UNIVERSIDAD CARLOS III DE MADRID

TESIS DOCTORAL

SYMMETRIES AND CONSTRAINTS IN CLASSICAL AND QUANTUM MECHANICS: LIE–JORDAN BANACH ALGEBRAS AND THEIR APPLICATIONS

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DEPARTAMENTO DE MATEMÁTICAS

Leganés, Octubre 2013

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Ph.D. Thesis

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Per Ervin e Lea

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ACKNOWLEDGMENTS

For decades, psychological research has identified two main factors which tie happines: gratefulness and appreciation. People who are happier tend to be more grateful and appreciative for what they have. But very often we are reluctant to externalize our gratitude. I think this is the right place where to be explictly thankful and grateful to all the people who contributed and supported me during the time of this thesis work.

First of all I would like to express my gratitude to my advisor Alberto. I sincerely admire him. He is like an artist of mathematics: like a painter who manages several techniques is able to creat new paintings, Alberto with his deep knowledge in matemathics and physics is able to create new knowledge out of them. Moreover he is an ”heartful” person of great humanity. Without his help this thesis would have not existed and I could have not finished it on time. I am really grateful to you!

Then I would like to thank Beppe for letting me work with him. It has been a great honor and pleasure for me to being part of his great family of ”mathematically-oriented” theoretical physicists. He is endowed with a super-fast mind, always ready to start discussions. As the ”father” of this family he convinced me not to give up.
Thanks to Fernando Falceto for his pleasant hospitality in Zaragoza where he gave me great lessons: I learnt a lot from his concrete way of think about abstract mathematics. Having many concrete examples in mind really gives a deeper understanding of abstract reasoning.

Thanks to Fernando Lledó: with his clearness and suggestions I definitely changed my way of preparing presentations. I will always use colors in my future slides!
Thanks to Franco for his kindness, for his gentle smiling and elegant manners.
Thanks to Saverio. I really appreciate his ability of being a great theoretical physicist with mathematical insights. His care for the details and humility are of great value for me.

A big thank to my friend Gabriele for being always present to help me with my questions and Gennaro for the many useful discussion over the years on many topics of mathematics, physics and life. Thanks to the friends of the UC3M for the time passed together: Alberto, Alfredo, Diego, Edi, Javi^2, Juanma, Natalia, Paulo. In particular I would like to thank Juanma for his careful reading of the manuscript and Javi for the help in the Spanish translation. Thanks also to Alessandro and Gennaro for their friendship in Zaragoza.

Thanks to my friends in Nocera for the good time spent together while I was at home: Antonio, Conci, Dario, Nicola, Peppe and Valentino; and to my friends in Napoli: Lorenzo, Luca, Marco and Mariano.

I would also like to express my gratitude to my family, Ale, Bove and Papy, for having always supported me during my studies.
My deepest gratitude goes to Monika for all the good time spent together in Madrid, for overcoming many difficulties and for having taught me to trust in myself. And thank you for your strenght and for having fighted to bring two little masterpieces to light.

Finally a big thank to Ervin and Lea for giving me the strenght to accomplish this thesis in short times. I hope you will appreciate this work one day and be a little bit proud of your father ☺.

RESUMEN

El objetivo principal de esta disertación es el estudio de la teoría de las álgebras de Lie–Jordan Banach y su papel en el marco de la mecánica clásica y cuántica y sus aplicaciones en diferentes ámbitos de las matemáticas.
La investigación algebraica de los fundamentos de la física surgió de la búsqueda de una formulación “excepcional” para la mecánica cuántica, que finalmente culminó en la formulación de las álgebras de Jordan y las posteriores C^*–álgebras.
Así el programa para descubrir un nuevo escenario algebraico para la mecánica cuántica propuesto por Jordan, identifica el producto simétrico

$$a \circ b \equiv \frac{1}{2}(ab + ba), \tag{0.0.1}$$

como la operación observable fundamental. Las propiedades fundamentales de este producto, además de su obvia conmutatividad, es la noción de asociatividad generalizada dada por:

$$a^2 \circ (b \circ a) = (a^2 \circ b) \circ a. \tag{0.0.2}$$

Un álgebra conmutativa real que satisface esta propiedad se llama un *álgebra de Jordan*. El propio Jordan, von Neumann y Wigner mostraron que, salvo para un álgebra de dimension 27, todas las álgebras de Jordan se derivan de un producto asociativo. Esto condujo a Segal al estudio de los fundamentos de la mecánica cuántica en términos de C^*–álgebras, que han tenido tenido una profunda influencia tanto en el desarrollo como en las aplicaciones de la física cuántica y de la teoría cuántica de campo.

Uno de los resultados de esta tesis es una nueva demostración del teorema que caracteriza a las álgebras de Jordan (Banach) que están en correspondencia con C^*–álgebras. La solución viene dada por la introducción de una estructura de Lie $[\cdot,\cdot]$ en el álgebra, que es compatible con la de de Jordan, en el sentido de que se verifica la identidad Leibniz:

$$[a, b \circ c] = [a, b] \circ c + b \circ [a, c] \,, \tag{0.0.3}$$

y el asociador del producto de Jordan está relacionado con la paréntesis de Lie a través de la ecuación

$$(a \circ b) \circ c - a \circ (b \circ c) = \kappa \, [b, [c, a]] \,, \tag{0.0.4}$$

donde κ es un número real positivo. Esto nos conduce al estudio de las álgebras de Lie–Jordan Banach.

Este formalismo resulta ser adecuado para describir al mismo tiempo la mecánica clásica y cuántica. Ademas podremos analizar los conceptos de ligaduras y simetrías de sistemas clásicos y cuánticos desde esta perspectiva. El enfoque algebraico a la mecánica cuántica y la teoría cuántica de campos convierte el problema de la reducción del sistema cuántico por simetrás o ligaduras en el problema de reducción de la C^*–álgebra del sistema. Este programa ha sido desarrollado con éxito para algunas teorías de gauge, en cuyo caso el correspondiente mecanismo de reducción se denomina T–reduction. En esta tesis hemos abordado este problema fundamental de la reducción de los sistemas clásicos y cuánticos usando la teoría de las algebras de Lie–Jordan Banach. Se obtiene en este contexto un procedimiento que consiste en la identificacion de un ideal del álgebra definida por las ligaduras del sistema, y se identifica el cociente de su normalizador de Lie por dicho ideal como el álgebra de Lie–Jordan Banach que determina el sistema físico correcto. Esta reducción se muestra además que es equivalente a la T–reducción abstracta.

Un procedimiento alternativo de tratar las simetrías y ligaduras de un sistema es la llamada simetría BRST. Se muestra a continuación en esta tesis como extender las ideas de la simetría BRST al marco de las álgebras de Lie–Jordan Banach.

En analogía con los sistemas clásicos, definimos una aplicación momento como un homomorfismo fuertemente continuo, que envía cada elemento del álgebra de Lie del grupo de simetría en derivaciones "skew–order" del álgebra Lie–Jordan Banach. Con el fin de obtener el adecuado sistema reducido cuántico en este marco, hemos de introducir además la noción de álgebras de Lie–Jordan supersimétricas, que son esencialmente álgebras de Lie–Jordan en un espacio vectorial graduado. Esta extensión de la noción de Lie–Jordan Banach álgebra permite tratar adecuadamente todos los grados de libertad de la teoría que ahora incluyen los correspondientes campos "(anti-) ghost". Finalmente se construye un complejo diferencial $(K^\bullet, \hat{D})$ con operador diferencial $\hat{D}$ determinado por la acción del grupo de simetría en la correspondiente álgebra de Lie–Jordan Banach. El álgebra reducida se identifica así con el grupo de cohomología cero del complejo BRST $(K^\bullet, \hat{D})$.

Una posible aplicación importante de la teoría de Lie–Jordan Banach álgebras la constituyen los sistemas dinámicos. Los sistemas dinámicos constituyeron originalmente una formulación matemática de la dinámica, de cómo el sistema físico evoluciona en el tiempo. Mediante la exploración de las propiedades algebraicas de la dinámica se ha reconocido el concepto de sistema dinámico como un marco común para el estudio de la evolución temporal y grupos de simetría en la mecánica clásica y cuántica. La evolución temporal de un sistema dinámico clásico está dado por una acción continua de $\mathbb{R}$ (visto como un grupo) en el espacio de estados. Si la acción está dada por un grupo general G, llegamos a una noción que nos permite estudiar por ejemplo la evolución temporal y las simetrías de forma simultánea. En mecánica cuántica, la acción del grupo viene dada por una representación unitaria en el espacio de Hilbert $\mathcal{H}$. Simetrías de un sistema mecánico cuántico deben ser biyecciones en el conjunto de los estados $\mathcal{S}(\mathcal{H})$. Estas simetrías deben preservar una estructura algebraica, identificada por Wigner con las transiciones de probabilidades. El célebre teorema de Wigner establece que una biyección que preserva las probabilidades de transición es necesariamente un operador unitario o antiunitario en $\mathcal{H}$. Esto significa que una representación π de una C^*– álgebra que describa un sistemas cuántico debe satisfacer una propiedad de covariancia del tipo

$$\pi(\alpha_t(a)) = U_t^* \pi(a) U_t, \tag{0.0.5}$$

es decir que (π, U) es una representación covariante del sistema dinámico $(\mathcal{A}, \mathbb{R}, \alpha)$. El estudio sistemático de las representaciones covariantes lo inicia Mackey en su estudio de la teoría de las representaciones inducidas de los grupos localmente compactos. Takesake extendió el formalismo de Mackey a las álgebras no conmutativas donde se introduce la noción de *crossed product algebra* $\mathcal{A} \ltimes_\alpha G$ y prueba que hay una relación uno a uno entre las representaciones no degeneradas de $\mathcal{A} \ltimes_\alpha G$ y las representaciones covariantes de $(\mathcal{A}, G, \alpha)$. Posteriormente las álgebras *crossed products* han aportado algunos de los ejemplos más importantes de C^*–álgebras y fueron fundamentales para el desarrollo de la geometría diferencial no conmutativa. Finalmente hemos desarrollado alguno de los fundamentos necesarios para la extensión de esta teoría al contexto de las álgebras de Lie–Jordan Banach y la correspondiente teoría de sistemas din 'amicos en álgebras de Lie–Jordan Banach.

1

INTRODUCTION

The main objective of this dissertation is the study of the theory of Lie–Jordan Banach algebras, their role in the framework of classical and quantum mechanics, and their applications to different branches of Mathematics.
The algebraic research in Physics arose from the search for an "exceptional" setting for quantum mechanics, which eventually culminated in the formulation of Jordan algebras and the subsequent theory of C^*–algebras.
In the usual interpretation of quantum mechanics (the "Copenhagen interpretation"), the physical observables are represented by self-adjoint operators on a Hilbert space or Hermitian matrices. The basic operations on operators are multiplication by a complex scalar, addition, multiplication of operators, and forming the adjoint operator. But these underlying operations are not "observable": the scalar multiple of a Hermitian matrix is not again Hermitian unless the scalar is real, the product is not Hermitian in general unless the factors happen to commute, and the adjoint is just the identity map on Hermitian matrices.

In 1933 the physicist Pascual Jordan proposed a program to discover a new

algebraic setting for quantum mechanics [Jor33], which would be free from the matrix structure but still enjoy all the same algebraic benefits as the Copenhagen model. He wished to study the intrinsic algebraic properties of Hermitian matrices and recast them in formal algebraic properties in order to see what other possible non-matrix systems satisfied these axioms.
Jordan decided that the fundamental observable operation was the symmetric product

$$a \circ b \equiv \frac{1}{2}(ab + ba), \tag{1.0.1}$$

now called *Jordan product* [HOS84],[McC04]. The key law governing this product, besides its obvious commutativity, is

$$a^2 \circ (b \circ a) = (a^2 \circ b) \circ a. \tag{1.0.2}$$

A real commutative algebra satisfying this property is now called *Jordan algebra*, and is called *special* if it can be realized as the Jordan algebra of an associative algebra as above (1.0.1), otherwise it is called *exceptional*.
Jordan's hopes were that by studying finite-dimensional algebras he could find families of simple exceptional algebras parameterized by natural numbers, so that in the infinite limit this would provide a suitable infinite-dimensional exceptional generalization for quantum mechanics.

In a fundamental paper in 1934 [JvNW34], Jordan, von Neumann and Wigner showed that there are only five basic types of simple finite-dimensional Jordan algebras: four types of Hermitian $n \times n$ matrix algebras $M_{n\times n}(\mathbb{K})$ where $\mathbb{K}$ can be the field of real numbers, complex numbers, quaternions and octonions (but for octonions only $n \leq 3$ is allowed), and the spin factors. The spin factors turn out to be realized as a subspace of Hermitian matrices, whereas the 27–dimensional Jordan algebra of 3×3 matrices with octonion entries $M_{3\times 3}(\mathcal{O})$ is an exceptional Jordan algebra, now called *Albert algebra*.
This result was quite disappointing to physicists since the only exceptional algebra $M_{3\times 3}(\mathcal{O})$ was too tiny to provide a generalization for quantum mechanics and the possible existence of infinite-dimensional exceptional algebras.
In 1979 the mathematician Zelmanov finally showed that even in infinite dimensions there are no simple exceptional Jordan algebras other than the Albert algebra

[Zel79]: it is an unavoidable fact of mathematical nature that simple algebraic systems obeying the basic Jordan identity (1.0.2) must (except in dimension 27) be derived from an associative structure.

However, the Jordan structure allows to recover most of the mathematical basis for the description of quantum systems, like the concept of compatible observables and the joint probability distribution for compatible observables [Emc84]. Eventually, the mathematical language becomes easier if one makes the technical assumption that the algebra $\mathcal{L}$ can be embedded in a complex extension $\mathcal{A} = \mathcal{L} \oplus i\mathcal{L}$ generated by complex linear combinations of elements of $\mathcal{L}$. This led Segal to the foundations of the so called today C^*–algebras [Seg47], which have had a profound influence on both the foundations and applications of quantum physics and quantum field theory [Haa96].
From the above discussion it is clear that special Jordan algebras admit such an extension. One of the main results of this thesis is the novel proof of a theorem which characterizes the Jordan (Banach) algebras that are in correspondence with C^*–algebras. The solution is given by introducing a Lie structure $[\cdot, \cdot]$ on the algebra, which is compatible with the Jordan one [FFIM13c], in the sense that Leibniz identity is verified:

$$[a, b \circ c] = [a, b] \circ c + b \circ [a, c] \,, \tag{1.0.3}$$

and the associator of the Jordan product is related to the Lie bracket by

$$(a \circ b) \circ c - a \circ (b \circ c) = \kappa \, [b, [c, a]] \,, \tag{1.0.4}$$

κ being a positive real number. This leads us to the study of Lie–Jordan Banach algebras [Emc84], [Lan98].
The problem of when a given Jordan–Banach algebra is the real part of a C^*–algebra had already been faced in the past by A. Connes on one side [Con74] and Alfsen and Shultz on the other [AS98]. The characterization obtained by Alfsen and Schultz in terms of the existence of a dynamical correspondence on a Jordan–Banach algebra amounts to state that the relevant structure to discuss the properties of the state space of a quantum system is exactly that of a Lie–Jordan Banach algebra. By making explicit this connection with the Lie structure also

the physical interpretation becomes clear since it reflects the dual role played by the observables: they are measurable quantities but also the generators of motions of the state space.

One of the merit of Lie–Jordan algebras is that they provide a neat algebraic framework common to classical and quantum mechanics. In the Hamiltonian picture of classical mechanics, one is naturally lead to the Lie algebraic structure of the Poisson brackets, which provide the equations of motion of the classical system. This could eventually help us to shed light on the intriguing problem of the classical limit of quantum mechanics and the quantization procedures. There have been in fact various ways in the literature of constructing quantum systems out of classical ones. All of them rely on a certain geometrical structure already present in the classical system, for example the Weyl quantization, the geometrical quantization and the deformation quantization. Following the previous ideas we would arrive to various descriptions of quantum systems, mainly of their algebra of observables, but the geometry that we used originally has faded out. However not all descriptions of quantum systems hide so thoroughly its geometrical structure. Because of a theorem by Kadison [Kad51] it is well-known that the C^*–algebra of observables of a given quantum system is isomorphic to the space of affine continuous functions on the convex space of states of the system. Thus, it would be convenient to identify the geometrical structures on the state space of a quantum system that will make Kadison correspondence more transparent. Such programme has been successfully developed along the last twenty years providing a consistent decription of the fundamental geometrical structures of quantum systems [CL84], [AS99], [CCGM07]. Moreover, a geometrical description of dynamical systems provides a natural setting to describe symmetries, and/or constraints. For instance, if the system carries a symplectic or Poisson structure, several procedures were introduced along the years to cope with them, like Marsden–Weinstein reduction, symplectic reduction, Poisson reduction, reduction of contact structures, etc. However, it was soon realized that the algebraic approach to reduction provided a convenient setting to deal with the reduction of classical systems [GLMV94], [IdLM97].

Whenever constraints are imposed on a quantum system or symmetries are present, both dynamical or gauge, some reduction on the state space must be considered either because not all states are physical and/or because families of states are equivalent. In the standard approach to quantum mechanics, constraints are imposed on the system by selecting subspaces determined by the quantum operators corresponding to the constraints of the theory, called Dirac states, and equivalence of quantum states was dealt with by using the representation theory of the corresponding group of symmetries. However many difficulties emerge when implementing this analysis for arbitrary singular Lagrangian systems or other singularities arise (like singular level sets of momentum maps for instance).

Taking as a departing point the algebraic approach to quantum mechanics and quantum field theory the problem of reduction of the quantum system becomes the problem of reducing the C^*–algebra of the system. Such programme was successfully developed for some gauge theories and was called T–reduction [GH85].

Another one of the main objectives of this thesis is to address the fundamental problem of reducing classical and quantum degenerate systems by using the theory of Lie–Jordan Banach algebras. It is the task of the physicist to extract the relevant physical (sub)system from such a degenerate one. Reduction means the procedure aiming to identify this physical algebra. In Chapter 3 we develop the algebraic framework for reducing systems in classical and quantum mechanics. This is done by identifying the ideal generated by the constraints and quotienting its Lie normalizer with respect to it. We also obtain a generalized reduction in classical mechanics which, apart from technical difficulties, is very promising at the quantum level since it encompasses the case of quantum anomalies. We then prove that our reduction procedure is equivalent to the T–reduction developed by Grundling *et al.*
One of the main outstanding problems of mathematical physics is to construct a C^*–algebra which describes a nonlinear field theory (higher than quadratic). An interesting feature of the reduced system in the classical realm is that it may turn out to be non-linear even if the starting one was linear. Motivated by this consideration, the quantum algebraic reduction theory could be also valuable in understanding how to provide mathematical descriptions of non-linear field theories.

In the subsequent chapters, we apply the reduction of Lie–Jordan Banach algebras to different and important fields of mathematics and physics. In Chapter 4 we extend the theory of the BRST symmetry to Lie-Jordan Banach algebras of observables. The BRST mechanism has been introduced to quantize constrained dynamical systems without having to explicitly solve for the constraints and hence lose covariance and locality. It introduces auxiliary degrees of freedom, called "ghosts" and "antighosts" and replaces the local gauge symmetry by a global supersymmetry generated by a single operator Q called the "supercharge", whose square vanishes, and transforming the symmetry in a (co-) homology theory. Then the role of this operation is to select the "true" physical states of the theory. The extension of the BRST symmetry to Lie–Jordan Banach algebras can be considered as an "intrinsic" quantum approach since we are not using any quantization scheme to do that. This led us to define a *strongly Hamiltonian action* of a symmetry group G (with Lie algebra $\mathfrak{g}$) on a quantum systems. This is a map $\hat{\rho}\colon \mathfrak{g} \to \mathrm{Der}\,\mathcal{L}$ such that the derivations are inner, that is there exists a map (4.5.4) $a\colon \mathfrak{g} \to \mathcal{L}$ such that

$$\hat{\rho}(\xi)x = [x, a(\xi)] \tag{1.0.5}$$

$\forall\, x \in \mathcal{L}$ and $\xi \in \mathfrak{g}$. The map a is the *quantum co-momentum map*. This enables us to construct the BRST complex $(S^\bullet, \hat{D})$, whose zeroth cohomology group is the quantum constrained algebra of the system.

Another important application of the ideas developed in this thesis are dynamical systems. Dynamical systems constitute a mathematical formulation of dynamics, of how a physical system changes in time, but replacing the group $\mathbb{R}$ by an arbitrary group G. By exploring the algebraic properties of dynamics it has been recognized the notion of dynamical system as a common framework for studying time evolutions and symmetry groups in classical and quantum mechanics. The time evolution in a classical dynamical system is given by a continuous action of $\mathbb{R}$ (seen as a group) on the state space. If we let the action being given by a general group G we arrive at a notion which allows us to study time evolution and symmetry simultaneously. In quantum mechanics the action of the group is given by a unitary representation on the Hilbert space $\mathcal{H}$. Symmetries of a quantum mechanical system should be bijections on the set of states $\mathcal{S}(\mathcal{H})$. One would expect these symmetries to preserve the algebraic structure we have imposed, and Wigner identified the transition probabilities as the key feature for that. Wigner's celebrated theorem states that a bijection which preserves transition probabilities is a unitary or antiunitary operator on $\mathcal{H}$ [Wig31]. In the context of C^*–algebraic quantum systems, a symmetry group is given by a strongly continuous action $\alpha\colon G \to \mathrm{Aut}(c\mathcal{A})$ of G by automorphisms of $\mathcal{A}$. In the Heisenberg picture the observables evolve according to the rule $\alpha\colon t \mapsto U_t^* \pi(a) U_t$. If we represent the algebra as bounded operators on the Hilbert space, then the representation π must satisfy

$$\pi(\alpha_t(a)) = U_t^* \pi(a) U_t, \tag{1.0.6}$$

that is (π, U) is a covariant representation of the dynamical system $(\mathcal{A}, \mathbb{R}, \alpha)$. Covariant representations arose first in Mackey's theory of induced representations of locally compact groups. Takesaki subsequently extended Mackey's machinery to noncommutative algebras and they were further developed by Green, who introduced the *crossed product* $\mathcal{A} \ltimes_\alpha G$ (which, interestingly, had been introduced earlier with physics applications in mind [DKR66]). This C^*–algebra is generated by a covariant representation of $(\mathcal{A}, G, \alpha)$ and there is a one-to-one correspondence between representations of $\mathcal{A} \ltimes_\alpha G$ and covariant representations of $(\mathcal{A}, G, \alpha)$ [RW98]. Thereafter crossed products have provided many of the most important examples of C^*–algebras and became fundamental for the development of noncommutative differential geometry [Con94]. We finally tried to give the basic fundamentals for the extension of such a theory to the context of Lie–Jordan Banach algebras.

Last, there is a remarkable similarity bewtween the theorem relating the reduction of Lie–Jordan Banach algebras and their counterpart C^*–algebras discussed in Chapter 3 and the Hitchin-Kobayashi correspondence. This relation will be considered by using a particular example in Section 5.1.

2

STATES AND OBSERVABLES IN CLASSICAL AND QUANTUM SYSTEMS: C^*–ALGEBRAS AND LJB–ALGEBRAS

2.1. The emergence of the algebraic approach

The idea of an algebraic approach to quantum mechanics was already present in the matrix formulation developed by Heisenberg, Born, Jordan, Dirac and others. At this stage von Neumann formulated the quantum theory as an eigenvalue problem in a Hilbert space [vN96] and analyzed the concept of state from the point of view of the theory of probability. The reader is assumed to be familiar with the usual mathematical formulation of quantum mechanics which we will review in this section in the form of three postulates.

Postulate 2.1.1. *To each observable A on a given physical system there corresponds a linear* **self-adjoint** *operator $\pi(A)$ acting on a Hilbert space $\mathcal{H}_\pi$ and conversely.*

Notice that the converse part of the postulate is now known to be untenable due to the existence of superselection rules [SS78]. However we will not be concerned with this possibility in the following.
If we denote by $\mathcal{L}$ the set of all observables on the physical system considered, we can already observe that the first postulate equips $\mathcal{L}$ with the structure of a **real vector space**.
Notice also that if A and B are two arbitrary elements of $\mathcal{L}$, $\pi(A)\pi(B)$ in general does not belong to $\pi(\mathcal{L})$, whereas the combinations $\pi(A)\pi(B) + \pi(B)\pi(A)$ and $\pi(A)\pi(B) - \pi(B)\pi(A)$ do.[1]
The **symmetrized** product

$$a \circ b = \frac{1}{2}(ab + ba), \quad \forall\, a, b \in \mathcal{L}, \tag{2.1.1}$$

satisfies a number of interesting properties. It is commutative and bilinear and its introduction does not require the knowledge of the ordinary product of two noncompatible observables (i.e., two observables such that the corresponding operators do not commute in the ordinary sense). By defining $a^2 = a \circ a$, we have indeed

$$a \circ b = \frac{1}{2}((a + b)^2 - a^2 - b^2), \tag{2.1.2}$$

which involves only operations like (2.1.1). This symmetrized product is not associative in general, that is

$$(a \circ b) \circ c - a \circ (b \circ c) \neq 0 \tag{2.1.3}$$

for arbitrary $a, b, c \in \mathcal{L}$, as can be seen by simple inspection. The product (2.1.1) is called the **Jordan product** [Jor33],[JvNW34] and will be further examined in Section 2.2 where we will give the axiomatic algebraic formulation of quantum mechanics motivated by the previous considerations.

The **state** of a physical system is understood intuitively as a way to express the maximal simultaneous knowledge of the expectation values of all observables on

[1] In the following we will indicate, for brevity, the operators $\pi(A), \pi(B), \ldots$ with lower case letters $a, b, \ldots$, and commit the sin of denoting $\pi(\mathcal{L})$ with $\mathcal{L}$ itself.

the physical system considered. From the standard theory of quantum mechanics we know that to each state is associated a density matrix:

Postulate 2.1.2. *To each state ψ of the physical system considered corresponds a positive self-adjoint operator ρ of trace 1, acting on the Hilbert space $\mathcal{H}_\pi$ of Postulate 2.1.1, and such that the expectation values $\psi(a)$ of the observable a in the state ψ, are given by $\psi(a) = \mathrm{Tr}\,(\rho a)$.*

Let us stress some properties of states that follow immediately from Postulate 2.1.2. For any linear combination of elements $a_i \in \mathcal{L}$

$$\psi\Big(\sum_i c_i a_i\Big) = \mathrm{Tr}\,\Big(\rho \sum_i c_i a_i\Big) = \sum_i c_i \mathrm{Tr}\,(\rho a_i) = \sum_i c_i \psi(a), \tag{2.1.4}$$

i.e. states act linearly on observables.

Finally concerning the dynamics we have the third postulate

Postulate 2.1.3. *The* **dynamical evolution** *of a closed quantum system described by a density state ρ is given by von Neumann's equation*

$$i\hbar \frac{\partial \rho}{\partial t} = [H, \rho], \tag{2.1.5}$$

where H is the Hamiltonian (observable) operator of the system.

This is *Schrödinger's equation* in the space of density states. If we require that the states do not evolve in time, then we can equivalently describe the dynamics by letting the observables evolve and obtaining the *Heisenberg equation* of motion:

$$\frac{d}{dt} a(t) = \frac{i}{\hbar}[H, a(t)] + \frac{\partial a(t)}{\partial t}. \tag{2.1.6}$$

Remark. Observe that the above dynamical equations are valid only for closed quantum systems, i.e. systems which do not interact with any external environment and do not have dissipative behaviour.

The commutator

$$[H, \rho] = H\rho - \rho H \tag{2.1.7}$$

which arises here because of Postulate 2.1.3, endows the physical observables with the role of generators of the motion on state space and satisfies a number of remarkable properties. Thus it is immediate to check that it is bilinear, antisymmetric and satisfies the Jacobi identity:

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0, \tag{2.1.8}$$

for all $a, b, c \in \mathcal{L}$. The antisymmetry of the bracket ensures that time-independent Hamiltonians are conserved [CS99]. The linearity guarantess that if $a(t)$ and $b(t)$ are two observables which only have dynamical evolution (i.e. without intrinsic time dependence), and λ_1, λ_2 two real constants, the observable $c_1(t) = \lambda_1 a(t) + \lambda_2 b(t)$ is also free of intrinsic time dependence. The Jacobi identity ensures that the observable $c_2(t) = [a(t), b(t)]$ also evolves dynamically:

$$\frac{dc_2}{dt} = \left[\frac{da}{dt}, b\right] + \left[a, \frac{db}{dt}\right] \tag{2.1.9}$$

$$= [[a, H], b] + [a, [b, H]] \tag{2.1.10}$$

$$= [[a, b], H] = [c_2, H]. \tag{2.1.11}$$

In particular this property also ensures the preservation of the canonical relations among canonical variables during time evolution.

2.1.1. Topological structure of the algebra of observables

With the wealth of information contained in their paper [JvNW34], Jordan, von Neumann, and Wigner demonstrated the power of a purely algebraic approach to quantum theories. However, there is a major weakness in their pioneering work, namely that they assumed that $\mathcal{L}$ has a finite linear basis. This had to be corrected by the introduction of an appropriate topological structure before the claim could be made that the theory provides a formalism general enough for the need of quantum problems. The aim would be to imitate the weak topology of operators acting on Hilbert spaces. Before proceeding to an axiomatic presentation of

LJB–algebras in Section 2.2 we show how to endow $\mathcal{L}$ with a natural topology in which the concept of state plays a significant role.

If we denote by $\mathcal{S}(\mathcal{L})$ the space of states associated to the quantum system, and define $\|a\| \equiv \sup_{\phi \in \mathcal{S}(\mathcal{L})} |\phi(a)|$, it follows immediately that for all $\lambda \in \mathbb{R}$, all a and b in $\mathcal{L}$, $\|\lambda a\| = |\lambda| \|a\|$, $\|a + b\| \leq \|a\| + \|b\|$ and that the vanishing of $\|a\|$ occurs only when $a = 0$. Therefore $\| \cdot \|$ is a norm for $\mathcal{L}$ and $\phi(a) \leq \|a\|$ for all $a \in \mathcal{L}$, $\phi \in \mathcal{S}(\mathcal{L})$.
As a result of these considerations, $\mathcal{L}$ is now equipped with the structure of a real Banach space relative to the natural norm introduced above, and the states ϕ in $\mathcal{S}$ are continuous (positive linear) functionals on $\mathcal{L}$ with respect to the topology induced by this norm.

From a phenomenological point of view we might remark at this point that one actually never deals in the laboratory with any observable a for which $\phi(a)$ is not finite; it is current practice nevertheless to consider in the theory “idealized observables” that are unbounded. There are probably novel approaches to get rid of this troubles by relaxing the Banach structure in favour of the more flexible structures like Frechet or Riesz structures.

2.1.2. The algebra of observables in classical mechanics

In classical mechanics one is usually introduced to the Newtonian formalism whose laws are shown to be equivalent to the more mathematically convenient Hamiltonian formalism. In Hamiltonian mechanics, we describe the state of a system by a point (q, p) in a symplectic manifold P, known as phase space. In all real physical systems, the position q and momentum p of the particle must remain bounded, and hence it is natural to assume that P is compact.
It is an experimental fact that we can never measure something with infinite precision. There are however quantities that we can, in principle, measure to an arbitrary precision. We call such quantities **classical observables**.
We would like to come up with a mathematically precise, physically motivated way to characterize classical observables. A first natural requirement is that observables depend on the state of the system, that is, observables are functions of

q and p. Moreover these functions must be real valued since we cannot measure complex quantities. Let us assume, as an experimental fact, that in the classical realm we can always measure q and p with arbitrary precision. Assume now that we want to measure the function f with error less than some $\epsilon > 0$. Since I can make the error in q and p arbitrarily small, there exist errors δ_q and δ_p such that $\forall\, q \in (q_0 - \delta_q, q_0 + \delta_q)$ and $p \in (p_0 - \delta_p, p_0 + \delta_p)$, the experimental value of $f(q,p)$ satisfies

$$f(q_0,p_0) - \epsilon < f(q,p) < f(q_0,p_0) + \epsilon.$$

But this is just the definition of a continuous function. We are then naturally lead to the characterization of observables in classical mechanics as the **continuous real-valued functions** on the phase space P.

We shall introduce in the algebra of observables one more operation, which is connected with the evolution of the mechanical system. For simplicity the discussion to follow is conducted using the example of a system with one degree of freedom. The equations of motion are given by **Hamilton's equations** which have the form:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \quad H = H(q,p), \tag{2.1.12}$$

with solutions $q(t)$ and $p(t)$. The equations above generate a one-parameter group of transformations of the phase space into itself and in turn they generate a family of transformations of the algebra of observables:

$$f(q,p,t) = f(q(t),p(t)). \tag{2.1.13}$$

The function $f(q,p,t)$ satisfies the differential equation

$$\frac{\partial f}{\partial t} = \frac{\partial H}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial f}{\partial p} = \{H, f\}, \tag{2.1.14}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket, which makes the classical observables a **Poisson algebra**. That is, it satisfies

i) $\{f,g\} = -\{f,g\}$,

ii) $\{f,\{g,h\}\}+\{h,\{f,g\}\}+\{g,\{h,f\}\}=0,$

iii) $\{f,gh\}=g\{f,h\}+\{f,g\}h,$

for all $f,g,h\in C^\infty(P)$. It is interesting to point it out that the Poisson bracket is not defined on the full algebra of classical observables $C^0(P)$ but only on a dense subalgebra, its smooth part $C^\infty(P)$. Notice again that the smooth part is determined by the choice of a smooth structure on the phase space P, however in more general terms, such smooth subalgebra can be determined by a dynamical system (we will come back to that in Section 5.2).

2.2. Lie–Jordan Banach algebras

Motivated by the previous considerations, we are ready to define in this section the abstract algebraic properties describing classical and quantum observables.

Let $\mathcal{L}$ be a real vector space on which it is defined a symmetric bilinear distributive product $\circ$, called **Jordan product** which satisfies the generalized associative law:

$$(a^2\circ b)\circ a=a^2\circ(b\circ a),\quad \forall\, a,b\in\mathcal{L}, \tag{2.2.1}$$

which is the usual replacement for associativity for Jordan algebras; and an antisymmetric **Lie product** $[\cdot,\cdot]$ satisfying the Jacobi identity

$$[[a,b],c]+[[c,a],b]+[[b,c],a]=0,\quad \forall\, a,b,c\in\mathcal{L}. \tag{2.2.2}$$

We require these two operations to be compatible in the sense that Leibniz identity is verified:

$$[a,b\circ c]=[a,b]\circ c+b\circ[a,c]\,, \tag{2.2.3}$$

or, in other words, the linear map $D_a(\cdot)\equiv[a,\cdot]$ is a derivation of the Jordan product $\circ$.

