UNIVERSIDAD CARLOS III DE MADRID
working papers

Working Paper 02-42 (11)
Statistics and Econometrics Series
September 2002

Departamento de Estadística y Econometría
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SINGULAR RANDOM MATRIX DECOMPOSITIONS: DISTRIBUTIONS.

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Abstract

Assuming that Y has a singular matrix variate elliptically contoured distribution with respect to the Hausdorff measure, the distributions of several matrices associated to QR, modified QR, SV and Polar decompositions of matrix Y are determined, for central and non-central, non-singular and singular cases, as well as their relationship to the Wishart and Pseudo-Wishart generalized singular and non-singular distributions. We present a particular example for the Karhunen-Lòeve decomposition. Some of these results are also applied to two particular subfamilies of elliptical distributions, the singular matrix variate normal distribution and the singular matrix variate symmetric Pearson type VII distribution.

Keywords: Noncentral singular distribution, generalized Wishart and Pseudo-Wishart distributions, elliptical distribution, SVD, QR, modified QR and polar decompositions, Spectral, L´DL, Cholesky decompositions, symmetric non-negative definite square root, Karhunen-Lòeve decomposition.

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1 Introduction

It has been a common practice in the past, to eliminate variables or individuals to correct for dependencies among columns or rows when we sample from a multivariate distribution. This solution in part, was due to the fact of not having a distribution theory to handle all those cases. In a more formal way, let $Y \in \mathbb{R}^{N\times m}$, be a sample of N individuals with m variables under study, if there exist dependencies among rows (individuals) or columns (variables), Y does not have a density with respect to the Lebesgue measure in $\mathbb{R}^{Nm}$. However, it is known that Y has a density on a subspace $\mathcal{M} \subset \mathbb{R}^{Nm}$ on which it is possible to define a measure called the Hausdorff measure, which coincides with the Lebesgue measure when it is define on $\mathcal{M}$. Details on this kind of problems can be found in Díaz-García et al. (1997) and Díaz-García and Gutiérrez-Jáimez (2001). They proposed expressions for the singular matrix variate Normal distribution and singular matrix variate Elliptically contoured distribution for which it is plausible to consider the dependencies among rows or columns. In other words, we count now with a solution for the classical multivariate statistical analysis when based on the Normal distributions and also for the more general case called the generalized multivariate statistical analysis based on the elliptical contoured distributions.

However, there still were other situations for which the problem of establishing associated distributions for Y remained, for example: the Wishart distribution, the matrix variate T or matrix variate F or Beta, among others. Two important references in this line are, Díaz-García et al. (1997) and Díaz-García and Gutiérrez-Jáimez (2001) since they solved the problem for the different ways of defining the Wishart and Pseudo-Wishart distributions either for the central or the noncentral cases. They worked the results for the Normal distribution as well as the Elliptically contoured distributions. Note that Uhlig (1994), found the central and nonsingular Pseudo-Wishart distributions based on the Normal case before them. The distribution for the matrix variate T is found indirectly when the singular matrix variate Elliptically contoured distribution is established since the matrix variate T is a member of this family. Also, Díaz-García and Gutiérrez-Jáimez (1997) and Uhlig (1994) provide expressions for the matrix variate F and Beta (central case), when only one type of dependency is present.

When Y has a distribution with respect to the Lebesgue measure we could find different ways of deriving the Wishart distribution. Some are base on the QR decomposition, Roy (1957), Srivastava and Khatri (1979) and Muirhead (1982), others on the single value decomposition (SVD), James (1954)

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and some others on the Polar decomposition, Herz (1955) and Cadet (1996). What all of these approaches through different factorizations are trying to do is to find an alternative coordinate system for the columns (or rows) for the matrix Y. For example, the coordinates obtained from the QR decomposition are called rectangular coordinates, Rao (1973, p. 597), for the Polar decomposition, polar coordinates, Cadet (1996), etc. These matrices of coordinates, besides of being the key part for establishing the Wishart, Pseudo-Wishart, F and Beta distributions, as well as distributions of $|Y'Y|$ and $\operatorname{tr} Y'Y$ among others, play an important role in other areas of knowledge, in particular on the Shape Theory and Pattern Recognition. As an example, if Y has a matrix variate normal distribution, it may be written as $Y = H_1 T$, the QR decomposition. In the context of Shape Theory, the distribution of T is called size-and-shape distribution, also known in the literature as the rectangular coordinates distribution, see Goodall and Mardia (1993), and Rao (1973, p. 597). In the same setting of shape theory, when considering the SV ($Y = V_1 D W_1$) or polar ($Y = P_1 R$) decompositions, the matrices (D, W_1) and R may both be thought of as an alternative coordinates system, in such a way that the corresponding distributions play the role of size-and-shape distributions, see Goodall (1991), and Le and Kendall (1993). Similarly, matrix D is considered as yet another coordinate system, and its corresponding distribution is called size-and-shape cone distribution, see Goodall and Mardia (1993), Díaz-García et al. (2000) and Díaz-García and Gutiérrez-Jáimez (2001). Some of these results were extended to the case in which Y has a singular Gaussian and a singular elliptically contoured distribution, see Díaz-García et al. (2000) and Díaz-García and Gutiérrez-Jáimez (2001). In the context of Pattern Recognition the role of some of these decomposition is also known, in particular the SVD decomposition knows as the Karhunen-Lòeve Expansion or Decomposition, Kotz and Johnson (1982).

