive decreasing function. Hence, $\alpha \ge \alpha' = f(R_n/r_n) \ge f(c_1/2) =: \alpha_0 > 0$ and by (5.2) we conclude that

$$\frac{d_{\partial F_n}(z,w)}{d_{\mathbb{R}^2}(z,w)} \le \frac{1}{\sin(\alpha/2)} \le \frac{1}{\sin(\alpha_0/2)}\,.$$

Therefore,

$$d_{\partial F_n}(z,w) \le \max\left\{\frac{c_1}{2}\,,\, \frac{1}{\sin(\alpha_0/2)}\right\} d_{\mathbb{R}^2}(z,w),$$

for every $z, w \in \partial F_n$ and for every n. Now, since $\operatorname{diam}_{\mathbb{R}^2} F_n \le k_1$ for every n, Lemma 5.1.1 (with $\mathcal{L}_1 = \varnothing$) finishes the proof. ∎

This theorem has the following consequence.

Corollary 5.1.9. *Suppose that a graph G is the 1-skeleton of a tessellation of $\mathbb{R}^2$ with tiles $\{F_n\}$. If every tile F_n is a regular polygon, then G is not hyperbolic.*

Proof. Let us fix n, and we define k_n as the number of edges in F_n, ω_n as the incircle of F_n, r_n as the inradius of F_n and R_n as the circumradius of F_n. It is well known that $r_n = R_n \cos(\pi/k_n)$, and since $k_n \ge 3$ we have $\cos(\pi/k_n) \ge 1/2$. Then, by Lemma 5.1.7 we have

$$L(\partial F_n) \le 2\pi R_n = \frac{2\pi}{\cos(\pi/k_n)}\, r_n \le 4\pi r_n.$$

Thus, by taking ω_n as B_n in Theorem 5.1.8, we obtain the result. ∎

Suppose that a graph G is the 1-skeleton of a tessellation of $\mathbb{R}^2$ with convex tiles $\{F_n\}$. Let us define

$$\begin{aligned} l_n &:= \min\{L_G(e) \mid e \in E(G),\, e \subset \partial F_n\}, \\ L_n &:= \max\{L_G(e) \mid e \in E(G),\, e \subset \partial F_n\}, \\ \alpha_n &:= \min\{\text{interior angles at the vertices in } \partial F_n\}, \\ N_n &:= \operatorname{card}\{e \in E(G) \mid e \subset \partial F_n\}. \end{aligned}$$

Theorem 5.1.10. *Suppose that a graph G is the 1-skeleton of a tessellation of $\mathbb{R}^2$ with convex tiles $\{F_n\}$. Let us assume that $L(\partial F_n) \le c_1 l_n$ and $\alpha_n \ge c_2$ for some positive constants c_1, c_2 and for every n. Then G is not hyperbolic.*

Proof. For each fixed n, let us consider two adjacent edges e_n^1, e_n^2 contained in ∂F_n such that α_n is attained at the point $V_n := e_n^1 \cap e_n^2$. We have that $N_n \le c_1$, since $L(\partial F_n) \le c_1 l_n$. So, we have that $\alpha_n \le (N_n - 2)\pi/N_n$, since F_n is a convex polygon. Let us consider $u_1 \in e_n^1$ and $u_2 \in e_n^2$ such that $d_{\mathbb{R}}(u_1, V_n) = d_{\mathbb{R}}(V_n, u_2) = l_n$. If A_n is the Euclidean convex hull in $\mathbb{R}^2$ of $\{u_1, V_n, u_2\}$ and B_n the incircle of A_n with radius r_n, then $L(\partial A_n) \le 6 r_n \operatorname{cotan}(\alpha/2)$

with $\alpha = \min\{c_2, \pi/c_1\}$, since $\min\{\textit{interior angles at the vertices of } \partial A_n\} \geq \min\{c_2, \pi/N_n\}$. Therefore, we have $L(\partial F_n) \leq c_1 l_n < (c_1/2)L(\partial A_n) \leq 3c_1 \operatorname{cotan}(\alpha/2)\, r_n$. Then Theorem 5.1.8 gives the result. □

From Theorem 5.1.10 we obtain directly the following result.

Corollary 5.1.11. *Suppose that a graph G is the 1-skeleton of a tessellation of $\mathbb{R}^2$ with convex tiles $\{F_n\}$. Let us assume that $L_n \leq c_1 l_n$, $N_n \leq c_1$ and $\alpha_n \geq c_2$ for some positive constants c_1, c_2 and for every n. Then G is not hyperbolic.*

We obtain the following results from Corollary 5.1.11.

Corollary 5.1.12. *Suppose that a graph G is the 1-skeleton of a tessellation of $\mathbb{R}^2$ with triangular tiles $\{F_n\}$. Let us assume that $\alpha_n \geq c_2$ for some positive constant c_2 and for every n. Then G is not hyperbolic.*

Corollary 5.1.13. *Suppose that a graph G is the 1-skeleton of a tessellation of $\mathbb{R}^2$ with rectangular tiles $\{F_n\}$. Let us assume that $L_n \leq c_1 l_n$ for some positive constant c_1 and for every n. Then G is not hyperbolic.*

Open problem. At the light of these results we conjecture that every tessellation graph of $\mathbb{R}^2$ with convex tiles is non-hyperbolic. The proof of this conjecture would use different arguments, since some tessellation graphs of $\mathbb{R}^2$ with convex tiles are not quasi-isometric to $\mathbb{R}^2$.

5.2 Hyperbolicity of dual graphs.

In this section we get results for a class of geodesic metric spaces wider than the tessellation graphs of the plane. First of all we give the precise definition of CW complex.

Definition 5.2.1. *Let D^n be the closed unit ball in $\mathbb{R}^n$. An n-cell ($n \geq 1$) is a space homeomorphic to the open n-ball $\operatorname{int}(D^n)$; a 0-cell is a single point. A cell is a space which is an n-cell for some $n \geq 0$.*

Note that $\operatorname{int}(D^m)$ and $\operatorname{int}(D^n)$ are homeomorphic if and only if $m = n$. Thus we can talk about the dimension of a cell. An n-cell will be said to have dimension n.

Definition 5.2.2. *A* cell-decomposition *of a space X is a family $\xi = \{e_\alpha | \alpha \in I\}$ of subspaces of X such that each e_α is a cell and*

$$X = \bigcup_{\alpha \in I} e_\alpha$$

(disjoint union of sets). The n-skeleton of X is the subspace $X^n = \cup_{\alpha \in I : dim(e_\alpha) \leq n}\, e_\alpha$.

Definition 5.2.3. *A pair (X, ξ) consisting of a Hausdorff space X and a cell-decomposition ξ of X is called a* CW-complex *if the following axioms are satisfied:*

Axiom 1 *(Characteristic Maps) For each n-cell $e \in \xi$ $(n \geq 1)$ there is a map $\Phi_e : D^n \longrightarrow X$ restricting to a homeomorphism $\Phi_e|_{\text{int}(D^n)} : \text{int}(D^n) \longrightarrow e$ and taking S^{n-1} into X^{n-1}.*

Axiom 2 *(Closure Finiteness) For any cell $e \in \xi$ the closure $\overline{e}$ intersects only a finite number of other cells in ξ.*

Axiom 3 *(Weak Topology) A subset $A \subseteq X$ is closed if and only if $A \cap \overline{e}$ is closed in X for each $e \in \xi$.*

If the largest dimension of any of the cells is n, then the CW complex is said CW n-complex.

We consider in this section a very large class of graphs which contains the tessellation graphs of complete Riemannian surfaces (with or without boundary): the set of all graphs G which are the 1-skeleton (the set of 0-cells and 1-cells) of some connected CW 2-complex. The *dual graph* G^* of such a graph G is a graph which has a vertex $p_j \in V(G^*)$ for each face (2-cell) P_j of the CW 2-complex, and an edge joining p_i and p_j for each edge of G in $\overline{P_i} \cap \overline{P_j}$ (if there are k edges in $\overline{P_i} \cap \overline{P_j}$, then $[p_i, p_j]$ is a multiple edge of order k). By definition, every edge of G^* has length 1.

Note that a CW 2-complex is a very general structure: if an edge e belongs to the closure of m_e faces, then m_e can be any non-negative integer number; also, two edges in the boundary of a face can be "identified" in the CW complex.

Next, we deal with the main result of this section.

Theorem 5.2.4. *Let G be the 1-skeleton of a connected CW 2-complex C and G^* be its dual graph. Let k_1, k_2, M, K be fixed positive constants. Assume that every edge $e \in E(G)$ is contained in the closure of a face of C and satisfies $k_1 \leq L(e) \leq k_2$, every face of C has at most M edges and if the closures of two faces in C have non-empty intersection then the corresponding vertices in G^* are at distance at most K. Then G is δ-hyperbolic if and only if G^* is δ^*-hyperbolic, quantitatively.*

Proof. Without loss of generality we can assume that $K \geq 1$. Let G_0 be a graph isomorphic to G such that every edge of G_0 has length 1. Note that any isomorphism $g : G \to G_0$ is a bijective $(\max\{k_2, k_1^{-1}\}, 0)$-quasi-isometry; therefore, by Lemma 1.3.5, without loss of generality we can assume that every edge of G has length 1.

Note that G^* is a connected graph, since G is connected, every edge in G is contained in the closure of a face of C and if the closures of two faces in C have non-empty intersection then the corresponding vertices in G^* are connected in G^*.

We obtain the result by proving that there exists a $(5/2)$-full $(\max\{2K+2, M/2\}, \max\{4K+4, M\})$-quasi-isometry $f : G \longrightarrow G^*$.

Since the graph G is the 1-skeleton of a CW 2-complex C with faces $\{P_n\}$, then $\{p_n\} = V(G^*)$. If an edge e is contained in $\overline{P_i} \cap \overline{P_j}$, $i \neq j$, we denote by $w^{(i,j)}(e)$ the midpoint of the edge in G^* corresponding to $e \in E(G)$, i.e., $w^{(i,j)}(e) \in [p_i, p_j]$ with $d_{G^*}(p_i, w^{(i,j)}(e)) = 1/2 = d_{G^*}(w^{(i,j)}(e), p_j)$. If e belongs to $\overline{P_i}$ for just one i, then we define $W(e) := p_i$; otherwise, we define $W(e)$ as the set of midpoints of the edges in G^* corresponding to $e \in E(G)$, i.e., $W(e) := \{w^{(i,j)}(e) \in [p_i, p_j] \mid e \subset \overline{P_i} \cap \overline{P_j}\}$. Note that, if there are $k \geq 2$ faces containing the edge e in their closures, then $|W(e)| = k(k-1)/2$. For each n, we write $\partial P_n = e_n^1 \cup \cdots \cup e_n^{j_n}$ (note that $j_n \leq M$ by hypothesis). We denote by $W_n(e_n^k)$ the set of midpoints of the edges in G^* starting in p_n and corresponding to e_n^k, note that $W_n(e_n^k) \subseteq W(e_n^k)$ and that $W_n(e_n^k) = \varnothing$ if and only if e_n^k belongs to the closure of just one face. We define now $P_n^* := \cup_{k=1}^{j_n} \big(\cup_{x \in W_n(e_n^k)} [p_n x] \big)$.

We define a function $f : G \longrightarrow G^*$ as follows: if e belongs to $\overline{P_i}$ for just one i, then we define $f(x) = p_i$ for every $x \in \operatorname{int}(e)$; otherwise, we choose two faces P_i,P_j with $e \subset \overline{P_i} \cap \overline{P_j}$ and we define $f(x) = w^{(i,j)}(e)$ for every $x \in \operatorname{int}(e)$; for each vertex $v \in V(G)$, let us choose an edge $e \in E(G)$ starting in v, and define $f(v)$ as the image via f of the interior of e.

Let us consider $x, y \in \partial P_n$. If $f(x), f(y) \in P_n^*$, then $d_{G^*}(f(x), f(y)) \leq \operatorname{diam}_{G^*}(P_n^*) = 1$. If we have $f(x) \in P_n^*$ and $f(y) \notin P_n^*$, then $y \in e$ with $f(y) \in W(e) \cap (P_n^*)^c$ or y is a vertex of G. Recall that, by hypothesis, if $\overline{P_i} \cap \overline{P_j} \neq \varnothing$ then $d_{G^*}(p_i, p_j) \leq K$. Therefore, if $f(x) \in P_n^*$, $f(y) \notin P_n^*$ and y is a vertex of G, then $d_{G^*}(f(x), f(y)) \leq K + 1$. Also, if $f(x) \in P_n^*$, $f(y) \notin P_n^*$ and $y \in e$ with $f(y) \in W(e) \cap (P_n^*)^c$, then $d_{G^*}(p_n, f(y)) = 3/2$ and $d_{G^*}(f(x), f(y)) \leq 2$. In both cases, we have $d_{G^*}(f(x), f(y)) \leq K + 1$, since $K \geq 1$. If $f(x) \notin P_n^*$, $f(y) \in P_n^*$, then we also have $d_{G^*}(f(x), f(y)) \leq K + 1$ by symmetry. If $f(x), f(y) \notin P_n^*$, then $d_{G^*}(f(x), f(y)) \leq d_{G^*}(f(x), p_n) + d_{G^*}(p_n, f(y)) \leq 2K + 2$. Hence,

$$d_{G^*}(f(x), f(y)) \leq 2K + 2\,, \qquad \text{for every } n \text{ and for all } x, y \in \partial P_n. \tag{5.3}$$

Fix now $x, y \in G$ and a geodesic γ in G joining x with y (then, $L_G(\gamma) = d_G(x, y)$). Let $\mathbb{P}$ be the set of collections of faces $P = \{P_{j_1}, P_{j_2}, \ldots, P_{j_r}\}$ with $\gamma \subset \cup_{m=1}^r \partial P_{j_m}$ and $\gamma \cap \partial P_{j_m}$ connected for every m; we say that r is the size of the collection P and we denote it by $s(P) = r$. Let us consider $P' \in \mathbb{P}$ with $s(P') = \min_{P \in \mathbb{P}} s(P) =: k$. Denote by $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ the faces in P'; without loss of generality we can assume that γ meets $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ in this order (with $x \in \partial P_{i_1}, y \in \partial P_{i_k}$); note that it is possible to have $i_a = i_b$ with $a \neq b$. Define γ_j as the connected subgeodesic of γ such that $\gamma_j \subseteq \gamma \cap \partial P_{i_j}$ $(1 \leq j \leq k)$, $\gamma_i \cap \gamma_{i+1} \neq \varnothing$ $(1 \leq j \leq k-1)$ if $k > 1$, and $\gamma = \gamma_1 \cup \cdots \cup \gamma_k$. Note that $L_G(\gamma_j) \geq 1$ for $1 < j < k$, $L_G(\gamma_1) > 0$ and $L_G(\gamma_k) > 0$.

If $k = 1$, then (5.3) gives $d_{G^*}(f(x), f(y)) \leq 2K + 2$.