By abstracting the previous properties, one says in general that a vector space with a symmetric operation $\circ$ and an antisymmetric one $[\cdot,\cdot]$ satisfying the properties (2.2.1),(2.2.2),(2.2.3), is called **"unlocked" Lie–Jordan algebra**. The complete definition of a ("locked") **Lie–Jordan algebra** requires that the associator of the structure product is related to the Lie product by:

$$(a \circ b) \circ c - a \circ (b \circ c) = \kappa \, [b, [c, a]] \, , \tag{2.2.4}$$

κ being a positive real number. Then we will call $(\mathcal{L}, \circ, [\cdot, \cdot])$ satisfying (2.2.4) a Lie–Jordan algebra with constant κ. The rational behind axiom (2.2.4) comes from the example discussed in the previous section oh physical observables as self-adjoint operators on a Hilbert space and that we will discuss again from this perspective. Thus if we consider for instance the real vector space of bounded self-adjoint operators on a Hilbert space $\mathcal{H}$, and the Jordan product $\circ$ defined by

$$a \circ b = \frac{1}{2}(ab + ba), \tag{2.2.5}$$

and the Lie product given by

$$[a, b] = \frac{i}{2\hbar}(ab - ba) = 0, \tag{2.2.6}$$

we obtain that (2.2.4) requires:

$$\kappa = \hbar^2, \tag{2.2.7}$$

if $\kappa \neq 0$. We have introduced an additional factor $\frac{i}{2\hbar}$ in (2.2.6) with respect to the familiar definition of commutator (2.1.7). In this way the space of observables actually acquires the structure of a Lie algebra. Moreover we insert the constant $\hbar$ for dimensional reasons and we actually see that the Lie–Jordan algebras thus defined depend on the physical constant $\hbar$.

Notice that if the Jordan product is associative and $\kappa \neq 0$ then, as it is proved in the next Theorem, the Lie structure becomes commutative, i.e. $[a, b] = 0 \, \forall \, a, b \in \mathcal{L}$.

Theorem 2.2.1. *A Lie–Jordan algebra $\mathcal{L}$ with constant $\kappa \neq 0$ is commutative if and only if the Jordan product is associative.*

Proof. Assume first that $\mathcal{L}$ is commutative. Then, trivially, from the associator identity (2.2.4) it follows that the Jordan algebra $\mathcal{L}$ is associative. Conversely, if the Jordan product is associative, then any triple commutator vanishes, so that $\forall\, a, b \in \mathcal{L}$

$$\begin{aligned} 0 &= [a,[b^2,a]] \\ &= [a, 2b\circ[b,a]] \\ &= 2b\circ[a,[b,a]] + [a,2b]\circ[b,a] \\ &= 2b\circ[a,[b,a]] - 2[a,b]^2 \\ &= -2[a,b]^2, \end{aligned}$$

where we used the Leibniz identity in the second and the third equality. In conclusion $[a, b] = 0,\ \forall\, a, b \in \mathcal{L}$. □

If we consider a classical carrier space, for instance a Poisson manifold, the algebra of smooth functions on the manifold becomes a Lie–Jordan algebra with constant $\kappa = 0$ when equipped with the (associative) pointwise product $f\circ g\,(x) = f(x)g(x)$, and Lie bracket $[f, g] = \{f, g\}$, with $\{\cdot,\cdot\}$ being the Poisson bracket defined on the manifold. Thus it follows that from an algebraic point of view it is quite appropriate to consider a Poisson algebra as a Lie–Jordan algebra with $\kappa = 0$. From this perspective we may consider the parameter κ a sort of deformation parameter connecting the classical and the quantum picture of a system. With this intuition in mind we may call Lie–Jordan algebras with $\kappa = 0$ classical. Notice that as we mentioned already there is no Lie–Jordan algebra structure on $C^0(P)$, however it could be a good idea to call them unbounded Lie–Jordan algebras.
In order to accomodate infinite dimensional systems in this formalism, we need to define a topological structure on the algebra.

Definition 2.2.2. A Lie–Jordan Banach algebra (or LJB–algebra for short) is Lie-Jordan algebra $(\mathcal{L}, \circ, [\cdot,\cdot])$ such that it carries a complete norm $\|\cdot\|$ verifying:

i) $\|a\circ b\| \leq \|a\|\,\|b\|$,

ii) $\|a^2\| = \|a\|^2$,

iii) $\|a^2\| \leq \|a^2 + b^2\|$,

$\forall\, a, b \in \mathcal{L}$.

In particular a LJB–algebra is a Jordan–Banach algebra (or JB–algebra) when considered with the Jordan product alone. On the other hand, if we are given a LJB–algebra $\mathcal{L}$, by taking combinations of the two products we can define an associative product on the complexification $\mathcal{L}^{\mathbb{C}} = \mathcal{L} \oplus i\mathcal{L}$. Specifically, we define:

$$ab = a \circ b - i\sqrt{\kappa}\,[a, b]\,, \quad \forall\, a, b \in \mathcal{L},$$

and extend it by linearity to $\mathcal{L}^{\mathbb{C}}$. Then $\mathcal{L}^{\mathbb{C}}$ becomes an associative *–algebra, where $(a + ib)^* = a - ib$. Such associative algebra equipped with the norm $\|x\| = \|x^* x\|^{1/2}$ where $x = a + ib$, is the unique C^*–algebra whose real part is precisely $\mathcal{L}$ (see Section 2.3).

Notice that if the LJB–algebra $\mathcal{L}$ is classical, i.e. $\kappa = 0$, its associated C^*–algebra is isomorphic to the space of continuous functions on a compact topological space with the supremum norm, hence if such space carries a differentiable structure the Lie bracket will define a family of unbounded derivations on the dense subspace of smooth functions, otherwise trivial. In other words we will need weaker topologies to accomodate classical LJB–algebras in the same picture. Then, from now on, we will just consider non–classical LJB–algebras, i.e., $\kappa \neq 0$. We must point out here that the study of unbounded LJB–algebras has never been started.

2.2.1. Spectrum and states of Lie–Jordan Banach algebras

The concept of spectrum of an observable is very important since it provides the possible outcomes of a measurement of the observable on the physical system. In this subsection we explore the definition of spectrum and states in algebraic terms.

Definition 2.2.3. Let $\mathcal{L}$ be a unital LJB–algebra. The spectrum $\sigma(a)$ of $a \in \mathcal{L}$ is defined as the set of those $\lambda \in \mathbb{R}$ for which $a - \lambda\mathbb{1}$ has no inverse in $\mathcal{L}$.

Note that a LJB–algebra $\mathcal{L}$ is a complete order unit space with respect to the positive cone [HOS84]:

$$\mathcal{L}^+ = \{\, a^2 \mid a \in \mathcal{L} \,\} \tag{2.2.8}$$

or equivalently an element is positive if and only if its spectrum is positive.[2]
We shall in this section prove some useful properties of the spectrum and then the Cauchy–Schwarz like inequalities.

Lemma 2.2.4.

$$\sigma(a_1^2 + a_2^2 + \rho[a_1, a_2]) \cup \{0\} = \sigma(a_1^2 + a_2^2 - \rho[a_1, a_2]) \cup \{0\} \tag{2.2.9}$$

$\forall\, a_1, a_2 \in \mathcal{L}$ *and* $\forall\, \rho \in \mathbb{R}$.

Proof. For $\lambda \neq 0$ the invertibility of $a_1^2 + a_2^2 + \rho[a_1, a_2] - \lambda\mathbb{1}$ implies the invertibility of $a_1^2 + a_2^2 - \rho[a_1, a_2] - \lambda\mathbb{1}$. Namely, one computes that

$$\begin{aligned}(a_1^2 + a_2^2 + \rho[a_1, a_2] - \lambda\mathbb{1})^{-1} \;&=\; \lambda^{-1}\{2a_1 \circ (b \circ a_1) - a_1^2 \circ b + 2a_2 \circ (b \circ a_2) + \\ &\quad -a_2^2 \circ b + 2[a_1, b \circ a_2] + 2a_1 \circ [b, a_2] - \mathbb{1}\}\end{aligned}$$

with $b = \{a_1^2 + a_2^2 + \rho[a_1, a_2] - \lambda\mathbb{1}\}^{-1}$. □

Lemma 2.2.5.

$$\sigma(a_1^2 + a_2^2 + \rho[a_1, a_2]) \subset \mathbb{R}^- \Rightarrow a_1^2 + a_2^2 + \rho[a_1, a_2] = 0 \tag{2.2.10}$$

$\forall\, a_1, a_2 \in \mathcal{L}$ *and* $\forall\, \rho \in \mathbb{R}$.

Proof. Note that $a_1^2 + a_2^2 - \rho[a_1, a_2] = 2a_1^2 + 2a_2^2 - (a_1^2 + a_2^2 + \rho[a_1, a_2])$ and then under the assumptions of the lemma $\sigma(a_1^2 + a_2^2 - \rho[a_1, a_2]) \subset \mathbb{R}^+$. This implies, by the previous lemma, that $\sigma(a_1^2 + a_2^2 - \rho[a_1, a_2]) = \{0\}$. □

Theorem 2.2.6.

$$X = a_1^2 + a_2^2 - \rho[a_1, a_2] \in \mathcal{L}^+ \tag{2.2.11}$$

$\forall\, a_1, a_2 \in \mathcal{L}$ *and* $\forall\, \rho \in \mathbb{R}$.

[2] An alternative way to express this would be: an element $a \in \mathcal{L}$ is positive if $\sigma(a) \subset \mathbb{R}^+$. It is possible to show that the cone of positive elements is given by $\mathcal{L}^+ = \{\, a^2 \mid a \in \mathcal{L} \,\}$. Moreover it is a consequence of the completeness of a LJB–algebra to show that it is a complete order unit space.

Proof. Every $X \in \mathcal{L}$ has the decomposition [Lan98] $X = X_+ + X_-$, where $X_+, X_- \in \mathcal{L}^+$ and $X_+ \circ X_- = [X_+, X_-] = 0$. It follows that $X_-^3 = -(b_1^2 + b_2^2 - \rho[b_1, b_2]) \geq 0$ with $b_1 = a_1 \circ X_- + \rho[a_2, X_-]$ and $b_2 = \rho[a_1, X_-] + a_2 \circ X_-$. But $X_-^3 = -2b_1^2 - 2b_2^2 + (b_1^2 + b_2^2 + \rho[b_1, b_2])$ which is a negative quantity and then in turn implies that $X_- = 0$ and then $X = X_+ \geq 0$. □

Motivated by the considerations of Section 2.1, we can define the space of **states** $\mathcal{S}(\mathcal{L})$ of a LJB–algebra as the set of all **real normalized positive linear functionals** on $\mathcal{L}$, i.e.

$$\rho\colon \mathcal{L} \to \mathbb{R} \tag{2.2.12}$$

such that $\rho(\mathbb{1}) = 1$ and $\rho(a^2) \geq 0$, $\forall\, a \in \mathcal{L}$. The state space is convex and compact with respect to the w*–topology.

We shall now prove the Lie–Jordan algebra version of the Cauchy–Schwarz inequalities. These are a very important "ingredient" for many subsequent proofs.

Theorem 2.2.7. *Let $\mathcal{L}$ be a unital LJB–algebra with constant $\hbar^2$ and ρ a state on $\mathcal{L}$. Then if $a, b \in \mathcal{L}$ we have*

$$\rho(a \circ b)^2 \leq \rho(a^2)\rho(b^2), \tag{2.2.13}$$

and

$$\rho([a, b])^2 \leq \frac{1}{\hbar^2}\rho(a^2)\rho(b^2). \tag{2.2.14}$$

Proof. Let $\lambda \in \mathbb{R}$, then we have

$$0 \leq \rho(\lambda a + b)^2) = \lambda^2\rho(a^2) + 2\lambda\rho(a \circ b) + \rho(b^2). \tag{2.2.15}$$

If $\rho(a^2) = 0$ then $\rho(a \circ b) = 0$ since λ is arbitrary. If $\rho(a^2) \neq 0$, let $\lambda = -\rho(a \circ b)\rho(a^2)^{-1}$, and the first proof is immediate.
The second inequality is proved similarly by using the positivity of $a_1^2 + a_2^2 + 2\hbar[a_1, a_2]$ as stated in Thm. (2.2.6). □

Example 2.2.8. As we have discussed beforem the self-adjoint subalgebra $\mathcal{B}_{sa} = \mathcal{L}(\mathcal{H})$ of the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ with the operator norm is a Lie–Jordan Banach algebra and the states are the positive linear functional on $\mathcal{B}_{sa}$. Let φ be a continuous state with respect to the ultrastrong topology on $\mathcal{B}(\mathcal{H})$ [vN36], i.e. the topology on $\mathcal{B}(\mathcal{H})$ given by the open neighbourhood base

$$N(a;(x_i)_1^\infty,\epsilon) = \{\, b \in \mathcal{B}(\mathcal{H})\colon \sum_{i=1}^{\infty} \|(a-b)x_i\|^2 < \epsilon \,\}, \tag{2.2.16}$$

for $a \in \mathcal{B}(\mathcal{H}),\ \epsilon > 0$ and any sequence $(x_i) \in \mathcal{H}$ satisfying $\sum_{i=1}^{\infty} \|x_i\|^2 < \infty$. Then there is a positive linear trace class operator $\rho \in \mathcal{B}_{sa}$ such that

$$\varphi(a) = \mathrm{Tr}(\rho a) \tag{2.2.17}$$

for all $a \in \mathcal{B}_{sa}$.
Conversely, if ρ is a positive trace class operator, then the functional $a \mapsto \mathrm{Tr}(\rho a)$ defines an ultrastrongly continuous positive linear functional on $\mathcal{B}_{sa}$.

We have shown that the algebra of observables of a quantum mechanical system is a LJB–algebra and described the quantum states as positive linear functionals on the algebra. As it is evident from the previous example not all the states on the algebra can be realized as density matrices. Those states realized as density matrices are called normal. It is nevertheless recognized the important of non-normal states in the mathematical approaches to quantum statistical mechanics [BR03].
A natural question may now arise. That is, can the algebraic framework accomodate something more general than the standard quantum theory we have seen? Is it possible to provide realizations of a LJB–algebra (or equivalently a C^*–algebra) different from the usual quantum mechanics? The answer to this question gives actually a solid background to the algebraic theory since it can be proved that LJB–algebras and C^*–algebras can always be represented as algebras of operators on a Hilbert space.

Theorem 2.2.9 (Gelfand, Naimark, Segal)**.** *Let $\mathcal{L}$ be a unital LJB–algebra. A* **representation** *of $\mathcal{L}$ on a complex Hilbert space $\mathcal{H}$, is a strongly continuous Lie–Jordan homomorphism π of $\mathcal{L}$ into the self-adjoint bounded operators on $\mathcal{H}$, i.e.*

$\forall\, a, b \in \mathcal{L}$

$$\pi(a \circ b) = \pi(a) \circ \pi(b) \tag{2.2.18}$$

$$[\pi(a), \pi(b)] = \pi([a, b]). \tag{2.2.19}$$

Moreover given a state ω of $\mathcal{L}$ there exists a Hilbert space $\mathcal{H}_\omega$ and a representation $\pi_\omega \colon \mathcal{L} \to \mathcal{B}_{sa}(\mathcal{H}_\omega)$ and a unit vector $|0\rangle \in \mathcal{H}_\omega$ such that for all $a \in \mathcal{L}$, $\omega(a) = \langle 0|\pi_\omega(a)|0\rangle$.

We will say that the representation π is *nondegenerate* if $\mathrm{span}\{\pi(a)|\phi\rangle \;\mid a \in L,\ |\phi\rangle \in \mathcal{H}\}$ is dense in $\mathcal{H}$.
Given two representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ of $\mathcal{L}$, we say that they are equivalent if there exists a unitary map $U \colon \mathcal{H}_1 \to \mathcal{H}_2$ such that $U \circ \pi_1(a) = \pi_2(a) \circ U$, $\forall\, a \in \mathcal{L}$.

Remark. If we replace the LJB–algebra by a Jordan–Banach algebra, as it was originally proposed by P. Jordan, then the above theorem is not true. It is in fact known (as discussed in the Introduction) that there is an “exceptional” Jordan algebras (i.e. a Jordan algebra which do not arise from an associative product) which cannot represented as an algebra of operators on a Hilbert space. This is the so called *Albert algebra* of 3×3 matrices with values in the Octonions.

2.3. C^*–algebras and dynamical correspondence

R. Haag’s algebraic approach to quantum systems [Haa96] has had a profound influence in both the foundations and applications of quantum physics.The background for that approach is to consider a quantum system as described by a C^*–algebra $\mathcal{A}$ whose real part are the observables of the system, and its quantum states ω are normalized positive complex functionals on it. However the state space $\mathcal{S}$ of the quantum system does not determine univocally the C^*–algebra structure of the system but only its Jordan–Banach real algebra part [JvNW34]–[Seg47]. In fact as Kadison’s theorem shows [Kad51], the real (or self-adjoint) part of a C^*–algebra $\mathcal{A}$, is isometrically isomorphic to the space of all w*– continuous affine functions on the state space of $\mathcal{A}$. A. Connes on one side [Con74] and Alfsen and Shultz on the other [AS98], solved the problem of when a given

Jordan–Banach algebra is the real part of a C^*–algebra. The characterization obtained by Alfsen and Schultz in terms of the existence of a dynamical correspondence on a Jordan–Banach algebra amounts to state that the relevant structure to discuss the properties of the state space of a quantum system is that of a LJB–algebra [Emc84]–[Lan98]. In fact the topological properties of the state space are completely captured by the Jordan–Banach algebra structure and the Lie algebra structure allows to construct the C^*–algebra setting for them, their GNS representations, etc.

In this section we are going to prove one of the main results in the theory of LJB–algebras, which we already anticipated in the previous sections. Namely, the equivalence between the category of C^*–algebras and the category of LJB–algebras. We will prove that a C^*–algebra is always a complexification of a LJB–algebra. In order to do this, we briefly give few definitions on C^*–algebras and derivations of LJB–algebras.

Definition 2.3.1. A C^*–algebra $\mathcal{A}$ is a Banach algebra over the field of complex numbers, together with an antilinear map $^*\colon \mathcal{A} \to \mathcal{A}$ called **involution**, which satisfies $(x^*)^* = x$ and

$$\|x^* x\| = \|x\| \|x^*\|, \quad \forall\, x \in \mathcal{A}. \tag{2.3.1}$$

Following [AS98] we will define a derivation of a JB–algebra $\mathcal{L}$ by focusing first only on the order structure with respect to the positive cone $\mathcal{L}^*$ defined before, ignoring for the moment the algebraic multiplicative aspect. All the proofs contained in [AS98] will be omitted.

Definition 2.3.2. A bounded linear operator δ on a JB–algebra $\mathcal{L}$ is called an order derivation if $\mathrm{e}^{t\delta}(\mathcal{L}^+) \subset \mathcal{L}^+$, $\forall\, t \in \mathbb{R}$.

We denote the Jordan multiplier determined by an element $b \in \mathcal{L}$ by δ_b. Thus for all $a \in \mathcal{L}$

$$\delta_b(a) = b \circ a.$$

Notice that $\mathrm{e}^{t\delta_b}$ is the multiplier associated to $\mathrm{e}^{tb} = (\mathrm{e}^{\frac{tb}{2}})^2 \in \mathcal{L}^+$. Then Jordan multipliers δ_b are order derivations $\forall\, b \in \mathcal{L}$.

Definition 2.3.3. An **order derivation** δ on a unital JB–algebra $\mathcal{L}$ is self-adjoint if there exists $a \in \mathcal{L}$ such that $\delta = \delta_a$ and is skew-adjoint if $\delta(\mathbb{1}) = 0$.

Again, it can be shown that if δ is an order derivation, then δ is skew if and only if δ is a **Jordan derivation**, i.e., it is a derivation with respect to the Jordan product:

$$\delta(a \circ b) = \delta a \circ b + a \circ \delta b, \quad \forall\, a, b \in \mathcal{L}. \tag{2.3.2}$$

We will establish now the main notion in [AS98].

Definition 2.3.4. A **dynamical correspondence** [AS98] on a unital JB–algebra $\mathcal{L}$ is a linear map

$$\psi\colon a \to \psi_a \tag{2.3.3}$$

from $\mathcal{L}$ into the set of skew order derivations on $\mathcal{L}$ which satisfies:

i) there exists $\kappa \in \mathbb{R}$ such that $\kappa\,[\psi_a, \psi_b] = -\,[\delta_a, \delta_b]\,, \quad \forall\, a, b \in \mathcal{L}$, and[3]

ii) $\psi_a a = 0, \quad \forall\, a \in \mathcal{L}$.

It follows immediately from the definitions that:

$$\psi_a b = -\psi_b a, \quad \forall\, a, b \in \mathcal{L}. \tag{2.3.4}$$

The dynamical correspondence then assigns a “skew order derivation” ψ_a to each element a of the given algebra $\mathcal{L}$. The skew order derivations are generators of one-parameter groups of unital order automorphisms of $\mathcal{L}$ [AS98], and by duality also of one-parameter groups of motions on the state space of $\mathcal{L}$. Thus a dynamical correspondence gives the elements of $\mathcal{L}$ a double identity, which reflects the dual role of physical variables as observables and as generators of a one-parameter group of motions of the state space.

Definition 2.3.5. Let $\mathcal{L}$ be a unital JB–algebra. A C^*–product compatible with $\mathcal{L}$ is an associative product on the complex linear space $\mathcal{L} \oplus i\mathcal{L}$ which induces the given Jordan product on $\mathcal{L}$ and makes $\mathcal{L} \oplus i\mathcal{L}$ into a C^*–algebra with involution $(a + ib)^* = a - ib$ and norm $\|x\| = \|x^* x\|^{1/2}$ where $x = a + ib$.

[3]The notations $[\psi_a, \psi_b]$ and $[\delta_a, \delta_b]$ are not related to any Lie bracket and stand for the commutator of the operators in the arguments, i.e. $[\psi_a, \psi_b] = \psi_a\psi_b - \psi_b\psi_a$, $[\delta_a, \delta_b] = \delta_a\delta_b - \delta_b\delta_a$.

Note that if a JB–algebra $\mathcal{L}$ is the self-adjoint part of a C^*–algebra $\mathcal{A}$, then there are a natural product and a norm induced in $\mathcal{L} \oplus i\mathcal{L}$ by using the representation $A = a + ib$ with $A \in \mathcal{A}$ and $a, b \in \mathcal{L}$. Such product and norm organize $\mathcal{L} \oplus i\mathcal{L}$ into a C^*–algebra. It follows that a JB–algebra is the self-adjoint part of a C^*–algebra if and only if there exists a C^*–product compatible with $\mathcal{L}$ on $\mathcal{L} \oplus i\mathcal{L}$. Yhe main result in [AS98] provides an explicit relation between JB–algebras and C^*–algebras provided that the former are equipped with a dynamical correspondence.

Theorem 2.3.6 ([AS98])**.** *A unital JB–algebra $\mathcal{L}$ is Jordan isomorphic to the self-adjoint part of a C^*–algebra if and only if there exists a dynamical correspondence on $\mathcal{L}$. Each dynamical correspondence ψ on $\mathcal{L}$ determines a unique associative C^*–product compatible with $\mathcal{L}$ defined as*

$$ab = a \circ b - i\sqrt{\kappa}\, \psi_a b \tag{2.3.5}$$

and each C^*–product compatible with $\mathcal{L}$ arises in this way from a unique dynamical correspondence ψ on $\mathcal{L}$.

We will now show that the existence of a dynamical correspondence on $\mathcal{L}$ is equivalent to the existence of a Lie product organizing $\mathcal{L}$ into a LJB–algebra. First we need the following lemmas:

Lemma 2.3.7. *Let $(\mathcal{L}, [\cdot,\cdot]_{\mathcal{L}}, \circ)$ be a LJB–algebra. Then there exists an associative bilinear product on $\mathcal{L} \times \mathcal{L}$ defined as*

$$a \cdot b = a \circ b - i\sqrt{\kappa}\,[a, b]_{\mathcal{L}}\,, \quad \forall\, a, b \in \mathcal{L}, \tag{2.3.6}$$

and extended linearly to $\mathcal{L} \oplus i\mathcal{L}$.

Proof. Bilinearity of the product follows directly from the bilinearity of the Jordan and Lie products. We have to prove the associativity, i.e.:

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c, \quad \forall\, a, b, c \in \mathcal{L}. \tag{2.3.7}$$

The l.h.s. of the previous equation leads to:

$$a\cdot(b\cdot c) = a\circ(b\circ c) - i\sqrt{\kappa}\, a\circ[b, c]_{\mathcal{L}} - i\sqrt{\kappa}\ [a, b]_{\mathcal{L}}\circ c - i\sqrt{\kappa}\, b\circ[a, c]_{\mathcal{L}} - \kappa\ [a, [b, c]_{\mathcal{L}}]_{\mathcal{L}}\,,$$

and the r.h.s.:

$$(a\cdot b)\cdot c = (a\circ b)\circ c - i\sqrt{\kappa}\, b\circ[a,c]_{\mathcal{L}} - i\sqrt{\kappa}\, a\circ[b,c]_{\mathcal{L}} - i\sqrt{\kappa}\ [a,b]_{\mathcal{L}}\circ c - \kappa\ [\,[\,a,b\,]_{\mathcal{L}}\,,c\,]_{\mathcal{L}}\,,$$

Then

$$\begin{aligned}
a\cdot(b\cdot c)-(a\cdot b)\cdot c &= a\circ(b\circ c)-(a\circ b)\circ c-\kappa\ [a,[b,c]_{\mathcal{L}}]_{\mathcal{L}}-\kappa\ [c,[a,b]_{\mathcal{L}}]_{\mathcal{L}}\\
&= \kappa\left([b,[c,a]_{\mathcal{L}}]_{\mathcal{L}}+[a,[b,c]_{\mathcal{L}}]_{\mathcal{L}}+[c,[a,b]_{\mathcal{L}}]_{\mathcal{L}}\right)\\
&= 0.
\end{aligned}$$

where we have used (2.2.2),(2.2.3) and (2.2.4). □

Note that the Jordan and Lie products can be obviously expressed in terms of the associative product as:

$$a\circ b = \frac{1}{2}(a\cdot b + b\cdot a), \tag{2.3.8}$$

$$[a,b]_{\mathcal{L}} = \frac{i}{2\sqrt{\kappa}}(a\cdot b - b\cdot a). \tag{2.3.9}$$

Lemma 2.3.8. *Let* $(\mathcal{L},[\cdot,\cdot]_{\mathcal{L}}\,,\circ)$ *be a LJB–algebra. Then* $\mathrm{e}^{[a,\cdot]_{\mathcal{L}}}$ *is a Jordan automorphism* $\forall\, a\in\mathcal{L}$.

Proof. We have to prove that

$$\mathrm{e}^{[a,\cdot]_{\mathcal{L}}}\ (b\circ c) = (\mathrm{e}^{[a,\cdot]_{\mathcal{L}}}\ b)\circ(\mathrm{e}^{[a,\cdot]_{\mathcal{L}}}\ c). \tag{2.3.10}$$

By Hadamard's formula [Ser65], the l.h.s. of the previous equation is:

$$\mathrm{e}^{[a,\cdot]_{\mathcal{L}}}\ (b\circ c) = \mathrm{e}^{a}\cdot(b\circ c)\cdot\mathrm{e}^{-a}\,.$$

By using formula (2.3.8), the r.h.s. of (2.3.10) becomes:

$$\begin{aligned}
(\mathrm{e}^{[a,\cdot]_{\mathcal{L}}}\ b)\circ(\mathrm{e}^{[a,\cdot]_{\mathcal{L}}}\ c) &= (\mathrm{e}^{a}\cdot b\cdot\mathrm{e}^{-a})\circ(\mathrm{e}^{a}\cdot c\cdot\mathrm{e}^{-a})\\
&= \frac{1}{2}\ \mathrm{e}^{a}\cdot(b\cdot c)\cdot\mathrm{e}^{-a}+\frac{1}{2}\ \mathrm{e}^{a}\cdot(c\cdot b)\cdot\mathrm{e}^{-a}\\
&= \mathrm{e}^{a}\cdot(b\circ c)\cdot\mathrm{e}^{-a}\,.
\end{aligned}$$

□

Lemma 2.3.9. *Let $(\mathcal{L}, [\cdot,\cdot]_{\mathcal{L}}, \circ)$ be a LJB–algebra. Then $[a,\cdot]_{\mathcal{L}}$ is an order derivation on $\mathcal{L}$ $\forall\, a \in \mathcal{L}$.*

Proof. From Definition 2.3.2, we have to prove that $\mathrm{e}^{t[a,\cdot]_{\mathcal{L}}}(\mathcal{L}^+) \subset \mathcal{L}^+$, $\forall\, a \in \mathcal{L}$ and $\forall\, t \in \mathbb{R}$. Since $\mathrm{e}^{t[a,\cdot]_{\mathcal{L}}}$ is a Jordan automorphism (Lemma 2.3.8), we have:

$$\mathrm{e}^{t[a,\cdot]_{\mathcal{L}}}(b \circ b) = (\mathrm{e}^{t[a,\cdot]_{\mathcal{L}}}\, b) \circ (\mathrm{e}^{t[a,\cdot]_{\mathcal{L}}}\, b),$$

$\forall\, a, b \in \mathcal{L}$ and $\forall\, t \in \mathbb{R}$, i.e. $\mathrm{e}^{t[a,\cdot]_{\mathcal{L}}}$ preserves the positive cone (2.2.8) $\mathcal{L}^+$. □

Then we can finally conclude:

Theorem 2.3.10 ([FFIM13c]). *Let $\mathcal{L}$ be a unital JB–algebra. There exists a dynamical correspondence ψ on $\mathcal{L}$ if and only if $\mathcal{L}$ is a LJB–algebra with Lie product $[\cdot,\cdot]_{\mathcal{L}}$ such that*

$$[a, b]_{\mathcal{L}} = \psi_a b \tag{2.3.11}$$

Proof. First assume that $\mathcal{L}$ is a LJB–algebra. From Definition 2.3.4 we have to check that $\forall\, a, b \in \mathcal{L}$

$$\kappa\, [\psi_a, \psi_b] = -\, [\delta_a, \delta_b]$$

that is

$$\kappa([a, [b, c]_{\mathcal{L}}]_{\mathcal{L}} - [b, [a, c]_{\mathcal{L}}]_{\mathcal{L}}) = b \circ (a \circ c) - a \circ (b \circ c)$$

which is an easy computation once the Jordan and Lie products are expressed as in (2.3.9) and (2.3.8). From the antisymmetry of the Lie product it is also true that $\psi_a a = [a, a]_{\mathcal{L}} = 0 \;\forall\, a \in \mathcal{L}$. Hence the linear map $a \to [a,\cdot]_{\mathcal{L}}$ from the LJB–algebra $\mathcal{L}$ to the skew-order derivations on $\mathcal{L}$ is a dynamical correspondence.

Conversely, assume $\mathcal{L}$ is a JB–algebra with a dynamical correspondence ψ. Then from (2.3.4) $\psi_a b = [a, b]_{\mathcal{L}}$ is antisymmetric. The Jacobi property (2.2.2) follows from the defining property i) of the dynamical correspondence (Definition 2.3.4), the Leibniz identity (2.2.3) follows from (2.3.2) and also the compatibility condition (2.2.4) is easy to check with a simple computation using the properties of the dynamical correspondence (Definition 2.3.4). Hence a JB–algebra with a dynamical correspondence is a LJB–algebra. □

Corollary 2.3.11. *A unital JB–algebra $\mathcal{L}$ is Jordan isomorphic to the self-adjoint part of a C^*–algebra if and only if it is a LJB–algebra.*

Proof. This is an obvious consequence of Theorems 2.3.6 and 2.3.10. □

This finally proves the equivalence between the category of C^*–algebras and that of LJB–algebras. We conclude with

Corollary 2.3.12. *Let $(\mathcal{L}, \circ, [\cdot,\cdot]_{\mathcal{L}})$ be a LJB–algebra and $\mathcal{A} = \mathcal{L}^{\mathbb{C}}$ the natural C^*–algebra defined by the complexification of $\mathcal{L}$. Then there is a natural identification between the states $\mathcal{S}(\mathcal{L})$ of $\mathcal{L}$ and the states $\mathcal{S}(\mathcal{A})$ of the C^*–algebra $\mathcal{A}$.*

Proof. Given a state ω of $\mathcal{L}$, we define a linear functional $\tilde{\omega}$ of $\mathcal{A}$ by extending it linearly. The linear functional $\tilde{\omega}$ is positive and normalized because ω is positive and normalized. Notice that if $x = a + ib \in \mathcal{L}^{\mathbb{C}}$, then $x^*x = a^2 + b^2$, then if $\widetilde{\omega}$ is a functional extending ω (chosen continuous by the Hahn–Banach theorem), then

$$\widetilde{\omega}(x^*x) = \widetilde{\omega}(a^2 + b^2) = \omega(a^2 + b^2) \geq 0, \tag{2.3.12}$$

because ω is positive. The converse is trivial. □

Notice in addition that if $\alpha\colon \mathcal{A} \to \mathcal{B}$ is a morphism of C^*–algebras, then $\alpha(a^*) = \alpha(a)^*$, thus α restricts to a morphism $\alpha_{sa}\colon \mathcal{A}_{sa} \to \mathcal{B}_{sa}$. Now let $\sigma\colon \mathcal{L}_1 \to \mathcal{L}_2$ be a morphism of Lie–Jordan algebras, i.e. $\forall\, a, b \in \mathcal{L}_1$

$$\sigma(a \circ b) = \sigma(a) \circ \sigma(b) \tag{2.3.13}$$
$$\sigma([a,b]) = [\sigma(a), \sigma(b)] \tag{2.3.14}$$

then we can define $\widetilde{\sigma}\colon \mathcal{L}_1^{\mathbb{C}} \to \mathcal{L}_2^{\mathbb{C}}$ as

$$\widetilde{\sigma}(a + ib) = \sigma(a) + i\sigma(b). \tag{2.3.15}$$

Then we have for all $a, b \in \mathcal{L}_1$

$$\begin{aligned} \widetilde{\sigma}(a \cdot b) &= \widetilde{\sigma}(a \circ b - i\sqrt{\kappa}[a,b]) = \sigma(a \circ b) - i\sqrt{\kappa}\sigma([a,b]) && (2.3.16)\\ &= \sigma(a) \circ \sigma(b) - i\sqrt{\kappa}[\sigma(a), \sigma(b)] && (2.3.17)\\ &= \widetilde{\sigma}(a) \cdot \widetilde{\sigma}(b). && (2.3.18) \end{aligned}$$

Therefore we have proved

Theorem 2.3.13. *Given a morphism* $\sigma\colon \mathcal{L}_1 \to \mathcal{L}_2$ *of LJB–algebras, there is a unique extension* $\widetilde{\sigma}$ *of* σ *to a morphism of the corresponding* C^**–algebras* $\widetilde{\sigma} = \mathcal{A}_1\colon \mathcal{L}_1^{\mathbb{C}} \to \mathcal{A}_2 = \mathcal{L}_2^{\mathbb{C}}$ *and* $\widetilde{\mathbb{1}_{\mathcal{L}}} = \mathbb{1}_{\mathcal{L}^{\mathbb{C}}}$*. Moreover the functors from the category* $\mathcal{LJB}$ *of Lie–Jordan Banach algebras in the category* $\mathcal{C}^*\mathcal{A}$ *of* C^**–algebras is an isomorphism of categories.*

3

REDUCTION OF LIE–JORDAN BANACH ALGEBRAS

A degenerate system is a physical system that when it is described mathematically it possesses extra, nonphysical degrees of freedom. Then a complete description of it is usually attained by adding supplementary conditions, by the action of a gauge group on it or by any other mean that allows to identify the true degrees of freedom of the theory. The task of the physicist is to extract the relevant physical subsystem from such a degenerate one. Indeed, physical information such as boundary conditions or constraints is often injected into a theory through the use of supplementary constraints. The treament of degenerate systems in classical mechanics was developed by Dirac [Dir01] who provided an algorithmic procedure and has now reached a high degree of mathematical maturity. It was formalized first by P. Bergmann [Ber61] and later on M. Gotay *et al.* set its geometrical foundation, being known as the presymplectic constraints algorithm [GNH78], [GN79], [GN80]. Later on Marmo, Mendella and Tulczyjew established its simplest geometrical structure by considering it as a consistency condition for implicit differential equations [MMT95]. Later on the algorithm has widespread to other areas like optimal control theory [LM00], [DTI03]. As for the quantum setting, these systems still remain within heuristic formulations without earning much from their classical rigor, due to the dubious nature of quantization. There is however a whole set of very successful ideas in tackling with degenerate systems in quantum field theory grouped under the name of BRST–BFV quantization theories. We will come back to this in the next chapter.