In the present work some results on distributions of random matrices, for which their density function exist with respect to the Lebesgue measure, will be extended to the case in which Y has a density with respect to the Hausdorff measure, and moreover, to the case in which Y has a singular matrix variate elliptically contoured distribution. In Section 2, the densities of matrices associated to the QR, modified QR, SVD and Polar decompositions are found with respect to the Hausdorff measure, both for the non-central (see Theorem 1) and central cases (see Corollary 3). These results are applied to the Karhunen-Lòeve expansion and for two subfamilies of elliptically contoured distributions, the matrix variate normal distribution and the matrix variate symmetric Pearson type VII distribution, (see Díaz-García and González-Farias, 1999). Finally, we present a discussion on Wishart and Pseudo-Wishart density functions defined with respect to different measures.

2 Density Functions

Notation. Let $\mathcal{L}_{m,N}^{+}(q)$ be the linear space of all $N \times m$ real matrices of rank $q \leq \min(N,m)$ with q distinct singular values. The set of matrices $H_1 \in \mathcal{L}_{m,N}$ such that $H_1'H_1 = I_m$ is a manifold denoted by $\mathcal{V}_{m,N}$, called Stiefel manifold. In particular, $\mathcal{V}_{m,m}$ is the group of orthogonal matrices $\mathcal{O}(m)$. Denote by $\mathcal{S}_m$, the homogeneous space of $m \times m$ positive definite symmetric matrices; $\mathcal{S}_m^{+}(q)$, the $(mq - q(q-1)/2)$-dimensional manifold of rank q positive semidefinite $m \times m$ symmetric matrices with q distinct positive eigenvalues; $\mathcal{T}_m$ denote the group of $m \times m$ upper triangular matrices and $\mathcal{T}_m^{+}$ is the group of $m \times m$ upper triangular matrices with positive diagonal elements; $\mathcal{T}_{m,N}^{+}$ the set of $N \times m$ upper quasi-triangular matrices such that $T = (T_1|T_2) \in \mathcal{T}_{m,N}^{+}$, with $T_1 \in \mathcal{T}_N^{+}$ and $T_2 \in \mathcal{L}_{m-N,N}(N)$; $\mathcal{T}_m^{1}$ and $\mathcal{T}_{m,N}^{1}$ denote the set of unit upper triangular or unit quasi-triangular matrices, respectively, such that $t_{ii} = 1$ for all i; $\mathcal{D}(m) \subset \mathcal{T}_m$ the diagonal matrices.

Now, let $Y \in I\!R^{N \times m}$ be a random matrix with rank $r(Y) = q \leq \min(N,m)$ and density function given by

$$\frac{1}{\left(\displaystyle\prod_{i=1}^{r} \lambda_i^{k/2}\right)\left(\displaystyle\prod_{j=1}^{k} \delta_j^{r/2}\right)} h\left(\operatorname{tr} \Sigma^{-}(Y-\mu)'\Theta^{-}(Y-\mu)\right) \tag{1}$$

$$\left.\begin{array}{r} E_1'(Y-\mu)M_2' = 0 \\ E_2'(Y-\mu)M_1' = 0 \\ E_2'(Y-\mu)M_2' = 0 \end{array}\right\} \text{a. s.} \tag{2}$$

where A^{-} is a symmetric generalized inverse of A, λ_i and δ_j are the nonzero eigenvalues of Σ and Θ, respectively, and $E_1 \in V_{k,N}$, $E_2 \in V_{N-k,N}$, $M_1' \in V_{r,m}$ and $M_2' \in V_{m-r,m}$. This is called Singular Elliptically Contoured Distribution and is denoted as;

$$Y \sim \mathcal{E}_{N \times m}^{k,r}(\mu, \Sigma, \Theta, h)$$

where $\Sigma : m \times m$, $r(\Sigma) = r \leq m$ and $\Theta : N \times N$, $r(\Theta) = k \leq N$, see Díaz-García and Gutiérrez-Jáimez (2001).

