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Title: DIRECT AND INVERSE RESULTS ON ROW SEQUENCES OF SIMULTANEOUS RATIONAL APPROXIMANTS

Author: Junot Cacoq

Director: Guillermo López Lagomasino
Co-director: Bernardo de la Calle Ysern

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Dedication

To God, for giving me health. To my family, my wife Darbis Vidal Vázquez and daughter Dayana Vidal Cacoq, my mother Gracieuse Paul and my father Saintilien Cacoq, and finally to my sister Gina Cacoq who unfortunately is not among us.

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For me it was a dream to come to Spain and defend a Ph.D thesis. I have to say that none of this would have been possible without the help of Guillermo López Lagomasino. For this reason, my first thanks go to him, above all for opening my eyes, for teaching me the real math, and making me aware. I have to admit, I have had to grow day by day.

There are not many people like Bernardo de la Calle Ysern. It was a privilege for me to know him, and I want to thank him, in particular, for one of the questions that made me learn a lot more, the question is: (why ...?).

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Thinking

Success is easy to obtain. The hard part is to deserve it.

Albert Camus

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Resumen

En la tesis se investiga la aproximación de funciones vectoriales mediante vectores de fracciones racionales que generalizan los llamados aproximantes de Padé correspondientes al caso de la aproximación de una función escalar. Se consideran dos tipos de aproximantes: los aproximantes simultaneos Hermite-Padé, que se construyen mediante criterios interpolatorios y los aproximantes Fourier-Padé basados en desarrollos en serie de Fourier a partir de un sistema de polinomios ortogonales.Los resultados obtenidos generalizan alcaso de la aproximación vectorial resultados muy conocidos para el caso escalar debidos a R. de Montessus de Ballore, A.A. Gonchar, S.P. Suetin, P.R. Graves-Morris, y E.B. Saff.

Abstract

In this Tesis we investigate the approximation of vector functions by vector rational function that generalizes Padé approximants. We consider two types of approximants: the simultaneous Hermite-Pade approximants, which are constructed by mean of interpolation criterion and Fourier-Padé approximants based on Fourier series expansions in terms of a system of orthogonal polynomials. The results obtained in terms of generalize to the vector case results well known for the scalar case due to R. of Montessus of Ballore, A.A. Gonchar, S.P. Suetin, P.R. Graves-Morris, and E. B. Saff.

Chapter 1

Introduction

1.1 Padé approximation

Let

$$f(z) = \sum_{n=0}^{\infty} \phi_n z^n, \qquad \phi_n \in \mathbb{C}, \tag{1.1}$$

denote a formal or convergent Taylor expansion about the origin. By $D_0(f)$ and $R_0(f)$ we denote the disk and radius of convergence, respectively, of the series (1.1). In [17], Jacques Hadamard introduced the notion of mth disk of meromorphy $D_m(f)$ of f. When $R_0(f) = 0$ this disk is defined to be the empty set. If $R_0(f) > 0$ then $D_m(f)$ is the largest disk centered at the origin to which the analytic element $(f, D_0(f))$ can be extended as a meromorphic function having no more than m poles. Let $R_m(f)$ denote the radius of $D_m(f)$. In the cited paper, Hadamard proves a beautiful formula which gives the values of the numbers $R_m(f)$ for all $m \in \mathbb{Z}_+$ using exclusively the data provided by the Taylor coefficients ϕ_n. For $m = 0$, it reduces to Cauchy's formula for the radius of convergence of a Taylor series. Hadamard's finding is intimately connected with the convergence theory of Padé approximations.

Definition 1.1.1. *Let f be the formal expansion* (1.1). *Let $n, m \in \mathbb{Z}_+, n \geq m$, be given. Then, there exist polynomials Q, P, satisfying*

a.1) $\deg P \leq n - m, \quad \deg Q \leq m, \quad Q \not\equiv 0,$

a.2) $[Qf - P](z) = Az^{n+1} + \cdots$.

Any pair (Q, P) *which satisfies* $a.1) - a.2)$ *defines a unique rational function* $\pi_{n,m} = P/Q$ *which is called the Padé approximation of type* (n, m) *of* f.

We have slightly modified (in an equivalent form) the usual definition of an (n, m) Padé approximation having in mind the aims of the thesis. Let $\pi_{n,m} = P_{n,m}/Q_{n,m}$ where $Q_{n,m}$ and $P_{n,m}$ are polynomials obtained eliminating all common factors and normalizing $Q_{n,m}$ to be monic (that is with leading coefficient equal to 1).

In his doctoral thesis, H. Padé studied the algebraic properties of the family of rational functions $\{\pi_{n,m}\}, m \in \mathbb{Z}_+, n \geq m$. Since then they are called Padé approximants. If m is fixed, the sequence $\{\pi_{n,m}\}, n \geq m$ is called the m-th row of the Padé table. The sequence $\{\pi_{2m,m}\}, m \geq 0$, is called the main diagonal. In this thesis we will restrict our attention to row sequences.

Robert de Montessus de Ballore, using Hadamard's work, proved the following result (see [20]). Let $\mathcal{Q}_m(f)$ stand for the monic polynomial whose zeros are the poles of f in $D_m(f)$ with multiplicity equal to the order of the corresponding pole. By $\mathcal{P}_m(f)$ we denote the set of distinct zeros of $\mathcal{Q}_m(f)$. Given a compact set $\mathcal{K} \subset \mathbb{C}$, $\|\cdot\|_{\mathcal{K}}$ denotes the sup norm on $\mathcal{K}$.

Montessus de Ballore's Theorem. *Assume that* $R_0(f) > 0$ *and that* f *has exactly* m *poles in* $D_m(f)$ *counting multiplicities, then*

$$\lim_{n\to\infty} \pi_{n,m} = f,$$

uniformly on each compact subset of $D_m(f) \setminus \mathcal{P}_m(f)$.

From this result it follows that if ζ is a pole of f in $D_m(f)$ of order τ, then for each $\varepsilon > 0$, there exists n_0 such that for $n \geq n_0$, $Q_{n,m}$ has exactly τ zeros in $\{z : |z-\zeta| < \varepsilon\}$. We say that each pole of f attracts as many zeros of $Q_{n,m}$ as its multiplicity.

An important improvement of Montessus' result is due to A.A. Gonchar.

Gonchar's Theorem. *Let* f *be a formal Taylor expansion about the origin and fix* $m \in \mathbb{Z}_+$. *Then, the following two assertions are equivalent.*

a) $R_0(f) > 0$ *and* f *has exactly* m *poles in* $D_m(f)$ *counting multiplicities.*

b) There is a polynomial Q_m *of degree* m, $Q_m(0) \neq 0$, *such that the sequence of denominators* $\{Q_{n,m}\}_{n \geq m}$ *of the Padé approximations of* f *satisfies*

$$\limsup_{n\to\infty} \|Q_m - Q_{n,m}\|^{1/n} = \theta < 1,$$

where $\|\cdot\|$ *denotes the coefficient norm in the space of polynomials.*

Moreover, if either a) or b) takes place then $Q_m \equiv \mathcal{Q}_m(f)$,

$$\theta = \frac{\max\{|\xi| : \xi \in \mathcal{P}_m(f)\}}{R_m(f)}, \tag{1.2}$$

and

$$\limsup_{n\to\infty} \|f - R_{n,m}\|_K^{1/n} = \frac{\|z\|_K}{R_m(f)}, \tag{1.3}$$

where K *is any compact subset of* $D_m(f) \setminus \mathcal{P}_m(f)$.

So stated this theorem does not appear in the literature and needs some comments. Under assumptions a), from Montessus' proof in [20] (because he did not express explicitly the geometric rate of convergence), one can derive that a) implies b) with $Q_m = \mathcal{Q}_m(f)$, show that $\theta \leq \max\{|\xi| : \xi \in \mathcal{P}_m(f)\}/R_m(f)$, and obtain (1.3) with equality replaced by $\leq$. These are the so called direct statements of the theorem. The inverse statements, b) implies a), $\theta \geq \max\{|\xi| : \xi \in \mathcal{P}_m(f)\}/R_m(f)$, and the inequality $\geq$ in (1.3) are immediate consequences of [12, Theorem 1].

The study of inverse problems of Padé approximation was suggested by A.A. Gonchar in [12, Subsection 12] where he presented some interesting conjectures. Some of them were solved in [39] and [40]. Other interesting papers on the convergence theory of row sequences of Padé approximants and its generalizations are [3], [4], [11], [12], [19], [26], [36], and [37].

The aim of this thesis is to extend the theorems of Montessus de Ballore and A.A. Gonchar to the case of vector rational approximants.

1.2 Hermite-Padé approximation

Let $\mathbf{f} = (f_1, \ldots, f_d)$ be a system of d formal or convergent Taylor expansions about the origin; that is, for each $k = 1, \ldots, d$, we have

$$f_k(z) = \sum_{n=0}^{\infty} \phi_{n,k} z^n, \qquad \phi_{n,k} \in \mathbb{C}. \tag{1.4}$$

Let $\mathbf{D} = (D_1, \ldots, D_d)$ be a system of domains such that, for each $k = 1, \ldots, d$, f_k is meromorphic in D_k. We say that the point ζ is a pole of $\mathbf{f}$ in $\mathbf{D}$ of order τ if there exists an index $k \in \{1, \ldots, d\}$ such that $\zeta \in D_k$ and it is a pole of f_k of order τ, and for $j \neq k$ either ζ is a pole of f_j of order less than or equal to τ or $\zeta \notin D_j$. When $\mathbf{D} = (D, \ldots, D)$ we say that ζ is a pole of $\mathbf{f}$ in D.

Let $R_0(\mathbf{f})$ be the largest disk in which all the expansions $f_k, k = 1, \ldots, d$ correspond to analytic functions. If $R_0(\mathbf{f}) = 0$, we take $D_m(\mathbf{f}) = \emptyset, m \in \mathbb{Z}_+$; otherwise, $R_m(\mathbf{f})$ is the radius of the largest disk $D_m(\mathbf{f})$ centered at the origin to which all the analytic elements $(f_k, D_0(f_k))$ can be extended so that $\mathbf{f}$ has at most m poles counting multiplicities. The disk $D_m(\mathbf{f})$ constitutes for systems of functions the analog of the mth disk of meromorphy defined by J. Hadamard in [17] for $d = 1$. Moreover, in that case both definitions coincide.

By $\mathcal{Q}_m(\mathbf{f})$ we denote the monic polynomial whose zeros are the poles of $\mathbf{f}$ in $D_m(\mathbf{f})$ counting multiplicities. The set of distinct zeros of $\mathcal{Q}_m(\mathbf{f})$ is denoted by $\mathcal{P}_m(\mathbf{f})$.

Definition 1.2.1. *Let $\mathbf{f} = (f_1, \ldots, f_d)$ be a system of d formal Taylor expansions as in* (1.4). *Fix a multi-index $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$ where $\mathbf{0}$ denotes the zero vector in $\mathbb{Z}_+^d$. Set $|\mathbf{m}| = m_1 + \cdots + m_d$. Then, for each $n \geq \max\{m_1, \ldots, m_d\}$, there exist polynomials $Q, P_k, k = 1, \ldots, d$, such that*

a.1) $\deg P_k \leq n - m_k,\ k = 1, \ldots, d, \quad \deg Q \leq |\mathbf{m}|, \quad Q \not\equiv 0,$

a.2) $Q(z) f_k(z) - P_k(z) = A_k z^{n+1} + \cdots .$

The vector rational function $\mathbf{R}_{n,\mathbf{m}} = (P_1/Q, \ldots, P_d/Q)$ is called an $(n, \mathbf{m})$ Hermite-Padé approximation of $\mathbf{f}$.

This vector rational approximation, in general, is not uniquely determined and in the sequel we assume that given $(n, \mathbf{m})$ one particular solution is taken. For that solution we write

$$\mathbf{R}_{n,\mathbf{m}} = (R_{n,\mathbf{m},1}, \ldots, R_{n,\mathbf{m},d}) = (P_{n,\mathbf{m},1}, \ldots, P_{n,\mathbf{m},d})/Q_{n,\mathbf{m}}, \tag{1.5}$$

where $Q_{n,\mathbf{m}}$ has no common zero simultaneously with all the $P_{n,\mathbf{m},k}$ and is normalized to be monic. A sequence $\{\mathbf{R}_{n,\mathbf{m}}\}$ for which $|\mathbf{m}|$ remains fixed when n varies is called a row sequence and a diagonal sequence when $m_1 = \cdots = m_d = m, n = (d+1)m, m \in \mathbb{Z}_+$.

The study of simultaneous Hermite-Padé approximations of systems of functions has a long tradition (see [18]) and they have been subject to renewed interest in the recent past (see, for instance, [10] and the references therein). Many papers deal with diagonal and close to diagonal sequences and their applications in different fields (number theory, random matrices, brownian motions, Toda lattices, to name a few). At the same time, few papers study row sequences. In this second direction a significant contribution is due to Graves-Morris/Saff in [14] where they prove an analog of the Montessus de Ballore theorem. See also [15]-[16] for different approaches to the same type of results as well as [27]-[33] for least-squares versions.

In [14], Graves-Morris and Saff proved an analog of the direct part of Gonchar's Theorem for simultaneous approximation with the aid of the concept of polewise independence of a system of functions. They also established upper bounds for the convergence rates corresponding to (1.2) and (1.3).

Definition 1.2.2. *A vector* $\mathbf{f} = (f_1, \ldots, f_d)$ *of functions meromorphic in some domain* D *is said to be polewise independent with respect to the multi-index* $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$ *in* D *if there do not exist polynomials* $p_1, \ldots, p_d$*, at least one of which is non-null, satisfying*

b.1) $p_j \equiv 0$ *if* $m_j = 0$,

b.2) $\deg p_j \leq m_j - 1, j = 1, \ldots, d,$ *if* $m_j \geq 1$, *b.3)* $\sum_{j=1}^{d} p_j f_j \in \mathcal{H}(D)$,

where $\mathcal{H}(D)$ *denotes the space of analytic functions in* D.

When $d = 1$ polewise independence merely expresses that the function has at least $m_1 = \mathbf{m}$ poles in D. We have

Graves-Morris/Saff Theorem. *Assume that* $R_0(\mathbf{f}) > 0$. *Fix a multi-index* $\mathbf{m} \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$ *and suppose that* $\mathbf{f}$ *is polewise independent with respect to* $\mathbf{m}$ *in* $D_{|\mathbf{m}|}(\mathbf{f})$, *then*

$$\limsup_{n\to\infty} \|f_k - R_{n,\mathbf{m},k}\|_K^{1/n} \leq \frac{\|z\|_K}{R_{|\mathbf{m}|}(\mathbf{f})}, \quad k = 1, \ldots, d, \tag{1.6}$$

where K *is any compact subset of* $D_{|\mathbf{m}|}(\mathbf{f}) \setminus \mathcal{P}_{|\mathbf{m}|}(\mathbf{f})$. *Additionally*

$$\limsup_{n\to\infty} \|\mathcal{Q}_{|\mathbf{m}|}(\mathbf{f}) - Q_{n,\mathbf{m}}\|^{1/n} \leq \frac{\max\{|\zeta| : \zeta \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{f})\}}{R_{|\mathbf{m}|}(\mathbf{f})}. \tag{1.7}$$

It also follows from this result that each pole of $\mathbf{f}$ in $D_{|\mathbf{m}|}(\mathbf{f})$ attracts exactly as many zeros of $Q_{n,\mathbf{m}}$ as its order when n tends to infinity.

The Graves-Morris/Saff Theorem was refined and complemented in [5, Theorem 4.4] by weakening the assumption of polewise independence, improving the upper bound given in [14] for the rate (1.7), and giving the exact one for (1.6). Until now, results of inverse type for row sequences of Hermite-Padé approximants are not available.

The main result of this thesis, which is contained in [6], establishes an analog of Gonchar's Theorem for simultaneous Hermite-Padé approximants, characterizing the exact rates of convergence of the $Q_{n,\mathbf{m}}$ and $\mathbf{R}_{n,\mathbf{m}}$. Therefore, it also improves [5, Theorem 4.4].

The underlying idea in inverse-type results is that a polynomial which is the limit of the denominators of the approximants must have as zeros the poles of the function being approximated, provided that the rate of convergence is geometric. However, the actual situation in simultaneous approximation may be rather complicated as the following example shows. Take $\mathbf{f} = (f_1, f_2)$, where

$$f_1 = \frac{1}{1-2z} + \sum_{n=0}^{\infty} z^{n!} + \frac{1}{z-2}, \qquad f_2 = \frac{1}{1-2z} + \sum_{n=0}^{\infty} z^{n!}, \tag{1.8}$$

and $\mathbf{m} = (1,1)$. It is clear that the unit circle is a natural boundary of definition for both functions f_1 and f_2 and thus $z = 2$ cannot be a pole of $\mathbf{f}$ in any system of domains. However, results contained in [5] (and Chapter 2) show that the denominators $Q_{n,\mathbf{m}}$ of the simultaneous Hermite-Padé approximants converge with geometric rate to the polynomial $(z-1/2)(z-2)$.

This kind of examples leads us to introduce the following concept which in part is inspired in that of polewise independence.

For each $r > 0$, set $D_r = \{z \in \mathbb{C} : |z| < r\}$, $\Gamma_r = \{z \in \mathbb{C} : |z| = r\}$, and $\overline{D}_r = \{z \in \mathbb{C} : |z| \leq r\}$.

Definition 1.2.3. *Given $\mathbf{f} = (f_1, \ldots, f_d)$ and $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$ we say that $\xi \in \mathbb{C} \setminus \{0\}$ is a system pole of order τ of $\mathbf{f}$ with respect to $\mathbf{m}$ if τ is the largest positive integer such that for each $s = 1, \ldots, \tau$ there exists at least one polynomial combination of the form*

$$\sum_{k=1}^{d} p_k f_k, \quad \deg p_k < m_k, \quad k = 1, \ldots, d, \tag{1.9}$$

which is analytic on a neighborhood of $\overline{D}_{|\xi|}$ except for a pole at $z = \xi$ of exact order s. If some component m_k equals zero the corresponding polynomial p_k is taken identically equal to zero.

The great advantage of this definition with respect to that of polewise independence is that we have liberated it from establishing a priori a region where the property should be verified. This turns out to be crucial.

We wish to underline that if some component m_k equals zero, that component places no restriction on Definition 1.2.1 and does not report any benefit in finding system poles; therefore, without loss of generality we can restrict our attention to multi-indices $\mathbf{m} \in \mathbb{N}^d$, and we will do so in the sequel except in reference to the convergence of the approximants themselves.

Notice that the definition of system pole strongly depends on the multi-index $\mathbf{m}$. During the proof of Theorem 1.2.4 below, carried out in Chapter 3, we give a procedure for finding in a finite number of steps all the system poles of $\mathbf{f}$ with respect to a multi-index $\mathbf{m}$ under appropriate conditions.

It is easy to see that a system pole may not be a pole of $\mathbf{f}$ or viceversa. For example, let $\mathbf{f}$ be the system given by (1.8) and $\mathbf{m} = (1,1)$. The point $z = 2$, which lies beyond the natural boundary of definition of f_1 and f_2 is not a pole; however it is a system pole of $\mathbf{f}$ since $f_1 - f_2$ has a pole at $z = 2$.

On the other hand, take $\mathbf{f} = (f_1, f_2)$ with

$$f_1 = \frac{1}{z-1} + \frac{1}{z-2}, \qquad f_2 = \frac{1}{z-3},$$

and $\mathbf{m} = (1,1)$. Then the points $z = 1$ and $z = 3$ are poles and system poles of $\mathbf{f}$ but $z = 2$ is only a pole because there is no way of eliminating the pole at $z = 1$ through linear combinations of f_1 and f_2 without eliminating the pole at $z = 2$.

To each system pole ξ of $\mathbf{f}$ with respect to $\mathbf{m}$ we associate several characteristic values. Let τ be the order of ξ as a system pole of $\mathbf{f}$. Let $\mathcal{G}(\mathbf{f}, \mathbf{m}, \xi, s)$ be the vector space of all functions g of type (1.9) that are analytic on $\{z : |z| \leq |\xi|\}$ except for a pole at $z = \xi$ of order s. For each $s = 1, \ldots, \tau$ define

$$r_{\xi,s}(\mathbf{f}, \mathbf{m}) = \max \{R_s(g) : g \in \mathcal{G}(\mathbf{f}, \mathbf{m}, \xi, s)\},$$

where $R_s(g)$ is the radius of the largest disk containing s poles of g. Since $\mathcal{G}(\mathbf{f}, \mathbf{m}, \xi, s)$ is finite dimensional, it is easy to see that the maximum is indeed attained. Set

$$R_{\xi,s}(\mathbf{f}, \mathbf{m}) = \min_{k=1,\ldots,s} r_{\xi,k}(\mathbf{f}, \mathbf{m}),$$

and

$$R_{\xi}(\mathbf{f}, \mathbf{m}) = R_{\xi,\tau}(\mathbf{f}, \mathbf{m}) = \min_{s=1,\ldots,\tau} r_{\xi,s}(\mathbf{f}, \mathbf{m}).$$

Obviously, if $d = 1$ and $(\mathbf{f}, \mathbf{m}) = (f, m)$, system poles and poles in $D_m(f)$ coincide. Also, $R_\xi(\mathbf{f}, \mathbf{m}) = R_m(f)$ for each pole ξ of f in $D_m(f)$.

By $\mathcal{Q}_{|\mathbf{m}|}(\mathbf{f}, \mathbf{m})$ we denote the monic polynomial whose zeros are the system poles of $\mathbf{f}$ with respect to $\mathbf{m}$ taking account of their order. The set of distinct zeros of $\mathcal{Q}_{|\mathbf{m}|}(\mathbf{f}, \mathbf{m})$ is denoted by $\mathcal{P}_{|\mathbf{m}|}(\mathbf{f}, \mathbf{m})$.

The following theorem constitutes the main result of the thesis and it is proved in Chapter 3 (see also [6]).

Theorem 1.2.4. *Let* $\mathbf{f}$ *be a system of formal Taylor expansions as in* (1.4) *and fix a multi-index* $\mathbf{m} \in \mathbb{N}^d$. *Then, the following two assertions are equivalent.*

a) $R_0(\mathbf{f}) > 0$ *and* $\mathbf{f}$ *has exactly* $|\mathbf{m}|$ *system poles with respect to* $\mathbf{m}$ *counting multiplicities.*

b) The sequence of denominators $\{Q_{n,\mathbf{m}}\}_{n \geq |\mathbf{m}|}$ *of simultaneous Padé approximations of* $\mathbf{f}$ *is uniquely determined for all sufficiently large* n *and there exists a polynomial* $Q_{|\mathbf{m}|}$ *of degree* $|\mathbf{m}|$, $Q_{|\mathbf{m}|}(0) \neq 0$, *such that*

$$\limsup_{n \to \infty} \|Q_{|\mathbf{m}|} - Q_{n,\mathbf{m}}\|^{1/n} = \theta < 1.$$

Moreover, if either a) *or* b) *takes place then* $Q_{|\mathbf{m}|} \equiv \mathcal{Q}_{|\mathbf{m}|}(\mathbf{f}, \mathbf{m})$, *and*

$$\theta = \max \left\{ \frac{|\xi|}{R_\xi(\mathbf{f}, \mathbf{m})} : \xi \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{f}, \mathbf{m}) \right\}. \tag{1.10}$$

If $d = 1$, $R_{n,m}$ and $Q_{n,m}$ are uniquely determined. Therefore, Theorem 1.2.4 contains Gonchar's Theorem except for (1.3) whose analog will be presented in Section 3.3 of Chapter 3 to avoid introducing new notation at this stage.

1.3 Simultaneous Fourier-Padé approximation

Let $\mathbb{T} = \{z : |z| = 1\}$ denote the unit circle and $\mathbb{D} = \{z : |z| < 1\}$ the open unit disk. By σ we denote a finite positive Borel measure whose support is contained in $\mathbb{T}$ and it satisfies $\sigma' > 0$ a.e. on $\mathbb{T}$. Let $\{\varphi_n\}$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients. That is,

$$\frac{1}{2\pi} \int \varphi_j(z) \overline{\varphi_k(z)} d\sigma(z) = \delta_{j,k}, \qquad j, k \in \mathbb{Z}_+,$$

where as usual $\delta_{j,k} = 0, j \neq k$ and $\delta_{k,k} = 1$. By $\mathcal{H}(\overline{\mathbb{D}})$ we denote the space of functions which are analytic on some neighborhood of $\overline{\mathbb{D}}$.

Definition 1.3.1. *Let* $\mathbf{f} = (f_1, \dots, f_d)$ *where* $f_k \in \mathcal{H}(\overline{\mathbb{D}}), k = 1, \dots, d$. *Fix a multi-index* $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$ *where* $\mathbf{0}$ *denotes the zero vector in* $\mathbb{Z}_+^d$. *Set* $|\mathbf{m}| = m_1 + \cdots + m_d$. *Then, for each* $n \geq \max\{m_1, \dots, m_d\}$, *there exist polynomials* $Q_{n,\mathbf{m}}, P_{n,\mathbf{m},j}, j = 1, \dots, d$, *such that*

a.1) $\deg P_{n,\mathbf{m},j} \leq n - m_j, j = 1, \dots, d, \quad \deg Q_{n,\mathbf{m}} \leq |\mathbf{m}|, \quad Q_{n,\mathbf{m}} \not\equiv 0,$

a.2) $[Q_{n,\mathbf{m}} f_j - P_{n,\mathbf{m},j}](z) = A_{n,n+1}^{(j)} \varphi_{n+1}(z) + A_{n,n+2}^{(j)} \varphi_{n+2}(z) + \cdots .$

We call the vector rational function $\mathbf{R}_{n,\mathbf{m}} = (P_{n,\mathbf{m},1}/Q_{n,\mathbf{m}}, \dots, P_{n,\mathbf{m},d}/Q_{n,\mathbf{m}})$ *an* $(n, \mathbf{m})$ *simultaneous Fourier-Padé approximation of* $\mathbf{f}$.

The numbers $A_{n,k}^{(j)}$ also depend on $\mathbf{m}$ but to simplify the notation we will not indicate it.

It is easy to see that for any pair $(n, \mathbf{m})$ there is at least one $\mathbf{R}_{n,\mathbf{m}}$ but, in general, it is not uniquely determined. In the sequel, we assume that given $(n, \mathbf{m})$, one solution is taken. We will normalize the common denominator to be monic.

As in the case of Hermite-Padé approximation we prove that under appropriate assumptions on $\mathbf{f}$

$$\lim_{n \to \infty} \mathbf{R}_{n,\mathbf{m}} = \mathbf{f}$$

uniformly on compact subsets of the largest disk centered at $z = 0$ containing at most $|\mathbf{m}|$ poles and that the zeros of the common denominator of the approximating rational functions point out the location and order of the poles of $\mathbf{f}$ in that disk.

In the scalar case, S.P. Suetin gives in [37] an extension of Montessus' result to Fourier expansions in terms of an orthonormal system of polynomials with respect to a measure supported on the real line.

Given $\mathbf{f} \in \mathcal{H}(\overline{\mathbb{D}})$ (that is, each component of $\mathbf{f}$ is analytic on a neighborhood of $\overline{\mathbb{D}}$) let $D_{|\mathbf{m}|}(\mathbf{f})$ denote the largest disk centered at the origin inside of which $\mathbf{f}$ has at most $|\mathbf{m}|$ poles and $R_{|\mathbf{m}|}(\mathbf{f})$ denotes its radius.

Let $Q_{|\mathbf{m}|}(\mathbf{f})$ be the monic polynomial whose zeros are the poles of $\mathbf{f}$ in $D_{|\mathbf{m}|}(\mathbf{f})$ counting multiplicities. The set of distinct zeros of $Q_{|\mathbf{m}|}(f)$ is denoted by $\mathcal{P}_{|\mathbf{m}|}(\mathbf{f})$.

Inspired in Suetin's version of the Montessus de Ballore theorem and the Graves-Morris/Saff theorem we prove the following.

Theorem 1.3.2. *Assume that* $\mathbf{f} \in \mathcal{H}(\overline{\mathbb{D}})$ *and* $\sigma' > 0$ *a.e. on* $\mathbb{T}$. *Fix a multi-index* $\mathbf{m} \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$ *and suppose that* $\mathbf{f}$ *is polewise independent with respect to* $\mathbf{m}$ *in* $D_{|\mathbf{m}|}(\mathbf{f})$, *Then,* $\mathbf{R}_{n,\mathbf{m}}$ *is uniquely determined for all sufficiently large* n. *For any compact subset* K *of* $D_{|\mathbf{m}|}(\mathbf{f}) \setminus \mathcal{P}_{|\mathbf{m}|}(\mathbf{f})$

$$\limsup_{n\to\infty} \|f_i - R_{n,\mathbf{m},i}\|_K^{1/n} \leq \frac{\|z\|_K}{R_{|\mathbf{m}|}(\mathbf{f})}, \quad i = 1, \ldots, d, \tag{1.11}$$

where $\|z\|_K$ *is replaced by* 1 *when* $K \subset \overline{\mathbb{D}}$. *Additionally,*

$$\limsup_{n\to\infty} \|Q_{|\mathbf{m}|}(\mathbf{f}) - Q_{n,\mathbf{m}}\|^{1/n} \leq \frac{\max\{|\zeta| : \zeta \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{f})\}}{R_{|\mathbf{m}|}(\mathbf{f})}. \tag{1.12}$$

This result is proved in Chapter 4 (see also [7]). An analog for simultaneous Fourier-Padé approximation defined in terms of an orthonormal system of polynomials with respect to a measure supported on a bounded subinterval of the real line is obtained in Chapter 5 (and [8]).

1.4 Structure of the thesis and methodology

In Chapter 2 we introduce a new type of scalar rational approximation which we call incomplete Padé approximation. Its main characteristic is that some of the free parameters in the construction are left completely undetermined and prove that the number of interpolation conditions imposed are sufficient to obtain convergence in some weak sense. The type of weak convergence obtained allows us to determine the behavior of some of the poles of the approximating rational functions. The freedom in the construction permits to show that the different components of a Hermite-Padé approximant are also incomplete Padé approximants for the corresponding component of the approximated vector function. In Chapter 3 we put together the partial information obtained from Chapter 2 for the different components of the vector rational function.

If the different components of the vector function are sufficiently different one from the other, one can complete the whole picture.

A similar strategy is followed in Chapters 4 and 5 for the study of simultaneous Fourier-Padé approximations on the unit circle and on a segment of the real line, respectively, but now the basis of the analysis are incomplete Fourier-Padé approximations.

The appendix contains some procedures implemented in Maple which can be used to construct simultaneous Padé approximations and test numerically some of our findings.

The results of this thesis have been presented at various international meetings: "11th International Symposium on Orthogonal Polynomials, Special Functions and Applications," August 29 to September 2, 2011, held at Universidad Carlos III de Madrid, Spain; "Workshop on Potential Theory and Applications," Szeged, Hungary, May 28 - 31, 2012; and "Journées Approximation" June 28-29, 2012, held at University of Lille 1, France.

The contents of Chapters 2 and 3 appear in [5] and [6]. Chapter 4, related with orthogonality on the unit circle, corresponds to [7] whereas Chapter 5, dedicated to simultaneous Fourier-Padé approximation on the real line, will appear in [8].

The methods used in our proofs involve a comprehensive knowledge of complex analysis, measure theory, and Fourier series as contained in [1] and [24] and rudiments of the asymptotic theory of orthogonal polynomials in the extent of Chapter 9 of [34]-[35].

Chapter 2

Incomplete Padé approximation

This chapter is dedicated to a new type of rational approximation which we have introduced in order to study simultaneous Padé approximation. We have called this construction incomplete Padé approximation. It may be used in other applications as well. The main idea is to leave completely undetermined a certain number of the free parameters at our disposal in order to adjust them at our convenience. Because of this freedom it is to be expected that the convergence of such approximants will be in some weaker sense which we will define immediately.