The aim of this chapter is to review first the classical treatment of constraints in the language of differential geometry and translate it in the more abstract algebraic framework, which turns out to be appropriate for both classical and quantum constraints. We then provide a quantum mechanical procedure for eliminating the degeneracy in a mathematically consistent way, by focusing on the algebra of observables and comparing this approach with the T-procedure in the C^*–algebra framework, developed along the years by Grundling and collaborators [GH85], [GH88], [GL00].

3.1. Symplectic reduction and Dirac's theory of constraints

In this section we rephrase Dirac's theory of constraints in the modern language of symplectic geometry. We start discussing the symplectic reduction with respect to a coisotropic submanifold of a symplectic submanifold. Then the parallel with Dirac's theory of constraints is naturally elucidated.

3.1.1. Symplectic reduction

Let (M, Ω) be a symplectic manifold, that is, Ω is a closed non-degenerate 2-form. Fix a point $p \in M$ and consider the vector space T_pM of tangent vectors to M at p. The symplectic form determines a non-degenerate antisymmetric form on T_pM, making it into a **symplectic vector space**. In a symplectic vector space V, we can define four kind of subspaces. Let W be a subspace of V, and denote by $W^\perp$ its **symplectic complement** relative to the symplectic form Ω:

$$W^\perp = \{X \in V \mid \Omega(X, Y) = 0 \quad \forall Y \in W\}. \tag{3.1.1}$$

Subspaces W obeying $W \subseteq W^{\perp}$ are called **isotropic** and they necessarily obey $\dim W \leq \frac{1}{2}\dim V$. On the other hand, if $W \supseteq W^{\perp}$, W is called **coisotropic** and it must obey $\dim W \geq \frac{1}{2}\dim V$. If W is both isotropic and coisotropic, then it is its own symplectic complement, it obeys $\dim W = \frac{1}{2}\dim V$ and it is called a **lagrangian** subspace. Finally, if $W \cap W^{\perp} = 0$, W is called **symplectic**.

Notice that if W is lagrangian, the restriction of Ω to W is identically zero; whereas if W is symplectic, Ω restricts to a symplectic form. In particular, symplectic subspaces are even dimensional. If W is coisotropic, Ω restricts to a nonzero antisymmetric bilinear form on W which, nevertheless, is degenerate since any vector in $W^{\perp} \subseteq W$ is symplectically orthogonal to all of W. But it then follows that the quotient $W/W^{\perp}$ inherits a well defined symplectic form and hence becomes a symplectic vector space. The passage from V to $W/W^{\perp}$ (which is a subquotient) is known as the **symplectic reduction** of V relative to the coisotropic subspace W. In the following we will make this procedure global by generalizing it to symplectic manifolds.

We similarly define a submanifold M_0 to be **isotropic**, **coisotropic**, **lagrangian**, or **symplectic** according to whetever at *all* points $p \in M_0$, the tangent spaces T_pM_0 are isotropic, coisotropic, lagrangian or symplectic subspaces of T_pM respectively.

Suppose now that a submanifold M_0 is a **coisotropic submanifold** of M, let $\iota\colon M_0 \hookrightarrow M$ denote the immersion and $\Omega_0 = i^*\Omega$ the pull back of the symplectic form of M onto M_0. This defines a distribution which we denote by $TM_0^{\perp}$, as follows. For $p \in M_0$ we let $(TM_0^{\perp})_p := (T_pM_0)^{\perp}$. This distribution is involutive: let $X, Y \in TM_0^{\perp}$, for all vector fields Z tangent to M_0, we have that

$$\begin{aligned} 0 &= d\Omega_0(X,Y,Z) && (3.1.2)\\ &= X\Omega_0(Y,Z) - Y\Omega_0(X,Z) + Z\Omega_0(X,Y) && (3.1.3)\\ &\quad -\Omega_0([X,Y],Z) + \Omega_0([X,Z],Y) - \Omega_0([Y,Z],X). && (3.1.4) \end{aligned}$$

But all terms except the fourth are automatically zero since they involve Ω_0 contractions between TM_0 and $TM_0^{\perp}$. Therefore the fourth term is also zero and this implies $[X,Y] \in TM_0^{\perp}\ \forall X,Y \in TM_0^{\perp}$. Therefore, by Frobenius' theorem, $TM_0^{\perp}$ are the tangent space to the leaves of a foliation and we denote by $\pi\colon M_0 \to \widetilde{M}$ the natural surjection mapping the points of M_0 to the unique connected leaf they belong to. Then if $\widetilde{M}$ is a smooth manifold, whose tangent space at a leaf would be isomorphic to $T_pM_0/T_pM_0^{\perp}$ for any point p lying in that leaf. We can therefore give $\widetilde{M}$ a symplectic structure $\widetilde{\Omega}$ by demanding that $\pi^*\widetilde{\Omega} = \Omega_0$. In other words, let $\widetilde{X}$, $\widetilde{Y}$ be vectors tangent to $\widetilde{M}$ at a leaf. To compute $\widetilde{\Omega}(\widetilde{X},\widetilde{Y})$ we merely lift $\widetilde{X}$ and $\widetilde{Y}$ to vectors X_0 and Y_0 tangent to M_0 at a point p in the leaf and then compute $\Omega_0(X_0,Y_0)$. The result is clearly independent of the particular lift since the difference of any two lifts belongs to $TM_0^{\perp}$ and is independent of the particular chosen point p of the leaf since, if Z is a tangent vector to the leaf, the Lie derivative of Ω_0 along Z

$$L_Z\Omega_0 = di_Z\Omega_0 + i_Zd\Omega_0 = 0. \tag{3.1.5}$$

Therefore $(\widetilde{M},\widetilde{\Omega})$ becomes a symplectic manifold and it is called the **symplectic reduction** of (M,Ω) relative to the coisotropic submanifold (M_0,Ω_0).

Suppose now that M_0 is a **symplectic submanifold** of M and let $i\colon M_0 \hookrightarrow M$ denote its inclusion. We can give M_0 a symplectic structure merely by pulling back Ω to M_0. Hence (M_0,Ω_0), $\Omega_0 = i^*\Omega$, becomes a symplectic manifold, called the **symplectic restriction** of M onto M_0. In this case we can obtain explicitly the Poisson bracket of M_0 in terms of the Poisson bracket of M, as we are going to show in the following.

Let f and g be smooth functions on M_0, and let us extend them to smooth functions on M, and we will use the abuse of notation of still calling them f and g. Let X_f and X_g be their respective hamiltonian vector fields on M (see Appendix A). Since M_0 is symplectic, the tangent space of M at every point $p \in M_0$ can be decomposed in the following direct sum

$$T_pM = T_pM_0 \oplus (T_pM_0)^{\perp}, \tag{3.1.6}$$

according to which a vector field X can be decomposed as the sum of two vectors: X_T, tangent to M_0, and $X^{\perp}$ symplectically perpendicular to M_0. The Poisson bracket of the two functions f and g on M_0 is simply given by

$$\{f, g\}_0 = \Omega(X_f - X_f^\perp, X_g - X_g^\perp). \tag{3.1.7}$$

Now suppose that $\{Z_\alpha\}$ is a local basis for $TM_0^\perp$. Then the normal part $X^\perp$ of any vector X can be written

$$X^\perp = \sum_\alpha \lambda_\alpha Z_\alpha. \tag{3.1.8}$$

Then we notice that

$$\Omega(X, Z_\alpha) = \Omega(X^\perp, Z_\alpha) = \sum_\beta \lambda_\beta \Omega(Z_\beta, Z_\alpha) \tag{3.1.9}$$

and define the square matrix M whose entries are $M_{\alpha\beta} = \Omega(Z_\alpha, Z_\beta)$, which is invertible since M_0 is a symplectic submanifold. Hence we call the inverse $M^{\alpha\beta}$ which satisfies

$$\sum_\beta M_{\alpha\beta} M^{\beta\gamma} = \delta_\alpha^\gamma. \tag{3.1.10}$$

It follows that the coefficients λ_α are given by

$$\lambda_\beta = \sum_\alpha \Omega(X, Z_\alpha) M^{\alpha\beta}. \tag{3.1.11}$$

Then by putting Eq. (3.1.11) into Eqs. (3.1.8) and (3.1.7) we obtain

$$\{f, g\}_0 = \{f, g\} - \sum_{\alpha,\beta} \Omega(X_f, Z_\alpha) M^{\alpha\beta} \Omega(Z_\beta, X_g). \tag{3.1.12}$$

If we further assume that the vector fields Z_α are hamiltonian vector fields associated (via Ω) to the funcions χ_α, then

$$\{f, g\}_0 = \{f, g\} - \sum_{\alpha,\beta} \{f, \chi_\alpha\} M^{\alpha\beta} \{\chi_\beta, g\}. \tag{3.1.13}$$

Therefore $\{\cdot, \cdot\}_0$ is nothing but the **Dirac bracket** associated to the constraints χ_α.

3.1.2. First and second class constraints

In this subsection we will show that the submanifold defined by a set of first/second class constraints is respectively coisotropic/symplectic.
Let (M, Ω) be a symplectic manifold on which it is defined a set of smooth functions $\{\psi_a\}$ which are called **constraints**. This means that the allowed "phase space" of the relevant dynamical system is the zero locus of the constraints

$$\{p \in M \mid \psi_a(p) = 0 \quad \forall\, a\}. \tag{3.1.14}$$

Any other set of functions with the same zero locus gives an equivalent description of the physics. This fact will be crucial in the algebraic description of constraints of the subsequent sections.

Following Dirac [Dir01] let us denote by Ψ the linear subspace generated by the $\{\psi_a\}$, and by $\mathcal{J}$ the ideal of $C^\infty(M)$ they generate, i.e. linear combinations of the $\{\psi_a\}$ whose coefficients are arbitrary smooth functions. Then let F be a maximal subspace of Ψ with the property that

$$\{F, \Psi\} \subset \mathcal{J}. \tag{3.1.15}$$

Let $\{\varphi_i\}$ be a basis of F: Dirac defines this functions as **first class constraints**.
Let now define the subspace $S \subset \Psi$ complementary to F to be spanned by the functions $\{\chi_\alpha\}$: Dirac calls these functions **second class constraints**.
Dirac proves that the matrix of functions $\{\chi_\alpha, \chi_\beta\}$ is nowhere degenerate, which is equivalent to the statment that the submanifold defined by the second class constraints is symplectic. In fact, let us define the function $\chi\colon M \to \mathbb{R}^k$ whose components are the second class constraints, i.e.

$$\chi(m) = (\chi_1(m), \ldots, \chi_k(m)) \tag{3.1.16}$$

and assume that the submanifold $N = \chi^{-1}(0)$ is a closed imbedded submanifold of M (see Section 4.1 for the sufficient conditions). Then the vectors tangent to N are precisely those vectors which are perpendicular to the gradients of the constraints. That is, X is a tangent vector to N if and only if $d\chi_\alpha(X) = 0$ for all $\alpha = 1, \dots k$. By definition of hamiltonian vector fields Z_α associated to the constraints, the above condition is

$$X \in TN \Longleftrightarrow \Omega(X, Z_\alpha) = 0 \quad \forall\, \alpha. \tag{3.1.17}$$

It follows that the Z_α span the symplectic complement of TN. Therefore we can restrict ourselves to the symplectic manifold N with the Poisson bracket given by (3.1.13).

We now restrict the first class constraints $\{\varphi_i\}$ to N where they are still first class constraints and we will denote them again by $\{\varphi_i\}$, with a little abuse of notation. We again put them together by defining the function $\varphi\colon N \to \mathbb{R}^l$ and assume that $N_0 \equiv \varphi^{-1}(0)$ is a closed imbedded submanifold. We will now show that N_0 is a coisotropic submanifold of N.
The tangent space to N_0 is again characterized by those vectors which are annihilated by the gradients of the constraints

$$X \in TN_0 \Longleftrightarrow d\varphi_i(X) = 0\, \forall\, i \tag{3.1.18}$$

which, by using the definition of the hamiltonian vector fields X_i associated to the constraints $\{\varphi_i\}$, it translates into

$$TN_0 = \langle X_i \rangle^\perp, \tag{3.1.19}$$

where $\langle X_i \rangle$ is the linear span of the X_is. Since the constraints now are first class, it follows

$$d\varphi_i(X_j) = \{\varphi_i, \varphi_j\} = c_{ij}^k \varphi_k, \tag{3.1.20}$$

which is zero on N_0. Therefore the X_i are tangent to N_0. This is equivalent to

$$TN_0^\perp \subset TN_0 \tag{3.1.21}$$

and hence N_0 is a coisotropic submanifold of N.

There is a slightly more geometrical version of the previous picture. It can be done by combining Gotay's presymplectic embedding theorem [GS81] with the previous discussion. This is, according to the presymplectic embedding theorem, given any presymplectic manifold (C, ω) where exists an essentially unique, symplectic manifold $(S, \widetilde{\Omega})$ such that C is embedded in S, $\iota\colon C \hookrightarrow S$ and $\omega = \iota^*\widetilde{\Omega}$. Then given a symplectic manifold (M, Ω) and a submanifold $C \subset M$, provided that the restriction ω of Ω to C is presymplectic (here we assume constant value of ω), then there exists a symplectic manifold $(S, \widetilde{\Omega})$ and a symplectic map $j\colon S \hookrightarrow M$ such that $j^*\Omega = \widetilde{\Omega}$ and $\iota^*\widetilde{\Omega} = \omega$. Notice that S is defined as a submanifold of M by the ideal $\mathcal{J}_S$ of functions vanishing at S. Because S is symplectic the ideal is generated by second class constraints. Moreover C is defined inside S by another ideal $\mathcal{J}_C$ and because C is coisotropic in S, this ideal consists of first class constraints.

3.2. Reduction of Poisson algebras

The power of the algebraic formalism is that it continues to make sense in situations where the geometry might be singular. The aim of this section is to show how it is possible to recast the symplectic and coisotropic reduction purely in the category of Poisson algebras.

Dual to a manifold M we have the commutative algebra $C^\infty(M)$ of its smooth functions which characterize it completely. To every point $p \in M$ there corresponds a closed maximal ideal $I(p)$ of $C^\infty(M)$ consisting of those functions vanishing at p. It turns out that these are all the maximal closed ideals. So that as a set, the manifold M is the set of maximal closed ideals of $C^\infty(M)$.

Similarly, if $\iota\colon M_0 \hookrightarrow M$ is a submanifold, it can be described by an ideal $I(M_0)$ consisting of the smooth functions vanishing on M_0. Clearly $I(M_0) = \bigcap\limits_{p \in M_0} I(p)$. For the submanifolds described by the regular zero locus of a set of smooth functions, the ideal $I(M_0)$ is generated by the constraints on the manifold. We have the following isomorphism:

$$C^\infty(M_0) \cong C^\infty(M)/I(M_0). \tag{3.2.1}$$

If (M, Ω) is a symplectic manifold and $M_0 \hookrightarrow M$ is a symplectic submanifold then $I(M_0)$ is generated by the second class constraints.

We now provide an algebraic description of the case in which M_0 is coisotropic. Recall that vector fields are derivations of the algebra of functions: $\mathfrak{X}(M) = \operatorname{Der} C^\infty(M)$. From the above isomorphism, a derivation of $C^\infty(M)$ gives ries to a derivation of $C^\infty(M_0)$ if and only if it preserves the ideal $I(M_0)$:

$$\operatorname{Der} C^\infty(M_0) = \{X \in \operatorname{Der} C^\infty(M) \mid X(I(M_0)) \subset I(M_0)\}. \tag{3.2.2}$$

As we have seen in the previous subsection, the vector fields in $TM_0^\perp$ are precisely the hamiltonian vector fields which arise from functions if $I(M_0)$, whence the coisotropy condition $TM_0^\perp \subset TM_0$ becomes the condition that the vanishing ideal is closed under the Poisson bracket: $\{\mathcal{J}, \mathcal{J}\} \subset \mathcal{J}$.
If we denote by Ω_0 the restriction of Ω to M_0, then $\ker \Omega_0$ is an integrable distribution (we assume that the rank of Ω_0 is constant). Then the quotient space $\widetilde{M}$ of M_0 with respect to the connected leaves of $\ker \Omega_0$, inherits a symplectic structure provided it is a manifold. Finally the functions on $\widetilde{M}$ are those functions on M_0 which are constant on the leaves of the foliation defined by $\ker \Omega_0$. Since the tangent vectors to the leaves are the hamiltonian vector fields of functions in $I(M_0)$, we have an isomorphism

$$C^\infty(\widetilde{M}) = \{f \in C^\infty(M_0) \mid \{f, I(M_0)\} = 0\}, \tag{3.2.3}$$

where $\{f, I(M_0)\} = 0$ on M_0. Extending f to a function on M, the isomorphism becomes

$$C^\infty(\widetilde{M}) = \{f \in C^\infty(M_0) \mid \{f, I(M_0)\} \subset I(M_0)\}/I(M_0). \tag{3.2.4}$$

By generalizing these constructions, if we have a Poisson algebra (i.e. an associative Lie–Jordan algebra) $(\mathcal{L}, \{\cdot,\cdot\})$ and an ideal $\mathcal{J}$ with respect to the Jordan product, we can work out the algebraic reduction by taking the normalizer $\mathcal{N}_\mathcal{J}$ with respect to the ideal $\mathcal{J}$

$$\mathcal{N}_\mathcal{J} = \{x \in \mathcal{L} \mid \{x, \mathcal{J}\} \subset \mathcal{J}\} \tag{3.2.5}$$

which is a Poisson subalgebra, and a straightforward computation shows $\mathcal{N}_\mathcal{J} \cap \mathcal{J}$ is its Poisson ideal. Therefore $\widetilde{\mathcal{L}} = \mathcal{N}_\mathcal{J}/(\mathcal{N}_\mathcal{J} \cap \mathcal{J})$ inherits the structure of a Poisson algebra.

The example in which (M, Ω) is a symplectic manifold, $\mathcal{L} = C^\infty(M)$ and $\mathcal{J}$ is the ideal

$$\mathcal{J} = \{f \in C^\infty(M) \ | \ f|_{M_0} = 0\} \tag{3.2.6}$$

shows the connection with the discussion before. By using the second isomorphism theorem for vector spaces

$$\mathcal{N}/(\mathcal{N} \cap \mathcal{J}) \simeq (\mathcal{N} + \mathcal{J})/\mathcal{J} \tag{3.2.7}$$

and taking into account that the quotient by $\mathcal{J}$ can be identified with the restriction to the submanifold M_0, the right hand side can be described as the restriction to M_0 of the functions in $\mathcal{N}_\mathcal{J} + \mathcal{J}$, but $\mathcal{N}_\mathcal{J} + \mathcal{J} = C^\infty(M)$ for second class constraints.

3.2.1. Reduction by symmetries

Suppose that we have a Lie group G acting on a symplectic manifold M and we want to restrict our Poisson algebra to functions that are invariant under the action of the group.

The infinitesimal action of the group induces a family of vector fields $E \subset \mathfrak{X}(M)$ that are an integrable distribution and actually the action of G on M induces a map $\hat{\rho}\colon \mathfrak{g} \to \operatorname{Der} C^\infty(M) = \mathfrak{X}(M)$ which is a Lie algebra homomorphism. Then $E = \hat{\rho}(\mathfrak{g})$. If the action of G on M is faithful then $E \cong \hat{\rho}(\mathfrak{g})$. With these geometric data we introduce the subspace

$$\mathcal{E} = \{f \in C^\infty(M) \mid Xf = 0, \ \forall\, X \in E\} \tag{3.2.8}$$

that is a Jordan subalgebra ($\mathcal{E} \circ \mathcal{E} \subset \mathcal{E}$), but not necessarily a Lie subalgebra. When this is the case, i.e. if

$$\{\mathcal{E}, \mathcal{E}\} \subset \mathcal{E}, \tag{3.2.9}$$

the restrictions of the operations to $\mathcal{E}$ endows it with the structure of a Poisson subalgebra.

From the algebraic point of view the action of vector fields on functions is a derivation of the Jordan algebra product $\circ$:

$$X(f \circ g) = Xf \circ g + f \circ Xg, \tag{3.2.10}$$

and if this derivation is also a Lie derivation:

$$X\{f,g\} = \{Xf,g\} + \{f,Xg\}, \tag{3.2.11}$$

then one easily sees that $\mathcal{E}$ is a Lie subalgebra.

An example of the previous situation happens when E is a family of Hamiltonian vector fields, i.e. there exists a Lie subalgebra $\mathcal{G} \subset \mathcal{C}^\infty(\mathcal{M})$ such that $X \in E$ if and only if there is a $g \in \mathcal{G}$ with $Xf = \{g, f\}$ for any $f \in C^\infty(M)$. This kind of derivations, defined through the Lie product, are called inner derivations, they are always Lie derivations and therefore they define a Lie-Jordan subalgebra with the procedure described above.
Then if $J\colon M \to \mathfrak{g}^*$ denotes the momentum map of the action, for any $\xi \in \mathfrak{g}$, $X_\xi \in E = \hat{\rho}(\mathfrak{g})$ is a Hamiltonian vector field with Hamiltonian $J_\xi = \langle J, \xi\rangle$. Thus we have $X_\xi(f) = \{J_\xi, f\}$ and in this case

$$\begin{aligned} X_\xi\{f,g\} &= \{J_\xi,\{f,g\}\} = \{\{J_\xi,f\},g\} + \{f,\{J_\xi,g\}\} \\ &= \{X_\xi f,g\} + \{f,X_\xi g\}. \end{aligned} \tag{3.2.12}$$

The submanifold $J^{-1}(0)$ (provided that 0 is a regular value of J) is coisotropic and the corresponding reduction is called Marsden–Weinstein reduction [MW74]. Actually Marsden–Weinstein reduction corresponds to reduce with respect to the manifold $J^{-1}(\mu)$, $\mu \in \mathfrak{g}^*$ which now is not coisotropic in general. As a particular instance of this situation consider a symplectic manifold (M, Ω) with a strongly Hamiltonian action of the connected Lie group G.

3.3. More general Poisson reductions

One attempt to combine the previous reductions (by constraints and by symmetries) to define a more general one is contained in [MR86]. We shall rephrase here in algebraic terms the original construction that was presented in geometric language.

The data are an embedded submanifold $\iota : N \to M$ of a Poisson manifold and a subbundle $B \subset T_N M := \iota^*(TM)$. With these data we define the Jordan ideal $\mathcal{I} = \{f \in C^\infty(M) \mid f|_N = 0\}$ as before, and the Jordan subalgebra

$\mathcal{B} = \{f \in C^\infty(M) \mid Xf = 0\,\forall\, X \in \Gamma(B)\}$. The goal is to define an associative Lie-Jordan structure in $\mathcal{B}/(\mathcal{B} \cap \mathcal{I})$.

Following [MR86] we assume that $\mathcal{B}$ is also a Lie subalgebra, then if $\mathcal{B} \cap \mathcal{I}$ is a Lie ideal of $\mathcal{B}$ the sought reduction is possible.

However, the condition that $\mathcal{B}$ is a subalgebra is a rather strong one [FZ08] and, consequently, the reduction procedure is much less general than initially expected. Actually, as we will show, it consists on a succesive application of the reductions introduced in the previous section. One can prove the following result.

Theorem 3.3.1 ([FZ08])**.** *With the previous definitions, if $\mathcal{B}$ is not the whole algebra, i.e. $B \neq 0$, and in addition it is a Lie subalgebra, then the following statements hold:*

a) $\mathcal{B} \subset \mathcal{N} := \{g \in C^\infty(M) \mid \{\mathcal{I}, g\} \subset \mathcal{I}\}$.

b) $\mathcal{B} \cap \mathcal{I}$ *is Poisson ideal of* $\mathcal{B}$.

c) $\mathcal{B}/(\mathcal{B} \cap \mathcal{I})$ *always inherits a Poisson bracket.*

d) Take another $0 \neq B' \subset T_N(M)$ *and define* $\mathcal{B}'$ *accordingly. If* $B \cap TN = B' \cap TN \Leftrightarrow \mathcal{B} + \mathcal{I} = \mathcal{B}' + \mathcal{I}$ *by the second isomorphism theorem we have*

$$\mathcal{B}/(\mathcal{B} \cap \mathcal{I}) \simeq (\mathcal{B} + \mathcal{I})/\mathcal{I} \simeq \mathcal{B}'/(\mathcal{B}' \cap \mathcal{I}) \tag{3.3.1}$$

and the two Poisson brackets induced on $(\mathcal{B} + \mathcal{I})/\mathcal{I}$ *coincide.*

Proof. We prove a) by contradiction. Assume that $\mathcal{B} \not\subset \mathcal{N}$ then there exist functions $f \in \mathcal{B}$, $g \in \mathcal{I}$ and an open set $U \subset N$, such that

$$\{g, f\}(p) \neq 0, \quad \text{for any } p \in U. \tag{3.3.2}$$

But certainly $g^2 \in \mathcal{B}$ as a simple consequence of the Leibniz rule for the action of vector fields. Therefore, using that $\mathcal{B}$ is a Lie subalgebra we have

$$\{g^2, f\} = 2g\{g, f\} \in \mathcal{B} \tag{3.3.3}$$

and due to the fact that $g \in \mathcal{I}$ and $\{g, f\}(p) \neq 0$ this implies $g \in \mathcal{B}_U$, where $\mathcal{B}_U$ is the set of functions whose restriction to U coincide with the restriction of someone in $\mathcal{B}$.

So far we know that $g \in \mathcal{B}_U \cap \mathcal{I}$ and therefore $hg \in \mathcal{B}_U \cap \mathcal{I}$ for any $h \in C^\infty(M)$. But using that $\mathcal{B}_U$ is a Lie subalgebra as it is $\mathcal{B}$ (due to the local character of the Poisson bracket) we have

$$\{hg, f\} = h\{g, f\} + g\{h, f\} \in \mathcal{B}_U \Rightarrow h\{g, f\} \in \mathcal{B}_U \Rightarrow h \in \mathcal{B}_U. \qquad (3.3.4)$$

But h is any function, then $\mathcal{B}_U = C^\infty(M)$ and $B|_U = 0$ which implies $B = 0$ as we assumed that it is a subbundle. This contradicts the hypothesis of the theorem and a) is proved.

b) follows immediately from a). Actually if $\mathcal{B} \subset \mathcal{N}$ we have $\{\mathcal{I}, \mathcal{B}\} \subset \mathcal{I}$ and moreover $\{\mathcal{B}, \mathcal{B}\} \subset \mathcal{B}$. Then $\{\mathcal{I} \cap \mathcal{B}, \mathcal{B}\} \subset \mathcal{I} \cap \mathcal{B}$.

c) is a simple consequence of the fact that $\mathcal{B}$ is a Lie-Jordan subalgebra and $\mathcal{B} \cap \mathcal{I}$ its Lie-Jordan ideal.

To prove d) take $f_i \in \mathcal{B}$ and $f_i' \in \mathcal{B}'$, $i = 1, 2$, such that $f_i + \mathcal{I} = f_i' + \mathcal{I}$. The Poisson bracket in $(\mathcal{B} + \mathcal{I})/\mathcal{I}$ is given by

$$\{f_1 + \mathcal{I}, f_2 + \mathcal{I}\} = \{f_1, f_2\} + \mathcal{I} \in (\mathcal{B} + \mathcal{I})/\mathcal{I}, \qquad (3.3.5)$$

where for simplicity we use the same notation for the Poisson bracket in the different spaces, which should not lead to confusion. We compute now the alternative expression $\{f_i' + \mathcal{I}, f_2' + \mathcal{I}\} = \{f_1', +f_2'\} + \mathcal{I}$. We assumed $f_i' = f_i + g_i$ with $g_i \in \mathcal{I} \cap (\mathcal{B} + \mathcal{B}')$ and therefore, as a consequence of a), we have $\{f_1, g_2\}, \{g_1, f_2\}, \{g_1, g_2\} \in \mathcal{I}$, which implies

$$\{f_1', +f_2'\} + \mathcal{I} = \{f_1, f_2\} + \mathcal{I} \qquad (3.3.6)$$

and the proof is completed. □

Last property implies that the reduction process does not depend effectively on B but only on $B \cap TN$. Actually one can show that this procedure is simply a successive application of the two previous reductions presented before: first we reduce the Poisson bracket by constraints to N and then by symmetries with $E = B \cap TN$.

For completeness we would like to comment on the situation when $B = 0$. In this case $\mathcal{B} = C^\infty(M)$ and, of course, it is always a Lie subalgebra. Under these premises the reduction is not possible unless $\mathcal{I}$ is a Lie ideal which is not the case in general. Anyhow, if the conditions to perform the reduction are met and we consider some $B' \neq 0$ such that $B' \cap TN = 0$ and $\mathcal{B}'$ is a Lie subalgebra, then we obtain again property d) of the theorem: the Poisson brackets induced by $B = 0$ and B' on $\mathcal{B}/\mathcal{I}$ are the same.

The question that arises here actually is, is this reduction the most general one that can be performed using $\mathcal{N}$ and $\mathcal{B}$? Or, in other words, if we are given $\mathcal{N}$ and $\mathcal{B}$ does there exists a more general way to obtain the desired associative Lie-Jordan structure in $\mathcal{B}/(\mathcal{B} \cap \mathcal{I})$ where $\mathcal{B}$ and $\mathcal{I}$ are defined as before?

To answer this question we will rephrase the problem in purely algebraic terms. We shall assume that together with an associative Lie-Jordan algebra we are given a Jordan ideal $\mathcal{I}$ and a Jordan subalgebra $\mathcal{B}$. Of course, a particular example of this is the geometric scenario discussed before. Under these premises $\mathcal{B} \cap \mathcal{I}$ is a Jordan ideal of $\mathcal{B}$ and $\mathcal{B} + \mathcal{I}$ is a Jordan subalgebra, then it is immediate to define Jordan structures on $\mathcal{B}/(\mathcal{B} \cap \mathcal{I})$ and on $(\mathcal{B} + \mathcal{I})/\mathcal{I}$ such that the corresponding projections π_B and π are Jordan homomorphisms. Moreover, the natural isomorphism between both spaces is also a Jordan isomorphism. The problem is whether or not we can also induce a Poisson bracket in the quotient spaces compatible with the Jordan product. One first step to carry out this program is contained in the following theorem.

Theorem 3.3.2 ([FFIM13a]). *Given an associative Lie-Jordan algebra,* $(\mathcal{L}, \circ, \{\ ,\ \})$, *a Jordan ideal* $\mathcal{I}$ *and a Jordan subalgebra* $\mathcal{B}$*, assume*

$$\text{a) } \{\mathcal{B}, \mathcal{B}\} \subset \mathcal{B} + \mathcal{I}, \qquad \text{b) } \{\mathcal{B}, \mathcal{B} \cap \mathcal{I}\} \subset \mathcal{I}, \tag{3.3.7}$$

then the following commutative diagram

$$\begin{tikzcd}
\mathcal{B} \times \mathcal{B} \arrow[rr, "\{\ ,\ \}"] \arrow[d, "\pi_B \times \pi_B"] & & \mathcal{B} + \mathcal{I} \arrow[d, "\pi"'] \\
\mathcal{B}/(\mathcal{B} \cap \mathcal{I}) \times \mathcal{B}/(\mathcal{B} \cap \mathcal{I}) \arrow[r] & \mathcal{B}/(\mathcal{B} \cap \mathcal{I}) \arrow[r, leftrightarrow, "\simeq"] & (\mathcal{B} + \mathcal{I})/\mathcal{I}
\end{tikzcd} \tag{3.3.8}$$

defines a unique bilinear, antisymmetric operation in $\mathcal{B}/(\mathcal{B} \cap \mathcal{I})$ *that satisfies the Leibniz rule.*

Proof. In order to show that we define uniquely an operation we have to check that π_B is onto and that $\ker(\pi_B) \times \mathcal{B}$ and $\mathcal{B} \times \ker(\pi_B)$ are mapped into $\ker(\pi) = \mathcal{I}$. But first property holds because π_B is a projection and the second one is a consequence of (3.3.7,b). The bilinearity of the induced operation follows form the linearity or bilinearity of all the maps involved in the diagram and its antisymmetry derives form that of $\{\ ,\ \}$. Finally Leibniz rule is a consequence of the same property for the original Poisson bracket and the fact that π and π_B are Jordan homomorphisms. □

The problem with this construction is that, in general, the bilinear operation does not satisfy the Jacobi identity as shown in the following example.