If $k = 2$ and $z \in \gamma_1 \cap \gamma_2$, then we have

$$d_{G^*}(f(x), f(y)) \leq d_{G^*}(f(x), f(z)) + d_{G^*}(f(z), f(y)) \leq 2K + 2 + 2K + 2 = 4K + 4.$$

If $k \geq 3$ and $z_j \in \gamma_j \cap \gamma_{j+1}$ for $1 \leq j \leq k-2$ then, we have $d_G(z_j, z_{j+1}) \geq 1$ and

$$\begin{aligned} d_{G^*}(f(x), f(y)) &\leq d_{G^*}(f(x), f(z_1)) + \sum_{j=1}^{k-2} d_{G^*}(f(z_j), f(z_{j+1})) + d_{G^*}(f(z_{k-1}), f(y)) \\ &\leq 2K+2 + \sum_{j=1}^{k-2}(2K+2) + 2K+2 \leq 4K+4 + (2K+2)\sum_{j=1}^{k-2} d_G(z_j, z_{j+1}) \\ &= 4K+4 + (2K+2)\, d_G(z_1, z_{k-1}) \leq 4K+4 + (2K+2)\, d_G(x, y). \end{aligned}$$

Let us consider a geodesic γ^* in G^* joining $f(x)$ with $f(y)$ (then, $L_{G^*}(\gamma^*) = d_{G^*}(f(x), f(y))$). Note that $f(x)$ (respectively, $f(y)$) is either a midpoint of one edge in $E(G^*)$ or a vertex in $V(G^*)$. If $f(x) = f(y)$ then there exists i such that $x, y \in \partial P_i$; since every face of G has at most M edges, we have $d_G(x, y) \leq M/2$. Then, $d_{G^*}(f(x), f(y)) = 0 \geq d_G(x, y) - M/2$. We assume now that there exist $p_{i_1}, p_{i_2}, \dots, p_{i_m} \in V(G^*)$ such that γ^* meets $f(x), p_{i_1}, \dots, p_{i_m}, f(y)$ in this order $\big($with $0 \leq d_{G^*}(f(x), p_{i_1}), d_{G^*}(p_{i_n}, f(y)) \leq 1/2\big)$ and we have $d_{G^*}(f(x), f(y)) = m$ (if $f(x)$, $f(y)$ are midpoints of edges), $d_{G^*}(f(x), f(y)) = m + 1/2$ (if just one is a midpoint of some edge) or $d_{G^*}(f(x), f(y)) = m - 1$ (if $f(x), f(y) \in V(G^*)$).

If $m = 1$, then we have that $f(x), f(y) \in P_i^*$ and $x, y \in \partial P_i$; so, we have $d_G(x, y) \leq M/2$. Therefore, $d_{G^*}(f(x), f(y)) \geq 0 \geq d_G(x, y) - M/2$.

Assume now that $m \geq 2$. Let $w_n := w^{(i_n, i_{n+1})} \in \gamma^*$ be the midpoint of the edge $[p_{i_n}, p_{i_{n+1}}]$, for $1 \leq n \leq m-1$. Let us consider an edge $e_n \subseteq P_{i_n} \cap P_{i_{n+1}}$, for $1 \leq n \leq m-1$; let z_n be the midpoint of e_n, for $1 \leq n \leq m-1$. Then, for $1 \leq n \leq m-1$, we have $d_G(z_n, z_{n+1}) \leq M/2$, $d_{G^*}(w_n, w_{n+1}) = 1$, and

$$\begin{aligned} d_{G^*}(f(x), f(y)) &= d_{G^*}(f(x), w_1) + \sum_{n=1}^{m-2} d_{G^*}(w_n, w_{n+1}) + d_{G^*}(w_{m-1}, f(y)) \\ &= d_{G^*}(f(x), w_1) + \sum_{n=1}^{m-2} 1 + d_{G^*}(w_{m-1}, f(y)) \\ &\geq d_G(x, z_1) - \frac{M}{2} + \frac{2}{M}\sum_{n=1}^{m-2}\frac{M}{2} + d_G(z_{m-1}, y) - \frac{M}{2} \\ &\geq \frac{2}{M}\, d_G(x, z_1) + \frac{2}{M}\sum_{n=1}^{m-2} d_G(z_n, z_{n+1}) + \frac{2}{M}\, d_G(z_{m-1}, y) - M \\ &\geq \frac{2}{M}\, d_G(x, y) - M. \end{aligned}$$

Consequently, f is a $(\max\{2K+2, M/2\}, \max\{4K+4, M\})$-quasi-isometric embedding. Furthermore, f is $(5/2)$-full, since for every $e \in E(G)$ we have

$$\operatorname{diam}_{G^*}(W(e)) \leq 2\,, \quad W(e) \cap f(G) \neq \varnothing \quad \text{and} \quad \sup_{x \in G^*} d_{G^*}(x, \cup_{e \in E(G^*)} W(e)) = 1/2.$$

This finishes the proof by Lemma 1.3.5. □

The following examples show that the conclusion of Theorem 5.2.4 does not hold if we remove any hypothesis from its statement.

Example 5.2.5. *Let us consider the sequence of wheel graphs $\{W_n\}_{n=4}^{\infty}$ (W_n has n vertices). Choose two vertices $a_n^n, b_n^n \in V(W_n)$ (different from the central vertex of W_n) with $d_{W_n}(a_n^n, b_n^n) = 1$ for $n \geq 5$, and two vertices $a_{n+1}^n, b_{n+1}^n \in V(W_n)$ (different from the central vertex of W_n) with $d_{W_n}(a_{n+1}^n, b_{n+1}^n) = 1$ for $n \geq 4$ and $\{a_{n+1}^n, b_{n+1}^n\} \cap \{a_n^n, b_n^n\} = \varnothing$ for $n \geq 5$. We define G as the union of $\{W_n\}_{n=4}^{\infty}$ obtained by identifying $[a_{n+1}^n, b_{n+1}^n]$ with $[a_{n+1}^{n+1}, b_{n+1}^{n+1}]$ for $n \geq 4$. Since the central vertex of each W_n has degree $n-1$ there are two faces with a common vertex (the central vertex of W_n) which their associated vertices in G^* are at a distance $\lfloor (n-1)/2 \rfloor$. It is clear that G is quasi-isometric to the graph G' obtained as the union of $\{W_n\}_{n=4}^{\infty}$ by identifying a_{n+1}^n with a_{n+1}^{n+1} for $n \geq 4$. Lemmas 1.3.7 and 1.3.19 give that G' is hyperbolic, since $\delta(G') = \sup_n \delta(W_n) = 3/2$. Hence, G is also hyperbolic by Lemma 1.3.5.*

Its dual graph G^* is isometric to a union of cycle graphs $\{C_n\}_{n=3}^{\infty}$ such that each C_n is joined with C_{n+1} by a graph isometric to the path graph P_2 for $n \geq 3$. Lemmas 1.3.7 and 1.3.19 give that G^* is not hyperbolic, since $\delta(G^*) = \sup_n \delta(C_n) = \sup_n n/4 = \infty$, although G is hyperbolic.

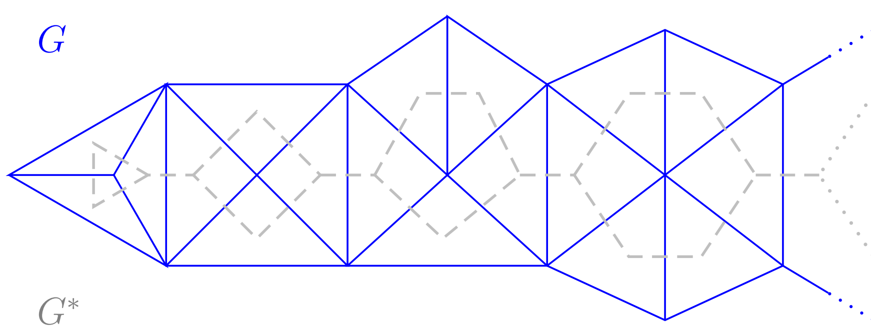


Figure 5.2: Infinite graph obtained by wheels graphs and its dual graphs.

Recall that given graphs G_1 and G_2, the Cartesian product of its graphs, denoted by $G_1 \square G_2$, is the graph with vertices $V(G_1 \square G_2) = V(G_1) \times V(G_2)$ and $[(u_1, u_2), (v_1, v_2)] \in$

$E(G_1 \square G_2)$ if and only if we have either $u_1 = v_1 \in V(G_1)$ and $[u_2, v_2] \in E(G_2)$ or $u_2 = v_2 \in V(G_2)$ and $[u_1, v_1] \in E(G_1)$.

Example 5.2.6. *Let us consider the sequence of graphs $\{C_n \square P_2\}_{n=3}^{\infty}$ represented in $\mathbb{R}^2$ by an "exterior" copy of C_n joined with an "interior" copy of C_n by n edges. Choose two vertices $a_n^n, b_n^n \in V(C_n \square P_2)$ (in the exterior copy of C_n) with $d_{C_n \square P_2}(a_n^n, b_n^n) = 1$ for $n \geq 4$, and two vertices $a_{n+1}^n, b_{n+1}^n \in V(C_n \square P_2)$ (in the exterior copy of C_n) with $d_{C_n \square P_2}(a_{n+1}^n, b_{n+1}^n) = 1$ for $n \geq 3$ and $\{a_{n+1}^n, b_{n+1}^n\} \cap \{a_n^n, b_n^n\} = \varnothing$ for $n \geq 4$. We define G as the union of $\{C_n \square P_2\}_{n=3}^{\infty}$ obtained by identifying $[a_{n+1}^n, b_{n+1}^n]$ with $[a_{n+1}^{n+1}, b_{n+1}^{n+1}]$ for $n \geq 3$. Note that the "central face" of each $C_n \square P_2$ (whose boundary is the interior copy of C_n) has n edges, and therefore there is not an upper bound for the number of edges of the faces in G. It is clear that G is quasi-isometric to the graph G' which is the union of $\{C_n \square P_2\}_{n=3}^{\infty}$ obtained by identifying a_{n+1}^n with a_{n+1}^{n+1} for $n \geq 3$. Since $C_n \square P_2$ has an isometric subgraph which is isomorphic to C_n, Lemma 1.3.2 gives that $\delta(C_n \square P_2) \geq \delta(C_n)$. Lemmas 1.3.7 and 1.3.19 give that G' is not hyperbolic, since $\delta(G') = \sup_n \delta(C_n \square P_2) \geq \sup_n \delta(C_n) = \sup_n n/4 = \infty$. Hence, G is not hyperbolic.*

Its dual graph G^* is isometric to a union of wheel graphs $\{W_n\}_{n=4}^{\infty}$ such that each W_n is joined with W_{n+1} by a graph isometric to the path graph P_2 for $n \geq 3$. Lemmas 1.3.7 and 1.3.19 give that G^* is hyperbolic, since $\delta(G^*) = \sup_n \delta(W_n) = 3/2$, although G is not hyperbolic.

Example 5.2.7. *Let us consider the CW 2-complex with only one 2-cell, the open unit square $\{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1, 0 < y < 1\}$, and with 1-skeleton equal to the Cayley graph of $\mathbb{Z}^2$, i.e., the planar graph G with $V(G) := \mathbb{Z}^2$ and unit edges defined by $E(G) := \{[(a, b), (c, d)] \bigm| |a - c| + |b - d| = 1\}$ (each edge is represented by a straight line). Let G^* be the dual graph of G. Since G is quasi-isometric to $\mathbb{R}^2$, G is not hyperbolic by Lemma 1.3.5. Note that just 4 edges of G belong to the closure of the single face. However, G^* is connected and 0-hyperbolic, since it is a graph with just one vertex.*

Example 5.2.8. *Take a plane hyperbolic graph G such that its faces can be 2-coloured like a chessboard in black and white (with triangular faces). Let L be a geodesic line in G. Let us consider the CW 2-complex C_0 consisting of G and all faces but those coloured in white that share an edge with L; in addition, we add two of the deleted white faces to C_0. Let us call these two faces P_1, P_2. The dual graph G_0^* of the 1-skeleton of C_0 is still connected and hyperbolic, but its hyperbolicity constant depends on the distance between the vertices corresponding to P_1 and P_2 in G_0^*.*

Consider now the CW 2-complex C consisting of G and all faces but those coloured in white that share an edge with L; in addition, we add $\{P_n\}_{n=1}^{\infty}$ of the deleted white faces to C, verifying the following: if p_n is the vertex in G^* corresponding to P_n, then $d_{G^*}(p_1, p_n) \geq 2^n$. Therefore, G^* is not hyperbolic, although G is hyperbolic.

Theorem 5.2.4 and Lemma 1.3.7 allow to deduce the following result.

Theorem 5.2.9. *Let G be the 1-skeleton of a connected CW 2-complex C and G^* be its dual graph. Let k_1, k_2, M, K be fixed positive constants. Assume that there exists a T-decomposition $\{G_n\}$ of G verifying the following properties: $\{G_n^*\}$ are the connected components of G^*, every edge $e \in E(G)$ is contained in the closure of a face of C and satisfies $k_1 \leq L(e) \leq k_2$, every face of C has at most M edges and if the closures of two faces in some G_n have non-empty intersection then the corresponding vertices in G_n^* are at distance at most K. Then G is δ-hyperbolic if and only if G_n^* is δ^*-hyperbolic for every n, quantitatively.*

Proof. Since $\{G_n\}$ is a T-decomposition of G, we have by Lemma 1.3.7 that $\delta(G) = \sup_n \delta(G_n)$.

Assume first that G is δ-hyperbolic. Then G_n is δ-hyperbolic for every n. Theorem 5.2.4 gives that G_n^* is δ^*-hyperbolic for every n, where δ^* depends just on k_1, k_2, K, M and δ.

Assume now that G_n^* is δ^*-hyperbolic for every n. Then Theorem 5.2.4 gives that G_n is δ-hyperbolic for every n, where δ depends just on k_1, k_2, K, M and δ^*. Then G is δ-hyperbolic. $\square$

Chapter 6

Chordal and Gromov hyperbolic graphs.

One of the main problems on the theory of hyperbolic graphs is to relate the hyperbolicity with other properties on graph theory. In this Chapter we extend in two ways (edge-chordality and path-chordality) the classical definition of chordal graphs in order to relate this property with Gromov hyperbolicity.

We prove in Section 6.1 that every edge-chordal graph is hyperbolic; in fact, Theorem 6.1.3 states

if G is a (k, m)-edge-chordal graph, then it is $(m + k/4)$-hyperbolic.

Also in this Section, we prove that every hyperbolic graph is path-chordal; in fact, Theorem 6.1.8 states

every δ-hyperbolic graph is 90δ-path-chordal.

Although the converse of these two Theorems do not hold (see Examples 6.1.6 and 6.1.9), the path-chordality is a very close condition to hyperbolicity, in the following sense: in Section 6.2 we prove that every path-chordal cubic graph (with small path-chordality constant) is hyperbolic (recall that, in order to study Gromov hyperbolicity, general graphs are equivalent to cubic graphs, see Chapter 1).

6.1 Edge-chordal and path-chordal graphs

Definition 6.1.1. *A* shortcut *in a cycle C of a graph G is a path σ joining two vertices $p, q \in C$ such that $L_G(\sigma) < d_C(p, q)$.*

An edge-shortcut *in a cycle C is an edge of G which is a shortcut of C.*

Given two constants $k, m \geq 0$, we say that a graph G is (k, m)-edge-chordal *if for any cycle C in G with length $L(C) \geq k$ there exists an edge-shortcut e with length $L(e) \leq m$. The graph G is* edge-chordal *if there exist constants $k, m \geq 0$ such that G is (k, m)-edge-chordal.*

We say that a graph G is r-path-chordal *if in every cycle C in G with $L_G(C) \geq r$ there exists at least a shortcut σ with $L(\sigma) \leq r/2$.*

Note that every (k, m)-edge-chordal graph G is $\max\{k, 2m\}$-path-chordal.