2.1 Convergence in h-content

Let B be a subset of the complex plane $\mathbb{C}$. By $\mathcal{U}(B)$ we denote the class of all coverings of B by at most a numerable set of disks. Set

$$h(B) = \inf \left\{ \sum_{i=1}^{\infty} |U_i| \, : \, \{U_i\} \in \mathcal{U}(B) \right\},$$

where $|U_i|$ stands for the radius of the disk U_i. The quantity $h(B)$ is called the 1-dimensional Hausdorff content of the set B. This set function is not a measure but it is semi-additive and monotonic, properties which will be used and are easy to prove. Clearly, if B is a disk then $h(B) = |B|$.

Definition 2.1.1. *Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of complex valued functions defined on a domain $D \subset \mathbb{C}$ and g another complex function defined on D. We say that $\{g_n\}_{n\in\mathbb{N}}$ converges in h-content to the function g on compact subsets of D if for every compact subset $\mathcal{K}$ of D and for each $\varepsilon > 0$, we have*

$$\lim_{n\to\infty} h\{z \in \mathcal{K} : |g_n(z) - g(z)| > \varepsilon\} = 0.$$

Such a convergence will be denoted by h-$\lim_{n\to\infty} g_n = g$ in D.

We wish to point out that the functions g_n and g are allowed to take the value ∞. We adopt the convention that $\infty \pm \infty = \infty$.

The next lemma proved by A. A. Gonchar in [11] allows us to derive uniform convergence on compact subsets of the region under consideration from convergence in h-content under appropriate assumptions.

Gonchar's Lemma. *Suppose that h-$\lim_{n\to\infty} \varphi_n = \varphi$ in D. Then the following assertions hold true:*

i) If the functions φ_n, $n \in \mathbb{N}$, are holomorphic in D, then the sequence $\{\varphi_n\}$ converges uniformly on compact subsets of D and φ is holomorphic in D (more precisely, it is equal to a holomorphic function in D except on a set of h-content zero).

ii) If each of the functions φ_n is meromorphic in D and has no more than $k < +\infty$ poles in this domain, then the limit function φ is (again except on a set of h-content zero) also meromorphic and has no more than k poles in D.

iii) If each function φ_n is meromorphic and has no more than $k < +\infty$ poles in D and the function φ is meromorphic and has exactly k poles in D, then all φ_n, $n \geq N$, also have k poles in D; the poles of φ_n tend to the poles $z_1, , \ldots, z_k$ of φ (taking account of their orders) and the sequence $\{\varphi_n\}$ tends to φ uniformly on compact subsets of the domain $D' = D \setminus \{z_1, , \ldots, z_k\}$.

2.2 Incomplete Padé approximants. Definition

In the following sections

$$f(z) = \sum_{n=0}^{\infty} \phi_n z^n, \qquad \phi_n \in \mathbb{C}, \tag{2.1}$$

denotes a formal or convergent Taylor expansion about the origin.

Definition 2.2.1. *Let f denote a formal Taylor expansion about the origin as in* (2.1)*. Fix $m^* \leq m$. Let $n \geq m$. We say that the rational function $R_{n,m}$ is an incomplete Padé approximation of type (n, m, m^*) corresponding to f if $R_{n,m}$ is the quotient of any two polynomials P and Q that verify*

d.1) $\deg P \leq n - m^*, \quad \deg Q \leq m, \quad Q \not\equiv 0,$

d.2) $[Qf - P](z) = Az^{n+1} + \cdots$.

Notice that given $(n, m, m^*), n \geq m \geq m^*$, any one of the Padé approximants $\pi_{n,m^*}, \ldots, \pi_{n,m}$ can be considered an incomplete Padé approximation of type (n, m, m^*) of f. The so-called Padé-type approximants (see [2]) where $m - m^*$ zeros of Q are fixed and m^* are left free are also incomplete Padé approximants. Moreover, from Definition 1.2.1 it follows that $R_{n,\mathbf{m},k}, k = 1, \ldots, d$, is an incomplete Padé approximation of type $(n, |\mathbf{m}|, m_k)$ corresponding to f_k.

In the sequel, for each $n \geq m \geq m^*$, we choose one candidate. After canceling out common factors between Q and P, we write

$$R_{n,m} = P_{n,m}/Q_{n,m},$$

where, additionally, $Q_{n,m}$ is normalized as follows

$$Q_{n,m}(z) = \prod_{|\zeta_{n,k}| \leq 1} (z - \zeta_{n,k}) \prod_{|\zeta_{n,k}| > 1} \left(1 - \frac{z}{\zeta_{n,k}}\right). \tag{2.2}$$

Suppose that Q and P have a common zero at $z = 0$ of order λ_n. From d.1)-d.2) it readily follows that d.3) $\deg P_{n,m} \leq n - m^* - \lambda_n, \quad \deg Q_{n,m} \leq m - \lambda_n, \quad Q_{n,m} \not\equiv 0,$

d.4) $[Q_{n,m} f - P_{n,m}](z) = Az^{n+1-\lambda_n} + \cdots .$

where A is, in general, a different constant from the one in d.2).

When f denotes a convergent series, it is well known by the specialists that any row sequence $\{\pi_{n,m}\}_{n \geq m}$, where $m \geq m^*$ is fixed, converges to f in h-content on compact subsets of $D_{m^*}(f)$. This is also true for any sequence of incomplete Padé approximations when $m \geq m^*$ is fixed. Before giving a formal statement of that result, let us introduce some additional definitions.

Take an arbitrary $\varepsilon > 0$ and define the open set J_ε as follows. For $n \geq m$, let $J_{n,\varepsilon}$ denote the $\varepsilon/6mn^2$-neighborhood of the set $\mathcal{P}_{n,m} = \{\zeta_{n,1}, \ldots, \zeta_{n,m_n}\}$ of finite zeros of $Q_{n,m}$. If $R_0(f) > 0$, let $J_{m-1,\varepsilon}$ denote the $\varepsilon/6m$-neighborhood of the set of poles of f in $D_m(f)$. Otherwise, $J_{m-1,\varepsilon} = \emptyset$. Set $J_\varepsilon = \cup_{n \geq m-1} J_{n,\varepsilon}$. We have $h(J_\varepsilon) < \varepsilon$ and $J_{\varepsilon_1} \subset J_{\varepsilon_2}$ for $\varepsilon_1 < \varepsilon_2$. For any set $B \subset \mathbb{C}$ we put $B(\varepsilon) = B \setminus J_\varepsilon$.

Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a sequence of functions defined on a domain D and φ another function also defined on D. Clearly, if $\{\varphi_n\}_{n \in \mathbb{N}}$ converges uniformly to φ on $\mathcal{K}(\varepsilon)$ for every compact $\mathcal{K} \subset D$ and every $\varepsilon > 0$, then $h\text{-}\lim_{n \to \infty} \varphi_n = \varphi$ in D.

Due to the normalization (2.2), for any compact set $\mathcal{K}$ of $\mathbb{C}$ and for every $\varepsilon > 0$, there exist constants C_1, C_2, independent of n, such that

$$\|Q_{n,m}\|_{\mathcal{K}} \leq C_1, \qquad \min_{z \in \mathcal{K}(\varepsilon)} |Q_{n,m}(z)| \geq C_2 n^{-2m}, \tag{2.3}$$

where the second inequality is meaningful when $\mathcal{K}(\varepsilon)$ is a non-empty set.

In the sequel, C will denote positive constants, generally different, that are independent of n but may depend on all the other parameters involved in each formula where they appear.

2.3 Direct results for incomplete Padé approximants

2.3.1 Convergence in h-content of incomplete Padé approximants

Proposition 2.3.1. *Let $R_0(f) > 0$. Fix m and m^* nonnegative integers, $m \geq m^*$. For each $n \geq m$, let $R_{n,m}$ be an incomplete Padé approximant of type (n, m, m^*) for f. Then*

$$h\text{-}\lim_{n\to\infty} R_{n,m} = f \quad in \quad D_{m^*}(f).$$

Proof. Let Q_{m^*} denote the monic polynomial whose zeros are the poles of f in $D_{m^*}(f)$. Using d.3), we have

$$\left[Q_{m^*}Q_{n,m}f - Q_{m^*}P_{n,m}\right](z) = Az^{n+1-\lambda_n} + \cdots ,$$

which implies that

$$\frac{\left[Q_{m^*}Q_{n,m}f - Q_{m^*}P_{n,m}\right](z)}{z^{n+1-\lambda_n}} \in \mathcal{H}(D_{m^*}(f)).$$

Set $|z| < r < R_{m^*}(f)$ with r arbitrarily close to $R_{m^*}(f)$ and let $\Gamma_r = \{z \in \mathbb{C} : |z| = r\}$. By Cauchy's integral formula we obtain

$$\begin{aligned}\frac{\left[Q_{m^*}Q_{n,m}f - Q_{m^*}P_{n,m}\right](z)}{z^{n+1-\lambda_n}} &= \frac{1}{2\pi i}\int_{\Gamma_r} \frac{[Q_{m^*}Q_{n,m}f](\zeta)}{\zeta^{n+1-\lambda_n}}\frac{d\zeta}{\zeta - z} + \\ -\int_{\Gamma_r} \frac{Q_{m^*}(\zeta)P_{n,m}(\zeta)}{\zeta^{n+1-\lambda_n}}\frac{d\zeta}{\zeta - z} &= \frac{1}{2\pi i}\int_{\Gamma_r} \frac{[Q_{m^*}Q_{n,m}f](\zeta)}{\zeta^{n+1-\lambda_n}}\frac{d\zeta}{\zeta - z},\end{aligned} \tag{2.4}$$

where the second integral after the first equality is zero due to the fact that the integrand is an analytic function outside Γ_r with a zero of multiplicity at least two at infinity (see d.3)).

Fix an arbitrary compact set $K \subset D_{m^*}(f)$ and take $0 < r < R_{m^*}(f)$ such that K and all of the poles of f in $D_{m^*}(f)$ are contained in the disk $\{z \in \mathbb{C} : |z| < r\}$. We also select an arbitrarily small $\varepsilon > 0$. From (2.4) it follows that

$$[Q_{m^*}(f - R_{n,m})](z) = \frac{z^{n+1-\lambda_n}}{2\pi i}\int_{\Gamma_r} \frac{[Q_{m^*}Q_{n,m}f](\zeta)}{Q_{n,m}(z)\,\zeta^{n+1-\lambda_n}}\frac{d\zeta}{\zeta - z},$$

for all $z \in K(\varepsilon)$. Using this last formula, (2.3), and the continuity of $Q_{m^*}f$ on Γ_r, we obtain

$$\|Q_{m^*}(f - R_{n,m})\|_{K(\varepsilon)} \leq C\,\frac{\|z\|_K^n}{r^n}\,\frac{\|Q_{n,m}\|_{\Gamma_r}}{\min\limits_{\zeta\in K(\varepsilon)}|Q_{n,m}(\zeta)|} \leq C\,\frac{\|z\|_K^n}{r^n}\,n^{2m}.$$

Taking n-th root, making n tend to infinity, and letting r approach $R_{m^*}(f)$, we arrive at

$$\limsup_{n\to\infty}\|Q_{m^*}(f - R_{n,m})\|_{K(\varepsilon)}^{1/n} \leq \frac{\|z\|_K}{R_{m^*}(f)} < 1.$$

As $\varepsilon > 0$ is arbitrary, we have proved that $h - \lim_{n\to\infty} Q_{m^*}R_{n,m} = Q_{m^*}f$ in $D_{m^*}(f)$, which is equivalent to the statement we wanted to prove. □

2.3.2 The disk $D_m^*(f)$

Let us find the radius of the largest disk centered at the origin in compact subsets of which the sequence $\{R_{n,m}\}_{n\geq m}$ converges to f in h-content. This number, which depends on the specific sequence of incomplete Padé approximants considered, lies between $R_{m^*}(f)$ and $R_m(f)$. We need some formulas.

Lemma 2.3.2. *Let a formal power series* (1.1) *be given. Fix $m \geq m^*$ two positive integers. Consider a corresponding sequence of incomplete Padé approximations. For each $n \geq m$, we have*

$$R_{n+1,m}(z) - R_{n,m}(z) = \frac{A_{n,m}z^{n+1-\lambda_n-\lambda_{n+1}}q^*_{n,m-m^*}(z)}{Q_{n,m}(z)Q_{n+1,m}(z)},$$

where $A_{n,m}$ is some constant and $q^*_{n,m-m^*}$ is a polynomial of degree less than or equal to $m - m^*$ normalized as in (2.2).

Proof. Using d.4) we have

$$z^{\lambda_n}[Q_{n,m}f - P_{n,m}](z) = Az^{n+1} + \cdots$$

and

$$z^{\lambda_{n+1}}[Q_{n+1,m}f - P_{n+1,m}](z) = A'z^{n+2} + \cdots .$$

Multiplying the first equation by $z^{\lambda_{n+1}}Q_{n+1,m}$, the second by $z^{\lambda_n}Q_{n,m}$, and deleting one of the equations so obtained from the other, it follows that

$$z^{\lambda_n+\lambda_{n+1}}[Q_{n,m}P_{n+1,m} - Q_{n+1,m}P_{n,m}](z) = Bz^{n+1} + \cdots .$$

Taking into consideration d.3) we see that on the left hand side we have a polynomial of degree $\leq n+1+m-m^*$. Consequently,

$$z^{\lambda_n+\lambda_{n+1}}[Q_{n,m}P_{n+1,m} - Q_{n+1,m}P_{n,m}](z) = z^{n+1}\widetilde{q}_{n,m-m^*},$$

where $\deg \widetilde{q}_{n,m-m^*} \leq m-m^*$. Dividing by $z^{\lambda_n+\lambda_{n+1}}Q_{n,m}Q_{n+1,m}$ and normalizing $\widetilde{q}_{n,m-m^*}$ as in (2.2) we obtain the desired formula. □

Take an arbitrary $\varepsilon > 0$ and define the open set J'_ε as follows. For $n \geq m$, let $J'_{n,\varepsilon}$ denote the $\varepsilon/6mn^2$-neighborhood of the set of zeros of $q^*_{n,m-m^*}$. Set $J'_\varepsilon = \cup_{n\geq m} J'_{n,\varepsilon}$. For any compact set $K \subset \mathbb{C}$ we put $K'(\varepsilon) = K \setminus J'_\varepsilon$.

Due to the fact that the polynomial $q^*_{n,m-m^*}$ is normalized as in (2.2), for any compact set K of $\mathbb{C}$ and for every $\varepsilon > 0$, there exist constants M_1, M_2, independent of n, such that

$$\|q^*_{n,m-m^*}\|_K \leq M_1, \qquad \min_{z\in K'(\varepsilon)} |q^*_{n,m-m^*}(z)| \geq M_2 n^{-2m}, \tag{2.5}$$

where the second inequality is meaningful when $K'(\varepsilon)$ is a non-empty set.

Define

$$R^*_m(f) = \frac{1}{\limsup\limits_{n\to\infty} |A_{n,m}|^{1/n}}, \qquad D^*_m(f) = \{z : |z| < R^*_m(f)\}. \tag{2.6}$$

Theorem 2.3.3. *Let f be a formal power series as in* (1.1). *Fix m and m^* nonnegative integers, $m \geq m^*$. Let $\{R_{n,m}\}_{n\geq m}$ be a sequence of incomplete Padé approximants of type (n, m, m^*) for f. If $R^*_m(f) > 0$ then $R_0(f) > 0$. Moreover,*

$$D_{m^*}(f) \subset D^*_m(f) \subset D_m(f)$$

and $D^*_m(f)$ is the largest disk in compact subsets of which $h - \lim_{n\to\infty} R_{n,m} = f$. Moreover, the sequence $\{R_{n,m}\}_{n\geq m}$ is pointwise divergent in $\{z : |z| > R^*_m(f)\}$ except on a set of h-content zero.

Proof. According to Lemma 2.3.2

$$R_{n+1,m}(z) - R_{n,m}(z) = \frac{A_{n,m} z^{n+1-\lambda_n-\lambda_{n+1}} q^*_{n,m-m^*}(z)}{Q_{n,m}(z)\, Q_{n+1,m}(z)}. \tag{2.7}$$

Considering telescopic sums, it follows that the sequence $\{R_{n,m}\}_{n\geq m}$ converges or diverges with the series

$$\sum_{n\geq n_0} \frac{A_{n,m} z^{n+1-\lambda_n-\lambda_{n+1}} q^*_{n,m-m^*}(z)}{Q_{n,m}(z)\, Q_{n+1,m}(z)},$$

where n_0 is chosen conveniently so that $Q_{n_0,m}(z) \neq 0$ at the specific point under consideration.

Let $R^*_m(f) > 0$ and $K \subset D^*_m(f)$. Fix $\varepsilon > 0$. Using (2.3) and (2.5), we have

$$\limsup_{n\to\infty} \left\| \frac{A_{n,m} z^{n+1-\lambda_n-\lambda_{n+1}} q^*_{n,m-m^*}(z)}{Q_{n,m}(z)\, Q_{n+1,m}(z)} \right\|^{1/n}_{K(\varepsilon)} \leq \frac{\|z\|_K}{R^*_m(f)} < 1. \tag{2.8}$$

Therefore, the series converges uniformly on $K(\varepsilon)$ for every $K \subset D^*_m(f)$ and every $\varepsilon > 0$. Thus $h - \lim_{n\to\infty} R_{n,m} = \varphi$ in $D^*_m(f)$, where, according to Gonchar's Lemma, φ is (except on a set of h-content zero) a meromorphic function with at most m poles in $D^*_m(f)$. On the other hand, if $|z| > R^*_m(f)$ and $z \notin J'_\varepsilon$ from (2.3) and (2.5) it follows that

$$\limsup_{n\to\infty} \left| \frac{A_{n,m} z^{n+1-\lambda_n-\lambda_{n+1}} q^*_{n,m-m^*}(z)}{Q_{n,m}(z)\, Q_{n+1,m}(z)} \right|^{1/n} \geq \frac{|z|}{R^*_m(f)} > 1, \tag{2.9}$$

and the series diverges. Therefore, the sequence $\{R_{n,m}\}_{n\geq m^*}$ pointwise diverges in $\{z : |z| > R^*_m(f)\}$ except on a set of h-content zero (namely, $\cap_{\varepsilon>0} J'_\varepsilon$).

Now, assume that there is a disk larger than $D^*_m(f)$ in compact subsets of which we have $h - \lim_{n\to\infty} R_{n,m} = f$, then there exist $\varepsilon_0 > 0$ and $r_0 > R^*_m(f)$ such that $|f(z) - R_{n,m}(z)| \leq \varepsilon_0$ for all z with $|z| = r_0$ and sufficiently large n. Hence

$$\limsup_{n\to\infty} |R_{n+1,m}(z) - R_{n,m}(z)|^{1/n} \leq 1,$$

for all z with $|z| = r_0$, which is absurd because of (2.9) and the fact that $\cap_{\varepsilon>0} J'_\varepsilon$ is a set of h-content zero.

We conclude the proof of the theorem if we show that $R^*_m(f) > 0$ implies that $R_0(f) > 0$. Indeed, if this is true, then necessarily $\varphi = f$ in $D^*_m(f)$ since by Proposition

2.3.1, f is the h-limit of $\{R_{n,m}\}_{n\geq m}$ at least in compact subsets of $D_{m^*}(f)$. Since $D_m^*(f)$ is the largest disk centered at the origin in compact subsets of which $\{R_{n,m}\}_{n\geq m}$ converges to f in h-content, we get that $D_{m^*}(f) \subset D_m^*(f)$. On the other hand, $D_m(f)$ is the largest disk centered at the origin in which f admits a meromorphic extension with no more than m poles, therefore $D_m^*(f) \subset D_m(f)$.

Let $R_m^*(f) > 0$, then h-$\lim_{n\to\infty} R_{n,m} = \varphi$ in $D_m^*(f)$, where φ has at most m poles in this disk. Choose a subsequence of indices $\Lambda \subset \mathbb{N}$ such that for all $n \in \Lambda$ the number of poles of $R_{n,m}$ is exactly equal to m_0, $m_0 \leq m$, and $\lim_{n\in\Lambda} \zeta_{n,j} = z_j, j = 1, \ldots, m_0$. Suppose that ℓ of the points z_j equal zero and let U be a neighborhood of $z = 0$ that does not contain any z_j other than zero and is contained in $D_m^*(f)$. From Gonchar's Lemma it follows that $\lim_{n\in\Lambda} R_{n,m} = \varphi$ uniformly on each compact subset of $U^* = U \setminus \{0\}$, where φ is holomorphic in U^* and its Laurent expansion in U^* has the form

$$\varphi(z) = \sum_{k=-\ell}^{\infty} \varphi_k z^k.$$

If we show that $\varphi_k = 0, k = -\ell, \ldots, -1$, and $\varphi_k = \phi_k, k \geq 0$, then the function φ is analytic in U and coincides with f in that set. In consequence, $R_0(f) > 0$.

Choose $r > 0$ such that $\Gamma = \{z : |z| = r\}$ belongs to U^*. For all sufficiently large $n \in \Lambda$ the points $\zeta_{n,j}, j = 1, \ldots, \ell$, are inside Γ and the points $\zeta_{n,j}, j = \ell + 1, \ldots, m^*$, are outside this curve. From now on we only consider such n's. Let us compare the Taylor expansion of $R_{n,m}$ about $z = 0$

$$R_{n,m}(z) = \sum_{k=0}^{\infty} \alpha_{n,k} z^k,$$

with its Laurent expansion on Γ,

$$R_{n,m}(z) = \sum_{k=-\infty}^{\infty} \beta_{n,k} z^k.$$

For notational convenience we set $\phi_k = 0$ and $\alpha_{n,k} = 0$ for $k = -1, -2, \ldots$ and $\varphi_k = 0$ for $k = -\ell-1, -\ell-2, \ldots$ We restrict our attention to the case when all $\zeta_{n,k},\ k = 1, \ldots, \ell$, are distinct. The general case is proved analogously with some additional technical difficulties.

Let $c_{n,j}, j = 1, \ldots, \ell$, be the residue of $R_{n,m}$ at $\zeta_{n,j}$. The Taylor expansion of $R_{n,m}$ about $z = 0$ and its Laurent expansion on Γ differ only because of the expansion of the fractions $c_{n,j}/(z - \zeta_{n,j}), j = 1, \ldots, \ell$. Therefore, it is easy to verify that

$$\beta_{n,k} - \alpha_{n,k} = \sum_{j=1}^{\ell} \frac{c_{n,j}}{\zeta_{n,j}^{k+1}}, \qquad k \in \mathbb{Z}. \tag{2.10}$$

By the definition of $R_{n,m}$ (in particular, see d.4)), $\alpha_{n,k} = \phi_k$ for $k < n + m - \lambda_n$; therefore, $\lim_{n\in\Lambda} \alpha_{n,k} = \phi_k, k \in \mathbb{Z}$. On the other hand, from the uniform convergence of $R_{n,m}$ to φ on Γ we also have $\lim_{n\in\Lambda} \beta_{n,k} = \varphi_k, k \in \mathbb{Z}$. We obtain

$$\lim_{n\in\Lambda}(\beta_{n,k} - \alpha_{n,k}) = \varphi_k - \phi_k, \qquad k \in \mathbb{Z}. \tag{2.11}$$

Set $\varepsilon_{n,k} = \beta_{n,k} - \alpha_{n,k}$ and

$$L_n(z) = \prod_{j=1}^{\ell}(1 - \zeta_{n,j} z) = 1 + \gamma_{n,1} z + \cdots + \gamma_{n,\ell} z^{\ell}.$$

Using (2.10), for arbitrary $k \in \mathbb{Z}$, we obtain

$$\varepsilon_{n,k} + \gamma_{n,1}\varepsilon_{n,k+1} + \cdots + \gamma_{n,\ell}\varepsilon_{n,k+\ell} = \sum_{j=1}^{\ell} \frac{c_{n,j}}{\zeta_{n,j}^{k+1}} L_n(\zeta_{n,j}^{-1}) = 0. \tag{2.12}$$

Since $\lim_{n\in\Lambda} \gamma_{n,j} = 0, j = 1, \ldots, \ell$, and $\lim_{n\in\Lambda} \varepsilon_{n,k+j} = \varphi_{k+j} - \phi_{k+j}, j = 1, \ldots, \ell$, from (2.12) it follows that $\lim_{n\to\infty} \varepsilon_{n,k} = 0$. Using (2.11) we obtain $\varphi_k = \phi_k, k \in \mathbb{Z}$, as we wanted to prove. $\square$

2.3.3 Attraction of poles

Next, we will prove that each pole of the function f in $D_m^*(f)$ attracts, with geometric rate, at least as many zeros of $Q_{n,m}$ as its order. For this purpose, let us define two indicators of the asymptotic behavior of the poles of the incomplete Padé approximants. These indicators were first introduced by A.A. Gonchar in [12] for the study of inverse-type theorems for row sequences of Padé approximants. Let

$$\mathcal{P}_{n,m} = \{\zeta_{n,1}, \ldots, \zeta_{n,\nu_n}\}, \quad n \in \mathbb{N}, \quad \nu_n \leq m,$$

denote the collection of zeros of $Q_{n,m}$ (repeated according to their multiplicity). It is easy to verify that $|\cdot|_1 : \mathbb{C}^2 \longrightarrow \mathbb{R}_+$ given by

$$|z-\omega|_1 = \min\{1, |z-\omega|\}, \qquad z, \omega \in \mathbb{C},$$

defines a distance in $\mathbb{C}$ (although $|\cdot|_1$ is not a norm in $\mathbb{C}$).

Choose a point $a \in \mathbb{C}$. The first indicator is defined by

$$\Delta(a) = \limsup_{n\to\infty} \prod_{j=1}^{\nu_n} |\zeta_{n,j} - a|_1^{1/n} = \limsup_{n\to\infty} \prod_{|\zeta_{n,j}-a|<1} |\zeta_{n,j} - a|^{1/n}.$$

Obviously, $0 \le \Delta(a) \le 1$ (when $\nu_n = 0$ the product is taken to be 1). The second indicator, a nonnegative integer $\mu(a)$, is defined as follows. We suppose that for each n the points in $\mathcal{P}_{n,m}$ are enumerated in nondecreasing distance to the point a. We put

$$\delta_j(a) = \limsup_{n\to\infty} |\zeta_{n,j} - a|_1^{1/n}. \tag{2.13}$$

These numbers are defined by (2.13) for $j = 1, \ldots, m', m' = \liminf_{n\to\infty} \nu_n$; for $j = m'+1, \ldots, m$ we define $\delta_j(a) = 1$. We have $0 \le \delta_j(a) \le 1$. If $\Delta(a) = 1$ (in that case all $\delta_j(a) = 1$), then $\mu(a) = 0$. If $\Delta(a) < 1$, then for some $\mu, 1 \le \mu \le m$, we have that $\delta_1(a) \le \cdots \le \delta_\mu(a) < 1$ and $\delta_{\mu+1}(a) = 1$ or $\mu = m$; in this case we take $\mu(a) = \mu$.

Clearly, $\Delta(a) < 1 \Leftrightarrow \mu(a) \ge 1$ and $\sum_{a\in\mathbb{C}} \mu(a) \le m$. We shall need $\Delta(a)$ and $\mu(a)$ only for points $a \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. It is easy to verify that

$$\Delta(a) = \limsup_{n\to\infty} |Q_{n,m}(a)|^{1/n}. \tag{2.14}$$

Theorem 2.3.4. *Let $R_0(f) > 0$. Fix m and m^* nonnegative integers, $m \ge m^*$. For each $n \ge m$, let $R_{n,m}$ be an incomplete Padé approximant of type (n, m, m^*) for f. Let a be a pole of f in $D_m^*(f)$ of order τ. Then*

$$\Delta(a) \le \frac{|a|}{R_m^*(f)} \quad \text{and} \quad \mu(a) \ge \tau.$$

Proof. Let a be a pole of f in $D_m^*(f)$ of order τ and take $r > 0$ sufficiently small so that the disk of center a and radius r, denoted by $D_{a,r}$, contains no other pole of f. It follows from Gonchar's Lemma that the approximants $R_{n,m}$ have at least τ poles in $D_{a,r}$ for sufficiently large $n \in \mathbb{N}$. If this were not so, from Theorem 2.3.3, there exists a subsequence $\{R_{n,m}\}_{n\in\Lambda}$ converging in h-content to f in compact subsets of $D_{a,r}$ with each approximant having less than τ poles in $D_{a,r}$ and part ii) of Gonchar's Lemma would imply that f has less than τ poles in $D_{a,r}$, which is absurd. As $r > 0$ is arbitrarily small, we have proved that each pole of f in $D_m^*(f)$ attracts at least as many zeros of $Q_{n,m}$ as its order.

Fix $\varepsilon > 0$ arbitrarily small and take again $r > 0$ sufficiently small so that $D_{a,r}$ contains no other pole of f. Since $h(J_\varepsilon) < \varepsilon$, we can choose r such that $\Gamma_{a,r} = \{z : |z-a| = r\} \subset D_{m^*}(f) \setminus J_\varepsilon$. Let $\zeta_{n,1}, \ldots, \zeta_{n,\mu_n}$ be the zeros of $Q_{n,m}$ in $D_{a,r}$ indexed in non-decreasing distance from a. That is,

$$|a - \zeta_{n,1}| \leq |a - \zeta_{n,2}| \leq \cdots \leq |a - \zeta_{n,\mu_n}|.$$

For all sufficiently large n we know that $\zeta_{n,\tau} \in D_{a,r}$. We will only consider such n's. Consequently, we have $\tau \leq \mu_n \leq m$. Set

$$Q_{n,a}(z) = \prod_{j=1}^{\mu_n} (z - \zeta_{n,j}).$$

For any ρ with $|a| + r < \rho < R_m^*(f)$, it follows from (2.7) and (2.8) that

$$\|f - R_{n,m}\|_{\Gamma_{a,r}} < Cq^n, \qquad q = \frac{|a| + r}{\rho} < 1, \tag{2.15}$$

for sufficiently large n.

Let $p(z)/(z-a)^\tau$ be the principal part of the function f at the point a and $p_n/Q_{n,a}$ the sum of the principal parts of $R_{n,m}$ corresponding to its poles in $D_{a,r}$. We have $\deg p < \tau$, $p(a) \neq 0$, and $\deg p_n < \mu_n$. It is known that the norm of the holomorphic component of a meromorphic function may be bounded in terms of the norm of the function and the number of poles (see Theorem 1 in [13]). Thus, using (2.15), we obtain

$$\left\| \frac{p(z)}{(z-a)^\tau} - \frac{p_n(z)}{Q_{n,a}(z)} \right\|_{\Gamma_{a,r}} < Cq^n,$$

for sufficiently large n. Therefore, getting rid of the denominators and applying the maximum principle, we have

$$\|p(z)\, Q_{n,a}(z) - (z-a)^\tau p_n(z)\|_{\overline{D}_{a,r}} < Cq^n, \tag{2.16}$$

for sufficiently large n. All the factors in $Q_{n,m}$ that contribute to the limit value $\Delta(a)$ are present in $Q_{n,a}$, see (2.14) and (2.2). So, making $z = a$ in (2.16) and taking limits as n tends to infinity gives the inequality $\Delta(a) \leq q$. As r, ε, and ρ are arbitrary we have proved that $\Delta(a) \leq |a|/R_m^*(f)$. To conclude the proof we must show that $\mu(a) \geq \tau$. We will prove it by induction.