Example 3.3.3. Consider $M = \mathbb{R}^3 \times \mathbb{R}^3$, with coordinates $(\mathbf{x}, \mathbf{y})$ and Poisson bracket given by the bivector $\Pi = \sum_{i=1}^{3} \frac{\partial}{\partial_{x_i}} \wedge \frac{\partial}{\partial_{y_i}}$. take $N = \{(0, 0, x_3, \mathbf{y})\}$ and for a given $\lambda \in C^\infty(N)$ define $B = \mathrm{span}\{\partial_{x_1}, \partial_{x_2} - \lambda\partial_{y_1}\} \subset T_N M$ and

$$\mathcal{B} = \{f \in C^\infty(M),\ |\ Xf|_N = 0, \forall\, X \in \Gamma(B)\}. \tag{3.3.9}$$

Notice that $T_N M$ is a direct sum of B and TN, therefore we immediately get

$$\{\mathcal{B}, \mathcal{B}\} \subset \mathcal{B} + \mathcal{I} = C^\infty(M) \quad \text{and} \quad \{\mathcal{B}, \mathcal{B} \cap \mathcal{I}\} \subset \mathcal{I}, \tag{3.3.10}$$

and we meet all the requirements to define a bilinear, antisymmetric operation on $\mathcal{B}/(\mathcal{B} \cap \mathcal{I}) \simeq C^\infty(N)$.

Using coordinates $(x^3, \mathbf{y})$ for N the bivector field is

$$\Pi_N = \frac{\partial}{\partial_{x_3}} \wedge \frac{\partial}{\partial_{y_3}} + \lambda \frac{\partial}{\partial_{y_1}} \wedge \frac{\partial}{\partial_{y_2}} \tag{3.3.11}$$

that does not satisfy the Jacobi identity unless $\partial_{x_3}\lambda = \partial_{y_3}\lambda = 0$.

Now the problem is to supplement (3.3.7) with more conditions to guarantee that the induced operation satisfies all the requirements for a Poisson bracket. We do not know a simple description of the minimal necessary assumption but a rather general scenario is the following proposition:

Proposition 3.3.4 ([FFIM13a]). *Suppose that in addition to the conditions of theorem 3.3.2 we have two Jordan subalgebras $\mathcal{B}_+$, $\mathcal{B}_-$*

$$\mathcal{B}_- \subset \mathcal{B} \subset \mathcal{B}_+ \quad and \quad \mathcal{B}_\pm + \mathcal{I} = \mathcal{B} + \mathcal{I}, \tag{3.3.12}$$

such that

$$\text{a) } \{\mathcal{B}_-, \mathcal{B}_-\} \subset \mathcal{B}_+, \quad \text{b) } \{\mathcal{B}_-, \mathcal{B}_+ \cap \mathcal{I}\} \subset \mathcal{I}. \tag{3.3.13}$$

Then the antisymmetric, bilinear operation induced by (3.3.8) is a Poisson bracket, i.e. it fulfils the Jacobi identity.

Proof. To prove this statement consider any two functions $f_1, f_2 \in \mathcal{B}$ and, for $i = 1, 2$, denote by $f_{i,-}$ a function in $\mathcal{B}_-$ such that $f_i + \mathcal{I} = f_{i,-} + \mathcal{I} \subset \mathcal{B} + \mathcal{I}$. Due to (3.3.7) we know that

$$\{f_{1,-}, f_{2,-}\} + \mathcal{I} = \{f_1, f_2\} + \mathcal{I}, \tag{3.3.14}$$

but if (3.3.13,a) also holds,

$$\{f_{1,-}, f_{2,-}\} \in \mathcal{B}_+, \tag{3.3.15}$$

in addition we have that

$$\{f_{1,-}, f_{2,-}\}_- - \{f_{1,-}, f_{2,-}\} \in \mathcal{B}_+ \cap \mathcal{I}, \tag{3.3.16}$$

and using (3.3.13,b)

$$\{\{f_{1,-}, f_{2,-}\}_-, f_{3,-}\} + \mathcal{I} = \{\{f_{1,-}, f_{2,-}\}, f_{3,-}\} + \mathcal{I}. \tag{3.3.17}$$

Therefore the Jacobi identity for the reduced antisymmetric product derives from that of the original Poisson bracket. □

Notice that the whole construction has been made in algebraic terms and therefore it will have an immediate translation to the quantum realm. But before going to that scenario we reexamine the example before, Ex. 3.3.3, to show how it fits into the general result.

Example 3.3.5. We take definitions and notations from example 3.3.3. Now let $\tilde{\lambda}$ be an arbitrary smooth extension of λ to M, i.e. $\tilde{\lambda} \in C^\infty(M)$ such that $\tilde{\lambda}|_N = \lambda$, we define $E = \mathrm{span}\{\partial_{x_1}, \partial_{x_2} - \tilde{\lambda}\partial_{y_1}\} \subset TM$ and $\mathcal{B}_- = \{f \in C^\infty(M) \mid Xf = 0,\ \forall X \in \Gamma(E)\}$.

If we define $\mathcal{B}_+ = \mathcal{B}$, it is clear that $\mathcal{B}_- \subset \mathcal{B} \subset \mathcal{B}_+$, $\mathcal{B}_\pm + \mathcal{I} = \mathcal{B} + \mathcal{I}$ and $\{\mathcal{B}_-, \mathcal{B}_+ \cap \mathcal{I}\} \subset \mathcal{I}$. But $\{\mathcal{B}_-, \mathcal{B}_-\} \subset \mathcal{B}_+$ if and only if $\partial_{x_3}\lambda = \partial_{y_3}\lambda = 0$.

Therefore, in our construction we can accommodate the most general situation in which the example provides a Poisson bracket. We believe that this is not always the case, but we do not have any further counterexamples.

We want to end this section with a comment on the possible application of the reduction described in this section to quantum systems. In this case the Lie-Jordan algebra is non-associative and due to the associator identity there is a deeper connection between the Jordan and Lie products. As a result the different treatment between the Jordan and the Lie part, that we considered in the case of associative algebras, is not useful any more and the natural thing to do is to consider a more *symmetric* prescription.

3.3.1. Generalized reduction of Lie–Jordan algebras

We propose in this subsection a generalisation of the standard reduction procedure (the quotient of subalgebras by ideals) for Lie–Jordan algebras along similar lines to those followed in the associative case.

The statement of the problem is the following: given a Lie-Jordan algebra $\mathcal{L}$ and two subspaces $\mathcal{B}, \mathcal{S}$ the goal is to induce a Lie-Jordan structure in the quotient space $\mathcal{B}/(\mathcal{B} \cap \mathcal{S})$.

If we assume the following conditions:

$$\mathcal{B} \circ \mathcal{B} \subset \mathcal{B} + \mathcal{S}, \qquad [\mathcal{B}, \mathcal{B}] \subset \mathcal{B} + \mathcal{S}, \tag{3.3.18a}$$

$$\mathcal{B} \circ (\mathcal{B} \cap \mathcal{S}) \subset \mathcal{S}, \qquad [\mathcal{B}, \mathcal{B} \cap \mathcal{S}] \subset \mathcal{S}, \tag{3.3.18b}$$

then a diagram similar to the one in Theorem 3.3.2 allows to induce commutative and antisymmetric bilinear operations in the quotient. Now, in order to fulfil the ternary properties (Jacobi, Leibniz and associator identity) we need more conditions. We can show that, again, it is enough to have two more subspaces $\mathcal{B}_- \subset \mathcal{B} \subset \mathcal{B}_+$ such that $\mathcal{B}_\pm + \mathcal{S} = \mathcal{B} + \mathcal{S}$ and moreover we get the conditions substituting (3.3.18a) and (3.3.18b):

$$\mathcal{B}_- \circ \mathcal{B}_- \subset \mathcal{B}_+, \qquad [\mathcal{B}_-, \mathcal{B}_-] \subset \mathcal{B}_+, \tag{3.3.19a}$$

$$\mathcal{B}_- \circ (\mathcal{B}_+ \cap \mathcal{S}) \subset \mathcal{S}, \qquad [\mathcal{B}_-, (\mathcal{B}_+ \cap \mathcal{S})] \subset \mathcal{S}. \tag{3.3.19b}$$

Then, under these conditions, one can correctly induce a Lie-Jordan structure in the quotient. Conditions (3.3.19a) constitute a weaker version of the notion of Lie–Jordan subalgebra. Actually if $\mathcal{B}_- = \mathcal{B} = \mathcal{B}_+$, then we are just claiming that $\mathcal{B}_-$ is a Lie–Jordan subalgebra. On the other hand conditions (3.3.19b) constitute a weaker version of the notion of ideal. If $\mathcal{B}_- = \mathcal{B} = \mathcal{B}_+$ then (3.3.19b) just implies that S is an ideal of $\mathcal{B}$. Because of this we will say that the pair $\mathcal{B}_-$, $\mathcal{B}_+$ is a **weak Lie–Jordan subalgebra**, and that S is a **weak Lie–Jordan ideal** of $(\mathcal{B}_-, \mathcal{B}_+)$. Then we have proved:

Theorem 3.3.6. *Let $\mathcal{L}$ be a Lie–Jordan algebra and $\mathcal{B}_- \subset \mathcal{B} \subset \mathcal{B}_+$ a weak Lie–Jordan subalgebra and S a weak Lie–Jordan ideal of $(\mathcal{B}_-, \mathcal{B}_+)$. Then $\mathcal{B}/\mathcal{B} \cap S$ inherits a canonical Lie–Jordan structure*

There are at least two aspects of this construction that need more work. The first one is to find examples in which this reduction procedure is relevant, similarly to what we did for the classical case in the previous section. The second problem is of topological nature: given a Banach space structure in the big algebra $\mathcal{L}$, compatible with its operations, we can correctly induce a norm in the quotient provided $\mathcal{B}$ and $\mathcal{S}$ are closed subspaces. However, the induced operations need not to be continuous in general; though they are, if $\mathcal{B}$ is a subalgebra and $\mathcal{S}$ an ideal. The study of more general conditions for continuity and compatibility of the norm will be the subject of further research.

3.4. Quantum constraints and reduction of Lie–Jordan Banach algebras

In this section we show how to deal with **quantum constraints** in the LJB–algebra setting.
In quantum physics the set of constraints is a set $\{A_i,\ i \in I\}$ (I an index set) of operators on some Hilbert space together with a selection condition for the subspace of physical states:

$$\mathcal{H}^C \equiv \overline{\{\psi \mid A_i\psi = 0 \quad \forall\, i \in I\}}. \tag{3.4.1}$$

The set of physical observables is then the set of observables which preserve $\mathcal{H}^C$. The final constrained system is the restriction of this algebra to the subspace $\mathcal{H}^C$. By going into the abstract level of LJB–algebras, one starts with an algebra $\mathcal{L}$ containing all physical observables, and assume that the constraints should appear in $\mathcal{L}$ as a subset $\mathcal{C}$. This assumption is justified by the fact that $\ker A_i = \ker A_i^* A_i$, hence we can assume the constraint algebra to be self-adjoint, this is it belongs to the LJB–algebra $\mathcal{L}$.

Remark. Note that in physical relevant situations the operators A_i are not bounded. Then, by following the seminal works of Grundling *et al.* [GH85], [GL00] some possibilities may arise which we can handle within the algebraic framework:

1. if the A_i are unbounded but essentially selfadjoint, we can take the unitaries $\mathcal{U} \equiv \{\exp(itA_j) \mid t \in \mathbb{R}\}$ and identify the constraint set with $\mathcal{C} = \{U - \mathbb{1} \mid U \in \mathcal{U}\}_{sa}$, i.e. taking the selfadjoint part defined by those unitaries.

2. If the A_i are unbounded and normal, we can identify $\mathcal{C}$ with $\{f(A_j) \mid j \in I\}$, where f is a bounded real valued Borel function with $f^{-1}(0) = \{0\}$.

3. If the A_i are unbounded, closable and not normal, we can replace each A_i with the essentially selfadjoint operator $A_i^* A_i$, which is justified by the fact that as mentioned above $\ker A_i = \ker A_i^* A_i$, reducing then to the case of essentially selfadjoint constraints.

The constraint set $\mathcal{C}$ select the physical state space, also called **Dirac states**

$$\mathcal{S}_D = \{\, \omega \in \mathcal{S}(\mathcal{L}) \mid \omega(c^2) = 0, \quad \forall\, c \in \mathcal{C}\,\}$$

where $\mathcal{S}(\mathcal{L})$ is the state space of $\mathcal{L}$. This is the analogue of selecting the constraint submanifold in the classical reduction. Now, in the algebraic setting, we also need a kind of "generalized" constraint subalgebra, which is a subalgebra of $\mathcal{L}$ which gives rise to the same set of Dirac states. Hence we define the **vanishing subalgebra** $\mathcal{V}$ as:

$$\mathcal{V} = \{\, a \in \mathcal{L} \mid \omega(a^2) = 0, \quad \forall\, \omega \in \mathcal{S}_D \,\}.$$

Proposition 3.4.1. *$\mathcal{V}$ is a non-unital LJB–subalgebra.*

Proof. Let $a, b \in \mathcal{V}$. From (2.2.4) it follows:

$$(a \circ b)^2 = \kappa\, [b, [a \circ b, a]\,] + a \circ (b \circ (a \circ b)\,). \tag{3.4.2}$$

If we introduce $c = [a \circ b, a]$ and $d = b \circ (a \circ b)$, Eq. (3.4.2) becomes:

$$(a \circ b)^2 = \kappa\, [b, c\,] + (a \circ d\,). \tag{3.4.3}$$

From the inequalities (2.2.13)(2.2.14) it is easy to show that if $\omega(a^2) = 0$ then

$$\omega(a \circ b) = 0 = \omega([a, b]) \quad \forall\, b \in \mathcal{L}. \tag{3.4.4}$$

Then if we apply the state ω to the expression (3.4.3), from (3.4.4) it follows:

$$\omega(\,(a \circ b)^2\,) = \kappa\, \omega([b, c\,]) + \omega(a \circ d\,) = 0. \tag{3.4.5}$$

By definition of $\mathcal{V}$, this means that $\forall\, a, b \in \mathcal{V}$, $a \circ b \in \mathcal{V}$.
By applying the state ω to the relation

$$(a \circ b)^2 - \kappa\, [a, b]^2 = a \circ (b \circ (a \circ b)) - \kappa\, a \circ [b, [a, b]\,],$$

we obtain $\omega(\,[a, b]^2\,) = \omega((a \circ b)^2) = 0$, that is $\forall\, a, b \in \mathcal{V}$, $[a, b] \in \mathcal{V}$. Hence $\mathcal{V}$ is a Lie–Jordan subalgebra.
$\mathcal{V}$ also inherits the Banach structure since it is defined as the intersection of closed subspaces. □

We can use the vanishing subalgebra to give an alternative description of the Dirac states that will be useful in the sequel.

Proposition 3.4.2. *With the previous definitions we have*

$$\mathcal{S}_D = \{\omega \in \mathcal{S}(\mathcal{L}) \mid \omega(a) = 0, \quad \forall a \in \mathcal{V}\}$$

Proof. As $\mathcal{V}$ is a subalgebra and it contains $\mathcal{C}$ it is clear that the right hand side is included into $\mathcal{S}_D$.

To see the other inclusion it is enough to consider that for any state $\omega(a)^2 \leq \omega(a^2)$, therefore any Dirac state should vanish on $\mathcal{V}$. □

Define now the Lie normalizer as

$$\mathcal{N}_{\mathcal{V}} = \{\, a \in \mathcal{L} \mid [a, \mathcal{V}] \subset \mathcal{V} \,\} \tag{3.4.6}$$

which corresponds roughly to Dirac's concept of "first class variables" [Dir01].

Proposition 3.4.3. *$\mathcal{N}_{\mathcal{V}}$ is a unital LJB–algebra and $\mathcal{V}$ is a Lie–Jordan ideal of $\mathcal{N}_{\mathcal{V}}$.*

Proof. Let $a, b \in \mathcal{N}_{\mathcal{V}}$ and $v \in \mathcal{V}$. Then by definition of normalizer it immediately follows:

$$[[a, b], v] = [[a, v], b] + [[v, b], a] \in \mathcal{V}. \tag{3.4.7}$$

Let us now prove that $\forall\, v \in \mathcal{V},\ v \circ a \in \mathcal{V}$, this is $\mathcal{V}$ is a Jordan ideal of $\mathcal{N}_{\mathcal{V}}$:

$$\omega((v \circ a)^2) = \kappa\, \omega([a, [v \circ a, v]]) + \omega(v \circ (a \circ (a \circ v))) \tag{3.4.8}$$

which gives zero by repeated use of properties (2.2.13) and (2.2.14).
Then it becomes easy to prove that $\mathcal{N}_{\mathcal{V}}$ is a Jordan subalgebra:

$$[a \circ b, v] = [a, v] \circ b + a \circ [b, v] \in \mathcal{V}. \tag{3.4.9}$$

Finally, since the Lie bracket is continuous with respect to the Banach structure, it also follows that $\mathcal{N}_{\mathcal{V}}$ inherits the Banach structure by completeness. □

In the spirit of Dirac, the physical algebra of observables in the presence of the constraint set $\mathcal{C}$ is represented by the LJB–algebra $\mathcal{N}_{\mathcal{V}}$ which can be reduced by the closed Lie–Jordan ideal $\mathcal{V}$ which induces a canonical Lie–Jordan algebra structure in the quotient:

$$\widetilde{\mathcal{L}} = \mathcal{N}_{\mathcal{V}} / \mathcal{V}. \tag{3.4.10}$$

We will denote in the following the elements of $\widetilde{\mathcal{L}}$ by $\widetilde{a}$.
The quotient Lie-Jordan algebra $\widetilde{\mathcal{L}}$ carries the quotient norm,

$$\|\widetilde{a}\| = \|\,[a]\,\| = \inf_{b\in\mathcal{V}} \|a+b\|,$$

where $a \in \mathcal{N}_{\mathcal{V}}$ is an element of the equivalence class $[a]$ of $\mathcal{N}_{\mathcal{V}}$ with respect to the ideal $\mathcal{V}$. The quotient norm provides a LJB–algebra structure to $\widetilde{\mathcal{L}}$.

Hence the reduction of the Lie–Jordan algebra $\mathcal{L}$ with respect to the constraint set $\mathcal{C}$ is given by the short exact sequence

$$0 \to \mathcal{V} \to \mathcal{N}_{\mathcal{V}} \to \widetilde{\mathcal{L}} \to 0. \tag{3.4.11}$$

In the Subsection 3.4.2 we will prove that the states on the reduced LJB–algebra $\widetilde{\mathcal{L}}$ are exactly the Dirac states restricted to the physical algebra of observables $\mathcal{N}_{\mathcal{V}}$.

3.4.1. Reduction of Lie–Jordan Banach algebras and constraints in C^*–algebras

Following [GH85], [GL00] we briefly recall how to deal with quantum constraints in a C^*–algebra setting. The aim of this section is to prove that the reduction procedure of C^*–algebras used to analyze quantum constraints, also called T–reduction, can be equivalently described by using the theory of reduction of LJB–algebras discussed above.

A quantum system with constraints is a pair $(\mathcal{F}, \mathcal{C})$ where now the field algebra $\mathcal{F}$ is a unital C^*–algebra containing the self–adjoint constraint set $\mathcal{C}$, i.e. $C = C^*\ \forall\, C \in \mathcal{C}$. The constraints select the Dirac states

$$\mathcal{S}_D \equiv \{\, \omega \in \mathcal{S}(\mathcal{F}) \mid \omega(C^2) = 0, \quad \forall\, C \in \mathcal{C}\,\}$$

where $\mathcal{S}(\mathcal{F})$ is the state space of $\mathcal{F}$.

Define $\mathcal{D} = [\mathcal{F}\mathcal{C}] \cap [\mathcal{C}\mathcal{F}]$ where the notation $[\cdot]$ denotes the closed linear space generated by its argument and satisfies the following

Theorem 3.4.4. *$\mathcal{D}$ is the largest non-unital C^*–algebra in* $\displaystyle\bigcap_{\omega\in\mathcal{S}_D} \ker\omega$.

For any set $\Omega \subset \mathcal{F}$, define as before its normalizer or "weak commutant" as

$$\Omega_W = \{\, F \in \mathcal{F} \mid [F, H] \subset \Omega, \quad \forall\, H \in \Omega \,\}. \tag{3.4.12}$$

Consider now the multiplier algebra of Ω as

$$\mathcal{M}(\Omega) = \{\, F \in \mathcal{F} \mid FH \in \Omega \text{ and } HF \in \Omega, \quad \forall\, H \in \Omega \,\} \tag{3.4.13}$$

i.e. the largest set for which Ω is a bilateral ideal. $\mathcal{M}(\Omega)$ is clearly an unital C^*–algebra and we have the following

Theorem 3.4.5. $\mathcal{O} \equiv \mathcal{D}_W = \mathcal{M}(\mathcal{D})$.

That is, the weak commutant of $\mathcal{D}$ is also the largest set for which $\mathcal{D}$ is a bilateral ideal and it will be denoted by $\mathcal{O}$. It follows that the maximal (and unital) C^*–algebra of physical observables determined by the constraints $\mathcal{C}$ is given by:

$$\widetilde{\mathcal{F}} = \mathcal{O}/\mathcal{D}. \tag{3.4.14}$$

To show that this procedure is equivalent to the reduction of the corresponding LJB–algebra (as discussed in Section 3.4), we need to prove some simple statements.

Lemma 3.4.6. *Let $\mathcal{Z}$ and $\mathcal{I}$ be two Lie-Jordan subalgebras of a LJB–algebra $\mathcal{L}$. Then $\mathcal{Z}^{\mathbb{C}} = \mathcal{Z} \oplus i\mathcal{Z}$ is the weak commutant (or Lie normalizer) of $\mathcal{I}^{\mathbb{C}} = \mathcal{I} \oplus i\mathcal{I}$ if and only if $\mathcal{Z}$ is the Lie normalizer of $\mathcal{I}$, i.e. $\mathcal{Z} = \mathcal{N}_{\mathcal{I}}$.*

Proof. Assume first $\mathcal{Z}^{\mathbb{C}}$ is the weak commutant of $\mathcal{I}^{\mathbb{C}}$ and let $a + ib \in \mathcal{Z}^{\mathbb{C}}$ with $a, b \in \mathcal{Z}$. By definition

$$[a + ib, \mathcal{I} \oplus i\mathcal{I}] \subset \mathcal{I} \oplus i\mathcal{I}$$

that is

$$[a + b, \mathcal{I}] \subset \mathcal{I} \quad \text{and} \quad [a - b, \mathcal{I}] \subset \mathcal{I}.$$

Since the normalizer is a vector space, this implies

$$[a, \mathcal{I}] \subset \mathcal{I} \quad \text{and} \quad [b, \mathcal{I}] \subset \mathcal{I}, \quad \forall\, a, b \in \mathcal{Z},$$

that is $\mathcal{Z}$ is the Lie normalizer of $\mathcal{I}$. Conversely assume $\mathcal{Z}$ is the Lie normalizer of $\mathcal{I}$:

$$[a, \mathcal{I}] \subset \mathcal{I}, \quad \forall\, a \in \mathcal{Z},$$

then it follows

$$[a + ib, x + iy] \in \mathcal{I}, \quad \forall\, a, b \in \mathcal{Z} \quad \text{and} \quad \forall\, x, y \in \mathcal{I},$$

that is $\mathcal{Z}^{\mathcal{C}}$ is the weak commutant (or Lie normalizer) of $\mathcal{I}^{\mathcal{C}}$. □

Lemma 3.4.7. *Let $\mathcal{Z}$ and $\mathcal{I}$ be two Lie-Jordan subalgebras of $\mathcal{L}$. Then $\mathcal{I}$ is a Lie–Jordan ideal of $\mathcal{Z}$ if and only if $\mathcal{I}^{\mathcal{C}} = \mathcal{I} \oplus i\mathcal{I}$ is an associative bilateral ideal of $\mathcal{Z}^{\mathcal{C}} = \mathcal{Z} \oplus i\mathcal{Z}$.*

Proof. Using the expressions provided by eqs. (2.3.8) and (2.3.9), the statement becomes an easy computation. □

Let us define $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ such that $\mathcal{F} = \mathcal{L} \oplus i\mathcal{L}$ and $\widetilde{\mathcal{F}} = \widetilde{\mathcal{L}} \oplus i\widetilde{\mathcal{L}}$, i.e. they are the self-adjoint part of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ respectively. From Corollary 2.3.11 it follows that $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are unital LJB–algebras. Similarly define the LJB–algebras $\mathcal{N}_{\mathcal{J}}$ and $\mathcal{J}$ as the self-adjoint parts of $\mathcal{O}$ and $\mathcal{D}$ respectively, i.e. $\mathcal{O} = \mathcal{N}_{\mathcal{J}} \oplus i\mathcal{N}_{\mathcal{J}}$, $\mathcal{D} = \mathcal{J} \oplus i\mathcal{J}$.

Theorem 3.4.8. *With the notations above, let $\mathcal{F} = \mathcal{L} \oplus i\mathcal{L}$ be the field algebra of the quantum system and $\mathcal{C}$ a real constraint set. Let $\mathcal{D} = [\mathcal{F}\mathcal{C}] \cap [\mathcal{C}\mathcal{F}]$, $\mathcal{O} = \mathcal{D}_W$ be as in Thm. 3.4.5, and $\widetilde{\mathcal{F}} = \mathcal{O}/\mathcal{D} = \widetilde{\mathcal{L}} \oplus i\widetilde{\mathcal{L}}$ be the reduced field algebra. Then:*

$$\widetilde{\mathcal{L}} = \mathcal{N}_{\mathcal{V}}/\mathcal{V},$$

with $\mathcal{V}$ and $\mathcal{N}_{\mathcal{V}}$ being the vanishing subalgebra of $\mathcal{L}$ and its Lie normalizer respectively.

Proof. Observe that the space of states on $\mathcal{F}$ is the space of states on $\mathcal{L}$ extended linearly by complexification and conversely $\mathcal{S}(\mathcal{L}) = \mathcal{S}(\mathcal{F})|_{\mathcal{L}}$. Then from Thm. 3.4.4 it follows that $\mathcal{D}$ is exactly the vanishing subalgebra for $\mathcal{S}_D$, that is $\mathcal{D} = \mathcal{V} \oplus i\mathcal{V}$. Then from the Lemmas 3.4.6 and 3.4.7 everything goes straightforward and the two procedures are clearly equivalent. □

The equivalence of the two approaches can be illustrated pictorially by the following "functorial" diagramme:

$$\begin{tikzcd}
\mathcal{L} \arrow[r, dash] \arrow[d, "{\mathcal{J}, \mathcal{N}_{\mathcal{J}}}"'] & \mathcal{F} = \mathcal{L} \oplus i\mathcal{L} \arrow[d, "{\mathcal{D}, \mathcal{O}}"] \\
\widetilde{\mathcal{L}} \arrow[r, dash] & \widetilde{\mathcal{F}} = \widetilde{\mathcal{L}} \oplus i\widetilde{\mathcal{L}}
\end{tikzcd}$$

3.4.2. The space of states of the reduced LJB–algebra

The purpose of the remaining two sections is to discuss the structure of the space of states and the GNS construction of reduced states for reduced LJB–algebras with respect to the space of states of the unreduced one.

As it was discussed in the previous section, let $\mathcal{A}$ be a C^*–algebra, $\mathcal{L} = \mathcal{A}_{\rm sa}$ its real part and $\mathcal{V}$ the vanishing subalgebra of $\mathcal{L}$ with respect to a constraint set $\mathcal{C}$ and let $\mathcal{N}_{\mathcal{V}}$ be the Lie normalizer of $\mathcal{V}$. Then we will denote as before by $\widetilde{\mathcal{L}}$ the reduced Lie–Jordan Banach algebra $\mathcal{N}_{\mathcal{V}}/\mathcal{V}$ and its elements by $\widetilde{a}$.

Let $\widetilde{\mathcal{S}} = \mathcal{S}(\widetilde{\mathcal{L}})$ be the state space of the reduced LJB–algebra $\widetilde{\mathcal{L}}$, i.e. $\widetilde{\omega} \in \widetilde{\mathcal{S}}$ means that $\widetilde{\omega}(\widetilde{a}^2) \geq 0 \; \forall \widetilde{a} \in \widetilde{\mathcal{L}}$, and $\widetilde{\omega}$ is normalized. Notice that if $\mathcal{L}$ is unital, then $\mathbb{1} \in \mathcal{N}_{\mathcal{V}}$ and $\mathbb{1} + \mathcal{V}$ is the unit element of $\widetilde{\mathcal{L}}$. We will denote it by $\widetilde{\mathbb{1}}$.
We have the following:

Lemma 3.4.9. *There is a one-to-one correspondence between normalized positive linear functionals on $\widetilde{\mathcal{L}}$ and normalized positive linear functionals on $\mathcal{N}_{\mathcal{V}}$ vanishing on $\mathcal{V}$.*

Proof. Let $\omega' \colon \mathcal{N}_{\mathcal{V}} \to \mathbb{R}$ be positive. The positive cone on $\widetilde{\mathcal{L}}$ consists of elements of the form $\widetilde{a}^2 = (a + \mathcal{V})^2 = a^2 + \mathcal{V}$, i.e.

$$\mathcal{K}^+_{\widetilde{\mathcal{L}}} = \{\, a^2 + \mathcal{V} \mid a \in \mathcal{N}_{\mathcal{V}} \,\} = \mathcal{K}^+_{\mathcal{N}_{\mathcal{V}}} + \mathcal{V}.$$

Thus if ω' is positive on $\mathcal{N}_{\mathcal{V}}$, $\omega'(a^2) \geq 0$, hence:

$$\omega'(a^2 + \mathcal{V}) = \omega'(a^2) + \omega'(\mathcal{V})$$

and if ω' vanishes on the closed ideal $\mathcal{V}$, then ω' induces a positive linear functional on $\widetilde{\mathcal{L}}$. Clearly ω' is normalized then the induced functional is normalized too because $\widetilde{\mathbb{1}} = \mathbb{1} + \mathcal{V}$.

Conversely, if $\widetilde{\omega}\colon \widetilde{\mathcal{L}} \to \mathbb{R}$ is positive and we define

$$\omega'(a) = \widetilde{\omega}(a + \mathcal{V})$$

then ω' is well-defined, positive, normalized and $\omega'|_{\mathcal{V}} = 0$. □

Notice also that given a positive linear functional on $\mathcal{N}_{\mathcal{V}}$ there exists an extension of it to $\mathcal{L}$ which is positive too.

Lemma 3.4.10. *Given a closed Jordan subalgebra $\mathcal{Z}$ of a LJB–algebra $\mathcal{L}$ such that $\mathbb{1} \in \mathcal{Z}$ and ω' is a normalized positive linear functional on $\mathcal{Z}$, then there exists $\omega\colon \mathcal{L} \to \mathbb{R}$ such that $\omega(a) = \omega'(a),\ \forall\, a \in \mathcal{Z}$ and $\omega \geq 0$.*

Proof. Since $\mathcal{L}$ is a JB–algebra, it is also a Banach space. Due to the Hahn–Banach extension theorem, there exists a continuous extension ω of ω', i.e. $\omega(a) = \omega'(a),\ \forall\, a \in \mathcal{Z}$, and moreover $\|\omega\| = \|\omega'\|$.