Usually a graph (with edges of length 1) is said chordal if it is $(4, 1)$-edge-chordal according to Definition 6.1.1. In [13] the authors prove that chordal graphs are hyperbolic. In [81] the authors introduce k-chordal graphs generalizing the chordality (a graph with edges of length 1 is k-chordal if it does not contain any induced n-cycle for $n > k$; then chordal graphs are 3-chordal) and they prove that k-chordal graphs are hyperbolic. Our concept of edge-chordality generalizes the k-chordality; in fact, k-chordal graphs are $(k + 1, 1)$-edge-chordal. We prove in this Section that every edge-chordal graph is hyperbolic (see Theorem 6.1.3) and that every hyperbolic graph is path-chordal (see Theorem 6.1.8).

We need some previous lemmas.

Lemma 6.1.2. *Given a (k, m)-edge-chordal graph G, a cycle C in G with length $L(C) \geq k$ and a geodesic $[ab] \subset C$ with $L([ab]) \geq k/2$, there exist two vertices $v \in V(G) \cap ([ab] \setminus \{a, b\})$ and $w \in V(G) \cap (C \setminus [ab])$ with $e = [v, w] \in E(G)$, $L(e) < d_C(v, w)$ and $L(e) \leq m$.*

Proof. Since $[ab]$ is a geodesic contained in C, we have $L(C \setminus [ab]) \geq L([ab])$ and $L(C) \geq 2L([ab])$.

Assume first that $L(C) = 2L([ab])$. In this case $L(C \setminus [ab]) = L([ab])$ and then $(C \setminus [ab]) \cup \{a, b\}$ is also a geodesic joining a and b. Since $L(C) \geq k$ and G is a (k, m)-edge-chordal graph, there exists an edge $e = [x, y]$ with $x, y \in V(G) \cap C$ such that $L(e) < d_C(x, y)$ and $L(e) \leq m$. It is not possible for e to join two vertices of $[ab]$, since $[ab]$ is a geodesic. Similarly, it is not possible for e to join two vertices of $(C \setminus [ab]) \cup \{a, b\}$, since $(C \setminus [ab]) \cup \{a, b\}$ is also a geodesic. Therefore, the conclusion of the lemma holds in this case.

Assume now that $L(C) > 2L([ab])$. Since $L(C) \geq k$ and G is a (k, m)-edge-chordal graph, there exists an edge $e = [x, y]$ with $x, y \in V(G) \cap C$ such that $L(e) < d_C(x, y)$ and $L(e) \leq m$. It is not possible for e to join two vertices of $[ab]$, since $[ab]$ is a geodesic. If either x or y belongs to $[ab] \setminus \{a, b\}$, then the conclusion of the lemma also holds in this case. If $x, y \notin [ab] \setminus \{a, b\}$, then we consider the cycle C_1 obtained by pasting e with the connected component of $C \setminus \{x, y\}$ which contains $[ab] \setminus \{a, b\}$. It is clear that $L(C_1) < L(C)$, $[ab] \subset C_1$ and $V(G) \cap C_1 \subseteq V(G) \cap C$.

Now we can apply the previous argument to C_1. If we do not obtain the conclusion of the Lemma, then we obtain a new cycle C_2 with $L(C_2) < L(C_1) < L(C)$, $[ab] \subset C_2$ and $V(G) \cap C_2 \subseteq V(G) \cap C$. Iterating this process we obtain either the conclusion of the Lemma or a sequence of cycles $C_1, C_2, \dots, C_j, \dots$ with $[ab] \subset C_j$, $V(G) \cap C_j \subseteq V(G) \cap C$ for every $j \geq 1$, and

$$L(C_j) < \cdots < L(C_2) < L(C_1) < L(C)\,. \tag{6.1}$$

Since G is locally finite and we have (6.1), this process must stop in some finite step by compactness; therefore, the conclusion of the Lemma holds. □

The following results give a necessary and a sufficient condition (which are close) for the hyperbolicity of graphs. In fact, we prove that edge-chordality implies hyperbolicity and that hyperbolicity implies path-chordality.

Theorem 6.1.3. *If G is a (k, m)-edge-chordal graph, then it is $(m + k/4)$-hyperbolic.*

Proof. Let us consider any fixed geodesic triangle $T = \{x, y, z\}$ in G. By Corollary 1.3.4, in order to compute $\delta(G)$ we can assume that T is a cycle. Without loss of generality we can assume also that there exists $p \in [xy]$ with $\delta(T) = d(p, [xz] \cup [yz])$. If $\delta(T) \le k/4$, then there is nothing to prove. If $\delta(T) > k/4$, then $d(p, \{x, y\}) > k/4$, $L([xy]) \ge k/2$ and $L(T) \ge k$. Let us consider $a, b \in [xy]$, with $a \ne b$ and $d(a, p) = d(b, p) = k/4$; then $p \in [ab] \subset [xy]$ and $L([ab]) = k/2$. Lemma 6.1.2 gives that there exist two vertices $v \in V(G) \cap ([ab] \setminus \{a, b\})$ and $w \in V(G) \cap (T \setminus [ab])$ with $e = [v, w] \in E(G)$, $L(e) < d_T(v, w)$ and $L(e) \le m$. Note that $w \notin [xy]$ since $L([v, w]) < d_T(v, w)$. Therefore,

$$d(p, [xz] \cup [yz]) \le d(p, w) \le d(p, v) + d(v, w) \le \frac{k}{4} + m\,.$$

Then G is $(m + k/4)$-hyperbolic. □

The inequality in Theorem 6.1.3 is sharp, Example 6.1.4 provides a family of graphs for which the equality is attained for $m \le k/4$.

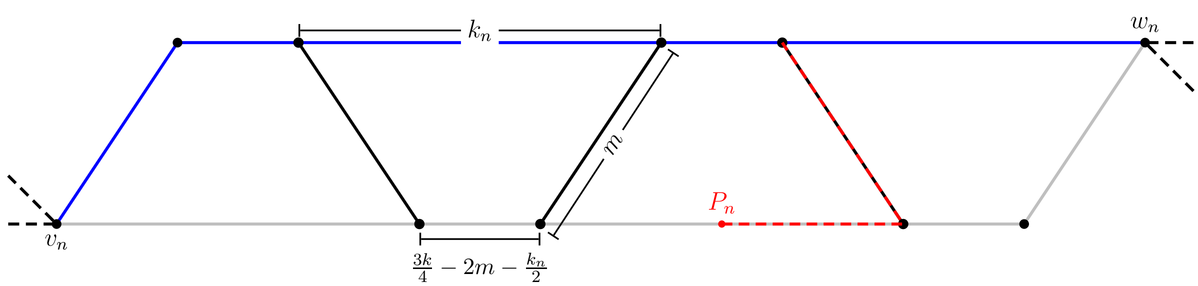


Figure 6.1: Infinite (k, m)-edge-chordal graph G with $\delta(G) = m + k/4$.

Example 6.1.4. *Let us fix a positive number k. Consider an increasing sequence of positive numbers $\{k_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} k_n = k/2$ and $m \le k/4$. Let G_n be the graph obtained from graph $P_5 \square P_2$ distorting its edges as follows: any edge of G_n which is a copy of P_2 has length m and both copies of P_5 in G alternating edges of length k_n and $3k/4 - 2m - k_n/2$ such that all cycle with length 4 has three edges of different length (see Figure 6.1). Consider the vertices $v_n, w_n \in V(G_n)$ for $n \ge 1$ such that $d_{G_n}(v_n, w_n) = \operatorname{diam} G_n$. We define G as the union of $\{G_n\}_{n=1}^{\infty}$ obtained by identifying v_n with w_n and v_{n+1} with w_{n+1}, respectively for $n \ge 1$.*

Clearly, G is a (k,m)-edge-chordal graph. By Theorem 1.3.7 $\delta(G) = \sup_n \delta(G_n)$ since $\{G_n\}_{n=1}^{\infty}$ is the canonical T-decomposition of G. Besides, Theorem 6.1.3 gives that $\delta(G) \leq m + k/4$. Consider γ, γ' the geodesics in G_n joining v_n and w_n such that $B := \gamma \cup \gamma'$ is a cycle (these geodesics are drawn blue and gray, respectively in Figure 6.1). So, we have $\delta(G_n) \geq \delta(B) \geq k_n/2 + m$ since there is $P_n \in \gamma$ (shown red in Figure 6.1) with $d_G(p, \gamma') = k_n/2 + m$. Thus, $\delta(G) = k/4 + m$.

We have the following consequences of Theorem 6.1.3.

Corollary 6.1.5. *If G is a (k,m)-edge-chordal graph with edges of integer length, then G is $\big(\lfloor (k-1)/2 \rfloor /2 + m\big)$-hyperbolic.*

Proof. Let us consider any fixed geodesic triangle $T = \{x, y, z\}$ in G. By Corollary 1.3.4 we can assume that T is a cycle. Consider γ a geodesic in T, without loss of generality we can suppose that $\gamma = [xy]$. We may list the vertices in $[xy]$ with edge-shortcuts of length at most m as a sequence $\{v_i\}_{i=1}^{n}$ such that $[xy] = [xv_1] \cup \big(\cup_{i=1}^{n-1} [v_i v_{i+1}]\big) \cup [v_n y]$. Denote $v_0 = x$, $v_{n+1} = y$ and $S_{[xy]} = \{v_i\}_{i=0}^{n+1}$. Notice that if $L([xy]) < k$ then $S_{[xy]}$ may be has exactly two elements x and y. By Lemma 6.1.2 we have $L([v_i v_{i+1}]) < k/2$ for $0 \leq i \leq n$. Then, for every $p \in [xy]$ there is i such that such that $p \in [v_i v_{i+1}]$, and so, we have $d_G(p, [yz] \cup [zx]) \leq d_G(p, \{v_i, v_{i+1}\}) + m \leq \big(\lfloor (k-1)/2 \rfloor /2 + m\big)$. This finish the proof since $[xy]$ is an arbitrary geodesic of T. □

The following example shows that the converse of Theorem 6.1.3 does not hold, i.e., hyperbolicity does not imply edge-chordality.

Example 6.1.6. *Let P_3 be the path graph with (adjacent) vertices v_1, v_2, v_3, and G the Cartesian product graph $G = \mathbb{Z} \square P_3$ with $L(e) = 1$ for every $e \in E(G)$. Since G and $\mathbb{Z}$ are quasi-isometric, G is hyperbolic. One can check that G is 5-path-chordal, but it is not edge-chordal, since for every natural number $r \geq 2$ the geodesic squares with vertices $(0, v_1), (r, v_1), (r, v_3), (0, v_3)$ do not have edge-shortcuts (see Figure 6.2).*

Theorem 6.1.8 below is a kind of reciprocal of Theorem 6.1.3. In order to prove it, we need the following technical result.

Theorem 6.1.7. *[37, p.92] Let us consider constants $\delta \geq 0, r > 0$, a δ-hyperbolic geodesic metric space X and a finite sequence $\{x_j\}_{0 \leq j \leq n}$ in X with*

$$d_X(x_{j-1}, x_{j+1}) \geq \max\{d_X(x_{j-1}, x_j), d_X(x_j, x_{j+1})\} + 18\delta + r$$

for every $0 < j < n$. Then $d_X(x_0, x_n) \geq rn$.

Theorem 6.1.8. *Every δ-hyperbolic graph is 90δ-path-chordal.*

Proof. Seeking for a contradiction, assume that G is a δ-hyperbolic graph which is not 90δ-path-chordal. Then there exists a cycle C in G with $L(C) \geq 90\delta$ without shortcuts σ with

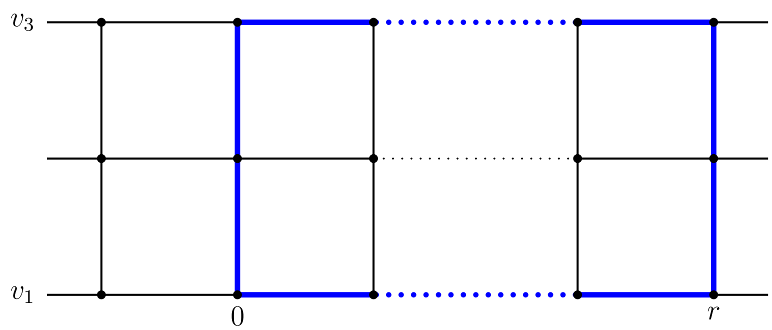


Figure 6.2: Cartesian product graph $G = \mathbb{Z}\square P_3$.

$L(\sigma) \leq 45\delta$. Consequently, any subcurve g of C with $L(g) \leq 45\delta$ is a geodesic in G. Let us define an integer n and a positive number ℓ by

$$n := \left\lceil \frac{2L(C)}{45\delta} \right\rceil, \qquad \ell := \frac{L(C)}{n}.$$

Since

$$\frac{2L(C)}{45\delta} \leq \left\lceil \frac{2L(C)}{45\delta} \right\rceil < \frac{2L(C)}{45\delta} + 1,$$

we deduce that

$$18\delta < \ell \leq \frac{45\delta}{2}.$$

Choose a finite sequence $\{x_j\}_{0\leq j\leq n}$ in C with $d_X(x_j, x_{j+1}) = d_C(x_j, x_{j+1}) = \ell$ for every $0 \leq j < n$, and $d_X(x_{j-1}, x_{j+1}) = 2\, d_C(x_j, x_{j+1}) = 2\ell$ for every $0 < j < n$; then $x_0 = x_n$.

If we define $r := \ell - 18\delta$, then

$$\begin{aligned} 2\ell &= \ell + 18\delta + r, \\ d_X(x_{j-1}, x_{j+1}) &= \max\{d_X(x_{j-1}, x_j), d_X(x_j, x_{j+1})\} + 18\delta + r, \end{aligned}$$

for every $0 < j < n$. Then Theorem 6.1.7 gives $0 = d_X(x_0, x_n) \geq rn > 0$, which is the contradiction we were looking for; hence, we conclude that G is 90δ-path-chordal. $\square$

The following example shows that the reciprocal of Theorem 6.1.8 does not hold, i.e., path-chordality does not imply hyperbolicity.

Example 6.1.9. *First of all, we assume that $0 \in \mathbb{N}$. Let $\sum_{n=0}^{\infty} a_n$ be a fixed convergent series of positive real numbers such that $a_0 = 1$ and $\sum_{n=0}^{\infty} a_n = S < \infty$. Now, Let us consider $\{S_n\}_{n=0}^{\infty}$ the sequence of partial sums. Let G be the Cartesian product graph $G = \mathbb{N} \times \mathbb{N}$ with $L([(p,q),(p+1,q)]) = S_{p+q} = L([(p,q),(p,q+1)])$.*

Note that G is a path chordal graph, since each cycle C of G with $L(C) > 4S$ has a vertex $v = (p+1, q+1) \in C$ such that $[(p+1, q), v], [(p, q+1), v] \in E(G)$ are contained in C (i.e., v is an upper-right vertex of C); then C has a shortcut $\sigma \subseteq [(p+1, q), (p, q)] \cup [(p, q), (p, q+1)]$, since $S_{p+q} < S_{p+q+1}$.