Alternatively, this density may be expressed as

$$\frac{1}{\left(\displaystyle\prod_{i=1}^{r} \lambda_i^{k/2}\right)\left(\displaystyle\prod_{j=1}^{k} \delta_j^{r/2}\right)} h\left(\operatorname{tr} \Sigma^{-}(Y-\mu)'\Theta^{-}(Y-\mu)\right) \nu(dY), \tag{3}$$

where $\nu()$ is the Hausdorff measure, which coincides with that of Lebesgue when it is defined on the subspace $\mathcal{M}$ given by the hyperplane (2), see Díaz-García and Gutiérrez-Jáimez (2001), Cramér (1945, p. 297) and Billingsley (1979, p. 209).

Now let $Y \sim \mathcal{E}^{k,r}_{N\times m}(\mu, \Sigma, \Theta, h)$, and define the generalized Wishart $(N \geq m)$ or Pseudo-Wishart $(N < m)$ matrix as $S = Y'\Theta^{-}Y$. Let $Q \in \mathcal{L}_{N,k}$, such that $\Theta = Q'Q$, and define $X = (Q^{-})'Y$. Then

$$X \sim \mathcal{E}^{k,r}_{k\times m}(\mu_x, \Sigma, I_k, h)$$

with $\mu_x = (Q^{-})'\mu$ in such a way that

$$S = Y'\Theta^{-}Y = ((Q^{-})'Y)'(Q^{-})'Y = X'X.$$

In this section, assuming that $X \sim \mathcal{E}^{k,r}_{k\times m}(\mu_x, \Sigma, I_k, h)$ and that h is expanded in power series, the densities of matrices T, R, (N, Ω), (D, W_1) and D associated with the QR, modified QR, SV, and Polar decompositions of matrix X are found.

Theorem 1 *(1) For $k \geq m$ or $k < m$, with $q = \min(k, r)$, the density of T is given by*

$$\frac{2^q \pi^{qk/2} \displaystyle\prod_{i=1}^{q} t_{ii}^{k-i}}{\Gamma_q\left[\frac{1}{2}k\right]\left(\displaystyle\prod_{i=1}^{r}\lambda_i^{k/2}\right)} \sum_{t=0}^{\infty}\sum_{\kappa} \frac{h^{(2t)}(\operatorname{tr}(\Sigma^{-}T'T + \Omega))}{t!} \frac{C_\kappa(\Omega\Sigma^{-}T'T)}{\left(\frac{1}{2}k\right)_\kappa}$$

$$(T - T_{\mu_x})M_2' = 0 \qquad a.s.$$

where $\mu_x = H_{1\mu_x}T_{\mu_x}$ is the QR decomposition of μ_x.

(2) Assuming that $k \geq m$, with $q = \min(k, r)$, the density of R is,

$$\frac{2^q \pi^{qk/2} |D|^{k-q} \displaystyle\prod_{i<j}^{q} (D_{ii} + D_{jj})}{\Gamma_q\left[\frac{1}{2}k\right]\left(\displaystyle\prod_{i=1}^{r}\lambda_i^{k/2}\right)} \sum_{t=0}^{\infty}\sum_{\kappa} \frac{h^{(2t)}(\operatorname{tr}(\Sigma^{-}R^2 + \Omega))}{t!} \frac{C_\kappa(\Omega\Sigma^{-}R^2)}{\left(\frac{1}{2}k\right)_\kappa}$$

$$(R - R_{\mu_x})M_2' = 0 \qquad a.s.$$

where $\mu_x = P_{1\mu_x}R_{\mu_x}$ is the polar decomposition of μ_x.

(3) For $k \geq m$ or $k < m$, with $q = \min(k, r)$, the density of $(N, \mathcal{G})$ is given by

$$\frac{2^q \pi^{qk/2} \displaystyle\prod_{i=1}^{q} n_{ii}^{k+m-2i}}{\Gamma_q\left[\frac{1}{2}k\right]\left(\displaystyle\prod_{i=1}^{r}\lambda_i^{k/2}\right)} \sum_{t=0}^{\infty}\sum_{\kappa} \frac{h^{(2t)}(\operatorname{tr}(\Sigma^{-}\mathcal{G}'N^2\mathcal{G} + \Omega))}{t!} \frac{C_\kappa(\Omega\Sigma^{-}\mathcal{G}'N^2\mathcal{G})}{\left(\frac{1}{2}k\right)_\kappa}$$

$$(N\mathcal{G} - N\mathcal{G}_{\mu_x})M_2' = 0 \qquad a.s.$$

where $\mu_x = H_{1\mu_x} N_{\mu_x} \mathcal{G}_{\mu_x}$ *is the modified QR decomposition of* μ_x.