From the equality of norms and the fact that ω' is positive we have $\|\omega\| = \omega'(\mathbb{1})$, but ω is an extension of ω' then $\|\omega\| = \omega(\mathbb{1})$, which implies that ω is a positive functional and satisfies all the requirements stated in the lemma. □

We can now prove the following:

Theorem 3.4.11. *The set $\mathcal{S}_D(\mathcal{N}_{\mathcal{V}})$ of Dirac states on $\mathcal{L}$ restricted to $\mathcal{N}_{\mathcal{V}}$ is in one-to-one correspondence with the space of states of the reduced LJB–algebra $\widetilde{\mathcal{L}}$.*

Proof. In Prop. 3.4.2 we characterised the Dirac states as those that vanish on $\mathcal{V}$. Combining this result with that of Lemma 3.4.9 the proof follows. □

3.4.3. The GNS representation of reduced states

Finally, we will describe the GNS representation of a reduced state in terms of data from the unreduced LJB–algebra. Let $\widetilde{\mathcal{L}}$ be, as before, the reduced LJB–algebra of $\mathcal{L}$ with respect to the constraint set $\mathcal{C}$. Denote by $\widetilde{\mathcal{A}} = \widetilde{\mathcal{L}} \oplus i\widetilde{\mathcal{L}}$ the corresponding C^*–algebra and by $\widetilde{\mathcal{S}}$ its state space. Let $\widetilde{\omega} \in \widetilde{\mathcal{S}}$ be a normalized state on $\widetilde{\mathcal{A}}$. The GNS representation of $\widetilde{\mathcal{A}}$ associated to the state $\widetilde{\omega}$, denoted by

$$\pi_{\widetilde{\omega}} \colon \widetilde{\mathcal{A}} \to \mathcal{B}(\mathbf{H}_{\widetilde{\omega}}),$$

is defined as

$$\pi_{\widetilde{\omega}}(\widetilde{A})(\widetilde{B} + \mathcal{J}_{\widetilde{\omega}}) = \widetilde{A}\widetilde{B} + \mathcal{J}_{\widetilde{\omega}}, \quad \forall\, \widetilde{A}, \widetilde{B} \in \widetilde{\mathcal{A}},$$

where the Hilbert space $\mathbf{H}_{\widetilde{\omega}}$ is the completion of the pre–Hilbert space defined on $\widetilde{\mathcal{A}}/\mathcal{J}_{\widetilde{\omega}}$ by the inner product

$$\langle \widetilde{A} + \mathcal{J}_{\widetilde{\omega}}, \widetilde{B} + \mathcal{J}_{\widetilde{\omega}} \rangle \equiv \widetilde{\omega}(\widetilde{A}^* \widetilde{B})$$

and $\mathcal{J}_{\widetilde{\omega}} = \{\widetilde{A} \in \widetilde{\mathcal{A}} \mid \widetilde{\omega}(\widetilde{A}^* \widetilde{A}) = 0\}$ is the Gelfand left-ideal of $\widetilde{\omega}$. Let ω be a state on $\mathcal{A} = \mathcal{L} \oplus i\mathcal{L}$ that extends the state ω' on $\mathcal{N}_{\mathcal{V}}^{\mathbb{C}}$ induced by $\widetilde{\omega}$ according to Lemmas 3.4.9 and 3.4.10. Notice that ω vanishes on $\mathcal{V}$, thus the Gelfand ideal $\mathcal{J}_\omega$ of ω contains $\mathcal{V}$. We will have then:

Theorem 3.4.12. *There is a unitary equivalence between $\mathbf{H}_{\widetilde{\omega}}$ and the completion of the pre–Hilbert space:*

$$\mathbf{H}' = \mathcal{N}_{\mathcal{V}}^{\mathbb{C}} \big/ \mathcal{N}_{\mathcal{V}}^{\mathbb{C}} \cap \mathcal{J}_\omega$$

with the inner product defined by

$$\langle A + \mathcal{N}_{\mathcal{V}}^{\mathbb{C}} \cap \mathcal{J}_\omega, B + \mathcal{N}_{\mathcal{V}}^{\mathbb{C}} \cap \mathcal{J}_\omega \rangle' \equiv \omega(A^* B), \quad \forall\, A, B \in \mathcal{N}_{\mathcal{V}}^{\mathbb{C}}.$$

Proof. Notice first that $\langle \cdot, \cdot \rangle'$ is well defined because of the properties of the Gelfand ideal $\mathcal{J}_\omega$. Morover we have that

$$\mathbf{H}_{\widetilde{\omega}} = \widetilde{\mathcal{A}}/\mathcal{J}_{\widetilde{\omega}}$$

and from Thm. 3.4.8, $\widetilde{\mathcal{A}} = \mathcal{N}_{\mathcal{V}}^{\mathbb{C}}/\mathcal{V}^{\mathcal{C}}$ and $\mathcal{J}_{\widetilde{\omega}} = \mathcal{J}_{\omega'}/\left(\mathcal{J}_{\omega'} \cap \mathcal{V}^{\mathcal{C}}\right)$.
Hence because $\mathcal{J}_{\omega'} = \mathcal{N}_{\mathcal{V}}^{\mathbb{C}} \cap \mathcal{J}_{\omega}$ and $\mathcal{V}^{\mathcal{C}} \subset \mathcal{J}_{\omega'}$, we have:

$$\begin{aligned}\mathbf{H}_{\widetilde{\omega}} &= \widetilde{\mathcal{A}}/\mathcal{J}_{\widetilde{\omega}} = \left(\mathcal{N}_{\mathcal{V}}^{\mathbb{C}}\big/\mathcal{V}^{\mathcal{C}}\right)\Big/\left(\mathcal{N}_{\mathcal{V}}^{\mathbb{C}} \cap \mathcal{J}_{\omega}\big/\mathcal{V}^{\mathcal{C}}\right) \\ &\cong \mathcal{N}_{\mathcal{V}}^{\mathbb{C}}\Big/\mathcal{N}_{\mathcal{V}}^{\mathbb{C}} \cap \mathcal{J}_{\omega}\,.\end{aligned}$$

□

Notice that

$$\mathbf{H}' = \mathcal{N}_{\mathcal{V}}^{\mathbb{C}}\Big/\mathcal{N}_{\mathcal{V}}^{\mathbb{C}} \cap \mathcal{J}_{\omega} \cong \left(\mathcal{N}_{\mathcal{J}}^{\mathcal{C}} + \mathcal{J}_{\omega}\right)\Big/\mathcal{J}_{\omega}\,.$$

Thus the reduced GNS construction corresponding to the state $\widetilde{\omega}$ is the GNS construction of any extension ω of $\widetilde{\omega}$ restricted to $\mathcal{N}_{\mathcal{V}}^{\mathbb{C}} + \mathcal{J}_{\omega}$. Notice that $\widetilde{\omega}$ will be a pure state if and only if $\pi_{\widetilde{\omega}}$ is irreducible, i.e. if the representation of π_{ω} of ω restricted to $\mathcal{N}_{\mathcal{V}}^{\mathbb{C}} + \mathcal{J}_{\omega}$ is irreducible. Then if $\mathcal{N}_{\mathcal{V}}^{\mathbb{C}} + \mathcal{J}_{\omega} = \mathcal{A}$, π_{ω} will be irreducible if ω is a pure state. If $\mathcal{N}_{\mathcal{V}}^{\mathbb{C}} + \mathcal{J}_{\omega} \subsetneq \mathcal{A}$, then the state ω extending $\widetilde{\omega}$ might be non pure.

4

BRST symmetry and reduction of supersymmetric Lie–Jordan Banach algebras

Since Becchi, Rouet, Stora [BRS74] and Tyutin [Tyu75] introduced today's so called BRST symmetry, it has been proven to be very useful to quantize degenerate systems with gauge symmetries. An important property of the BRST symmetry is that it can be described geometrically and as we will see in this chapter algebraically in the setting of Lie–Jordan Banach algebras. In its classical setting the BRST symmetry describes the structure of first class constraints in phase space. We will extend in this chapter such ideas to the context of Lie-Jordan Banach algebras of observables where the notion of first class submanifold is substituted by a closed ideal of the LJB–structure.

The fundamental idea of this mechanism consists in introducing auxiliary degrees of freedom, called "ghosts" and "antighosts" and replace the local gauge symmetry by a global supersymmetry generated by a single operator called the "supercharge", whose square vanishes, and transforming the symmetry in a (co-)homology theory. Then the role of this operation is to select the "true" physical

states of the theory as we will see later on.
We will be here interested in extending the standard BRST symmetry to the setting of Lie–Jordan Banach algebras. In this sense this can be considered an "intrinsic" quantum approach to BRST where we are not using any quantization scheme. The aim of this effort is to emphasize the algebraic nature of the theory and make it useful in other areas of Mathematics and Physics.
There are various presentations of the BRST techniques. We will adopt here a generalization of the Hamiltonian approach (see for instance [Hen85], [HT88]), but we will reduce the construction to its bare essentials in order to make clear the basic ingredients. Actually, as it will be shown, the Hamiltonian structure is not needed at the classical level and the construction is purely geometric. The Hamiltonian structure is substituted by a group action. However the Hamiltonian picture will surface again in the LJB–algebra setting where it is intrinsic to the model.
Thus we will review first the well-established BRST symmetry and then we will move to the abstract realm of operator algebras.

4.1. The classical BRST symmetry

We will consider first, as in the Hamiltonian setting, a smooth manifold M and a submanifold $C \subset M$ defined by certain set of constraint functions. We will assume that C is a regular submanifold, or in other words, that there is a smooth function $\Phi \colon M \to V^*$, where V is a finite dimensional vector space (so it is V^*) such that $C = \Phi^{-1}(0)$ and 0 a regular value of Φ. Let us recall that 0 is a regular value of Φ if $\forall m \in C$, the tangent map $T\Phi(m)\colon\ T_mM \to T_0V^* \cong V^*$ is surjective. Under these conditions, C is an embedded submanifold of M and the tangent space to C at $m \in C$ is just the kernel of the map $T\Phi(m)$

$$T_mC = \ker T\Phi(m) \tag{4.1.1}$$

Remark. The regularity condition for $0 \in V^*$ it is often too strong, however it can be replaced by a weaker condition, just asking that 0 is a weakly regular value of Φ, that is:

i) $C = \Phi^{-1}(0)$ is a regular submanifold of M, and

ii) $\forall\, m \in C, T_m C = \ker T\Phi(m)$, i.e. Eq. (4.1.1) holds.

Now we may consider that there is a Lie group G acting on M leaving invariant Φ, i.e. $\Phi(g \cdot m) = \Phi(m)\,\forall\, g \cdot m \in G,\ \forall\, m \in G$. We want to describe the space of orbits C/G. Even better, we will try to describe the space of G–invariant functions on G. That will provide a description of the space C/G suitable for algebraic generalizations.

4.2. The classical BRST complex

We will assume as before that we have a smooth manifold M and a submanifold $C = \Phi^{-1}(0)$ described as the zero level set of a map $\Phi\colon\ M \to V^*$, such that 0 is a weak regular value of Φ. We will also assume that there is a Lie group G acting smoothly on M and such that it preserves the submanifold C, i.e. if $x \in C$ then $g \cdot x \in C \quad \forall\, g \in G$. We will show in what follows that the algebra of smooth functions on C/G can be identified with the zeroth order cohomology group for a differential operator $D = d_1 + d_2$ defined on a double complex (S, d_1, d_2) which is obtained as the tensor product of the **Chevalley complex** $(\Lambda^\bullet \mathfrak{g}^*, d_2)$ of the Lie group G and a certain **Koszul complex** $(K^\bullet, d_1)$ described below.
This construction combines both algebraic and geometric aspects. Let us begin with the algebraic ones.

Let $\mathfrak{g}$ be the finite dimensional Lie algebra of the Lie group G and $\Lambda^\bullet \mathfrak{g}^* = \bigoplus_{k\geq 0} \Lambda^k \mathfrak{g}^*$ be the exterior or Grassmanian algebra over its dual space $\mathfrak{g}^*$. Elements in $\Lambda^\bullet \mathfrak{g}^*$ will be denoted by $\alpha, \beta, \ldots$ and they are in one-to-one correspondence with left (or right) invariant forms on the group G. Clearly $\Lambda^\bullet \mathfrak{g}^*$ is a graded commutative associative algebra of dimension $2^{\dim \mathfrak{g}}$ with respect to the exterior product $\wedge$. Consider now the unique derivation of degree one over $\Lambda^\bullet \mathfrak{g}^*$

$$d_2\colon\ \Lambda^k \mathfrak{g}^* \to \Lambda^{k+1} \mathfrak{g}^* \tag{4.2.1}$$

such that

$$(d_2\alpha)(\xi,\eta) = -\langle \alpha, [\xi,\eta]\rangle \tag{4.2.2}$$

where $\alpha \in \Lambda^1\mathfrak{g}^*$ and $\xi,\eta \in \mathfrak{g}$. Notice that and if α is an homogeneous element of $\Lambda^\bullet\mathfrak{g}^*$ of degree $|\alpha|$

$$d_2(\alpha\wedge\beta) = d_2\alpha\wedge\beta + (-1)^{|\alpha|}\alpha\wedge d_2\beta. \tag{4.2.3}$$

Then it is easy to check that $d_2^2 = 0$, hence $(\Lambda^\bullet\mathfrak{g}^*, d_2)$ defines a differential complex, depicted by the sequence:

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{g}^* \xrightarrow{d_2} \Lambda^2\mathfrak{g}^* \xrightarrow{d_2} \Lambda^3\mathfrak{g}^* \xrightarrow{d_2} \cdots \xrightarrow{d_2} \Lambda^k\mathfrak{g}^* \xrightarrow{d_2} \cdots \xrightarrow{d_2} \Lambda^r\mathfrak{g}^* \xrightarrow{d_2} 0, \tag{4.2.4}$$

where $r = \dim\mathfrak{g}$ and $d_2^2 = 0$ (i.e. $\operatorname{im} d_2 \subset \ker d_2$ at each step).
The cohomology of this complex is by definition the **Chevalley's cohomology** of the Lie algebra $\mathfrak{g}$, i.e.

$$H^k(\mathfrak{g}) \equiv \frac{\ker d_2\colon\ \Lambda^k\mathfrak{g}^* \to \Lambda^{k+1}\mathfrak{g}^*}{\operatorname{im} d_2\colon\ \Lambda^{k-1}\mathfrak{g}^* \to \Lambda^k\mathfrak{g}^*}. \tag{4.2.5}$$

Notice for instance that $H^0(\mathfrak{g}) = \mathbb{R}$. Then we say that this cohomology is *connected*. The graded space

$$H^\bullet(\mathfrak{g}) = \bigoplus_{k\geq 0} H^k\mathfrak{g} = H^0(\mathfrak{g})\oplus H^1(\mathfrak{g})\oplus\ldots\oplus H^r(\mathfrak{g}) \cong \mathbb{R}\oplus H^1(\mathfrak{g})\oplus\ldots\oplus H^r(\mathfrak{g}) \tag{4.2.6}$$

inherits a graded commutative algebraic associative structure by defining

$$[\alpha]\wedge[\beta] \equiv [\alpha\wedge\beta]. \tag{4.2.7}$$

Remark. It can be shown that any differential complex (S,d) with $S = \bigoplus_{k\geq 0} S^k$ satisfying the previous properties is the exterior algebra complex of a Lie algebra. We define $\mathfrak{g}^* = S^1$ and the Lie algebra structure by using Eq. (4.2.2).

Suppose now that $(K^\bullet, d_1)$ is another differential complex where $K^\bullet = \bigoplus_k K^k$ is a commutative graded algebra and d_1 is an operator of degree one such that

$d_1^2 = 0$. We will assume that we have a representation ρ_K of G on $K^\bullet$ preserving the graduation of $K^\bullet$, i.e.

$$\rho_K\colon\ G \to \mathrm{Aut}(K^\bullet) \tag{4.2.8}$$

such that $\rho_K(gh) = \rho_K(g) \circ \rho_K(h)$ and $\rho_K(g)K^k \subset K^k$, $\forall\, g, h \in G$, $\forall\, k$.
Actually we will only need a representation $\hat{\rho}_K$ of the Lie algebra $\mathfrak{g}$ of G on $K^\bullet$ by derivations of degree 0, i.e.

$$\hat{\rho}_K\colon\ \mathfrak{g} \to \mathrm{Der}(K^\bullet) \tag{4.2.9}$$

such that

$$\hat{\rho}_K([\xi, \eta]) = [\hat{\rho}_K(\xi), \hat{\rho}_K(\eta)], \quad \forall\, \xi, \eta \in \mathfrak{g}. \tag{4.2.10}$$

Moreover we require that $\hat{\rho}_K K^k \subset K^k$ and

$$\hat{\rho}_K(\xi)(f \cdot g) = (\hat{\rho}_K(\xi) f) \cdot g + f \cdot (\hat{\rho}_K(\xi) g), \quad \forall\, f, g \in K^\bullet. \tag{4.2.11}$$

If we have a representation ρ_K of the group G on $K^\bullet$ which is "regular" enough, we can differentiate it to obtain the representation $\hat{\rho}_K$ of its Lie algebra with the properties above. In general such map will be defined only on the "smooth" part of $K^\bullet$ (see later on in Section 5.2).
Now we will assume that the operator d_1 is $\hat{\rho}_K$–invariant, i.e.

$$\hat{\rho}_K(\xi)(d_1 f) = d_1(\hat{\rho}_K(\xi) f), \quad \forall f \in K^\bullet,\ \xi \in \mathfrak{g}. \tag{4.2.12}$$

We will call a complex $(K^\bullet, d_1)$ such that $K^\bullet$ is a graded commutative algebra together with a representation of a Lie algebra by zero order derivations of $K^\bullet$, a **Koszul complex**.[1]

Definition 4.2.1. A Koszul complex $(K^\bullet, d_1)$ is said to be **acyclic** if its cohomology is trivial except, possibly, at order 0, i.e.:

$$H^k_{d_1}(K^\bullet) = 0, \quad \forall\, k \neq 0. \tag{4.2.13}$$

[1] This is not the most general definition of a Koszul complex, but it is an adapted one for the needs of this work.

Consider now the graded tensor product of an $\hat{\rho}_K$–invariant Koszul complex $K^\bullet$ (or $(K^\bullet, d_1, \hat{\rho}_K)$) and the Chevalley complex $(\Lambda^\bullet \mathfrak{g}^*, d_2)$. We will get then the double graded commutative algebra $S^{\bullet\bullet} = K^\bullet \otimes \Lambda^\bullet \mathfrak{g}^*$. The algebra $S^{\bullet\bullet}$ is equipped with a double graduation

$$S^{\bullet\bullet} = \bigoplus_{k,j} (K^k \otimes \Lambda^j \mathfrak{g}^*) = \bigoplus_{k,j} S^{kj}, \tag{4.2.14}$$

where $S^{kj} = K^k \otimes \Lambda^j \mathfrak{g}^*$. The graded commutative associative product is defined in the obvious way extending by linearity the product of monomials:

$$(f \otimes \alpha) \cdot (g \otimes \beta) = (-1)^{|\alpha||g|} (f \cdot g) \otimes (\alpha \wedge \beta), \tag{4.2.15}$$

with $f, g \in K^\bullet$ homogeneous elements and $\alpha, \beta \in \Lambda^\bullet \mathfrak{g}^*$.
Then the double graded algebra $S^{\bullet\bullet}$ inherits a single graded structure $S^\bullet$ (called anti-diagonal) defined by:

$$S^\bullet = \bigoplus_{k \geq 0} S^k, \quad S^k = \bigoplus_{i+j=k} S^{ij}. \tag{4.2.16}$$

Moreover both d_1, d_2 can be extended in a unique way to operators of bidegree $(1,0)$ and $(0,1)$ respectively over $S^{\bullet\bullet}$ by means of:

$$d_1(f \otimes \alpha) = d_1 f \otimes \alpha \tag{4.2.17}$$

and d_2:

$$d_2(f \otimes \mathbb{1}) \in K^\bullet \otimes \mathfrak{g}^*, \quad d_2(f \otimes \mathbb{1})(\xi) = (-1)^{|f|} \hat{\rho}_K(\xi) f \otimes \mathbb{1} \tag{4.2.18}$$

or

$$d_2(f \otimes \alpha) = d_2 f \otimes \alpha + (-1)^{|f|} \otimes d_2 \alpha, \tag{4.2.19}$$

with $(d_2 f)(\xi) = (-1)^{|f|} \hat{\rho}_K(\xi) f$. When we consider the double degree of $S^{\bullet\bullet}$ reduced to the antidiagonal degree above we will denote it by $S^\bullet$.

Proposition 4.2.2. *If $(K^\bullet, d_1)$ is a $\hat{\rho}_k$–invariant complex, the operators d_1, d_2 induced in the graded tensor product algebra $S^\bullet = K^\bullet \otimes \Lambda^\bullet \mathfrak{g}^*$ are such that the operator $D = d_1 + d_2$ is a cohomology operator, i.e. $D^2 = 0$.*

Each one of the three operators d_1, d_2, D on $S^\bullet$ define its own cohomology $H^\bullet_{d_1}(S), H^\bullet_{d_2}(S), H^\bullet_D(S)$ which are not independent. However, even if the relation between d_1, d_2 and D is very simple, this is not so in the case of their cohomology algebras. The relation between them is obtained explicitely by using the so called **associated spectral sequence** (see for instance [BT82], Ch. 3, Sec. 14). However the fact that $(K^\bullet, d_1)$ is acyclic greatly simplifies the problem because in such case it can be shown that the associated spectral sequence degenerates at the term E_2 (again [BT82], Ch. 3, Sec. 14 pp. 166 and ff.) and then:

$$H^k_D(S^\bullet) = H^k_{d_2}(H^0_{d_1}(S^\bullet)), \quad k \geq 0. \tag{4.2.20}$$

The equality here is abstract, this is, up to isomorphisms, or in other words, once the isomorphism has been established, finding the element in $H^k_D(S^\bullet)$ that corresponds to a given element in $H^k_{d_2}(H^0_{d_1}(S^\bullet))$ is done by solving a family of equations (whose solutions are not unique in general) known as *descending equations*. We will call the complex $(S^\bullet, D)$ thus obtained the **BRST complex** corresponding to $(K^\bullet, d_1)$ and $\mathfrak{g}$, and D the abstract BRST cohomology operator.
Let us emphasize here that except for supporting a representation of $\mathfrak{g}$, the Koszul complex $(K^\bullet, d_1)$ is arbitrary. We will use this fact later on to construct the BRST complex of a Lie–Jordan Banach algebra supporting a representation of a Lie group.
Notice again that we have not made use of any Hamiltonian structure so far like Poisson brackets, momentum map, etc. We could have done that by assuming that M is a Poisson manifold. In such case we will end up with $(S^\bullet, D)$ being a Poisson superalgebra and the BRST operator being Hamiltonian with odd Hamiltonian function Q, i.e. $D = \{\cdot, Q\}$. Then because $D^2 = 0$, it must be satisfied that $\{Q, Q\} = 0$.
In general the complex $(K^\bullet, d_1)$ is obtained from simpler data, suggested by the concrete problem at hand. Now we will describe the construction of $(K^\bullet, d_1)$ in the classical situation.

4.3. The classical Koszul complex $(K^\bullet, d_1)$

Let V be a finite dimensional linear space and let R be a commutative algebra with unit $\mathbb{1}$ (not graded). We assume that both V and R support representations of $\mathfrak{g}$ that will be denoted by $\hat\rho_V$ and $\hat\rho_R$ respectively. This is $\hat\rho_V\colon\ \mathfrak{g} \to \mathrm{End}(V)$ such that

$$[\hat\rho_V(\xi), \hat\rho_V(\eta)] = \hat\rho_V([\xi,\eta]), \quad \forall\, \xi, \eta \in \mathfrak{g} \tag{4.3.1}$$

and $\hat\rho_R\colon\ \mathfrak{g} \to \mathrm{Der}(R)$ such that

$$[\hat\rho_R(\xi), \hat\rho_R(\eta)] = \hat\rho_R([\xi,\eta]). \tag{4.3.2}$$

Consider now a linear map $a\colon\ V \to R$ such that

$$\hat\rho_R(\xi)(a(v)) = a(\hat\rho_V(\xi)v), \quad \forall\, v \in V,\ \forall\, \xi \in \mathfrak{g}; \tag{4.3.3}$$

i.e., a is $\mathfrak{g}$–equivariant. We will define now $K^\bullet$ as the graded commutative algebra $K^\bullet = R \otimes \Lambda^\bullet V$, with $K^k = R \otimes \Lambda^k V$ and $K^\bullet = \bigoplus_{k\geq 0} K^k$. Moreover

$$(x \otimes v)\cdot(y \otimes u) = (x\cdot y) \otimes (v \wedge u), \tag{4.3.4}$$

where $x, y \in R$ and $v, u \in \Lambda^\bullet V$.
Now we define d_1 as the unique derivation of degree -1 of $K^\bullet$ that extends a and that acts trivially on R, thus $d_1\colon\ K^k \to K^{k-1}$ with[2]:

$$d_1(x \otimes v) = x \cdot a(v), \quad \forall\, x \in R,\ \forall\, v \in V. \tag{4.3.5}$$

Moreover, $d_1 x = 0$ if $x \in K^0 = R$ and, if $v \in V$, then $d_1(v) = a(v) \in R$. This induces the sequence

$$\cdots \longrightarrow \underbrace{R \otimes \Lambda^k V}_{K^k} \xrightarrow{d_1} \underbrace{R \otimes \Lambda^{k-1} V}_{K^{k-1}} \longrightarrow \cdots \xrightarrow{d_1} \underbrace{R \otimes V}_{K^1} \xrightarrow{d_1} \underbrace{R}_{K^0} \xrightarrow{d_1} 0 \tag{4.3.6}$$

[2]Notice that in order that d_1 will have degree $+1$ instead of -1, we should consider $K^\bullet$ with the opposite graduation: $K^i = R \otimes \Lambda^{-i} V$.

Notice that d_1 being a derivation means that

$$d_1(x \otimes v \wedge u) = x \otimes d_1(v \wedge u) = x \otimes (d_1 v \wedge u + (-1)^{|v|} v \wedge d_1 u). \quad (4.3.7)$$

Then if $v \in V$, we have $d_1(x \otimes v \wedge u) = x \cdot a(v) \otimes u - x \otimes v \wedge d_1 u$, but then

$$\begin{aligned} d_1^2(x \otimes v \wedge u) &= x \cdot a(v) \otimes d_1 u - x \cdot d_1 v \wedge d_1 u + x \otimes v \wedge d_1^2 u \\ &= x \cdot a(v) \otimes d_1 u - x \cdot a(v) d_1 u + x \otimes v \wedge d_1^2 u \quad (4.3.8) \end{aligned}$$

i.e., $d_1^2(x \otimes v \wedge u) = x \otimes v \wedge d_1^2 u$. But again repeating the argument, if $u = w \wedge z$, with $w \in V$

$$d_1^2(x \otimes v \wedge w \wedge z) = x \otimes v \wedge w \wedge d_1^2 z, \quad (4.3.9)$$

but eventually always the last term will be $d_1^2 z$ with $z \in V$, then $d_1^2 z = d_1(a(z)) = 0$, since $a(z) \in R$. This proves that $d_1^2 = 0$.

A simple computation shows that

$$H^0_{d_1}(K^\bullet) = \frac{\ker d_1 \colon\ K^0 \to 0}{\operatorname{im} d_1 \colon K^1 \to K^0} = {R}\big/{R \cdot a(V)}, \quad (4.3.10)$$

or what is the same $H^0_{d_1}(K^\bullet) = R/\mathcal{J}_a$, where $\mathcal{J}_a$ is the ideal of elements generated by the image of the map a.
In order to apply the result expressed by Eq. (4.2.20) we have to be able to show that the complex $(K^\bullet, d_1)$ so constructed is acyclic, i.e. $H^k_{d_1}(K^\bullet) = 0$, $\forall\, k > 0$, that in general will not be.[3]

In the next section we will apply this construction to describe the space of G–invariant functions $C^\infty(C/G)$ on a constraint submanifold C and along the way we will provide sufficient conditions for the complex $(K^\bullet, d_1)$ before to be acyclic, with R being the algebra of smooth functions on a manifold M.

[3]Even if $(K^\bullet, d_1)$ were not acyclic, it can be shown that there always exists an extension $(\tilde{K}^\bullet, \tilde{d}_1)$ of $(K^\bullet, d_1)$ which is acyclic, however we will not use this argument here.

4.4. Classical BRST reduction

Let G be as before a connected Lie group with Lie algebra $\mathfrak{g}$ and let M be a smooth manifold on which G acts (see Appendix A). Let R be the algebra of smooth functions $C^\infty(M)$ on M. The action of G on M induces a representation of $\mathfrak{g}$ on R given by $\hat{\rho}_R\colon\ \mathfrak{g} \to \mathrm{Der}(R)$ associating to each element $\xi \in \mathfrak{g}$ the fundamental vector field X_ξ. Let V be a finite dimensional vector space supporting a linear representation of G, $\rho_V\colon\ G \to \mathrm{GL}(V)$. Such representation defines a representation $\hat{\rho}_V$ of $\mathfrak{g}$ on V as

$$\hat{\rho}_V(\xi)(v) = \frac{d}{dt}\rho_V(\mathrm{e}^{-t\xi}(v))|_{t=0}. \tag{4.4.1}$$

Finally we assume that there is a map $\Phi\colon\ M \to V^*$ which is $\mathfrak{g}$–equivariant in the sense that:

$$\langle \Phi(x), \hat{\rho}_V(\xi)v\rangle = \langle (\hat{\rho}_R(\xi)\Phi)(x), v\rangle, \quad \forall\, v \in V,\ \xi \in \mathfrak{g},\ x \in M. \tag{4.4.2}$$

Notice that Φ defines a map

$$a\colon\ V \to C^\infty(M) = R \tag{4.4.3}$$

by means of $a(v)(x) \equiv \langle \Phi(x), v\rangle$ which is $\mathfrak{g}$–equivariant, that is

$$\hat{\rho}_R(\xi)(a(v)) = a(\hat{\rho}_V(\xi)(v)). \tag{4.4.4}$$

So, the map $a\colon V \to R$ above satisfies the requirement discussed in the previous section to construct the Koszul complex $(K^\bullet, d_1)$, $K^\bullet = R \otimes \Lambda^\bullet V$.

Let us denote the zero level set $\Phi^{-1}(0)$ by C. We have three questions that arise naturally: under which conditions we will have that

i) C is a submanifold of M,

ii) $H^0_{d_1}(K) = C^\infty(C)$, and

iii) the complex $(R \otimes \Lambda^\bullet V, d_1)$ is acyclic?

The answer to the first question has already been anticipated: 0 must be a weakly regular value of Φ. Regarding the other questions, we have the following theorems that provide a complete answer.

Theorem 4.4.1. *If* 0 *is a weakly regular value of* Φ*, then*

$$H^0_{d_1}(K^\bullet) = C^\infty(C) \tag{4.4.5}$$

with $(K^\bullet, d_1)$ *being the Koszul complex* $K^\bullet = C^\infty(M) \otimes \Lambda^\bullet V$ *defined before.*

Proof. If 0 is weakly regular, then for any $x \in C$ there exist local coordinates $(x^1, \ldots, x^p, x^{p+1}, \ldots, x^{p+q})$ with $p+q = \dim M$ and $q = \dim C$ and the submanifold C is defined by $C = \{x^1 = \ldots = x^p\}$, where we are assuming $\dim V \geq p$. The coordinates $x^1, \ldots, x^p$ are chosen from the components Φ^i of the map Φ in some linear basis $\{v_i\}$ of V.
Notice that $H^0_{d_1}(K^\bullet) = C^\infty(M) \big/ C^\infty(M) \cdot a(V) \cdot$
It is clear that $a(v) \in \mathcal{J}_C$, with $\mathcal{J}_C$ the ideal of smooth functions vanishing at C:

$$\mathcal{J}_C = \{ f \in C^\infty(M) \ \mid \ f \mid_C = 0 \} \tag{4.4.6}$$

and

$$a(v)(x) = \langle \Phi(x), v \rangle = 0, \quad \forall\, x \in C = \Phi^{-1}(0). \tag{4.4.7}$$

Thus $a(v) \subset \mathcal{J}_C$ but locally $\mathcal{J}_C$ is generated by $x^1 \ldots . x^p$ which are obtained from the components of Φ, i.e. they have the form $a(v^i) = x^i$. It follows $\mathcal{J}_C = C^\infty(M) \cdot a(V)$. □

Theorem 4.4.2. *Suppose that* 0 *is a weakly regular value of* Φ*. Then the Koszul complex* $K^\bullet = C^\infty(M) \otimes \Lambda^\bullet V$ *is acyclic iff* 0 *is a regular value.*

Proof. We will assume first that 0 is a regular value of Φ. Notice that we can compute the cohomology of $K^\bullet = C^\infty(M) \otimes \Lambda^\bullet V$ with respect to d_1 locally because d_1 acts trivially on $C^\infty(M)$, this if α is a k-cocycle, $d_1\alpha = 0$, $\alpha \in C^\infty(M) \otimes \Lambda^k V$, then for any open set $U \subset M$ we can find a smaller open set V and a compact set W such that $V \subset K \subset U$, and a "bump" function σ adapted to it, that is, σ is smooth and

$$\sigma(x) = \begin{cases} 1 & x \in V \\ 0 & x \notin W. \end{cases} \tag{4.4.8}$$

Then $\sigma\alpha$ is a d_1–cocycle with support contained in U (actually in $V \subset U$) because $d_1(\sigma\alpha) = \sigma d_1\alpha = 0$.
Similarly if β is a d_1–coboundary, i.e. $\exists\, \gamma$ such that $\beta = d_1\gamma$, then $\sigma\beta$ is a d_1–coboundary too, because $d_1(\sigma\gamma) = \sigma d_1\gamma = \sigma\beta$. We will distinguish two cases, depending if $U \cap C = \emptyset$ or $U \cap C \neq \emptyset$. However in both cases we will prove the result by showing that there exists a homotopic contraction for d_1 on U_1 i.e. a linear map

$$h\colon C^\infty(U) \otimes \Lambda^k V \to C^\infty(U) \otimes \Lambda^{k+1} V \tag{4.4.9}$$

such that the operator $hd_1 + d_1h$ is the identity in the kernel of d_1, then $\alpha = (hd_1 + d_1h)\alpha = d_1(h\alpha)$ if $d_1\alpha = 0$, then α is a coboundary and $H^k_{d_1}(K^\bullet) = 0$, i.e.

$$(hd_1 + d_1h)\alpha = \alpha \tag{4.4.10}$$

and $\alpha \in C^\infty(U) \otimes \Lambda^+ V$, with $\Lambda^+ V = \bigoplus\limits_{k>0} \Lambda^k V$.
If $U \cap C \neq \emptyset$, then using local coordinates $x^1, \ldots, x^p, x^{p+1}, \ldots, x^{p+q}$ like in Theorem 4.4.2, and now $\dim V = p$ (0 is regular) we define $h\alpha$ by using “Poincare’s trick”:

$$(h\alpha)(x) = \int_0^1 ds\; s^k \partial_i \alpha^{i_1 \ldots i_k}(s \cdot x) v_i \wedge v_{i_1} \wedge \ldots \wedge v_{i_k} \tag{4.4.11}$$

where $\alpha \in C^\infty(U) \otimes \Lambda^k V$ and x belongs to a star-shaped open neighborhood contained in U. Now after some long and tedious computations we check that h is the homotopy contraction we were looking for.

If $U \cap C = \emptyset$ we may just use

$$ha = \frac{1}{\sum\limits_i (\phi^i)^2} \sum_i \phi^i v_i \wedge \alpha. \tag{4.4.12}$$

Let us prove now the converse. Assume that 0 is a weakly regular value of Φ but *not* regular. This implies that $p < \dim V$. Then the differentials $d\phi^i$ must be linearly dependent over C. Then there will exist functions f_i on C (not all of them vanishing at the same time) such that

$$\sum_i f_i d\Phi^i \,|_C = 0 \tag{4.4.13}$$

which implies that $\sum_i f_i \Phi^i \in \mathcal{J}_C^1$, i.e. the ideal of functions that vanish on C as well as its first derivative.
However a theorem by Tougeron [Tou68] on ideals of smooth functions, shows that

$$\mathcal{J}_C^1 = \mathcal{J}_C \cdot \mathcal{J}|_C \tag{4.4.14}$$

then $\exists\, h_i,\ g_j \in C^\infty(M)$ such that

$$\underbrace{\sum_i f_i \Phi_i}_{\in \mathcal{J}_C^1} = \underbrace{(\sum_j h_j \Phi_j)}_{\in \mathcal{J}_C} \underbrace{(\sum_l g_l \Phi_l)}_{\in \mathcal{J}_C} \tag{4.4.15}$$

i.e. $\sum_i (f_i - \sum_j g_j h_i \Phi_j) v_i$ is a d_1–cocycle, but it cannot be a coboundary because

$$f_i - \sum_i g_i h_i \Phi_j|_C \neq 0 \tag{4.4.16}$$

at least for some i. □

Then it follows from the previous theorem the result we were looking for:

Corollary 4.4.3. *If* 0 *is a regular value of* Φ*, then*

$$H_D^0(K^\bullet) = C^\infty(C/G). \tag{4.4.17}$$

4.5. BRST symmetry for Lie–Jordan Banach algebras

As it was pointed out in the previous sections, most of the constructions leading to the BRST symmetry can be cast in a broader setting, this is the algebra R that we used to construct the Koszul complex $K^\bullet$, even the Koszul complex, is largely undetermined. We also require representations of a Lie algebra and a $\mathfrak{g}$–equivariant map $a\colon\ \mathfrak{g} \to R$. What we will explore now is what happens when we will choose a Lie–Jordan Banach setting for R. We will arrive easily to a BRST symmetry that reproduces the theory of reduction of LJB–algebras if the Koszul complex that will be constructed satisfies the appropriate conditions.
Notice that the BRST symmetry reproduces (under the conditions discussed in the previous section) faithfully the theory of reduction at the classical level, i.e. when $R = C^\infty(M)$. Then the BRST symmetry has been used as a tool to quantize gauge symmetries. The idea was that quantizing the BRST operator D is sometimes easier, and then the physical states of the theory would be "defined" to be those such that $\hat{D}|\phi\rangle = 0$ module those of the form $|\psi\rangle = \hat{D}|\chi\rangle$, with $\hat{D}$ the quantum operator associated to D acting on some Hilbert space $\mathcal{H}$. This is, the physical states of the quantum theory will be defined as $H(\hat{D}) = {\ker \hat{D}}\big/{\operatorname{im} \hat{D}}$, thinking that this is the quantum counterpart of $H^0_D(S) = C^\infty(C/G)$.
This argument is, as one may see, not rigorous at all, even if it has proved to be very successful in concrete applications and it has been widely used in dealing with gauge symmetries in quantum field theories. In these section we will explore a different road. LJB–algebras represent the algebra of observables of quantum systems by themselves and the reduction theory we have already constructed in Chapter 2 should provide the algebra of quantum observables of the reduced quantum system. Thus if $\widetilde{\mathcal{L}}$ is a reduced Lie–Jordan Banach algebra, we want to understand if we can obtain $\widetilde{\mathcal{L}}$ by using a "quantum" BRST symmetry, i.e. under what conditions $\widetilde{\mathcal{L}}$ is the cohomology of a BRST operator $\hat{D}$.