Let G_0 be the Cartesian product graph $G = \mathbb{N} \times \mathbb{N}$ with $L(e) = 1$ for every $e \in E(G_0)$. Since G_0 and G are quasi-isometric, G is not hyperbolic.

6.2 Chordality in cubic graphs

We want to remark that by Theorems 1.3.5, 1.3.20 and 1.3.21, the study of the hyperbolicity of graphs can be reduced to the study of the hyperbolicity of cubic graphs. Along this Section we just consider (finite or infinite) graphs with edges of length 1.

In this section we obtain several results which guarantee the hyperbolicity of many path-chordal cubic graphs (see Theorems 6.2.4 and 6.2.9).

A *proper shortcut* in C is a shortcut σ joining two vertices $p, q \in C \cap V(G)$ such that $\sigma \cap C = \{p, q\}$ and σ is a geodesic. Note that in any cycle C of a r-path-chordal graph G such that $L(C) \geq r$ there is a proper shortcut with length at least $r/2$. Therefore, we may replace proper shortcut by shortcut in the definition of path-chordal graph.

Note that, since we just consider graphs with edges of length 1, every edge-shortcut is a proper shortcut.

Theorem 6.2.1. *Let G be any cubic graph. Then G is 4-path-chordal if and only if it is a chordal.*

Proof. If G is a chordal graph, then it is 4-path-chordal.

Assume now that G is a 4-path-chordal graph. Seeking for a contradiction, assume that there exists a cycle C in G with $L(C) \geq 4$ and such that C has no shortcut with length 1. Since $L(C) \geq 4$ and G is 4-path-chordal, the set $V_C := \{(u, v) |\, u, v \in V(G) \cap C$ and $[uv]$ is a shortcut in C with length 2$\}$ is non-empty. Let $(x, y) \in V_C$ with $d_C(x, y) = \min\{d_C(u, v) |\, (u, v) \in V_C\}$. Let g_1 be a path joining x and y contained in C such that $L(g_1) = d_C(x, y)$. Define $C_1 := g_1 \cup [xy]$; then $L(C_1) \geq 2L([xy]) \geq 4$ and there exists a proper shortcut $\rho = [zw]$ in C_1. Since it is not possible to have $\{z, w\} \subset [xy]$ or $\{z, w\} \subset g_1$, without loss of generality we can assume that $z \in g_1 \setminus \{x, y\}$ and $w \in [xy] \setminus \{x, y\}$; since $L([xy]) = 2$, then w is the midpoint of $[xy]$.

Note that we have either $L(\rho) = 1$ or $L(\rho) = 2$.

If $L(\rho) = 1$, then $d_C(z, x) \leq 2$ and $d_C(z, y) \leq 2$, since $[x, w] \cup [w, z]$ and $[y, w] \cup [w, z]$ are not shortcuts in C. We prove now that $d_C(z, x) = d_C(z, y) = 1$. Otherwise, by symmetry, we can assume that $d_C(z, x) = 2$; then the cycle $C_2 = [x, w] \cup \rho \cup [zx]$ has length 4 and there exists a shortcut in C_2; but since x, z, w have "full degree", there is just one vertex in

C_2 that can be an endpoint of the shortcut. This is a contradiction and we conclude that $d_C(z,x) = d_C(z,y) = 1$. Then $d_C(x,y) = 2 = L([xy])$ and $[xy]$ is not a shortcut in C, which is a contradiction.

If $L(\rho) = 2$, then we have a shortcut in each of the two induced cycles on C_1 by ρ; if v is the midpoint of ρ, then v is an endpoint of the two shortcuts. Since G is a cubic graph, the two shortcuts are $[v,v_0] \cup [v_0,v_1]$ and $[v,v_0] \cup [v_0,v_2]$ for some vertices $v_0 \in V(G)$ and $v_1, v_2 \in V(G) \cap g_1$. If g_2 is the path contained in g_1 joining v_1 and v_2, then $\gamma = [v_1,v_0] \cup [v_0,v_2] \cup g_2$ is a cycle with $L(\gamma) \geq 4$. Since γ does not have a shortcut, we obtain a contradiction.

Hence, we conclude that G is a chordal graph. $\square$

Lemma 6.2.2. *Let G be a 4-path-chordal cubic graph and let C be any cycle in G with two different shortcuts with length 1. Then, G is isomorphic to the complete graph with 4 vertices K_4.*

Proof. By Theorem 6.2.1 any cycle of G with length grater than 3 has an edge-shortcut. Let $\sigma_1 := [x,x']$ and $\sigma_2 := [y,y']$ be two different edge-shortcuts in C. Let g (respectively, g') be a subcurve of C joining x and y (respectively, x' and y') such that $g \cap g' = \emptyset$; then $C_1 := \sigma_1 \cup g' \cup \sigma_2 \cup g$ is a cycle with $L(C_1) \geq 4$. The cycle C can be oriented either by:

$$(1) \quad x \to y \to y' \to x',$$

or

$$(2) \quad x \to y \to x' \to y'.$$

Assume that C is oriented by (1). Then C_1 has an edge-shortcut e_1 joining $g \setminus \{x,y\}$ and $g' \setminus \{x',y'\}$. Let C_2 be a cycle obtained by joining e_1 with a path contained in C_1. Proceeding this way, we obtain a finite sequence of cycles $C_1, C_2, \dots, C_k$ such that $L(C) > L(C_1) > L(C_2) > \cdots > L(C_k) = 4$ and the four vertices of C_k have full degree; then there is no shortcut in C_k, which is a contradiction.

Assume now that C is oriented by (2). Let γ_1, γ_2 be two curves with $\gamma_1 \cup \gamma_2 = C$ and $\gamma_1 \cap \gamma_2 = \{x,x'\}$. If $\max\{L(\gamma_1), L(\gamma_2)\} > 2$, then without loss of generality we can assume that $L(\gamma_1) > 2$; hence, $\gamma_1 \cup [x,x']$ is a cycle with $L(\gamma_1 \cup [x,x']) \geq 4$ and there is an edge-shortcut e_1 in $\gamma_1 \cup [x,x']$; since x and x' have full degree, $[x,x'] \cap e_1 = \emptyset$; consequently, $[x,x']$ and e_1 are two edge-shortcuts in C in the case (1), and we have proved that this is a contradiction. Therefore, $\max\{L(\gamma_1), L(\gamma_2)\} \leq 2$; we conclude that $L(\gamma_1) = L(\gamma_2) = 2$, and then G is isomorphic to K_4. $\square$

Corollary 6.2.3. *If G is a 4-path-chordal cubic graph, then G does not have cycles with length greater than 4.*

Proof. Seeking for a contradiction, assume that there exists a cycle C with length $r > 4$.

If $r = 5$, then there is an edge-shortcut $[x,y]$ with $d_C(x,y) = 2$. If g is the path in C joining x and y with length 3, then $[x,y] \cup g$ is a cycle with length 4 and there is no shortcut in it since x and y have full degree. This is the contradiction we were looking for.

If $r > 5$, then there is an edge-shortcut $[x, y]$. Let g_1, g_2 be two paths with $g_1 \cup g_2 = C$ and $g_1 \cap g_2 = \{x, y\}$. Without loss of generality we can assume that $L(g_1) \geq L(g_2)$; then $[x, y] \cup g_1$ is a cycle with $L([x, y] \cup g_1) \geq 4$ and there exists an edge-shortcut e in $[x, y] \cup g_1$. Hence, $[x, y]$ and e are two edge-shortcuts in C, and G is isometric to K_4 by Lemma 6.2.2. This is a contradiction. □

Corollary 6.2.3 and Proposition 3.2.5 have the following consequence.

Theorem 6.2.4. *If G is a 4-path-chordal cubic graph, then G is 1-hyperbolic.*

The following result provides a simple and explicit formula for the hyperbolicity constant of the 4-path-chordal cubic graphs.

Theorem 6.2.5. *If G is a 4-path-chordal cubic graph, then $\delta(G) = c(G)/4$.*

Proof. By Proposition 3.2.5, $\delta(G) \leq c(G)/4$.

Let us prove the converse inequality. By Corollary 6.2.3 we have $c(G) \leq 4$. If $c(G) \leq 3$, then $\delta(G) \geq c(G)/4$ by Lemma 3.2.2. Assume now that $c(G) = 4$ and consider a cycle g with length 4. Let x, y be midpoints of edges in g with $d(x, y) = 2$ and paths g_1, g_2 with $g_1 \cup g_2 = g$ and $g_1 \cap g_2 = \{x, y\}$. Then $\{g_1, g_2\}$ is a geodesic bigon in G. If p is the midpoint of g_1, then $\delta(G) \geq d(p, g_2) = d(p, \{x, y\}) = 1 = c(G)/4$. □

Recall that given an edge $e = [u, v]$ in a graph G the *edge contraction* of G (relative to e) is the graph obtained as follows: the edge e is removed and its two incident vertices are merged into a new vertex w, where the edges incident to w each correspond to an edge incident to either u or v. The following result characterizes in a simple and precise way the 4-path-chordal cubic graphs.

Theorem 6.2.6. *G is a 4-path-chordal cubic graph if and only if G is isomorphic to one of the following graphs:*

1. *a complete graph with 4 vertices K_4,*
2. *a graph with exactly 2 vertices and a 3-multiple edge joining them,*
3. *a graph obtained from any tree with vertices of degree at most 3 such that we replace*
   - *each vertex of degree 1 by a loop or a cycle graph with 3 vertices and a double edge,*
   - *each vertex of degree 2 by a complete graph with 4 vertices without one edge, or a graph with two vertices and two multiple edges,*
   - *each vertex in an arbitrary subset of vertices with degree 3 by a cycle graph C_3.*

Proof. Assume that G is a 4-path-chordal cubic graph. If G is isomorphic to the graphs in (1) or (2), then we have finished. Assume now that G is not isomorphic to the graphs in (1) or (2). By Corollary 6.2.3 we have $c(G) \leq 4$. Since G is a 4-path-chordal cubic graph, the cycles with length 4 are pairwise disjoint, and the induced graph by the vertices of each cycle with length 4 is isomorphic to a complete graph with 4 vertices without one edge.

Let G_1 be the graph obtained by the contraction of every edge in every cycle with length 4; then G_1 has a vertex of degree 2 corresponding to each cycle in G with length 4. Since G_1 is a graph with vertices of degree at most 3 and $c(G) \leq 3$, its cycles with length 3 are pairwise disjoint (except the cycles with different edges of the same double edge).

Let G_2 be the graph obtained by the contraction of every edge in every cycle with length 3; then G_2 has a vertex of degree 1 corresponding to each cycle graph with 3 vertices and a double edge, and a vertex of degree 3 corresponding to each cycle graph with 3 vertices and simple edges. Since G_2 is a graph with vertices of degree at most 3 and $c(G) \leq 2$, its cycles with length 1 or 2 are pairwise disjoint.

Let G_3 be the graph obtained by the contraction of every double edge and every loop; then G_3 has a vertex of degree 1 corresponding to each loop, and a vertex of degree 2 corresponding to each double edge. Then G_3 is a tree with vertices of degree at most 3.

One can check easily the converse implication. $\square$

Lemma 6.2.7. *If G is a 5-path-chordal cubic graph, then there are not proper shortcuts with length 2 in any cycle of G.*

Proof. We prove the Lemma by complete induction. It is clear that on every cycle in G with length 5 the proper shortcuts have length 1. Now, we assume that

any cycle in G with length at most k does not have proper shortcuts with length 2.

Consider a cycle C in G with $k+1$ vertices. Seeking for a contradiction, assume that C has a proper shortcut $\sigma := [xy]$ with length 2, and let v be the midpoint of σ. Let g_1, g_2 be two paths in G joining x and y such that $C = g_1 \cup g_2$ and $g_1 \cap g_2 = \{x, y\}$. Consider the cycles $C_1 := g_1 \cup \sigma$ and $C_2 := g_2 \cup \sigma$. Let ρ_1 be a proper shortcut in C_1; by hypothesis, $L(\rho_1) = 1$. If ρ_1 joins two vertices u and v in g_1, then denote by g_1' the path joining u, v contained in C and which contains g_2; the cycle $\rho_1 \cup g_1'$ verifies $L(\rho_1 \cup g_1') \leq k$ and has the proper shortcut σ with length 2, which is a contradiction. Hence, ρ_1 does not join two vertices in g_1. Since x and y have full degree, $\rho_1 = [v, z]$ with $z \in g_1 \setminus \{x, y\}$. In a similar way, there exists another shortcut $[v, w]$ with $w \in g_2 \setminus \{x, y\}$. Hence, $\deg(v) \geq 4$; this is the contradiction we were looking for and we conclude that C does not have proper shortcuts with length 2. $\square$

By Lemma 6.2.7 any 5-path-chordal cubic graph G is $(5, 1)$-edge-chordal, and Corollary 6.1.5 gives that $\delta(G) \leq 2$. However, Theorem 6.2.9 below improves this inequality.

In order to obtain the hyperbolicity of any 5-path-chordal cubic graph we prove the following result.

Lemma 6.2.8. *Let C be a cycle in a 5-path-chordal cubic graph G and $[xy]$ a geodesic contained in C. If there are two edge-shortcuts $\rho_1 := [x,u]$, $\rho_2 := [y,v]$ in C and there is no other edge-shortcut in C starting in $[xy]$, then $[x,y],[u,v] \in E(G)$.*

Furthermore, the cycle obtained by joining the shortcuts ρ_1 and ρ_2 with paths contained in C has length 4.

Proof. Denote by γ the path contained in C which joins u and v such that $x, y \notin \gamma$. Denote by C_1 the cycle $C_1 := \rho_1 \cup [xy] \cup \rho_2 \cup \gamma$. Notice that is suffices to prove that $L(C_1) = 4$. Seeking for a contradiction, assume that $L(C_1) > 4$. Then, there is an edge-shortcut $\sigma := [u_1, v_1]$ in C_1 joining two points of γ such that $d_\gamma(u_1, v_1)$ is maximum. Without loss of generality we can suppose that γ can be oriented by $u \to u_1 \to v_1 \to v$. Since G is a cubic graph, we have $\rho_i \cap \sigma = \varnothing$ for $i \in \{1,2\}$; then we have that the cycle $C_2 := \rho_1 \cup [uu_1] \cup \sigma \cup [v_1 v] \cup \rho_2 \cup [xy]$ has length grater than 5; since C_2 does not have edge-shortcuts, we obtain the contradiction we were looking for. Therefore, we conclude that $L(C_1) = 4$ and $[x,y],[u,v] \in E(G)$. ☐

Theorem 6.2.9. *If G is a 5-path-chordal cubic graph, then G is $(3/2)$-hyperbolic.*

Proof. Fix a geodesic triangle $T = \{x,y,z\}$ in G. By Theorem 1.3.23, in order to study $\delta(G)$ we can assume that T is a cycle with $x,y,z \in J(G)$. If $L(T) \le 6$, then the three geodesic sides of T have length at most 3 and, consequently, $\delta(T) \le 3/2$. Assume now that $L(T) \ge 7$. By Lemma 6.2.7 there exists an edge-shortcut in T. By symmetry, it suffices to prove that for every $p \in [xy]$ we have $d_G(p, [yz] \cup [zx]) \le 3/2$.

