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On the inefficiency of Brownian motions and heavier tailed price processes

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Abstract. Recent literature has shown the existence of pathologies if one combines the most important models for pricing and hedging derivatives and coherent risk measures. There may exist portfolios (good deals) whose $(return,\ risk)$ is as close as desired to $(\infty,\ -\infty)$. This paper goes beyond existence properties and looks for explicit constructions and empirical tests. It will be shown that the good deal above may be a combination of European and digital options, very easy to replicate in practice. This theoretical finding will enable us to implement empirical experiments involving three international stock indices ($S\&P\_500$, $Eurostoxx\_50$ and DAX) and three commodity futures ($Gold$, $Brent$ and $DJ - UBSCI$). According to the empirical results, the good deal always outperforms the underlying index/commodity.

The good deal is built in full compliance with the standard Derivative Pricing Theory. Properties of classical pricing models totally inspire and lead the good deal construction. This is a very interesting difference with respect to previous literature attempting to outperform a benchmark. Besides, the selected pricing models satisfy the existence of risk neutral probabilities such that self-financing price processes become martingales. According to recent results, while local martingales characterize the absence of arbitrage, martingales characterize the existence of equilibrium. However, this equilibrium is difficult to imagine, because for every portfolio traders can build a new one with identical price, higher return and lower risk. Perhaps dynamic arbitrage free pricing models contradict other important achievements of Financial Economics related to efficiency and equilibrium, and further research is required to recover consistency.

Key words. Market Efficiency; Derivative Pricing; Risk Measure; Good Deal.

J.E.L. Classification. G11, G13, G14, G32.

1 Introduction

The History of Science shows that there are no immovable ideas, however consolidated they may seem. Internal contradictions may arise when dealing with a scientific theory, or empirical findings may contradict the hitherto accepted analysis. Some examples of the 20^{th} century may be found in Mathematics, where contradictions in Set Theory led to a new crisis of foundations and the Axiomatic Set Theory. Or in Physics, where new empirical phenomena, not explained in the framework of Classical Mechanics, led to the development of the Theory of Relativity. In general, when facing such a challenge, scientific disciplines eventually develop new approaches outperforming the previous ones, significantly improving our knowledge of a given phenomenon.

Financial Economics is not unaware of this reality. So, one of its paradigms, the capital markets efficiency, has been repeatedly addressed over the last fifty years. The existence of efficiency has been widely discussed and argued (Fama, 1970), but there have been detractors, even from the Theory of Economic Equilibrium (Green, 1977, or Grossman and Stiglitz, 1980, among others). This paper addresses two main questions:

$Question\_1$). Is the financial theory of capital markets consistent? Can we guarantee the absence of internal paradoxes (contradictions)?

$Question\_2$). If there are paradoxes, can empirical evidence support further discussions?

The first question is partially answered in Balbás *et al.* (2010a) and (2013). The authors find pathological results when combining the most important arbitrage free pricing models involving derivatives (Black and Scholes, stochastic volatility, etc.) and many measures of risk. Indeed, if z_π is the stochastic discount factor (SDF) of the pricing model, and $Log\left(z_\pi\right)$ is unbounded, then, for every coherent (Artzner *et al.,* 1999) and expectation bounded (Rockafellar *et al.,* 2006) risk measure ρ one can construct sequences $\left(y_n\right)_{n=1}^{\infty}$ of investment strategies whose expected return tends to plus infinite while their risk $\left(\rho\left(y_n\right)\right)_{n=1}^{\infty}$ remains bounded or tends to minus infinite. Moreover, the presence of frictions does not modify this result, unless $Log\left(z_\pi\right)$ becomes bounded (Balbás *et al.*, 2013). The obvious consequence is that the sequence $\left(y_n\right)_{n=1}^{\infty}$ will outperform every investment strategy. In other words, the existence of an Efficient Market Portfolio cannot hold for coherent risk measures. Furthermore, if traders can build in practice every y_n then they will do that (to become as rich as desired), and the existence of Equilibrium is difficult to imagine. In this sense, dynamic arbitrage free pricing models seem to contradict other very important achievements of Financial Economics.

Needless to say, $Question\_1$ is not totally solved in the papers above. Furthermore, the authors only prove existence theorems, but they do not give any way to construct the sequence $(y_n)_{n=1}^{\infty}$ of portfolios. In other words, $Question\_1$ can be simplified and rephrased:

$Question\_1^*$). Can one remain in full compliance with the standard Derivative Pricing Theory and simultaneously construct explicit portfolios whose $(return, risk)$ is as close as desired to $(+\infty, -\infty)$**?**

Section 2 will be devoted to answering this question. Specifically, in Section 2.1 we will present the general framework of the paper. Theorems 1 and 2, along with their remarks, will partially prove the existence of a positive answer to $Question\_1^*$ if the SDF of the pricing model is unbounded and the sub-gradient of the risk measure is composed of bounded pay-offs. In some sense these results are less general than those in Balbás *et al.* (2013), but they are much more easily applied in practice. Indeed, we cannot include every coherent and expectation bounded risk measure (the sub-gradient must be composed of bounded elements) or every pricing model whose SDF has unbounded logarithm (if the SDF is unbounded then so is its logarithm, but the converse implication may fail), but there are still many risk measures and pricing models satisfying the new required conditions, and we intend to go beyond existence properties. We are looking for effective constructions and empirical tests. Actually, Sections 2.2 and 2.3 will deal with important cases satisfying the assumptions we need. The risk measure is the Conditional Value at Risk ($CVaR$) with a given level of confidence $0 < \mu < 1 = 100\%$. The selected arbitrage-free pricing model is the Black and Scholes one, although we will point out that every model with heavier tails (for instance, most of the stochastic volatility models) can be similarly studied. In fact, the heavier the tail the faster the convergence of the couple $(return, risk)$ to $(+\infty, -\infty)$, but choosing the Black and Scholes model we will simplify many aspects of the paper exposition. The most important results in Sections 2.2 and 2.3 are Theorem 4 and its remarks. They will give explicit portfolios composed of European and binary options showing that the response to $Question\_1^*$ is "yes". Theorem 4 and its remarks generate some numerical algorithms that will be presented in Appendix I.

We have chosen the $CVaR$ for several reasons. Firstly, it is coherent and therefore sub-additive, which facilitates diversification in many risk minimization problems (recall that the Value at Risk or VaR is not sub-additive, Artzner *et al.,* 1999). Secondly, $CVaR$ is easily interpreted in practice, since, at least for continuous distributions, it coincides with the expected losses beyond a percentile. Thirdly, $CVaR$ measures risk in terms of potential capital losses, and therefore it provides us with monetary information which is not captured by volatilities. Fourthly, $CVaR$ is consistent with the Second Order Stochastic Dominance, and once again volatilities and standard deviations do not satisfy this property in presence of fat tails and asymmetries (Ogryczak and Ruszczynski, 1999). Needless to say, asymmetries and stochastic dominance are becoming more and more important in financial markets (Constantinides *et al.*, 2011**,** Hirshleifer, *et al.*, 2011, Neuberger, 2012 or Christoffersen, *et al.,* 2012, among others). In some sense $CVaR$ may solve some drawbacks related to the standard deviation, and it has been selected in many recent theoretical and empirical analyses (Basak and Shapiro, 2001, Agarwal and Naik, 2004, or Annaert *et al.*, 2009, among others), though, in general, risk measurement is still an unsolved problem provoking intense discussions and research (recent approaches may be found in Aumann and Serrano, 2008, Brown and Sim, 2009, Foster and Hart, 2009, or Bali *et al.*, 2011, among others).

In Section 3 we deal with $Question\_2$. In Section 3.1 we draw on Monte Carlo simulation so as to generate trajectories of a Geometric Brownian Motion (GBM). For every simulated trajectory we draw on Theorem 4 and its consequences so as to create the portfolio of derivatives solving $Question\_1^*$. This is done in a self-financing setting. The robust conclusion of this numerical experiment is that the strategy of Theorem 4 (henceforth "good deal") clearly outperforms the GBM, even if one deals with the standard deviation as the risk measure. The classical Sharpe ratio of the good deal is much higher than the Sharpe ratio of the simulated sample for the GBM. A GBM can never be an efficient strategy.

Section 3.2 deals with real data rather than simulated paths. We have selected three international index futures ($S\&P\_500$, $Eurostoxx\_50$ and DAX) and three commodity futures ($Gold$, $Brent$, and the Dow Jones-UBS Commodity Index $DJ - UBSCI$). The methodology is exactly the same as in Section 3.1, but the role of the simulated paths is plaid by a database of daily quotes. Once again the good deal clearly outperforms the underlying index/commodity, and this is still true if the $CVaR$ is replaced by the VaR or the standard deviation. In fact, the distance between the good deal performance and the performance of the underlying index/commodity becomes much higher than it was in Section 3.1. The reason is clear. Indeed, real data tails are much heavier than the tails of a GBM, and the tail of the SDF becomes much heavier too, which implies that the SDF becomes really far from bounded. Recall that a bounded SDF is a necessary condition to prevent the the good deal existence (remarks of Theorem 2).