We will assume as in Chapter 2 that $\mathcal{L}$ is a LJB–algebra with unit, i.e. the self-adjoint or real part of a unital C^*–algebra $\mathcal{A}$. We will also assume that G is a Lie group represented on the LJB–algebra $\mathcal{L}$. This is, there is a strongly continuous[4]

[4]Strongly continuous means that the map $\rho_x\colon\ G \to \mathcal{L}$ sending $g \to \rho(g)x$ is continuous

homomorphism

$$\rho\colon G \to \mathrm{Aut}(\mathcal{L}) \tag{4.5.1}$$
$$g \mapsto \rho(g)\colon \mathcal{L} \to \mathcal{L} \tag{4.5.2}$$

such that $\rho(e) = \mathbb{1}$ and $\rho(g)\rho(h) = \rho(gh)$.
We will also assume that for every $\xi \in \mathfrak{g}$ there exists:

$$\hat{\rho}(\xi) \equiv \lim_{t\to 0} \frac{\rho(\mathrm{e}^{t\xi}) - \mathbb{1}}{t}, \tag{4.5.3}$$

so that $\hat{\rho}\colon \mathfrak{g} \to \mathrm{Der}\,\mathcal{L}$ is a continuous Lie algebra homomorphism[5]. Finally we will require that the derivation $\hat{\rho}(\xi)$ associated to each Lie algebra element ξ is skew-symmetric, i.e. we will assume that $\forall\,\xi \in \mathfrak{g}$, there exists an element $a(\xi) \in \mathcal{L}$ such that

$$\hat{\rho}(\xi)x = [x, a(\xi)] \tag{4.5.4}$$

$\forall\, x \in \mathcal{L}$, where $[\cdot,\cdot]$ is the Lie bracket in $\mathcal{L}$. The map $a\colon \mathfrak{g} \to \mathcal{L}$ mapping each ξ into $a(\xi)$ will be called the *(co)-momentum* map and it is exactly the analogue of the map a dual to Φ discussed in the Section 4.4. We will assume that the map a is $\mathfrak{g}$–equivariant, or in other words that it is a Lie–algebra homomorphism:

$$[a(\xi), a(\eta)] = a([\xi,\eta]), \quad \forall\,\xi,\eta \in \mathfrak{g} \tag{4.5.5}$$

Definition 4.5.1. Let $\mathcal{L}$ be a LJB algebra and G a connected Lie group acting on $\mathcal{L}$. We will say that the action of G on $\mathcal{L}$ is **strongly Hamiltonian** if there exists a map $a\colon \mathfrak{g} \to \mathcal{L}$ such that Eqs. (4.5.4) and (4.5.5) are satisfied. We will call this map the **(co-)momentum map** of the action[6].

Now if $(\mathcal{L}, G, a)$ is a LJB algebra with a strongly Hamiltonian action a, we can construct easily its Koszul complex $(K^\bullet, d_1)$, as the graded LJB–algebra

$$K^\bullet = \mathcal{L} \otimes \Lambda^\bullet \mathfrak{g} \tag{4.5.6}$$

$\forall\, x \in \mathcal{L}$.

[5]This restriction could be removed because such map always exists restricting the LJB–algebra $\mathcal{L}$ to its smooth part $\mathcal{L}_\infty$ which is dense, but we will not consider this generality here.

[6]See the comments about the comomentum map at the end of Appendix A.

$$K^\bullet = \bigoplus_{k\leq 0} K^k; \quad K^k = \mathcal{L}\otimes\Lambda^{-k}\mathfrak{g}, \tag{4.5.7}$$

and the cohomology operator $d_1 \colon K^k \to K^{k+1}$ is the unique derivation of degree one that acts trivially on $\mathcal{L}$ and restricts to a on $\mathfrak{g}$, i.e.

$$d_1(x\otimes\xi) = x\circ a(\xi), \quad \forall\, x\in\mathcal{L}, \forall\xi\in\mathfrak{g}. \tag{4.5.8}$$

and

$$d_1(x) = 0, \quad \forall\, x\in\mathcal{L}. \tag{4.5.9}$$

The operator d_1 induces the following sequence:

$$\cdots \underset{d_1}{\longrightarrow} \underbrace{\mathcal{L}\otimes\Lambda^k\mathfrak{g}}_{K^k} \underset{d_1}{\longrightarrow} \underbrace{\mathcal{L}\otimes\Lambda^{k+1}\mathfrak{g}}_{K^{k+1}} \underset{d_1}{\longrightarrow} \cdots \xrightarrow{d_1} \underbrace{\mathcal{L}\otimes\mathfrak{g}\wedge\mathfrak{g}}_{K^{-2}} \xrightarrow{d_1} \underbrace{\mathcal{L}\otimes\mathfrak{g}}_{K^{-1}} \xrightarrow{d_1} \underbrace{\mathcal{L}}_{K^0} \xrightarrow{d_1} 0 \tag{4.5.10}$$

Now the BRST complex $S^\bullet$ is constructed by taking the graded tensor product (see Section 4.8):

$$S^{\bullet\bullet} = K^\bullet\otimes\Lambda^\bullet\mathfrak{g}^* \tag{4.5.11}$$

and restricting to the diagonal grading:

$$S^\bullet = \bigoplus_{k\geq 0} S^k, \quad S^k = \bigoplus_{i+j=k}\Lambda^i\mathfrak{g}^*\otimes K^{-j} \tag{4.5.12}$$

the BRST operator is given by

$$\hat{D} = d_1 + d_2 \tag{4.5.13}$$

Notice that the BRST complex $S^\bullet$ is a graded LJB algebra with the natural norm induced from the norm $\|\cdot\|$ of $\mathcal{L}$.

4.6. Reduction of LJB–algebras and BRST symmetry

Let $(\mathcal{L}, \circ, [\cdot,\cdot])$ be a LJB–algebra and $\rho\colon G \to \mathrm{Aut}(\mathcal{L})$ a strongly Hamiltonian action of G on $\mathcal{L}$ with (co-)momentum map $a\colon \mathfrak{g} \to \mathcal{L}$. If we pick-up a linear basis ξ_a for $\mathfrak{g}$, then $\hat{J}_\alpha \equiv a(\xi_a) \in \mathcal{L}$ are real observables that could be interpreted as the components of a quantum momentum map or as the quantum conserved quantities of the system. Notice that because of the assumption that the action is strongly Hamiltonian then

$$[\hat{J}_\alpha, \hat{J}_\beta] = c_{\alpha\beta}^{\gamma} \hat{J}_\gamma, \tag{4.6.1}$$

where $c_{\alpha\beta}^{\gamma}$ are the structure constants of the Lie algebra $\mathfrak{g}$, i.e. $[\xi_\alpha, \xi_\beta] = c_{\alpha\beta}^{\gamma}\xi_\gamma$, so no anomalies are permitted in our theory (which is again an unnecessary restriction that can be released in a slightly more general presentation of the subject that will be conducted elsewhere).
Then the analogue of the zero level set $\Phi^{-1}(0) = C$ constraint in the classical theory is given here by the family of constraints $\mathcal{C}$ determined by the components $\hat{J}_\alpha$,

$$\mathcal{C} = \{\hat{J}_\alpha,\ \alpha = 1, \ldots, r\}. \tag{4.6.2}$$

The reduced LJB–algebra $\mathcal{L}//G$ is defined here as

$$\mathcal{L}//G = \mathcal{N}_{\mathcal{J}}/\mathcal{J}, \tag{4.6.3}$$

where $\mathcal{N}_{\mathcal{J}}$ is the Lie normalizer of the ideal $\mathcal{J} = \mathcal{L} \circ \mathcal{C}$ generated by the constraints $\mathcal{C}$. Notice that in general $\mathcal{L}//G$ is not the same as $\widetilde{\mathcal{L}} = \mathcal{N}/\mathcal{V}$ where $\mathcal{V}$ is the vanishing subalgebra defined by $\mathcal{C}$, i.e. the annihilator of the Dirac states of $\mathcal{C}$, and $\mathcal{N}$ its normalizer, as discussed in Section 3.4. As in the classical case some "regularity" condition must be demanded to the action a in order to have $\widetilde{\mathcal{L}} = \mathcal{L}//G$. We will discuss this in the next section.
We have the following theorem whose proof is the same as in the classical situation:

Theorem 4.6.1. *Let $\mathcal{L}$ be a LJB–algebra supporting a strongly Hamiltonian action of the connected Lie group G. If $a\colon \mathfrak{g} \to \mathcal{L}$ denotes its (co-)momentum map and if the nontrivial Koszul complex associated to it $K^\bullet = \mathcal{L} \otimes \Lambda^\bullet \mathfrak{g}$ is acyclic,* *then the zeroth cohomology group of the BRST operator $\hat{D}$ is exactly $\mathcal{L}//G$:*

$$H^0_{\hat{D}}(S^\bullet) \cong \mathcal{L}//G. \tag{4.6.4}$$

The natural question that arises here is under what conditions it is possible to guarantee that the Koszul complex $(K^\bullet, d_1)$ is acyclic? In order to provide an answer to this question we need to introduce first a few notions of modules on LJB–algebras.

4.7. Modules and Lie–Jordan Banach algebras

4.7.1. Topological modules and topological algebras

Let R be a topological associative algebra with unit, i.e. R is a Banach space with a complete norm $\|\cdot\|$.
Let M be a topological left R–module, i.e. M is a linear space with a norm $\|\cdot\|$ such that it is a Banach space (this condition could be weakened) and there is a continuous action of R on M by continuous linear operators, i.e. there is a continuous map

$$\begin{array}{rcl} \cdot\colon R\times M & \to & M \\ (x,m) & \mapsto & x\cdot m \end{array} \tag{4.7.1}$$

such that it is distributive $x\cdot(m+m') = x\cdot m + x\cdot m'$, $(\lambda \mathbb{1}_R)\cdot m = \lambda m$, $\lambda \in \mathbb{R}, \mathbb{C}$[7] and associative $(x\cdot y)\cdot m = x\cdot(y\cdot m), \forall\, x,y\in R,\ m\in M$.
The linear map $\phi_m\colon R\to M$ given by $\phi_m x = x\cdot m$ ($m\in M$ fixed) is continuous, i.e. $\exists\, k>0$ such that $\|\phi_m x\| \leq k\|x\|$ since $\|x\cdot m\| \leq \widetilde{k}\|x\|\|m\|$ for some $\widetilde{k}$. In this setting all algebraic notions to the theory of R–modules pass without change into the category of topological modules over topological algebras.

If $\mathcal{A}$ is a topological *–algebra, there is a natural notion of *–module M over $\mathcal{A}$. It is a topological $\mathcal{A}$–bimodule with a * operation

$$^*\colon M\to M \tag{4.7.2}$$

[7] Depending whether R is a real or complex algebra.

such that $(a \cdot m)^* = m^* \cdot a^*$ and $m^{**} = m$, $\forall\, x \in \mathcal{A}$, $m \in M$. Notice that if $\lambda \in \mathbb{C}$, then $(\lambda m)^* = m^* \lambda^* = \lambda^* m^*$. Moreover $((ab)m)^* = m^*(ab)^* = m^*(b^* a^*) = (m^* b^*) a^* = (bm)^* a^* = (a(bm))^*$, $\forall\, a, b \in \mathcal{A}$, $m \in M$.
Thus a topological $\mathcal{A}$–module over a unital C^*–algebra is a *–module M over $\mathcal{A}$ which is a topological bimodule over $\mathcal{A}$. Notice that in this case

$$(a^* a) \cdot m = (a^* a)^* \cdot m = (a^* a) \cdot (m^*)^* = (m^* \cdot (a^* a))^* \tag{4.7.3}$$

then $\|(a^* a) \cdot m\| = \|m^* \cdot (a^* a)\|$ and taking $a = \mathbb{1}_{\mathcal{A}}$, it follows $\|m\| = \|m^*\|$. Thus * is a unitary morphism.
In what follows we will omit "topological" whenever we consider modules over topological algebras understanding that we are always in the appropriate category.

4.7.2. Topological modules over LJB–algebras

Let $\mathcal{L}$ be a unital LJB–algebra with unit and $\mathcal{A} = \mathcal{L}^{\mathbb{C}}$ its associated C^*–algebra. Given a *–module M over $\mathcal{A}$, we will say that an element $m \in M$ is real if $m^* = m$. We will denote by M_{sa} the real elements of M. Now it is clear that the real elements of M form a real linear subspace but not an $\mathcal{A}$–submodule because $(a \cdot m)^* = m^* \cdot a^* = m \cdot a^*$ that in general will be different from $a \cdot m$.
Consider an $\mathcal{A}$–bimodule M, then we can define the composition law $a \circ m = m \circ a = \frac{1}{2}(a \cdot m + m \cdot a)$. Then $(M, \mathcal{A}, \circ)$ is a Jordan $\mathcal{A}$–module.
Now suppose that $a \in \mathcal{A}$ and $m \in M$ are real, i.e. $a^* = a$ and $m^* = m$, then am is not real but $a \circ m$ is, as it can be checked by simple inspection. Thus the LJB–algebra $\mathcal{L} = \mathcal{A}_{sa}$ acts on M_{sa}. Notice also that because $a \circ m = m \circ a$ and that:

$$(x^2 \circ m) \circ x = (x^2 \circ m) \circ x, \quad m \in M_{sa},\ x \in \mathcal{L}, \tag{4.7.4}$$

which is the identity replacing the associativity property $(a(bm)) = (ab)m$ in an $\mathcal{A}$–module.
In addition to the composition $\circ$, there is another operation induced in the real part of a *–bimodule over a *–algebra. We define

$$[a, m] \equiv \lambda(am - ma), \quad \lambda \in \mathbb{R}. \tag{4.7.5}$$

Then we can recover the left (and right) module structure out of $x \circ m$ and $[x, m]$, $x \in \mathcal{L},\ m \in M_{sa}$. The real part inherits a double structure too.
The two operations $\circ$ and $[\cdot, \cdot]$ in the linear space M_{sa} satisfy the following axioms

$$[x, [y, m]] = [[x, y], m] + [y, [x, m]], \tag{4.7.6a}$$

$$[x, y \circ m] = [x, y] \circ m + y \circ [x, m], \tag{4.7.6b}$$

$$x \circ (y \circ m) - (x \circ y) \circ m = \tau[y, [x, m]], \tag{4.7.6c}$$

$\forall\, x, y \in \mathcal{L},\ \forall\, m \in M_{sa}$ and where $\tau = -\frac{1}{4\lambda^2} \in \mathbb{R}$

Definition 4.7.1. A topological real linear space E is a Lie–Jordan module over the Lie–Jordan algebra $\mathcal{L}$ if there are two continuous bilinear operations:

$$\begin{array}{rcl} \circ\colon \mathcal{L} \times E & \to & E \\ (x, u) & \mapsto & x \circ u \end{array} \tag{4.7.7}$$

and

$$\begin{array}{rcl} [\cdot, \cdot]\colon \mathcal{L} \times E & \to & E \\ (x, u) & \mapsto & [x, u] \end{array} \tag{4.7.8}$$

satisfying the above axioms (4.7.6a), (4.7.6b) and (4.7.6c). Moreover if E is a Banach space, $\mathcal{L}$ a LJB–algebra and the operations $\circ$, $[\cdot, \cdot]$ are bounded, we will say that E is a normed Lie–Jordan module over $\mathcal{L}$ (or just a LJ–module for short).

Example 4.7.2. a) If M is a *–$\mathcal{A}$–bimodule, where $\mathcal{A}$ is a C^*–algebra, M_{sa} is a Lie–Jordan module over $\mathcal{A}_{sa}$.

b) Consider a LJB algebra $\mathcal{L}$ and $\mathcal{J} \lhd \mathcal{L}$ a closed Lie–Jordan ideal of $\mathcal{L}$, then $\mathcal{J}$ is a Lie–Jordan module.

c) Consider the direct product $E = \underbrace{\mathcal{L} \times \mathcal{L} \times \cdots \times \mathcal{L}}_{n}$ of $\mathcal{L}$ with itself n-times. Then E is a free Lie–Jordan module over $\mathcal{L}$.
It is clear that a morphism of LJ–modules $E,\ E'$ over $\mathcal{L}$ is a continuous linear map $\varphi\colon E \to E'$ such that $\varphi(a \circ m) = a \circ \varphi(m)$ and $\varphi([a, m]) = [\varphi(a), m]$.

This notion can be extended to LJ–modules over different LJB–algebras $\mathcal{L}$, $\mathcal{L}'$ if needed in the obvious way.
A LJ–submodule $V \subset E$ of a LJ–module E is a closed subspace such that $[a, V] \subset V$, $a \circ V \subset V$, $\forall\, a \in \mathcal{L}$. Then clearly V is a LJ–module over $\mathcal{L}$ by itself. The quotient space E/V equipped with the quotient topology becomes a LJ–module over $\mathcal{L}$ with the standard definitions

$$[a, m + V] = [a, m] + V \tag{4.7.9a}$$
$$a \circ (m + V) = a \circ m + V. \tag{4.7.9b}$$

The canonical projection $\pi \colon E \to E/V$ is a LJ–morphism.

d) If F is a free Lie–Jordan module over $\mathcal{L}$ and if $F \stackrel{\pi}{\longrightarrow} U \longrightarrow 0$ is a sequence of Lie–Jordan morphisms, then U is a projective Lie–Jordan module over $\mathcal{L}$ if there exists a Lie–Jordan morphism $\sigma \colon U \to F$ such that $\pi \circ \sigma = id$.

4.7.3. Regular actions on Lie–Jordan Banach algebras

Given a R–module M there always exists a left resolution of it, i.e. an exact sequence of R–modules and morphisms

$$\cdots \longrightarrow M_k \stackrel{\phi_k}{\longrightarrow} M_{k-1} \stackrel{\phi_{k-1}}{\longrightarrow} \cdots \stackrel{\phi_1}{\longrightarrow} M_0 \stackrel{\phi_0}{\longrightarrow} M \longrightarrow 0 \tag{4.7.10}$$

with all the M_k modules over R and $\mathrm{im}\phi_k = \ker\phi_{k-1}$, $\forall\, k = 0, 1, \ldots$
In particular if $I \subset R$ is an ideal generated by elements $\{x_i\} \subset R$, then we may think of R as an I–module and the previous statement tells that there exists a left resolution of R which is just the Koszul complex discussed before. If the number of generators is finite and they correspond to a basis of a Lie algebra represented on R, then the Koszul complex associated to R is just the Koszul complex constructed before ($K^\bullet = R \otimes \Lambda^\bullet \mathfrak{g}, d_1$).

Definition 4.7.3. We will say that $x \in R$ such that $x \neq 0$ is a zero divisor in the R–module M if there exists $m \in M$, $m \neq 0$ such that $x \cdot m = 0$, otherwise we say that x is a non-zero divisor.

Definition 4.7.4. We will say that $\{x_i\}_{i=1}^{\infty} \subset R$ is a **regular sequence** if x_k is a non-zero divisor for the module ${}^{M}\!/_{\mathcal{J}_k \cdot M}$, for all k, where $\mathcal{J}_k = (x_i,\ i = 1, \ldots, k-1)$ denotes the ideal generated by the elements $x_i,\ i = 1, \ldots, k-1$ and $\mathcal{J}_k \cdot M$ is the submodule of M of the form $\sum_{j=1}^{r} w_j m_j,\, w_j \in \mathcal{J}_k,\ m_j \in M$.

Theorem 4.7.5. *Suppose that M is an R–module and R is generated by the family $\{x_i\}$, then the Koszul resolution of the R–module M is acyclic if $\{x_i\}$ is a regular sequence.*

Proof. The proof of the theorem is standard. It proceeds by induction on the order k of the left resolution of M. We will do it for $k = 1$, and it is obvious how to proceed for $k > 1$. Consider the free R–module over the generators $\{x_i\}$ of R. We denote it by $\widetilde{R}$. Then an element of $\widetilde{R}$ has the form $\sum_i r_i \widetilde{x}_i$, where the sum is finite, $r_i \in R$ and $\widetilde{x}_i$ are the generators of $\widetilde{R}$. Then $M_1 = M \otimes_R \widetilde{R}$, $M_2 = M \otimes_R \Lambda^2 \widetilde{R}$, etc.
The map $\phi_1 \colon M_1 = M \otimes \widetilde{R} \to M$ is given by $\phi_1(\sum_i m_i \otimes \widetilde{x}_i) = \sum_i x_i \cdot m_i$ and $\phi_2 \colon M_2 \to M_1$ is given by $\phi_2(\sum_{ij} m_{ij} \widetilde{x}_i \wedge \widetilde{x}_j) = \sum_{ij} (x_i m_{ij} \widetilde{x}_j - x_j m_{ij} \widetilde{x}_i)$. Clearly $\phi_1 \circ \phi_2 = 0$.

$$\cdots \longrightarrow M_2 \xrightarrow{\phi_2} M_1 \xrightarrow{\phi_1} M \longrightarrow 0. \tag{4.7.11}$$

Clearly the map ϕ_1 is surjective. Now let $\sum_i m_i \otimes \widetilde{x}_i \in M_1$ be such that $\phi_1(\sum_i m_i \widetilde{x}_i) = 0$, i.e. $\sum_i x_i m_i = 0$. We will assume that $\sum_i m_i \widetilde{x}_i$ is not of the form $\phi_2(x)$ i.e. $m_i \neq \sum_j x_j m_{ij}$. Then let $m_1 \neq 0$ (we can reorder if we want the elements $m_1, m_2, \ldots$ to satisfy such condition). Hence $x_1 m_1 = -x_2 m_2 - \cdots$. If all $m_j = 0$ for $j \geq 2$, then it would imply $x_1 m_1 = 0$ which is impossible because $\{x_i\}$ is a regular sequence and hence x_1 cannot be a non-zero divisor in M. Let $m_2 \neq 0$, then because $m_1 \neq x_2 m_{21} + x_j m_{j1} + \cdots$ and $x_2 \neq x_1 m_{12} + x_j m_{j2}, \cdots$ we have that $x_1 m_1 + x_2 m_2 \neq 0$, but now $x_2 m_2 + x_j m_j + \cdots = 0$ on $M/\mathcal{J}_1 M$. Iterating the argument we see that eventually $x_n m_n$ will be in $M/\mathcal{J}_{n-1} M$ but this is impossible because $\{x_i\}$ is a regular sequence. $\square$

All previous ideas can be translated "mutatis mutandis" to the category of Lie–Jordan modules. Then the previous theorem can be applied to the instance of $M_0 = \mathcal{L}$ being a LJB–algebra which is a $\mathcal{J}$–module where $\mathcal{J}$ is the ideal generated by the elements $J_\alpha = a(\xi_\alpha)$, $\alpha = 1, \dots, r$, ξ_α being a basis of $\mathfrak{g}$. Then we will have the following theorem:

Theorem 4.7.6. *Given a LJB–algebra $\mathcal{L}$ with a strongly Hamiltonian action of a connected Lie group G and (co-)momentum map $a\colon \mathfrak{g} \to \mathcal{L}$, the Koszul complex $K^\bullet = \mathcal{L} \otimes \Lambda^\bullet \mathfrak{g}$ is acyclic if the sequence of elements $J_\alpha \in \mathcal{L}$, $\alpha = 1, \dots, r$ is* **regular***, this is J_α is a non-zero divisor of $\mathcal{L} \big/ (J_\beta,\ \beta = 1, \dots, r-1) \cdot \mathcal{L}$*

We will say that the action of G on the LJB–algebra is regular if the sequence of elements $a(\xi_\alpha)$, ξ_α basis of $\mathfrak{g}$, is a regular sequence. Then we have

Corollary 4.7.7. *Let G be a connected Lie group acting with a regular strongly Hamiltonian action on the LJB–algebra $\mathcal{L}$ with (co-)momentum map a. Then*

$$H^0_D(S^\bullet) = \mathcal{L}//G, \tag{4.7.12}$$

where $(S^\bullet, D)$ is the BRST complex of the action and $\mathcal{L}//G$ is the reduced Lie–Jordan Banach algebra defined by Eq. (4.6.3).

We expect that the regularity of the action a is the required condition in order to have $\widetilde{\mathcal{L}} = \mathcal{L}//G$. However we still do not have a proof for this and prefer to leave this open problem as a conjecture:

Conjecture. *If the action of G on $\mathcal{L}$ is regular then $\widetilde{\mathcal{L}} = \mathcal{L}//G$ where $\widetilde{\mathcal{L}}$ is the reduced LJB–algebra of $\mathcal{L}$ with respect to the constraint set $\mathcal{C} = \{J_\alpha,\ \alpha = 1, \dots, r\}$.*

4.8. Super Lie–Jordan Banach algebras

The construction of the BRST symmetry in the setting of LJB–algebras leads naturally to the introduction of the notion of a super–LJB algebra. We can proceed as follows:

Definition 4.8.1. A **super** Lie–Jordan algebra is a graded algebra together with two bilinear operations preserving the grading, the graded Jordan product:

$$\circ\colon \mathcal{L}^S \times \mathcal{L}^S \to \mathcal{L}^S, \tag{4.8.1}$$

and the graded Lie product (or supercommutator):

$$[\cdot,\cdot] : \mathcal{L}^S \times \mathcal{L}^S \to \mathcal{L}^S, \tag{4.8.2}$$

such that $\mathcal{L}^m \circ \mathcal{L}^n \subset \mathcal{L}^{m+n}$. We require the graded Jordan product to be graded commutative:

$$a \circ b = (-1)^{|a||b|} b \circ a, \tag{4.8.3}$$

whereas the graded Lie product defines a Lie superalgebra

$$[a, b] = -(-1)^{|a||b|}[b, a], \tag{4.8.4}$$

$$(-1)^{|a||c|}[a, [b, c]] + (-1)^{|b||c|}[c, [a, b]] + (-1)^{|a||b|}[b, [c, a]] = 0. \tag{4.8.5}$$

For the compatibility conditions we require them to satisfy the superderivation property:

$$[a, b \circ c] = [a, b] \circ c + (-1)^{|a||b|} b \circ [a, c], \tag{4.8.6}$$

and the usual associator identity

$$(a \circ b) \circ c - a \circ (b \circ c) = k\ [b, [c, a]], \tag{4.8.7}$$

for some $k \in \mathbb{R}^+$. The weak associativity of the graded Jordan product follows of course from the associator identity.

Regarding the topological structure, we define a super Lie–Jordan Banach algebra if it carries a complete norm $\| \cdot \|$ verifying the usual conditions (see Definition 2.2.2:

i) $\|a \circ b\| \leq \|a\|\ \|b\|$,

ii) $\|a^2\| = \|a\|^2$,

iii) $\|a^2\| \leq \|a^2 + b^2\|$,

$\forall\, a, b \in \mathcal{L}^S$.

A trivial example of super Lie–Jordan algebra of degree 0 is a (non-graded) Lie–Jordan Banach algebra $\mathcal{L}^0$.
A more interesting example is given when we construct from a Lie group G, with Lie algebra $\mathfrak{g}$, the exterior algebra $\Lambda(\mathfrak{g} \oplus \mathfrak{g}^*)$, which possess a super(–associative) LJB–algebra structure with associative Jordan multiplication given by the wedge product and the Lie bracket defined for $X, Y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^*$ by

$$[\alpha, X] = \alpha(X) = [X, \alpha] \quad [X, Y] = 0 = [\alpha, \beta]. \tag{4.8.8}$$

We then extend it to all of $\Lambda(\mathfrak{g} \oplus \mathfrak{g}^*)$ as an odd derivation.
This is a $\mathbb{Z}$–graded SLJB–algebra with elements of $\mathfrak{g}$ having degree -1 and elements of $\mathfrak{g}^*$ having degree $+1$.

4.8.1. Graded differential algebras

Given two super LJB–algebras $\mathcal{P}$ and $\mathcal{Q}$, their tensor product $\mathcal{P} \otimes \mathcal{Q}$ can be given the structure of a super LJB–algebra. According to the Eqs. (B.2.3 and (B.2.4) of the Section B.2, $\forall\, a, b \in P$ and $u, v \in Q$ we define

$$(a \otimes u) \circ (b \otimes v) = (-1)^{|u||b|}(a \circ b \otimes u \circ v + \sqrt{k_1 k_2}\,[a, b] \otimes [u, v]), \tag{4.8.9}$$

$$[a \otimes u, b \otimes v] = (-1)^{|u||b|}(\sqrt{k_2}\, a \circ b \otimes [u, v] + \sqrt{k_1}\,[a, b] \otimes u \circ v), \tag{4.8.10}$$

where k_1 and k_2 are the constants appearing in the associator identity of the algebras $\mathcal{P}$ and $\mathcal{Q}$ respectively. These operations satisfy the axioms of a graded (super) LJB–algebra (4.8.3)-(4.8.7).
Notice that if we require that the canonical immersions $P \hookrightarrow P \otimes Q$, $Q \hookrightarrow P \otimes Q$ to be morphisms, then we arrive to the condition that $k_1 = k_2$ (see Section B.2).
The topology in $P \otimes Q$ will be the one induced from the corresponding C^*–algebra (see Appendix B). For nuclear LJB–algebras it would be unique. An interesting example is given by the tensor product $\mathcal{C} = \mathcal{L} \otimes \Lambda(\mathfrak{g} \oplus \mathfrak{g}^*)$, where $\Lambda(\mathfrak{g})$ is the exterior algebra over the Lie algebra $\mathfrak{g}$. It is a $\mathbb{Z}$–graded LJB–algebra:

$$\mathcal{C} = \bigoplus_n \mathcal{C}^n = \bigoplus_{i-j=n} \mathcal{C}^{i,j} = \bigoplus_{i-j=n} \Lambda^i(\mathfrak{g}^*) \otimes \Lambda^j(\mathfrak{g}) \otimes \mathcal{L}. \tag{4.8.11}$$

Although the bigrading is preserved by the exterior product, the Lie bracket does not preserve it, in fact

$$[\mathcal{C}^{i,j}, \mathcal{C}^{k,l}] \subset \mathcal{C}^{i+k,j+l} \oplus \mathcal{C}^{i+k-1,j+l-1}, \tag{4.8.12}$$

but the total degree is preserved.

A **superderivation** of degree k is a linear map $D\colon \mathcal{C}^n \to \mathcal{C}^{n+k}$ such that

$$D(a \circ b) = (Da) \circ b + (-1)^{k|a|} a \circ (Db), \tag{4.8.13}$$
$$D[a, b] = [Da, b] + (-1)^{k|a|}[a, Db]. \tag{4.8.14}$$

The map $a \to [Q, a]$ for some $Q \in \mathcal{C}^k$ is a superderivation of degree k, called *inner*.
The total differential or BRST operator is an inner superderivation of degree -1, given by $D = [Q, \cdot]$, where $Q \in \mathcal{C}^1$ is explicitly written as:

$$Q = J_\alpha \theta^\alpha - \frac{1}{2} c^\alpha_{\beta\gamma} \theta^\beta \wedge \theta^\gamma \wedge \xi_\alpha, \tag{4.8.15}$$

where $\{\xi_\alpha\}$ (antighosts) are a basis of $\mathfrak{g}$ which satisfies $[\xi_\alpha, \xi_\beta] = c^\gamma_{\alpha\beta}\xi_\gamma$ and $\{\theta^\alpha\}$ (ghosts) are the dual basis for $\mathfrak{g}^*$. It is also verified that $[Q, Q] = 0$ which is equivalent to $D^2 = 0$.

4.8.2. Reduction of super Lie–Jordan Banach algebras

It is clear that we can define the reduction of super LJB–algebras very much as we did for standard LJB–algebras. Thus if $\mathcal{L}^S$ is a super-LJB algebra we define a state as a normalized positive linear function ρ on $\mathcal{L}^S$. Then given a family of constraints $\mathcal{C}$ we define its space of Dirac states in the standard way:

$$\mathcal{S}_D(\mathcal{C}) = \{\rho \in \mathcal{S}_D(\mathcal{L}^S) \mid \rho(c^2) = 0, \quad \forall\, c \in \mathcal{C}\}, \tag{4.8.16}$$

and the super–LJB vanishing subalgebra as:

$$\mathcal{V} = \{a \in \mathcal{L}^S \ \mid \ \rho(a^2) = 0, \quad \forall\, \rho \in \mathcal{S}_D(\mathcal{C})\}. \tag{4.8.17}$$

Then the reduced super–LJB algebra is given by:

$$\widetilde{\mathcal{L}}^S = \mathcal{N}_{\mathcal{V}}/\mathcal{V}, \tag{4.8.18}$$

where $\mathcal{N}_{\mathcal{V}}$ is the normalizer of $\mathcal{V}$ in $\mathcal{L}^S$ with respect to the supercommutator $[\cdot,\cdot]$.

5

APPLICATIONS AND FURTHER DEVELOPMENTS: THE HITCHIN–KOBAYASHI CORRESPONDENCE AND DYNAMICAL SYSTEMS

We have seen in the previous chapter how to give a description of the reduced LJB–algebra in terms of a BRST symmetry and left an open problem, this is the proof of the Conjecture 4.7.3 establishing the condition under which $\widetilde{\mathcal{L}} = \mathcal{L}//G$.

