Efficient random variable generation: ratio of uniforms and polar rejection sampling

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Monte Carlo techniques, which require the generation of samples from some target density, are often the only alternative for performing Bayesian inference. Two classic sampling techniques to draw independent samples are the ratio of uniforms (RoU) and rejection sampling (RS). An efficient sampling algorithm is proposed combining the RoU and polar RS (i.e. RS inside a sector of a circle using polar coordinates). Its efficiency is shown in drawing samples from truncated Cauchy and Gaussian random variables, which have many important applications in signal processing and communications.

Introduction: Bayesian inference has become very popular in signal processing and communications during the past few decades. Monte Carlo techniques, such as Markov chain Monte Carlo (MCMC) methods or particle filters, which are often necessary for their implementation, require the generation of samples from some target density [1]. Two classic sampling techniques, that are often used together, are the ratio of uniforms (RoU) and rejection sampling (RS). In this Letter we propose an efficient sampling algorithm combining the RoU and polar RS (i.e. RS inside a sector of a circle using polar coordinates), instead of the usual rectangular RS approach (i.e. RS inside a rectangular area using Cartesian coordinates), which can be inefficient in some cases. We show the efficiency of the algorithm in drawing samples from truncated Cauchy and Gaussian random variables (RVs), which have many important applications in signal processing and communications [2 4].

Ratio of uniforms and rejection sampling: RoU is a classic technique for generating samples from an arbitrary probability density function (PDF), $p_0(x) \quad kp(x)$ with $k > 0$ [5]. Given a pair of independent RVs, (u,v), uniformly distributed inside

$$C_p \quad \left\{(u, v) : 0 \leq u \leq \sqrt{p(v/u)}\right\} \tag{1}$$

then $x = v/u$ is distributed exactly according to $p_0(x)$, i.e. $x \sim p_0(x)$. Hence, the RoU method allows us to draw samples from any PDF, as long as the resulting region C_p is bounded, simply by generating a couple of independent uniform RVs.

Unfortunately, the efficiency and applicability of the RoU depends on the ability of generating uniform samples inside the region C_p, which is often not straightforward. The usual approach is embedding C_p inside the rectangular region R_p (Fig. 1*b*) and applying rectangular rejection sampling (RS). RS is another standard technique for generating samples from an arbitrary target $p_0(x) \quad kp(x)$ PDF, by using an alternative, simpler proposal PDF, $\pi(x)$, such that $p(x)/\pi(x) \leq L$ [5]. RS works by generating samples from the proposal density, $w \sim \pi(x)$, accepting them when $z \leq p(w)/[L\pi(w)]$, with $z \sim U([0, 1])$ uniformly distributed, and rejecting them otherwise.

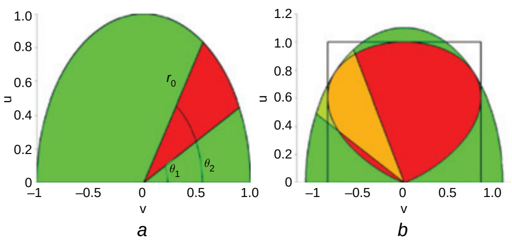

Fig. 1 *Region C_p and embedding regions R_p and S_p*
a $C_p = S_p$ for Cauchy and truncated Cauchy RVs
b C_p, R_p *and* S_p for a Gaussian and S_p for a truncated Gaussian

The combination of the RoU and rectangular RS yields the following algorithm for generating each sample $x \sim p_0(x)$: draw two independent uniform RVs, (u,v), inside R_p; accept the (u,v) pair if it belongs to C_p (i.e. if the inequality in (1) is fulfilled), obtaining the generated RV as $x = v/u$; otherwise, discard (u,v) and keep generating pairs of samples until $(u,v) \in C_p$. The key performance measure for this algorithm is the acceptance rate (i.e. the percentage of candidate samples accepted), which is given by $\Gamma_R \quad |C_p|/|R_p|$, where $|C_p|$ and $|R_p|$ denote the areas of the regions C_p and R_p, respectively. Unfortunately, for many RVs of interest Γ_R may be too low, leading to a large number of candidate samples being discarded. Therefore, alternative embedding areas which can be easily sampled, such as combinations of rectangles [6] or triangles [7], have been proposed to improve the acceptance rate.

RoU and polar RS: Another geometric area where uniform samples can be easily generated and that has not been exploited for RS is a sector of a circle, i.e. the region bounded by two radii of the circle and the arc of the circumference lying between them (Fig. 1*a*). Uniform sampling inside the sector delimited by angles θ_1 and θ_2 and radius r_0 can be achieved by drawing two independent samples from:

$$p_e(\theta) \quad \begin{cases} \dfrac{1}{\theta_2 \quad \theta_1}, & \theta_1 \leq \theta \leq \theta_2 \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

$$p_R(r) \quad \begin{cases} \dfrac{2r}{r_0^2}, & 0 \leq r \leq r_0 \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