(4) The joint density of D *and* W_1 *is*

$$\frac{2^{-q}\pi^{q(k-m)/2}\Gamma_q\left[\frac{1}{2}m\right]|D|^{k+m-2q}\prod_{i<j}^{q}(D_{ii}^2 - D_{jj}^2)}{\Gamma_q\left[\frac{1}{2}k\right]\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)}$$

$$\sum_{t=0}^{\infty}\sum_{\kappa}\frac{h^{(2t)}(\operatorname{tr}(\Sigma^- W_1 D^2 W_1' + \Omega))}{t!}\frac{C_\kappa(\Omega\Sigma^- W_1 D^2 W_1')}{\left(\frac{1}{2}k\right)_\kappa}$$

$$(DW_1' - D_{\mu_x}W_{1\mu_x}')M_2' = 0 \qquad a.s.$$

where $\mu_x = V_{1\mu_x} D_{\mu_x} W_{1\mu_x}'$ *is the SVD of* μ_x, $(dD) = \bigwedge_{i=1}^{q} dD_{ii}$ *and* $(dW_1) = \dfrac{(W_1' dW_1)}{\operatorname{Vol}(\mathcal{V}_{q,m})}$.

(5) The density of D *is given by*

$$\frac{2^q\pi^{q(k+m)/2}\prod_{i=1}^{q}D_{ii}^{k+m-2q}\prod_{i<j}^{q}(D_{ii}^2 - D_{jj}^2)}{\Gamma_q\left[\frac{1}{2}k\right]\Gamma_q\left[\frac{1}{2}m\right]\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)}$$

$$\sum_{\theta,\kappa}^{\infty}\sum_{\phi\in\theta,\kappa}\frac{h^{(2t+l)}(\operatorname{tr}\Omega))}{t!l!}\frac{\Delta_\phi^{\theta,\kappa}C_\phi(D^2)C_\phi^{\theta,\kappa}(\Sigma^-, \Omega\Sigma^-)}{\left(\frac{1}{2}k\right)_\kappa C_\phi(I_m)}$$

$$(D - D_{\mu_x})M_2' = 0 \qquad a.s.$$

where $\mu_x = H_{1\mu_x} D_{\mu_x} W_{1\mu_x}$ *is the SVD of* μ_x.

with $\Omega = \Sigma^-\mu'\Theta^-\mu$, $\left(\frac{1}{2}k\right)_\kappa$ *is the generalized hypergeometric coefficient and* $C_\kappa(.)$ *is the zonal polynomial, see James (1964), Farrell (1985) and Muirhead (1982)). The multiple addition operators* $\Delta_\phi^{\theta,\kappa}$ *and* $C_\phi^{\theta,\kappa}$ *are given in Davis (1980), see also Chikuse (1980).*

Proof.

(1) Considering the non-degenerated part of the density of X we have

$$\frac{1}{\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)}h(\operatorname{tr}\Sigma^-(X-\mu_x)'(X-\mu_x))(dX)$$

or

$$\frac{1}{\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)}h(\operatorname{tr}\Sigma^-(X'X + \mu_x'\mu_x) - 2\operatorname{tr}\Sigma^- X'\mu_x)(dX).$$

Factoring, $X = H_1 T$, from Theorem 11 (see Díaz-García and González-Farias, 2002) we have that the joint density (non-degenerated part) of H_1 and T is given by

$$\frac{\left(\prod_{i=1}^{q} t_{ii}^{k-i}\right)}{\left(\prod_{i=1}^{r} \lambda_i^{k/2}\right)} h(\operatorname{tr}(\Sigma^- T'T + \Omega) - 2\operatorname{tr}\Sigma^- T' H_1' \mu_x)(H_1' dH_1)(dT)$$

where $\Omega = \Sigma^- \mu_x' \mu_x = \Sigma^- \mu' \Theta^- \mu$. Assuming that $h(\cdot)$ can be expanded in power series, (see Fan, 1990a), i.e.,

$$h(v) = \sum_{t=0}^{\infty} a_t v^t$$

and expanding the binomial, we have

$$\frac{\left(\prod_{i=1}^{q} t_{ii}^{k-i}\right)}{\left(\prod_{i=1}^{r} \lambda_i^{k/2}\right)} \sum_{t=0}^{\infty} a_t \sum_{\eta=0}^{t} \binom{t}{\eta} (\operatorname{tr}(\Sigma^- T'T + \Omega))^{t-\eta}$$
$$(\operatorname{tr}(-2\mu_x \Sigma^- T' H_1'))^{\eta} (H_1' dH_1)(dT).$$