Since $\Delta(a) < 1$, we have $\delta_1(a) < 1$. Let $\delta_1(a) \leq \cdots \leq \delta_k(a) < 1$ and $k < \tau$. We differentiate the polynomial inside the norm in (2.16) k times. As this polynomial has degree bounded by $2m - 1$, its kth derivative satisfies an inequality similar to (2.16) by virtue of Bernstein's inequality (see, for instance, Section 4.4.2 in [25]). If we put $z = a$ in the corresponding inequality, we obtain

$$\left| \left(p(z) \prod_{j=1}^{\mu_n} (z - \zeta_{n,j}) \right)^{(k)} (a) \right| < Cq^n, \tag{2.17}$$

for sufficiently large n. Now

$$\left(p(z) \prod_{j=1}^{\mu_n} (z - \zeta_{n,j}) \right)^{(k)} (a) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \, p^{(\beta)}(a) \prod_{j=1}^{\mu_n} (z - \zeta_{n,j})^{(\alpha_j)}(a), \tag{2.18}$$

where $\alpha = (\beta, \alpha_1, \ldots, \alpha_{\mu_n}) \in \mathbb{Z}_+^{\mu_n+1}$, $\alpha! = \beta! \cdot \alpha_1! \cdot \ldots \cdot \alpha_{\mu_n}!$, and $|\alpha| = \beta + \alpha_1 + \cdots + \alpha_{\mu_n}$. By $\sum_{|\alpha|=k}$ we mean that the sum is taken over all the multi-indices α such that $|\alpha| = k$. The total amount of such multi-indices is bounded independently of n. One of them is $(0, 1, \ldots, 1, 0, \ldots, 0)$ corresponding to the term

$$k! \, p(a) \prod_{j=k+1}^{\mu_n} (z - \zeta_{n,j}).$$

The remaining terms must necessarily contain one factor $(z - \zeta_{n,j})$, $j \in \{1, 2, \ldots, k\}$. Since we have assumed that $\delta_j(a) < 1$ for $j = 1, \ldots, k$, it follows from (2.17) and (2.18) that

$$\limsup_{n \to \infty} \prod_{j=k+1}^{\mu_n} |z - \zeta_{n,j}|^{1/n} < 1,$$

which in turn implies $\limsup_{n \to \infty} |z - \zeta_{n,k+1}|^{1/n} < 1$, that is, $\delta_{k+1}(a) < 1$. Therefore it holds that $\mu(a) \geq \tau$ and we are done. □

The estimate $\Delta(a) \le |a|/R_m^*(f)$ can be sharpened if one knows that a given pole attracts exactly as many zeros of $Q_{n,m}$ as its order.

Theorem 2.3.5. *Let $R_0(f) > 0$ and let a be a pole of f in $D_m^*(f)$ of order τ. Assume that* $\liminf_{n\to\infty} |a - z_{n,\tau+1}| > 0$. *Then*

$$\delta_1(a) \le \cdots \le \delta_\tau(a) \le \left(\frac{|a|}{R_m^*(f)}\right)^{1/\tau}.$$

In particular, $\delta_1(a) = \cdots = \delta_\tau(a) = (|a|/R_m^*(f))^{1/\tau}$ if and only if $\Delta(a) = |a|/R_m^*(f)$.

Proof. Let us maintain the notation used in the proof of Theorem 2.3.4. We may assume that

$$Q_{n,a}(z) = \prod_{j=1}^{\tau}(z - \zeta_{n,j}).$$

Recall that $p(a) \neq 0$. So, taking $z = a$ in (2.16), we obtain $|Q_{n,a}(a)| < Cq^n$, for sufficiently large n. From this, (2.17), and the formula

$$(p\,Q_{n,a})^{(k)}(a) = p(a)\,Q_{n,a}^{(k)}(a) + \sum_{j=0}^{k-1}\begin{pmatrix} k \\ j \end{pmatrix} p^{(k-j)}(a)\,Q_{n,a}^{(j)}(a)$$

it readily follows by induction that

$$|Q_{n,a}^{(k)}(a)| \le Cq^n, \qquad k = 0, 1, \dots, \tau - 1, \tag{2.19}$$

for sufficiently large n. These inequalities and the expression

$$Q_{n,a}(z) = (z-a)^\tau + \sum_{k=0}^{\tau-1} \frac{Q_{n,a}^{(k)}(a)}{k!}(z-a)^k \tag{2.20}$$

give $\|(z-a)^\tau - Q_{n,a}(z)\|_{\overline{D}_{a,r}} < Cq^n$, for $n \ge N \in \mathbb{N}$. If we put here $z = \zeta_{n,\tau}$ we obtain

$$|\zeta_{n,\tau} - a|^\tau < Cq^n, \qquad n \ge N,$$

which implies $\delta_\tau(a)^\tau \le q$. As $q = (|a| + r)/\rho$ and $r > 0$ and $\rho < R_m^*(f)$ are arbitrary, we have

$$\delta_\tau(a) \le \left(\frac{|a|}{R_m^*(f)}\right)^{1/\tau},$$

which is all we need to show since $\delta_1(a) \le \cdots \le \delta_\tau(a)$ is trivial.

On the other hand, according to Theorem 2.3.4, $\Delta(a) \le \frac{|a|}{R_m^*(f)}$ is always true and the last statement readily follows. $\square$

2.4 Inverse results for incomplete Padé approximants

The first difficulty encountered in dealing with inverse-type results is to justify in terms of the data that the formal series corresponds to an analytic element which does not reduce to a polynomial. In our aid comes the next result, which provides such information depending on whether the zeros of the polynomials $Q_{n,m}$ remain away or not from 0 and/or ∞ as n grows. Let

$$\mathcal{P}_{n,m} = \{\zeta_{n,1}, \ldots, \zeta_{n,m_n}\}, \quad n \geq m, \quad m_n \leq m,$$

denote the collection of zeros of $Q_{n,m}$ repeated according to their multiplicity, where $\deg Q_{n,m} = m_n$. Put

$$S = \sup_{N \geq m} \inf \{|\zeta_{n,k}| : n \geq N, m_n \geq 1, 1 \leq k \leq m_n\}$$

and

$$G = \inf_{N \geq m} \sup \{|\zeta_{n,k}| : n \geq N, m_n \geq 1, 1 \leq k \leq m_n\}.$$

Finally, set

$$\tau_n = \min\{n - m^* - \lambda_n - \deg P_{n,m}, m - \lambda_n - m_n\}, \quad n \geq m.$$

From d.3) we know that $\tau_n \geq 0$, $n \geq m$.

Theorem 2.4.1. *Let f be a formal power series as in* (2.1)*. Fix $m \geq m^* \geq 1$. The following assertions hold.*

i) If $|\lambda_n - \lambda_{n-1}| \leq m^* - 1, n \geq n_0$, and $S > 0$ then $R_0(f) > 0$.

ii) If $|(m_n + \lambda_n + \tau_n) - (m_{n-1} + \lambda_{n-1} + \tau_{n-1})| \leq m^* - 1, n \geq n_0$, and $G < \infty$ then either f is a polynomial or $R_0(f) < \infty$. If, additionally, there exists a sequence of indices Λ such that $\deg Q_{n,m} \geq 1, n \in \Lambda$, then $R_0(f) < \infty$.

Proof. From definition

$$(Q_{n,m}f - P_{n,m})(z) = Az^{n+1-\lambda_n} + \cdots, \tag{2.21}$$

and $Q_{n,m}(0) \neq 0$.

Let us assume that $\inf\{|\zeta_{n,k}| : n \geq n_0, m_n \geq 1, 1 \leq k \leq m_n\} > 0$ and $|\lambda_n - \lambda_{n-1}| \leq m^* - 1$, $n \geq n_0$. If $\deg Q_{n,m} = m_n \geq 1$ we renormalize $Q_{n,m}$ to have independent term equal to 1 and write

$$q_{n,m}(z) = \prod_{k=1}^{m_n} \left(1 - \frac{z}{\zeta_{n,k}}\right) = a_{n,0} + a_{n,1}z + \cdots + a_{n,m_n} z^{m_n}, \quad a_{n,0} = 1.$$

Otherwise $q_{n,m}(z) \equiv 1 = a_{n,0}$. The corresponding numerator of $R_{n,m}$ is then denoted $p_{n,m}$.

Using the Vieta formulas connecting the coefficients of a polynomial and its zeros it follows that there exists $C_1 \geq 1$ such that

$$\sup\{|a_{n,k}| : 0 \leq k \leq m_n,\ n \geq n_0\} \leq C_1 < \infty. \tag{2.22}$$

The coefficient corresponding to $z^k, k \in \{n - m^* - \lambda_n + 1, \ldots, n - \lambda_n\}$ in the left hand side of (2.21) equals

$$\phi_k + a_{n,1}\phi_{k-1} + \cdots + a_{n,m_n}\phi_{k-m_n} = 0, \tag{2.23}$$

since $\deg p_{n,m} \leq n - m^* - \lambda_n$.

If $m_n \geq 1$, (2.22) and (2.23) imply that

$$|\phi_k| \leq C_1(|\phi_{k-1}| + \cdots + |\phi_{k-m_n}|).$$

Therefore, for each $k \in \{n - m^* - \lambda_n + 1, \ldots, n - \lambda_n\}$ there exists $k' \in \{k-1, \ldots, k-m\}$ $(m_n \leq m)$ such that

$$|\phi_k| \leq C_1 m |\phi_{k'}|. \tag{2.24}$$

Should $m_n = 0$, for the same values of k, we have $\phi_k = 0$ and (2.24) is trivially verified. Substituting n by $n-1$, we deduce that for each $k \in \{n - m^* - \lambda_{n-1}, \ldots, n - \lambda_{n-1} - 1\}$ there exists $k' \in \{k-1, \ldots, k-m\}$ such that

$$|\phi_k| \leq C_1 m |\phi_{k'}|. \tag{2.25}$$

As $n \geq n_0$, we have

$$n - \lambda_{n-1} \geq n - \lambda_n - m^* + 1$$

and

$$n - \lambda_{n-1} - m^* \leq n - \lambda_n - 1,$$

because $|\lambda_n - \lambda_{n-1}| \leq m^* - 1$. Consequently, the range of values taken by k due to relations (2.24) and (2.25) are either contiguous or overlapping for $n \geq n_0$. Since $n - \lambda_n$ tends to ∞ as n goes to ∞, we conclude that for all $n \geq n_0$ there exists $n' \in \{n-1, \ldots, n-m\}$ such that

$$|\phi_n| \leq C_1 m |\phi_{n'}|. \tag{2.26}$$

Let Λ be a sequence of indices such that

$$\lim_{n\in\Lambda} |\phi_n|^{1/n} = \limsup_{n\to\infty} |\phi_n|^{1/n} = 1/R_0(f).$$

Choose $n \in \Lambda$. Due to (2.26) there exist indices $n_1 > n_2 > \cdots > n_{r_n}$, $n_{r_n} \leq n_0$, where $r_n \leq n - n_0$, such that

$$|\phi_n| \leq C_1 m |\phi_{n_1}| \leq \cdots \leq (C_1 m)^{r_n} |\phi_{n_{r_n}}|.$$

Consequently,

$$\frac{1}{R_0(f)} = \lim_{n\in\Lambda} |\phi_n|^{1/n} \leq \limsup_{n\to\infty} (C_1 m)^{r_n/n} \leq C_1 m.$$

Therefore, $R_0(f) \geq (C_1 m)^{-1} > 0$, which proves i).

As for ii), assume that $\sup\{|\zeta_{n,k}| : n \geq n_0, m_n \geq 1, 1 \leq k \leq m_n\} < \infty$ and $|(m_n + \lambda_n + \tau_n) - (m_{n-1} + \lambda_{n-1} + \tau_{n-1})| \leq m^* - 1$, $n \geq n_0$. Now, we renormalize $Q_{n,m}$ to be monic and denote it $q_{n,m}$ and the corresponding numerator $p_{n,m}$. Set $t_n(z) = (z-1)^{\tau_n}$. Define $\tilde{q}_{n,m} = t_n q_{n,m}$ and $\tilde{p}_{n,m} = t_n p_{n,m}$. Normalize $\tilde{q}_{n,m}$ as follows. If $m_n + \tau_n \geq 1$ take

$$\tilde{q}_{n,m}(z) = \prod_{k=1}^{m_n+\tau_n} (z - \zeta_{n,k}) = b_{n,0} z^{m_n+\tau_n} + \cdots + b_{n,m_n+\tau_n-1} z + b_{n,m_n+\tau_n},$$

where $b_{n,0} = 1$. Should $m_n + \tau_n = 0$ we set $\tilde{q}_{n,m} \equiv 1 = b_{n,0}$. Using the Vieta formulas, it follows that there exists $C_2 \geq 1$ such that

$$\sup\{|b_{n,k}| : 0 \leq k \leq m_n,\ n \geq n_0\} \leq C_2 < \infty. \tag{2.27}$$

The coefficient corresponding to $z^k, k \in \{n - m^* - \lambda_n + 1, \ldots, n - \lambda_n\}$, in the left hand side of (2.21) equals

$$\phi_{k-m_n-\tau_n} + b_{n,1}\phi_{k-m_n-\tau_n+1} + \cdots + b_{n,m_n+\tau_n}\phi_k = 0, \tag{2.28}$$

since $\deg \tilde{p}_{n,m} \leq n - m^* - \lambda_n$.

Should $m_n + \tau_n \geq 1$, (2.27) and (2.28) imply that

$$|\phi_{k-m_n-\tau_n}| \leq C_2(|\phi_{k-m_n-\tau_n+1}| + \cdots + |\phi_k|),$$

or, what is the same, for each $k \in \{n - m^* - \lambda_n - m_n - \tau_n + 1, \ldots, n - \lambda_n - m_n - \tau_n\}$, we have

$$|\phi_k| \leq C_2(|\phi_{k+1}| + \cdots + |\phi_{k+m_n+\tau_n}|).$$

Therefore, for each $k \in \{n - m^* - \lambda_n - m_n - \tau_n + 1, \ldots, n - \lambda_n - m_n - \tau_n\}$ there exists $k' \in \{k+1, \ldots, k+m\}$ $(m_n + \tau_n \leq m)$ such that

$$|\phi_{k'}| \geq \frac{|\phi_k|}{C_2 m}. \tag{2.29}$$

In case that $m_n + \tau_n = 0$ we have $\phi_k = 0$ for the same values of k and (2.29) is also true.

Using the assumption that $|\lambda_n + m_n + \tau_n - \lambda_{n-1} - m_{n-1} - \tau_{n-1}| \leq m^* - 1$, it is easy to check, similar to the previous case, that the range of values taken by the parameter k for consecutive values of n are either contiguous or overlapping. Also, $n - \lambda_n - m_n - \tau_n$ tends to ∞ as n goes to ∞. Consequently, from (2.29) we have that for all $n \geq n_0$ there exists $n' \in \{n+1, \ldots, n+m\}$ such that

$$|\phi_{n'}| \geq \frac{|\phi_n|}{C_2 m} \tag{2.30}$$

Using (2.30) we can find an increasing sequence of multi-indices $\{n_s\}_{s\in\mathbb{Z}_+}$, $n_{s+1} \in \{n_s+1,\dots,n_s+m\}$ and $n_1 \in \{n_0,\dots,n_0+m\}$ such that

$$|\phi_{n_{s+1}}| \geq \frac{|\phi_{n_1}|}{(C_2 m)^s}.$$

Should f be a polynomial there is nothing to prove. Otherwise, changing the value of n_0 if necessary, without loss of generality we can assume that $\phi_{n_1} \neq 0$. Then,

$$\liminf_{s\to\infty} |\phi_{n_{s+1}}|^{1/n_{s+1}} \geq \frac{1}{\limsup\limits_{s\to\infty}(C_2 m)^{s/n_{s+1}}} \geq \frac{1}{C_2 m},$$

since

$$\limsup_{s\to\infty} \frac{s}{n_{s+1}} \leq \limsup_{s\to\infty} \frac{s}{n_1+s} = 1.$$

It follows that

$$R_0(f) = \frac{1}{\limsup\limits_{n\to\infty} |\phi_n|^{1/n}} \leq \frac{1}{\liminf\limits_{s\to\infty} |\phi_{n_{s+1}}|^{1/n_{s+1}}} \leq C_2 m < \infty,$$

as we needed to prove.

Finally, if f is a polynomial, say of degree N, we would have that for all $n \geq N+m$, $f \equiv p_{n,m}/q_{n,m}$ and $q_{n,m} \equiv 1$. Consequently, if there exists Λ such that $\deg q_{n,m} \geq 1, n \in \Lambda$, f cannot be a polynomial and, therefore, only $R_0(f) < \infty$ is possible. $\square$

Lemma 2.4.2. *A sufficient condition to have $|\lambda_n - \lambda_{n-1}| \leq m^* - 1$ and $|(m_n + \lambda_n + \tau_n) - (m_{n-1} + \lambda_{n-1} + \tau_{n-1})| \leq m^* - 1$ is that*

$$\min\{m_n + \tau_n,\, m_{n-1} + \tau_{n-1}\} \geq m - m^* + 1.$$

Proof. In fact, for $k = n-1$ and $k = n$, if $m_k + \tau_k \geq m - m^* + 1$ then $0 \leq \lambda_k \leq m^* - 1$ because $\lambda_k + m_k + \tau_k \leq m$ and the first inequality readily follows. On the other hand,

$$|(m_n + \lambda_n + \tau_n) - (m_{n-1} + \lambda_{n-1} + \tau_{n-1})|$$

$$= |(m_n + \lambda_n + \tau_n - m + m^* - 1) - (m_{n-1} + \lambda_{n-1} + \tau_{n-1} - m + m^* - 1)|$$

and $0 \leq m_k + \lambda_k + \tau_k - m + m^* - 1 \leq m^* - 1$ for $k = n-1$ and $k = n$. Therefore, the second inequality also holds. $\square$

Applied to Padé approximation ($m^* = m$), Theorem 2.4.1 and Lemma 2.4.2 imply that if $\deg Q_{n,m} \geq 1$ and its zeros remain uniformly bounded away from 0 and ∞, for all sufficiently large n, then $0 < R_0(f) < \infty$. This result has not been stated elsewhere.

Let us see some consequences of Theorem 2.4.1 and Lemma 2.4.2 on the extendability of a formal power series and the location of some of its poles in terms of the behavior of the zeros of the approximants.

Recall that

$$R_{m^*}(f) \leq R_m^*(f) \leq R_m(f), \tag{2.31}$$

that $R_m^*(f) > 0$ implies $R_0(f) > 0$, and that each pole of the function f in $D_m^*(f)$ attracts, with geometric rate, at least as many zeros of $Q_{n,m}$ as its order (see Theorems 2.3.3 and 2.3.4).

In the rest of this section we write

$$R_{n,m} = p_{n,m}/q_{n,m},$$

where $p_{n,m}$ and $q_{n,m}$ are relatively prime and $q_{n,m}$ is monic.

Corollary 2.4.3. *Let f be a formal power series as in* (2.1)*. Fix $m \geq m^* \geq 1$. Assume that there exists a polynomial q_m of degree greater than or equal to $m - m^* + 1$, $q_m(0) \neq 0$, such that* $\lim_{n\to\infty} q_{n,m} = q_m$*. Then $0 < R_0(f) < \infty$ and the zeros of q_m contain all the poles, counting multiplicities, that f has in $D_m^*(f)$.*

We need a relaxed version of Corollary 2.4.3 for the proof of Theorem 1.2.4.

Lemma 2.4.4. *Let f be a formal power series as in* (2.1) *that is not a polynomial. Fix $m \geq m^* \geq 1$. Let $R_{n,m} = \tilde{p}_{n,m}/\tilde{q}_{n,m}$ be an incomplete Padé approximant of type (n, m, m^*) corresponding to f, where $\tilde{p}_{n,m}$ and $\tilde{q}_{n,m}$ are obtained from Definition 2.2.1 and common factors between them are allowed. Assume that there exists a polynomial $\tilde{q}_m$ of degree m, $\tilde{q}_m(0) \neq 0$, such that* $\lim_{n\to\infty} \tilde{q}_{n,m} = \tilde{q}_m$*. Then $0 < R_0(f) < \infty$ and the zeros of $\tilde{q}_m$ contain all the poles, counting multiplicities, that f has in $D_m^*(f)$.*

Proof. Let us show that the assumptions of Lemma 2.4.2 are verified for the incomplete approximant $R_{n,m}$. Let $R_{n,m} = p_{n,m}/q_{n,m}$, where the polynomials $p_{n,m}$ and $q_{n,m}$ are relatively prime. Since $\tilde{q}_m(0) \neq 0$, then $\tilde{q}_{n,m}(0) \neq 0$, $n \geq n_0$. Thus, $\tilde{p}_{n,m}$ and $\tilde{q}_{n,m}$ do not have a common zero at $z = 0$ and $\lambda_n = 0$ for all $n \geq n_0$. As before, set $m_n = \deg q_{n,m}$ and

$$\tau_n = \min\{n - m^* - \deg p_{n,m},\, m - m_n\}, \qquad n \geq n_0.$$

Notice that $\tau_n = m - m_n$, $n \geq n_0$, because the polynomials $q_{n,m}$ and $p_{n,m}$ are obtained eliminating possible common factors between $\tilde{q}_{n,m}$ and $\tilde{p}_{n,m}$ and by assumption

$$\min\{n - m^* - \deg \tilde{p}_{n,m},\, m - \deg \tilde{q}_{n,m}\} = 0, \qquad n \geq n_0.$$

Therefore, we have

$$m_n + \tau_n = m \geq m - m^* + 1, \qquad n \geq n_0,$$

and Lemma 2.4.2 is applicable.

From Theorem 2.4.1 we obtain $0 < R_0(f) < \infty$. Now, from the fact that each pole of f in $D_m^*(f)$ attracts as many zeros of $q_{n,m}$ as its order it follows that the zeros of $\tilde{q}_m$ contain all the poles, counting multiplicities, that f has in $D_m^*(f)$. □

In case that there exists $R > R_{m^*}(f)$ inside of which f is meromorphic then D_R contains at least m^*+1 poles of f since $D_{m^*}(f)$ is the largest disk where f is meromorphic with at most m^* poles. We can prove the following inverse-type result.

Theorem 2.4.5. *Fix $m \geq m^* \geq 1$. Let f be a formal power series as in* (2.1) *that is not a rational function with at most $m^* - 1$ poles. Let $R_{n,m} = \tilde{p}_{n,m}/\tilde{q}_{n,m}$ be an incomplete Padé approximant of type (n, m, m^*) corresponding to f, where $\tilde{p}_{n,m}$ and $\tilde{q}_{n,m}$ are obtained from Definition 2.2.1 and common factors between them are allowed. Suppose that there exists a polynomial $\tilde{q}_m$, of degree m, $\tilde{q}_m(0) \neq 0$, such that*

$$\limsup_{n\to\infty} \|\tilde{q}_{n,m} - \tilde{q}_m\|^{1/n} = \theta < 1. \tag{2.32}$$

Then, either f has exactly m^* poles in $D_{m^*}(f)$, which are zeros of $\tilde{q}_m$ counting multiplicities, or $R_0(\tilde{q}_m f) > R_{m^*}(f)$.

Proof. From Lemma 2.4.4 we have $R_0(f) > 0$. So, f is analytic in a neighborhood of $z = 0$. We also know that $R_0(\tilde{q}_m f) \geq R_{m^*}(f)$ since the zeros of $\tilde{q}_m$ contain all the poles that f has in $D_{m^*}(f)$. Assume that $R_0(\tilde{q}_m f) = R_{m^*}(f)$. Let us show that then f has exactly m^* poles in $D_{m^*}(f)$. To the contrary, suppose that f has in $D_{m^*}(f)$ at most $m^* - 1$ poles. Then there exists a polynomial q_{m^*}, with $\deg q_{m^*} < m^*$, such that

$$R_0(q_{m^*} f) = R_{m^*}(f) = R_0(q_{m^*} \tilde{q}_m f).$$

Let

$$q_{m^*}(z)\, \tilde{q}_m(z)\, f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

then

$$R_{m^*}(f) = R_0(q_{m^*} \tilde{q}_m f) = 1/\limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

The n-th Taylor coefficient of $q_{m^*}[\tilde{q}_{n,m} f - \tilde{p}_{n,m}]$ is equal to zero. Therefore, the n-th Taylor coefficients of $q_{m^*}\tilde{q}_m f$ and $q_{m^*}\tilde{q}_m f - q_{m^*}\tilde{q}_{n,m} f + q_{m^*}\tilde{p}_{n,m}$ coincide. Take $0 < r < R_{m^*}(f)$ and recall that $\Gamma_r = \{z \in \mathbb{C} : |z| = r\}$. Hence

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_{\Gamma_r} \frac{[q_{m^*}\tilde{q}_m f - q_{m^*}\tilde{q}_{n,m} f + q_{m^*} p_{n,m}](\omega)}{\omega^{n+1}}\, d\omega \\ &= \frac{1}{2\pi i} \int_{\Gamma_r} \frac{[\tilde{q}_m - \tilde{q}_{n,m}](\omega)\, q_{m^*}(\omega) f(\omega)}{\omega^{n+1}}\, d\omega. \end{aligned}$$

Making use of (2.32) it readily follows that

$$\frac{1}{R_{m^*}(f)} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \leq \frac{\theta}{r}.$$

Letting r tend to $R_{m^*}(f)$ we have

$$\frac{1}{R_{m^*}(f)} \leq \frac{\theta}{R_{m^*}(f)}, \qquad \theta < 1,$$

which implies that $R_{m^*}(f) = \infty$. Let us show that this is not possible.

In fact,

$$[q_{m^*}\, \tilde{q}_{n,m}\, f - q_{m^*}\, \tilde{p}_{n,m}]\,(z) = A_n z^{n+1} + \cdots,$$

and $\deg q_{m^*}\tilde{p}_{n,m} \leq n-1$. It follows that $(q_{m^*}\tilde{p}_{n,m})/\tilde{q}_{n,m} = p_{n,m}/q_{n,m}$ is an incomplete Padé approximant of the function $q_{m^*}f$ of type $(n,m,1)$, where the polynomials $p_{n,m}$ and $q_{n,m}$ are relatively prime. As $\tilde{q}_{n,m}(0) \neq 0$, $n \geq n_0$, the polynomials $q_{m^*}\tilde{p}_{n,m}$ and $\tilde{q}_{n,m}$ do not have a common zero at $z=0$ and $\lambda_n = 0$ for all $n \geq n_0$. Again, set $m_n = \deg q_{n,m}$ and

$$\tau_n = \min\left\{n-1-\deg p_{n,m},\, m-m_n\right\}.$$

Notice that $\tau_n = m - m_n$, $n \geq n_0$, because

$$\min\left\{n-1-\deg q_{m^*}\tilde{p}_{n,m},\, m-\deg\tilde{q}_{n,m}\right\} = 0, \quad n \geq n_0.$$

Thus, $m_n + \tau_n = m$, $n \geq n_0$. Using Lemma 2.4.2 (for $m^* = 1$) and Theorem 2.4.1 we conclude that either $R_0(q_{m^*}f) < \infty$ or $q_{m^*}f$ is a polynomial. However, the latter is not possible by hypotheses. On the other hand, $R_0(q_{m^*}f) < \infty$ contradicts $R_{m^*}(f) = \infty$. As claimed, f has exactly m^* poles in $D_{m^*}(f)$. □

We wish to mention that apart from the application of incomplete Padé approximation to simultaneous Padé approximation, which will be seen in the next chapter, there are other possibilities. For example, we may have a priori knowledge of the location of some of the poles of f and we can use this information to fix some of the zeros of $Q_{n,m}$ at such points. This is typical in Padé-type approximation (see [2]). Another possibility is to combine two (or more) approximation criteria to define the rational functions; for example, interpolation at some points and least squares as considered in [27]-[33]. Such least squares vector-valued approximations are also incomplete in the sense we defined above. Therefore, for them convergence in h-content immediately follows.

2.5 Examples

The following simple examples illustrate our plan for the next chapter.

2.5.1 On the values of $R^*_m(f)$

The purpose of this example is to show that $R^*_m(f)$ may take any value between $R_{m^*}(f)$ and $R_m(f)$ depending on the sequence of incomplete Padé approximants considered. Take $m^* = 1$, $m = 2$, and

$$f(z) = \frac{1}{1-z^2}.$$

Then, $R_1(f) = 1$, $R_2(f) = +\infty$. Consider

$$g(z) = \frac{z}{1+z^2}, \quad h(z) = \frac{1}{1+z}, \quad w_p(z) = \frac{1}{1+z} + \frac{1}{1-z/p}, \quad p > 1.$$

Fix $\mathbf{m} = (1,1)$ and set $\mathbf{f} = (f,g)$. It is clear that $R_2(\mathbf{f}) = 1$ and the system $\mathbf{f}$ is not polewise independent with respect to $\mathbf{m}$ in $D_2(\mathbf{f})$. On the other hand, $R_1(f) = R_1(g) = 1$ and $R_2(f) = R_2(g) = +\infty$. It is very easy to see that $Q_{n,\mathbf{m}} = 1 - z^2$ if n is even and $Q_{n,\mathbf{m}} = 1 + z^2$ when n is odd. So, $R^*_2(f) = 1$ since $R^*_2(f) \geq R_1(f) = 1$ and $R^*_2(f)$ cannot be greater than 1. Otherwise, from part iii) of Gonchar's Lemma, it follows that the polynomial $Q_{n,\mathbf{m}}$ tends to $1 - z^2$, which is not true. An analogous argument proves that $R^*_2(g) = 1$.

Now, take $\mathbf{f} = (f,h)$ with the same multi-index $\mathbf{m}$. Obviously, $R^*_2(h) = +\infty$ since $R_1(h) = +\infty$. The system $\mathbf{f}$ is polewise independent respect to $\mathbf{m}$ in $D_2(\mathbf{f}) = \mathbb{C}$. Therefore, $R^*_2(f) = +\infty$ as a consequence of (1.6) in the Graves-Morris/Saff theorem and the fact that $R^*_2(f)$ is the radius of the largest disk inside of which there is convergence in h-content.

Finally, consider $\mathbf{f} = (f, w_p)$ and fix $\mathbf{m} = (1,1)$. We have $R_2(\mathbf{f}) = p$ and the system $\mathbf{f}$ is polewise independent with respect to $\mathbf{m}$ in $D_2(\mathbf{f})$. As $R^*_2(w_p) \geq R_1(w_p) = p$, necessarily $R^*_2(f) \geq p$ using again (1.6) and Theorem 2.3.3. Then $Q_{n,\mathbf{m}}$ tends to $1 - z^2$ and $R^*_2(w_p) = p$ due to Gonchar's Lemma. An easy calculation shows that

$$Q_{n,\mathbf{m}}(z) = \begin{cases} \lambda_n \left(z^2 + \dfrac{p^2-1}{p^n - p} z - 1 \right), & \text{if } n \text{ is even,} \\ z^2 - \dfrac{p^n - p^2}{p^n - 1}, & \text{if } n \text{ is odd,} \end{cases}$$

with $\lim_{n\to\infty} \lambda_n = 1$. Now, $R_2^*(f)$ may be worked out by means of formula (2.6) according to Lemma 2.3.2. Keeping in mind the notation adopted there and using the expression of $Q_{n,\mathbf{m}}$ calculated before, it turns out that

$$|A_{n,2}| = \lambda_n \frac{p(p^2-1)}{p^{n+1}-1}, \qquad n \text{ even}.$$

Then, $\lim_{n=2\mathbb{Z}_+} |A_{n,2}|^{1/n} = 1/p$, which implies

$$p \le R_2^*(f) = \frac{1}{\limsup_{n\to\infty} |A_{n,2}|^{1/n}} \le p.$$

Thus, we have proved that $R_2^*(f) = p$ may take any value between $R_1(f) = 1$ and $R_2(f) = \infty$, both ends included. The problem with (2.6) is that in more general cases it is very difficult to derive from it the value of $R_m^*(f)$. A simpler characterization in terms of the analytic properties of the components of the vector function is desirable.

2.5.2 Limitations of the Graves-Morris/Saff Theorem

First, let us see that there are very simple systems $\mathbf{f}$ that are not polewise independent in $D_{|\mathbf{m}|}(\mathbf{f})$ and still convergence takes place. Set

$$f_1(z) = \frac{1}{1-z} + \frac{1}{2-z}, \qquad f_2(z) = \frac{1}{3-z}, \tag{2.33}$$

and fix the multi-index $\mathbf{m} = (1,1)$. Put $\mathbf{f} = (f_1, f_2)$. It is clear that $R_2(\mathbf{f}) = 3$ and, as $0f_1 + f_2$ is analytic in $D_2(\mathbf{f})$, the system $\mathbf{f}$ is not polewise independent in $D_2(\mathbf{f})$. Also, as $R_1(f_2) = \infty$, we have $R_2^*(f_2) = \infty$ and one of the poles of $Q_{n,\mathbf{m}}$ is attracted by the point $z = 3$. On the other hand, $R_1(f_1) = 2$, so $R_2^*(f_1) \ge 2$ but $R_2^*(f_1)$ cannot be greater than 2 since in that case two other poles of $Q_{n,\mathbf{m}}$ would be attracted by the points $z = 1$ and $z = 2$, which is absurd. Then $R_2^*(f_1) = 2$. Using what we have proved before for incomplete Padé approximants we know that the first component of the simultaneous Padé approximants will converge to f_1 uniformly on compact subsets of $\{z : |z| < 2\} \setminus \{1\}$ and the second component to f_2 on compact subsets of $\mathbb{C} \setminus \{3\}$.