Sampling from (2) only requires generating $u \sim U([0, 1])$ and scaling it, obtaining the suitable angle as $\theta \quad \theta_1 + (\theta_2 \quad \theta_1)u$. Sampling from (3) is slightly more involved, requiring the generation of two independent RVs, $v, w, \sim U([0, 1])$. The desired radius is then:

$$r \quad \begin{cases} r_0 v, & w \leq v; \\ r_0(1 \quad v), & w > v. \end{cases} \tag{4}$$

In this Letter we propose to combine the RoU and polar RS (i.e. RS using a sector of a circle, S_p, as the embedding region where we sample uniformly using polar coordinates) to improve the efficiency in the generation of some RVs, such as truncated Cauchy and Gaussian. The algorithm for generating a sample from the target PDF is the following:

1. Given a target PDF, $p_0(x) \quad kp(x)$ with $k > 0$, find a sector of a circle, $S_p \quad \{(\theta, r) : \theta_1 \leq \theta \leq \theta_2, 0 \leq r \leq r_0\}$, such that C_p is embedded inside S_p, i.e. $C_p \subseteq S_p$.
2. Draw a uniform sample pair inside S_p, (u,v), using polar coordinates as described by (2) (4).
3. Accept it when $(u,v) \in C_p$, i.e. $0 \leq u \leq \sqrt{p(v/u)}$. In this case, the generated sample is $x = v/u$.
4. Otherwise, discard it and repeat steps 2 4.

Step 1 corresponds to the initialisation (i.e. it is performed only once), whereas the core of the algorithm (steps 2 4) only requires drawing three independent uniform RVs per generated sample. The acceptance rate in this case is $\Gamma_S \quad |C_p|/|S_p|$, with $|S_p|$ indicating the area of S_p.

Results: To show the applicability of the proposed algorithm, consider the generation of standard Cauchy RVs, $x \sim p(x) \quad 1/[\pi(1 + x^2)]$, truncated or not. In this case, for the full Cauchy RV $C_p \quad \{(u, v) : 0 \leq u^2 + v^2 \leq 1\}$ is the half circle shown in Fig. 1*a*, whereas for the truncated Cauchy, $x \in [x_1, x_2]$, C_p is the sector shown also in Fig. 1*a* with $r_0 \quad 1$, $\theta_1 \quad \pi/2 \quad \arctan(x_1)$ and $\theta_2 \quad \pi/2 \quad \arctan(x_2)$. In both cases the proposed algorithm provides samples from the exact target PDF without any rejection (i.e. $\Gamma_S = 1$). We remark that the inversion method [5], another popular sampling technique, can also be used here, but requires evaluating a tangent, thus leading to a higher computational cost than the proposed algorithm.

As a second example, we address the generation of standard Gaussian RVs, $x \sim p(x) \quad \frac{1}{\sqrt{2\pi}}\exp(\quad x^2/2)$. For a full Gaussian RV we obtain the region C_p shown in Fig. 1*b*, which can be embedded inside a rectangle R_p (with $\Gamma_R \approx 0.72$) or a circle S_p (with $\Gamma_S \simeq 0.65$). Although the acceptance rate of rectangular RS is higher than the one for polar RS in this case, that is no longer true for truncated Gaussians. Fig. 2 shows the acceptance rate (averaged over 20 000 runs) for different values of x_1 and x_2 and five RS algorithms: RS with constant upper bound (triangle down marker), RS with a triangular linear proposal (diamond marker), RS with a half Gaussian proposal (triangle up marker), RS with a truncated exponential PDF as proposal [8] (rectangular marker), and our polar RS method (circle marker). Our method provides the best acceptance rate when $0 < x_1 < 1$ and $x_2 \geq 3$ (Fig. 2*b*), and the second best for $x_1 > 1$ (Fig. 2*a*), only surpassed by the method from [8], which has a higher complexity. Note also that the first two approaches cannot be applied inside an infinite domain (i.e. when $x_1 \rightarrow \ \infty$ or $x_2 \rightarrow \ \infty$), whereas our approach is always feasible, regardless of the values of x_1 and x_2. Finally, we remark that the inverse cumulative function of a Gaussian is not known analytically. Hence, the commonly used inversion method [5] can only be applied approxi mately, whereas our method always provides samples from the exact target PDF.

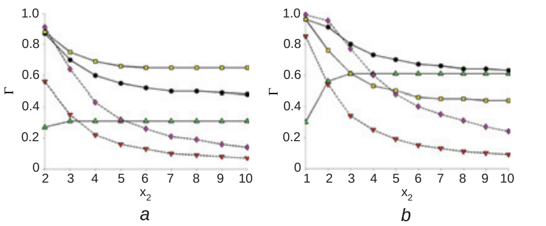

Fig. 2 *Acceptance rates with different sampling methods*
a $x_1 = 1$, x_2 variable
b $x_1 = 0.5$, x_2 variable

Conclusions: An efficient algorithm for drawing samples from arbitrary univariate distributions based on the ratio of uniforms and a novel polar rejection sampling technique has been proposed. Its applicability and good performance has been demonstrated in the generation of full and truncated Cauchy and Gaussian RVs with the exact PDF and higher acceptance rates than other proposed methods. Adaptive versions of this method and extensions to multivariate PDFs can be easily implemented.

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