Integrating on $H_1 \in \mathcal{V}_{q,k}$, noting that this integral equals zero when η is odd (see James, 1964, eqs.(34)-(36)), the marginal (non-degenerated) density of T may be expressed as

$$\frac{\left(\prod_{i=1}^{q} t_{ii}^{k-i}\right)}{\left(\prod_{i=1}^{r} \lambda_i^{k/2}\right)} \sum_{t=0}^{\infty} a_t \sum_{\eta=0}^{[\frac{t}{2}]} \binom{t}{\eta} (\operatorname{tr}(\Sigma^- T'T + \Omega))^{t-\eta}$$
$$\int_{H_1 \in \mathcal{V}_{q,k}} (\operatorname{tr}(-2\mu_x \Sigma^- T' H_1')^{2\eta} (H_1' dH_1)(dT).$$

Integrating, see Muirhead (1982, Lemma 9.5.3, p. 397) and James (1964, eq. 22), we have

$$\begin{aligned}
\int_{H_1 \in \mathcal{V}_{q,k}} & (\operatorname{tr}(-2\mu_x \Sigma^- T' H_1'))^{2\eta} (H_1' dH_1) \\
&= \frac{2^q \pi^{qk/2}}{\Gamma_q[\frac{1}{2}k]} \sum_{\kappa} \frac{\left(\frac{1}{2}\right)_\eta C_\kappa(4\mu_x \Sigma^- T'T \Sigma^- \mu_x')}{\left(\frac{1}{2}k\right)_\kappa} \\
&= \frac{2^q \pi^{qk/2}}{\Gamma_q[\frac{1}{2}k]} \sum_{\kappa} \frac{\left(\frac{1}{2}\right)_\eta 4^\eta}{\left(\frac{1}{2}k\right)_\kappa} C_\kappa(\Omega \Sigma^- T'T).
\end{aligned}$$

Observing that $4^\eta \left(\frac{1}{2}\right)_\eta = \dfrac{(2\eta)!}{\eta!} = 2^\eta(2\eta-1)!!$, the non-degenerated part is obtained (see Teng, Fang, and Deng, 1989).
The degenerated part is still considering the QR decomposition of $\mu_x = H_{1_x} T_{\mu_x}$.

(2) Proof is similar to the one given in Theorem 1(1), considering in this case the Jacobian given in Theorem 18 (1) in Díaz-García and González-Farias (2002).

(3) For proof of Theorem 1(3) and Theorem 1(4) use the Theorem 1 in Díaz-García and González-Farias (2002) and see Díaz-García and Gutiérrez-Jáimez (2001).

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Remark 2 *Observe that, if function $h(\cdot)$ is not easily expandable in power series, an integral expression for the densities of T, R, (D, W_1), $(N, \mathcal{G})$ and D may be found in an analogous form to the one given by Fan (1990b), for the generalized Wishart matrix case.*

From the Wishart matrix (or generalized Wishart matrix), the $S = T'T$ factorization is known in the literature as Bartlett decomposition (central or non-central). The density of T has been studied by different authors for the central non-singular case ($q = m \leq N$), as a function of both the density of S and the density of X ($S = X'X$), see Srivastava and Khatri (1979, p. 74), Muirhead (1982, p. 99), Eaton (1983, p. 314), Fang and Zhang (1990, p. 119), amongs others. In the normal, non-central, non-singular case, Goodal and Mardia (1992) and Goodall and Mardia (1993) study the density of T when $q = \min(k, m)$, with $k \geq m$ and $k < m$, in the shape theory setting. Later Dahel and Giri (1994), also under normal theory, find the density of T for the case when $r(\mu_x) = 1$.

Also, Olkin and Rubin (1964) study the density of R under a non-singular central normal distribution, expressing the eigenvalues of R as a function of the elements of S, for the case when $q = m = 2$. Díaz-García et al. (1997), under normal theory, find the non-central density of D^2, when $q = \min(k, m)$. This result is extended to the case of a singular non- central elliptical model by Díaz-García and Gutiérrez-Jáimez (2001). Among other results, Díaz-García et al. (2000) show that the density of $D/||D||$ in the central case, is invariant under all the elliptically contoured distributions.

Next, the densities of T, R, $(N, \mathcal{G})$, (D, W_1) and D are presented for the central case, $\mu_x = 0$.