Overall, the paper finds positive answers to $Question\_1^*$ and $Question\_2$. Thus, one can design simple strategies of derivatives (good deals) outperforming the most important international indices and every alternative investment. Furthermore, it is worth noticing that the good deal is built in full compliance with Financial Theory, in the sense that the properties of classical pricing models totally inspire the good deal construction. From Financial Theory one outperforms the Market. This is a very interesting difference with respect to previous literature attempting to outperform a benchmark. Besides, the selected pricing models satisfy the existence of risk neutral probabilities such that the self-financing price processes become martingales. According to recent results, while local martingales characterize the absence of arbitrage (Delbaen and Schachermayer, 1994), martingales characterize the existence of equilibrium leading to the given pricing model (Jarrow and Larsson, 2012). However, as said above, this equilibrium is difficult to imagine, because for every portfolio traders can easily construct a new one with identical price, higher return and lower risk. Perhaps dynamic arbitrage free pricing models contradict other important achievements of Financial Economics related to Efficiency and Equilibrium, and therefore an extra effort is required to recover consistency and integration. This challenge cannot be addressed without bearing in mind the empirical evidence. According to our empirical tests, the convergence of $(return, risk)$ to $(+\infty, -\infty)$ becomes very slow if $return$ is positive enough and $risk$ is negative enough, but Derivative Pricing Models provide us with the instruments to outperform the benchmark. This might suggest that the integration between Derivative Pricing and Efficiency and Equilibrium should incorporate ideas coming from both approaches.

The most important conclusions of the paper are summarized in Section 4, some complex proofs are presented in Appendix I, and Appendix II yields some illustrative figures, which are useful to understand the empirical findings.

2 The Theory

2.1 General approach

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ composed of the set of "states of the world" Ω, the $\sigma-$algebra $\mathcal{F}$ and the probability measure $\mathbb{P}$. Consider a time interval $[0, T]$, a subset $\mathcal{T} \subset [0, T]$ of trading dates containing 0 and T, and a filtration $(\mathcal{F}_t)_{t \in \mathcal{T}}$ providing the arrival of information and such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. We will deal with final (at T) pay-offs belonging to the space L^2 (or $L^2(\Omega, \mathcal{F}, \mathbb{P})$), *i.e.*, whose expectation and variance are finite.

Let us assume that the market is complete, *i.e.*, every pay-off y in L^2 may be replicated by means of self-financing portfolios. In other words, there is a price process of $(S_t)_{t \in \mathcal{T}}$, adapted to the filtration $(\mathcal{F}_t)_{t \in \mathcal{T}}$, related to a self-financing strategy, and such that $S_T = y$, *a.s.* Furthermore, there is a continuous pricing rule

$$\Pi : L^2 \longrightarrow \mathbb{R}$$

providing us with the current (at 0) price $\Pi(y)$ of every pay-off y.

Denote by $r_f \geq 0$ the risk-free rate, and the equality

$$\Pi(k) = ke^{-r_f T} \tag{1}$$

must hold for every $k \in \mathbb{R}$.

According to the Riesz Representation Theorem, there exists a unique $z_\pi \in L^2$ such that

$$\Pi(y) = e^{-r_f T} \mathbb{E}(y z_\pi) \tag{2}$$

for every pay-off y, $\mathbb{E}()$ representing the mathematical expectation. Moreover, to prevent the existence of arbitrage, the strict inequality $z_\pi > 0$ *a.s.* must hold (Duffie, 1988). z_π is usually called SDF, and it is closely related to the Market Portfolio of the $CAPM$ (Chamberlain and Rothschild. 1983, or Duffie, 1988). Expressions (1) and (2) imply that $ke^{-r_f T} = \Pi(k) = e^{-r_f T} k \mathbb{E}(z_\pi)$, which leads to

$$\mathbb{E}(z_\pi) = 1. \tag{3}$$

Let $\rho : L^2 \longrightarrow \mathbb{R}$ be a coherent (Artzner *et al.,* 1999) and expectation bounded (Rockafellar *et al.,* 2006) risk measure that a trader uses in order to control the risk level of his final wealth at T. Among many others, significant examples are the $CVaR$ and the Weighted Conditional Value at Risk or $WCVaR$ (Rockafellar *et al.* 2006). A very important technical property of these measures is that they are the envelope of some linear functions. More accurately, there exists a set of random variables $\Delta_\rho \subset L^2$, called the sub-gradient of ρ, such that

$$\rho(y) = Max \ \{-\mathbb{E}(yz) : z \in \Delta_\rho\} \tag{4}$$

holds for every $y \in L^2$. Furthermore,

$$\Delta_\rho \subset \left\{z \in L^2; \ \mathbb{P}(z \geq 0) = 1 \ and \ \mathbb{E}(z) = 1\right\}. \tag{5}$$

Suppose that the random pay-off $y_0 \in L^2$ represents a trader's final wealth. Its final risk will be given by $\rho(y_0)$, and suppose that the trader is going to invest

$$Ce^{-r_f T} > 0 \tag{6}$$

more dollars so as to reduce the risk $\rho(y_0)$. If $Ce^{-r_f T}$ were invested in the riskless asset, then, according to the translation invariance property of every coherent risk measure (Artzner *et al.,* 1999), the new risk level would be

$$\rho(y_0 + C) = \rho(y_0) - C. \tag{7}$$

However, since it is not obvious that the highest risk reduction will be generated by the riskless asset, the investor could select the solution y of the optimization problem

$$Min \ \{\rho(y + y_0 - \mathbb{E}(yz_\pi)); \ \mathbb{E}(yz_\pi) \leq C, \ y \geq 0\}. \tag{8}$$

Problem (8) considers the global risk level $\rho(y + y_0 - \mathbb{E}(yz_\pi))$ that the trader is facing, so it has to incorporate the value $\mathbb{E}(yz_\pi)$ of the added portfolio that will have to be paid and will reduce the trader wealth. One may also accept negative pay-offs if they do not make the global risk grow, which will lead to the problem

$$Min \ \{\rho(y + y_0 - \mathbb{E}(yz_\pi)); \ \mathbb{E}(yz_\pi) \leq C\}. \tag{9}$$

The saddle point methods of Balbás *et al.* (2010*b*) easily lead to the dual problems of (8) and (9), as well as to the optimality conditions below.

Theorem 1. *The dual problems of (8) and (9) are*

$$Max \ \{-C\lambda - \mathbb{E}(y_0 z) ; \ z \leq (1+\lambda) z_\pi, \ \lambda \in \mathbb{R}, \lambda \geq 0, \ z \in \Delta_\rho\} \tag{10}$$

and

$$Max \ \{-\mathbb{E}(y_0 z) ; \ z = z_\pi, \ z \in \Delta_\rho\} \tag{11}$$

respectively, $\lambda \in \mathbb{R}$ and $z \in \Delta_\rho$ being decision variables. There is no duality gap between (8) and (10), and there is no duality gap between (9) and (11) either. Furthermore, the solution of (10) (respectively, (11)) is attainable if Problem (8) (respectively, (9)) is bounded.

Suppose that $y^* \in L^2$ and $(\lambda^*, z^*) \in \mathbb{R} \times L^2$. Then, they solve (8) and (10) if and only if the following Karush-Kuhn-Tucker conditions

$$\begin{cases} \lambda^* (C - \mathbb{E}(y^* z_\pi)) = 0 \\ C - \mathbb{E}(y^* z_\pi) \geq 0 \\ \mathbb{E}((y^* + y_0) z) \geq \mathbb{E}((y^* + y_0) z^*), \qquad \forall z \in \Delta_\rho \\ \mathbb{E}(((1+\lambda^*) z_\pi - z^*) y^*) = 0 \\ (1+\lambda^*) z_\pi - z^* \geq 0 \\ y^* \in L^p, \ y^* \geq 0, \ \lambda \in \mathbb{R}, \lambda \geq 0, \ z^* \in \Delta_\rho \end{cases} \tag{12}$$

are fulfilled. □

Theorem 2. *Suppose that y^* solves (8) and (λ^*, z^*) solves (10). Then,*

a) If $\lambda^* = 0$ then $z^* = z_\pi$.

b) If $\mathbb{P}(y^* > 0) = 1$ then $\lambda^* = 0$ and $z^* = z_\pi$.