Now, fix again $\mathbf{m} = (1,1)$ and take $\mathbf{g} = (g_1, g_2)$, where

$$g_1(z) = \frac{1}{1-z} + \log(3-z), \qquad g_2(z) = \frac{1}{2-z} + \log(10-z).$$

Obviously, $R_1(g_1) = R_2^*(g_1) = R_2(g_1) = 3$ and $R_1(g_2) = R_2^*(g_2) = R_2(g_2) = 10$. The system $\mathbf{g}$ is polewise independent in $D_2(\mathbf{g})$ with $R_2(\mathbf{g}) = 3$. The Graves-Morris/Saff Theorem gives

$$\limsup_{n\to\infty} \|g_2 - R_{n,\mathbf{m},2}\|_K^{1/n} \leq \frac{\|z\|_K}{3},$$

for any compact subset K of $\{z : |z| < 3\}$ and

$$\limsup_{n\to\infty} \|\mathcal{Q}_{\mathbf{m}}(\mathbf{g}) - Q_{n,\mathbf{m}}\|^{1/n} \leq 2/3,$$

where $\mathcal{Q}_{\mathbf{m}}(\mathbf{g})(z) = (z-1)(1-z/2)$. On the other hand, our result on the convergence of incomplete Padé approximation gives

$$\limsup_{n\to\infty} \|g_2 - R_{n,\mathbf{m},2}\|_K^{1/n} \leq \frac{\|z\|_K}{10},$$

for any compact subset K of $\{z : |z| < 10\}$ and

$$\limsup_{n\to\infty} \|\mathcal{Q}_{\mathbf{m}}(\mathbf{g}) - Q_{n,\mathbf{m}}\|^{1/n} \leq \max\{1/3, 1/5\} = 1/3.$$

Are the improved bounds which we have given here exact?

2.5.3 Linear transformations

Let $\mathbf{f} = (f_1, \ldots, f_d)$ be a system of functions and $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$ a fixed multi-index. Consider two indices $j, k \in \{1, \ldots, d\}$ such that $m_j \leq m_k$. Let us construct the system $\hat{\mathbf{f}} = (\hat{f}_1, \ldots, \hat{f}_d)$ given by $\hat{f}_j = af_j + bf_k$ and $\hat{f}_i = f_i$, $i \neq j$. We say that the system $\hat{\mathbf{f}}$ is obtained from $\mathbf{f}$ by means of a linear transformation. It is very easy to see that the $(n, \mathbf{m})$ Hermite-Padé approximants of the systems $\mathbf{f}$ and $\hat{\mathbf{f}}$ have the same common denominator $Q_{n,\mathbf{m}}$. Also, $D_{|\mathbf{m}|}(\mathbf{f}) = D_{|\mathbf{m}|}(\hat{\mathbf{f}})$, and the system $\mathbf{f}$ is polewise independent in $D_{|\mathbf{m}|}(\mathbf{f})$ if and only if the system $\hat{\mathbf{f}}$ is polewise independent in $D_{|\mathbf{m}|}(\hat{\mathbf{f}})$. Obviously, the same properties are shared by two systems of functions related to each other by means of a finite number of the transformations described above.

The following example shows how to linearly transform a system of functions to prove geometric convergence of the denominators $Q_{n,\mathbf{m}}$. Take $\mathbf{m} = (1,1)$ and set

$$\hat{f}_1(z) = \frac{1}{1-z} + \frac{1}{2-z}, \qquad \hat{f}_2(z) = \frac{1}{1-z} + \frac{1}{2-z} + \frac{1}{3-z}.$$

Put $\hat{\mathbf{f}} = (\hat{f}_1, \hat{f}_2)$. It is clear that $\hat{f}_1 = f_1$ and $\hat{f}_2 = f_1 + f_2$, where $\mathbf{f} = (f_1, f_2)$ is the system given by (2.33). The $(n, \mathbf{m})$ Hermite-Padé approximants of the systems $\mathbf{f}$ and $\hat{\mathbf{f}}$ have the same common denominator $Q_{n,\mathbf{m}}$. Thus, according to what was said in Subsection 2.5.2, the zeros of the polynomials $Q_{n,\mathbf{m}}$ are attracted by the points $z = 1$ and $z = 3$. Then, necessarily $R_2^*(\hat{f}_1) = 2$ and $R_2^*(\hat{f}_2) = 2$. Otherwise, the point $z = 2$ would attract zeros of $Q_{n,\mathbf{m}}$ which is not possible. The system $\hat{\mathbf{f}}$ has only one pole in $\{z : |z| < 2\}$. Nevertheless, the knowledge of the behavior of the zeros of $Q_{n,\mathbf{m}}$ allows us to deduce that

$$\limsup_{n\to\infty} \|\hat{f}_k - R_{n,\mathbf{m},k}\|_K^{1/n} \le \frac{\|z\|_K}{2}, \quad k = 1, 2,$$

due to Gonchar's Lemma, where K is any compact subset of $\{z : |z| < 2\} \setminus \{z = 1\}$. Also,

$$\limsup_{n\to\infty} \|\mathcal{Q}_{\mathbf{m}}(\mathbf{f}) - Q_{n,\mathbf{m}}\|^{1/n} \le \max\{0, 1/2\} = 1/2,$$

where $\mathcal{Q}_{\mathbf{m}}(\mathbf{f})(z) = (z-1)(1-z/3)$. Are these bounds exact?

Now, we show how to improve the bounds on the rate of convergence using linear transformations. Fix $\mathbf{m} = (1, 1)$ and consider the system $\mathbf{h} = (h_1, h_2)$, where

$$h_1(z) = \frac{1}{1-z} + \frac{1}{2-z} + \log(3-z), \quad h_2(z) = \frac{1}{1-z} + \log(3-z) + \log(4-z).$$

Obviously $R_2(\mathbf{h}) = 3$ and the system $\mathbf{h}$ is polewise independent in $D_2(\mathbf{h})$ and the Grave-Morris/Saff Theorem gives us

$$\limsup_{n\to\infty} \|\mathcal{Q}_{\mathbf{m}}(\mathbf{h}) - Q_{n,\mathbf{m}}\|^{1/n} \le \max\{1/3, 2/3\} = 2/3,$$

where $\mathcal{Q}_{\mathbf{m}}(\mathbf{h})(z) = (z-1)(1-z/2)$.

Now, using our results on incomplete Padé approximation we know that $z = 1$ attracts one zero of $Q_{n,\mathbf{m}}$ with rate $1/3$. Consider the system $\hat{\mathbf{h}} = (\hat{h}_1, \hat{h}_2)$, where $\hat{h}_1 = h_1 - h_2$ and $\hat{h}_2 = h_2$. The $(n, \mathbf{m})$ Hermite-Padé approximants of the systems $\mathbf{h}$ and $\hat{\mathbf{h}}$ have the same common denominator $Q_{n,\mathbf{m}}$. Then, using again our results on incomplete Padé approximation we know that $z = 2$ attracts one zero of $Q_{n,\mathbf{m}}$ with rate 1/2. Therefore

$$\limsup_{n\to\infty} \|\mathcal{Q}_{\mathbf{m}}(\mathbf{h}) - Q_{n,\mathbf{m}}\|^{1/n} \leq \max\{1/3, 1/2\} = 1/2,$$

which gives a better estimate. Is 1/2 the exact value of this limit?

Let us make a computational experiment using $\hat{\mathbf{h}}$ to see how the zeros of $Q_{n,\mathbf{m}}$ converge. Let $z_{n,1}^*$ and $z_{n,2}^*$ be the zeros of $Q_{n,\mathbf{m}}$.

n	$z_{n,1}^*$	$z_{n,2}^*$
5	1.009688457	1.818732351
10	1.000006348	1.998306144
15	1.000000013	1.999969057
20	.9999999998	1.999999318
25	1.000000000	1.999999984
35	1.000000000	2.000000000

In the calculations we employed the procedure VRMPADe06 written in Maple which is included in the Appendix of the thesis.

Chapter 3

Hermite-Padé approximation

Throughout this chapter, $\mathbf{f} = (f_1, \ldots, f_d)$ denotes a system of formal power expansions as in (1.4) and $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$ is a fixed multi-index. We are concerned with the simultaneous approximation of $\mathbf{f}$ by sequences of vector rational functions defined according to Definition 1.2.1 taking account of (1.5). That is, for each $n \in \mathbb{N}, n \geq |\mathbf{m}|$, let $(R_{n,\mathbf{m},1}, \ldots, R_{n,\mathbf{m},d})$ be a Hermite-Padé approximation of type $(n, \mathbf{m})$ corresponding to $\mathbf{f}$.

In this chapter we use the notation

$$\mathbf{R}_{n,\mathbf{m}} = (R_{n,\mathbf{m},1}, \ldots, R_{n,\mathbf{m},d}) = (P_{n,\mathbf{m},1}, \ldots, P_{n,\mathbf{m},d})/Q_{n,\mathbf{m}},$$

where $Q_{n,\mathbf{m}}$ has no common zero simultaneously with all the $P_{n,\mathbf{m},k}$ and is normalized to be monic.

3.1 Polynomial independence

As mentioned before, $R_{n,\mathbf{m},k}$ is an incomplete Padé approximant of type $(n, |\mathbf{m}|, m_k)$ with respect to f_k, $k = 1, \ldots, d$. Thus, from (2.31) we have

$$D_{m_k}(f_k) \subset D^*_{|\mathbf{m}|}(f_k) \subset D_{|\mathbf{m}|}(f_k), \quad k = 1, \ldots, d.$$

Definition 3.1.1. *A vector* $\mathbf{f} = (f_1, \ldots, f_d)$ *of formal power expansions is said to be* *polynomially independent with respect to* $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$ *if there do not exist polynomials* $p_1, \ldots, p_d$*, at least one of which is non-null, such that*

c.1) $\deg p_k \leq m_k - 1,\ k = 1, \ldots, d,$

c.2) $\sum_{k=1}^{d} p_k f_k$ *is a polynomial.*

In particular, polynomial independence implies that for each $k = 1, \ldots, d$, f_k is not a rational function with at most $m_k - 1$ poles. Notice that polynomial independence may be verified solely in terms of the coefficients of the formal Taylor expansions defining the system $\mathbf{f}$.

Given $\mathbf{f} = (f_1, \ldots, f_d)$ and $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$, we consider the associated system $\overline{\mathbf{f}}$ of formal power expansions

$$\overline{\mathbf{f}} = (f_1, \ldots, z^{m_1 - 1} f_1, f_2, \ldots, z^{m_d - 1} f_d) = (\bar{f}_1, \ldots, \bar{f}_{|\mathbf{m}|}).$$

We also define an associated multi-index $\overline{\mathbf{m}}$ given by $\overline{\mathbf{m}} = (1, 1, \ldots, 1)$ with $|\overline{\mathbf{m}}| = |\mathbf{m}|$. The systems $\mathbf{f}$ and $\overline{\mathbf{f}}$ share most properties. In particular, the poles of $\mathbf{f}$ and $\overline{\mathbf{f}}$ coincide and $R_m(\mathbf{f}) = R_m(\overline{\mathbf{f}})$, $m \in \mathbb{Z}_+$.

From the definition it readily follows that $\mathbf{f}$ is polynomially independent with respect to $\mathbf{m}$ if and only if there do not exist constants $c_k, k = 1, \ldots, |\mathbf{m}|$, not all zero, such that

$$\sum_{k=1}^{|\mathbf{m}|} c_k \bar{f}_k$$

is a polynomial. That is, $\mathbf{f}$ is polynomially independent with respect to $\mathbf{m}$ if and only if $\overline{\mathbf{f}}$ is polynomially independent with respect to $\overline{\mathbf{m}}$. By the same token, the system poles of $\mathbf{f}$ with respect to $\mathbf{m}$ (see Definition 1.2.3) are the same as the system poles of $\overline{\mathbf{f}}$ with respect to $\overline{\mathbf{m}}$.

Finally, it is very easy to check that, for all $n \geq |\mathbf{m}|$, the equations that define the common denominator $Q_{n,\mathbf{m}}$ for $(\mathbf{f}, \mathbf{m})$ are the same as those defining $Q_{n,\overline{\mathbf{m}}}$ for $(\overline{\mathbf{f}}, \overline{\mathbf{m}})$ and, consequently, both classes of polynomials coincide.

Lemma 3.1.2. *Let* $\mathbf{f} = (f_1, \ldots, f_d)$ *be a system of formal Taylor expansions as in* (1.4) *and fix a multi-index* $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$*. Suppose that for all* $n \geq n_0$ *the polynomial* $Q_{n,\mathbf{m}}$ *is unique and* $\deg Q_{n,\mathbf{m}} = |\mathbf{m}|$*. Then, the system* $\mathbf{f}$ *is polynomially independent with respect to* $\mathbf{m}$.

Proof. Because of what was said just before the statement of Lemma 3.1.2, we can assume without loss of generality that $\mathbf{m} = (1, 1, \ldots, 1)$ and $d = |\mathbf{m}|$. We argue by contradiction. Suppose that there exist constants c_k, $k = 1, \ldots, d$, not all zero, such that $\sum_{k=1}^{d} c_k f_k$ is a polynomial. Should $d = 1$, $Q_{n,\mathbf{m}} \equiv 1$ for all n sufficiently large and $\deg Q_{n,\mathbf{m}} < 1 = |\mathbf{m}|$. If $d > 1$, without loss of generality, we can suppose that $c_1 \neq 0$. Then

$$f_1 = p - \sum_{k=2}^{d} c_k f_k,$$

where p is a polynomial, say of degree N.

On the other hand, for each $n \geq d-1$, there exist polynomials $Q_n, P_{n,k}$, $k = 2, \ldots, d$, such that

- $\deg P_{n,k} \leq n-1$, $k = 2, \ldots, d$, $\quad \deg Q_n \leq d-1$, $\quad Q_n \not\equiv 0$,
- $Q_n(z)\, f_k(z) - P_{n,k}(z) = A_k z^{n+1} + \cdots\,, \quad k = 2, \ldots, d.$

Therefore,

$$Q_n(z)\left(p(z) - \sum_{k=2}^{d} c_k f_k(z)\right) - \left(Q_n(z)\, p(z) - \sum_{k=2}^{d} c_k P_{n,k}(z)\right) = A z^{n+1} + \ldots$$

and, for $n \geq d+N$, the polynomial $P_{n,1} = Q_n\, p - \sum_{k=2}^{d} c_k P_{n,k}$ verifies $\deg P_{n,1} \leq n-1$. Thus, for all n sufficiently large, the polynomials $P_{n,k}$, $k = 1, \ldots, d$, satisfy Definition 1.2.1 with respect to $\mathbf{f}$ and $\mathbf{m}$. Naturally, Q_n gives rise to a polynomial $Q_{n,\mathbf{m}}$ with $\deg Q_{n,\mathbf{m}} < d = |\mathbf{m}|$ against our assumption on $Q_{n,\mathbf{m}}$. $\square$

Set

$$\mathbf{D}^*_{\mathbf{m}}(\mathbf{f}) = \left(D^*_{|\mathbf{m}|}(f_1), \ldots, D^*_{|\mathbf{m}|}(f_d)\right).$$

The following corollaries are straightforward consequences of Corollary 2.4.3 and Theorem 2.4.5, respectively, together with the fact that, for each $k = 1, \ldots, d$, $R_{n,\mathbf{m},k} = P_{n,\mathbf{m},k}/Q_{n,\mathbf{m}}$ is an incomplete Padé approximant of type $(n, |\mathbf{m}|, m_k)$ with respect to f_k.

Corollary 3.1.3. *Let $\mathbf{f} = (f_1, \ldots, f_d)$ be a system of formal Taylor expansions as in* (1.4) *and fix a multi-index $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$. Assume that $\mathbf{f}$ is polynomially independent with respect to $\mathbf{m}$ and there exists a polynomial $Q_{|\mathbf{m}|}$ of degree $|\mathbf{m}|$, $Q_{|\mathbf{m}|}(0) \neq 0$, such that $\lim_{n\to\infty} Q_{n,\mathbf{m}} = Q_{|\mathbf{m}|}$. Then $R_0(\mathbf{f}) > 0$, the zeros of $Q_{|\mathbf{m}|}$ contain all the poles that $\mathbf{f}$ has in $\mathbf{D}^*_{\mathbf{m}}(\mathbf{f})$, and $R_0(f_k) < \infty$ for each $k = 1, \ldots, d$.*

Corollary 3.1.4. *Let $\mathbf{f} = (f_1, \ldots, f_d)$ be a system of formal Taylor expansions as in* (1.4) *and fix a multi-index $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$. Assume that $\mathbf{f}$ is polynomially independent with respect to $\mathbf{m}$ and there exists a polynomial $Q_{|\mathbf{m}|}$ of degree $|\mathbf{m}|$, $Q_{|\mathbf{m}|}(0) \neq 0$, such that*

$$\limsup_{n\to\infty} \|Q_{|\mathbf{m}|} - Q_{n,\mathbf{m}}\|^{1/n} = \theta < 1.$$

Then, for each $k = 1, \ldots, d$, either f_k has exactly m_k poles in $D_{m_k}(f_k)$ or $R_0(Q_{|\mathbf{m}|} f_k) > R_{m_k}(f_k)$.

Before proving the main Theorem we wish to describe some properties of system poles.

Lemma 3.1.5. *Given $\mathbf{f} = (f_1, \ldots, f_d)$ and $\mathbf{m} \in \mathbb{Z}^d_+ \setminus \{\mathbf{0}\}$, $\mathbf{f}$ can have at most $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ (counting their order). Moreover, if the system has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ and ξ is a system pole of order τ then for all $s > \tau$ there can be no polynomial combination of the form* (1.9) *analytic on a neighborhood of $\overline{D}_{|\xi|}$ except for a pole at $z = \xi$ of exact order s.*

Proof. Notice that the polynomial combinations of the form (1.9) generate a vector space of dimension less than or equal to $|\mathbf{m}|$. On the other hand, the set of functions which determine the system poles and their order are linearly independent. Consequently, there may be at most $|\mathbf{m}|$ such functions. Thus, the number of system poles counting their order is at most $|\mathbf{m}|$.

Assume that there are exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ and let ξ be one of them of order τ. Take $s > \tau$. Obviously, for $s = \tau + 1$ there can be no polynomial combination of the form (1.9) analytic on a neighborhood of $\overline{D}_{|\xi|}$ except for a pole at $z = \xi$ of exact order s because the order of the system pole would be at least $\tau + 1$. For $s \geq \tau + 2$ no such combination can exist either because that would give another function which is linearly independent to the rest of the functions which determine the system poles and their order which by assumption are already $|\mathbf{m}|$. $\square$

3.2 Proof of the main Theorem

Let us prove first that b) implies a). From Lemma 3.1.2 it follows that $\mathbf{f}$ is polynomially independent with respect to $\mathbf{m}$ and, in turn, from Corollary 3.1.3 we know that $R_0(\mathbf{f}) > 0$. So, it is enough to prove that $\mathbf{f}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ and, without loss of generality, we can assume that $\mathbf{m} = (1, 1, \ldots, 1)$.

We divide the proof into two parts. First, we collect a set of $|\mathbf{m}|$ candidates to be system poles of $\mathbf{f}$ and prove that they are the zeros of $Q_{|\mathbf{m}|}$. In the second part we prove that all these points previously collected are actually system poles of $\mathbf{f}$.

In the disk $D_0(\mathbf{f})$ there cannot be system poles of $\mathbf{f}$ since all the functions f_k are analytic. Now, for each $k = 1, \ldots, d$, by Corollaries 3.1.4 and 3.1.3, either the disk $D_1(f_k)$ contains exactly one pole of f_k, and it is a zero of $Q_{|\mathbf{m}|}$, or $R_0(Q_{|\mathbf{m}|} f_k) > R_1(f_k)$. Therefore, $D_0(\mathbf{f}) \neq \mathbb{C}$ and $Q_{|\mathbf{m}|}$ contains as zeros all the poles of f_k on the boundary of $D_0(f_k)$ counting their order for $k = 1, \ldots, d = |\mathbf{m}|$. Moreover, the functions f_k cannot have on the boundary of $D_0(f_k)$ singularities other than poles.

According to this, the poles of $\mathbf{f}$ on the boundary of $D_0(\mathbf{f})$ are all zeros of $Q_{|\mathbf{m}|}$ counting multiplicities and the boundary contains no other singularity except poles. Let us call them candidate system poles of $\mathbf{f}$ and denote them by $a_1, \ldots, a_{n_1}$ taking account of their order. Obviously, any system pole of $\mathbf{f}$ on the boundary of $D_0(\mathbf{f})$ must be one of the candidates since no linear combination of the functions in $\mathbf{f}$ can produce poles at any other point of that set.

Since $\deg Q_{|\mathbf{m}|} = |\mathbf{m}|$ we have $n_1 \leq |\mathbf{m}|$. Should $n_1 = |\mathbf{m}|$ we have found all the candidates we were looking for. Let us assume that $n_1 < |\mathbf{m}|$. We can find coefficients $c_1, \ldots, c_{|\mathbf{m}|}$ such that

$$\sum_{k=1}^{|\mathbf{m}|} c_k f_k$$

is analytic in a neighborhood of $\overline{D_0(\mathbf{f})}$. Finding the coefficients c_k reduces to solving a linear homogeneous system of n_1 equations with $|\mathbf{m}|$ unknowns. In fact, if $z = a$ is a candidate system pole of $\mathbf{f}$ with multiplicity τ we obtain τ equations choosing the coefficients c_k so that

$$\int_{|\omega-a|=\delta} (\omega - a)^i \left(\sum_{k=1}^{|\mathbf{m}|} c_k f_k(\omega) \right) d\omega = 0, \qquad i = 0, \ldots, \tau - 1. \tag{3.1}$$

where δ is sufficiently small. We do the same with each distinct candidate on the boundary of $D_0(\mathbf{f})$. The linear homogeneous system of equations so obtained has at least $|\mathbf{m}| - n_1$ linearly independent solutions which we denote by $\mathbf{c}_j^1$, $j = 1, \ldots, |\mathbf{m}| - n_1^*$, $n_1^* \leq n_1$.

Set

$$\mathbf{c}_j^1 = (c_{j,1}^1, \ldots, c_{j,|\mathbf{m}|}^1), \quad j = 1, \ldots, |\mathbf{m}| - n_1^*.$$

Construct the $(|\mathbf{m}| - n_1^*) \times |\mathbf{m}|$ dimensional matrix

$$C^1 = \begin{pmatrix} \mathbf{c}_1^1 \\ \vdots \\ \mathbf{c}_{|\mathbf{m}|-n_1^*}^1 \end{pmatrix}.$$

Define the system $\mathbf{g}_1$ of $|\mathbf{m}| - n_1^*$ functions by means of

$$\mathbf{g}_1^t = C^1 \mathbf{f}^t = (g_{1,1}, \ldots, g_{1,|\mathbf{m}|-n_1^*})^t,$$

where $(\cdot)^t$ means taking transpose. We have

$$g_{1,j} = \sum_{k=1}^{|\mathbf{m}|} c_{j,k}^1 f_k, \qquad j = 1, \ldots, |\mathbf{m}| - n_1^*.$$

As the rows of C^1 are non-null, none of the functions $g_{1,j}$ are polynomials because of the polynomial independence of $\mathbf{f}$ with respect to $\mathbf{m} = (1, 1, \ldots, 1)$.

Consider the region

$$D_0(\mathbf{g}_1) = \bigcap_{j=1}^{|\mathbf{m}|-n_1^*} D_0(g_{1,j}).$$

Obviously, by construction, $D_0(\mathbf{f})$ is strictly included in $D_0(\mathbf{g}_1)$

It is easy to see that

$$\sum_{k=1}^{|\mathbf{m}|} c_{j,k}^1 \frac{P_{n,\mathbf{m},k}}{Q_{n,\mathbf{m}}}$$

is an $(n, |\mathbf{m}|, 1)$ incomplete Padé approximant of $g_{1,j}$. Using Theorem 2.4.5 with $m^* = 1$, for each $j = 1, \ldots, |\mathbf{m}| - n_1^*$, either the disk $D_1(g_{1,j})$ contains exactly one pole of $g_{1,j}$, and it is a zero of $Q_{|\mathbf{m}|}$, or $R_0(Q_{|\mathbf{m}|} g_{1,j}) > R_1(g_{1,j})$. In particular, $D_0(\mathbf{g}_1) \neq \mathbb{C}$ and all the singularities of $\mathbf{g}_1$ on the boundary of $D_0(\mathbf{g}_1)$ are poles which are zeros of $Q_{|\mathbf{m}|}$ counting their order. They constitute the next layer of candidate system poles of $\mathbf{f}$ (now, it is possible that some candidates are not poles of $\mathbf{f}$ since the functions f_k intervene in the linear combination as we saw in example (1.8)).

Let us denote these new candidates by $a_{n_1+1}, \ldots, a_{n_1+n_2}$. Of course $n_1 + n_2 \leq |\mathbf{m}|$. Should $n_1 + n_2 = |\mathbf{m}|$, we are done. Otherwise, $n_2 < |\mathbf{m}| - n_1 \leq |\mathbf{m}| - n_1^*$ and we can repeat the process. In order to eliminate the n_2 poles we have $|\mathbf{m}| - n_1^*$ functions which are analytic on $D_0(\mathbf{g}_1)$ and meromorphic on a neighborhood of $\overline{D_0(\mathbf{g}_1)}$. The corresponding homogeneous linear system of equations, similar to (3.1), has at least $|\mathbf{m}| - n_1^* - n_2$ linearly independent solutions $\mathbf{c}_j^2, j = 1, \ldots, |\mathbf{m}| - n_1^* - n_2^*$, $n_2^* \leq n_2$. Set

$$\mathbf{c}_j^2 = (c_{j,1}^2, \ldots, c_{j,|\mathbf{m}|-n_1^*}^2), \quad j = 1, \ldots, |\mathbf{m}| - n_1^* - n_2^*.$$

Construct the $(|\mathbf{m}| - n_1^* - n_2^*) \times (|\mathbf{m}| - n_1^*)$ dimensional matrix

$$C^2 = \begin{pmatrix} \mathbf{c}_1^2 \\ \vdots \\ \mathbf{c}_{|\mathbf{m}|-n_1^*-n_2^*}^2 \end{pmatrix}.$$

Define the system $\mathbf{g}_2$ of $|\mathbf{m}| - n_1^* - n_2^*$ functions by means of

$$\mathbf{g}_2^t = C^2 \mathbf{g}_1^t = C^2 C^1 \mathbf{f}^t = (g_{2,1}, \ldots, g_{2,|\mathbf{m}|-n_1^*-n_2^*})^t.$$

The rows of C^2C^1 are of the form $\mathbf{c}_j^2C^1, j = 1, \ldots, |\mathbf{m}| - n_1^* - n_2^*$, where C^1 has rank $|\mathbf{m}| - n_1^*$ and the vectors $\mathbf{c}_k^2$ are linearly independent. Therefore, the rows of C^2C^1 are linearly independent; in particular, they are non-null. Consequently, the components of $\mathbf{g}_2$ are not polynomials because of the polynomial independence of $\mathbf{f}$ with respect to $\mathbf{m} = (1, 1, \ldots, 1)$. Thus, we can apply again Theorem 2.4.5. The proof is completed using finite induction.

Notice that the numbers $n_1, n_2, \ldots$ which so arise are greater than or equal to 1 and on each iteration their sum is less than or equal to $|\mathbf{m}|$. Therefore, in a finite number of steps, Say $N - 1$, their sum must equal $|\mathbf{m}|$. Consequently, the number of candidate system poles of $\mathbf{f}$ in some disk, counting their multiplicities, is exactly equal to $|\mathbf{m}|$ and they are precisely the zeros of $Q_{|\mathbf{m}|}$ as we wanted to prove. Summarizing, in the $N - 1$ steps we have taken we have produced N layers of candidate system poles. Each layer contains n_k candidates, $k = 1, \ldots, N$. At the same time, at each step $k, k = 1, \ldots, N-1$, we have solved a linear system of equations with $|\mathbf{m}| - n_1^* - \cdots - n_k^*$, $n_k^* \leq n_k$, linearly independent solutions. We find ourselves on the N-th layer with n_N candidates.

Let us try to eliminate these poles. As before we write the corresponding system of linear homogeneous equations as in (3.1) and we get

$$n_N = |\mathbf{m}| - n_1 - \cdots - n_{N-1} \leq |\mathbf{m}| - n_1^* - \cdots - n_{N-1}^* =: n_N^*$$

equations with n_N^* unknowns. For each candidate system pole a of multiplicity τ on the N-th layer we impose the equations

$$\int_{|\omega - a| = \delta} (\omega - a)^i \left(\sum_{k=1}^{n_N^*} c_k g_{N-1,k}(\omega) \right) d\omega = 0, \qquad i = 0, \ldots, \tau - 1. \tag{3.2}$$

where δ is sufficiently small and the $g_{N-1,k}, k = 1, \ldots, n_N^*$, are the functions associated with the linearly independent solutions produced on step $N - 1$.

Assume that $n_N < n_N^*$ (this occurs, for example, if $n_k < n_k^*$ for some $k \in \{1, \ldots, N-1\}$). In this case, there exists at least one nontrivial solution of the system. The corresponding function g can be written as a linear combination of the components of $\mathbf{f}$ and it cannot reduce to a polynomial because $\mathbf{f}$ is polynomially independent. Using

Theorem 2.4.5, we obtain that g has on the boundary of its disk of analyticity a pole which is a zero of $Q_{|\mathbf{m}|}$ but this is clearly impossible because all the zeros of $Q_{|\mathbf{m}|}$ are strictly contained in that disk. Consequently, $n_N = n_N^*$ and $n_k = n_k^*$, $k = 1, \ldots, N-1$.

What we have proved implies that in all the N homogeneous systems which we have solved (including the last one) there are no redundant equations. In turn, this implies that if in any one of those systems of equations we equate one of its equations to 1, instead of zero (see (3.1) or (3.2)), the corresponding non-homogeneous linear system of equations has a solution. Applying the definition of system pole this means that each candidate system pole is a system pole of order at least equal to its multiplicity as zero of $Q_{|\mathbf{m}|}$. But, as we saw in Lemma 3.1.5, $\mathbf{f}$ can have at most $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$; therefore, all candidate system poles are indeed system poles and their order coincides with the multiplicity of that point as a zero of $Q_{|\mathbf{m}|}$.

Thus, the proof of the inverse-type result is complete and we have $Q_{|\mathbf{m}|} = \mathcal{Q}(\mathbf{f}, \mathbf{m})$ as well.

Let us prove now that a) implies b). Except for details related to the numbers $R_\xi(\mathbf{f}, \mathbf{m})$, where ξ is a system pole of $\mathbf{f}$, the arguments are similar to those employed in [14] to prove the Graves-Morris/Saff Theorem. Despite of this, for completeness, we give the entire proof.

For each $n \geq |\mathbf{m}|$, let $q_{n,\mathbf{m}}$ be the polynomial $Q_{n,\mathbf{m}}$ normalized so that

$$\sum_{k=1}^{|\mathbf{m}|} |\lambda_{n,k}| = 1, \qquad q_{n,\mathbf{m}}(z) = \sum_{k=1}^{|\mathbf{m}|} \lambda_{n,k} z^k. \tag{3.3}$$

Due to this normalization, the polynomials $q_{n,\mathbf{m}}$ are uniformly bounded on each compact subset of $\mathbb{C}$.