In the first section of this chapter we will deal with the Hitchin-Kobayashi correspondence and show in a simple example that this is nothing but the correspondence between the reduction of C^*– and LJB– algebras. Further work is however still needed in order to understand if this correspondence still holds true as in the general case of Theorem 5.1.1.

Then in the second section we laid the foundations for the theory of dynamical systems and crossed product algebras in the LJB–algebra setting. One main motivation for this study is given by the paper of Doplicher, Kastler and Robinson [DKR66]. The main idea there was that the field algebra of a relativistic quantum system should encode all the algebraic relations of the operators describing the system and its representation theory should comprise only of those covariant representations with respect to the Poincaré group. Here we follow a rather abstract approach by constructing the crossed product LJB–algebra for a locally compact group G. We leave for future work the task of understanding the role of the ∞–dimensional algebra arising from finite dimensional groups with physics applications in mind, for instance when G is the Poincaré group.

This chapter ultimately shows interesting applications of the theory of LJB–algebras but at the same time leaves open space for further developments and generalizations.

5.1. The Hitchin–Kobayashi correspondence

The Hitchin–Kobayashi correspondence establishes the equivalence of two reduction proccesses: symplectic reduction of Kähler manifolds and holomorphic reduction. This is, let (M, ω, g) be a compact Kähler manifold and G a connected compact Lie group acting on M possessing an equivariant momentum map:

$$J\colon M \to \mathfrak{g}^* \tag{5.1.1}$$

and preserving the Kähler structure. If 0 is a regular value of the momentum map and the action is proper, the quotient space $\tilde{M} = J^{-1}(0)/G$ inherits not only a symplectic structure but a Kähler one, becoming in this way a reduced Kähler manifold $(\widetilde{M}, \widetilde{\omega}, \widetilde{g})$.

The Kähler manifold structure on the quotient space was already known in the community of algebraic geometry and was sometimes called the Mumford quotient [MFK65]. It was obtained as follows: let $G^{\mathbb{C}}$ be the complexification of G. Then there is a natural holomorphic action of $G^{\mathbb{C}}$ on M. We will consider now the semistable points M^{ss} on M, i.e. the unstable submanifold of $J^{-1}(0)$ with respect to the gradient flow of $|J|^2$ and the Kählerian metric. Then we have:

Theorem 5.1.1 (Atiyah-Hitchin-Kirwan-Kobayashi-Mumford)**.** *The symplectic reduction of M with respect to the zero level set of J and the holomorphic reduction are isomorphic:*

$$J^{-1}(0)\big/G \cong M^{ss}/G^{\mathbb{C}}, \tag{5.1.2}$$

isomorphism means as Kähler structures.

The simplest example of this situation is provided by $G = U(1)$ acting on $\mathbb{C}^{n+1} = M$ (in this case M is not compact but we may consider $M = \mathbb{C}\mathrm{P}^n$ instead). In this case $J(z) = |z|^2$ and taking $J^{-1}(1)$ (0 is not a regular value of J, however 1 is), the symplectic reduction of $\mathbb{C}^{n+1}$ by the level set 1 of the momentum map J is just $\mathbb{C}\mathrm{P}^n$:

$$J^{-1}(1)\big/U(1) = S^{2n+1}\big/U(1) \cong \mathbb{C}\mathrm{P}^n. \tag{5.1.3}$$

On the other hand the semistable points of $J^{-1}(1)$ under the gradient flow of J^2 are just $\mathbb{C}^{n+1} \setminus \{0\}$. Note that $U(1)^{\mathbb{C}} \cong \mathbb{C}^*$, the multiplication group of complex numbers, hence $\mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^* \cong \mathbb{C}\mathrm{P}^n$ as it should be according to the previous theorem.

The description of the previous correspondence using the theory of LJB or C^*–algebras will be done by using the geometrization approach to quantum mechanics proposed by A. Ashtekar [AS99] and developed by G. Marmo *et al.* [EMM10], [CCGM07], [GMK05]. In the next subsection we review very briefly the geometrical formulation of quantum mechanics starting with a standard Hilbert space formulation.

5.1.1. The geometrical formulation of quantum mechanics

The physical carrier space of our formulation (the space of pure states) should be identified with the space of rays of the Hilbert space $\mathcal{H}$. The true space of pure quantum states is the complex projective Hilbert space $\mathbb{P}(\mathcal{H})$. Any vector $\phi \in \mathcal{H}$ defines a constant vector field $X_\phi \colon \mathcal{H} \to T\mathcal{H} \cong \mathcal{H} \times \mathcal{H}$ by $X_\phi(\psi) := (\psi, \phi)$.

Similarly, a bounded linear operator $A \colon \mathcal{H} \to \mathcal{H}$ gives rise to a bundle morphism $T_A \colon T\mathcal{H} \to T\mathcal{H}$ defined as follows $T_A(\psi, \phi) := (\psi, A\phi)$. Moreover, one introduces a complex structure $J \colon T\mathcal{H} \to T\mathcal{H}$ defined by the (1, 1)–tensor field $J(\psi, \phi) := (\psi, i\phi)$, and a linear structure $\Delta \colon \mathcal{H} \to T\mathcal{H}$ defined by the Liouville vector field $\Delta(\psi) := (\psi, \psi)$. Finally the so called phase–vector field $\Gamma \colon \mathcal{H} \to T\mathcal{H}$ is defined by $\Gamma := J \circ \Delta$, i.e. $\Gamma(\psi) = (\psi, i\psi)$.

The Hermitian product $\langle\psi|\psi\rangle$ on $\mathcal{H}$ is replaced by an Hermitian tensor field on $\mathcal{H}$:

$$h(X_{\phi_1}, X_{\phi_2})(\psi) := \langle\phi_1|\phi_2\rangle \tag{5.1.4}$$

on the corresponding real differential manifold $\mathcal{H}^{\mathbb{R}}$. The real part of h is a Riemannian metric tensor g, while its imaginary part is a symplectic tensor field ω on $\mathcal{H}^{\mathbb{R}}$. Then $h = g + i\omega$, and $\omega(X, Y) = g(JX, Y)$. The above tensor fields endow $\mathcal{H}$ with the structure of a Kähler manifold. If we denote by G and Λ the contravariant counterpart of the covariant tensors g and ω respectively, they give rise to two bi-differential operators which may be used to define two brackets on the space of one-forms.

These tensor fields may also be used to define the metric structure and Poisson bracket on the space of rays $\mathbb{P}(\mathcal{H})$. Note, however, that neither G nor Λ can be directly projected from $\mathcal{H}$ to $\mathbb{P}(\mathcal{H})$. It is easy to show that the corresponding tensor fields which are projectable are given by

$$\tilde{G} := e^{\sigma} G - \Delta \otimes \Delta - \Gamma \otimes \Gamma; \quad \tilde{\Lambda} := e^{\sigma}\Lambda - (\Delta \otimes \Gamma - \Gamma \otimes \Delta), \tag{5.1.5}$$

where the conformal factor $e^{\sigma} \geq 0$ is defined by $\sigma(\psi) := \ln\langle\psi|\psi\rangle$. The projected tensor fields, that will be denoted again as G and Λ without creating confussion, allow for the definition of two products in the space of functions on $\mathbb{P}(\mathcal{H})$: the symmetric bracket

$$f_1 \circ f_2 := G(df_1, df_2) + f_1 \cdot f_2, \tag{5.1.6}$$

and the antisymmetric Poisson bracket

$$\{f_1, f_2\} := \Lambda(df_1, df_2), \tag{5.1.7}$$

with f_1, f_2 arbitrary real valued functions in $\mathcal{F}(\mathbb{P}(\mathcal{H}))$. In this formulation quantum observables are defined to be functions f whose Hamiltonian vector fields are also Killing vector fields, i.e.

$$\mathcal{K}(\mathbb{P}(\mathcal{H})) := \{f \in \mathcal{F}(\mathbb{P}(\mathcal{H})) \mid \mathcal{L}_{X_f} G = 0\}, \tag{5.1.8}$$

where $X_f = \Lambda(df)$. We call such function f a Kählerian function. A complex valued function is Kählerian if and only if both real and imaginary parts are Kählerian. On the space $\mathcal{A}(\mathbb{P}(\mathcal{H}))$ of complex Kählerian functions we may define an associative bilinear product $f \star g$ associated to the contravariant Hermitian tensor $H = G + i\Lambda$:

$$f \star g := f \cdot g + H(df, dg). \tag{5.1.9}$$

One shows that for any two Kählerian functions f and g the nonlocal product $f \star g$ defines a Kählerian function. We observe that any complex valued Kählerian function on $\mathbb{P}(\mathcal{H})$ corresponds to an operator $A \in B(H)$ by means of:

$$A \mapsto f_A([\psi]) \equiv \frac{\langle \psi \mid A\psi \rangle}{\langle \psi \mid \psi \rangle}, \tag{5.1.10}$$

that is, f_A is an expectation value function. It is easy to show that $f_A \star f_B = f_{AB}$. Thus quantum observables correspond to real valued Kählerian functions and hence they are represented by Hermitian operators on $\mathcal{H}$.

The space $\mathcal{A}(\mathbb{P}(\mathcal{H}))$ equipped with the above associative noncommutative product provides a realization of a C^*–algebra isomorphic to the C^*–algebra of bounded operators on the Hilbert space $\mathcal{H}$, with the supremum norm

$$\|f\|_\infty = \sup_{[\psi] \in \mathbb{P}\mathcal{H}} \|f([\psi])\|, \quad \forall\, f \in \mathcal{A}(\mathbb{P}(\mathcal{H})). \tag{5.1.11}$$

Consider now a general Kähler manifold (M, h) not necessarily a projective Hilbert space. It is clear that one may define a nonlocal $\star$–product as before: $f \star g :=$

$f \cdot g + H(df, dg)$, for arbitrary Kählerian functions f, g on M. Now, for an arbitrary manifold M the corresponding space of complex valued Kählerian functions is not closed under the $\star$–product. The Poisson bracket $\{f, g\} = \frac{i}{2}(f \star g - g \star f)$ is again Kählerian, however, the symmetric bracket $f \circ g = \frac{1}{2}(f \star g + g \star f)$, in general is not. The condition that the space of Kählerian function over M is closed with respect to the symmetric bracket puts strong conditions on the Kähler structure, namely the holomorphic sectional curvature of M is constant [CL84]. This in turn implies that M is a projective Hilbert space $\mathbb{P}\mathcal{H}$ or the covering space of the symplectic orbit in $\mathfrak{u}(\mathcal{H})$. Thus only orbits of the unitary group are associated with C^*–algebras under this correspondence.

5.1.2. Correspondence between symplectic and Kähler reduction

Now consider a compact group G acting by Kähler transformations on $\mathbb{P}\mathcal{H}$, i.e. symplectic and isometric transformations. For instance we may imagine $G = U(1)$ acting on $\mathbb{P}\mathcal{H}$ as

$$\mathrm{e}^{i\varphi}[z_1, z_2, \ldots, z_n, \ldots] = [\mathrm{e}^{i\varphi}\, z_1, \ldots, e^{i\varphi} z_n, z_{n+1}, \ldots], \quad n \geq 2 \qquad (5.1.12)$$

i.e. it acts just on the first n homogeneous coordinates z_k, where $|\phi\rangle = \sum_{k=1}^{\infty} z_k |k\rangle$ and $|k\rangle$ is an orthonormal basis on $\mathcal{H}$. The symplectic reduction of $M = \mathbb{C}\mathrm{P}^n$ by the action of $U(1)$ before in the first homogeneous coordinate will be $\mathbb{C}\mathrm{P}^{n-1}$. In infinite dimensions the reduction is going to be $\mathbb{P}\mathcal{H}'$, where $\mathcal{H}' \subset \mathcal{H}$ is a closed subspace (still infinite dimensional) of $\mathcal{H}$. Now this reduction can be performed either using the LJB algebra $(\mathcal{K}(\mathbb{P}\mathcal{H}), \circ, [\cdot, \cdot])$ or the C^*–algebra $(\mathcal{A}(\mathbb{P}\mathcal{H}), \star)$. We will obtain that the reduced LJB–algebra because:

$$\widetilde{\mathcal{K}}(\mathbb{P}\mathcal{H}) = \mathcal{K}(\mathbb{P}\mathcal{H}'), \qquad (5.1.13)$$

which is the self-adjoint part of the reduced C^*–algebra

$$\widetilde{\mathcal{A}(\mathbb{P}\mathcal{H})} = \mathcal{K}(\mathbb{P}\mathcal{H}') = \mathcal{K}(J^{-1}(1)/U(1)), \qquad (5.1.14)$$

indicating that the equivalence between symplectic and Kähler reduction is just the correspondence discussed in Section 3.4.1, Thm. 3.4.8 between reduced LJB–algebras and C^*–algebras.

5.2. Dynamical systems on LJB–algebras

Dynamical systems are the mathematical formulation of dynamics, this is, how physical systems change in time. By exploring the algebraic properties of dynamics, it has been recognized that C^*–algebras provide the algebraic framework for studying the time evolution of physical systems. Moreover the theory of dynamical systems on C^*–algebras plays an important role in the applications of the theory and in the development and foundations of noncommutative differential geometry. In this section we will develop some of the foundations of the corresponding theory for LJB–algebras and derive some of its basic results.

5.2.1. The LJB–algebra of a group G

In this subsection we discuss how to construct a LJB–algebra from generators and relations. Given a generic set S whose elements will be viewed as generators, we can consider the free *–algebra generated by $S \cup S^*$, where S^* is a collection of elements $a^*,\ a \in S$. This free *–algebra $\mathcal{F}(S)$ is the family of all noncommutative polynomials with complex coefficients in the variables a and a^* with a *–operation defined in the evident way:

$$(ab)^* = b^* a^*, \quad (a^*)^* = a, \quad \forall\, a, b \in S. \tag{5.2.1}$$

We will call the free Lie–Jordan algebra $\mathcal{L}(S)$ generated by S, the selfadjoint part of $\mathcal{F}(S)$ with the classical Lie–Jordan algebraic structure induced from $\mathcal{F}(S)$, i.e.

$$\mathcal{L}(S) = \{\xi \in \mathcal{F}(S) \ \mid \ \xi^* = \xi\}, \tag{5.2.2}$$

$$\xi \circ \zeta = \frac{1}{2}(\xi\zeta + \zeta\xi); \quad [\xi, \zeta] = \lambda(\xi\zeta - \zeta\xi), \tag{5.2.3}$$

for some real number λ. Notice that $a \notin \mathcal{L}(S)$, however $a+a^*$, $i(a-a^*) \in \mathcal{L}(S)$.

Relations are polynomials in the generators $a \in S \cup S^*$. We will denote by $\mathcal{R} \subset \mathcal{F}(S)$ a set of relations. Let $\mathcal{J}(S,\mathcal{R})$ be the Lie–Jordan ideal generated by $\mathcal{R}$ in $\mathcal{L}(S)$. Then we define the Lie–Jordan algebra generated by S with relations $\mathcal{R}$ the quotient Lie–Jordan algebra:

$$\mathcal{L}(S,\mathcal{R}) \equiv \mathcal{L}(S)\Big/\mathcal{J}(S,\mathcal{R})\,. \tag{5.2.4}$$

Notice that a relation could be for instance $a = a^*$, this would imply that $a \in S$ would become a real element and it would belong to $\mathcal{L}(S,\mathcal{R})$ (notice that $\frac{1}{2}(a+a^*)$ is always in $\mathcal{L}(S,R)$).

Now we can look for representations of $\mathcal{L}(S,\mathcal{R})$ in a Hilbert space $\mathcal{H}$ as discussed in Subsection 2.2.1. We define the *universal* Lie–Jordan Banach norm for any $x \in \mathcal{L}(S,\mathcal{R})$ as:

$$\|x\|_{LJB} \equiv \sup\{\|\pi(x)\|_{\mathcal{H}} \;\mid\; (\pi,\mathcal{H}) \text{ is a representation of } \mathcal{L}(S,\mathcal{R})\}. \tag{5.2.5}$$

Then we define the LJB–algebra generated by the set S and the relations $\mathcal{R}$ as the closure of $\mathcal{L}(S,\mathcal{R})$ with respect to the universal LJB–norm $\|\cdot\|_{LJB}$. We will keep in the following the same notation for $\mathcal{L}(S,\mathcal{R})$ and its closure.

Remark. Notice that the universal norm can be infinite, for instance if $S = \{x\}$, then $\pi(x) \in \mathbb{R}$ can be any number, and hence $\sup\|\pi(x)\| = +\infty$. We will say that the LJB–algebra generated by S and $\mathcal{R}$ does not exist if for at least one element $x \in \mathcal{L}(S,\mathcal{R})$, $\|x\|_{LJB} = +\infty$.

Example 5.2.1. Let $S = \{\mathbb{1}, u\}$ and $\mathcal{R} = \{uu^* = \mathbb{1} = u^*u,\ \mathbb{1}u = u\mathbb{1} = u,\ 1^* = 1\}$. Then $\mathcal{L}(S,\mathcal{R})$ is the set of polynomials $f = \sum_{k=-N}^{N} c_k u^k$ such that $c_k^* = c_{-k}$. Notice that any Lie–Jordan representation π of $\mathcal{L}(S,\mathcal{R})$ can be extended to a *–representation of $\mathcal{F}(S,\mathcal{R}) = \mathcal{F}(S)/\mathcal{J}(S,\mathcal{R})$. Denoting with the same symbol such representation we get then that

$$\pi(u)^* = \pi(u^*) = \pi(u)^{-1} \tag{5.2.6}$$

and $\pi(u)$ is a unitary operator. Then $\|u\|_{C^*} = 1$, where $\|\cdot\|_{C^*}$ is the universal C^*–norm of the C^*–algebra generated by $\mathbb{1}$ and u with relations $\mathcal{R}$. Then this algebra, denotedy $\mathcal{A}(S, \mathcal{R})$, is the closure with respect to the norm $\|\cdot\|_{C^*}$ of commutative polynomials on the variable u, that is complex continuous functions on the circle S^1, i.e. $\mathcal{A}(S, \mathcal{R}) = C(S^1, \mathbb{C})$. The corresponding LJB–algebra $\mathcal{L}(S, \mathcal{R})$ is given by the real continuous functions on the circle with the sup-norm.

Let G be a group. Consider the set $S = G$ and the collection of relations:

$$\mathcal{R} = \{gh = k,\ g^* = g^{-1},\ g^*g = 1 = gg^* \mid g, h, k \in G\}. \tag{5.2.7}$$

We will call the LJB–algebra of the group G the LJB–algebra generated by S and R and it will be denoted by $\mathcal{L}(G)$.
Then any representation π of the Lie–Jordan algebra $\mathcal{L}(G) = \mathcal{L}(\mathcal{R}, S)$ can be extended to a *–representation of the *–algebra generated by S and $\mathcal{R}$. But then any such representation is unitary: $\pi(g)^* = \pi(g^{-1}) = \pi(g)^{-1}$. If we denote by $\mathcal{A}(G)$ the *–algebra generated by G, by $\mathcal{L}(G)$ the Lie–Jordan algebra of G, and by $C^*(G)$ the C^*–algebra generated by G, i.e. $C^*(G) = \overline{\mathcal{A}(G)}^{\|\cdot\|_{C^*}}$, then the LJB–algebra generated by G is the self-adjoint part of $C^*(G)$

$$\mathcal{L}(G) = C^*(G)_{sa}. \tag{5.2.8}$$

If our group G is a locally compact topological group, then there exists a left-invariant measure on it, called the Haar measure μ_G. We consider then the Hilbert space $L^2(G, \mu_G)$ and the left-regular representation L of G on $L^2(G, \mu_G)$:

$$(L(g)\psi)(h) = \psi(g^{-1}h) \quad \forall\, \psi \in L^2(G, \mu_G),\ g,\ h \in G. \tag{5.2.9}$$

When we consider $\mathcal{A}(G)$, its elements are (noncommutative) polynomials on elements of g, i.e. $a = \sum_{g\in G} a(g)g,\ a(g) \in \mathbb{C}$.
Then we define the norm of $a \in \mathcal{A}(G)$ as the norm of an operator on $L^2(G)$, i.e.

$$\|a\|_L \equiv \|L(a)\|_{L^2(G)}, \tag{5.2.10}$$

with $L(a)\psi = \sum a(g)L(g)\psi$. We can also consider the closure of $\mathcal{A}(G)$, and in the same way of $\mathcal{L}(G)$, with respect to the norm induced by the left regular representation of G. Notice that in this case both $\overline{\mathcal{A}(G)}^{\|\cdot\|_L}$ and $\overline{\mathcal{L}(G)}^{\|\cdot\|_L}$ will be closed algebras inside $\mathcal{B}(L^2(G))$. The question then is when is $\|a\|_L = \|a\|_{C^*}$? We can actually use this as a definition of "amenability".

Definition 5.2.2. We will say that the locally compact topological group G is **amenable** if $\|\cdot\|_L = \|\cdot\|_{C^*}$. In such case the closed algebras generated by $\mathcal{A}(G)$ and $\mathcal{L}(G)$ via the left regular representation (also called the reduced algebras) coincide with the universal C^*– and LJB– algebras $C^*(G)$ and $\mathcal{L}(G)$ generated by G.

Many groups of common use are amenable, for instance finite groups, compact groups, nilpotent and solvable discrete groups, etc. However free groups, $\mathrm{SL}(2,\mathbb{Z})$, etc. are not amenable.

5.2.2. Dynamical systems on Lie–Jordan Banach algebras

Let G be a discrete group. Suppose now that $\rho\colon G \to \mathrm{Aut}(\mathcal{L})$ is a morphism where $\mathcal{L}$ is a LJB–algebra and $\mathrm{Aut}(\mathcal{L})$ denotes its group of automorphisms. We will call $(G,\rho,\mathcal{L})$ a discrete **dynamical system** on $\mathcal{L}$.

Example 5.2.3. If $\mathcal{L} = C(M)$, the space of continuous real functions on a cpmpact manifold M, then for every $g \in G$, $\rho(g)$ induces a homomorphism of M, i.e. a discrete dynamical system on M. Conversely any discrete group G acting by continuous maps on M defines a discrete dynamical system in the sense of the previous definition.

5.2.3. The crossed-product algebra $\mathcal{L} \ltimes_\rho G$

Let $(G, \rho, \mathcal{L})$ be a dynamical system on the LJB–algebra $\mathcal{L}$ and G a discrete group. Consider the set $C_c(G, \mathcal{L})$ of all real functions $f\colon G \to \mathcal{L}$ which are zero except on a finite subset of G, i.e.

$$f = \sum_{\substack{g \in S \\ S \subset G \text{ finite}}} f_g x_g. \tag{5.2.11}$$

Now suppose that we have a unitary representation U of G on a Hilbert space $\mathcal{H}$, i.e. $U\colon G \to \mathcal{U}(\mathcal{H})$; then a representation π of $\mathcal{L}$ on $\mathcal{H}$ will be said to be covariant with respect to the dynamical system $(G, \rho, \mathcal{L})$ if:

$$\pi(\rho_g(x)) = U_g \pi(x) U_g^{-1}, \quad \forall\, g \in G,\, \forall\, x \in \mathcal{L}. \tag{5.2.12}$$

Remark. This is exactly the covariance condition for quantum fields, i.e. a quantum field Φ is an operator-valued distribution on a space-time X, i.e. $\Phi\colon C(X) \to \mathcal{B}(\mathcal{H})$. Then suppose there is a group G acting on X with action ρ and represented unitarily on $\mathcal{H}$ by U. The covariance property is writtern:

$$\Phi(\rho_g(x)) = U_g \Phi(x) U_g^{-1}, \quad \forall\, g \in G,\, \forall\, x \in X. \tag{5.2.13}$$

See for instance [BIMM11] for a recent discussion on this topic.

Now if we have a covariant representation π of the LJB–algebra $\mathcal{L}$ we can consider the LJB–algebra generated by the operators $\pi(x) \in \mathcal{B}(\mathcal{H})$, $x \in \mathcal{L}$ and $U_g \in \mathcal{B}(\mathcal{H})$, $g \in G$, satisfying the covariance relation

$$U_g \pi(x) = \pi(\rho_g(x)) U_g. \tag{5.2.14}$$

Notice that because of the previous expression, words in this algebra can always be rearranged to be of the form $\pi(x)U_g$ (i.e. we can always pass $\pi(x)$ over U_g to the left by using (5.2.14)). Thus finite sums of them become

$$\sum_{g \in G} f_g \pi(x_g) U_g = \sum_{g \in G} \pi(f_g x_g) U_g, \tag{5.2.15}$$

In other words each function $f \in C_c(G, \mathcal{L})$ determines an element of the LJB–algebra generated by G and $\mathcal{L}$ as before:

$$\xi_f = \sum_{g \in G} \pi(f(g)) U_g. \tag{5.2.16}$$

Now if $f, h \in C_c(G, \mathcal{L})$, then we can define two products among them: $\circ$ and $[\cdot, \cdot]$ as

$$\xi_f \circ \xi_h \equiv \xi_{f \circ h} \tag{5.2.17}$$

$$[\xi_f, \xi_h] = \xi_{[f,h]}. \tag{5.2.18}$$

A simple computation shows:

$$\begin{aligned}
\xi_f \circ \xi_h |\phi\rangle &= \sum_g \pi(f(g)) U_g \circ \sum_{g'} \pi(h(g')) U_{g'} |\phi\rangle \\
&= \sum_{g,g'} \pi(f(g)) \circ \pi(\rho_g h(g')) U_g U_{g'} |\phi\rangle \\
&= \sum_g \Big(\Big(\sum_{g'} \pi(f(g)) \pi(\rho_g (h(g'' g^{-1}) \Big) U_{g''} |\phi\rangle \Big) \\
&= \sum_{g''} \Big(\sum_g \pi(f(g) \circ \rho_g (h(g'' g^{-1})) \Big) U_{g''} |\phi\rangle \\
&= \sum_{g''} \pi(f \circ h(g'')) U_{g''} |\phi\rangle \\
&= \xi_{f \circ h} |\phi\rangle,
\end{aligned} \tag{5.2.19}$$

that is

$$(f \circ h)(g) = \sum_{g'} f(g') \circ \rho_{g'} h(g g'^{-1}), \tag{5.2.20}$$

where we have used $gg' = g''$ and $g' = g'' g^{-1}$.
In a similar way we can prove that

$$[f, g](g) = \sum_{g'} [f(g'), \rho_{g'} h(g g'^{-1})] \tag{5.2.21}$$

and the combination $f * g = f \circ g + i\,[f, g]$ is the standard convolution product on the C^*–algebra $\mathcal{L}^{\mathbb{C}}$.

Moreover notice that

$$\|\xi_f\| \leq \sum_{g\in G} \|f(g)\|\;\|U_g\| = \sum_{g\in G} \|f(g)\| = \|f\|_1. \tag{5.2.22}$$

Then $\|f \circ g\|_1 \leq \|f\|_1 \|g\|_1$. Similarly we may prove that $[f, g]$ is continuous.
Let us define now the norm $\|\cdot\|_\rho$ on $C_c(G, \mathcal{L})$ as

$$\|f\|_\rho = \sup\{\|\xi_f\|_{\mathcal{H}} \;\mid\; \xi_f = \sum_g f_g \pi(x_g) U_g,\; (\pi, U) \text{ cov. repr. of } (G, \mathcal{L}, \rho)\}. \tag{5.2.23}$$

Then we will denote then the LJB–algebra generated in this way as:

$$\mathcal{L} \ltimes_\rho G, \tag{5.2.24}$$

and will be called either the covariance LJB–algebra or the crossed product LJB–algebra for $(\mathcal{L}, G, \rho)$.
It is clear that the previous construction can be repeated "mutatis mutandi", for a locally compact group G replacing functions $f : G \to \mathcal{L}$ defined on finite sets by functions with compact support and replacing the sums $\sum_{g\in G}$ with the integrals $\int_G d\mu_G$. So in what follows we will assume that G is a locally compact group.

It is natural now to investigate the representations of $\mathcal{L} \ltimes_\rho G$.
Consider any representation $(\pi_0, \mathcal{H}_0)$ of $\mathcal{L}$, and let $\mathcal{H}$ be now the Hilbert space of square integrable functions on G with values in $\mathcal{H}_0$, i.e.

$$\mathcal{H} = L^2(G; \mathcal{H}_0). \tag{5.2.25}$$

Let $U : G \to \mathcal{U}(\mathcal{H})$ be defined as

$$(U_g\psi)(g') = \psi(g^{-1}g') \tag{5.2.26}$$

and $\pi\colon \mathcal{L} \ltimes_\rho G \to \mathcal{B}(\mathcal{H})$ as

$$(\pi(x)\psi)(g) = \underbrace{\pi_0(\underbrace{\rho_{g^{-1}}(x)}_{\in\mathcal{L}})}_{\in\mathcal{B}(\mathcal{H}_0)} \underbrace{\psi(g)}_{\in\mathcal{H}_0}. \tag{5.2.27}$$

For any element $f \in C_c(G, \mathcal{L})$ as before we get:

$$\begin{aligned}(\pi(f)\psi)(g) &= \Big(\pi\Big(\int_G d\mu_G(g')f(g')x_{g'}\Big)\psi\Big)(g)\\ &= \int_G d\mu_G(g')f(g')(\pi(x_{g'})\psi)(g)\\ &= \int_G d\mu_G(g')f(g')\pi_0(\rho_{g^{-1}}(x_{g'}))\psi(g). \end{aligned}\tag{5.2.28}$$

Notice that

$$\begin{aligned}(U_g(\pi(x)\psi))(g') &= (\pi(x)\psi)(g^{-1}g') = \pi_0(\rho_{g'^{-1}g}(x))\psi(g^{-1}g')\\ &= \pi_0(\rho_{g'^{-1}}(\rho_g(x)))(U_g\phi)(g') = (\pi(\rho_g(x))U_g)(g'),\end{aligned}$$

that is

$$U_g\pi(x) = \pi(\rho_g(x))U_g. \tag{5.2.29}$$

We will call these representations **natural representations** of $\mathcal{L} \ltimes_\rho G$ (and they play a similar role as the regular left representation for $\mathcal{L}(G)$).

So we may set for $f \in C_c(G, \mathcal{L})$

$$\|f\|_r = \sup\{\|\pi(f)\|,\ \pi \text{ natural representation}\}. \tag{5.2.30}$$

In general this norm is different from the universal norm on $\mathcal{L} \ltimes_\rho G$ defined before: $\|f\|_r \neq \|f\|_\rho$.

Then again as in the case of $C^*(G)$ and $\mathcal{L}(G)$, we will say that the dynamical system $(\mathcal{L}, G, \rho)$ is *amenable* if $\|f\|_r = \|f\|_\rho$.
It is clear that if G is amenable then $(\mathcal{L}, G, \rho)$ is amenable. However the converse is not true.

5.2.4. A simple example

Let G be a finite group, $G = \{g_1 = e, g_2, \dots, g_N\}$. Then it is clear that the C^*–algebra generated by G is just the group algebra

$$\mathbb{C}\,[G] = \{f = \sum_{g \in G} f(g)g = \sum_{i=1}^{N} f(g_i)g_i\}, \tag{5.2.31}$$

where $g^* = g^{-1}$. The algebraic product is the convolution product

$$\begin{aligned} f * h &= \sum_g f(g)g \sum_{g'} h(g')g' = \sum_{g,g'} f(g)h(g') \underbrace{gg'}_{g''} \\ &= \sum_{g''} \Big(\sum_g f(g)h(g^{-1}g'') \Big) g'' = \sum_{g''} (f*h)(g'')g'' \end{aligned} \tag{5.2.32}$$

i.e.

$$(f*h)(g'') = \sum_g f(g)h(g^{-1}g^*), \tag{5.2.33}$$

where the involution * is given by

$$f^* = \sum_g \overline{f(g)}g^* = \sum_g \overline{f(g)}g^{-1} = \sum_g \overline{f(g^{-1})}g, \tag{5.2.34}$$

i.e. $f^*(g) = \overline{f(g')}$.
Now the LJB–algebra generated by G is the real part of $\mathbb{C}\,[G]$, i.e.

$$\mathcal{L}(G) = \{f \ \mid f^* = f\} = \{f \ \mid \overline{f(g^{-1})} = f(g)\}. \tag{5.2.35}$$

This is,

$$\mathcal{L}(G) = \{f = \sum_{g \in G} f(g)g \ \mid \ \overline{f(g)} = f(g^{-1})\}. \tag{5.2.36}$$

The LJB–algebra $\mathcal{L}(G)$ carries the Jordan product

$$g \circ g' = \frac{1}{2}(gg' + g'g), \quad \forall\, g, g' \in G, \tag{5.2.37}$$

or, in other words:

$$(f \circ h) = \sum_g f(g)g \circ \sum_{g'} h(g')g' = \sum_{g,g'} \frac{1}{2} f(g)h(g')(gg' + g'g). \qquad (5.2.38)$$

If G is abelian $gg' = g'g$, then $f \circ h = \sum_{g,g'} f(g)h(g')gg' = f * g$.
Similarly, the Lie product is given by

$$[g, g'] = \lambda(gg' - g'g), \quad \forall\, g, g' \in G. \qquad (5.2.39)$$

Then the LJB algebra $\mathcal{L}(G)$ becomes a Lie algebra of dimension $|G|$. Again if G is abelian then $[g, g'] = 0$. Now suppose that $U\colon G \to \mathcal{U}(\mathcal{H})$ is any unitary representation of G on some finite dimensional Hilbert space $\mathcal{H}$ and let χ be its character, i.e.

$$\begin{aligned} \chi\colon G &\longrightarrow \mathbb{C} & (5.2.40)\\ g &\mapsto \operatorname{Tr} U(g). & (5.2.41) \end{aligned}$$

Then $\chi \in \mathbb{C}[G]$ and $\chi = \sum_g \chi(g)g$, but

$$\chi^* = \sum_g \overline{\chi(g^{-1})} g = \sum_g \chi(g)g = \chi, \qquad (5.2.42)$$

because $\chi(g^{-1}) = \operatorname{Tr} U(g^{-1}) = \operatorname{Tr} U(g)^* = \overline{\operatorname{Tr} U(g)} = \overline{\chi(g)}$. This implies that $\chi \in \mathcal{L}(G)$.
Notice that $\chi(gg') = \chi(g'g)\ \forall\, g, g' \in G$, i.e. $\chi(gg' - g'g) = \chi([g, g']) = 0$, which implies $\chi([f, h]) = 0, \forall\, f, h \in \mathcal{L}(G)$.
Now it is clear that:

Lemma 5.2.4. *If F is a central function on $\mathcal{L}(G)$ then $[F, H] = 0$, $\forall\, H \in \mathcal{L}(G)$.*

Proof.

$$(F * H)(g) = \sum_{g'} F(g')H(g'^{-1}g), \qquad (5.2.43)$$

then if F is central $F(g'g^{-1}) = F(g^{-1}g')$ or $F(gg'g^{-1}) = F(g')$. This implies

$$\begin{aligned}\sum_{g'} F(g')H(g'^{-1}g) &= \sum_{g'} F(gg'g^{-1})H(\underbrace{g'^{-1}g}_{g''}) = \sum_{g''} F(gg''^{-1})H(g'') \\ &= \sum_{g''} H(g'')F(g''^{-1}g) = (H * F)(g), \end{aligned} \tag{5.2.44}$$

hence $F * H = H * F$ and $[F, H] = 0$. □

Corollary 5.2.5. *The center of the Lie algebra structure of $\mathcal{L}(G)$ is given by the central functions of G.*

Notice that the center of $(\mathcal{L}(G), [\cdot, \cdot])$ are the Casimirs of the Lie bracket and then because of the previous lemma they are determined by the orbits $gg'g^{-1}$ of the group G. The characters of G are central functions and actually the characters of irreducible representations form an (orthonormal) basis of the space of central functions. Then we have that

$$[\chi_\alpha, \chi_\beta] = 0. \tag{5.2.45}$$

Finally it is easy to check that

$$\chi_\alpha \circ \chi_\beta = \delta_{\alpha\beta}\chi_\beta. \tag{5.2.46}$$

A similar analysis could be performed also in the case of compact Lie groups.