Proof. If $\lambda^* = 0$ then the dual constraint leads to $z^* \leq (1+\lambda^*) z_\pi = z_\pi$, and therefore $z^* = z_\pi$ because both random variables have the same expectation (see (3) and (5)). Besides, if $\mathbb{P}(y^* > 0) = 1$ then the fourth equation in (12) implies that $z^* = (1+\lambda^*) z_\pi$. Taking expectations and bearing in mind (3) and (5) we have that $1 = 1 + \lambda^*$. □

Remark 1 *The solution y^* of (8) will frequently be a risky asset. Indeed, if it were the riskless asset $y^* = k$ then (1), (6), and (8) would imply that $0 \leq k \leq C$, and (7) would imply that $y^* = C > 0$. Hence, Theorem 2b) would lead to $z_\pi \in \Delta_\rho$ which does not hold for many important risk measures and pricing models. For instance, Rockafellar et al. (2006)*

show that if $\rho = CVaR_\mu$, $\mu \in (0,1)$ being the level of confidence, then

$$\Delta_{CVaR_\mu} = \left\{ z \in L^2; \mathbb{E}(z) = 1,\ 0 \leq z \leq \frac{1}{1-\mu} \right\}. \tag{13}$$

and therefore the elements in Δ_{CVaR_μ} are essentially bounded. But there are many pricing models whose SDF is not essentially bounded. For instance, the Black and Scholes model, in which case z_π has a log-normal distribution, or most of the stochastic volatility models, in which case z_π usually reflects a heavier tail. □

Remark 2 *Theorem* 1 *shows that $z_\pi \notin \Delta_\rho$ will make (11) infeasible and (9) unbounded. In other words, one can construct sequences of pay-offs $(y_n)_{n=1}^{\infty}$ such that*

$$\rho(y_n + y_0 - \mathbb{E}(y_n z_\pi)) \longmapsto -\infty$$

and $\mathbb{E}(y_n z_\pi) \leq C$, $n = 1, 2, ...$ Since ρ is expectation bounded, the inequality

$$\rho(y) \geq -\mathbb{E}(y) \tag{14}$$

holds for every reachable pay-off y (Rockafellar et al., 2006), and therefore

$$\mathbb{E}(y_n + y_0 - \mathbb{E}(y_n z_\pi)) \geq -\rho(y_n + y_0 - \mathbb{E}(y_n z_\pi)) \longmapsto \infty.$$

Thus, if $z_\pi \notin \Delta_\rho$ then one can construct sequences of portfolios whose prices are bounded, whose expected returns tend to plus infinite, and whose risks tend to minus infinite. □

Remark 3 *Suppose that $z_\pi \notin \Delta_\rho$. The remarks above show that (8) is not solved by the riskless asset and (9) is unbounded. Suppose that we find an algorithm to solve (8), and let us show how to build the sequence $(y_n)_{n=1}^{\infty}$ of Remark* 2*.*

Consider $n \in \mathbb{N}$, along with an approximation of (9) given by Problem

$$Min\ \ \{\rho(y + y_0 - \mathbb{E}(y z_\pi));\ \mathbb{E}(y z_\pi) \leq C,\ y \geq -n\}. \tag{15}$$

Then, due to (3) and (7), it is easy to see that the change of variable $x_n = y + n$ leads to

$$Min\ \ \{\rho(x_n + y_0 - \mathbb{E}(x_n z_\pi));\ \mathbb{E}(x_n z_\pi) \leq C + n,\ x_n \geq 0\}, \tag{16}$$

which is a new problem analogous to (8). Consider the sequence $(y_n^*)_{n=1}^{\infty} = (x_n^* - n)_{n=1}^{\infty}$ of solutions of (15), $(x_n^*)_{n=1}^{\infty}$ denoting the solutions of (16). It is easy to see that $(y_n^*)_{n=1}^{\infty}$ is the sequence we were looking for. The algorithm to solve (8) (and therefore (16)) will be given in Appendix I. □

2.2 Dealing with the CVaR

Throughout this subsection let us assume that $\rho = CVaR_{\mu}$, $\mu \in (0,1)$ being the level of confidence. This risk measure is important for researchers, practitioners and regulators for several reasons. Firstly, it is compatible with the second order stochastic dominance, whereas the standard deviation does not satisfy this property in presence of asymmetric returns (Ogryczak and Ruszczynski, 1999 and 2002). Secondly, it provides the risk in terms of potential capital losses (and, therefore, capital requirements), whereas deviation measures do not satisfy this property. Thirdly, it is coherent and expectation bounded, whereas the VaR_{μ} is not sub-additive and therefore it does not facilitate diversification in many optimization problems (Artzner *et al.,* 1999). Actually, there are many practical problems that are more properly addressed when the standard deviation is replaced by an alternative risk measure such as $CVaR_{\mu}$ (Basak and Shapiro, 2001, Agarwal and Naik, 2004, Bali *et al.,* 2013, etc.).

Remark 4 *According to (13), every element in the* $CVaR_{\mu}$ *sub-gradient is an essentially bounded random variable. Hence, given a pricing model without bounded* SDF *(Black and Scholes, stochastic volatility, etc.) Remarks* 1, 2 *and* 3 *apply. Consequently, the solution of (8) is not a riskless asset, and bearing in mind that*

$$CVaR_{\mu}(y) \geq VaR_{\mu}(y) \tag{17}$$

holds for every pay-off y*, there exists a sequence* $(y_n)_{n=1}^{\infty}$ *of pay-offs such that*

$$\begin{cases} \mathbb{E}\left(y_n + y_0 - \mathbb{E}\left(y_n z_\pi\right)\right) \longmapsto +\infty \\ CVaR_{\mu}\left(y_n + y_0 - \mathbb{E}\left(y_n z_\pi\right)\right) \longmapsto -\infty \\ VaR_{\mu}\left(y_n + y_0 - \mathbb{E}\left(y_n z_\pi\right)\right) \longmapsto -\infty \\ \mathbb{E}\left(y_n z_\pi\right) \leq C, \qquad\qquad n = 1, 2, ... \end{cases} \tag{18}$$

hold. Henceforth the sequence $(y_n)_{n=1}^{\infty}$ *above will be called good deal.*

Actually, the sequence of investment strategies $(y_n)_{n=1}^{\infty}$ *of (18) may be independent of the confidence level* μ*, i.e.,* $(y_n)_{n=1}^{\infty}$ *may be selected in such a manner that (18) holds for every* $0 < \mu < 1$*. Indeed, choose an increasing sequence*

$$0 < \mu_1 < \mu_2 < \mu_3 < .... \longmapsto 1.$$

Then, (18) shows that there exists a pay-off y_n such that

$$\begin{cases} \mathbb{E}\left(y_n + y_0 - \mathbb{E}\left(y_n z_\pi\right)\right) > n, & n = 1, 2, ... \\ VaR_{\mu_n}\left(y_n + y_0 - \mathbb{E}\left(y_n z_\pi\right)\right) \leq CVaR_{\mu_n}\left(y_n + y_0 - \mathbb{E}\left(y_n z_\pi\right)\right) < -n, & n = 1, 2, ... \\ \mathbb{E}\left(y_n z_\pi\right) \leq C, & n = 1, 2, ... \end{cases}$$

and therefore it is easy to see that (18) holds for $(y_n)_{n=1}^{\infty}$ and every $0 < \mu < 1$. □

The expression "good deal" has been adopted from Cochrane and Saa-Requejo (2000), where this notion is introduced as an investment strategy providing traders with a very high Sharpe ratio in comparison with the Market Portfolio. In our case a good deal is something different, although closely related. Indeed, our risk measure is the $CVaR$ rather than the standard deviation, and both risk and return must be unbounded, which does not hold in the paper above. However, as will be seen in Section 3, the good deal (18) usually satisfies the conditions of Cochrane and Saa-Requejo (2000) too.

The third condition in (12) may be replaced by others that make it easier to solve (8) and (10) in practice if $\rho = CVaR_\mu$.

Lemma 3. *If $y^* \in L^2$ and $z^* \in \Delta_{CVaR_\mu}$ then z^* satisfies the third condition in (12) if and only if there exist $\alpha \in \mathbb{R}$, $\alpha_1, \alpha_2 \in L^2$ and a measurable partition $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ such that*

$$\begin{cases} y^* + y_0 = \alpha - \alpha_1 + \alpha_2 & \\ \alpha_i \geq 0 & i = 1, 2 \\ \alpha_1 = \alpha_2 = 0 & \text{on } \Omega_0 \\ z^* = \dfrac{1}{1-\mu} \text{ and } \alpha_2 = 0 & \text{on } \Omega_1 \\ z^* = 0 \text{ and } \alpha_1 = 0 & \text{on } \Omega_2 \end{cases} \tag{19}$$

hold.

Proof. See Appendix I. □

2.3 Dealing with the Black and Scholes model

Let us focus on the Black and Scholes model and suppose that y_0 is the final value (at T) of a Geometric Brownian Motion (GBM). Then, it is known that y_0 has a log-normal distribution. Without loss of generality we can simplify the structure of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Indeed, assume that $\Omega = (0,1)$ and $\mathbb{P}$ is the Lebesgue measure on the Borel $\sigma-$algebra of this set. Then we can take

$$y_0(\omega) = W_0 Exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\Phi^{-1}(\omega)\right) \tag{20}$$

for $\omega \in (0,1)$, $W_0 > 0$ denoting the current price of y_0, and r and σ denoting the drift and the volatility of the GBM, respectively. Obviously, $\Phi : \mathbb{R} \longmapsto (0,1)$ is the cumulative distribution function of the standard normal distribution.

The simplification above cannot be implemented when pricing path dependent or American style derivatives. In both situations the dynamic evolution of the GBM plays a critical role, as well as the notion of "stopping time" in the second case. Thus, when we choose the simple probability space $(\Omega, \mathcal{F}, \mathbb{P})$ above we are aware of the fact that we are missing information, and the performance of the solution y^* of our Theorem 4 below might be improved by including some path dependent or American derivative. However, our simplification is interesting because the exposition is shortened and becomes much easier. We will still obtain an explicit solution y^* of (8) that outperforms the risk-free security, and we will build concrete sequences of European style derivatives satisfying (18) for the Black and Scholes model.