Let ξ be a system pole of order τ of $\mathbf{f}$ with respect to $\mathbf{m}$. Consider a polynomial combination g_1 of type (1.9) that is analytic on a neighborhood of $\overline{D}_{|\xi|}$ except for a simple pole at $z = \xi$ and verifies that $R_1(g_1) = R_{\xi,1}(\mathbf{f}, \mathbf{m}) = r_{\xi,1}((\mathbf{f}, \mathbf{m}))$. Then, we have

$$g_1 = \sum_{k=1}^{|\mathbf{m}|} p_{k,1} f_k, \quad \deg p_{k,1} < m_k, \quad k = 1, \ldots, |\mathbf{m}|,$$

and

$$q_{n,\mathbf{m}}(z)\, h_1(z) - (z-\xi) \sum_{k=1}^{|\mathbf{m}|} p_{k,1}(z)\, P_{n,\mathbf{m},k}(z) = Az^{n+1} + \ldots,$$

where $h_1(z) = (z-\xi)\, g_1(z)$. Hence, the function

$$\frac{q_{n,\mathbf{m}}(z)\, h_1(z)}{z^{n+1}} - \frac{z-\xi}{z^{n+1}} \sum_{k=1}^{|\mathbf{m}|} p_{k,1}(z)\, P_{n,\mathbf{m},k}(z)$$

is analytic on $D_1(g_1)$. Take $0 < r < R_1(g_1)$ and set $\Gamma_r = \{z \in \mathbb{C} : |z| = r\}$. Using Cauchy's formula, we obtain

$$q_{n,\mathbf{m}}(z) h_1(z) - (z-\xi) \sum_{k=1}^{|\mathbf{m}|} p_{k,1}(z) P_{n,\mathbf{m},k}(z) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{z^{n+1}}{\omega^{n+1}} \frac{q_{n,\mathbf{m}}(\omega)\, h_1(\omega)}{\omega - z}\, d\omega,$$

for all z with $|z| < r$, since $\deg \sum_{k=1}^{|\mathbf{m}|} p_{k,1} P_{n,\mathbf{m},k} < n$. In particular, taking $z = \xi$ in the above formula, we arrive at

$$q_{n,\mathbf{m}}(\xi)\, h_1(\xi) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{\xi^{n+1}}{\omega^{n+1}} \frac{q_{n,\mathbf{m}}(\omega)\, h_1(\omega)}{\omega - \xi}\, d\omega. \tag{3.4}$$

Straightforward calculations lead to

$$\limsup_{n\to\infty} |h_1(\xi) q_{n,\mathbf{m}}(\xi)|^{1/n} \leq \frac{|\xi|}{r}.$$

Using that $h_1(\xi) \neq 0$ and making r tend to $R_1(g_1)$ we obtain

$$\limsup_{n\to\infty} |q_{n,\mathbf{m}}(\xi)|^{1/n} \leq \frac{|\xi|}{R_{\xi,1}(\mathbf{f}, \mathbf{m})} < 1.$$

Now, we employ induction. Suppose that

$$\limsup_{n\to\infty} \left|q_{n,\mathbf{m}}^{(j)}(\xi)\right|^{1/n} \leq \frac{|\xi|}{R_{\xi,j+1}(\mathbf{f}, \mathbf{m})}, \quad j = 0, 1, \ldots, s-2 \tag{3.5}$$

(recall that $R_{\xi,j+1}(\mathbf{f}, \mathbf{m}) = \min_{k=1,\ldots,j+1} R_{\xi,k}(\mathbf{f}, \mathbf{m})$), with $s \leq \tau$, and let us prove that formula (3.5) holds for $j = s-1$.

Consider a polynomial combination g_s of the type (1.9) that is analytic on a neighborhood of $\overline{D}_{|\xi|}$ except for a pole of order s at $z = \xi$ and verifies that $R_s(g_s) = r_{\xi,s}(\mathbf{f}, \mathbf{m})$. Then, we have

$$g_s = \sum_{k=1}^{|\mathbf{m}|} p_{k,s} f_k, \quad \deg p_{k,s} < m_k, \quad k = 1, \ldots, |\mathbf{m}|.$$

Set $h_s(z) = (z-\xi)^s g_s(z)$. Reasoning as in the previous case, the function

$$\frac{q_{n,\mathbf{m}}(z)\, h_s(z)}{z^{n+1}(z-\xi)^{s-1}} - \frac{z-\xi}{z^{n+1}} \sum_{k=1}^{|\mathbf{m}|} p_{k,s}(z)\, P_{n,\mathbf{m},k}(z)$$

is analytic on $D_s(g_s) \setminus \{\xi\}$. Put $P_s = \sum_{k=1}^{|\mathbf{m}|} p_{k,s} P_{n,\mathbf{m},k}$. Fix an arbitrary compact set $K \subset (D_s(g_s) \setminus \{\xi\})$. Take $\delta > 0$ sufficiently small and $0 < r < R_s(g_s)$ with $K \subset D_r$. Using Cauchy's integral formula and the residue theorem, for all $z \in K$, we have

$$\frac{q_{n,\mathbf{m}}(z) h_s(z)}{(z-\xi)^{s-1}} - (z-\xi)\, P_s(z) = I_n(z) - J_n(z), \tag{3.6}$$

where

$$I_n(z) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{z^{n+1}}{\omega^{n+1}} \frac{q_{n,\mathbf{m}}(\omega)\, h_s(\omega)}{(\omega-\xi)^{s-1}(\omega-z)}\, d\omega$$

and

$$J_n(z) = \frac{1}{2\pi i} \int_{|\omega-\xi|=\delta} \frac{z^{n+1}}{\omega^{n+1}} \frac{q_{n,\mathbf{m}}(\omega)\, h_s(\omega)}{(\omega-\xi)^{s-1}(\omega-z)}\, d\omega.$$

We have used in (3.6) that $\deg P_s < n$. The first integral I_n is estimated as in (3.4) to obtain

$$\limsup_{n\to\infty} \|I_n(z)\|_K^{1/n} \le \frac{\|z\|_K}{R_s(g_s)} = \frac{\|z\|_K}{r_{\xi,s}(\mathbf{f},\mathbf{m})}. \tag{3.7}$$

As for J_n, write

$$q_{n,\mathbf{m}}(\omega) = \sum_{j=0}^{|\mathbf{m}|} \frac{q_{n,\mathbf{m}}^{(j)}(\xi)}{j!} (\omega-\xi)^j.$$

Then

$$J_n(z) = \sum_{j=0}^{s-2} \frac{1}{2\pi i} \int_{|\omega-\xi|=\delta} \frac{z^{n+1}}{\omega^{n+1}} \frac{q_{n,\mathbf{m}}^{(j)}(\xi)}{j!(\omega-z)} \frac{h_s(\omega)}{(\omega-\xi)^{s-1-j}}\, d\omega. \tag{3.8}$$

Using the inductive hypothesis (3.5), estimating the integral in (3.8), and making ε tend to zero, we obtain

$$\limsup_{n\to\infty} \|J_n(z)\|_K^{1/n} \le \frac{\|z\|_K}{|\xi|} \frac{|\xi|}{R_{\xi,s-1}(\mathbf{f},\mathbf{m})} = \frac{\|z\|_K}{R_{\xi,s-1}(\mathbf{f},\mathbf{m})},$$

which, together with (3.7) and (3.6), gives

$$\limsup_{n\to\infty} \|q_{n,\mathbf{m}}(z) h_s(z) - (z-\xi)^s\, P_s(z)\|_K^{1/n} \le \frac{\|z\|_K}{R_{\xi,s}(\mathbf{f},\mathbf{m})}. \tag{3.9}$$

As the function inside the norm in (3.9) is analytic in $D_s(g_s)$, inequality (3.9) also holds for any compact set $K \subset D_s(g_s)$. Besides, we can differentiate $s-1$ times that function and the inequality still holds true by virtue of Cauchy's integral formula. So, taking $z = \xi$ in (3.9) for the differentiated version, we obtain

$$\limsup_{n\to\infty} \left| (q_{n,\mathbf{m}} h_s)^{(s-1)}(\xi) \right|^{1/n} \leq \frac{|\xi|}{R_{\xi,s}(\mathbf{f},\mathbf{m})}.$$

Using the Leibnitz formula for higher derivatives of a product of two functions and the induction hypothesis (3.5), we arrive at

$$\limsup_{n\to\infty} \left| q_{n,\mathbf{m}}^{(s-1)}(\xi) \right|^{1/n} \leq \frac{|\xi|}{R_{\xi,s}(\mathbf{f},\mathbf{m})},$$

since $h_s(\xi) \neq 0$. This completes the induction.

Let $\xi_1, \ldots, \xi_p$ be the distinct system poles of $\mathbf{f}$ and let τ_i be the order of ξ_i as a system pole, $i = 1, \ldots, p$. By assumption, $\tau_1 + \cdots + \tau_p = |\mathbf{m}|$. We have proved that, for $i = 1, \ldots, p$ and $j = 0, 1, \ldots, \tau_i - 1$,

$$\limsup_{n\to\infty} \left| q_{n,\mathbf{m}}^{(j)}(\xi_i) \right|^{1/n} \leq \frac{|\xi_i|}{R_{\xi_i,j+1}(\mathbf{f},\mathbf{m})} \leq \frac{|\xi_i|}{R_{\xi_i}(\mathbf{f},\mathbf{m})}. \tag{3.10}$$

Recall that $\mathcal{Q}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m})$ is the monic polynomial whose zeros are the system poles of $\mathbf{f}$. Denote by $L_{i,j}$, $i = 1, \ldots, p$; $j = 0, 1, \ldots \tau_i - 1$, the fundamental interpolating polynomials at the zeros of $\mathcal{Q}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m})$; that is, for each $i = 1, \ldots, p$ and $j = 0, 1, \ldots \tau_i - 1$, $\deg L_{i,j} \leq |\mathbf{m}| - 1$ and

$$L_{i,j}^{(\nu)}(b_\kappa) = \delta_{i\kappa}\delta_{j\nu}, \quad \kappa = 1, \ldots, p, \quad \nu = 0, 1, \ldots, \tau_i - 1.$$

Then

$$q_{n,\mathbf{m}}(z) = \lambda_{n,|\mathbf{m}|}\mathcal{Q}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m}) + \sum_{i=1}^{p}\sum_{j=0}^{\tau_i - 1} q_{n,\mathbf{m}}^{(j)}(\xi_i)\, L_{i,j}(z). \tag{3.11}$$

From (3.10) and (3.11) it follows that

$$\limsup_{n\to\infty} \| q_{n,\mathbf{m}} - \lambda_{n,|\mathbf{m}|}\mathcal{Q}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m}) \|_K^{1/n} \leq \theta < 1,$$

for any compact $K \subset \mathbb{C}$, where

$$\theta = \max\left\{ \frac{|\xi|}{R_\xi(\mathbf{f},\mathbf{m})} : \xi \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m}) \right\}. \tag{3.12}$$

As all norms in finite dimensional spaces are equivalent, we obtain

$$\limsup_{n\to\infty} \|q_{n,\mathbf{m}} - \lambda_{n,|\mathbf{m}|}\mathcal{Q}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m})\|^{1/n} \leq \theta < 1. \tag{3.13}$$

Now, necessarily we have

$$\liminf_{n\to\infty} |\lambda_{n,|\mathbf{m}|}| > 0, \tag{3.14}$$

since if there exists a subsequence $\Lambda \subset \mathbb{N}$ such that $\lim_{n\in\Lambda} \lambda_{n,|\mathbf{m}|} = 0$, then from (3.13) we have $\lim_{n\in\Lambda} \|q_{n,\mathbf{m}}\| = 0$, contradicting (3.3).

As $q_{n,\mathbf{m}} = \lambda_{n,|\mathbf{m}|}Q_{n,\mathbf{m}}$, we have proved

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}} - \mathcal{Q}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m})\|^{1/n} \leq \theta < 1, \tag{3.15}$$

where θ is given by (3.12). In particular, for $n \geq n_0$, $\deg Q_{n,\mathbf{m}} = |\mathbf{m}|$. The difference of any two non-collinear solutions Q_1 and Q_2 of Definition 1.2.1 with the same degree and equal leading coefficient produces a new solution of smaller degree, but we have proved that any solution must have degree $|\mathbf{m}|$. Hence, the polynomial $Q_{n,\mathbf{m}}$ is uniquely determined for sufficiently large n. With this we conclude the proof of the direct result.

Let us prove that the upper bound in (3.15) actually gives the exact rate of convergence to obtain (1.10). To the contrary, suppose that

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}} - \mathcal{Q}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m})\|^{1/n} = \theta' < \theta. \tag{3.16}$$

Let ζ be a system pole of $\mathbf{f}$ of order τ such that

$$\frac{|\zeta|}{R_\zeta(\mathbf{f},\mathbf{m})} = \theta = \max\left\{\frac{|\xi|}{R_\xi(\mathbf{f},\mathbf{m})} \,:\, \xi \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m})\right\}.$$

Naturally, if there is inequality in (3.16) then $R_\zeta(\mathbf{f},\mathbf{m}) < \infty$.

Choose a polynomial combination

$$g = \sum_{k=1}^{d} p_k f_k, \quad \deg p_k < m_k, \quad k = 1,\ldots,d, \tag{3.17}$$

that is analytic on a neighborhood of $\overline{D}_{|\zeta|}$ except for a pole of order $s, 1 \leq s \leq \tau$, at $z = \zeta$ with $R_s(g) = R_\zeta(\mathbf{f},\mathbf{m})$. On the boundary of $D_s(g)$ the function g must have a singularity which is not a system pole. In fact, if all the singularities were system poles we could find a different polynomial combination g_1 of type (3.17) for which $R_s(g_1) > R_s(g) = R_\zeta(\mathbf{f}, \mathbf{m})$ against our definition of $R_\zeta(\mathbf{f}, \mathbf{m})$. For short, put $\mathcal{Q}_{|\mathbf{m}|}(\mathbf{f}, \mathbf{m}) = Q_{|\mathbf{m}|}$. Consequently,

$$G(z) := Q_{|\mathbf{m}|}(z) g(z)$$

can be represented as a power series $\sum_{j=0}^{\infty} c_j z^j$ with radius of convergence $R_\zeta(\mathbf{f}, \mathbf{m})$. So

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = 1/R_\zeta(\mathbf{f}, \mathbf{m}). \tag{3.18}$$

On the other hand, by virtue of (3.17), we have

$$H_n(z) := Q_{n,\mathbf{m}}(z)\, g(z) - \sum_{k=1}^{d} p_k(z)\, P_{n,\mathbf{m},k}(z) = B_n z^{n+1} + \ldots$$

and this function is analytic at least in $D_{|\zeta|}$ with a zero of multiplicity at least $n+1$ at $z=0$. Taking $r < |\zeta|$, we obtain

$$\frac{1}{2\pi i} \int_{\Gamma_r} \frac{H_n(\omega)}{\omega^{n+1}}\, d\omega = 0.$$

Set $P_n = \sum_{k=1}^{d} p_k P_{n,\mathbf{m},k}$. Clearly

$$G(z) \equiv [Q_{|\mathbf{m}|}(z) - Q_{n,\mathbf{m}}(z)] g(z) + P_n(z) + H_n(z)$$

and, since $\deg P_n \leq n-1$, we arrive at

$$c_n = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{G(\omega)}{\omega^{n+1}}\, d\omega = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{\left[Q_{|\mathbf{m}|}(\omega) - Q_{n,\mathbf{m}}(\omega)\right] g(\omega)}{\omega^{n+1}} d\omega.$$

Taking (3.18) and (3.16) into consideration, estimating the integral, and letting r tend to $|\zeta|$, it follows that

$$\frac{1}{R_\zeta(\mathbf{f}, \mathbf{m})} = \limsup_{n\to\infty} \sqrt[n]{|c_n|} \leq \frac{\theta'}{|\zeta|} < \frac{\theta}{|\zeta|} = \frac{1}{R_\zeta(\mathbf{f}, \mathbf{m})},$$

which is absurd. We have completed the proof of Theorem 1.2.4. $\square$

3.3 On the rate of convergence

The following result is in some sense the analog of the formula displayed just after (58) in [12] written in different terms.

Corollary 3.3.1. *Assume that either a) or b) in Theorem 1.2.4 takes place. If ξ is a system pole of order τ of $\mathbf{f}$ in $D_{|\mathbf{m}|}(\mathbf{f},\mathbf{m})$, then*

$$\max_{j=0,\dots,\overline{s}} \limsup_{n\to\infty} \left|Q_{n,\mathbf{m}}^{(j)}(\xi)\right|^{1/n} = \frac{|\xi|}{R_{\xi,\overline{s}+1}(\mathbf{f},\mathbf{m})}, \quad \overline{s}=0,1,\dots,\tau-1. \tag{3.19}$$

Proof. Let ξ be as indicated. From (3.10) and (3.14) we have

$$\max_{j=0,\dots,\overline{s}} \limsup_{n\to\infty} \left|Q_{n,\mathbf{m}}^{(j)}(\xi)\right|^{1/n} \leq \frac{|\xi|}{R_{\xi,\overline{s}+1}(\mathbf{f},\mathbf{m})}, \quad \overline{s}=0,1,\dots,\tau-1.$$

Assume that there is strict inequality for some $\overline{s}\in\{0,\dots,\tau-1\}$ and fix $\overline{s}$.

Choose a polynomial combination

$$g=\sum_{k=1}^{d} p_k f_k, \quad \deg p_k < m_k, \quad k=1,\dots,d,$$

that is analytic on a neighborhood of $\overline{D}_{|\xi|}$ except for a pole of order $s\,(\leq \overline{s}+1)$ at $z=\xi$ with $R_s(g)=R_{\xi,\overline{s}+1}(\mathbf{f},\mathbf{m})$. As before, on the boundary of $D_s(g)$ the function g must have a singularity which is not a system pole. Set $\mathcal{Q}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m})=Q_{|\mathbf{m}|}$. Consequently, the function $Q_{|\mathbf{m}|}\,g$ can be represented as a power series $\sum_{j=0}^{\infty} c_j z^j$ with radius of convergence $R_{\xi,\overline{s}+1}(\mathbf{f},\mathbf{m})$. So

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = 1/R_{\xi,\overline{s}+1}(\mathbf{f},\mathbf{m}). \tag{3.20}$$

On the other hand, by virtue of (3.17), we have

$$H_n(z) := Q_{n,\mathbf{m}}(z)\,g(z) - \sum_{k=1}^{d} p_k(z)\,P_{n,\mathbf{m},k}(z) = B_n z^{n+1}+\dots$$

and this function is analytic in $D_s(g)\setminus\{\xi\}$. Take r smaller than but sufficiently close to $R_{\xi,\overline{s}+1}(\mathbf{f},\mathbf{m})$ and $\delta>0$ sufficiently small. Let $\Gamma_{\delta,r}$ be the positively oriented curve determined by $\gamma_\delta=\{\omega:|\omega-\xi|=\delta\}$ and Γ_r. We have

$$\frac{1}{2\pi i}\int_{\Gamma_{\delta,r}} \frac{H_n(\omega)}{\omega^{n+1}}\,d\omega = 0.$$

Set $P_n = \sum_{k=1}^d p_k P_{n,\mathbf{m},k}$ and $h(\omega) = (\omega - \xi)^s g(\omega)$. Obviously,

$$Q_{|\mathbf{m}|}\, g \equiv (Q_{|\mathbf{m}|} - Q_{n,\mathbf{m}})\, g + P_n + H_n$$

and, since $\deg P_n \leq n-1$, we obtain

$$c_n = \frac{1}{2\pi i}\int_{\Gamma_{\delta,r}} \frac{Q_{|\mathbf{m}|}(\omega)g(\omega)}{\omega^{n+1}}\,d\omega = \frac{1}{2\pi i}\int_{\Gamma_{\delta,r}} \frac{[Q_{|\mathbf{m}|} - Q_{n,\mathbf{m}}](\omega)h(\omega)}{(\omega-\xi)^s\omega^{n+1}}d\omega$$

$$= \frac{1}{2\pi i}\int_{\Gamma_r} \frac{[Q_{|\mathbf{m}|} - Q_{n,\mathbf{m}}](\omega)h(\omega)}{(\omega-\xi)^s\omega^{n+1}}d\omega - \sum_{\nu=0}^{|\mathbf{m}|}\frac{1}{2\pi i}\int_{\gamma_\delta} \frac{[Q^{(\nu)}_{|\mathbf{m}|} - Q^{(\nu)}_{n,\mathbf{m}}](\xi)h(\omega)}{\nu!(\omega-\xi)^{s-\nu}\omega^{n+1}}d\omega$$

$$= \frac{1}{2\pi i}\int_{\Gamma_r} \frac{[Q_{|\mathbf{m}|} - Q_{n,\mathbf{m}}](\omega)h(\omega)}{(\omega-\xi)^s\omega^{n+1}}d\omega - \sum_{\nu=0}^{s-1}\frac{1}{2\pi i}\int_{\gamma_\delta} \frac{Q^{(\nu)}_{n,\mathbf{m}}(\xi)h(\omega)}{\nu!(\omega-\xi)^{s-\nu}\omega^{n+1}}d\omega.$$

Estimating these integrals, using (1.10) and the temporary assumption that

$$\max_{j=0,\ldots,\overline{s}}\limsup_{n\to\infty}\left|Q^{(j)}_{n,\mathbf{m}}(\xi)\right|^{1/n} = \frac{|\xi|}{\kappa} < \frac{|\xi|}{R_{\xi,\overline{s}+1}(\mathbf{f},\mathbf{m})},$$

we obtain

$$\limsup_{n\to\infty}|c_n|^{1/n} \leq \max\left\{\frac{1}{\kappa}, \frac{\theta}{R_{\xi,\overline{s}+1}(\mathbf{f},\mathbf{m})}\right\} < \frac{1}{R_{\xi,\overline{s}+1}(\mathbf{f},\mathbf{m})},$$

which contradicts (3.20). Hence, (3.19) takes place. □

Now, we are ready to give the analog of (1.3) for simultaneous approximation. We need to introduce some notation. Fix $k \in \{1,\ldots,d\}$. Let $D_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m})$ be the largest disk centered at $z=0$ in which all the poles of f_k are system poles of $\mathbf{f}$ with respect to $\mathbf{m}$, their order as poles of f_k does not exceed their order as system poles, and f_k has no other singularity. By $R_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m})$ we denote the radius of this disk. Let $\xi_1,\ldots,\xi_N$ be the poles of f_k in $D_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m})$. For each $j=1,\ldots,N$, let $\widetilde{\tau}_j$ be the order of ξ_j as a pole of f_k and τ_j its order as a system pole. By assumption $\widetilde{\tau}_j \leq \tau_j$. Set

$$R^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m}) = \min\left\{R_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m}), \min_{j=1,\ldots,N} R_{\xi_j,\widetilde{\tau}_j}(\mathbf{f},\mathbf{m})\right\}$$

and let $D^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m})$ be the disk centered at $z=0$ with this radius.

Recall that $h(B)$ stands for the 1-dimensional Hausdorff content of the set B.

Definition 3.3.2. *We say that a compact set $K \subset \mathbb{C}$ is h-regular if for each $z_0 \in K$ and for each $\delta > 0$, we have $h\{z \in K : |z - z_0| < \delta\} > 0$.*

In the following, $R^*_{|\mathbf{m}|}(f_k)$ represents the radius of the largest disk centered at the origin in which $h\text{-}\lim_{n\to\infty} R_{n,\mathbf{m},k} = f_k$ taking $m = |\mathbf{m}|$ and $m^* = m_k$. Set

$$D^*_{|\mathbf{m}|}(f_k) = \left\{z : |z| < R^*_{|\mathbf{m}|}(f_k)\right\}, \quad k = 1, \dots, d.$$

Theorem 3.3.3. *Let $\mathbf{f}$ be a system of formal Taylor expansions as in* (1.4) *and fix a multi-index $\mathbf{m} \in \mathbb{Z}^d_+ \setminus \{\mathbf{0}\}$. Suppose that either* a) *or* b) *in Theorem 1.2.4 takes place. Then,*

$$\limsup_{n\to\infty} \|f_k - R_{n,\mathbf{m},k}\|_K^{1/n} \le \frac{\|z\|_K}{R^*_{|\mathbf{m}|}(f_k)}, \quad k = 1, \dots, d, \tag{3.21}$$

where K is any compact subset of $D^*_{|\mathbf{m}|}(f_k) \setminus \mathcal{P}_{|\mathbf{m}|}(\mathbf{f}, \mathbf{m})$. If, additionally, K is h-regular, then we have equality in (3.21)*. Moreover,*

$$R^*_{|\mathbf{m}|}(f_k) = R^*_{|\mathbf{m}|,k}(\mathbf{f}, \mathbf{m}), \qquad k = 1, \dots, d. \tag{3.22}$$

Proof. Let us fix $k \in \{1, \dots, d\}$ and maintain the notation introduced above. In particular, $\xi_1, \dots, \xi_N$ denote the poles of f_k in $D_{|\mathbf{m}|,k}(\mathbf{f}, \mathbf{m})$ and $\widetilde{\tau}_1, \dots, \widetilde{\tau}_N$ denote their orders, respectively. Let K be a compact subset contained in $D^*_{|\mathbf{m}|,k}(\mathbf{f}, \mathbf{m}) \setminus \mathcal{P}_{|\mathbf{m}|}(\mathbf{f}, \mathbf{m})$. Take r smaller than but sufficiently close to $R^*_{|\mathbf{m}|,k}(\mathbf{f}, \mathbf{m})$, and $\delta > 0$ sufficiently small so that K is in the region bounded by Γ_r and the circles $\gamma_{\delta,j} = \{z : |z - \xi_j| = \delta\}, j = 1, \dots, N$. We assume that the circles $\gamma_{\delta,j}$ are also contained in $D_{|\mathbf{m}|,k}(\mathbf{f}, \mathbf{m})$ and do not intersect. Let $\Gamma_{\delta,r}$ be the curve with positive orientation determined by Γ_r and those circles. On account of Definition 1.2.1, using Cauchy's integral formula we have

$$(Q_{n,\mathbf{m}} f_k - P_{n,\mathbf{m},k})(z) = \frac{1}{2\pi i} \int_{\Gamma_{\delta,r}} \frac{z^{n+1}}{\omega^{n+1}} \frac{(Q_{n,\mathbf{m}} f_k)(\omega)}{\omega - z} d\omega =$$

$$\frac{1}{2\pi i} \int_{\Gamma_r} \frac{z^{n+1}}{\omega^{n+1}} \frac{(Q_{n,\mathbf{m}} f_k)(\omega)}{\omega - z} d\omega - \sum_{j=1}^N \frac{1}{2\pi i} \int_{\gamma_{\delta,j}} \frac{z^{n+1}}{\omega^{n+1}} \frac{(Q_{n,\mathbf{m}} f_k)(\omega)}{\omega - z} d\omega =$$

$$\frac{1}{2\pi i} \int_{\Gamma_r} \frac{z^{n+1}}{\omega^{n+1}} \frac{(Q_{n,\mathbf{m}} f_k)(\omega)}{\omega - z} d\omega - \sum_{j=1}^N \sum_{\nu=0}^{\widetilde{\tau}_j - 1} \frac{1}{2\pi i} \int_{\gamma_{\delta,j}} \frac{z^{n+1}}{\omega^{n+1}} \frac{Q^{\nu}_{n,\mathbf{m}}(\xi_j)(\omega - \xi_j)^{\nu} f_k(\omega)}{\nu!(\omega - z)} d\omega.$$

Now, estimating each one of these integrals separately using (3.19) and $\lim_{n\to\infty} Q_{n,\mathbf{m}} = Q_{|\mathbf{m}|}$, we obtain

$$\limsup_{n\to\infty} \|f_k - R_{n,\mathbf{m},k}\|_K^{1/n} \leq \frac{\|z\|_K}{R^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m})}. \tag{3.23}$$

This last relation implies that h-$\lim_{n\to\infty} R_{n,\mathbf{m},k} = f_k$ inside $D^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m})$. Since $R^*_{|\mathbf{m}|}(f_k)$ is the largest disk inside of which such convergence takes place it readily follows that $R^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m}) \leq R^*_{|\mathbf{m}|}(f_k)$. Should $D^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m})$ contain on its boundary some singularity which is not a system pole then necessarily $R^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m}) = R^*_{|\mathbf{m}|}(f_k)$ because h-convergence implies that all singularities inside the region of convergence must be zeros of $Q_{|\mathbf{m}|}$, but the zeros of this polynomial are all system poles as we proved in Theorem 1.2.4. Assume that $R^*_{|\mathbf{m}|}(f_k) > R^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m})$. Then, we have $R^*_{|\mathbf{m}|}(f_k) > \min_{j=1,\ldots,N} R_{\xi_j,\widetilde{\tau}_j}(\mathbf{f},\mathbf{m})$. From the proof of Theorem 2.3.5 (see (2.19)) we know that for each pole ξ of order $\widetilde{\tau}$ of f_k inside $D^*_{|\mathbf{m}|}(f_k)$

$$\limsup_{n\to\infty} \left|Q^{(j)}_{n,\mathbf{m}}(\xi)\right|^{1/n} \leq \frac{|\xi|}{R^*_{|\mathbf{m}|}(f_k)}, \quad j = 0, 1, \ldots, \widetilde{\tau} - 1.$$

This contradicts (3.19). Consequently $R^*_{|\mathbf{m}|}(f_k) = R^*_{|\mathbf{m}|,k}(\mathbf{f},\mathbf{m})$ as claimed. Due to (3.23), we have also proved (3.21).

Suppose now that the compact set $K \subset D^*_{|\mathbf{m}|}(f_k) \setminus \mathcal{P}_{\mathbf{m},k}(\mathbf{f})$ is h-regular. Let us consider the constants $A_{n,\mathbf{m},k}$ and the polynomials $q^*_{n,m-m^*,k}$ defined according to Lemma 2.3.2 for the incomplete Padé approximant $R_{n,\mathbf{m},k}$, where $m = |\mathbf{m}|$ and $m^* = m_k$. Denote the denominator of $R_{n,\mathbf{m},k}$, after canceling out all common factors with the numerator, by $Q_{n,\mathbf{m},k}$. Put $J'_0 = \cap_{\varepsilon>0} J'_\varepsilon$ and take $z_0 \in K$ such that $\|z\|_K = |z_0| > 0$. As J'_0 is a set of h-content zero and the compact set K is h-regular, there exists a sequence $\{z_j\}_{j\in\mathbb{N}} \subset K \setminus J'_0$ verifying $\lim_{j\to\infty} z_j = z_0$. We may assume that $|z_j| > 0$ for all $j \in \mathbb{N}$.

From Lemma 2.3.2, it follows that

$$|A_{n,\mathbf{m},k}| = \frac{|(Q_{n+1,\mathbf{m},k}\, Q_{n,\mathbf{m},k})(z_j)|\, |R_{n+1,\mathbf{m},k}(z_j) - R_{n,\mathbf{m},k}(z_j)|}{|z_j|^{n+1-\lambda_n-\lambda_{n+1}} |q^*_{n,m-m^*,k}(z_j)|}.$$

We may write

$$|R_{n+1,\mathbf{m},k}(z_j) - R_{n,\mathbf{m},k}(z_j)| \leq \|f_k - R_{n+1,\mathbf{m},k}\|_K + \|f_k - R_{n,\mathbf{m},k}\|_K.$$

So, taking into account the formulas (2.3) and (2.5), we arrive at

$$\frac{1}{R^*_{|\mathbf{m}|}(f_k)} = \limsup_{n\to\infty} |A_{n,\mathbf{m},k}|^{1/n} \leq \frac{1}{|z_j|} \limsup_{n\to\infty} \|f_k - R_{n,\mathbf{m},k}\|_K^{1/n}.$$

Taking limits in the above expression as j tends to infinity, it follows that the inequality (3.21) is actually an equality when K is a h-regular compact set, as we wanted to prove. □

As a direct consequence of Theorems 1.2.4 and 3.3.3 we obtain improved versions of the Graves-Morris/Saff Theorem and Theorem 4.4 in [5] which we state as corollaries.