FINAL CONCLUSIONS

In this dissertation we developed the theory of Lie–Jordan Banach algebras and its applications in mathematical physics. The main idea driving this research is the algebraic approach to symmetries and constraints in classical and quantum mechanics. We will summarize briefly the main results of this thesis and underline the topics which deserve further investigation.

In Chapter 2 we have shown that the concept of dynamical correspondence [AS98] on Jordan–Banach algebras is equivalent to the definition of Lie–Jordan Banach algebras, thus providing a theorem on the correspondence between C^*–algebras and Lie–Jordan Banach algebras: a C^*–algebra $\mathcal{A}$ is always the complexification of a Lie–Jordan Banach algebra $\mathcal{L}$, $\mathcal{A} = \mathcal{L} \oplus i\mathcal{L}$, and conversely a Lie–Jordan Banach algebra is always the self-adjoint part of a C^*–algebra.
Then in Chapter 3 we have cast the geometric reduction of classical mechanics in the algebraic language of associative Lie–Jordan algebras. Then we have addressed the problem of quantum constraints in algebraic terms and obtained a theory of reduction of Lie–Jordan Banach algebras, which is equivalent to the T-reduction of C^*–algebras developed by Grundling *et al.* [GH85]. In this context it would be interesting to extend the generalized Poisson reduction developed in Section 3.3 to the quantum case. As explained at the end of the same section, it is not trivial to induce a Banach structure in the quotient space $\mathcal{B}/(\mathcal{B} \cap S)$. This generalized reduction is very appealing in the quantum setting since conditions 3.3.18a very much resembles the presence of quantum anomalies. Then a notion of generalized Lie–Jordan Banach reduction would provide a good framework to

study quantum anomalous systems.

In Chapter 4 we have extended the classical theory of BRST symmetry to the quantum case. This led us to define a strongly Hamiltonian action of a symmetry group G (with Lie algebra $\mathfrak{g}$) on a quantum systems as a map $\hat{\rho}\colon \mathfrak{g} \to \mathrm{Der}\,\mathcal{L}$ such that the derivations are inner, that is there exists a map (4.5.4) $a\colon \mathfrak{g} \to \mathcal{L}$ such that

$$\hat{\rho}(\xi)x = [x, a(\xi)] \tag{5.2.47}$$

$\forall\, x \in \mathcal{L}$ and $\xi \in \mathfrak{g}$. We call the map *a quantum co-momentum map*. This enables us to construct the BRST complex $(S^\bullet, \hat{D})$, whose zeroth cohomology group is the quantum constrained algebra (4.6.3) $\mathcal{L}//G$. Then further work is needed to understand under which conditions this reduction is the same as the one developed in Section 3.4.
Finally in the last chapter, we deal with the Hitchin-Kobayashi correspondence and show in a simple example that this is nothing but the correspondence between the reduction of C^*– and LJB– algebras as stated by Theorem 3.4.8. Further investigation is however still needed in order to understand if this correspondence still holds true as in the general case of Theorem 5.1.1.
Then we develop the foundations for the theory of dynamical systems and crossed product algebras in the LJB–algebra setting. We follow a rather abstract approach by constructing the crossed product LJB–algebra for a locally compact group G. We leave for future work the task of understanding the role of the ∞–dimensional algebra arising from finite dimensional groups with physics applications in mind, for instance when G is the Poincaré group. We expect in this case to obtain the field algebras of relativistic quantum systems which encode the appropriate algebraic relations and whose representation theory should be covariant with respect to the Poincaré group.

In conclusion, the message we tried to convey in this thesis is that Lie–Jordan Banach algebras represent an original algebraic approach with many applications in Mathematics and Physics, and a fertile ground for exploration and better understanding of consolidated theories.

A

DIFFERENTIABLE AND SYMPLECTIC GROUP ACTIONS

Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Suppose G acts smoothly on a differentiable manifold M, i.e. there is a smooth map $\rho\colon\ G\times M \to M$ such that $\rho(g,m) \equiv g\cdot m$, and $g\cdot(h\cdot m) = (gh)\cdot m$, $\forall\, g,h \in G$, $m \in M$ and $e\cdot m = m \,\forall\, m \in M$, with e being the identity element of the group.

Let $\mathfrak{X}(M)$ denote the Lie algebra of vector fields on M. Given the action ρ we have a map

$$\begin{aligned} \hat{\rho}\colon \mathfrak{g} &\to \mathfrak{X}(M) \\ \xi &\mapsto X_\xi \end{aligned} \tag{A.0.1}$$

associating to each $\xi \in \mathfrak{g}$ a vector field X_ξ on M, called **fundamental vector field** defined by its action on the functions $f \in C^\infty(M)$:

$$X_\xi f(m) = \frac{d}{dt} f(\mathrm{e}^{-t\xi}\, m)\ |_{t=0}\ . \tag{A.0.2}$$

The map (A.0.1) is a Lie algebra homomorphism

$$X_{[\xi,\zeta]} = [X_\xi, X_\zeta], \tag{A.0.3}$$

where in the rhs we have the Lie bracket of vector fields.
If $G)\mathbb{R}$, then an action of $\mathbb{R}$ on M provides a one-parameter family of diffeomorphisms $\phi_S \colon M \to M$, which we call an autonomous dynamical system. Such terminology can be extended to a general action of G on M, however this is only done in the setting of C^*–algebras and, by extension, as we do for actions of groups on LJB–algebras, as described in Chapter 5.
If $Y \in \mathfrak{X}(M)$, then $\mathfrak{g}$ acts on it via the Lie bracket $[X_\xi, Y]$.
Similarly, if $\theta \in \Omega^1(M)$ is a one-form, then for all $\xi \in \mathfrak{g}$ we have the action of ξ on θ given by:

$$L_{X_\xi}\theta(Y) = X_\xi\theta(Y) - \theta([X_\xi, Y]). \tag{A.0.4}$$

In general if $\omega \in \Omega^p(M)$ is a p-form, we define:

$$L_{X_\xi}\omega \equiv (di_{X_\xi} + i_{X_\xi}d)\omega, \tag{A.0.5}$$

where d is the exterior derivative and i_{X_ξ} is the contraction operator defined by

$$(i_{X_\xi}\omega)(Y_1, \ldots, Y_{p-1}) = \omega(X, Y_1, \ldots, Y_{p-1}). \tag{A.0.6}$$

By simple inspection, notice that its action agrees on functions and on one-forms (recall that the operator $L_X = di_X + i_X d$ is called the Lie derivative).

Now let (M, Ω) be a symplectic manifold, that is, Ω is a closed non-degenerate 2-form. In other words, $d\Omega = 0$ and the natural map $\hat{\Omega} \colon TM \to T^*M$ is an isomorphism. Thus on a symplectic manifold there is a natural map between vector fields and one-forms:

$$\begin{aligned} \beta \colon \mathfrak{X} \quad &\to \quad \Omega^1(M) && \text{(A.0.7)} \\ X \quad &\mapsto \quad i_X\Omega = \hat{\Omega}(X), && \text{(A.0.8)} \end{aligned}$$

which is an isomorphism with inverse $\beta^{-1} \colon \Omega^1(M) \to \mathfrak{X}(M)$. In local coordinates,

$$\Omega = \frac{1}{2}\Omega_{ij}dx^i \wedge dx^j, \tag{A.0.9}$$

and nondegeneracy of Ω implies that $\det(\Omega_{ij}) \neq 0$.

We now take a connected Lie group G acting on M via *symplectomorphisms*, i.e. diffeomorphisms which preserve Ω. Infinitesimally, this means that if $\xi \in \mathfrak{g}$ then

$$\begin{aligned} 0 &= L_X\Omega && \text{(A.0.10)} \\ &= di_X\Omega + i_X d\Omega && \text{(A.0.11)} \\ &= di_X\Omega, && \text{(A.0.12)} \end{aligned}$$

where X is the vector field associated to ξ. The one-form $i_X\Omega$ is closed. A vector field X such that $i_X\Omega$ is closed is said to be a **symplectic vector field**. It is clear that the symplectic vector fields are the image of closed forms under β^{-1}.
If $\beta(X)$ is exact, we say that X is a **Hamiltonian vector field**. This means that there exists $f_X \in C^\infty(M)$ such that

$$\beta(X) + df_X = 0. \tag{A.0.13}$$

This function is not unique because we can add to it a locally-constant function and still satisfy the above equation. We have that the Hamiltonian vector fields are the images of exact form under β^{-1}. A G-action on M said to be **Hamiltonian** if

to every $\xi \in \mathfrak{g}$ we can assign an Hamiltonian vector field X_ξ.
In a symplectic manifold, the functions define a **Poisson algebra**. If $f, g \in C^\infty(M)$, we define the Poisson bracket by

$$\{f, g\} \equiv \Omega(X_f, X_g), \tag{A.0.14}$$

where X_f is the Hamiltonian vector field such that $\beta(X_f) + df = 0$. The Poisson bracket is clearly skew-symmetric and obeys the Jacobi identity (since $d\Omega = 0$), and is a derivation on functions. Hence it gives $C^\infty(M)$ the structure of a Lie algebra. A Hamiltonian action is said to be Poisson or **strongly Hamiltonian** if there is a Lie algebra homomorphism $\mathfrak{g} \to C^\infty(M)$ sending X to f_X in such a way that $\beta(X) + df_X = 0$ and that $f_{[X,Y]} = \{f_X, f_Y\}$. In such case we can define the **momentum map** $J\colon M \to \mathfrak{g}^*$ of the action by:

$$\langle J, \xi \rangle = f_{X_\xi}. \tag{A.0.15}$$

The map

$$\begin{aligned} a\colon \mathfrak{g} &\to C^\infty(M) \\ \xi &\mapsto f_\xi \end{aligned} \tag{A.0.16}$$

is sometimes called the **comomentum map**, however in the setting of LJB–algebras described in this work (see Chapters 4 and 5) we will not distinguish them.

B

TENSOR PRODUCTS OF C^*–ALGEBRAS AND LIE–JORDAN BANACH ALGEBRAS

B.1. Tensor products of C^*–algebras

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be C^*–algebras. We can define their *–algebra tensor product as the standard algebraic tensor product of algebras $A_1 \otimes \mathcal{A}_2$ with product $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ and involution $(a \otimes b)^* = a^* \otimes b^*$.
The norm of a C^*–algebra is unique in the sense that on a given $*$–algebra $\mathcal{A}$ there is at most one norm which makes $\mathcal{A}$ into a C^*–algebra. Still, on a $*$–algebra $\mathcal{A}$ there may exist different norms satisfying the C^*–property. The completion with respect to any of such norms results in a C^*–algebra which contains $\mathcal{A}$ as a dense subalgebra. This is precisely what happens when the tensor product of C^*–algebra is considered: in the general case there are many different norms on the algebraic tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2$ (which is a $*$–algebra) with the C^*–property.

For example we may define

$$\left\|\sum a_i \otimes b_i\right\|_\wedge = \sum \|a_i\|\|b_i\|. \tag{B.1.1}$$

This seminorm becomes a norm on $\mathcal{A}_1 \otimes \mathcal{A}_2$ on an appropriate subspace, and its completion is denoted $\mathcal{A}_1\hat{\otimes}\mathcal{A}_2$ and called the projective tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$. We also have

$$\begin{aligned} \left\|\left(\sum a_i \otimes b_i\right)^*\right\|_\wedge &= \left\|\sum a_i^* \otimes b_i^*\right\|_\wedge = \sum \|a_i^*\|\|b_i^*\| \\ &= \sum \|a_i\|\|b_i\| = \left\|\sum a_i \otimes b_i\right\|, \end{aligned} \tag{B.1.2}$$

so $\mathcal{A}_1\hat{\otimes}\mathcal{A}_2$ is a Banach *–algebra. But it fails to satisfy the C^*–axiom ($\|x^*x\| = \|x\|^2$):

$$\begin{array}{rcl} \|(\sum a_i \otimes b_i)^* (\sum a_i \otimes b_i)\| &=& \|(\sum a_i^* \otimes b_i^*)(\sum a_i \otimes b_i)\| \\ &=& \|\sum a_i^* a_j \otimes b_i^* b_j\| \\ &=& \sum \|a_i^* a_j\|\|b_i^* b_j\| \\ &\leq& \sum \|a_i\|\|a_j\|\|b_i\|\|b_j\| \\ &=& (\sum \|a_i\|\|b_i\|)^2 \\ &=& \|\sum a_i \otimes b_i\|^2 \end{array} . \tag{B.1.3}$$

It turns out that representations on $\mathcal{A}_1$ and $\mathcal{A}_2$ allow us to define norms on $\mathcal{A}_1 \otimes \mathcal{A}_2$ that make it a C^*–algebra.

Definition B.1.1. Let $\rho_{\mathcal{A}_1} : \mathcal{A}_1 \to \mathcal{B}(\mathcal{H}_1)$ and $\rho_{\mathcal{A}_2} : \mathcal{A}_2 \to \mathcal{B}(\mathcal{H}_2)$ be representations on $\mathcal{A}_2$ and $\mathcal{A}_2$. We define the **product representation** $\rho = \rho_{\mathcal{A}_1} \otimes \rho_{\mathcal{A}_2}$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ as

$$\rho(a \otimes b) = \rho_{\mathcal{A}_1}(a) \otimes \rho_{\mathcal{A}_2}(b) \in \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2). \tag{B.1.4}$$

Since we always have the trivial representations, the set of representations of $\mathcal{A}_1$ on $\mathcal{H}_1$ and $\mathcal{A}_2$ on $\mathcal{H}_2$ are never empty.

Definition B.1.2. We define the **minimal** C^*–norm on $\mathcal{A}_1 \otimes \mathcal{A}_2$ by

$$\begin{aligned} \left\|\sum a_i \otimes b_i\right\|_{\text{min}} &= \sup_{\rho_A, \rho_B} \left\|\rho\left(\sum a_i \otimes b_i\right)\right\| \\ &= \sup_{\rho_A, \rho_B} \left\|\sum \rho_A(a_i) \otimes \rho_B(b_i)\right\| \end{aligned} \tag{B.1.5}$$

where the two norms on the right are operator norms.

This is clearly finite (hence a norm) and satisfies the C^*–axiom. The completion of $\mathcal{A}_1 \otimes \mathcal{A}_2$ with this norm is a C^*–algebra called the **minimal** (or **spatial**) tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$ and will be denoted by $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$.

Definition B.1.3. Let $\rho_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a representation and $N \subseteq \mathcal{H}$ be the largest subspace of $\mathcal{H}$ such that $\rho(a)(x) = 0$ for all $a \in \mathcal{A}$ and $x \in N$. Then $N^{\perp}$ is called the *essential subspace* of $\mathcal{H}$, and we will denote it $E(\mathcal{H})$. If $E(\mathcal{H}) = \mathcal{H}$, then ρ_A is said to be **nondegenerate**.

Proposition B.1.4. *If $\rho_{\mathcal{A}_1} : \mathcal{A}_1 \to \mathcal{B}(\mathcal{H})$ is a nondegenerate representation, then there are unique nondegenerate representations $\rho_{\mathcal{A}_1} : \mathcal{A}_1 \to \mathcal{B}(\mathcal{H})$ and $\rho_{\mathcal{A}_2} : \mathcal{A}_2 \to \mathcal{B}(\mathcal{H})$ such that $\rho(a \otimes b) = \rho_{\mathcal{A}_1}(a)\rho_{\mathcal{A}_2}(b) = \rho_{\mathcal{A}_2}(b)\rho_{\mathcal{A}_1}(a)$.*

But arbitrary representations of the tensor product of algebras cannot be broken into pieces. This gives us the following.

Definition B.1.5. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A}_1$,$\mathcal{A}_2$ be C^*–algebra. We define the **maximal** C^*–norm on $\mathcal{A}_1 \otimes \mathcal{A}_2$ as

$$\left\|\sum a_i \otimes b_i\right\|_{\text{max}} = \sup_{\rho} \left\|\rho\left(\sum a_i \otimes b_i\right)\right\| \tag{B.1.6}$$

where $\rho : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{B}(\mathcal{H})$. This is also a C^*–norm, and the completion of $\mathcal{A}_1 \otimes \mathcal{A}_2$ under this norm is a C^*–algebra called the **maximal** tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$ and will be denoted by $\mathcal{A}_1 \otimes_{\max} \mathcal{A}_2$.

An important result [Bla06] is that

$$\|\cdot\|_{\text{min}} \leq \|\cdot\|_* \leq \|\cdot\|_{\text{max}} \leq \|\cdot\|_{\wedge} \tag{B.1.7}$$

where $\|\cdot\|_*$ is any C^*–norm. It follows that $\|(a\otimes b)\|_* = \|a\|\|b\|$. Then clearly the natural map $\mathcal{A}_1 \otimes_{\max} \mathcal{A}_2 \to \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$ is continuous.

We conclude by defining *nuclear* C^*–algebras.

Definition B.1.6. A C^*–algebra $\mathcal{A}_1$ is **nuclear** if for every C^*–algebra $\mathcal{A}_2$, there is a unique C^*–norm on $\mathcal{A}_1 \otimes \mathcal{A}_2$, i.e. $\mathcal{A}_1 \otimes_{\max} \mathcal{A}_2 = \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$.

For instance if G is discrete, $C^*(G)$ is nuclear if and only if G is amenable (however this is not true if G is not discrete). Examples of non-nuclear algebras for discrete groups are given for instance by $C^*_r(F_2)$, the reduced C^*–algebra of the free group generated by two elements [Tak64].

B.2. Tensor products of Lie–Jordan Banach algebras and the uniqueness of the Planck constant

The tensor product of two LJB–algebra describes the interaction of two quantum systems, which should result in a composite system describable within the same framework. We are here seeking to find a Lie–Jordan structure on the tensor products of two Lie–Jordan algebras. This can be easily obtained by the rules of the algebraic tensor product of two C^*–algebras $\mathcal{A}_1$ and $\mathcal{A}_2$. We can in fact define an associative $*$–product on $\mathcal{A}_1 \otimes \mathcal{A}_2$ by

$$(x_1 \otimes y_1)\cdot(x_2 \otimes y_2) = (x_1 \cdot x_2) \otimes (y_1 \cdot y_2), \tag{B.2.1}$$

where $x_1, x_2 \in \mathcal{A}_1$ and $y_1, y_2 \in \mathcal{A}_2$ and choosing an appropriate norm as explained in the previous section in order to make $\mathcal{A}_1 \otimes \mathcal{A}_2$ into a C^*–algebra. From our Theorem 2.3.6 we know we can then obtain the Lie and Jordan product by requiring

$$(x_1 \otimes y_1)\cdot(x_2 \otimes y_2) = (x_1 \otimes y_1)\circ(x_2 \otimes y_2) - i\sqrt{k}\,[x_1 \otimes y_1, x_2 \otimes y_2]. \tag{B.2.2}$$

Hence, given the Lie–Jordan algebras $(\mathcal{L}_1, \circ, [\cdot,\cdot], k_1)$ and $(\mathcal{L}_1, \circ, [\cdot,\cdot], k_2)$, we can endow the tensor product $\mathcal{L}_1 \otimes \mathcal{L}_2$ with a Lie–Jordan structure with constant k by the following definitions:

$$(a_1 \otimes b_1)\circ(a_2 \otimes b_2) = (a_1 \circ a_2)\otimes(b_1 \circ b_2) + \sqrt{k_1 k_2}\,[a_1, a_2]\otimes[b_1, b_2], \tag{B.2.3}$$

$$[a_1 \otimes b_1, a_2 \otimes b_2] = \sqrt{\frac{k_2}{k}}\,(a_1 \circ a_2) \otimes [b_1, b_2] + \sqrt{\frac{k_1}{k}}\,[a_1, a_2] \otimes (b_1 \circ b_2), \quad \text{(B.2.4)}$$

where $a_1, a_2 \in \mathcal{L}_1$ and $b_1, b_2 \in \mathcal{L}_2$. It can be checked by simple inspection that these products B.2.3 and B.2.4 satisfy all the axioms 2.2.1-2.2.4. Regarding the Banach structure, we know from the theory of tensor products of C^*–algebras (see previous section) that is always possible to define a compatible Banach structure on the tensor algebra.

Remark. Notice that the tensor product of Lie–Jordan algebras so far defined is well "categorically" defined, in the sense that given a third Lie–Jordan algebra $\mathcal{L}_3$, for all $c_1, c_2 \in \mathcal{L}_3$ the tensor product composition is associative with respect to the Jordan

$$\{(a_1 \otimes b_1) \otimes c_1\} \circ \{(a_2 \otimes b_2) \otimes c_2\} - \{a_1 \otimes (b_1 \otimes c_1)\} \circ \{a_2 \otimes (b_2 \otimes c_2)\} = 0 \quad \text{(B.2.5)}$$

and Lie product

$$[(a_1 \otimes b_1) \otimes c_1, (a_2 \otimes b_2) \otimes c_2] - [a_1 \otimes (b_1 \otimes c_1), a_2 \otimes (b_2 \otimes c_2)] = 0. \quad \text{(B.2.6)}$$

We will now require another natural property, that the restriction of the products on $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$ to the two subalgebras $\mathcal{L}_1$ and $\mathcal{L}_2$ must be compatible. From the subalgebra immersions

$$\mathcal{L}_1 \hookrightarrow \mathcal{L}_1 \otimes \mathbb{1}, \quad \mathcal{L}_2 \hookrightarrow \mathbb{1} \otimes \mathcal{L}_2, \quad \text{(B.2.7)}$$

the restriction requirement means that

$$(a_1 \otimes \mathbb{1}) \circ (a_2 \otimes \mathbb{1}) = (a_1 \circ a_2) \otimes \mathbb{1} \quad \text{(B.2.8)}$$

and

$$[a_1 \otimes \mathbb{1}, a_2 \otimes \mathbb{1}] = [a_1, a_2] \otimes \mathbb{1}, \quad \text{(B.2.9)}$$

and the same for b_1, b_2. In particular the Eqs. B.2.9 and B.2.4 imply that

$$k_1 = k_2 = k, \quad \text{(B.2.10)}$$

that is all the Lie–Jordan algebras must have the same defining constant. Remember from Eq. 2.2.7 that for quantum systems $k = \hbar^2$, which proves the **uniqueness** of the Planck's constant $\hbar$ for composite systems from purely algebraic considerations.

BIBLIOGRAPHY

[Ara00] H. Araki, *Mathematical Theory of Quantum Fields*, Oxford University Press, Oxford, 2000.

[AS98] E. M. Alfsen and F. W. Shultz, *On Orientation and Dynamics in Operator Algebras. Part I*, Commun. Math. Phys. **194** (1998), 87.

[AS99] A. Ashtekar and T. A. Schilling, *Geometrical formulation of quantum mechanics*, On Einstein's Path edited by A. Harvey (Springer-Verlag, Berlin) (1999).

[Ber61] P. G. Bergmann, *Observables in general relativity*, Rev. Mod. Phys. **33** (1961), no. 4, 510–514.

[BIMM11] A. P. Balachandran, A. Ibort, G. Marmo, and M. Martone, *Covariant quantum fields on noncommutative spacetimes*, JHEP **2011** (2011), no. 3, 1–18.

[Bla06] B. Blackadar, *Operator Algebras*, vol. 122, Encyclopedia of Mathematical Sciences, Springer-Verlag, 2006.

[BR03] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics 1: C*-and W*-algebras. Symmetry groups. Decomposition of states*, vol. 1, Springer, 2003.

[BRS74] C. Becchi, A. Rouet, and R. Stora, *The abelian Higgs Kibble model, unitarity of the S-operator*, Physics Letters B **52** (1974), no. 3, 344–346.

[BT82] R. Bott and L. W. Tu, *Differential forms in algebraic topology*, Springer, 1982.

[CCGM07] J. F. Cariñena, J. Clemente-Gallardo, and G. Marmo, *Geometrization of Quantum Mechanics*, Theor. Math. Phys. **152** (2007), 895.

[CF04] I. Calvo and F. Falceto, *Poisson reduction and branes in Poisson-Sigma models*, Lett. Math. Phys. **70** (2004), 231.

[CGM09] J. Clemente-Gallardo and G. Marmo, *Towards a definition of quantum integrability*, Int. J. Geom. Meth. Mod. Phys. **6** (2009), 129.

[CL84] R. Cirelli and P. Lanzavecchia, *Hamiltonian vector fields in Quantum Mechanics*, Nuovo Cimento B **79** (1984), 271.

[CM09] D. Chruscinski and G. Marmo, *Remarks on the GNS Representation and the Geometry of Quantum States*, Open Syst. Info. Dyn. **16** (2009), 157.

[Con74] A. Connes, *Caractérisation des espaces vectoriels ordonnés sous-jacents aux algèbres de von Neumann*, Annales de l'institut Fourier, vol. 24, 1974, pp. 121–155.

[Con90] J. B. Conway, *A course in functional analysis*, vol. 96, Springer, 1990.

[Con94] A. Connes, *Noncommutative geometry*, Academic press, San Diego, CA, 1994.

[CS99] J. Caro and L. L. Salcedo, *Impediments to mixing classical and quantum dynamics*, Physical Review A **60** (1999), no. 2, 842.

[Dir01] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Dover Publications, 2001.

[DKR66] S. Doplicher, D. Kastler, and D. W. Robinson, *Covariance algebras in field theory and statistical mechanics*, Comm. Math. Phys. **3** (1966), no. 1, 1–28.

[DTI03] M. Delgado-Téllez and A. Ibort, *On the geometry and topology of singular optimal control problems and their solutions*, Discrete and Continuous Dynamical Systems (2003), 223–333.

[Emc84] G. G. Emch, *Foundations of 20th century Physics*, North Holland, Amsterdam, 1984.

[EMM10] E. Ercolessi, G. Marmo, and G. Morandi, *From the equations of motion to the canonical commutation relations*, Riv. Nuovo Cim. **033** (2010), 401–590.

[FFIM13a] F. Falceto, L. Ferro, A. Ibort, and G. Marmo, *Reduction of Lie–Jordan algebras: Classical*, Il Nuovo Cimento C **36** (2013), no. 03, 117–125.

[FFIM13b] ———, *Reduction of Lie–Jordan algebras: Quantum*, Il Nuovo Cimento C **36** (2013), no. 03, 107–115.

[FFIM13c] ———, *Reduction of Lie–Jordan Banach algebras and quantum states*, J. Phys. A: Math. Theor. **46** (2013), 015201.

[FFMP13] P. Facchi, L. Ferro, G. Marmo, and S. Pascazio, *Defining quantumness via the Jordan product*, preprint arXiv:1309.4635 (2013).

[FI12] L. Ferro and A. Ibort, *Folding and unfolding quantum states*, Int. J. Geom. Methods Mod. Phys. **9** (2012), no. 2, 1260028.

[FZ08] F. Falceto and M. Zambon, *An extension of the Marsden–Ratiu reduction for Poisson manifolds*, Lett. Math. Phys. **85** (2008), 203.

[GH85] H. Grundling and C. A. Hurst, *Algebraic Quantization Of Systems With A Gauge Degeneracy*, Commun. Math. Phys. **98** (1985), 369.

[GH88] ———, *The quantum theory of second class constraints: Kinematics*, Comm. Math. Phys. **119** (1988), no. 1, 75–93.

[GKM06] J. Grabowski, M. Kus, and G. Marmo, *Symmetries, group actions, and entanglement*, Open Syst. Info. Dyn. **13** (2006), 343.

[GL00] H. Grundling and F. Lledó, *Local quantum constraints*, Rev. Math. Phys. **12** (2000), 1159.

[GLMV94] J. Grabowski, G. Landi, G. Marmo, and G. Vilasi, *Generalized Reduction Procedure: Symplectic and Poisson formalism*, Forts. Phys. **42** (1994), 393.

[GMK05] J. Grabowski, G. Marmo, and M. Kus, *Geometry of quantum systems: Density states and entanglement*, J. Phys. A: Math. Theor. **38** (2005), 10217.

[GMS05] G. Giachetta, L. Mangiarotti, and G. Sardanashvily, *Geometric and Algebraic Topological Methods in Quantum Mechanics*, World Scientific, Heidelberg, 2005.

[GN79] M. J. Gotay and J. M. Nester, *Presymplectic Lagrangian systems. I: the constraint algorithm and the equivalence theorem*, Annales de l'institut Henri Poincaré (A) Physique théorique, vol. 30, 1979, pp. 129–142.

[GN80] ———, *Presymplectic Lagrangian systems. II: the second-order equation problem*, Annales de l'institut Henri Poincaré (A) Physique théorique, vol. 32, 1980, pp. 1–13.

[GNH78] M. J. Gotay, J. M. Nester, and G. Hinds, *Presymplectic manifolds and the Dirac–Bergmann theory of constraints*, J. Math. Phys. **19** (1978), 2388.

[GS81] M. J. Gotay and J. Ŝniatycki, *On the quantization of presymplectic dynamical systems via coisotropic imbeddings*, Comm. Math. Phys. **82** (1981), no. 3, 377–389.

[Haa96] R. Haag, *Local quantum physics: Fields, particles, algebras*, Springer-Verlag Berlin, Heidelberg, 1996.

[Hen85] M. Henneaux, *Hamiltonian form of the path integral for theories with a gauge freedom*, Physics Reports **126** (1985), no. 1, 1–66.

[HOS84] H. Hanche-Olsen and E. Stormer, *Jordan Operator Algebras*, Pitman Advanced Publishing Program, 1984.

[HT88] M. Henneaux and C. Teitelboim, *BRST cohomology in classical mechanics*, Communications in Mathematical Physics **115** (1988), no. 2, 213–230.

[IdLM97] A. Ibort, M. de León, and G. Marmo, *Reduction of Jacobi manifolds*, J. Phys. A: Math. Gen. **30** (1997), 2783.

[Jor33] P. Jordan, Zschr. f. Phys. **80** (1933), 285.

[JvNW34] P. Jordan, J. von Neumann, and E. P. Wigner, *On an algebraic generalization of the quantum mechanical formalism*, Ann. Math. **35** (1934), 297.

[Kad51] R. V. Kadison, *A representation theory for commutative topological algebra*, Mem. Amer. Math. Soc. **7** (1951).

[Lan93] N. P. Landsman, *Quantization and classicization: from Jordan-Lie algebras of observables to gauge fields*, Classical and Quantum Gravity **10** (1993), S101.

[Lan98] ______, *Mathematical Topics Between Classical and Quantum Mechanics*, Springer Mathematical Monographs, New York, 1998.

[LM00] C. López and E. Martínez, *Sub-Finslerian metric associated to an optimal control system*, SIAM Journal on Control and Optimization **39** (2000), no. 3, 798–811.

[McC04] K. McCrimmon, *A taste of Jordan algebras*, Springer, 2004.

[MFK65] D. Mumford, J. Fogarty, and F. C. Kirwan, *Geometric invariant theory*, vol. 965, Springer, 1965.

[MMT95] G. Mendella, G. Marmo, and W. M. Tulczyjew, *Integrability of implicit differential equations*, J. Phys. A: Math. Gen. **28** (1995), no. 1, 149.

[MR86] J. E. Marsden and T. Ratiu, *Reduction of Poisson manifolds*, Lett. Math. Phys. **11** (1986), 161.

[MW74] J. Marsden and A. Weinstein, *Reduction of symplectic manifolds with symmetry*, Reports on mathematical physics **5** (1974), no. 1, 121–130.

[RW98] I. Raeburn and D. P. Williams, *Morita equivalence and continuous-trace C^*-algebras*, no. 60, American Mathematical Society, 1998.

[Seg47] I. E. Segal, *Postulates for general quantum mechanics*, Annals of Math. **48** (1947), 930.

[Ser65] J. P. Serre, *Lie algebras and Lie groups*, Springer-Verlag, 1965.

[SS78] T. N. Sherry and E. C. G. Sudarshan, *Interaction between classical and quantum systems: A new approach to quantum measurement. I*, Physical Review D **18** (1978), no. 12, 4580.

[Tak64] M. Takesaki, *On the cross-norm of the direct product of C^*–algebras*, Tohoku Mathematical Journal **16** (1964), no. 1, 111–122.

[Tak03] ______, *Theory of operator algebras*, vol. I, Springer, Berlin, 2003.

[Tou68] J. C. Tougeron, *Idéaux et fonctions differéntiables I*, Annales de l'Institut Fourier, vol. 18, 1968, pp. 177–240.

[Tyu75] I. V. Tyutin, *Gauge invariance in field theory and statistical physics in operator formalism*, preprint of P.N. Lebedev Physical Institute (1975), no. 39.

[vN36] J. von Neumann, *On a Certain Topology for Rings of Operators*, Annals of Mathematics **37** (1936), no. 1, 111.

[vN96] ______, *Mathematical foundations of quantum mechanics*, Princeton university press (Translation from German edition), New Jersey, 1996.

[Wig31] E. P. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren*, F. Vieweg & Sohn Akt.-Ges., 1931.

[Zel79] E. Zelmanov, *Primary Jordan algebras*, Algebra and Logic **18** (1979), 103–111.