Taking into account (20), it may be immediately verified that y_0 is a continuous and strictly increasing function (with respect to ω) such that

$$Lim_{\omega \to 0} y_0(\omega) = 0, \tag{21}$$

and $Lim_{\omega \to 1} y_0(\omega) = \infty$. It is also known that z_π is also log-normal and in our framework it is the first derivative of the one to one increasing and convex function

$$(0,1) \ni \omega \longmapsto g(\omega) = \Phi\left(a + \Phi^{-1}(\omega)\right) \in (0,1), \tag{22}$$

where

$$a = \frac{r - r_f}{\sigma}\sqrt{T} \tag{23}$$

is positive because we assume, as usual, that $r > r_f$. Computing the derivative in (22) we have that

$$z_\pi(\omega) = Exp\left(-\frac{a^2}{2} - a\Phi^{-1}(\omega)\right) \tag{24}$$

$\omega \in (0,1)$, which easily allows us to verify that z_π is continuous, strictly decreasing,

$$Lim_{\omega \to 0} z_\pi(\omega) = \infty, \tag{25}$$

and

$$Lim_{\omega \to 1} z_\pi(\omega) = 0. \tag{26}$$

Theorem 2, Remark 3 and Remark 4 have shown the existence of an alternative investment y^* outperforming the riskless asset, as well as the existence of sequences of strategies satisfying (18) which are easy to compute if one knows y^*. Next, let us compute y^*.

Theorem 4. *Under assumptions and notations above, if* $\rho = CVaR_\mu$ *and* y^* *solves (8) then there exist* $\alpha, \beta \in \mathbb{R}$ *such that* $0 < \beta < \alpha$ *and*

$$y^* = \begin{cases} 0 & if\ y_0 \leq \beta\ or\ y_0 > \alpha \\ \alpha - y_0 & if\ \beta < y_0 \leq \alpha \end{cases} \tag{27}$$

Proof. See Appendix I. □

Remark 5 *Notice that the solution* y^* *above may be given by*

$$y^* = y^*_\alpha - y^*_\beta - (\alpha - \beta)\, y^*_{D\beta},$$

y^*_α *denoting the European put option with maturity at* T *and strike* α*,* y^*_β *denoting the similar put with strike* β*, and* $y^*_{D\beta}$ *denoting the digital put option with maturity at* T *and strike* β*. Then, the solution of (8) is a combination of three put options whose underlying asset is* y_0*.* □

Remark 6 *The proof of the theorem applies for models much more general than Black and Scholes. Actually, if* y_0 *may be understood as a continuous increasing function of* $0 < \omega < 1$ *such that (21) holds and* z_π *is decreasing and satisfies (25) and (26), then the same proof applies and Theorem* 4 *remains true. Notice that these are quite general properties because (20) may be extended to* $y_0(\omega) = F^{-1}(\omega)$*,* $0 < \omega < 1$ *being an uniform random variable and* F *denoting the cumulative distribution function of* y_0*.* □

Remark 7 *In order to apply Theorem* 4 *in practice we have to provide the values of* β *and* α*. See Appendix I for an appropriate algorithm. Notice that the provided algorithm also applies for models more complex than Black and Scholes.* □

Remark 8 *As indicated in Remarks* 3 *and* 4*, the solution* y^* *of Theorem* 4 *permits us to compute good deals, i.e., sequences of portfolios satisfying (18). Moreover, a unique sequence may satisfy (18) for every level of confidence, i.e., the good deal does not depend on* μ*. Appendix I presents an adequate algorithm to build this good deal. Once again, the algorithm applies for models beyond Black and Scholes.* □

Remark 9 *The risk measure* $CVaR_\mu$ *may be also given by (Rockafellar et al., 2006)*

$$CVaR_\mu\left(y\right) = \frac{1}{1-\mu}\int_0^{1-\mu} VaR_{1-t}\left(y\right)dt,$$

for every $y \in L^2$*. Accordingly, since* $VaR_\mu\left(y\right)$ *only focuses on "the worst" values of* y *(on the left tail of* y*), so does* $CVaR_\mu\left(y\right)$*. Thus, it is not so surprising that* y^* *vanishes if* y_0 *achieves high values, since they are not affecting the global risk level.*

A little bit more shocking is that y^* *also vanishes if* y_0 *achieves its lowest values. The proof of Theorem* 4 *leads to* $\beta = y_0\left(\gamma_1\right)$ *with* $0 < \gamma_1 < 1-\mu$*. Therefore,* $Lim_{\mu \longmapsto 1}\gamma_1 = 0$*, which, along with (21), imply that* $Lim_{\mu_0 \longmapsto 1}\beta = 0$*. Thus, for high levels of confidence low values of* y_0 *become very important, and* y^* *almost becomes the European put option* y^*_α*.* □

Remark 10 *There are several classical strategies providing "portfolio insurance". Maybe the most popular one is the purchase of an appropriate European put option. Theorem* 4 *highlights that for large levels of confidence the use of portfolio insurance strategies may be adequate to hedge the investor's risk. It is consistent with some empirical recent findings. For instance, the test implemented by Annaert et al. (2009) reveals that some put option-linked portfolio insurance strategies are not outperformed by other hedging methods. The authors use stochastic dominance criteria and* VaR *and* $CVaR$ *in their empirical test. This is also consistent with the findings of Ahn et al. (1999).* □

Remark 11 *If we assume that the riskless rate is lower than the drift of the* GBM $(r < r_f)$ *then the solution* y^* *of (8) is different, but its computation may be addressed with similar methods. The major difference is that* z_π *is not necessarily decreasing, which implies that (27) may fail. However, an explicit solution of (8) may be given again. We will not study this problem in order to shorten the exposition, since, as said above, the analytical technics are analogous.* □

3 Numerical and empirical experiments

Theorem 4 and and Remark 3 enable us to build sequences of good deals satisfying (18) for the Black and Scholes model. We have constructed these sequences in order to verify their performance in practice. Two different analyses have been implemented. Firstly, we have considered a GBM and have generated dynamic trajectories by means of Monte Carlo simulations. Secondly, we have dealt with real databases involving both international stock indices and commodity futures.

3.1 Good deals with Monte Carlo simulation

Remarks 1, 2 and 3 show that the fulfillment of (18) critically depends on the fact $z_\pi \notin \Delta_{CVaR_\mu}$, where z_π is given by (24) and unbounded, and Δ_{CVaR_μ} is given by (13) and composed of bounded pay-offs. Besides, Theorem 4 and Remarks 3 and 5 imply that the construction of the sequence of good deals satisfying (18) must incorporate digital options, whose pay-off obviously jumps if the GBM attains at T the digital option strike.

Algorithm II (see Appendix I) provides us with a practical method so as to select a strategy y_N^* in the sequence $(y_n^*)_{n=1}^\infty$ of good deals. N is chosen in such a manner that the values of $CVaR_\mu(y_N + y_0 - \mathbb{E}(y_N z_\pi))$ and $VaR_\mu(y_N + y_0 - \mathbb{E}(y_N z_\pi))$ are "negative enough", whereas $\mathbb{E}(y_N + y_0 - \mathbb{E}(y_N z_\pi))$ is "large enough". The word "enough" means that the desired goals (A_1, A_2, A_3) for VaR, $CVaR$ and expected return are achieved.

According to our numerical experiments, if the goals (A_1, A_2, A_3) are "big" then "N becomes enormous", because the convergence of the good deal $(y_n^*)_{n=1}^\infty$ in (18) is really slow for a GBM. The main reason is that Property $z_\pi \notin \Delta_{CVaR_\mu}$ "almost fails", in the sense that z_π is log-normal and its tails are not heavy enough (although z_π is unbounded). In order to make N decrease one has to select small levels of confidence μ, which provokes a new drawback. Indeed, if μ is far from one then the jump of (27) is reached for values of β which may be close to the current value W_0 of the GBM (see (20) and Remark 7). Therefore, the usual $\delta-$hedging of y_N^* may be very unstable as T is approaching .δ, sensitivity of the price of y_N^* with respect to the GBM, is unstable because the second Greek Γ may be close to infinite for digital options. Thus, small modifications of the GBM may imply huge modifications of the replica of y_N^*. The obvious consequence is that it might be impossible in practice to replicate the pay-off y_N^* by combining the riskless asset and a GBM in a dynamic and self-financing setting. Nevertheless, the instability of δ may be mitigated by choosing very long time periods (large values of T).

In order to overcome the caveats above we have proceed as follows:

i) We have selected small confidence levels (low values for μ).

ii) We have fixed very long time periods (large values for T).

iii) Day by day we have simulated a new value of the GBM. Thus, we have obtained a global path of daily values of the GBM that will be denoted $(GBM_i)_i$, where GBM_i is the obtained value of the GBM at the $i-th$ simulated day. It is very easy to simulate trajectories of a GBM if one bears in mind (20) and $0 < \omega < 1$ is the uniform distribution.

iv) Day by day we have applied Theorem 4 and Algorithm I (see Appendix I) and have computed the pay-off y^* solving (8) with $\rho = CVaR_\mu$. We did not fix the horizon in order to prevent unstable $\delta-$hedging strategies of y^*. On the contrary, day by day we extended the horizon for one more day. Thus, the value of T remained the same for the whole period, and it was not decreasing "as maturity was approaching". This method allowed us to obtain the sequence of solutions $(d_i^*)_i$ of (8). Furthermore, at the $i-th$ day we computed δ_i, sensitivity between d_i^* and GBM_i. Thus, the replica of d_i^* is given by δ_i units of GBM_i and a position in the riskless asset.

v) At the initial day the invested amount $C = C_0$ of (6) was randomly fixed "*ad hoc* " in order to obtain d_0^*. Then, at the $i-th$ day the invested amount $C = C_i$ was being adapted so as to create a dynamic and self-financing strategy. More accurately, after simulating the new value GBM_i, C_i was selected as $C_i = Min\left\{C_0, \tilde{C}_i\right\}$, $\tilde{C}_i$ being the price at the $i-th$ day of the previous strategy d_{i-1}^*. This criterion permitted us to rebalance the position in the risky asset (the GBM) according to the variation $\delta_i - \delta_{i-1}$, and the position in the riskless security so as to reach a self-financing strategy.