Corollary 3.3.4. *Assume that $R_0(\mathbf{f}) > \mathbf{0}$ and $\mathbf{f}$ is polewise independent with respect to $\mathbf{m}$ in $D_{|\mathbf{m}|}(\mathbf{f})$. Then, $\mathbf{f}$ has $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ which coincide in order with the poles of $\mathbf{f}$ in $D_{|\mathbf{m}|}(\mathbf{f})$. Consequently,*

$$\limsup_{n\to\infty} \|\mathcal{Q}_{|\mathbf{m}|} - Q_{n,\mathbf{m}}\|^{1/n} = \max\left\{\frac{\zeta}{R_\zeta(\mathbf{f},\mathbf{m})} : \zeta \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m})\right\}.$$

and

$$\limsup_{n\to\infty} \|f_k - R_{n,\mathbf{m},k}\|_K^{1/n} \leq \frac{\|z\|_K}{R^*_{|\mathbf{m}|}(f_k)}, \qquad k = 1, \ldots, d,$$

where K is any compact subset of $D^*_{|\mathbf{m}|}(f_k) \setminus \mathcal{P}_{|\mathbf{m}|}(\mathbf{f},\mathbf{m})$, with equality if K is h-regular.

Proof. Let us show that polewise independence implies that $\mathbf{f}$ has exactly $|\mathbf{m}|$ system poles which coincide in order with the poles of $\mathbf{f}$ in $D_{|\mathbf{m}|}(\mathbf{f})$. The rest of the statements follow immediately from Theorems 1.2.4 and 3.3.3.

In fact, let $z_1, \ldots, z_N$ be the poles of $\mathbf{f}$ in $D_{|\mathbf{m}|}(\mathbf{f})$ and $\tau_1, \ldots, \tau_N$ their orders, respectively. Obviously, $\tau_1 + \cdots + \tau_N = |\mathbf{m}|$, otherwise it is easy to find polynomials $p_1, \ldots, p_d, \deg p_i \leq m_i - 1$, such that $\sum_{i=1}^d p_i f_i \in \mathcal{H}(D_{|\mathbf{m}|}(\mathbf{f}))$. The problem reduces to solving a linear homogeneous system of equations with more unknowns (the coefficients of the polynomials) than equations (see below).

Let us prove that $z_k, k = 1, \ldots, N$, is a system pole of order τ_k of $\mathbf{f}$ with respect to $\mathbf{m}$. Take $s \in \{1, \ldots, \tau_k\}$ and find polynomials $p_1, \ldots, p_d$ such that for each $j \neq k$,

$$\int_{|z-z_j|=\varepsilon} (z - z_j)^\ell \sum_{i=1}^d p_i(z) f_i(z) dz = 0, \qquad \ell = 0, \ldots, \tau_j - 1,$$

and

$$\int_{|z-z_k|=\varepsilon} (z-z_k)^\ell \sum_{i=1}^d p_i(z) f_i(z) dz = 0, \quad \ell = 0, \dots, \tau_k - 1, \quad \ell \neq s-1,$$

where $\varepsilon > 0$ is sufficiently small. These integrals give us a homogeneous linear system of $|\mathbf{m}| - \mathbf{1}$ equations on $|\mathbf{m}|$ unknowns (the coefficients of the polynomials p_i). We have

$$\int_{|z-z_k|=\varepsilon} (z-z_k)^{s-1} \sum_{i=1}^d p_i(z) f_i(z) dz \neq 0,$$

since if it were equal to zero $\sum_{i=1}^d p_i f_i$ would be holomorphic in $D_{|\mathbf{m}|}(\mathbf{f})$ against our assumption of polewise independence. We have constructed for each $s = 1, \dots, \tau_k$ a function which is holomorphic in $\overline{D}_{|z_k|}$ except for a pole of exact order s at z_k, and there can be no polynomial combination with those properties for $s > \tau_k$ because the corresponding system of linear equations to be solved can only have the trivial solution. □

The example in Section 2.5.3 shows that the bounds we give in Corollary 3.3.4 improve those given by Graves-Morris and Saff.

Recall that

$$\mathbf{D}^*_{\mathbf{m}}(\mathbf{f}) = \left(D^*_{|\mathbf{m}|}(f_1), \dots, D^*_{|\mathbf{m}|}(f_d) \right).$$

Corollary 3.3.5. *Assume that $R_0(\mathbf{f}) > \mathbf{0}$ and $\mathbf{f}$ has exactly $|\mathbf{m}|$ poles in $\mathbf{D}^*_{\mathbf{m}}(\mathbf{f})$. Then, $\mathbf{f}$ has $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ which coincide in order with the poles of $\mathbf{f}$ in $\mathbf{D}^*_{\mathbf{m}}(\mathbf{f})$. Consequently,*

$$\limsup_{n\to\infty} \| \mathcal{Q}_{|\mathbf{m}|} - Q_{n,\mathbf{m}} \|^{1/n} = \max \left\{ \frac{\zeta}{R_\zeta(\mathbf{f}, \mathbf{m})} : \zeta \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{f}, \mathbf{m}) \right\}.$$

and

$$\limsup_{n\to\infty} \| f_k - R_{n,\mathbf{m},k} \|_K^{1/n} \leq \frac{\|z\|_K}{R^*_{|\mathbf{m}|}(f_k)}, \quad k = 1, \dots, d,$$

where K is any compact subset of $D^*_{|\mathbf{m}|}(f_k) \setminus \mathcal{P}_{|\mathbf{m}|}(\mathbf{f}, \mathbf{m})$, with equality if K is h-regular.

Proof. Indeed, according to Theorem 2.3.4 each pole of $\mathbf{f}$ in $\mathbf{D}^*_{\mathbf{m}}(\mathbf{f})$ attracts with geometric rate at least as many zeros of $Q_{n,\mathbf{m}}$ as its order. Since there are $|\mathbf{m}|$ of such poles (counting multiplicities) it readily follows that there exists a polynomial $Q_{|\mathbf{m}|}$ of degree $|\mathbf{m}|, Q_{|\mathbf{m}|}(0) \neq 0$, such that

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}} - Q_{|\mathbf{m}|}\|^{1/n} = \theta < 1$$

and the conclusions follow at once. □

The hypothesis of Corollaries 3.3.4 and 3.3.5 may be easier to verify than those of Theorem 1.2.4. A variant of Corollary 3.3.5 would be to assume that $\mathbf{f}$ has $|\mathbf{m}|$ poles in

$$(D_{m_1}(f_1), \ldots, D_{m_d}(f_d))$$

with the same conclusion (and proof).

We wish to emphasize that formula (3.22) in Theorem 3.3.3 offers a characterization of the values $R^*_{|\mathbf{m}|}(f_k)$ in terms of the analytic properties of the functions in the system instead of the coefficients of their Taylor expansions. An open question is to obtain an analogous characterization for these values when the assumptions of Theorem 3.3.3 do not take place; that is when the number of system poles is less than $|\mathbf{m}|$.

It would be interesting to study inverse problems for row sequences of Hermite-Padé approximation, when only the limit behavior of some of the zeros of the polynomials $Q_{n,\mathbf{m}}$ is known, in the spirit of the conjectures proposed by A.A. Gonchar in [12].

Chapter 4

Simultaneous Fourier-Padé approximation on $\mathbb{D}$

4.1 Statement of the problem

Let $\mathbb{T} = \{z : |z| = 1\}$ denote the unit circle and $\mathbb{D} = \{z : |z| < 1\}$ the open unit disk. By σ we denote a finite positive Borel measure whose support is contained in $\mathbb{T}$ and $\sigma' > 0$ a.e. on $\mathbb{T}$. Let $\{\varphi_n\}$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients. That is,

$$\frac{1}{2\pi}\int \varphi_j(z)\overline{\varphi_k(z)}d\sigma(z) = \delta_{j,k}, \qquad j,k \in \mathbb{Z}_+,$$

where as usual $\delta_{j,k} = 0, j \neq k$ and $\delta_{k,k} = 1$. By $\mathcal{H}(\overline{\mathbb{D}})$ we denote the space of functions which are analytic on some neighborhood of $\overline{\mathbb{D}}$.

Definition 4.1.1. *Let* $\mathbf{f} = (f_1, \ldots, f_d)$ *where* $f_k \in \mathcal{H}(\overline{\mathbb{D}}), k = 1, \ldots, d$. *Fix a multi-index* $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$ *where* $\mathbf{0}$ *denotes the zero vector in* $\mathbb{Z}_+^d$. *Set* $|\mathbf{m}| = m_1 + \cdots + m_d$. *Then, for each* $n \geq \max\{m_1, \ldots, m_d\}$, *there exist polynomials* $Q_{n,\mathbf{m}}, P_{n,\mathbf{m},j}, j = 1, \ldots, d$, *such that*

a.1) $\deg P_{n,\mathbf{m},j} \leq n - m_j, j = 1, \ldots, d, \quad \deg Q_{n,\mathbf{m}} \leq |\mathbf{m}|, \quad Q_{n,\mathbf{m}} \not\equiv 0,$

a.2) $[Q_{n,\mathbf{m}}f_j - P_{n,\mathbf{m},j}](z) = A_{n,n+1}^{(j)}\varphi_{n+1}(z) + A_{n,n+2}^{(j)}\varphi_{n+2}(z) + \cdots .$

We call the vector rational function $\mathbf{R}_{n,\mathbf{m}} = (P_{n,\mathbf{m},1}/Q_{n,\mathbf{m}}, \ldots, P_{n,\mathbf{m},d}/Q_{n,\mathbf{m}})$ *an* $(n, \mathbf{m})$ *simultaneous Fourier-Padé approximation of* $\mathbf{f}$ *(on the unit circle).*

For any pair $(n, \mathbf{m})$ there is at least one $R_{n,\mathbf{m}}$ but, in general, it is not uniquely determined. We assume that given $(n, \mathbf{m})$, one solution is taken. We will normalize the common denominator in terms of its zeros $z_{n,k}$ as follows

$$Q_{n,\mathbf{m}}(z) = \prod_{|z_{n,k}| \leq 1} (z - z_{n,k}) \prod_{|z_{n,k}| > 1} \left(1 - \frac{z}{z_{n,k}}\right). \tag{4.1}$$

Notice that here we have taken a normalization different from the one adopted in Section 1.3.

Obviously, $Q_{n,\mathbf{m}} f_j - P_{n,\mathbf{m},j} \in \mathcal{H}(\overline{\mathbb{D}})$. Due to the asymptotic properties satisfied by $\{\varphi_n\}$ it is easy to verify that the Fourier expansion of this function converges uniformly on compact subsets of a neighborhood of $\overline{\mathbb{D}}$. (This will be justified later.)

We will prove, under appropriate assumptions on $\mathbf{f}$, that

$$\lim_{n \to \infty} \mathbf{R}_{n,\mathbf{m}} = \mathbf{f}$$

uniformly on compact subsets of the largest disk centered at $z = 0$ containing at most $|\mathbf{m}|$ poles of $\mathbf{f}$ and that the zeros of the common denominator of the approximating rational functions point out the location and order of the poles of $\mathbf{f}$ in that disk. We will use the concept of polewise independence due to Graves-Morris/Saff presented in Definition 1.2.2 of Chapter 1.

Polewise independence of $\mathbf{f}$ with respect to $\mathbf{m}$ in D implies that $\mathbf{f}$ has at least $|\mathbf{m}|$ poles in $\mathbf{D} = (D, \ldots, D)$ counting multiplicities, see [14, Lemma 1].

Given $\mathbf{f} \in \mathcal{H}(\overline{\mathbb{D}})$ (that is, each component of $\mathbf{f}$ is analytic in a neighborhood of $\overline{\mathbb{D}}$) let $D_{|\mathbf{m}|}(\mathbf{f})$ denote the largest disk centered at the origin inside of which $\mathbf{f}$ has at most $|\mathbf{m}|$ poles and $R_{|\mathbf{m}|}(\mathbf{f})$ denotes its radius.

Let $Q_{|\mathbf{m}|}(\mathbf{f})$ be the polynomial whose zeros are the poles of $\mathbf{f}$ in $D_{|\mathbf{m}|}(\mathbf{f})$ counting multiplicities normalized as in (4.1). This set of poles is denoted by $\mathcal{P}_{|\mathbf{m}|}(\mathbf{f})$. Taking account of the normalization, Theorem 1.3.2 adopts the form

Theorem 4.1.2. *Assume that* $\mathbf{f} \in \mathcal{H}(\overline{\mathbb{D}})$ *and* $\sigma' > 0$ *a.e. on* $\mathbb{T}$. *Fix a multi-index* $\mathbf{m} \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$ *and suppose that* $\mathbf{f}$ *is polewise independent with respect to* $\mathbf{m}$ *in* $D_{|\mathbf{m}|}(\mathbf{f})$, *Then,* $\mathbf{R}_{n,\mathbf{m}}$ *is uniquely determined for all sufficiently large* n. *For any compact subset* K *of* $D_{|\mathbf{m}|}(\mathbf{f}) \setminus \mathcal{P}_{|\mathbf{m}|}(\mathbf{f})$

$$\limsup_{n\to\infty} \|f_i - R_{n,\mathbf{m},i}\|_K^{1/n} \leq \frac{\|z\|_K}{R_{|\mathbf{m}|}(\mathbf{f})}, \quad i = 1, \ldots, d, \tag{4.2}$$

where $\|z\|_K$ *is replaced by* 1 *when* $K \subset \overline{\mathbb{D}}$. *Additionally,*

$$\limsup_{n\to\infty} \|Q_{|\mathbf{m}|}(\mathbf{f}) - Q_{n,\mathbf{m}}\|^{1/n} \leq \frac{\max\{|\zeta| : \zeta \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{f})\}}{R_{|\mathbf{m}|}(\mathbf{f})}. \tag{4.3}$$

In the space of polynomials of degree less than or equal to $|\mathbf{m}|$ all norms are equivalent so in (4.3) any norm can be taken.

Sections 2 and 3 of this chapter are dedicated to the study of incomplete Fourier-Padé approximation and it is shown that they converge in h-content. Based on this, in Section 4 we prove Theorem 4.1.2.

4.2 Incomplete Fourier-Padé approximants on $\mathbb{D}$

Definition 4.2.1. *Let* $f \in \mathcal{H}(\overline{\mathbb{D}})$. *Fix* $m^* \leq m$. *Let* $n \geq m$. *We say that the rational function* $R_{n,m} = P_{n,m}/Q_{n,m}$ *is an incomplete Fourier-Padé approximation of type* (n, m, m^*) *(on the unit circle) corresponding to* f *(and* σ*) if* $P_{n,m}$ *and* $Q_{n,m}$ *are polynomials that verify*

c.1) $\deg P_{n,m} \leq n - m^*, \quad \deg Q_{n,m} \leq m, \quad Q_{n,m} \not\equiv 0,$

c.2) $[Q_{n,m} f - P_{n,m}](z) = a_{n,n+1}\varphi_{n+1}(z) + a_{n,n+2}\varphi_{n+2} + \cdots$.

The polynomials $P_{n,m}, Q_{n,m}$ depend on m^* and the numbers $a_{n,k}$ depend on (m, m^*) but we do not indicate it to reduce the notation.

From Definitions 4.1.1 and 4.2.1 it follows that $R_{n,\mathbf{m},k} = P_{n,\mathbf{m},k}/Q_{n,\mathbf{m}}, k = 1, \ldots, d$, is an incomplete Fourier Padé approximation of type $(n, |\mathbf{m}|, m_k)$ with respect to f_k.

Given $n \geq m \geq m^*$, $R_{n,m}$ is not unique so we choose one candidate. The denominator $Q_{n,m}$ will be normalized as in (4.1).

In the next section we will prove that $h - \lim_{n\to\infty} R_{n,m} = f$ in the largest disk $D_{m^*}(f)$ inside of which f has at most m^* poles. Having this in mind, we introduce some auxiliary sets.

Take an arbitrary $\varepsilon > 0$ and define the open set J_ε as we did in Chapter 2. For $n \geq m$, let $J_{n,\varepsilon}$ denote the $\varepsilon/6mn^2$-neighborhood of the set $\mathcal{P}_{n,m} = \{\zeta_{n,1}, \ldots, \zeta_{n,m_n}\}$ of finite zeros of $Q_{n,m}$ and let $J_{m-1,\varepsilon}$ denote the $\varepsilon/6m$-neighborhood of the set of poles of f in $D_m(f)$. Set $J_\varepsilon = \cup_{n\geq m-1} J_{n,\varepsilon}$. For any set $B \subset \mathbb{C}$ we put $B(\varepsilon) := B \setminus J_\varepsilon$.

Due to the normalization (4.1), for any compact set K of $\mathbb{C}$ and for every $\varepsilon > 0$, there exist positive constants C_1, C_2, independent of n, such that

$$\|Q_{n,m}\|_K < C_1, \qquad \min_{z\in K(\varepsilon)} |Q_{n,m}(z)| > C_2 n^{-2m}, \tag{4.4}$$

where the second inequality is meaningful when $K(\varepsilon)$ is a non-empty set.

In the sequel, C_k will be used to denote positive constants, generally different, that are independent of n but may depend on all the other parameters involved in each formula where they appear.

Given σ and φ_n denote

$$\psi_n(z) = \frac{1}{2\pi}\int \frac{\overline{\varphi_n(\zeta)}d\sigma(\zeta)}{z-\zeta}$$

If $\sigma' > 0$ a.e. on $\mathbb{T}$, E.A. Rakhmanov's theorem (see, for example, [23] and [35, Chapter 9]) implies that

$$\lim_{n\to\infty} \frac{\varphi_{n+k}(z)}{\varphi_n(z)} = z^k, \qquad k \in \mathbb{Z}, \tag{4.5}$$

$$\lim_{n\to\infty} \frac{\psi_{n+k}(z)}{\psi_n(z)} = \frac{1}{z^k}, \qquad k \in \mathbb{Z}, \tag{4.6}$$

uniformly on each compact subset of $\mathbb{C} \setminus \overline{\mathbb{D}}$. In turn, these relations easily give that

$$\lim_{n\to\infty} |\varphi_n(z)|^{1/n} = |z|, \tag{4.7}$$

$$\lim_{n\to\infty} |\psi_n(z)|^{1/n} = |z|^{-1}, \tag{4.8}$$

uniformly on each compact subset of $\mathbb{C}\setminus\overline{\mathbb{D}}$. Formulas (4.5)-(4.8) are basic in our proofs.

Let $g \in \mathcal{H}(\overline{\mathbb{D}})$. Take $r > 1$ so that $g \in \mathcal{H}(\{z : |z| \leq r\})$. Set $T_r = \{z : |z| = r\}$. Using Cauchy's integral formula and Fubini's theorem, we obtain

$$\langle g, \varphi_k \rangle = \frac{1}{2\pi} \int g(\zeta)\overline{\varphi_k(\zeta)}d\sigma(\zeta) = \frac{1}{2\pi} \int \frac{1}{2\pi i} \int_{T_r} \frac{g(z)dz}{z-\zeta}\overline{\varphi_k(\zeta)}d\sigma(\zeta) =$$

$$\frac{1}{2\pi i} \int_{T_r} g(z)\psi_k(z)dz. \tag{4.9}$$

Using (4.7)-(4.9), it readily follows that

$$\sum_{k=0}^{\infty} \langle g, \varphi_k \rangle \varphi_k(z)$$

converges uniformly on each compact subset of $D_r = \{z : |z| < r\}$ and the limit must be an analytic function in D_r. On the other hand, this is the Fourier series of g with respect to the orthonormal system $\{\varphi_n\}$; consequently, its norm-2 limit on $\mathbb{T}$ is g. By the principle of analytic continuation, the series converges uniformly to g on compact subsets of D_r. This justifies that the right hand sides of a.2) in Definition 4.1.1 and c.2) in Definition 4.2.1 converge uniformly to the corresponding left-hand on each compact subset of a neighborhood of $\overline{\mathbb{D}}$.

4.3 Convergence of incomplete Fourier-Padé approximants on $\mathbb{D}$

Theorem 4.3.1. *Let $f \in \mathcal{H}(\overline{\mathbb{D}})$ and $\sigma' > 0$ a.e. on $\mathbb{T}$. Fix m and m^* nonnegative integers, $m \geq m^*$. For each $n \geq m$, let $R_{n,m}$ be an incomplete Padé approximant of type (n, m, m^*) for f. Then, for each $\varepsilon > 0$ and every compact subset K of $D_{m^*}(f)$*

$$\limsup_{n\to\infty} \|f - R_{n,m}\|_{K(\varepsilon)}^{1/n} \leq \frac{\|z\|_K}{R_{m^*}(f)}, \tag{4.10}$$

where $\|z\|_K$ should be replaced by 1 when $K \subset \overline{\mathbb{D}}$. In particular,

$$h\text{-}\lim_{n\to\infty} R_{n,m} = f \ \ in \ \ D_{m^*}(f).$$

Finally, for each pole z_j of f in $D_{m^*}(f)$, and every $\varepsilon > 0$, there exists n_0 such that for all $n \geq n_0$ the polynomials $Q_{n,m}$ have at least τ_j zeros in the disk $\{z : |z - z_j| < \varepsilon\}$, where τ_j denotes the order of the pole z_j.

Proof. Let $z_1, \ldots, z_N$ be the distinct poles of f in $D_{m^*}(f)$ and $\tau_1, \ldots, \tau_N$ their orders, respectively. Consequently, $\sum_{k=1}^{N} \tau_k = \widetilde{m} \leq m^*$. Put

$$w(z) = \prod_{k=1}^{N} (z - z_k)^{\tau_k}.$$

Using c.2) we obtain

$$(w_m Q_{n,m} f - w_m P_{n,m})(z) = \sum_{k \geq n+1} a_{n,k} w_m(z) \varphi_k(z) = \sum_{\nu \geq 0} b_{n,\nu} \varphi_\nu(z). \tag{4.11}$$

Notice that $w_m Q_{n,m} f - w_m P_{n,m} \in H(D_{m^*}(f))$ and $\deg w_m P_{n,m} \leq n$.

We have two ways of calculating the Fourier coefficients $b_{n,\nu}$. On one hand, for each $\nu \geq 0$

$$b_{n,\nu} = \sum_{k=n+1}^{\infty} a_{n,k} \langle w_m \varphi_k, \varphi_\nu \rangle = \sum_{k=n+1}^{\infty} \frac{a_{n,k}}{2\pi} \int w_m(z) \varphi_k(z) \overline{\varphi_\nu(z)} d\sigma(z). \tag{4.12}$$

On the other hand

$$b_{n,\nu} = \begin{cases} \langle w_m Q_{n,m} f - w_m P_{n,m^*}, \varphi_\nu \rangle, & \nu = 0, \cdots, n, \\ \langle w_m Q_{n,m} f, \varphi_\nu \rangle, & \nu \geq n+1. \end{cases}$$

Since $w_m Q_{n,m} f$ is analytic in $D_{m^*}(f)$ taking $1 < R < R_{m^*}(f)$, we obtain (see (4.9))

$$b_{n,\nu} = \frac{1}{2\pi i} \int_{T_R} (w_m Q_{n,m} f - w_m P_{n,m})(z) \psi_\nu(z) dz, \qquad \nu = 0, ..., n,$$

and, similarly,

$$b_{n,\nu} = \frac{1}{2\pi i} \int_{T_R} (w_m Q_{n,m} f)(z) \psi_\nu(z) dz. \qquad \nu \geq n+1. \tag{4.13}$$

We will show that $\sum_{\nu \geq 0} b_{n,\nu} \varphi_\nu(z)$ converges uniformly to zero on each compact subset of $D_{m^*}(f)$ as n tends to ∞ with geometric rate. To this end, due to the maximum principle, without loss of generality we assume that the compact sets contain the closed unit disk. Let us separate the series in two

$$\sum_{\nu\geq 0} b_{n,\nu}\varphi_\nu(z) = \sum_{\nu=0}^{n} b_{n,\nu}\varphi_\nu(z) + \sum_{\nu\geq n+1} b_{n,\nu}\varphi_\nu(z). \tag{4.14}$$

Fix a compact subset $K, \overline{\mathbb{D}} \subset K \subset D_{m^*}(f)$.

We begin with the infinite series to the right of (4.14) which is easier to handle. Take $1 < R < R_{m^*}(f)$ such that K and the poles of f in $D_{m^*}(f)$ are surrounded by T_R. According to (4.13) and the first inequality in (4.4)

$$|b_{n,\nu}| \leq C\|\psi_\nu\|_{T_R},$$

for some constant C independent of n. Choose $\delta > 0$ sufficiently small so that $\|z\|_K + \delta < R - \delta$. According to (4.7)-(4.8), there exists n_0 such that

$$\|\psi_\nu\|_{T_R} \leq \frac{1}{(R-\delta)^\nu}, \quad \|\varphi_\nu\|_K \leq (\|z\|_K + \delta)^\nu, \qquad \nu \geq n_0.$$

Set $q = \frac{\|z\|_K+\delta}{R-\delta} (< 1)$. If $n \geq n_0$, we obtain

$$\sum_{\nu\geq n+1} |b_{n,\nu}|\|\varphi_\nu(z)\|_K \leq C \sum_{\nu\geq n+1} q^\nu = C\frac{q^{n+1}}{1-q}.$$

Taking $\limsup_n$ of the nth root, then making δ tend to zero and R to $R_{m^*}(f)$ it readily follows that

$$\limsup_{n\longrightarrow\infty} \|\sum_{\nu\geq n+1} b_{n,\nu}\varphi_\nu\|_K^{1/n} \leq \frac{\|z\|_K}{R_{m^*}(f)}. \tag{4.15}$$

Should K be contained in $\overline{\mathbb{D}}$ one substitutes $\|z\|_K$ by 1 in the formula.

To deal with the first sum, we begin estimating the values $a_{n,k}$. The trick we are about to exhibit was borrowed from [37]. Choose $r > 1$ such that $T_r \subset D_0(f)$ and $r < R < R_{m^*}(f)$ such that all the poles of f in $D_{m^*}(f)$ and the compact set K are surrounded by T_R. Set

$$a_{n,k} = \frac{1}{2\pi i}\int_{T_r} (Q_{n,m} f \psi_k)(z) dz, \qquad k \geq n - m^* + 1,$$

$$\gamma_{n,k} = \frac{1}{2\pi i}\int_{T_R} (Q_{n,m} f \psi_k)(z) dz, \qquad k \geq n - m^* + 1.$$

Notice that $a_{n,k} = 0, k = n - m^* + 1, \ldots, n$, and for $k \geq n+1$ the $a_{n,k}$ are precisely the Fourier coefficients on the right hand side of c.2) (see (4.9)).

Since f is meromorphic in $D_{m^*}(f)$, using the residue theorem we have

$$\gamma_{n,k} - a_{n,k} = \sum_{j=1}^{N} \operatorname{Res}(Q_{n,m} f \psi_k, z_j), \quad k \geq n - m^* + 1, \tag{4.16}$$

where $\operatorname{Res}(Q_{n,m} f \psi_k(z), z_j)$ is the residue of $Q_{n,m} f \psi_k$ at z_j. At z_j the function $Q_{n,m} f \psi_k$ has a pole of order $\leq \tau_j$; therefore,

$$\operatorname{Res}(Q_{n,m} f \psi_k, z_j) = \frac{1}{(\tau_j - 1)!} \lim_{z \to z_j} \left[(Q_{n,m} \psi_n)(z) \frac{(z - z_j)^{\tau_j} f(z) \psi_k(z)}{\psi_n(z)} \right]^{(\tau_j - 1)}.$$

Using the Leibnitz formula, it follows that

$$\left[(Q_{n,m} \psi_n)(z) \frac{(z - z_j)^{\tau_j} f(z) \psi_k(z)}{\psi_n(z)} \right]^{(\tau_j - 1)} =$$

$$\sum_{\ell=0}^{\tau_j - 1} \binom{\tau_j - 1}{\ell} [(Q_{n,m} \psi_n)(z)]^{(\tau_j - 1 - \ell)} \left[\frac{(z - z_j)^{\tau_j} f(z) \psi_k(z)}{\psi_n(z)} \right]^{(\ell)}. \tag{4.17}$$

Define

$$\alpha_n(j, \ell) := \frac{1}{(\tau_j - 1)!} \binom{\tau_j - 1}{\ell} \lim_{z \to z_j} (Q_{n,m} \psi_n)^{(\tau_j - 1 - \ell)}(z). \tag{4.18}$$

By (4.17) and (4.18), we obtain

$$\operatorname{Res}(Q_{n,m} f \psi_k, z_j) = \sum_{\ell=0}^{\tau_j - 1} \alpha_n(j, \ell) \left[\frac{(z - z_j)^{\tau_j} f(z) \psi_k(z)}{\psi_n(z)} \right]_{z = z_j}^{(\ell)}. \tag{4.19}$$

Notice that $\alpha_n(j, \ell)$ does not depend on k. Then, for each $k \geq n - m^* + 1$, (4.16) and (4.19) give

$$a_{n,k} = \gamma_{n,k} - \sum_{j=1}^{N} \sum_{\ell=0}^{\tau_j - 1} \alpha_n(j, \ell) \left[\frac{(z - z_j)^{\tau_j} f(z) \psi_k(z)}{\psi_n(z)} \right]_{z = z_j}^{(\ell)}. \tag{4.20}$$

Since $a_{n,k} = 0$, for $k = n - m^* + 1, \ldots, n$ we can write

$$\gamma_{n,k} = \sum_{j=1}^{N} \sum_{\ell=0}^{\tau_j - 1} \alpha_n(j, \ell) \left[\frac{(z - z_j)^{\tau_j} f(z) \psi_k(z)}{\psi_n(z)} \right]_{z = z_j}^{(\ell)}. \tag{4.21}$$

Recall that $\sum_{j=1}^{d} \tau_j = \widetilde{m} \leq m^*$. Thus we have obtained a system of $\widetilde{m}$ equations on $\widetilde{m}$ unknowns (the quantities $\alpha_n(j, \ell)$).

The determinant Δ_n of the system has the form

$$\Delta_n = \left| \begin{array}{cccc} \left[\frac{(z-z_j)^{\tau_j} f(z)\psi_{n-\widetilde{m}+1}(z)}{\psi_n(z)}\right]_{z=z_j} & \cdots & \left[\frac{(z-z_j)^{\tau_j} f(z)\psi_{n-\widetilde{m}+1}(z)}{\psi_n(z)}\right]^{(\tau_j-1)}_{z=z_j} \\ \left[\frac{(z-z_j)^{\tau_j} f(z)\psi_{n-\widetilde{m}+2}(z)}{\psi_n(z)}\right]_{z=z_j} & \cdots & \left[\frac{(z-z_j)^{\tau_j} f(z)\psi_{n-\widetilde{m}+2}(z)}{\psi_n(z)}\right]^{(\tau_j-1)}_{z=z_j} \\ \vdots & \vdots & \vdots \\ \left[\frac{(z-z_j)^{\tau_j} f(z)\psi_{n}(z)}{\psi_n(z)}\right]_{z=z_j} & \cdots & \left[\frac{(z-z_j)^{\tau_j} f(z)\psi_{n}(z)}{\psi_n(z)}\right]^{(\tau_j-1)}_{z=z_j} \end{array} \right|_{j=1,\ldots,N},$$

where the subindex on the determinant means that the indicated group of columns are successively written for $j = 1, 2, \ldots, N$. Due to (4.7) we have

$$\lim_{n\longrightarrow\infty} \Delta_n = \Delta,$$

where

$$\Delta = \left| \begin{array}{cccc} [(z-z_j)^{\tau_j} z^{\widetilde{m}-1} f(z)]_{z=z_j} & \cdots & [(z-z_j)^{\tau_j} z^{\widetilde{m}-1} f(z)]^{(\tau_j-1)}_{z=z_j}(z_j) \\ [(z-z_j)^{\tau_j} z^{\widetilde{m}-2} f(z)]_{z=z_j} & \cdots & [(z-z_j)^{\tau_j} z^{\widetilde{m}-2} f(z)]^{(\tau_j-1)}_{z=z_j}(z_j) \\ \vdots & \vdots & \ddots & \vdots \\ [(z-z_j)^{\tau_j} f(z)]_{z=z_j} & \cdots & [(z-z_j)^{\tau_j} f(z)]^{(\tau_j-1)}_{z=z_j}. \end{array} \right|_{j=1,2,\cdots,N}.$$

Notice that $\Delta \neq 0$. In fact, should this determinant be equal to zero that would mean that there exists a linear combination of its rows giving the zero vector. In turn, this implies that there exists a polynomial of degree $\leq \widetilde{m}-1$ which multiplied times f eliminates the $\widetilde{m}$ poles which f has in $D_{m^*}(f)$ which is clearly impossible. Therefore, $|\Delta_n| \geq C > 0$ for all sufficiently large n. In the sequel we only consider such n's.