Assume first that there is no edge-shortcut in T starting in $[xy]$. Since G is $(5,1)$-edge-chordal, by Lemma 6.1.2 we have that $L([xy]) \le 2$; therefore, we have for every $p \in [xy]$,

$$d_G(p, [yz] \cup [zx]) \le d_G(p, \{x,y\}) \le 1.$$

Assume now that there is an edge-shortcut in T joining $[xy]$ and $[xz]$, but there is no edge-shortcut joining $[xy]$ and $[yz]$. Let σ_1 be an edge-shortcut in T joining P_1 and Q_1, where $P_1 \in [xy]$, $Q_1 \in [xz]$ and P_1 is the closest vertex to x with an edge-shortcut. Consider the cycle $C := [xP_1] \cup \sigma_1 \cup [Q_1 x]$. Then, C does not have edge-shortcuts; therefore, $L(C) \le 4$ and $L([xP_1]) + L([xQ_1]) \le 3$. Hence, since $L([xP_1]) \le 1 + L([xQ_1])$, we have $L([xP_1]) \le 2$, $L([xP_1]) + L([P_1 Q_1]) \le 3$ and we obtain $d_G(p, [yz] \cup [zx]) \le d_G(p, [zx]) \le d_G(p, \{x, Q_1\}) \le 3/2$ for every $p \in [xP_1]$. Let n be the exact number of edge-shortcuts in T joining $[xy]$ and $[xz]$. Let $P_1, \dots, P_n \in [xy]$, $Q_1, \dots, Q_n \in [xz]$ with $[P_i, Q_i] \in E(G)$ for $1 \le i \le n$ and $L([xP_i]) < L([xP_{i+1}])$ for $1 \le i < n$. Hence, by Lemma 6.2.8 we have that $[P_i, P_{i+1}], [Q_i, Q_{i+1}] \in E(G)$ for every $1 \le i < n$; thus, for every $p \in [P_i P_{i+1}]$, we obtain $d_G(p, [yz] \cup [zx]) \le d_G(p, [zx]) \le d_G(p, \{Q_i, Q_{i+1}\}) \le 3/2$.

Furthermore, since there is no edge-shortcut in T from $[P_n y]$, by Lemma 6.1.2 we have that $L([P_n y]) \le 2$; therefore, for every $p \in [P_n y]$ we have $d_G(p, [yz] \cup [zx]) \le d_G(p, \{Q_n, y\}) \le 3/2$. Hence, we obtain

$$d_G(p, [yz] \cup [zx]) \le 3/2, \qquad \text{for every } p \in [xy].$$

Finally, assume that there are shortcuts in T joining $[xy]$ with $[xz]$, and $[xy]$ with $[yz]$. Let m be the exact number of edge-shortcuts in T joining $[xy]$ and $[yz]$. Let $R_1, \dots, R_m \in [xy]$, $S_1, \dots, S_m \in [yz]$ with $[R_i, S_i] \in E(G)$ for $1 \le i \le m$ and $L([yR_i]) < L([yR_{i+1}])$ for $1 \le i < m$. Let $1 \le k \le m$ with $[P_n R_k] \cap \{R_1, \dots, R_m\} = R_k$; by Lemma 6.2.8 we have that $[P_n R_k]$ is an edge. So, a similar argument to the one in the previous case gives

$$d_G(p, [yz] \cup [zx]) \le 3/2, \qquad \text{for every } p \in [xy].$$

□

The equality in Theorem 6.2.9 is attained by the Cartesian product graphs $P_2 \square P_n$ for $n \ge 4$ with the appropriated addition of two multiple edges (see Figure 6.3).

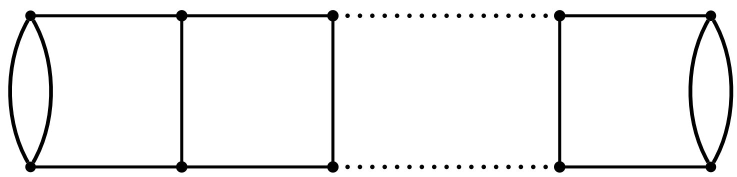


Figure 6.3: Graphs $P_2 \square P_n$ with appropriated addition of two multiple edges

The following example shows that the converse of Theorem 6.1.8 does not hold even for cubic graphs, i.e., path-chordality does not imply hyperbolicity in cubic graphs.

Example 6.2.10. *Consider a graph G which is the 1-skeleton of the semiregular tessellation of the plane obtained by octagons and squares, see Figure 6.4.*

Clearly, G is a cubic graph; we show now that G is a 18-path-chordal graph. Let us consider a cycle C in G with length greater than 17. Let R_C be the compact region in $\mathbb{R}^2$ whose boundary is C. We pay attention to the relative position of the octagons contained in R_C. Notice that we have either:

1. there is an octagon E in R_C intersecting C such that either (a) neither of the two octagons which are horizontal neighbors of E are contained in R_C or (b) neither of the two octagons which are vertical neighbors of E are contained in R_C or (c) both of the above, simultaneously,

2. there are three octagons in R_C intersecting C to form a "right angle" (i.e., there is a octagon E in R_C intersecting C, the "corner", such that one of the octagons which are horizontal neighbors of E is contained in R_C and the other one is not contained in R_C, and one of the octagons which are vertical neighbors of E is contained in R_C and the other one is not contained in R_C) and (1) does not hold,

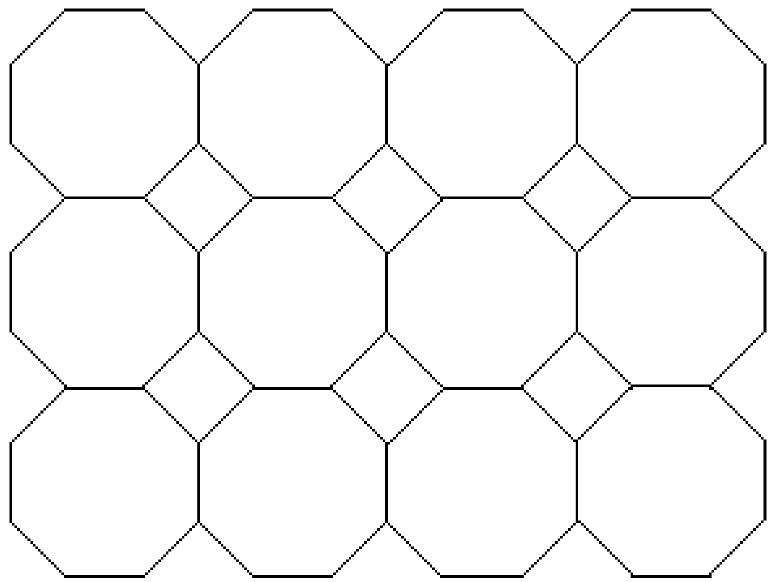


Figure 6.4: Semiregular tessellation of $\mathbb{R}^2$ whose it 1-skeleton is a cubic 18-path-chordal graph.

3. there are four octagons in R_C intersecting C to form a "right angle without the corner", and (1) and (2) do not hold.

If (1) holds, then $E \setminus C$ is a shortcut in C with length at most 3. If (2) holds, then C has a shortcut of length at most 5 (delimiting the octagon at the corner). If (3) holds, then C has a shortcut of length at most 9 (delimiting the two octagons closest to the corner). This prove that G is 18-path-chordal. Finally, by Theorem 5.1.6 we have that G is not hyperbolic.

Chapter 7

Hyperbolicity in graph join and corona of graphs.

Throughout this Chapter, $G = (V, E)$ denotes a simple graph (not necessarily connected) such that every edge has length 1. These properties guarantee that any connected graph (or any connected component) is a geodesic metric space. We denote the degree of a vertex $v \in V$ in G by $\deg(v) \leq \infty$, and the maximum degree of G by $\Delta_G := \sup_{v \in V} \deg(v)$. If x, y are in different connected components of G, we define $d_G(x, y) = \infty$.

Since deciding whether or not a graph is hyperbolic is usually very difficult, it is interesting to study the hyperbolicity of particular classes of graphs. The papers [7, 13, 16, 17, 24, 59, 63, 65, 69, 77] study the hyperbolicity of, respectively, complement of graphs, chordal graphs, strong product graphs, lexicographic product graphs, line graphs, Cartesian product graphs, cubic graphs, tessellation graphs, short graphs and median graphs.

In [16, 17, 59] the authors characterize the hyperbolic product graphs (for strong product, lexicographic product and Cartesian product) in terms of properties of the factor graphs. In this Chapter we characterize the hyperbolic product graphs for graph join $G_1 \uplus G_2$ and the corona $G_1 \diamond G_2$: $G_1 \uplus G_2$ is always hyperbolic, and $G_1 \diamond G_2$ is hyperbolic if and only if G_1 is hyperbolic (see Corollaries 7.2.1 and 7.3.3). Furthermore, we obtain simple formulae for the hyperbolicity constant of the graph join $G_1 \uplus G_2$ and the corona $G_1 \diamond G_2$ (see Theorems 7.2.14 and 7.3.2). In particular, Theorem 7.3.2 states that $\delta(G_1 \diamond G_2) = \max\{\delta(G_1), \delta(G_2 \uplus E_1)\}$, where E_1 is a graph with just one vertex. We want to remark that it is not usual at all to obtain explicit formulae for the hyperbolicity constant of large classes of graphs.

7.1 Distance in graph join

In order to estimate the hyperbolicity constant of the graph join $G_1 \uplus G_2$ of G_1 and G_2, we will need an explicit formula for the distance between two arbitrary points. We will use the definition given by Harary in [41].

Definition 7.1.1. *Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ two graphs with $V(G_1) \cap V(G_2) = \varnothing$. The* graph join $G_1 \uplus G_2$ *of G_1 and G_2 has $V(G_1 \uplus G_2) = V(G_1) \cup V(G_2)$ and two different vertices u and v of $G_1 \uplus G_2$ are adjacent if $u \in V(G_1)$ and $v \in V(G_2)$, or $[u, v] \in E(G_1)$ or $[u, v] \in E(G_2)$.*

From the definition, it follows that the graph join of two graphs is commutative. Figure 7.1 shows the graph join of two graphs.

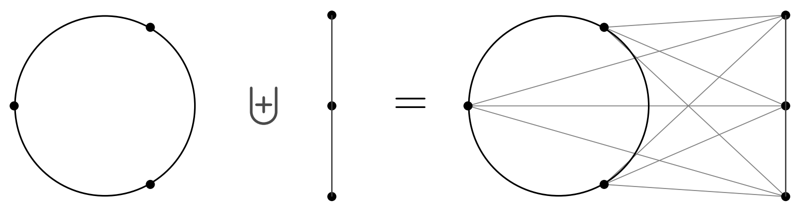


Figure 7.1: Graph join of two graphs $C_3 \uplus P_3$.

Remark 7.1.2. *For every graphs G_1, G_2 we have that $G_1 \uplus G_2$ is a connected graph with a subgraph isomorphic to a complete bipartite graph with $V(G_1)$ and $V(G_2)$ as its parts.*

Note that, from a geometric viewpoint, the graph join $G_1 \uplus G_2$ is obtained as an union of the graphs G_1, G_2 and the complete bipartite graph $K(G_1, G_2)$ linking the vertices of $V(G_1)$ and $V(G_2)$.

The following result allows to compute the distance between any two points in $G_1 \uplus G_2$. Furthermore, this result provides information about the geodesics in the graph join.

Proposition 7.1.3. *For every graphs G_1, G_2 we have:*

(a) If $x, y \in G_i$ $(i \in \{1, 2\})$, then

$$d_{G_1 \uplus G_2}(x, y) = \min \left\{ d_{G_i}(x, y), d_{G_i}\big(x, V(G_i)\big) + 2 + d_{G_i}\big(V(G_i), y\big) \right\}.$$

(b) If $x \in G_i$ and $y \in G_j$ with $i \neq j$, then

$$d_{G_1 \uplus G_2}(x, y) = d_{G_i}\big(x, V(G_i)\big) + 1 + d_{G_j}\big(V(G_j), y\big).$$

(c) If $x \in G_i$ and $y \in K(G_1, G_2)$, then

$$d_{G_1 \uplus G_2}(x, y) = \min \left\{ d_{G_i}(x, Y_i) + d_{G_1 \uplus G_2}(Y_i, y), d_{G_i}\big(x, V(G_i)\big) + 1 + d_{G_1 \uplus G_2}(Y_j, y) \right\},$$

where $y \in [Y_1, Y_2]$ with $Y_i \in V(G_i)$ and $Y_j \in V(G_j)$.

(d) If $x, y \in K(G_1, G_2)$, then

$$d_{G_1 \uplus G_2}(x, y) = \min\{d_{K(G_1,G_2)}(x, y), M\},$$

where $x \in [X_1, X_2]$, $y \in [Y_1, Y_2]$ with $X_1, Y_1 \in V(G_1)$ and $X_2, Y_2 \in V(G_2)$, and $M = \min\{d_{G_1 \uplus G_2}(x, X_1) + d_{G_1}(X_1, Y_1) + d_{G_1 \uplus G_2}(Y_1, y), d_{G_1 \uplus G_2}(x, X_2) + d_{G_2}(X_2, Y_2) + d_{G_1 \uplus G_2}(Y_2, y)\}$.

Proof. We will prove each item separately. In item (a), if $i \neq j$, we may consider the two shortest possible paths to go from x to y such that either is contained in G_i or intersects G_j (and then it intersects G_j just in a single vertex). In item (b), since any path in $G_1 \uplus G_2$ joining x and y contains at less one edge in $K(G_1, G_2)$, we have a geodesic when the path contains an edge joining a closest vertex to x in $V(G_i)$ and a closest vertex to y in $V(G_j)$. In item (c) we may consider the two shortest possible paths from x to y that contain either Y_1 or Y_2. Finally, in item (d) we may consider the three shortest possible paths from x to y such that either is contained in $K(G_1, G_2)$ or contains at lest an edge in $E(G_1)$ or contains at lest an edge in $E(G_2)$. $\square$

Proposition 7.1.3 gives the following result.

Proposition 7.1.4. *Let G_1, G_2 be two graph and let Γ_1, Γ_2 be isometric subgraphs to G_1 and G_2, respectively. Then, $\Gamma_1 \uplus \Gamma_2$ is an isometric subgraph to $G_1 \uplus G_2$.*

The following result allows to compute the diameter of the set of vertices in a graph join.

Proposition 7.1.5. *For every graphs G_1, G_2 we have $1 \leq \operatorname{diam} V(G_1 \uplus G_2) \leq 2$. Furthermore, $\operatorname{diam} V(G_1 \uplus G_2) = 1$ if and only if G_1 and G_2 are complete graphs.*

Proof. Since $V(G_1), V(G_2) \neq \emptyset$, $\operatorname{diam} V(G_1 \uplus G_2) \geq 1$. Besides, if $u, v \in V(G_1 \uplus G_2)$, we have $d_{G_1 \uplus G_2}(u, v) \leq d_{K(G_1,G_2)}(u, v) \leq 2$.