The methodology above generated many trajectories $(GBM_i)_i$ and $(d_i^*)_i$. We will only present three numerical illustrative results because the remaining ones do not add any relevant information. The reported results are for 1000, 3500 and 20000 simulated days, *i.e.*, more or less 3.8, 13.5 and 77 years. Figure 1 (see Appendix II) illustrates the evolution of $(GBM_i)_i$ and $(d_i^*)_i$ with 20000 simulations. The three examples are simulated with the riskless rate $r_f = 2\%$, drift $r = 4.5\%$, volatility $\sigma = 20\%$, maturity $T = 35$ years and level of confidence $\mu = 0.35 = 35\%$. Obviously, the obtained VaR, $CVaR$ and expected return for every d_i^* show that (18) holds, despite the fact that this expression is unrealistic and inconsistent with equilibrium. It is not surprising because Remarks 3 and 4 theoretically predict this pathological behavior.

More surprisingly, the GBM is still inefficient and outperformed by the sequence $(d_i^*)_i$ if the role of VaR and $CVaR$ is played by the standard deviation, the usual risk measure in Portfolio Theory. Indeed, in the first numerical experiment with 1000 simulations the realized annual return, annual volatility and annual Sharpe ratio of $(GBM_i)_i$ are 1.25%, 20% and -0.0375 respectively, while these values become 3.62%, 22.48% and 0.0718 for the self-financing sequence $(d_i^*)_i$. Similarly, for 3500 and 20000 simulations the table below compares the performances of $(GBM_i)_i$ and $(d_i^*)_i$.

	GBM_1000	$(d_i^*)_i$.	GBM_3500	$(d_i^*)_i$.	GBM_20000	$(d_i^*)_i$.
$\mathrm{Re}\,turn$	1.25%	3.62%	4.25%	5.46%	3.17%	3.17%
$Volatility$	20%	22.48%	20.47%	20.87%	19.85%	10.49%
$Sharpe$	-0.0375	0.0718	0.11	0.1657	0.0588	0.1116

The robust conclusion of this numerical experiment is that the strategy of Theorem 4 clearly outperforms every GBM, even if one deals with the standard deviation as the risk measure. A good deal built with the $CVaR$ is also a good deal for the standard deviation, *i.e.*, in the sense of Cochrane and Saa-Requejo (2000), who introduced this notion. A GBM can never be efficient because the performance of the solution of (8) is much better.

3.2 Good deals with real market data

As stated in the introduction, we have selected three international index futures ($S\&P\_500$, $Eurostoxx\_50$ and DAX) and three commodity futures (*Gold*, *Brent*, and the Dow Jones-UBS Commodity Index $DJ-UBSCI$). The methodology is exactly the same as in Section 3.1, but the role of the simulated paths is now plaid by a database of daily quotes, and there are minor modifications caused by the the heavier tails of these daily quotes. The tail of the SDF is fat enough and the confidence level may increase without slowing down the good deal convergence. Consequently, one can also shorten the good deal maturity. Summarizing, we have:

i) We have selected confidence levels within the spread $70\% \le \mu \le 99\%$.

ii) We have fixed good deal maturities within the spread $six\_months - one\_year$.

iii) Day by day we have considered a new quotation of the underlying future. Thus, we have a global path of daily values that will be denoted $(S_i)_i$, where S_i is the quotation at the $i-th$ analyzed day.

iv) Day by day we have applied Theorem 4 and Algorithm I (see Appendix I) and have computed the pay-off y^* solving (8) with $\rho = CVaR_\mu$. We did not fix the horizon. On the contrary, day by day we extended the horizon for one more day. Thus, the value of T remained the same for the whole period, and it was not decreasing "as maturity was approaching". This method allowed us to obtain the sequence of solutions $(d_i^*)_i$ of (8). Furthermore, at the $i-th$ day with computed δ_i, sensitivity between d_i^* and S_i.

v) At the initial day the invested amount $C = C_0 = 50000$ of (6) was fixed in order to obtain d_0^* and its delta δ_0. The selected portfolio invested the amount C_0 in the riskless asset and bought δ_0 futures. Then, at the $i-th$ day C_i was selected as $C_i = Min\left\{C_0, \tilde{C}_i\right\}$, $\tilde{C}_i$ being the price at the $i-th$ day of the previous strategy d_{i-1}^*. This criterion permitted us to rebalance the position in the underlying future according to the variation $\delta_i - \delta_{i-1}$. The position in the riskless security was never rebalanced, so the global strategy was self-financing.

In order to address $Step\_iv$ above we need the drift r and the volatility σ of the underlying asset. We have dealt with historical estimations. Though there are market linked alternatives, our empirical results show that the good deal performance is better if one draws on the historical volatility rather than implied volatilities or volatility indices such as VIX or $VDAX$ (see Figure 2). Estimations of both r and σ have been implemented with samples of 30 daily quotations and several decay rates. Every day along the tested period we have estimated values for r and σ and have computed the good deal d_i^* for these estimations. In order words, drift and volatility were dynamic and modified every day, despite the fact that the good deal was computed according to Theorem 4 and therefore in compliance with the Black and Scholes model.

In order to prevent the effect of the sample size we have incorporated ambiguity with respect both r and σ. More accurately, for every $i-th$ tested day we have computed these parameters with samples composed of 30, 50 and 70 daily quotations, and we have computed the good deal under the three scenarios for r and σ. Then strategy d_i^* is not the riskless security if and only if the three obtained good deals have a delta with the same sign (positive or negative). If so, we selected δ_i according to the minimum absolute value, and d_i^* was the purchase (sale) of $|\delta_i|$ futures if $\delta_i > 0$ ($\delta_i < 0$). The empirical evidence indicates that ambiguous parameters do not outperform the non-ambiguous framework, though both ambiguous and non-ambiguous good deals outperform the underlying index/commodity. Figures 2 and 3 illustrate this finding for both $S\&P\_500$ and DAX.

The robust conclusion is that the good deal performs much better than the underlying asset. Furthermore, the difference between both performances is magnified with respect to the GBM of Section 3.1. This is not surprising because the tail of the SDF is heavier, and therefore the property $z_\pi \notin \Delta_{CVaR_\mu}$ of Remarks 2 and 3 is more properly fulfilled. Figures $2-6$ illustrate this finding.

Diversified good deals have also been tested, *i.e.*, portfolios composed of several good deals with different underlying assets. Since the results do not reflect important differences with respect to those related to non-diversified good deals, we will not report any empirical finding.

Conservative good deal. Finally, we have implemented a very conservative strategy in order to verify whether market imperfections might alter the good deal performance. This new strategy respects the following constraints:

i) 20% of the traded futures nominal value is a margin with null interest rate.

ii) The good deal initial price equals 10 million dollars.

iii) The traded futures nominal value must remain lower than 7.5 million dollars.

iv) One can only trade an integer number of futures, so the value of δ_i is rounded every day. For instance, for $\delta_i = 2.7$ one buys 3 futures and for $\delta_i = -3.1$ one sells 3 futures.

Figure 7 summarizes the results for the index $S\&P\_500$ and shows how the incorporation of market imperfections does not imply significant modifications of the main conclusions. During the period 1998, January 6^{th} $-$ 2011, December 21^{st} the index future realized annual return and volatility were 1.93% and 22.45%, while these values became 3.97% and 9.92% for the "conservative good deal" (Figure 7).

4 Conclusions

The effective construction of good deals is easy to implement if one bears in mind the main properties of the classical Derivative Pricing Models. The developed theory shows that the good deals may be sequences of portfolios composed of European and binary options, and they satisfy that

$$(return, CVaR_\mu, VaR_\mu)$$

is as close as desired to $(+\infty, -\infty, -\infty)$. Moreover, the same sequence may apply for every level of confidence $0 < \mu < 1$.

The empirical evidence reveals that the convergence of

$$(return, CVaR_\mu, VaR_\mu)$$

to $(+\infty, -\infty, -\infty)$ becomes slower and slower as $return$ grows and $CVaR_\mu$ and VaR_μ fall. Nevertheless, these good deals allow us to outperform the underlying benchmark in a simple manner. The good deal performs much better than the underlying benchmark even if one deals with the standard deviation as the selected risk measure. The classical Sharpe ratio clearly shows that.

The existence of good deals is a pathology reflecting some inconsistencies between the standard Derivative Pricing Models and other fields of Financial Economics related to Efficiency or Equilibrium. This caveat becomes more obvious if one takes into account the good properties of the $CVaR$ (compatibility with the Second Order Stochastic Dominance and information about risk in monetary terms, amongst others). Since Efficiency and Equilibrium are major notions that we cannot abandon, maybe they should be reached by means of models incorporating ideas coming from the Theory of Derivatives. Otherwise, this theory may inspire and lead the construction of good deals with empirical performance better than that of the Market Portfolio. Besides, the valuation of derivatives should be also affected by equilibrium arguments.