Let $\Delta_n(j,\ell)$ denote the determinant which is obtained substituting in the determinant of the system the column with index $q = \sum_{i=1}^{j-1} \tau_i + \ell + 1$ with the column vector $(\gamma_{n,n-\widetilde{m}+1}, \ldots, \gamma_{n,n})^t$ formed with the independent terms of equations (4.21). By Cramer's rule

$$\alpha_n(j,\ell) = \frac{\Delta_n(j,\ell)}{\Delta_n} = \frac{1}{\Delta_n} \sum_{s=1}^{\widetilde{m}} \gamma_{n,n-\widetilde{m}+s} M_n(s,q). \tag{4.22}$$

where $M_n(s,q)$ is the cofactor corresponding to row s and column q of $\Delta_n(j,\ell)$. Making use of the fact that the $\alpha_n(j,\ell)$ do not depend on k from (4.20) and (4.22) it follows that

$$a_{n,k} = \gamma_{n,k} - \frac{1}{\Delta_n} \sum_{j=1}^{N} \sum_{\ell=0}^{\tau_j-1} \sum_{s=1}^{m^*} \gamma_{n,n-\widetilde{m}+s} M_n(s,q) \left(\frac{\psi_k}{\psi_n}\right)^{(\ell)} (z_j), \tag{4.23}$$

for $k \geq n - m^* + 1$.

Choose $\varepsilon > 0$ so that $|z_j| - \varepsilon > r$ for all $j = 1, \ldots, N$. Recall that r was chosen greater than 1. Using Cauchy's integral formula

$$\left(\frac{\psi_k}{\psi_n}\right)^{(\ell)}(z_j) = \frac{\ell!}{2\pi i}\int_{|z-z_j|=\varepsilon} \frac{\psi_k(z)dz}{\psi_n(z)(z-z_j)^{\ell+1}}.$$

On account of (4.6), there exists a constant C_1 such that

$$\left|\left(\frac{\psi_k}{\psi_n}\right)^{(\ell)}(z_j)\right| \leq C_1\frac{1}{r^{k-n}}, \qquad k \geq n - m^* + 1,$$

for all $j = 1\ldots, N, l = 0, 1, \cdots, \tau_j - 1$, and n sufficiently large. Consequently,

$$|M_n(s,q)| \leq C_2.$$

Using (4.23), it follows that there exists a constant C_3 such that

$$|a_{n,k}| \leq |\gamma_{n,k}| + \frac{C_3}{r^{k-n}}\sum_{s=1}^{m^*}|\gamma_{n-\widetilde{m}+s}|, \qquad k \geq n+1. \tag{4.24}$$

From the integral which defines $\gamma_{n,k}$, the first inequality in (4.4), and using (4.7), given $\delta > 0, R - \delta > r$, for all sufficiently large n, we have

$$|\gamma_{n,k}| \leq \frac{1}{(R-\delta)^k}, \qquad k \geq n - m^* + 1,$$

and taking into consideration (4.24), we obtain

$$|a_{n,k}| \leq \frac{C_4}{r^{k-n}(R-\delta)^n}, \quad k \geq n+1, \tag{4.25}$$

for some constant C_4. Since $|\langle w_m\varphi_k, \varphi_\nu\rangle| \leq \|w_m\|_{\mathbb{T}}$, due to (4.12) we can find a constant C_5 for which

$$|b_{n,\nu}| \leq \frac{C_5}{(R-\delta)^n}. \tag{4.26}$$

Finally, let us estimate $\sum_{\nu=0}^{n} b_{n,\nu}\varphi_\nu(z)$. Fix a compact subset $K \subset D_{m^*}(f)$. As we did for the other sum, we can assume without loss of generality that $K \supset \overline{\mathbb{D}}$. We also assume that R and δ chosen previously satisfy $\|z\|_K + \delta < R - \delta$. From (4.7) it follows that there exists some constant C_6 such that

$$\|\varphi_\nu\|_K \leq C_6(\|z\|_K + \delta)^\nu, \qquad \nu \geq 0.$$

Therefore, making use of (4.26), we obtain

$$\| \sum_{\nu=0}^{n} b_{n,\nu}\varphi_\nu \|_K \leq \frac{C_5 C_6}{(R-\delta)^n} \sum_{\nu=0}^{n} (\|z\|_K + \delta)^\nu = \frac{C_5 C_6}{(R-\delta)^n} \frac{(\|z\|_K + \delta)^{n+1} - 1}{\|z\|_K + \delta - 1}.$$

Taking $\limsup_n$ of the nth root, then making R tend to $R_{m^*}(f)$ and δ to zero, we arrive at

$$\limsup_{n\to\infty} \| \sum_{\nu=0}^{n} b_{n,\nu}\varphi_\nu \|_K^{1/n} \leq \frac{\|z\|_K}{R_{m^*}(f)}. \tag{4.27}$$

Again, if K is contained in $\overline{\mathbb{D}}$ one must write 1 on the right hand of this formula in place of $\|z\|_K$. Formulas (4.11) and (4.14) together with inequalities (4.15) and (4.27) easily render

$$\limsup_{n\to\infty} \|w_m(Q_{n,m}f - P_{n,m})\|_K^{1/n} \leq \frac{\|z\|_K}{R_{m^*}(f)}. \tag{4.28}$$

Fix $\varepsilon > 0$ and take any compact subset $K \subset D_{m^*}(f)$. For $z \in K(\varepsilon) = K \setminus J_\varepsilon$, according to the second inequality in (4.4), we have (notice that $J(\varepsilon)$ leaves out an $\varepsilon/6m$ neighborhood of the zeros of w_m)

$$\|f - R_{n,m}\|_{K(\varepsilon)} \leq \frac{n^{2m}}{C_7} \|w_m(Q_{n,m}f - P_{n,m})\|_K$$

for some constant C_7, and applying (4.28), we obtain (4.10). As mentioned in the introduction of the sets $J(\varepsilon)$, (4.10) (and much less) implies convergence in Hausdorff content in $D_{m^*}(f)$ as claimed. The statement concerning the asymptotic behavior of some of the zeros of $Q_{n,m}$ is a direct consequence of the convergence in Hausdorff content and Gonchar's Lemma stated in Section 2.1 (see [11, Lemma 1]). With this we conclude the proof. $\square$

We wish to point out that from (4.28) one can derive that the τ_j poles of $Q_{n,m}$ closest to z_j in fact converge to z_j with geometric rate not greater than $|z_j|/R_{m^*}(f)$. We will return to this later.

4.4 Convergence of simultaneous Fourier-Padé approximation on $\mathbb{D}$

Given $\mathbf{f} = (f_1, \ldots, f_d) \in \mathcal{H}(\overline{\mathbb{D}})$ and $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$ recall that for each $k = 1, \ldots, d$ the rational function $R_{n,\mathbf{m},k}$ is an $(n, |\mathbf{m}|, m_k)$ incomplete Fourier-Padé approximation to f_k. Let $\mathcal{P}_{n,\mathbf{m}}$ be the collection of zeros of $Q_{n,\mathbf{m}}$. A direct consequence of Theorem 4.3.1 is the following corollary.

Corollary 4.4.1. *Let $\mathbf{f} \in \mathcal{H}(\overline{\mathbb{D}})$ and $\sigma' > 0$ a.e. on $\mathbb{T}$. Fix $\mathbf{m} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$. For each $n \geq |\mathbf{m}|$, let $\mathbf{R}_{n,\mathbf{m}}$ be a Fourier-Padé approximant of type $(n, \mathbf{m})$ for $\mathbf{f}$. Then, for each $i = 1, \ldots, d, K \subset D_{m_k}(f_k)$, and $\varepsilon > 0$*

$$\limsup_{n\to\infty} \|f_i - R_{n,\mathbf{m},i}\|_{K(\varepsilon)}^{1/n} \leq \frac{\|z\|_K}{R_{m_i}(f_i)}, \tag{4.29}$$

where $\|z\|_K$ should be replaced by 1 when $K \subset \overline{\mathbb{D}}$. In particular,

$$h\text{-}\lim_{n\to\infty} R_{n,\mathbf{m},i} = f_i \text{ in } D_{m_i}(f_i).$$

Finally, for each $i = 1, \ldots, d$, and pole z_j of f_i in $D_{m_i}(f_i)$, for any $\varepsilon > 0$, there exists n_0 such that for all $n \geq n_0$ the polynomials $Q_{n,\mathbf{m}}$ have at least τ_j zeros in $\{z : |z - z_j| < \varepsilon\}$, where τ_j denotes the order of the pole z_j.

Now let us prove Theorem 4.1.2. We will combine arguments used to prove Theorem 4.3.1 and ideas from [14].

Proof of Theorem 4.1.2. First of all, notice that $\mathbf{f}$ must have exactly $|\mathbf{m}|$ poles in $D_{|\mathbf{m}|}(\mathbf{f})$. If this were not the case, it is easy to show that $\mathbf{f}$ is not polewise independent with respect to $\mathbf{m}$ in $D_{|\mathbf{m}|}(\mathbf{f})$. By $z_1, \ldots, z_N$ we denote the distinct poles of $\mathbf{f}$ in $D_{|\mathbf{m}|}(\mathbf{f})$ and let $\tau_1, \ldots, \tau_N$ be their orders, respectively.

According to Definition 4.1.1, for each $i = 1, \ldots, d$ we have

$$(Q_{n,\mathbf{m}} f_i - P_{n,\mathbf{m},i})(z) = A_{n,n+1}^{(i)} \varphi_{n+1}(z) + \cdots,$$

and $\deg P_{n,\mathbf{m},i} \leq n - m_i$. Therefore,

$$A_{n,k}^{(i)} = \langle Q_{n,\mathbf{m}} f_i, \varphi_k \rangle, \qquad k \geq n - m_i + 1,$$

and $A_{n,k}^{(i)} = 0, k = n - m_i + 1, \ldots, n$. Our first goal will be to estimate the values $A_{n,k}^{(i)}$. The procedure is similar to the one employed to estimate the quantities $a_{n,k}$ in the proof of Theorem 4.3.1, so we will not go through all the details, but there are some important aspects to single out.

Take $r > 1$ such that $T_r \subset D_0(\mathbf{f})$ and $R < R_{|\mathbf{m}|}(\mathbf{f})$ such that T_R surrounds all the poles $z_1, \ldots, z_N$ of $\mathbf{f}$ in $D_{|\mathbf{m}|}(\mathbf{f})$. Obviously, for $i = 1, \ldots, d$

$$A_{n,k}^{(i)} = \frac{1}{2\pi i} \int_{T_r} (Q_{n,\mathbf{m}} f_i \psi_k)(z) dz, \qquad k \geq n - m_i + 1.$$

Define

$$\gamma_{n,k}^{(i)} = \frac{1}{2\pi i} \int_{T_R} (Q_{n,\mathbf{m}} f_i \psi_k)(z) dz, \qquad k \geq n - m_i + 1.$$

Using the residue theorem it follows that

$$\gamma_{n,k}^{(i)} - A_{n,k}^{(i)} = \sum_{j=1}^{N} \mathrm{Res}(Q_{n,\mathbf{m}} f_i \psi_k, z_j), \quad k \geq n - m^* + 1, \tag{4.30}$$

and

$$\mathrm{Res}(Q_{n,m} f_i \psi_k, z_j) = \frac{1}{(\tau_j - 1)!} \lim_{z \to z_j} \left[(Q_{n,\mathbf{m}} \psi_n)(z) \frac{(z - z_j)^{\tau_j} f_i(z) \psi_k(z)}{\psi_n(z)} \right]^{(\tau_j - 1)}.$$

Using the Leibnitz formula, it follows that

$$\left[(Q_{n,\mathbf{m}} \psi_n)(z) \frac{(z - z_j)^{\tau_j} f_i(z) \psi_k(z)}{\psi_n(z)} \right]^{(\tau_j - 1)} =$$

$$\sum_{l=0}^{\tau_j - 1} \binom{\tau_j - 1}{\ell} [(Q_{n,\mathbf{m}} \psi_n)(z)]^{(\tau_j - 1 - \ell)} \left[\frac{(z - z_j)^{\tau_j} f_i(z) \psi_k(z)}{\psi_n(z)} \right]^{(\ell)}. \tag{4.31}$$

Define

$$\alpha_n(j, \ell) := \frac{1}{(\tau_j - 1)!} \binom{\tau_j - 1}{\ell} \lim_{z \to z_j} [Q_{n,m} \psi_n(z)]^{(\tau_j - 1 - \ell)}. \tag{4.32}$$

These quantities do not depend on i or k. By (4.31) and (4.32), we obtain

$$\mathrm{Res}(Q_{n,m} f_i \psi_k, z_j) = \sum_{\ell=0}^{\tau_j - 1} \alpha_n(j, \ell) \left[\frac{(z - z_j)^{\tau_j} f_i(z) \psi_k(z)}{\psi_n(z)} \right]_{z = z_j}^{(\ell)}. \tag{4.33}$$

Then, for each $k \geq n - m^* + 1$ and $i = 1, \ldots, d$, (4.30) and (4.33) give

$$A_{n,k}^{(i)} = \gamma_{n,k}^{(i)} - \sum_{j=1}^{N} \sum_{\ell=0}^{\tau_j - 1} \alpha_n(j, \ell) \left[\frac{(z - z_j)^{\tau_j} f_i(z) \psi_k(z)}{\psi_n(z)} \right]_{z = z_j}^{(\ell)}.$$

For $k = n - m_i + 1, \ldots, n, i = 1, \ldots, d$

$$\gamma_{n,k}^{(i)} = \sum_{j=1}^{N} \sum_{\ell=0}^{\tau_j - 1} \alpha_n(j,\ell) \left[\frac{(z-z_j)^{\tau_j} f_i(z) \psi_k(z)}{\psi_n(z)} \right]_{z=z_j}^{(\ell)}, \tag{4.34}$$

since $A_{n,k}^{(i)} = 0$ for these values of k. Thus, we have obtained a system of $|\mathbf{m}|$ equations on $|\mathbf{m}|$ unknowns (the quantities $\alpha_n(j,\ell)$).

The determinant Δ_n of the system has the form

$$\begin{vmatrix} \left[\frac{(z-z_j)^{\tau_j} f_i(z)\psi_{n-m_i+1}(z)}{\psi_n(z)}\right]_{z=z_j} & \cdots & \left[\frac{(z-z_j)^{\tau_j} f_i(z)\psi_{n-m_i+1}(z)}{\psi_n(z)}\right]_{z=z_j}^{(\tau_j-1)} \\ \left[\frac{(z-z_j)^{\tau_j} f_i(z)\psi_{n-m_i+2}(z)}{\psi_n(z)}\right]_{z=z_j} & \cdots & \left[\frac{(z-z_j)^{\tau_j} f_i(z)\psi_{n-m_i+2}(z)}{\psi_n(z)}\right]_{z=z_j}^{(\tau_j-1)} \\ \vdots & \vdots & \vdots \\ \left[\frac{(z-z_j)^{\tau_j} f_i(z)\psi_{n}(z)}{\psi_n(z)}\right]_{z=z_j} & \cdots & \left[\frac{(z-z_j)^{\tau_j} f_i(z)\psi_{n}(z)}{\psi_n(z)}\right]_{z=z_j}^{(\tau_j-1)} \end{vmatrix}_{j=1,\ldots,N, i=1,\ldots,d},$$

where the subindex on the determinant means that the indicated group of columns are successively written for $j = 1, 2, \ldots, N$ and the rows repeated for $i = 1, \ldots, d$. Due to (4.7) we have

$$\lim_{n \longrightarrow \infty} \Delta_n = \Delta,$$

where

$$\Delta = \begin{vmatrix} [(z-z_j)^{\tau_j} z^{m_i-1} f_i(z)]_{z=z_j} & \cdots & [(z-z_j)^{\tau_j} z^{m_i-1} f_i(z)]_{z=z_j}^{(\tau_j-1)}(z_j) \\ [(z-z_j)^{\tau_j} z^{m_i-2} f_i(z)]_{z=z_j} & \cdots & [(z-z_j)^{\tau_j} z^{m_i-2} f_i(z)]_{z=z_j}^{(\tau_j-1)}(z_j) \\ \vdots & \vdots & \vdots \\ [(z-z_j)^{\tau_j} f_i(z)]_{z=z_j} & \cdots & [(z-z_j)^{\tau_j} f_i(z)]_{z=z_j}^{(\tau_j-1)}. \end{vmatrix}_{j=1,2,\cdots,N, i=1,\ldots,d}.$$

Let us show that $\Delta \neq 0$. Assume the contrary. Then there exists a linear combination of rows giving the zero vector. This means that there exist polynomials $p_1, \ldots, p_d, \deg p_i \leq m_i - 1$, such that

$$\sum_{i=1}^{d} [(z-z_j)^{\tau_j} p_i(z) f_i(z)]_{z=z_j}^{(\ell)} = 0, \quad j = 1, \ldots, d, \quad \ell = 0, \ldots, \tau_j - 1,$$

but this contradicts the assumption that $\mathbf{f}$ is polewise independent with respect to $\mathbf{m}$ in $D_{|\mathbf{m}|}(\mathbf{f})$. Consequently, $|\Delta_n| \geq C > 0$ for all sufficiently large n and we restrict our attention to such n's.

Using Cramer's rule, the system of equations (4.34) allows us to express the $\alpha_n(j,l)$ in terms of the $\gamma_{n,k}^{(i)}, i = 1,\ldots,d, k = n-m_i+1,\ldots,n$. Arguing as in the proof of Theorem 4.3.1 we arrive at the bounds (compare with (4.25))

$$|A_{n,k}^{(i)}| \le \frac{C}{r^{k-n}(R-\delta)^n}, \quad k \ge n+1, \quad i = 1,\ldots,d. \tag{4.35}$$

Fix $i \in \{1,2,\ldots,d\}$. We have

$$Q_{n,\mathbf{m}}(z)Q_{|\mathbf{m}|}(z)f_i(z) - Q_{|\mathbf{m}|}(z)P_{n,\mathbf{n},i}(z) = \sum_{\ell\ge n+1} A_{n,\ell}^{(i)} Q_{|\mathbf{m}|}(z)\varphi_\ell(z) =$$

$$\sum_{\nu=0}^{n+|\mathbf{m}|-m_i} B_{n,\nu}^{(i)}\varphi_\nu(z) + \sum_{\nu\ge n+|\mathbf{m}|-m_i+1} B_{n,\nu}^{(i)}\varphi_\nu(z), \tag{4.36}$$

where $Q_{|\mathbf{m}|}(z)P_{n,\mathbf{n},i}(z)$ has degree at most $n+|\mathbf{m}|-m_i$.

Fix a compact set $K \subset D_{|\mathbf{m}|}(\mathbf{f})$. In the sequel we assume that R was chosen so that T_R also surrounds K. For the series in (4.36), as in the proof of Theorem 4.3.1, it is easy to show that

$$\limsup_{n\to\infty} \|\sum_{\nu\ge n+|\mathbf{m}|-m_i+1} B_{n,\nu}^{(i)}\varphi_\nu(z)\|_K^{1/n} \le \frac{\|z\|_K}{R_{|\mathbf{m}|}(\mathbf{f})}.$$

Here and below $\|z\|_K$ is replaced by 1 when $K \subset \overline{\mathbb{D}}$. In order to prove that

$$\limsup_{n\to\infty} \|\sum_{\nu=0}^{n+|\mathbf{m}|-m_i} B_{n,\nu}^{(i)}\varphi_\nu(z)\|_K^{1/n} \le \frac{\|z\|_K}{R_{|\mathbf{m}|}(\mathbf{f})}$$

one employs (4.35) and the equality

$$B_{n,\nu}^{(i)} = \sum_{\ell\ge n+1} A_{n,\ell}^{(i)} \langle Q_{|\mathbf{m}|}(z)\varphi_\ell(z), \varphi_\nu\rangle$$

in a similar fashion as in Theorem 4.3.1. Consequently, for each $i = 1,\ldots,d$

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}}Q_{|\mathbf{m}|}f_i - Q_{|\mathbf{m}|}P_{n,\mathbf{n},i}\|_K^{1/n} \le \frac{\|z\|_K}{R_{|\mathbf{m}|}(\mathbf{f})} \tag{4.37}$$

and

$$\limsup_{n\to\infty} \|f_i - R_{n,\mathbf{n},i}\|_{K(\varepsilon)}^{1/n} \le \frac{\|z\|_K}{R_{|\mathbf{m}|}(\mathbf{f})}. \tag{4.38}$$

From here convergence in Hausdorff content readily follows. Therefore, using Gonchar's lemma, each pole of $\mathbf{f}$ in $D_{|\mathbf{m}|}(\mathbf{f})$ attracts as many zeros of $Q_{n,\mathbf{m}}$ as its order. Since $\deg Q_{n,\mathbf{m}} \leq |\mathbf{m}|$ and the total number of poles of $\mathbf{f}$ in $D_{|\mathbf{m}|}(\mathbf{f})$ equals $|\mathbf{m}|$ we have $\deg Q_{n,\mathbf{m}} = |\mathbf{m}|$ for all sufficiently large n. This implies that $\mathbf{R}_{n,\mathbf{m}}$ is unique for such n's. In fact, if this were not the case we could find an infinite subsequence of indices for which Definition 4.1.1 has solutions with $\deg Q_{n,\mathbf{m}} < |\mathbf{m}|$, which contradicts what was proved. In the sequel, we only consider such n's. As the poles of $\mathbf{f}$ have absolute value greater than 1, we obtain

$$Q_{n,\mathbf{m}}(z) = \prod_{j=1}^{|\mathbf{m}|} \left(1 - \frac{z}{z_{n,j}}\right)$$

and

$$\lim_{n\to\infty} Q_{n,\mathbf{m}}(z) = \prod_{k=1}^{N} \left(1 - \frac{z}{z_k}\right)^{\tau_k} = Q_{|\mathbf{m}|}(z).$$

Since $\mathcal{P}_{|\mathbf{m}|}(\mathbf{f})$ is the set of accumulation points of the zeros of $Q_{n,\mathbf{m}}$, (4.2) follows at once from (4.38).

Let us prove (4.3). To this end we start by proving that for $k = 1, \ldots, N$

$$\limsup_{n\to\infty} |Q_{n,\mathbf{m}}^{(j)}(z_k)|^{1/n} \leq |z_k|/R_{|\mathbf{m}|}(\mathbf{f}), \qquad j = 0, \ldots, \tau_k - 1. \tag{4.39}$$

Suppose that the pole z_k attains its order with the function f_i. Let $\varepsilon > 0$ be sufficiently small so that the closed disk $C_{k,\varepsilon} = \{z : |z - z_k| \leq \varepsilon\}$ is contained in $D_{|\mathbf{m}|}(\mathbf{f})$ and contains no other pole of $\mathbf{f}$. On account of (4.37)

$$\limsup_{n\to\infty} \|(z - z_k)^{\tau_k} f_i Q_{n,\mathbf{m}} - (z - z_k)^{\tau_k} P_{n,\mathbf{m},i}\|_{C_{k,\varepsilon}}^{1/n} \leq \frac{\|z\|_{C_{k,\varepsilon}}}{R_{|\mathbf{m}|}(\mathbf{f})},$$

and using Cauchy's integral formula for the derivative, we have

$$\limsup_{n\to\infty} \|[(z - z_k)^{\tau_k} f_i Q_{n,\mathbf{m}} - (z - z_k)^{\tau_k} P_{n,\mathbf{m},i}]^{(j)}\|_{C_{k,\varepsilon}}^{1/n} \leq \frac{\|z\|_{C_{k,\varepsilon}}}{R_{|\mathbf{m}|}(\mathbf{f})}, \tag{4.40}$$

for all $j \geq 0$. In particular, taking $z = z_k$ and $j = 0$, we obtain

$$\limsup_{n\to\infty} |AQ_{n,\mathbf{m}}(z_k)|^{1/n} \leq \frac{|z_k|}{R_{|\mathbf{m}|}(\mathbf{f})},$$

where $A = \lim_{z\to z_k}(z-z_k)^{\tau_k} f_i(z) \neq 0$ since z_k is a pole of f_i of order τ_k. Therefore

$$\limsup_{n\to\infty} |Q_{n,\mathbf{m}}(z_k)|^{1/n} \leq \frac{|z_k|}{R_{|\mathbf{m}|}(\mathbf{f})}.$$

Proceeding by induction, take $s \leq \tau_k$ and assume that

$$\limsup_{n\to\infty} |(Q^{(j)}_{n,\mathbf{m}}(z_k)|^{1/n} \leq \frac{|z_k|}{R_{|\mathbf{m}|}(\mathbf{f})}, \qquad j = 0,\dots,s-2, \tag{4.41}$$

and let us show that (4.41) holds for $j = s-1$. As $s-1 < \tau_k$, using (4.40) we deduce that

$$\limsup_{n\to\infty} |[(z-z_k)^{\tau_k} f_i Q_{n,\mathbf{m}}]^{(s-1)}(z_k)| \leq \frac{|z_k|}{R_{|\mathbf{m}|}(\mathbf{f})}. \tag{4.42}$$

Applying the Leibnitz formula it follows that

$$[(z-z_k)^{\tau_k} f_i Q_{n,\mathbf{m}}]^{(s-1)}(z_k) = \sum_{\ell=0}^{s-1} \binom{s-1}{\ell} Q^{(s-1-\ell)}_{n,\mathbf{m}}(z_k)[(z-z_k)^{\tau_k} f_i]^{(\ell)}(z_k).$$

Using (4.41), (4.42), and that $A \neq 0$, we conclude that

$$\limsup_{n\to\infty} |(Q^{(s-1)}_{n,\mathbf{m}}(z_k)|^{1/n} \leq \frac{|z_k|}{R_{|\mathbf{m}|}(\mathbf{f})}.$$

Consider a basis of polynomials $\{q_{k,s} : k = 1,\dots,N, s = 0,\dots,\tau_k - 1\}$ such that $\deg q_{k,s} \leq |\mathbf{m}| - 1$ for all k, s and

$$q^{(j)}_{k,s}(z_i) = \delta_{i,k}\delta_{j,s}, \quad 1 \leq i \leq N, \quad 0 \leq j \leq \tau_i - 1.$$

Then

$$Q_{n,\mathbf{m}}(z) = \sum_{k=1}^{N} \sum_{s=0}^{\tau_k - 1} Q^{(s)}_{n,\mathbf{m}}(z_k) q_{k,s}(z) + C_n Q_{|\mathbf{m}|}(z),$$

where $C_n = \prod_{k=1}^N z_k^{\tau_k} / \prod_{j=1}^{|\mathbf{m}|} z_{n,j}$. From (4.39) it readily follows that

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}} - C_n Q_{|\mathbf{m}|}\|^{1/n} \leq \frac{\max\{|\zeta| : \zeta \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{f})\}}{R_{|\mathbf{m}|}(\mathbf{f})}.$$

Evaluating at zero, we obtain

$$\limsup_{n\to\infty} |1 - C_n|^{1/n} \leq \frac{\max\{|\zeta| : \zeta \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{f})\}}{R_{|\mathbf{m}|}(\mathbf{f})},$$

which combined with the previous estimate gives us (4.3). We are done. $\square$

Corollary 4.3.1 complements Theorem 4.1.2 because under the assumptions of the latter it may still occur that $R_{m_i}(f_i) > R_{|\mathbf{m}|}(\mathbf{f})$ for some i so that (4.29) gives a better estimate than (4.2) for that particular i. As in section 2.5.2, it is also possible to construct examples where f is not polewise independent with respect to $\mathbf{m}$ and using Corollary 4.3.1 one can derive uniform convergence on compact subsets of the region obtained deleting from $D_{m_i}(f_i)$ the poles of f_i.

Chapter 5

Incomplete Fourier-Padé approximants on I

5.1 Statement of the problem

The results of this chapter have a lot in common with those in Chapter 4 but correspond to a different setting. All the results can be formulated for the case of measures supported on any bounded subinterval of the real line but, without loss of generality, we will restrict our attention to measures supported on the interval $[-1,1]$ which we will denote I.

By μ we denote a finite positive Borel measure whose support is contained in I and $\mu' > 0$ a.e. on I. Let $\{F_n\}$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients. That is,

$$\int_I F_j(x)F_k(x)d\mu(x) = \delta_{j,k}, \qquad j,k \in \mathbb{Z}_+,$$

where as usual $\delta_{j,k} = 0, j \neq k$ and $\delta_{k,k} = 1$. By $\mathcal{H}(I)$ we denote the space of functions which are analytic on some neighborhood of I.

Definition 5.1.1. *Let* $\mathbf{f} = (f_1,\ldots,f_d)$ *where* $f_k \in \mathcal{H}(I), k = 1,\ldots,d$. *Fix a multi-index* $\mathbf{m} = (m_1,\ldots,m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$ *where* $\mathbf{0}$ *denotes the zero vector in* $\mathbb{Z}_+^d$. *Set*

$|\mathbf{m}| = m_1 + \ldots + m_d$. Then, for each $n \geq \max\{m_1, \ldots, m_d\}$, there exist polynomials $Q_{n,\mathbf{m}}, P_{n,\mathbf{m},j}, j = 1, \ldots, d$, such that

a.1) $\deg P_{n,\mathbf{m},j} \leq n - m_j, j = 1, \ldots, d, \quad \deg Q_{n,\mathbf{m}} \leq |\mathbf{m}|, \quad Q_{n,\mathbf{m}} \not\equiv 0,$

a.2) $[Q_{n,\mathbf{m}} f_j - P_{n,\mathbf{m},j}](z) = A^{(j)}_{n,n+1} F_{n+1}(z) + A^{(j)}_{n,n+2} F_{n+2}(z) + \ldots.$

We call the vector rational function $\mathbf{R}_{n,\mathbf{m}} = (P_{n,\mathbf{m},1}/Q_{n,\mathbf{m}}, \ldots, P_{n,\mathbf{m},d}/Q_{n,\mathbf{m}})$ an $(n, \mathbf{m})$ simultaneous Fourier-Padé approximation of $\mathbf{f}$ (on the real line).

It is easy to see that for any pair $(n, \mathbf{m})$ there is at least one $R_{n,\mathbf{m}}$ but, in general, it is not uniquely determined. In the sequel, we assume that given $(n, \mathbf{m})$, one solution is taken. We will normalize the common denominator in terms of its zeros $z_{n,k}$ conveniently.

Let ϕ be the conformal representation of $\overline{\mathbb{C}} \setminus I$ onto the exterior of the closed unit disk normalized so that $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$. We extend ϕ by continuity to I assuming that I has two sides $I_\pm$. Fix $R > 1$. Let $\Gamma_R = \{z : |\phi(z)| = R\}$ and D_R denotes the region bounded by Γ_R. This region is an ellipse with foci ± 1 having R as the sum of its semi-axes. The region D_R will be called canonical.

Given $\mathbf{f} \in \mathcal{H}(I)$ (that is, each component of $\mathbf{f}$ is analytic in a neighborhood of I), let $D_{|\mathbf{m}|}(\mathbf{f})$ denote the largest canonical region inside of which $\mathbf{f}$ has at most $|\mathbf{m}|$ poles. Let $R_{|\mathbf{m}|}$ be such that $D_{R_{|\mathbf{m}|}} = D_{|\mathbf{m}|}(\mathbf{f})$ and we define $R_{|\mathbf{m}|}(\mathbf{f}) := R_{|\mathbf{m}|}$.

Let $z_1, \ldots, z_N$ be the poles of $\mathbf{f}$ in $D_{|\mathbf{m}|}(\mathbf{f})$ and set

$$M = \left(1 + \min_{j=1,\ldots,N} |\phi(z_j)|\right)/2.$$

If $\mathbf{f}$ has no pole in $D_{|\mathbf{m}|}(\mathbf{f})$, take $M = 3/2$. Notice that $M > 1$. We normalize $Q_{n,\mathbf{m}}$ as follows

$$Q_{n,\mathbf{m}}(z) = \prod_{|\phi(z_{n,k})| \leq M} (z - z_{n,k}) \prod_{|\phi(z_{n,k})| > M} \left(1 - \frac{z}{z_{n,k}}\right). \tag{5.1}$$

Obviously, $Q_{n,\mathbf{m}} f_j - P_{n,\mathbf{m},j} \in \mathcal{H}(I)$. Due to the asymptotic properties satisfied by $\{F_n\}_{n \geq 0}$ it is easy to verify that the Fourier expansion of $Q_{n,\mathbf{m}} f_j - P_{n,\mathbf{m},j}$ in terms $\{F_n\}_{n \geq 0}$ converges uniformly on compact subsets of a neighborhood of I.