In order to finish the proof note that on the one hand, if G_1 and G_2 are complete graphs, then $G_1 \uplus G_2$ is a complete graph with at least 2 vertices and $\operatorname{diam} V(G_1 \uplus G_2) = 1$. On the other hand, if $\operatorname{diam} V(G_1 \uplus G_2) = 1$, then for every two vertices $u, v \in V(G_1)$ we have $[u, v] \in E(G_1)$; by symmetry, we have the same result for every $u, v \in V(G_2)$. $\square$

Since $\operatorname{diam} V(G) \leq \operatorname{diam} G \leq \operatorname{diam} V(G) + 1$ for every graph G, the previous proposition has the following consequence.

Corollary 7.1.6. *For every graphs G_1, G_2 we have $1 \leq \operatorname{diam} G_1 \uplus G_2 \leq 3$.*

Proposition 7.1.3 and Corollary 7.1.6 give the following results. Given a graph G, we say that $x \in G$ is a midpoint (of an edge) if $d_G(x, V(G)) = 1/2$.

Corollary 7.1.7. *Let G_1, G_2 be two graphs. If $d_{G_1 \uplus G_2}(x, y) = 3$, then x, y are two midpoints in G_i with $d_{G_i}(x, y) \geq 3$ for some $i \in \{1, 2\}$.*

Corollary 7.1.8. *Let G_1, G_2 be two graphs. Then, $\operatorname{diam} G_1 \uplus G_2 = 3$ if and only if there are two midpoints x, y in G_i with $d_{G_i}(x, y) \geq 3$ for some $i \in \{1, 2\}$.*

7.2 Hyperbolicity constant of the graph join of two graphs

In this section we obtain some bounds for the hyperbolicity constant of the graph join of two graphs. These bounds allow to prove that the joins of graphs are always hyperbolic with a small hyperbolicity constant.

We have the following consequence of Corollary 7.1.6 and Theorem 1.3.8.

Corollary 7.2.1. *For every graphs G_1, G_2, the graph join $G_1 \uplus G_2$ is hyperbolic with $\delta(G_1 \uplus G_2) \le 3/2$, and the inequality is sharp.*

Theorem 7.2.13 characterizes the graph join of two graphs for which the equality in the previous corollary is attained.

Theorem 7.2.2. *For every graphs G_1, G_2, we have*

$$\delta(G_1 \uplus G_2) = \max\{\delta(\Gamma_1 \uplus \Gamma_2) : \Gamma_i \text{ is isometric to } G_i \text{ for } i = 1, 2\}.$$

Proof. By Proposition 7.1.4 and Lemma 1.3.2 we have $\delta(G_1 \uplus G_2) \ge \delta(\Gamma_1 \uplus \Gamma_2)$ for any isometric subgraph Γ_i of G_i for $i = 1, 2$. Besides, since any graph is an isometric subgraph of itself we obtain the equality by taking $\Gamma_1 = G_1$ and $\Gamma_2 = G_2$. □

We have the following consequence for the hyperbolicity constant of the joins of graphs.

Proposition 7.2.3. *For every graphs G_1, G_2 the graph join $G_1 \uplus G_2$ is hyperbolic with hyperbolicity constant $\delta(G_1 \uplus G_2)$ in $\{0, 3/4, 1, 5/4, 3/2\}$.*

If G_1 and G_2 are *isomorphic*, then we write $G_1 \simeq G_2$. It is clear that if $G_1 \simeq G_2$, then $\delta(G_1) = \delta(G_2)$. The n-vertex edgeless graph ($n \ge 1$) or *empty graph* is a graph without edges and with n vertices, and it is commonly denoted as E_n.

The following result allows to characterize the joins of graphs with hyperbolicity constant less than one in terms of its factor graphs.

Theorem 7.2.4. *Let G_1, G_2 be two graphs.*

(1) $\delta(G_1 \uplus G_2) = 0$ *if and only if* G_1 *and* G_2 *are empty graphs and one of them is isomorphic to* E_1.

(2) $\delta(G_1 \uplus G_2) = 3/4$ *if and only if* $G_1 \simeq E_1$ *and* $\Delta_{G_2} = 1$, *or* $G_2 \simeq E_1$ *and* $\Delta_{G_1} = 1$.

Proof.

(1) By Theorem 1.3.16 it suffices to characterize the joins of graphs which are trees. If G_1 and G_2 are empty graphs and one of them is isomorphic to E_1, then it is clear that $G_1 \uplus G_2$ is a tree. Assume now that $G_1 \uplus G_2$ is a tree. If G_1 and G_2 have at least two vertices then $G_1 \uplus G_2$ has a cycle with length four. Thus, G_1 or G_2 is isomorphic to E_1. Without loss of generality we can assume that $G_1 \simeq E_1$. Note that if G_2 has at least one edge then $G_1 \uplus G_2$ has a cycle with length three. Then, $G_2 \simeq E_n$ for some $n \in \mathbb{N}$.

(2) By Theorem 1.3.16 it suffices to characterize the joins of graphs with circumference three. If $G_1 \simeq E_1$ and $\Delta_{G_2} = 1$, or $G_2 \simeq E_1$ and $\Delta_{G_1} = 1$, then it is clear that $c(G_1 \uplus G_2) = 3$. Assume now that $c(G_1 \uplus G_2) = 3$. If G_1, G_2 both have at least two vertices then $G_1 \uplus G_2$ contains a cycle with length four and so $c(G_1 \uplus G_2) \geq 4$. Therefore, G_1 or G_2 is isomorphic to E_1. Without loss of generality we can assume that $G_1 \simeq E_1$. Note that if $\Delta_{G_2} \geq 2$ then there is an isomorphic subgraph to $E_1 \uplus P_3$ in $G_1 \uplus G_2$; thus, $G_1 \uplus G_2$ contains a cycle with length four. So, we have $\Delta_{G_2} \leq 1$. Besides, since G_2 is a non-empty graph by (1), we have $\Delta_{G_2} \geq 1$.

□

Theorems 1.3.19 and 7.2.4 show that the family of graphs $E_1 \uplus G$ when G belongs to the set of graphs is a representative collection of joins of graphs since their hyperbolicity constants take all possible values.

Theorem 1.3.18 has the following consequence for joins of graphs.

Lemma 7.2.5. *Let G_1, G_2 be two graphs. If $\delta(G_1) > 1$, then $\delta(G_1 \uplus G_2) > 1$.*

Proof. By Theorem 1.3.18, there exist a cycle σ in $G_1 \uplus G_2$ (contained in G_1) with length $L(\sigma) \geq 5$ and a vertex $w \in \sigma$ such that $\deg_\sigma(w) = 2$. Thus, Theorem 1.3.18 gives $\delta(G_1 \uplus G_2) > 1$. □

Note that the converse of Lemma 7.2.5 does not hold, since $\delta(E_1) = \delta(P_4) = 0$ and we can check that $\delta(E_1 \uplus P_4) = 5/4$.

Corollary 7.2.6. *Let G_1, G_2 be two graphs. Then $\delta(G_1 \uplus G_2) \geq \min\big\{5/4, \max\{\delta(G_1), \delta(G_2)\}\big\}$.*

Proof. By symmetry, it suffices to show $\delta(G_1 \uplus G_2) \geq \min\{5/4, \delta(G_1)\}$. If $\delta(G_1) > 1$, then the inequality holds by Lemma 7.2.5. If $\delta(G_1) = 1$, then there exists a cycle isomorphic to C_4 in $G_1 \subset G_1 \uplus G_2$; hence, $\delta(G_1 \uplus G_2) \geq 1$. If $\delta(G_1) = 3/4$, then there exists a cycle isomorphic to C_3 in $G_1 \subset G_1 \uplus G_2$; hence, $\delta(G_1 \uplus G_2) \geq 3/4$. The inequality is direct if $\delta(G_1) = 0$. □

The following results allow to characterize the joins of graphs with hyperbolicity constant one in terms of G_1 and G_2.

Lemma 7.2.7. *Let G be any graph. Then, $\delta(E_1 \uplus G) \leq 1$ if and only if every path η joining two vertices of G with $L(\eta) = 3$ satisfies $\deg_\eta(w) \geq 2$ for every vertex $w \in V(\eta)$.*

Note that if every path η joining two vertices of G with $L(\eta) = 3$ satisfies $\deg_\eta(w) \geq 2$ for every vertex $w \in V(\eta)$, then the same result holds for $L(\eta) \geq 3$ instead of $L(\eta) = 3$.

Proof. Let v be the vertex in E_1.

Assume first that $\delta(E_1 \uplus G) \leq 1$. Seeking for a contradiction, assume that there is a path η joining two vertices of G with $L(\eta) = 3$ and one vertex $w' \in V(\eta)$ with $\deg_\eta(w') = 1$. Consider now the cycle σ obtained by joining the endpoints of η with v. Note that $w' \in \sigma$ and $\deg_\sigma(w') = 2$; therefore, Theorem 1.3.18 gives $\delta(E_1 \uplus G) > 1$, which is a contradiction.

Assume now that every path η joining two vertices of G with $L(\eta) = 3$ satisfies $\deg_\eta(w) \geq 2$ for every vertex $w \in V(\eta)$. Note that if G does not have paths isomorphic to P_4 then there is no cycle in $E_1 \uplus G$ with length greater than 4 and so, $\delta(E_1 \uplus G) \leq 1$. We are going to prove now that for every cycle σ in G with $L(\sigma) \geq 5$ we have $\deg_\sigma(w) \geq 3$ for every vertex $w \in V(\sigma)$. Let σ be any cycle in $E_1 \uplus G$ with $L(\sigma) \geq 5$. If $v \in \sigma$, then $\sigma \cap G$ is a subgraph of G isomorphic to P_n for $n = L(\sigma) - 1$, and $\deg_\sigma(v) = n \geq 4$. Since $L(\sigma \cup G) \geq 3$, $\deg_{\sigma \cap G}(w) \geq 2$ for every $w \in V(\sigma \cap G)$ by hypothesis, and we conclude $\deg_\sigma(w) \geq 3$ for every $w \in V(\sigma) \setminus \{v\}$. If $v \notin \sigma$, let w be any vertex in σ and let $P(w)$ be a path with length 3 contained in σ and such that w is an endpoint of $P(w)$. By hypothesis $\deg_{P(w)}(w) \geq 2$; since w has a neighbor $w' \in V(\sigma \setminus P(w))$, $\deg_\sigma(w) \geq 3$ for any $w \in V(\sigma)$. Then, Theorem 1.3.18 gives the result. □

Note that if a graph G verifies $\operatorname{diam} G \leq 2$ then every path η joining two vertices of G with $L(\eta) = 3$ satisfies $\deg_\eta(w) \geq 2$ for every vertex $w \in V(\eta)$. The converse does not hold, since in the disjoint union $C_3 \cup C_3$ of two cycles C_3 any path with length 3 is a cycle and $\operatorname{diam} C_3 \cup C_3 = \infty$. However, these two conditions are equivalent if G is connected.

If G is a graph with connected components $\{G_j\}$, we define

$$\operatorname{diam}^* G := \sup_j \operatorname{diam} G_j.$$

Note that $\operatorname{diam}^* G = \operatorname{diam} G$ if G is connected; otherwise, $\operatorname{diam} G = \infty$. Also, $\operatorname{diam}^* G > 1$ is equivalent to $\Delta_G \geq 2$. We also have the following result:

Lemma 7.2.8. *Let G be any graph. Then $\operatorname{diam}^* G \leq 2$ if and only if every η joining two vertices of G with $L(\eta) = 3$ satisfy $\deg_\eta(w) \geq 2$ for every $w \in V(G)$.*

Lemma 7.2.9. *Let G_1 and G_2 be two graphs with at least two vertices. Then, $\delta(G_1 \uplus G_2) = 1$ if and only if $\operatorname{diam} G_i \leq 2$ or G_i is an empty graph for $i = 1, 2$.*

Proof. Assume that $\delta(G_1 \uplus G_2) = 1$. Seeking for a contradiction, assume that $\operatorname{diam} G_1 \geq 5/2$ and G_1 is a non-empty graph or $\operatorname{diam} G_2 \geq 5/2$ and G_2 is a non-empty graph. By symmetry, without loss of generality we can assume that $\operatorname{diam} G_1 \geq 5/2$ and G_1 is a non-empty graph;

hence, there are a vertex $v \in V(G_1)$ and a midpoint $p \in [w_1, w_2]$ with $d_{G_1}(v, p) \geq 5/2$. Consider a cycle σ in $G_1 \uplus G_2$ containing the vertex v, the edge $[w_1, w_2]$ and two vertices of G_2, with $L(\sigma) = 5$. We have $\deg_\sigma(v) = 2$. Thus, Theorem 1.3.18 gives $\delta(G_1 \uplus G_2) > 1$. This contradicts our assumption, and so, we obtain $\operatorname{diam} G_1 \leq 2$.

Assume now that $\operatorname{diam} G_i \leq 2$ or G_i is an empty graph for $i = 1, 2$. Since G_1 and G_2 have at least two vertices, there exists a cycle isomorphic to C_4 in $G_1 \uplus G_2$.

First of all, if G_1 and G_2 are empty graphs then Theorem 1.3.19 gives $\delta(G_1 \uplus G_2) = 1$.

Without loss of generality we can assume that G_1 is a non-empty graph, then G_1 satisfies $\operatorname{diam} G_1 \leq 2$.

Assume that G_2 is an empty graph. Let σ be any cycle in $G_1 \uplus G_2$ with $L(\sigma) \geq 5$. Since σ contains at least three vertices in G_1, we have $\deg_\sigma(v) = |V(G_1) \cap \sigma| \geq 3$ for every $v \in V(G_2) \cap \sigma$. Besides, if $|V(G_2) \cap \sigma| \geq 3$ then $\deg_\sigma(w) \geq |V(G_2) \cap \sigma| \geq 3$ for every $w \in V(G_1) \cap \sigma$. If $|V(G_2) \cap \sigma| = 1$, then $\eta := \sigma \cap G_1$ is a path in G_1 with $L(\eta) \geq 3$, and so, $\deg_\eta(w) \geq 2$ and $\deg_\sigma(w) \geq 3$ for every $w \in V(\eta)$. If $|V(G_2) \cap \sigma| = 2$, then $\sigma \cap G_1$ is the union of two paths and $|V(G_1) \cap \sigma| \geq 3$; since $\operatorname{diam} G_1 \leq 2$, we have $\deg_{G_1 \cap \sigma}(w) \geq 1$ for every $w \in V(G_1) \cap \sigma$ (otherwise there are a vertex $w \in V(G_1) \cap \sigma$ and a midpoint $p \in G_1 \cap \sigma$ with $d_{G_1}(w, p) > 2$). Then, we have $\deg_\sigma(v) \geq 3$ for every $v \in V(\sigma)$ and so, we obtain $\delta(G_1 \uplus G_2) = 1$ by Theorem 1.3.17.

Finally, assume that $\operatorname{diam} G_2 \leq 2$. By Theorem 1.3.23 it suffices to consider geodesic triangles $T = \{x, y, z\}$ in $G_1 \uplus G_2$ that are cycles with $x, y, z \in J(G_1 \uplus G_2)$. So, since $\operatorname{diam} G_1, \operatorname{diam} G_2 \leq 2$, Proposition 7.1.3 gives that $L([xy]), L([yz]), L([zx]) \leq 2$; thus, for every $\alpha \in [xy]$, $d_{G_1 \uplus G_2}(\alpha, [yz] \cup [zx]) \leq d_{G_1 \uplus G_2}(\alpha, \{x, y\}) \leq L([xy])/2$. Hence, $\delta(T) \leq \max\{L([xy]), L([yz]), L([zx])\}/2 \leq 1$ and so, $\delta(G_1 \uplus G_2) \leq 1$. Since G_1 and G_2 have at least two vertices, by Theorem 1.3.16 we have $\delta(G_1 \uplus G_2) \geq 1$ and we conclude $\delta(G_1 \uplus G_2) = 1$. $\square$

The following result characterizes the joins of graphs with hyperbolicity constant one.