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5 Appendix I (proofs and algorithms)

Proof of Lemma 3. Problem

$$Min \left\{ \mathbb{E}\left(\left(y^* + y_0\right) z\right);\ 0 \leq z \leq \frac{1}{1-\mu},\ \mathbb{E}\left(z\right) = 1 \right\} \tag{28}$$

is obviously linear, in the sense that both the objective function and the constraints are linear. Thus, its Karush-Kuhn-Tucker conditions are sufficient optimality conditions. Besides, since its first and second constraints may be valued in the space of essentially bounded random variables L^∞, and the natural cone of this space has non void interior, the Slater Qualification holds (Luenberger, 1969), since there are random variables z that are feasible and such that $\mathbb{P}(0 < z < \frac{1}{1-\mu}) = 1$ (for instance, take the zero-variance random variable $z = 1$). Thus the Karush-Kuhn-Tucker conditions of (28) are also necessary optimality conditions (Luenberger, 1969). Furthermore, the dual space of L^∞ is composed of those finitely additive measures that have bounded variation and are $\mathbb{P}-$continuous (Luenberger, 1969). Thus, according to Luenberger (1969), the Karush-Kuhn-Tucker conditions of (28) hold if and only if there exist $\alpha \in \mathbb{R}$, and two measures α_1 and α_2 in the dual of L^∞ such that (19) is satisfied. In particular, $\alpha_1 = \alpha - (y^* + y_0)$ in Ω_1 and vanishes outside Ω_1, which proves that $\alpha_1 \in L^2$. Similarly, $\alpha_2 \in L^2$. □

Proof of Theorem 4. Consider the dual solution (λ^*, z^*). (13) implies that $\Delta_{CVaR_\mu} \subset L^\infty$, while (25) shows that z_π is not bounded. Then, Theorem 2 implies that $\lambda^* > 0$. Since $(1 + \lambda^*)\, z_\pi$ is continuous and strictly decreasing (25) and (26) show the existence of

$\gamma_1 \in (0,1)$ such that $(1+\lambda^*)\, z_\pi(\gamma_1) = \dfrac{1}{1-\mu}$, $(1+\lambda^*)\, z_\pi(\omega) > \dfrac{1}{1-\mu}$ for $\omega \in (0,\gamma_1)$ and $(1+\lambda^*)\, z_\pi(\omega) < \dfrac{1}{1-\mu}$ for $\omega \in (\gamma_1, 1)$. In particular, $z^*(\omega) < (1+\lambda^*)\, z_\pi(\omega)$ in $(0,\gamma_1)$, which, along with the fourth and fifth equations in (12), imply that $y^*(\omega) = 0$ in $(0,\gamma_1)$. On the other hand, y_0 being continuous and strictly increasing, take $\beta = y_0(\gamma_1)$ and we have that $y_0 \le \beta$ if and only if $(0,\gamma_1] \ni \omega$, *i.e.*, the third part of (27) has been proved.

Consider the partition $(0,1) = \Omega_0 \cup \Omega_1 \cup \Omega_2$ of (19). Notice that the fourth equation in (19) and the fifth one in (12) lead to $\Omega_1 \subset (0,\gamma_1]$. Notice also that $y_0 = \alpha - \alpha_1$ in Ω_1, whereas $y_0 = \alpha + \alpha_2$ in $(0,\gamma_1] \setminus \Omega_1$, since α_1 vanishes outside Ω_1 and y^* vanishes in $(0,\gamma_1]$. Being $\alpha_1, \alpha_2 \ge 0$ we conclude that y_0 increases from Ω_1 to $(0,\gamma_1] \setminus \Omega_1$. Since y_0 is strictly increasing there will exist $\tilde{\gamma}_1 \le \gamma_1$ such that $\Omega_1 = (0, \tilde{\gamma}_1]$.

Let us see that $(\tilde{\gamma}_1, \gamma_1] \subset \Omega_2$. Indeed, otherwise in a non-null subset of $(\tilde{\gamma}_1, \gamma_1]$ we would have $y_0 = \alpha + \alpha_2 = \alpha$ (α_2 vanishes outside Ω_2), but this is a contradiction because y_0 is strictly increasing and cannot achieve any concrete value with strictly positive probability.

Assume for a few moments that Ω_0 is void. Then $\Omega_2 = (\tilde{\gamma}_1, 1)$ and $z^* = 0$ in $(\tilde{\gamma}_1, 1)$ (last condition in (19)). Since $(1+\lambda^*)\, z_\pi > 0$ (see (24)), the fourth equation in (12) implies $y^* = 0$ in $(0,1)$. Then $C > 0$ and $\lambda^* > 0$ provoke that the first equality in (12) does not hold, and we are facing a contradiction.

Consequently Ω_0 is not a null set. Let us see that $\tilde{\gamma}_1 = \gamma_1$. Indeed, we know that $\Omega_0 \subset (\gamma_1, 1)$. Fix λ^*. According to (10), z^* must solve

$$Min\ \left\{ \mathbb{E}\left(y_0 z\right);\ z \le (1+\lambda^*)\, z_\pi, z \in \Delta_\rho \right\}. \tag{29}$$

If $\tilde{\gamma}_1 < \gamma_1$ then take $v = Inf\ (\Omega_0)$, $u = Sup\ (\Omega_0)$ and

$$\tilde{z} = \begin{cases} z^*, & \omega \in \Omega_1 = (0, \tilde{\gamma}_1] \\ z^*\left(\omega + v - \tilde{\gamma}_1\right), & \tilde{\gamma}_1 < \omega < \tilde{\gamma}_1 + u - v \\ 0, & otherwise \end{cases}$$

$\tilde{z}$ trivially satisfies the constraints of (29) because so does z^*, z^* vanishes on Ω_2 and z_π is strictly decreasing. On the other hand, $\mathbb{E}\left(y_0 \tilde{z}\right) < \mathbb{E}\left(y_0 z^*\right)$ trivially holds because y_0 is strictly increasing, so z^* does not solve (29). Hence, $\tilde{\gamma}_1 = \gamma_1$.

Applying an analogous argument it is easy to show the existence of $\gamma_2 > \gamma_1$ such that $\Omega_0 = (\gamma_1, \gamma_2)$. Moreover, $y^* = \alpha - y_0$ in (γ_1, γ_2) implies that $y_0(\omega) \le \alpha$ for $\omega \in (\gamma_1, \gamma_2)$, because $y^* \geq 0$. Since y_0 is continuous and strictly increasing one has that

$$\alpha \geq y_0(\gamma_2) > y_0(\gamma_1) = \beta > 0.$$

Finally, if $y_0(\omega) > \alpha$ then $\omega > \gamma_2$, so $\omega \in \Omega_2$, $z^* = 0$ (last equation in (19)), the fifth equation in (12) holds in terms of strict inequality, and the fourth equation in (12) shows that y^* vanishes. □

Algorithm I to solve (8) for the Conditional Value at Risk and the Black and Scholes model. Suppose for a few moments that we know the value of the dual solution λ^*. Then proof of Theorem 4 and (20) show that β may be computed in practice by

$$\beta = W_0 Exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\Phi^{-1}(\gamma_1)\right),$$

where, according to the theorem's proof and (24),

$$\gamma_1 = z_\pi^{-1}\left(\frac{1}{(1-\mu)(1+\lambda^*)}\right) = \Phi\left(\frac{2L(1-\mu) + 2L(1+\lambda^*) - a^2}{2a}\right),$$

and a is given by (23).

Since the theorem's proof is constructive, it also yields and algorithm leading to the computation of λ^*. Indeed, take in a first iteration $\gamma_1 = 1 - \mu$ and

$$1 + \lambda^* = \frac{1}{(1-\mu)z_\pi(\gamma_1)}. \tag{30}$$

In the theorem's proof this choice means that we are taking

$$z^* = \begin{cases} \frac{1}{1-\mu} & \omega \leq \gamma_1 \\ 0 & otherwise \end{cases}$$

We know that this choice does not provide the dual solution because it implies that Ω_0 is void (see the theorem's proof). Anyway, we can compute the (minus) objective of (10) in the proposed solution,

$$C\lambda^* + \mathbb{E}(z^* y_0). \tag{31}$$

Then, choose a "small enough step" $\varepsilon > 0$ and consider $\gamma_1 = 1 - \mu - \varepsilon$. Take λ^* as in (30) and

$$z^* = \begin{cases} \frac{1}{1-\mu} & \omega \leq \gamma_1 \\ (1+\lambda^*)z_\pi & \gamma_1 < \omega \leq \gamma_2 \\ 0 & otherwise \end{cases},$$

where γ_2 must be selected so as to reach

$$\mathbb{E}(z^*) = \frac{\gamma_1}{1-\mu} + (1+\lambda^*) \int_{\gamma_1}^{\gamma_2} y_0(\omega) z_\pi(\omega) d\omega = 1.$$

Notice that the integral may be calculated by numerical methods. Then compute the (minus) objective of (10) as indicated in (31). If the value of (31) has decreased with respect to the previous one then we already reached the desired value λ^*. Otherwise take $\gamma_1 = 1 - \mu - 2\varepsilon$ and repeat a new iteration of the algorithm.