Corollary 3 *(1) The central density of T is*

$$\frac{2^q\pi^{kq/2}\displaystyle\prod_{i=1}^{q}t_{ii}^{k-i}}{\Gamma_q[\frac{1}{2}k]\left(\displaystyle\prod_{i=1}^{r}\lambda_i^{k/2}\right)}h(\operatorname{tr}\Sigma^{-}T'T)$$

$$TM_2' = 0 \quad a.s.$$

(2) The central density of R is

$$\frac{2^q\pi^{kq/2}|D|^{k-q}\displaystyle\prod_{i<j}^{q}(D_{ii}+D_{jj})}{\Gamma_q[\frac{1}{2}k]\left(\displaystyle\prod_{i=1}^{r}\lambda_i^{k/2}\right)}h(\operatorname{tr}\Sigma^{-}R^2)$$

$$RM_2' = 0 \quad a.s.$$

(3) The central density of $(N,\mathcal{G})$ is

$$\frac{2^q\pi^{kq/2}\displaystyle\prod_{i=1}^{q}t_{ii}^{k-i}}{\Gamma_q[\frac{1}{2}k]\left(\displaystyle\prod_{i=1}^{r}\lambda_i^{k/2}\right)}h(\operatorname{tr}\Sigma^{-}\mathcal{G}'N^2\mathcal{G})$$

$$N\mathcal{G}M_2' = 0 \quad a.s.$$

(4) The central density of (D,W_1) is

$$\frac{2^{-q}\pi^{q(k-m)/2}\Gamma_q[\frac{1}{2}m]|D|^{k+m-2q}\displaystyle\prod_{i<j}^{q}(D_{ii}^2-D_{jj}^2)}{\Gamma_q[\frac{1}{2}k]\left(\displaystyle\prod_{i=1}^{r}\lambda_i^{k/2}\right)}h(\operatorname{tr}\Sigma^{-}W_1D^2W_1')$$

$$DW_1'M_2' = 0 \quad a.s.$$

(5) The central density of D is

$$\frac{2^q\pi^{q(k+m)/2}\displaystyle\prod_{i=1}^{q}D_{ii}^{k+m-2q}\prod_{i<j}^{q}(D_{ii}^2-D_{jj}^2)}{\Gamma_q[\frac{1}{2}k]\Gamma_q[\frac{1}{2}m]\left(\displaystyle\prod_{i=1}^{r}\lambda_i^{k/2}\right)}\sum_{t=0}^{\infty}\sum_{\kappa}\frac{h^t(0)C_\kappa(\Sigma^{-})C_\kappa(D^2)}{t!C_\kappa(I_m)}$$

$$DM_2' = 0 \quad a.s.$$

Observe that when $q = k \leq m$, in the QR, QDR and SVD decomposition, and $q = k = m$ in the Polar decomposition, two cases may be distinguished by their respective forms of $H_1, P_1, V_1 \in \mathcal{O}(k)$. If we denote all those matrices, H_1, P_1, V_1 just as F, we get:

(1) F includes reflection, $F \in \mathcal{O}(k)$, $|F| = \pm 1$. In addition, for matrices T and D, $t_{kk} \geq 0$ and $D_{kk} \geq 0$.
(2) H_1 excludes reflection, $F \in \mathcal{SO}(k)$, $|F| = 1$, t_{kk} is not restricted, and in the case of SVD if $q = k = m$ $\operatorname{sign}(D_{kk}) = \operatorname{sign}(|X|)$. Matrices T, (D, W_1) and R are denoted as T^{NR}, $(D, W_1)^{NR}$ and R^{NR}, respectively, Goodall (1991, Section 4), Goodall and Mardia (1993) and Le and Kendall (1993, Section 4).

In this way, those densities with reflection of T, R, $(N, \mathcal{G})$, (D, W_1) and D for the central and non-central case, are given in 1 and Corollary 4, respectively. For the case were reflection is excluded, we have the following result.

Corollary 4 *When $q = k \leq m$ and $r(\mu) < k$, the densities of T^{NR}, R^{NR}, $(N, \mathcal{G})^{NR}$ and $(D, W_1)^{NR}$ are the same as those given in Theorem 1* (1), (2), (3) *and* (4), *respectively, divided by 2. In particular, for the density of T^{NR} ($(N, \mathcal{G})^{NR}$), $t_{ii} \geq 0$ ($n_{ii} \geq 0$), for $i = 1, \dots, (k-1)$ and t_{kk} (n_{kk}) non-restricted, similarly for the density of (D, W_1), if $q = k = m$, $\operatorname{sign}(d_{kk}) = \operatorname{sign}(|X|)$. When $k > m$, t_{kk} is not present, see Srivastava and Khatri (1979), Goodall and Mardia (1993) and Le and Kendall (1993). For the case of the distribution of D, the densities, including and excluding reflection, are equal, see Goodall and Mardia (1993, Section 7).*