Theorem 7.2.10. *Let G_1, G_2 be any two graphs. Then the following statements hold:*

- *Assume that $G_1 \simeq E_1$. Then $\delta(G_1 \uplus G_2) = 1$ if and only if $1 < \operatorname{diam}^* G_2 \leq 2$.*
- *Assume that G_1 and G_2 have at least two vertices. Then $\delta(G_1 \uplus G_2) = 1$ if and only if $\operatorname{diam} G_i \leq 2$ or G_i is an empty graph for $i = 1, 2$.*

Proof. We have the first statement by Theorem 7.2.4 and Lemmas 7.2.7 and 7.2.8. The second statement is just Lemma 7.2.9. $\square$

In order to compute the hyperbolicity constant of any graph join we are going to characterize the joins of graphs with hyperbolicity constant 3/2.

Lemma 7.2.11. *Let G_1, G_2 be any two graphs. If $\delta(G_1 \uplus G_2) = 3/2$, then each geodesic triangle $T = \{x, y, z\}$ in $G_1 \uplus G_2$ that is a cycle with $x, y, z \in J(G_1 \uplus G_2)$ and $\delta(T) = 3/2$ is contained in either G_1 or G_2.*

Proof. Seeking for a contradiction assume that there is a geodesic triangle $T = \{x, y, z\}$ in $G_1 \uplus G_2$ that is a cycle with $x, y, z \in J(G_1 \uplus G_2)$ and $\delta(T) = 3/2$ which contains vertices in both factors G_1, G_2. Without loss of generality we can assume that there is $p \in [xy]$ with $d_{G_1 \uplus G_2}(p, [yz] \cup [zx]) = 3/2$, and so, $L([xy]) \geq 3$. Hence, $d_{G_1 \uplus G_2}(x, y) = 3$ by Corollary 7.1.6, and by Corollary 7.1.7 we have that x, y are midpoints either in G_1 or in G_2, and so, p is a vertex in $G_1 \uplus G_2$. Without loss of generality we can assume that $x, y \in G_1$. Let V_x be the closest vertex to x in $[xz] \cup [zy]$. If $p \in V(G_2)$ then $d_{G_1 \uplus G_2}(p, [yz] \cup [zx]) \leq d_{G_1 \uplus G_2}(p, V_x) = 1$. This contradicts our assumption. If $p \in V(G_1)$ then since T contains vertices in both factors, we have $d_{G_1 \uplus G_2}(p, [yz] \cup [zx]) \leq d_{G_1 \uplus G_2}\big((p, V(G_2) \cap \{[yz] \cup [zx]\}\big) = 1$. This also contradicts our assumption, and so, we have the result. $\square$

Corollary 7.2.12. *Let G_1, G_2 be any two graphs. If $\delta(G_1 \uplus G_2) = 3/2$, then* $\max\{\delta(G_1), \delta(G_2)\} \geq 3/2$.

The following families of graphs allow to characterize the joins of graphs with hyperbolicity constant 3/2. Denote by C_n the cycle graph with $n \geq 3$ vertices and by $V(C_n) := \{v_1^{(n)}, \dots, v_n^{(n)}\}$ the set of their vertices such that $[v_n^{(n)}, v_1^{(n)}] \in E(C_n)$ and $[v_i^{(n)}, v_{i+1}^{(n)}] \in E(C_n)$ for $1 \leq i \leq n-1$. Let us consider $\mathcal{C}_6^{(1)}$ the set of graphs obtained from C_6 by addying a (proper or not) subset of the set of edges $\big\{[v_2^{(6)}, v_6^{(6)}],\ [v_4^{(6)}, v_6^{(6)}]\big\}$. Let us define the set of graphs

$$\mathcal{F}_6 := \{G \text{ containing, as induced subgraph, an isomorphic graph to some element of } \mathcal{C}_6^{(1)}\}.$$

Let us consider $\mathcal{C}_7^{(1)}$ the set of graphs obtained from C_7 by addying a (proper or not) subset of the set of edges $\big\{[v_2^{(7)}, v_6^{(7)}],\ [v_2^{(7)}, v_7^{(7)}],\ [v_4^{(7)}, v_6^{(7)}],\ [v_4^{(7)}, v_7^{(7)}]\big\}$. Define

$$\mathcal{F}_7 := \{G \text{ containing, as induced subgraph, an isomorphic graph to some element of } \mathcal{C}_7^{(1)}\}.$$

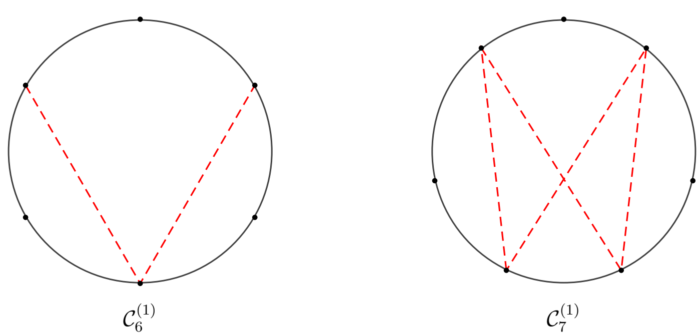


Figure 7.2: Generators of $\mathcal{C}_6^{(1)}$ and $\mathcal{C}_7^{(1)}$.

Let us consider $\mathcal{C}_8^{(1)}$ the set of graphs obtained from C_8 by addying a (proper or not) subset of the set $\big\{[v_2^{(8)}, v_6^{(8)}],\ [v_2^{(8)}, v_8^{(8)}],\ [v_4^{(8)}, v_6^{(8)}],\ [v_4^{(8)}, v_8^{(8)}]\big\}$. Also, consider $\mathcal{C}_8^{(2)}$ the set of graphs obtained from C_8 by addying a (proper or not) subset of $\{[v_2^{(8)}, v_8^{(8)}],\ [v_4^{(8)}, v_6^{(8)}],\ [v_4^{(8)}, v_7^{(8)}],\ [v_4^{(8)}, v_8^{(8)}]\}$. Define

$\mathcal{F}_8 := \{G$ containing, as induced subgraph, an isomorphic graph to some element of $\mathcal{C}_8^{(1)} \cup \mathcal{C}_8^{(2)}\}$.

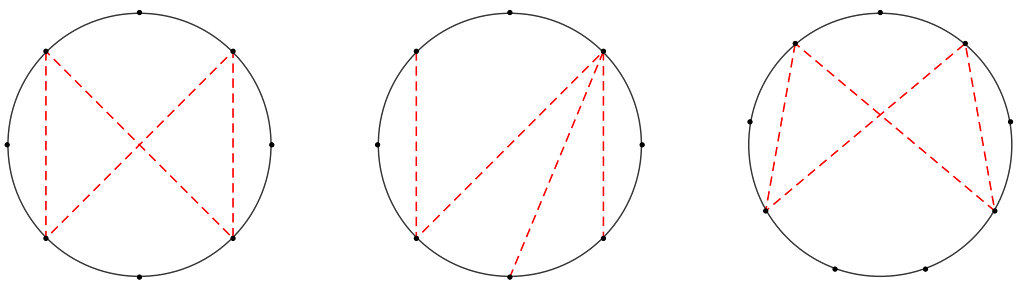
Let us consider $\mathcal{C}_9^{(1)}$ the set of graphs obtained from C_9 by addying a (proper or not) subset of the set of edges $\{[v_2^{(9)}, v_6^{(9)}],\ [v_2^{(9)}, v_9^{(9)}],\ [v_4^{(9)}, v_6^{(9)}],\ [v_4^{(9)}, v_9^{(9)}]\}$. Define

$\mathcal{F}_9 := \{G$ containing, as induced subgraph, an isomorphic graph to some element of $\mathcal{C}_9^{(1)}\}$.

Finally, we define the set $\mathcal{F}$ by

$$\mathcal{F} := \mathcal{F}_6 \cup \mathcal{F}_7 \cup \mathcal{F}_8 \cup \mathcal{F}_9.$$

Note that $\mathcal{F}_6$, $\mathcal{F}_7$, $\mathcal{F}_8$ and $\mathcal{F}_9$ are not disjoint sets of graphs.



$\mathcal{C}_8^{(1)}$ $\mathcal{C}_8^{(2)}$ $\mathcal{C}_9^{(1)}$

Figure 7.3: Generators of $\mathcal{C}_8^{(1)}$, $\mathcal{C}_8^{(2)}$ and $\mathcal{C}_9^{(1)}$.

The following theorem characterizes the joins of graphs G_1 and G_2 with $\delta(G_1 \uplus G_2) = 3/2$. For any non-empty set $S \subset V(G)$, the induced subgraph of S will be denoted by $\langle S \rangle$.

Theorem 7.2.13. *Let G_1, G_2 be any two graphs. Then, $\delta(G_1 \uplus G_2) = 3/2$ if and only if $G_1 \in \mathcal{F}$ or $G_2 \in \mathcal{F}$.*

Proof. Assume first that $\delta(G_1 \uplus G_2) = 3/2$. By Theorem 1.3.23 there is a geodesic triangle $T = \{x, y, z\}$ in $G_1 \uplus G_2$ that is a cycle with $x, y, z \in J(G_1)$ and $\delta(T) = 3/2$. By Lemma 7.2.11, T is contained either in G_1 or in G_2. Without loss of generality we can assume that T is contained in G_1. Without loss of generality we can assume that there is $p \in [xy]$ with $d_{G_1 \uplus G_2}(p, [yz] \cup [zx]) = 3/2$, and by Corollary 7.1.6, $L([xy]) = 3$. Hence, by Corollary 7.1.7 we have that x, y are midpoints in G_1, and so, $p \in V(G_1)$. Since $L([yz]) \le 3$, $L([zx]) \le 3$ and $L([yz]) + L([zx]) \ge L([xy])$, we have $6 \le L(T) \le 9$.

Assume that $L(T) = 6$. Denote by $\{v_1, \ldots, v_6\}$ the vertices in T such that $T = \bigcup_{i=1}^{6} [v_i, v_{i+1}]$ with $v_7 := v_1$. Without loss of generality we can assume that $x \in [v_1, v_2]$,

$y \in [v_4, v_5]$ and $p = v_3$. Since $d_{G_1 \uplus G_2}(x, y) = 3$, we have that $\langle\{v_1, \ldots, v_6\}\rangle$ contains neither $[v_1, v_4]$, $[v_1, v_5]$, $[v_2, v_4]$ nor $[v_2, v_5]$; besides, since $d_{G_1 \uplus G_2}(p, [yz] \cup [zx]) > 1$ we have that $\langle\{v_1, \ldots, v_6\}\rangle$ contains neither $[v_3, v_1]$, $[v_3, v_5]$ nor $[v_3, v_6]$. Note that $[v_2, v_6]$, $[v_4, v_6]$ may be contained in $\langle\{v_1, \ldots, v_6\}\rangle$. Therefore, $G_1 \in \mathcal{F}_6$.

Assume that $L(T) = 7$ and $G_1 \notin \mathcal{F}_6$. Denote by $\{v_1, \ldots, v_7\}$ the vertices in T such that $T = \bigcup_{i=1}^{7}[v_i, v_{i+1}]$ with $v_8 := v_1$. Without loss of generality we can assume that $x \in [v_1, v_2]$, $y \in [v_4, v_5]$ and $p = v_3$. Since $d_{G_1 \uplus G_2}(x, y) = 3$, we have that $\langle\{v_1, \ldots, v_7\}\rangle$ does not contain neither $[v_1, v_4]$, $[v_1, v_5]$, $[v_2, v_4]$, $[v_2, v_5]$; besides, since $d_{G_1 \uplus G_2}(p, [yz] \cup [zx]) > 1$ we have that $\langle\{v_1, \ldots, v_7\}\rangle$ does not contain neither $[v_3, v_1]$, $[v_3, v_5]$, $[v_3, v_6]$, $[v_3, v_7]$. Since $G_1 \notin \mathcal{F}_6$, $[v_1, v_6]$ and $[v_5, v_7]$ are not contained in $\langle\{v_1, \ldots, v_7\}\rangle$. Note that $[v_2, v_6]$, $[v_2, v_7]$, $[v_4, v_6]$, $[v_4, v_7]$ may be contained in $\langle\{v_1, \ldots, v_7\}\rangle$. Hence, $G_1 \in \mathcal{F}_7$.

Assume that $L(T) = 8$ and $G_1 \notin \mathcal{F}_6 \cup \mathcal{F}_7$. Denote by $\{v_1, \ldots, v_8\}$ the vertices in T such that $T = \bigcup_{i=1}^{8}[v_i, v_{i+1}]$ with $v_9 := v_1$. Without loss of generality we can assume that $x \in [v_1, v_2]$, $y \in [v_4, v_5]$ and $p = v_3$. Since $d_{G_1 \uplus G_2}(x, y) = 3$, we have that $\langle\{v_1, \ldots, v_8\}\rangle$ does not contain neither $[v_1, v_4]$, $[v_1, v_5]$, $[v_2, v_4]$, $[v_2, v_5]$; besides, since $d_{G_1 \uplus G_2}(p, [yz] \cup [zx]) > 1$ we have that $\langle\{v_1, \ldots, v_8\}\rangle$ does not contain neither $[v_3, v_1]$, $[v_3, v_5]$, $[v_3, v_6]$, $[v_3, v_7]$, $[v_3, v_8]$. Since $G_1 \notin \mathcal{F}_6 \cup \mathcal{F}_7$, $[v_1, v_6]$, $[v_1, v_7]$, $[v_5, v_7]$, $[v_5, v_8]$ and $[v_6, v_8]$ are not contained in $\langle\{v_1, \ldots, v_8\}\rangle$. Since T is a geodesic triangle we have that $z \in \{v_{6,7}, v_7, v_{7,8}\}$ with $v_{6,7}$ and $v_{7,8}$ the midpoints of $[v_6, v_7]$ and $[v_7, v_8]$, respectively. If $z = v_7$ then $\langle\{v_1, \ldots, v_8\}\rangle$ does not contain neither $[v_2, v_7]$, $[v_4, v_7]$. Note that $[v_2, v_6]$, $[v_2, v_8]$, $[v_4, v_6]$, $[v_4, v_8]$ may be contained in $\langle\{v_1, \ldots, v_8\}\rangle$. If $z = v_{6,7}$ then $\langle\{v_1, \ldots, v_8\}\rangle$ does not contain neither $[v_2, v_6]$, $[v_2, v_7]$. Note that $[v_2, v_8]$, $[v_4, v_6]$, $[v_4, v_7]$, $[v_4, v_8]$ may be contained in $\langle\{v_1, \ldots, v_8\}\rangle$. By symmetry, we obtain an equivalent result for $z = v_{7,8}$. Therefore, $G_1 \in \mathcal{F}_8$.