Once β has been computed one can calculate α because the price of y^* must equal $Ce^{-r_f T}$, *i.e.*, the following equation

$$\Pi(y^*_\alpha) = Ce^{-r_f T} + \Pi\left(y^*_\beta\right) + (\alpha - \beta) \Pi\left(y^*_{D\beta}\right)$$

must hold. □

Algorithm II to construct the sequence satisfying (18) for the Black and Scholes model. Consider $n \in \mathbb{N}$. As indicated in Remark 3, the sequence $(y^*_n)_{n=1}^\infty = (x^*_n - n)_{n=1}^\infty$ solves (15), $(x^*_n)_{n=1}^\infty$ denoting the solutions of (16), which can be computed by dealing with Algorithm I above. In practice one cannot solve Algorithm I infinitely many times, so one can proceed as follows:

Step_a) Fix a desired finite level (A_1, A_2, A_3) for $(VaR_\mu(y^*_n), CVaR_\mu(y^*_n), \; return(y^*_n))$.

Step_b) Apply Algorithm I in order to compute y^*_n for several values of $n \in \mathbb{N}$, and then stop once $n = N$ is large enough so as to imply $VaR_\mu(y^*_N) \leq A_1$, $CVaR_\mu(y^*_N) \leq A_2$, and $return(y^*_N) \geq A_3$. □

6 Appendix II (figures)

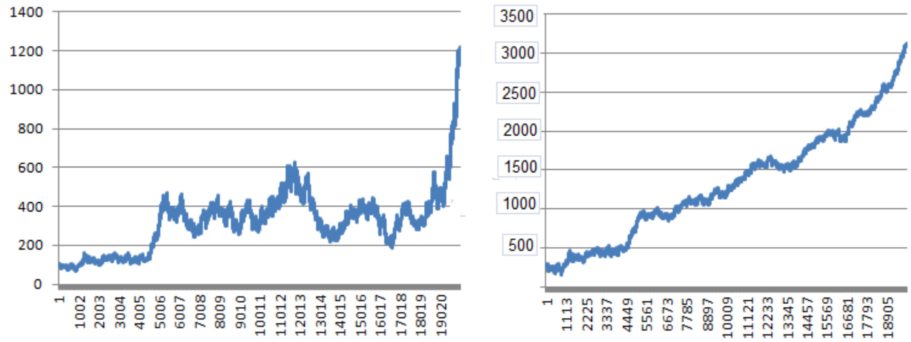

Figure 1. Simulated trajectory of a GBM (left) and trajectory of the associated good deal price (right). There are 20000 simulated days.

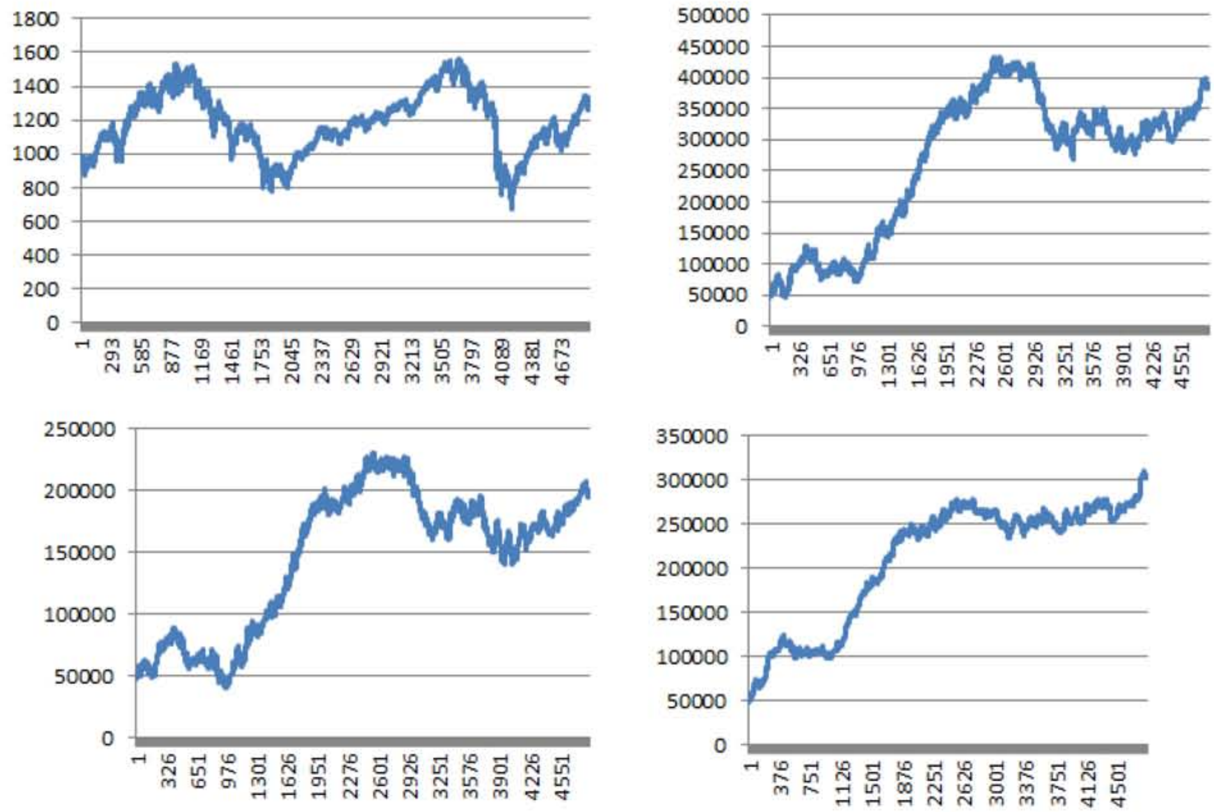

Figure 2. On the abscissa axis we represent tested days. On the vertical axis we represent index points for indices and money for good deals. The first row contains the $S\&P\_500$ evolution and the accumulated amount generated by the self-financing good deal, respectively. The initial investment $C_0 = \$50000$ generated a final wealth equaling \$388914.58. The tested period is 1997, $December$, 7^{th} $-$ 2011, $March$, 31^{st}. The first graph on the second row contains the amount generated by the self-financing good deal if one draws on the VIX index as a realized volatility predictor. This good deal is obviously outperformed by the previous one, since the accumulated amount is close to \$200000 only. The second graph is related to the good deal under ambiguity, and the accumulated amount is close to \$300000. Once again, the initial good deal, computed with historical parameters and without ambiguity, seems to be the best investment strategy. The three good deals are constructed with the confidence level $\mu = 70\%$.

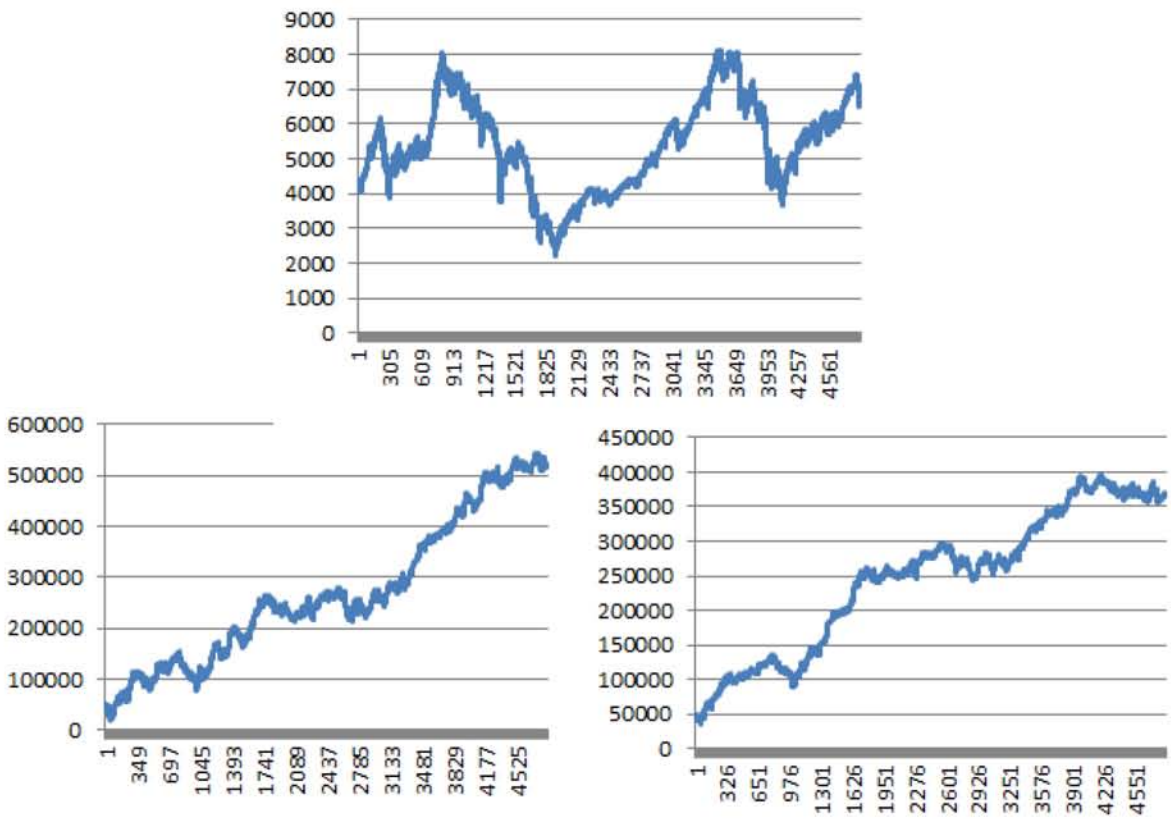

Figure 3. The first row contains the DAX evolution. The second row contains the amount accumulated by the good deal without and with ambiguity, respectively. The tested period is 1997, $December$, 7^{th} $-$ 2011, $March$, 31^{st}. It is worth pointing out that the DAX index generates the good deal with best performance among the tested indices. The two good deals are constructed with the confidence level $\mu = 70\%$.