Proof. Expanding the exponential in Goodall and Mardia (1993, eq. 2.10) in power series, and integrating term by term, it is established for $r(Z) < k$ that

$$\int_{\mathcal{SO}(k)} (\operatorname{tr} ZH)^{2t} (H'dH) = \frac{1}{2} \int_{\mathcal{O}(k)} (\operatorname{tr} ZH)^{2t} (H'dH),$$

from which the result is obtained. ∎

Remark 5 (Karhunen-Lòeve Decomposition) *Given a matrix $X \in \mathcal{L}^{+}_{m,N}(q)$ the Karhunen-Lòeve expansion or decomposition (also known in the literature as Proper Orthogonal Decomposition, Principal Component Analysis, the Singular Value Decomposition, Analysis by Empirical Eigenfunction and Hotelling Transform, among other names), it could be written as,*

$$X = AV'$$

were $A \in \mathcal{L}^{+}_{r,N}(r)$, $r < q$ and $V \in \mathcal{V}_{r,m}$, are matrices such that $E(A) = 0$, $A'U_1A = \Lambda$ and $R = E(X'U_1X)$ with U_1 a matrix $m \times m$ which represents a prior probability matrix. Moreover, if u_{ij} denote the ij-t́h element of U_1, it should be that $0 \leq u_{ii} < 1$,$u_{ij} = 0$, $i \neq j$, Kotz and Johnson (1982, p.357).

Observe that $X \sim \mathcal{E}^{k,r}_{m \times n}(0, \Sigma, \Theta, h)$, the distribution of A could be found from *the density given in Corollary 3(3). Note that such density includes a double-sided Karhunen-Lòeve Decomposition introduced by Fernando and Nicolson cited in Kotz and Johnson (1982, p.357), in which the correlation among rows and columns is considered.*

Now two particular cases of elliptically contoured distributions are considered, the matrix variate normal distribution, and the class of matrix variate symmetric Pearson type VII distributions, Gupta and Varga (1993, pp. 75-76), for which the density of R is found. The densities of T, (D, W_1) and D are obtained in a similar form.

Corollary 6 *Let $X \sim \mathcal{E}^{k,r}_{k\times m}(\mu_x, \Sigma, I_k, h)$, with h expanding in series of powers. Then,*

(1) if X has a matrix variate normal distribution, the density of R is

$$\frac{2^{(2q-kr)/2}\pi^{k(q-r)/2}|D|^{k-q}\prod_{i<j}^{q}(D_{ii}+D_{jj})}{\Gamma_q[\frac{1}{2}k]\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)}\operatorname{etr}(-\tfrac{1}{2}(\Sigma^- R^2+\Omega))_0F_1(\tfrac{1}{2}k;\tfrac{1}{4}\Omega\Sigma^- R^2)$$

$$(R-R_{\mu_x})M_2' = 0 \quad a.s.$$

(2) if X has a matrix variate symmetric Pearson type VII distribution, the density of R is

$$\frac{2^{q}\pi^{k(q-r)/2}\Gamma[b]|D|^{k-q}\prod_{i<j}^{q}(D_{ii}+D_{jj})}{\Gamma_q[\frac{1}{2}k]a^{kr/2}\Gamma[\frac{1}{2}(2b-kr)]\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)}\sum_{t=0}^{\infty}\sum_{\kappa}\frac{(b)_{2t}\left(1+\dfrac{\operatorname{tr}(\Sigma^- R^2+\Omega)}{a}\right)^{-(b+2t)}}{t!}$$
$$\frac{C_\kappa\left(\dfrac{1}{a^2}\Omega\Sigma^- R^2\right)}{(\frac{1}{2}k)_\kappa}$$

$$(R-R_{\mu_x})M_2' = 0 \quad a.s.$$

where ${}_0F_1()$ is a hypergeometric function of matrix argument, James (1964) and Muirhead (1982, p. 258).

Proof. The proof follows from Theorem 1(2), observing in addition that:

(1) For the normal case

$$h(v) = \frac{1}{(2\pi)^{kr/2}}\exp(-\tfrac{1}{2}v),$$

therefore

$$h^{(2t)}(v) = \frac{1}{2^{2t+kr/2}\pi^{kr/2}}\exp(-\tfrac{1}{2}v).$$

(2) For the Pearson type VII case

$$h(v) = \frac{\Gamma[b]}{(\pi\ a)^{kr/2}\Gamma[b-kr/2]}(1+v/a)^{-b},$$

then

$$h^{(2t)}(v) = \frac{\Gamma[b]}{(\pi\ a)^{kr/2}\Gamma[b-kr/2]}\frac{(b)_{2t}}{a^{2t}}(1+v/a)^{-(b+2t)}.$$