Assume that $L(T) = 9$ and $G_1 \notin \mathcal{F}_6 \cup \mathcal{F}_7 \cup \mathcal{F}_8$. Denote by $\{v_1, \ldots, v_9\}$ the vertices in T such that $T = \bigcup_{i=1}^{9}[v_i, v_{i+1}]$ with $v_{10} := v_1$. Without loss of generality we can assume that $x \in [v_1, v_2]$, $y \in [v_4, v_5]$ and $p = v_3$. Since $d_{G_1 \uplus G_2}(x, y) = 3$, we have that $\langle\{v_1, \ldots, v_9\}\rangle$ does not contain neither $[v_1, v_4]$, $[v_1, v_5]$, $[v_2, v_4]$, $[v_2, v_5]$; besides, since $d_{G_1 \uplus G_2}(p, [yz] \cup [zx]) > 1$ we have that $\langle\{v_1, \ldots, v_9\}\rangle$ does not contain neither $[v_3, v_1]$, $[v_3, v_5]$, $[v_3, v_6]$, $[v_3, v_7]$, $[v_3, v_8]$, $[v_3, v_9]$. Since T is a geodesic triangle we have that z is the midpoint of $[v_7, v_8]$. Since $d_{G_1 \uplus G_2}(y, z) = d_{G_1 \uplus G_2}(z, x) = 3$, we have that $\langle\{v_1, \ldots, v_9\}\rangle$ does not contain neither $[v_1, v_7]$, $[v_1, v_8]$, $[v_2, v_7]$, $[v_2, v_8]$, $[v_4, v_7]$, $[v_4, v_8]$, $[v_5, v_7]$, $[v_5, v_8]$. Since $G_1 \notin \mathcal{F}_6 \cup \mathcal{F}_7 \cup \mathcal{F}_8$, $[v_1, v_6]$, $[v_5, v_9]$, $[v_6, v_8]$, $[v_6, v_9]$ and $[v_7, v_9]$ are not contained in $\langle\{v_1, \ldots, v_9\}\rangle$. Note that $[v_2, v_6]$, $[v_2, v_9]$, $[v_4, v_6]$, $[v_4, v_9]$ may be contained in $\langle\{v_1, \ldots, v_9\}\rangle$. Hence, $G_1 \in \mathcal{F}_9$.

Finally, one can check that if $G_1 \in \mathcal{F}$ or $G_2 \in \mathcal{F}$, then $\delta(G_1 \uplus G_2) = 3/2$, by following the previous arguments. $\square$

These results allow to compute, in a simple way, the hyperbolicity constant of every graph join:

Theorem 7.2.14. *Let G_1, G_2 be any two graphs. Then,*

$$\delta(G_1 \uplus G_2) = \begin{cases} 0, & \text{if } G_1 \text{ and } G_2 \text{ are empty graphs and one of them is isomorphic to } E_1, \\ 3/4, & \text{if } G_1 \simeq E_1 \text{ and } \Delta_{G_2} = 1, \text{ or } G_2 \simeq E_1 \text{ and } \Delta_{G_1} = 1, \\ 1, & \text{if } G_1 \simeq E_1 \text{ and } 1 < \operatorname{diam}^* G_2 \le 2; \text{ or} \\ & G_2 \simeq E_1 \text{ and } 1 < \operatorname{diam}^* G_1 \le 2; \text{ or} \\ & |V(G_1)| \ge 2, |V(G_2)| \ge 2 \text{ and } \operatorname{diam} G_i \le 2 \text{ or} \\ & G_i \text{ is an empty graph for } i = 1, 2; \\ 3/2, & \text{if } G_1 \in \mathcal{F} \text{ or } G_2 \in \mathcal{F}, \\ 5/4, & \text{otherwise.} \end{cases}$$

Corollary 7.2.15. *Let G be any graph. Then,*

$$\delta(E_1 \uplus G) = \begin{cases} 0, & \text{if } \operatorname{diam}^* G = 0, \\ 3/4, & \text{if } \operatorname{diam}^* G = 1, \\ 1, & \text{if } 1 < \operatorname{diam}^* G \le 2, \\ 5/4, & \text{if } \operatorname{diam}^* G > 2 \text{ and } G \notin \mathcal{F}, \\ 3/2, & \text{if } G \in \mathcal{F}. \end{cases}$$

7.3 Hyperbolicity of corona of two graphs

In this section we study the hyperbolicity of the corona of two graphs, defined by Frucht and Harary in 1970, see [35].

Definition 7.3.1. *Let G_1 and G_2 be two graphs with $V(G_1) \cap V(G_2) = \emptyset$. The* corona *of G_1 and G_2, denoted by $G_1 \diamond G_2$, is defined as the graph obtained by taking one copy of G_1 and a copy of G_2 for each vertex $v \in V(G_1)$, and then joining each vertex $v \in V(G_1)$ to every vertex in the v-th copy of G_2.*

From the definition, it clearly follows that the corona product of two graphs is a non-commutative and non-associative operation. Figure 7.4 show the corona of two graphs.

Many authors deal just with corona of finite graphs; however, our results hold for finite or infinite graphs.

We remark that the corona $G_1 \diamond G_2$ of two graphs is connected if and only if G_1 is connected.

The following result characterizes the hyperbolicity of the corona of two graphs and provides the precise value of its hyperbolicity constant.

Theorem 7.3.2. *Let G_1, G_2 be any two graphs. Then $\delta(G_1 \diamond G_2) = \max\{\delta(G_1), \delta(E_1 \uplus G_2)\}$.*

Proof. Assume first that G_1 is connected. The formula follows from Theorem 1.3.7, since $\left\{G_1, \{\{v\} \uplus G_2\}_{v \in V(G_1)}\right\}$ is a T-decomposition of $G_1 \diamond G_2$. Finally, note that if G_1 is a non-connected graph, then we can apply the previous argument to each connected component. $\square$

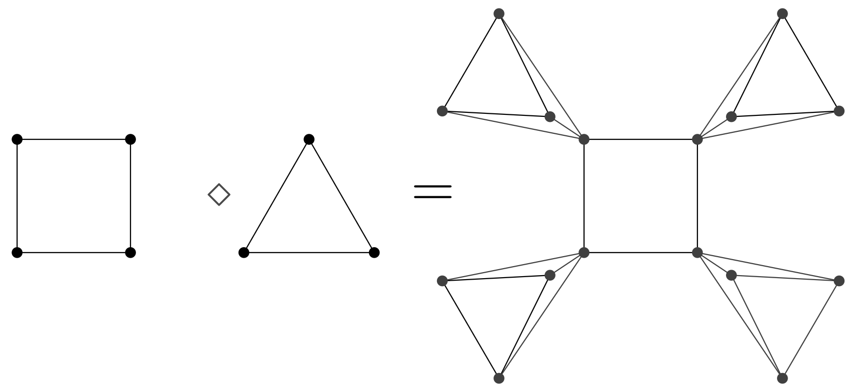


Figure 7.4: Corona of two graphs $C_4 \diamond C_3$.

Note that Corollary 7.2.15 provides the precise value of $\delta(E_1 \uplus G_2)$.

Corollary 7.3.3. *Let G_1, G_2 be any two graphs. Then $G_1 \diamond G_2$ is hyperbolic if and only if G_1 is hyperbolic.*

Proof. By Theorem 7.3.2 we have $\delta(G_1 \diamond G_2) = \max\{\delta(G_1), \delta(E_1 \uplus G_2)\}$. Then, by Corollary 7.2.1 we have $\delta(G_1) \leq \delta(G_1 \diamond G_2) \leq \max\{\delta(G_1), 3/2\}$. $\square$

Conclusions

Conclusions

In this PhD Thesis we obtain quantitative information about the distortion of the hyperbolicity constant of the graph $G \setminus e$ obtained from the graph G by deleting an arbitrary edge e from it. These inequalities allow to characterize in a quantitative way the hyperbolicity of any graph in terms of local hyperbolicity.

We also obtain information about the hyperbolicity constant of the line graph $\mathcal{L}(G)$ in terms of properties of the graph G. In particular, we prove qualitative results as the following: a graph G is hyperbolic if and only if $\mathcal{L}(G)$ is hyperbolic; if $\{G_n\}$ is a T-decomposition of G, the line graph $\mathcal{L}(G)$ is hyperbolic if and only if $\sup_n \delta(\mathcal{L}(G_n))$ is finite. Besides, we obtain quantitative results when k is the length of the edges of G and $\mathcal{L}(G)$. Two of them are quantitative versions of our qualitative results. We also prove that $g(G)/4 \le \delta(\mathcal{L}(G)) \le c(G)/4 + 2k$, where $g(G)$ is the girth of G and $c(G)$ is its circumference. We show that $\delta(\mathcal{L}(G)) \ge \sup\{L(g) : g \text{ is an isometric cycle in } G\}/4$. Besides, we obtain bounds for $\delta(G) + \delta(\mathcal{L}(G))$. Also, we characterize the graphs G with $\delta(\mathcal{L}(G)) < k$.

Furthermore, we consider G with edges of arbitrary lengths, and $\mathcal{L}(G)$ with edges of non-constant lengths. In particular, we prove that $\delta(G) \le \delta(\mathcal{L}(G)) \le 5\delta(G) + 3l_{max}$, where $l_{max} := \sup_{e\in E(G)} L(e)$. This result implies the monotony of the hyperbolicity constant under a non-trivial transformation (the line graph of a graph).

Also, we obtain criteria which allow us to decide, for a large class of graphs, whether they are hyperbolic or not. We are especially interested in the planar graphs which are the "boundary" (the 1-skeleton) of a tessellation of the Euclidean plane. Furthermore, we prove that a graph obtained as the 1-skeleton of a general CW 2-complex is hyperbolic if and only if its dual graph is hyperbolic.

Besides, we extend in two ways (edge-chordality and path-chordality) the classical definition of chordal graphs in order to relate this property with Gromov hyperbolicity. In fact, we prove that every edge-chordal graph is hyperbolic and that every hyperbolic graph is path-chordal. Furthermore, we prove that every path-chordal cubic graph (with small path-chordality constant) is hyperbolic.

Finally, we characterize the hyperbolic product graphs for graph join $G_1 \uplus G_2$ and the corona $G_1 \diamond G_2$: $G_1 \uplus G_2$ is always hyperbolic, and $G_1 \diamond G_2$ is hyperbolic if and only if G_1

is hyperbolic (see Corollaries 7.2.1 and 7.3.3). Furthermore, we obtain simple formulae for the hyperbolicity constant of the graph join $G_1 \uplus G_2$ and the corona $G_1 \diamond G_2$. In particular, $\delta(G_1 \diamond G_2) = \max\{\delta(G_1), \delta(G_2 \uplus E_1)\}$, where E_1 is a graph with just one vertex. We want to remark that it is not usual at all to obtain explicit formulae for the hyperbolicity constant of large classes of graphs.

Future Works

The papers [7, 13, 16, 17, 21, 24, 59, 63, 65, 69, 77] study the hyperbolicity of, respectively, complement of graphs, chordal graphs, strong product graphs, lexicographic product graph, join and corona of graphs, line graphs, Cartesian product graphs, cubic graphs, tessellation graphs, short graphs and median graphs. These results have as a natural continuation of our work in the following problems about hyperbolic graphs.

Characterization The main goal in this topic is to characterize the hyperbolic graphs in terms of some classical concept appearing in graph theory.

- In [13, 81] the authors study the chordal graphs. In Chapter 6 we extend in two ways (edge-chordality and path-chordality) the classical definition of *chordal graphs* in order to relate this property with Gromov hyperbolicity. In fact, we prove that every edge-chordal graph is hyperbolic and that every hyperbolic graph is path-chordal. However, the converses do not hold: we obtain a hyperbolic graph which is not edge-chordal and a path-chordal graph that is non-hyperbolic. We hope to find an appropriate generalization of chordality equivalent to hyperbolicity.
- In [69] the author characterizes the hyperbolic *short graphs*: an r-short graph G is hyperbolic if and only if $S_{9r}(G)$ is finite, where $S_R(G) := \sup\{L(C) : C$ is an R-isometric cycle in $G\}$ and we say that a cycle C is R-isometric if $d_C(x, y) \leq d_G(x, y) + R$ for every $x, y \in C$. We are trying to obtain a relation between $S_{9r}(G)$ and $S_0(G)$. This would improve the characterization in [69] and we think that this could give a characterization of general hyperbolic graphs.

Product graphs In [16, 17, 59] the authors characterize in a simple way the hyperbolicity of Cartesian product, strong product and lexicographic product of graphs. Besides, in Chapter 7 (see [21]) we study the hyperbolicity of graph join and corona of two graphs characterizing their hyperbolicity. We propose to study the hyperbolicity of other binary operations of graphs as:

- Cartesian sum of graphs.
- Kronecker product or tensor product of graphs.

- Zig-zag product of regular graphs.
- Rooted product of graphs.
- Some others products.

Contraction of an edge In [9] the authors study the distortion of the hyperbolicity constant when we delete loops and multi-edges. Besides, in Chapter 2 (see [18]) we study the distortion of the hyperbolicity constant of a graph when we delete an edge. So, it is natural to study the distortion of the hyperbolicity constant when we contract an edge of a graph.

Others operations In [7] the authors study the hyperbolicity of the complement of a graph. In Chapter 3 and Chapter 4 (see [24, 20]) we study the hyperbolicity of line graphs. In Chapter 5 (see [19]) we study some class of planar graphs and also we obtain results about their dual graphs. We propose to study the hyperbolicity for some unary operations of graphs, for instance:

- Graph minor.
- Power of graphs.
- Mycielskian graph.
- Some others unary operations.

Parameters of graphs In [58] the authors relate the hyperbolicity of a graph with its order and its girth, obtaining upper and lower bounds. In [72] the authors obtain an upper bound of the hyperbolicity constant of a graph in terms of its diameter. Besides, in Chapter 3 (see [24]) we obtain an upper bound in terms of its circumference. Furthermore, in [70] the authors relate the hyperbolicity constant of a graph with some known parameters of the graph, as its independence number, its maximum and minimum degree and its domination number. We are trying to relate the hyperbolicity with other natural parameter of graphs, as:

- Isoperimetric constants.
- Differential.

Convex tessellation of $\mathbb{R}^2$ In [65] and Chapter 5 (see [19]) the authors prove that a large class of 1-skeletons of tessellations of the Euclidean plane are not hyperbolic. They also find a non-hyperbolic 1-skeleton of a tessellation of $\mathbb{R}^2$ with non-convex tiles. They conjecture that the 1-skeleton is not hyperbolic for any tessellation with convex polygons. In [15] the authors show that in order to prove this conjecture it suffices to consider tessellations graphs of $\mathbb{R}^2$ such that every tile is a triangle. This is an interesting open problem.

Particular classes of graphs In [7, 13, 63, 65, 69, 77] the authors study the hyperbolicity of complement of graphs, chordal graphs, cubic graphs, tessellation graphs, short graphs and median graphs. Furthermore, Chapter 5 (see [19]) study some planar graphs and Chapter 6 (see [4]) study the hyperbolicity of some extended chordal graphs. We propose to study the hyperbolicity for several classes of graphs, for instance:

- Cage's graphs.
- Polytope graphs.
- Delaunay triangulations.
- Some other class of graphs.

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