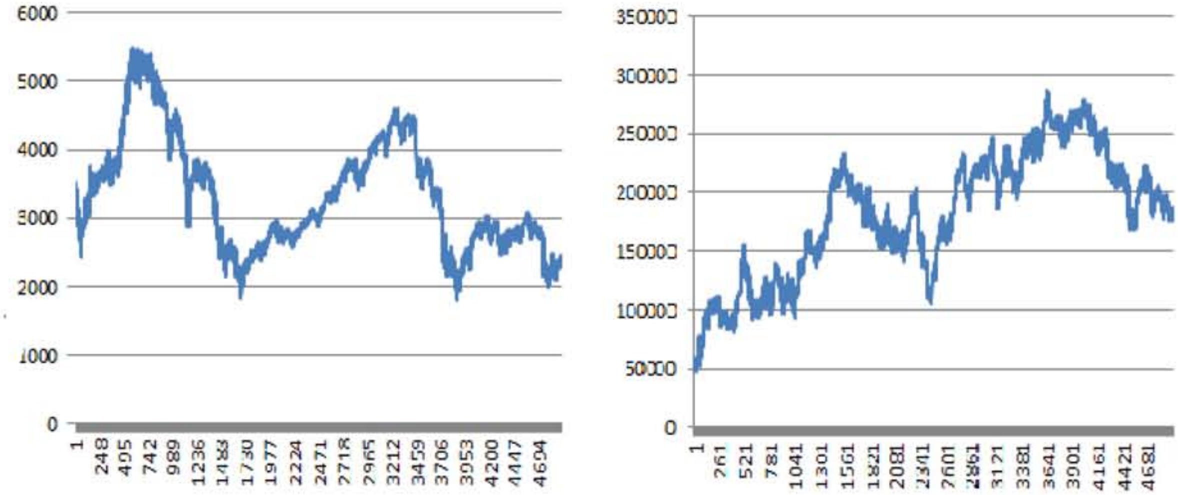

Figure 4. This figure contains the $Eurostoxx\_50$ evolution and the associated good deal. The tested period is 1998, $Augoust$, $2^{nd} - 2011$, $March$, 31^{st}. The good deal is constructed with the confidence level $\mu = 70\%$.

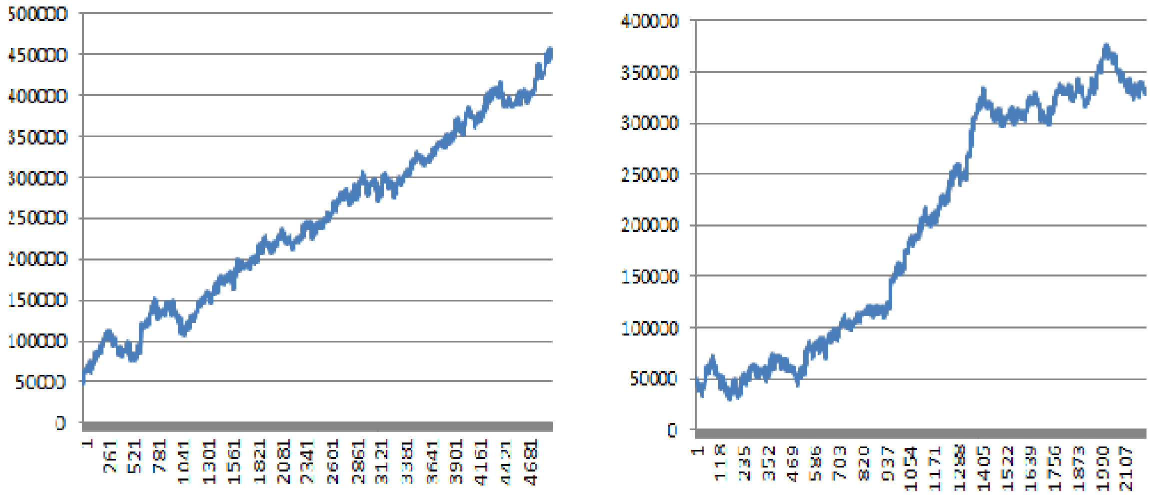

Figure 5. This figure contains the good deal accumulated amount for gold and Brent, respectively. It is worth pointing out that the gold future generates the good deal with best performance among the tested securities. The two good deals are constructed with the confidence level $\mu = 70\%$.

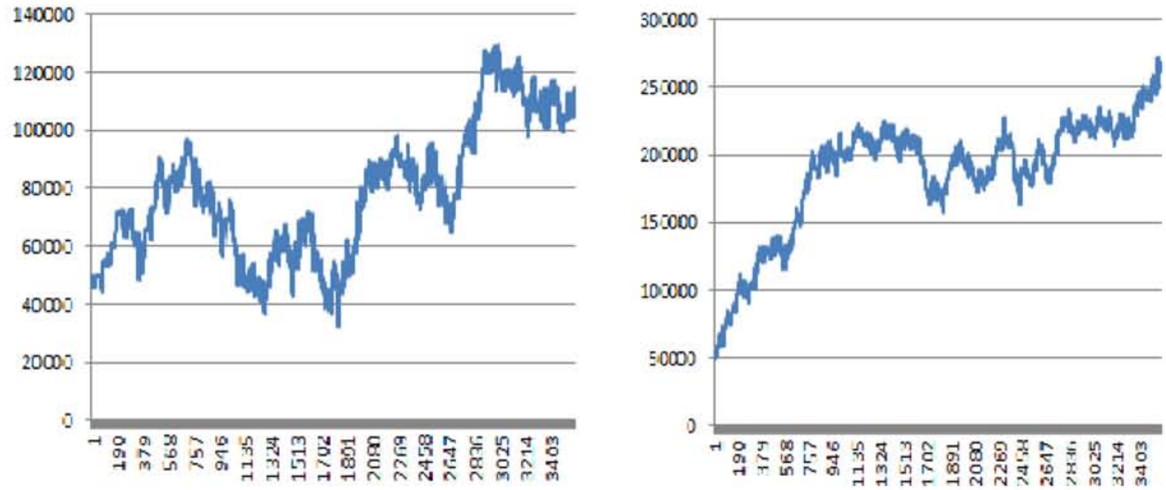

Figure 6. Good deals for the $DJ-UBSCI$ commodity index. The first graph is related to the risk measure $CVaR_{70\%}$, and its performance is the worst one among the tested securities, though it is still good enough. The second graph is related to the risk measure $CVaR_{99\%}$. The performance is much better, though we think that this is just a random finding. The higher level of confidence should imply the more conservative strategy, and therefore the lower realized return.

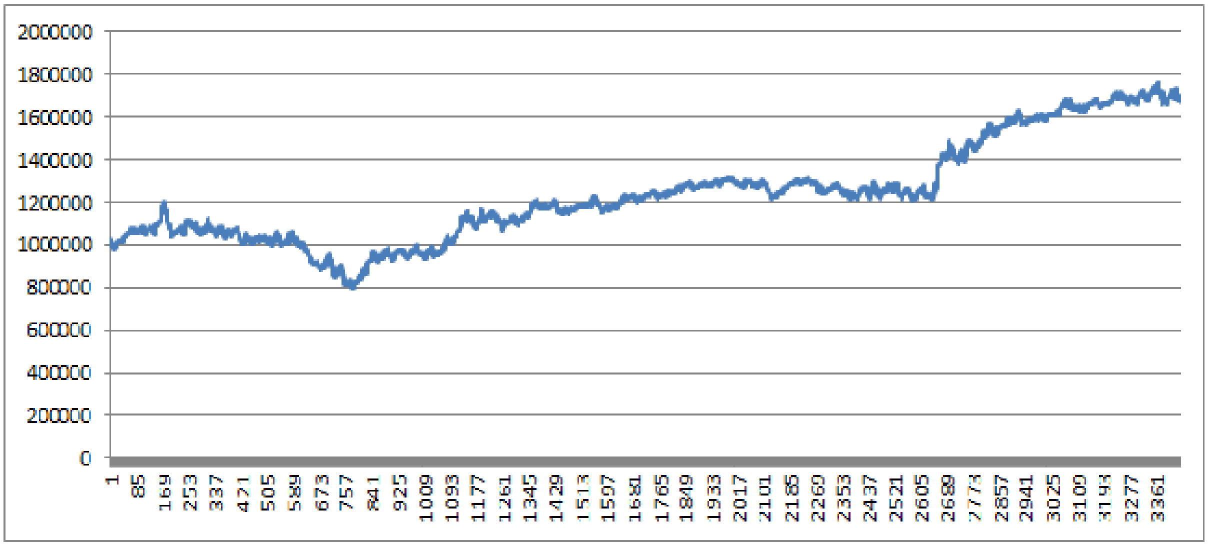

Figure 7. Accumulated amount for the "conservative good deal" of Section 3.2. The underlying asset is the $S\&P\_500$ index, and the confidence level is $\mu = 70\%$.

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