From which the results are obtained. ■

3 About Wishart and Pseudo-Wishart Distributions

Let $Y \sim \mathcal{E}^{k,r}_{N\times m}(\mu,\Sigma,\Theta,h)$, we want to find the distribution of the matix Wishart or the generalized Pseudo-Wishart $S = Y'\Theta^- Y$, were $\Theta^-$is a generalized inverse of Θ. The density of S could be found through those of T, R, (N,Ω) and (D,W_1), with the help of theorems 13(1), 16, 15(1) and 3(1) in Díaz-García and González-Farias (2002), respectively. Other approach would be by considering the density of X, $S = Y'\Theta^- Y = X'X$, were $X = QY$ is such that $Q \in \mathcal{L}^+_{N,k}(k)$ with $\Theta = Q'Q$, and using theorems 13(2), 18(2), 15(2) and 3(2) in Díaz-García and González-Farias (2002), respectively. However, note that depending on the factorization given to Y (X), there exist four measures (dS) and therefore, four expression for the density of S. The genral density form for any of those factorization, will be,

$$\frac{\pi^{qk/2}|\Psi|^{(k-m-1)/2}}{\Gamma_q(\frac{1}{2}k)\left(\prod_{i=1}^{r}\lambda^{k/2}\right)}\sum_{t=0}^{\infty}\sum_{\kappa}\frac{h^{(2t)}(\operatorname{tr}(\Sigma^- S+\Omega))}{t!}\frac{C_\kappa(\Omega\Sigma^- S)}{\left(\frac{1}{2}k\right)_\kappa}\ (dS) \quad (4)$$

$$P_2(S-\mu'\Theta^-\mu)P_2' = 0. \quad (5)$$

where A^- is a symmetric generalized inverse of A, $\Omega = \Sigma^-\mu'\Theta^-\mu$, $C_\kappa(B)$ are the zonal polynomials of B corresponding to the partition $\kappa = (t_1,\dots,t_l)$ of t, with $\sum_1^l t_i = t$, $(\frac{1}{2}k)_\kappa$ being the generalized hypergeometric coefficients, see James (1964), and $h^{(j)}(\cdot)$ is the j-th derivative of h with respect to $v = \operatorname{tr}\Sigma^- S$.

The matrix Ψ and the volumen (dS), are defined according to each factorization and excluding the Polar decomposition, the density S can be found for all the cases, i.e. when $N \geq m$ (Wishart distribution), $N < m$ (Pseudo-Wishart distribution) and $q = \min(k,r)$ (singular and non singular cases), with $k \geq r$ or $k < r$. In particular, we have:

(1) *QR decomposition.* In this case the matrix Ψ is defined by S_{11}, where if $T = \left(\underset{q\times q}{T_1} \vdots \underset{q\times m-q}{T_2} \right)$ then

$$S = \begin{pmatrix} \underset{q\times q}{S_{11}} & \underset{q\times m-q}{S_{12}} \\ \underset{m-q\times q}{S_{21}} & \underset{m-q\times m-q}{S_{22}} \end{pmatrix} = \begin{pmatrix} T_1' \\ T_2' \end{pmatrix} \left(T_1 \vdots T_2 \right) = \begin{pmatrix} T_1'T_1 & T_1'T_2 \\ T_2'T_1 & T_2'T_2 \end{pmatrix},$$

and, $|S_{11}| = |T_1'T_1| = |T_1|^2 = \prod_{i=1}^{q} t_{ii}^2$. In this way, the volumen with respect to which the density of $S = T'T$ exist, is given by,

$$(dS) = 2^q \prod_{i=1}^{q} t_{ii}^{m-i-1} (dT)$$

(2) *Polar decomposition.* Under this decomposition, the density of $S = R^2$ can be established only when $N \geq m$ (Wishart distribution) and $q = k \geq r$ (singular case). Here, the matrix Ψ is defined by $L = D^2$, with $R = Q_1'DQ_1$ and the volumen (dS), is defined as,

$$(dS) = 2^q |D|^{m-q+1} \prod_{i<j}^{q} (D_{ii} + D_{jj})(dR) = |D|^{m-q} \prod_{i\leq j}^{q} (D_{ii} + D_{jj})(dR)$$

(3) *QDR decomposition.* Ψ would be defined as O, and the corresponding volumen for which the density of $S = \Omega' O \Omega$ will exist, is given by,

$$(dS) = \prod_{i=1}^{q} o_{ii}^{m-i} (d\Omega)(dO)$$

(4) *Singular value decomposition.* Here, $S = W_1 L W_1'$, $D^2 = L = \Psi$, and the volumen (dS) is defined as,

$$(dS) = 2^{-q} |L|^{m-q} \prod_{i<j}^{q} (L_{ii} - L_{jj})(dL)(W_1' dW_1)$$

This case has been worked in detail by Díaz-García and Gutiérrez-Jáimez (2001), for the elliptical case and in Díaz-García et al. (1997) for the Normal distribution case.

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