We will prove, under appropriate assumptions on $\mathbf{f}$, that

$$\lim_{n\to\infty} \mathbf{R}_{n,\mathbf{m}} = \mathbf{f}$$

uniformly on compact subsets of the largest canonical region containing at most $|\mathbf{m}|$ poles. Moreover, we will see that under those assumptions the zeros of the common denominator of the approximating rational functions point out the location and order of the poles of $\mathbf{f}$ in that region.

For the proof of the corresponding version of the Montessus de Ballore theorem we need to adapt the notion of polewise independence to fit the type of regions in which convergence takes place (which are no longer circular).

Definition 5.1.2. *A vector $\mathbf{f} = (f_1, \ldots, f_d) \in \mathcal{H}(I)$ of functions meromorphic in some canonical region D_R is said to be polewise independent with respect to the multi-index $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$ in D_R if there do not exist polynomials $p_1, \ldots, p_d$, at least one of which is non-null, satisfying*

b.1) $\deg p_j \leq m_j - 1, j = 1, \ldots, d$, *if* $m_j \geq 1$,

b.2) $p_j \equiv 0$ *if* $m_j = 0$,

b.3) $\sum_{j=1}^d p_j \cdot (f_j \circ \phi^{-1}) \in \mathcal{H}(\xi : 1 < |\xi| < R)$.

Let $Q_{|\mathbf{m}|}(\mathbf{f})$ be the polynomial whose zeros $z_1, z_2, \ldots, z_N$ are the poles of $\mathbf{f}$ in $D_{|\mathbf{m}|}(\mathbf{f})$ counting multiplicities normalized as follows

$$Q_{|\mathbf{m}|}(\mathbf{f}) = \prod_{k=1}^{N} \left(1 - \frac{z}{z_k}\right).$$

The set of distinct zeros of $Q_{|\mathbf{m}|}(\mathbf{f})$ is denoted by $\mathcal{P}_{|\mathbf{m}|}(\mathbf{f})$. We prove the following analog of the Graves-Morris/Saff theorem contained in [14] (see also [5]).

Theorem 5.1.3. *Assume that $\mathbf{f} \in \mathcal{H}(I)$ and $\mu' > 0$ a.e. on I. Fix a multi-index $\mathbf{m} \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$ and suppose that $\mathbf{f}$ is polewise independent with respect to $\mathbf{m}$ in $D_{|\mathbf{m}|}(\mathbf{f})$,* *Then,* $\mathbf{R}_{n,\mathbf{m}}$ *is uniquely determined for all sufficiently large* n*. For any compact subset* K *of* $D_{|\mathbf{m}|}(\mathbf{f}) \setminus \mathcal{P}_{|\mathbf{m}|}(\mathbf{f})$

$$\limsup_{n\to\infty} \|f_i - R_{n,\mathbf{m},i}\|_K^{1/n} \leq \frac{\|\phi(z)\|_K}{R_{|\mathbf{m}|}(\mathbf{f})}, \quad i = 1, \ldots, d. \tag{5.2}$$

Additionally,

$$\limsup_{n\to\infty} \|Q_{|\mathbf{m}|}(\mathbf{f}) - Q_{n,\mathbf{m}}\|^{1/n} \leq \frac{\max\{|\phi(\zeta)| : \zeta \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{f})\}}{R_{|\mathbf{m}|}(\mathbf{f})}. \tag{5.3}$$

The chapter is organized as follows. Sections 2 and 3 are dedicated to the study of incomplete Fourier-Padé approximation on the real line and prove their convergence in h-content. In the last section we apply the results obtained in Sections 2 and 3 to the case of simultaneous Fourier-Padé approximation and prove Theorem 5.1.3.

5.2 Incomplete Fourier-Padé approximants on I

Now, let us introduce the auxiliary scalar approximants which are appropriate in the present situation.

Definition 5.2.1. *Let* $f \in \mathcal{H}(I)$*. Fix* $m^* \leq m$*. Let* $n \geq m$*. We say that the rational function* $R_{n,m} = P_{n,m}/Q_{n,m}$ *is an incomplete Fourier-Padé approximation of type* (n, m, m^*) *(on the real line) corresponding to* f *(and* μ*) if* $P_{n,m}$ *and* $Q_{n,m}$ *are polynomials that verify*

c.1) $\deg P_{n,m} \leq n - m^*, \quad \deg Q_{n,m} \leq m, \quad Q_{n,m} \not\equiv 0,$

c.2) $[Q_{n,m}f - P_{n,m}](z) = a_{n,n+1}F_{n+1}(z) + a_{n,n+2}F_{n+2} + \ldots.$

The polynomials $P_{n,m}, Q_{n,m}$ depend on m^* and the numbers $a_{n,k}$ depend on (m, m^*) but we do not indicate it.

From Definitions 5.1.1 and 5.2.1 it follows that $R_{n,\mathbf{m},k} = P_{n,\mathbf{m},k}/Q_{n,\mathbf{m}}, k = 1, \ldots, d,$ is an incomplete Padé approximation of type $(n, |\mathbf{m}|, m_k)$ with respect to f_k.

Given $n \geq m \geq m^*$, $R_{n,m}$ is not unique so we choose one candidate. The denominator $Q_{n,m}$ will be normalized as in (5.1).

In the next section we show that $h-\lim_{n\to\infty} R_{n,m} = f$ in the largest canonical region $D_{m^*}(f)$ inside of which f has at most m^* poles. For this we need to introduce some notation.

Take an arbitrary $\varepsilon > 0$ and define the open set J_ε as follows. For $n \geq m$, let $J_{n,\varepsilon}$ denote the $\varepsilon/6mn^2$-neighborhood of the set $\mathcal{P}_{n,m} = \{\zeta_{n,1}, \ldots, \zeta_{n,m_n}\}$ of finite zeros of $Q_{n,m}$ and let $J_{m-1,\varepsilon}$ denote the $\varepsilon/6m$-neighborhood of the set of poles of f in $D_m(f)$. Take $J_\varepsilon = \cup_{n\geq m-1} J_{n,\varepsilon}$. For any set $B \subset \mathbb{C}$ we put $B(\varepsilon) := B \setminus J_\varepsilon$.

Obviously, if $\{g_n\}_{n\in\mathbb{N}}$ converges uniformly to g on $K(\varepsilon)$ for every compact $K \subset D$ and $\varepsilon > 0$, then h-$\lim_{n\to\infty} g_n = g$ in D.

Due to the normalization (5.1), for any compact set K of $\mathbb{C}$ and for every $\varepsilon > 0$, there exist positive constants C_1, C_2, independent of n, such that

$$\|Q_{n,m}\|_K < C_1, \qquad \min_{z\in K(\varepsilon)} |Q_{n,m}(z)| > C_2 n^{-2m}, \tag{5.4}$$

where the second inequality is meaningful when $K(\varepsilon)$ is a non-empty set.

In the sequel, C_k will be used to denote positive constants, generally different, that are independent of n but may depend on all the other parameters involved in each formula where they appear.

Given μ and F_n denote

$$H_n(z) = \int_I \frac{F_n(\zeta)d\mu(\zeta)}{z-\zeta}.$$

If $\mu' > 0$ a.e. on I, E.A. Rakhmanov's theorem (see [21]-[23]) implies that

$$\lim_{n\to\infty} \frac{F_{n+k}(z)}{F_n(z)} = \phi^k(z), \qquad k \in \mathbb{Z}, \tag{5.5}$$

$$\lim_{n\to\infty} \frac{H_{n+k}(z)}{H_n(z)} = \frac{1}{\phi^k(z)}, \qquad k \in \mathbb{Z}, \tag{5.6}$$

uniformly on each compact subset of $\mathbb{C} \setminus I$. In turn, these relations easily imply that

$$\lim_{n\to\infty} |F_n(z)|^{1/n} = |\phi(z)|, \tag{5.7}$$

$$\lim_{n\to\infty} |H_n(z)|^{1/n} = |\phi(z)|^{-1}, \tag{5.8}$$

uniformly on each compact subset of $\mathbb{C} \setminus I$.

Let $g \in \mathcal{H}(I)$. Take $R > 1$ so that $g \in \mathcal{H}(\{z : |\phi(z)| \leq R\})$. Set $\Gamma_R = \{z : |\phi(z)| = R\}$. Using Cauchy's integral formula and Fubini's theorem, we obtain

$$\langle g, F_k \rangle = \int g(\zeta) F_k(\zeta) d\mu(\zeta) = \int_I \frac{1}{2\pi i} \int_{\Gamma_R} \frac{g(z)dz}{z - \zeta} F_k(\zeta) d\mu(\zeta) =$$

$$\frac{1}{2\pi i} \int_{\Gamma_R} g(z) H_k(z) dz. \tag{5.9}$$

Using (5.6)-(5.9), it readily follows that

$$\sum_{k=0}^{\infty} \langle g, F_k \rangle F_k(z)$$

converges uniformly on each compact subset of $D_R = \{z : |\phi(z)| < R\}$ and the limit must be an analytic function on D_R. On the other hand, this is the Fourier series of g with respect to the orthonormal system $\{F_n\}$; consequently, its limit on I in norm-2 is g. By the principle of analytic continuation, the series converges uniformly to g on compact subsets of D_R. This justifies that the right hand sides of a.2) in Definition 5.1.2 and c.2) in Definition 5.2.1 converge uniformly to the corresponding left hand on each compact subset of a neighborhood of I.

5.3 Convergence of incomplete Fourier-Padé approximants on I

Theorem 5.3.1. *Let $f \in \mathcal{H}(I)$ and $\mu' > 0$ a.e. on I. Fix m and m^* nonnegative integers, $m \geq m^*$. For each $n \geq m$, let $R_{n,m}$ be an incomplete Padé approximant of type (n, m, m^*) for f. Then, for each $\varepsilon > 0$ and every compact subset K of $D_{m^*}(f)$*

$$\limsup_{n\to\infty} \|f - R_{n,m}\|_{K(\varepsilon)}^{1/n} \leq \frac{\|\phi(z)\|_K}{R_{m^*}(f)}. \tag{5.10}$$

In particular,

$$h\text{-}\lim_{n\to\infty} R_{n,m} = f \quad in \quad D_{m^*}(f).$$

Finally, for each $\varepsilon > 0$, and each pole z_j of f in $D_{m^*}(f)$, there exists n_0 such that for all $n \geq n_0$ the polynomials $Q_{n,m}$ have at least τ_j zeros in the disk $\{z : |z - z_j| < \varepsilon\}$, where τ_j denotes the order of the pole z_j.

Proof. Let $z_1, \ldots, z_N$ be the distinct poles of f in $D_{m^*}(f)$ and $\tau_1, \ldots, \tau_N$ their orders, respectively. Consequently, $\sum_{k=1}^{N} \tau_k = \widetilde{m} \leq m^*$. Put

$$w(z) = \prod_{k=1}^{N} (z - z_k)^{\tau_k}.$$

Using c.2) we obtain

$$(w_m Q_{n,m} f - w_m P_{n,m})(z) = \sum_{k \geq n+1} a_{n,k} w_m(z) F_k(z) = \sum_{\nu \geq 0} b_{n,\nu} F_\nu(z). \tag{5.11}$$

Notice that $w_m Q_{n,m} f - w_m P_{n,m} \in \mathcal{H}(D_{m^*}(f))$ and $\deg w_m P_{n,m} \leq n$.

We have

$$b_{n,\nu} = \sum_{k=n+1}^{\infty} a_{n,k} \langle w_m F_k, F_\nu \rangle = \sum_{k=n+1}^{\infty} a_{n,k} \langle w_m F_\nu, F_k \rangle, \quad \nu \geq 0. \tag{5.12}$$

Therefore $b_{n,\nu} = 0$, for $\nu = 0, 1, 2, \ldots, n - m^*$, and (5.11) reduces to

$$(w_m Q_{n,m} f - w_m P_{n,m})(z) = \sum_{\nu \geq n - m^* + 1} b_{n,\nu} F_\nu(z).$$

We will show that $\sum_{\nu \geq n-m^*+1} b_{n,\nu} F_\nu$ converges uniformly to zero on each compact subset of $D_{m^*}(f)$ as $n \to \infty$ with geometric rate. Let us separate the series in two

$$\sum_{\nu \geq n-m^*+1} b_{n,\nu} F_\nu(z) = \sum_{\nu = n-m^*+1}^{n} b_{n,\nu} F_\nu(z) + \sum_{\nu \geq n+1} b_{n,\nu} F_\nu(z). \tag{5.13}$$

Fix a compact subset $K, K \subset D_{m^*}(f)$.

We begin with the infinite series to the right of (5.13) which is easier to handle. Take $1 < R < R_{m^*}(f)$ such that K and the poles of f in $D_{m^*}(f)$ are surrounded by Γ_R. Using (5.9), (5.11) and the first inequality in (5.4) it follows that

$$|b_{n,\nu}| = |w_m Q_{n,m} f F_\nu(\xi) d\mu(\xi)| \leq C \|H_\nu\|_{\Gamma_R}, \quad \nu \geq n+1,$$

for some constant C independent of n. Choose $\delta > 0$ sufficiently small so that $\|\phi\|_K + \delta < R - \delta$. According to (5.7)-(5.8), there exists n_0 such that

$$\|H_\nu\|_{\Gamma_R} \leq \frac{1}{(R-\delta)^\nu}, \quad \|F_\nu\|_K \leq (\|\phi\|_K + \delta)^\nu, \qquad \nu \geq n_0.$$

Set $q = \frac{\|\phi\|_K+\delta}{R-\delta}(< 1)$. If $n \geq n_0$, we obtain

$$\sum_{\nu \geq n+1} |b_{n,\nu}| \|F_\nu\|_K \leq C \sum_{\nu \geq n+1} q^\nu = C\frac{q^{n+1}}{1-q}.$$

Taking $\limsup_n$ of the nth root, then making δ tend to zero and R to $R_{m^*}(f)$ it readily follows that

$$\limsup_{n\to\infty} \| \sum_{\nu \geq n+1} b_{n,\nu} F_\nu \|_K^{1/n} \leq \frac{\|\phi\|_K}{R_{m^*}(f)}. \tag{5.14}$$

To deal with the first sum, we estimate the values $a_{n,k}$. Choose $r > 1, \Gamma_r \subset D_0(f)$, and $r < R < R_{m^*}(f)$ such that all the poles of f in $D_{m^*}(f)$ and the compact set K are surrounded by Γ_R. Set

$$a_{n,k} = \frac{1}{2\pi i}\int_{\Gamma_r} (Q_{n,m} f H_k)(z) dz, \qquad k \geq n - m^* + 1,$$

$$\gamma_{n,k} = \frac{1}{2\pi i}\int_{\Gamma_R} (Q_{n,m} f H_k)(z) dz, \qquad k \geq n - m^* + 1.$$

Notice that $a_{n,k} = 0, k = n-m^*+1, \ldots, n$, and for $k \geq n+1$ the $a_{n,k}$ are precisely the Fourier coefficients on the right hand side of c.2) (see (5.9)).

Since f is meromorphic in $D_{m^*}(f)$, using the residue theorem we have

$$\gamma_{n,k} - a_{n,k} = \sum_{j=1}^N \mathrm{Res}(Q_{n,m} f H_k, z_j), \quad k \leq n - m^* + 1, \ldots, \tag{5.15}$$

where $\mathrm{Res}(Q_{n,m} f H_k(z), z_j)$ is the residue of $Q_{n,m} f H_k$ at z_j. At z_j the function $Q_{n,m} f H_k$ has a pole of order $\leq \tau_j$; therefore,

$$\mathrm{Res}(Q_{n,m} f H_k, z_j) = \frac{1}{(\tau_j - 1)!} \lim_{z\to z_j} \left[(Q_{n,m} H_n)(z) \frac{(z-z_j)^{\tau_j} f(z) H_k(z)}{H_n(z)} \right]^{(\tau_j - 1)}.$$

Using the Leibnitz formula, it follows that

$$\left[(Q_{n,m} H_n)(z) \frac{(z-z_j)^{\tau_j} f(z) H_k(z)}{H_n(z)} \right]^{(\tau_j - 1)} =$$

$$\sum_{\ell=0}^{\tau_j-1} \binom{\tau_j-1}{\ell} [(Q_{n,m}H_n)(z)]^{(\tau_j-1-\ell)} \left[\frac{(z-z_j)^{\tau_j} f(z) H_k(z)}{H_n(z)}\right]^{(\ell)}. \tag{5.16}$$

Define

$$\alpha_n(j,\ell) := \frac{1}{(\tau_j-1)!}\binom{\tau_j-1}{\ell} \lim_{z\to z_j} (Q_{n,m}H_n)^{(\tau_j-1-\ell)}(z). \tag{5.17}$$

By (5.16) and (5.17), we obtain

$$\mathrm{Res}(Q_{n,m}fH_k, z_j) = \sum_{\ell=0}^{\tau_j-1} \alpha_n(j,\ell) \left[\frac{(z-z_j)^{\tau_j} f(z) H_k(z)}{H_n(z)}\right]^{(\ell)}_{z=z_j}. \tag{5.18}$$

Notice that $\alpha_n(j,\ell)$ does not depend on k. Then, for each $k = 0,1,2,\ldots$, (5.15) and (5.18) give

$$a_{n,k} = \gamma_{n,k} - \sum_{j=1}^{N}\sum_{\ell=0}^{\tau_j-1} \alpha_n(j,\ell) \left[\frac{(z-z_j)^{\tau_j} f(z) H_k(z)}{H_n(z)}\right]^{(\ell)}_{z=z_j}. \tag{5.19}$$

Since $a_{n,k} = 0$, for $k = n-m^*+1,\ldots,n$ we can write

$$\gamma_{n,k} = \sum_{j=1}^{N}\sum_{\ell=0}^{\tau_j-1} \alpha_n(j,\ell) \left[\frac{(z-z_j)^{\tau_j} f(z) H_k(z)}{H_n(z)}\right]^{(\ell)}_{z=z_j}. \tag{5.20}$$

Recall that $\sum_{j=1}^{d} \tau_j = \widetilde{m} \le m^*$. Thus we have obtained a system of $\widetilde{m}$ equations on $\widetilde{m}$ unknowns (the quantities $\alpha_n(j,\ell)$).

The determinant Δ_n of the system has the form

$$\Delta_n = \begin{vmatrix} \left[\frac{(z-z_j)^{\tau_j} f(z) H_{n-\widetilde{m}+1}(z)}{H_n(z)}\right]_{z=z_j} & \cdots & \left[\frac{(z-z_j)^{\tau_j} f(z) H_{n-\widetilde{m}+1}(z)}{H_n(z)}\right]^{(\tau_j-1)}_{z=z_j} \\ \left[\frac{(z-z_j)^{\tau_j} f(z) H_{n-\widetilde{m}+2}(z)}{H_n(z)}\right]_{z=z_j} & \cdots & \left[\frac{(z-z_j)^{\tau_j} f(z) H_{n-\widetilde{m}+2}(z)}{H_n(z)}\right]^{(\tau_j-1)}_{z=z_j} \\ \vdots & \vdots & \vdots \\ \left[\frac{(z-z_j)^{\tau_j} f(z) H_n(z)}{H_n(z)}\right]_{z=z_j} & \cdots & \left[\frac{(z-z_j)^{\tau_j} f(z) H_n(z)}{H_n(z)}\right]^{(\tau_j-1)}_{z=z_j} \end{vmatrix}_{j=1,\ldots,N},$$

where the subindex on the determinant means that the indicated groups of columns are successively written for $j = 1,2,\ldots,N$. Due to (5.7) we have

$$\lim_{n\longrightarrow\infty} \Delta_n = \Delta,$$

where

$$\Delta = \begin{vmatrix} [(z-z_j)^{\tau_j}\phi^{\widetilde{m}-1}(z)f(z)]_{z=z_j} & \cdots & [(z-z_j)^{\tau_j}\phi^{\widetilde{m}-1}(z)f(z)]^{(\tau_j-1)}_{z=z_j} \\ [(z-z_j)^{\tau_j}\phi^{\widetilde{m}-2}(z)f(z)]_{z=z_j} & \cdots & [(z-z_j)^{\tau_j}\phi^{\widetilde{m}-2}(z)f(z)]^{(\tau_j-1)}_{z=z_j} \\ \vdots & \vdots & \ddots & \vdots \\ [(z-z_j)^{\tau_j}f(z)]_{z=z_j} & \cdots & [(z-z_j)^{\tau_j}f(z)]^{(\tau_j-1)}_{z=z_j}. \end{vmatrix}_{j=1,2,\ldots,N}.$$

Notice that $\Delta \neq 0$. In fact, should this determinant be equal to zero that would mean that there exists a linear combination of its rows giving the zero vector. In turn, this implies that there exists a polynomial of degree $\leq \widetilde{m}-1$ in $\phi(z)$ which multiplied times f eliminates the $\widetilde{m}$ poles which f has in $D_{m^*}(f)$ which is clearly impossible because such polynomial can have at most $\widetilde{m}-1$ zeros since ϕ is one to one. Therefore, $|\Delta_n| \geq C > 0$ for all sufficiently large n. In the sequel we only consider such n's.

Let $\Delta_n(j,\ell)$ denote the determinant which is obtained substituting in the determinant of the system the column with index $q = \sum_{i=1}^{j-1} \tau_i + \ell + 1$ with the column vector $(\gamma_{n,n-\widetilde{m}+1}, \ldots, \gamma_{n,n})^t$ formed with the independent terms of equations (5.20). By Cramer's rule

$$\alpha_n(j,\ell) = \frac{\Delta_n(j,\ell)}{\Delta_n} = \frac{1}{\Delta_n} \sum_{s=1}^{\widetilde{m}} \gamma_{n,n-\widetilde{m}+s} M_n(s,q), \tag{5.21}$$

where $M_n(s,q)$ is the cofactor corresponding to row s and column q of $\Delta_n(j,\ell)$. Making use of the fact that the $\alpha_n(j,\ell)$ do not depend on k from (5.19) and (5.21) it follows that

$$a_{n,k} = \gamma_{n,k} - \frac{1}{\Delta_n} \sum_{j=1}^{N} \sum_{\ell=0}^{\tau_j-1} \sum_{s=1}^{m^*} \gamma_{n,n-\widetilde{m}+s} M_n(s,q) \left(\frac{H_k}{H_n}\right)^{(\ell)} (z_j), \tag{5.22}$$

for $k \geq n+1$.

Choose $\varepsilon > 0$ so that $|\phi(z_j)| - \varepsilon > r$ for all $j = 1, \ldots, N$. Recall that r was chosen greater than 1. Using Cauchy's integral formula we obtain

$$\left(\frac{H_k}{H_n}\right)^{(\ell)} (z_j) = \frac{\ell!}{2\pi i} \int_{|z-z_j|=\varepsilon} \frac{H_k(z)dz}{H_n(z)(z-z_j)^{\ell+1}}.$$

On account of (5.6), there exists a constant C_1 such that

$$\left| \left(\frac{H_k}{H_n}\right)^{(\ell)} (z_j) \right| \leq \frac{C_1}{r^{k-n}}, \qquad k \geq n - m^* + 1.$$

for all $j = 1 \ldots, N, l = 0, 1, \ldots, \tau_j - 1$, and n sufficiently large. Consequently,

$$|M_n(s,q)| \leq C_2.$$

Using (5.22), it follows that there exists a constant C_3 such that

$$|a_{n,k}| \leq |\gamma_{n,k}| + \frac{C_3}{r^{k-n}} \sum_{s=1}^{m^*} |\gamma_{n-\widetilde{m}+s}|, \quad k \geq n+1. \tag{5.23}$$

From the integral which defines $\gamma_{n,k}$, the first inequality in (5.4), and using (5.7), given $\delta > 0, R - \delta > r$, for all sufficiently large n, we have

$$|\gamma_{n,k}| \leq \frac{C_4}{(R-\delta)^k}, \qquad k \geq n+1.$$

and taking into consideration (5.23), we obtain

$$|a_{n,k}| \leq \frac{C_5}{r^{k-n}(R-\delta)^{n+1}}, \quad k \geq n+1, \tag{5.24}$$

for some constant C_5 since

$$|\langle w_m F_k, F_\nu \rangle| \leq \|w_m\|_I = \max_I |w_m|.$$

Due to (5.12) we can find a constant C_6 for which

$$|b_{n,\nu}| \leq \frac{C_6}{(R-\delta)^{n+1}}. \tag{5.25}$$

Finally, let us estimate $\displaystyle\sum_{\nu=n-m^*+1}^{n} b_{n,\nu} F_\nu(z)$. Fix a compact subset $K \subset D_{m^*}(f)$. We assume that R and δ chosen previously satisfy $\|\phi\|_K + \delta < R - \delta$. From (5.7) it follows that there exists some constant C_7 such that

$$\|F_\nu\|_K \leq C_7(\|\phi\|_K + \delta)^\nu, \qquad \nu \geq n - m^* + 1.$$

Therefore, making use of (5.25), we obtain

$$\| \sum_{\nu=n-m^*+1}^{n} b_{n,\nu} F_\nu \|_K \leq \frac{C_6 C_7}{(R-\delta)^{n+1}} \sum_{\nu=n-m^*+1}^{n} (\|\phi\|_K + \delta)^\nu \leq m^* C_6 C_7 \left(\frac{\|\phi\|_K + \delta}{R-\delta} \right)^{n+1}.$$

Taking $\limsup_n$ of the nth root, then making R tend to $R_{m^*}(f)$ and δ to zero, we arrive at

$$\limsup_{n\to\infty} \| \sum_{\nu=n-m^*+1}^{n} b_{n,\nu} F_\nu(z) \|_K^{1/n} \leq \frac{\|\phi\|_K}{R_{m^*}(f)}. \tag{5.26}$$

Formulas (5.11) and (5.13) together with inequalities (5.14) and (5.26) give

$$\limsup_{n\to\infty} \|w_m(Q_{n,m} f - P_{n,m})\|_K^{1/n} \leq \frac{\|\phi\|_K}{R_{m^*}(f)}. \tag{5.27}$$

Fix $\varepsilon > 0$ and take any compact subset $K \subset D_{m^*}(f)$. For $z \in K(\varepsilon) = K \setminus J_\varepsilon$, according to the second inequality in (5.4), we have (notice that $J(\varepsilon)$ leaves out an $\varepsilon/6m$ neighborhood of the zeros of w_m)

$$\|f - R_{n,m}\|_{K(\varepsilon)} \leq \frac{n^{2m}}{C_8}\|w_m(Q_{n,m}f - P_{n,m})\|_K$$

for some constant C_8, and applying (5.27), we obtain (5.10) which implies convergence in Hausdorff content in $D_{m^*}(f)$ as claimed. The statement concerning the asymptotic behavior of some of the zeros of $Q_{n,m}$ is a direct consequence of the convergence in Hausdorff content and Gonchar's Lemma. With this we conclude the proof. □

5.4 Convergence of simultaneous Fourier-Padé approximation on I

In this section $\mathbf{f} = (f_1, \ldots, f_d) \in \mathcal{H}(I)$ and $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$. We will study the convergence of $\mathbf{R}_{n,\mathbf{m}}$ to $\mathbf{f}$. Recall that for each $k = 1, \ldots, d$ the rational function $R_{n,\mathbf{m},k}$ is an $(n, |\mathbf{m}|, m_k)$ incomplete Fourier-Padé approximation to f_k. Let $\mathcal{P}_{n,\mathbf{m}}$ be the collection of zeros of $Q_{n,\mathbf{m}}$. A direct consequence of Theorem 5.3.1 is the following result.

Corollary 5.4.1. *Let $\mathbf{f} \in \mathcal{H}(I)$ and $\mu' > 0$ a.e. on I. Fix $\mathbf{m} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$. For each $n \geq |\mathbf{m}|$, let $\mathbf{R}_{n,\mathbf{m}}$ be a Fourier-Padé approximant of type $(n, \mathbf{m})$ for $\mathbf{f}$. Then, for each $i = 1, \ldots, d, K \subset D_{m_i}(f_i)$, and $\varepsilon > 0$*

$$\limsup_{n\to\infty} \|f_i - R_{n,\mathbf{m},i}\|_{K(\varepsilon)}^{1/n} \leq \frac{\|\phi\|_K}{R_{m_i}(f_i)}.$$

In particular.

$$h\text{-}\lim_{n\to\infty} R_{n,\mathbf{m},i} = f_i \ \text{ in } \ D_{m_i}(f_i).$$

Finally, for each pole z_j of f_i in $D_{m_i}(f_i), i = 1, \ldots, d$, and each $\varepsilon > 0$, there exists n_0 such that for all $n \geq n_0$ the polynomials $Q_{n,\mathbf{m}}$ have at least τ_j zeros in the disk $\{z : |z - z_j| < \varepsilon\}$, where τ_j denotes the order of the pole z_j.

Now let us prove Theorem 5.1.3.

Proof of Theorem 5.1.3. First of all, notice that $\mathbf{f}$ must have exactly $|\mathbf{m}|$ poles in $D_{|\mathbf{m}|}(\mathbf{f})$. If this were not the case, it is easy to show that $\mathbf{f}$ is not polewise independent with respect to $\mathbf{m}$ in $D_{|\mathbf{m}|}(\mathbf{f})$. By $z_1, \ldots, z_N$ we denote the distinct poles of $\mathbf{f}$ in $D_{|\mathbf{m}|}(\mathbf{f})$ and let $\tau_1, \ldots, \tau_N$ be their orders, respectively.

According to Definition 5.1.1, for each $i = 1, \ldots, d$ we have

$$(Q_{n,\mathbf{m}} f_i - P_{n,\mathbf{m},i})(z) = A_{n,n+1}^{(i)} F_{n+1}(z) + \ldots,$$

and $\deg P_{n,\mathbf{m},i} \leq n - m_i$. Therefore,

$$A_{n,k}^{(i)} = \langle Q_{n,\mathbf{m}} f_i, F_k \rangle, \qquad k \geq n - m_i + 1,$$

and $A_{n,k}^{(i)} = 0, k = n - m_i + 1, \ldots, n$. Let us estimate the values $A_{n,k}^{(i)}$. The procedure is similar to the one employed to estimate the quantities $a_{n,k}$ in the proof of Theorem 5.3.1, so we will not go through all the details.

Take $r > 1$ such that $\Gamma_r \subset D_0(\mathbf{f})$ and $R < R_{|\mathbf{m}|}(\mathbf{f})$ such that Γ_R surrounds all the poles $z_1, \ldots, z_N$. Obviously

$$A_{n,k}^{(i)} = \frac{1}{2\pi i} \int_{\Gamma_r} (Q_{n,\mathbf{m}} f H_k)(z) dz, \qquad k \geq n - m_i + 1.$$

Define

$$\gamma_{n,k}^{(i)} = \frac{1}{2\pi i} \int_{\Gamma_R} (Q_{n,\mathbf{m}} f H_k)(z) dz, \qquad k \geq n - m_i + 1.$$

Using the residue theorem it follows that

$$\gamma_{n,k}^{(i)} - A_{n,k}^{(i)} = \sum_{j=1}^{N} \operatorname{Res}(Q_{n,\mathbf{m}} f_i H_k, z_j), \quad k \geq n - m^* + 1, \tag{5.28}$$

and

$$\operatorname{Res}(Q_{n,m} f_i H_k, z_j) = \frac{1}{(\tau_j - 1)!} \lim_{z \to z_j} \left[(Q_{n,\mathbf{m}} H_n)(z) \frac{(z - z_j)^{\tau_j} f_i(z) H_k(z)}{H_n(z)} \right]^{(\tau_j - 1)}.$$

Using the Leibnitz formula, we have

$$\left[(Q_{n,\mathbf{m}} H_n)(z) \frac{(z - z_j)^{\tau_j} f_i(z) H_k(z)}{H_n(z)} \right]^{(\tau_j - 1)} =$$

$$\sum_{l=0}^{\tau_j-1}\binom{\tau_j-1}{\ell}\left[(Q_{n,\mathbf{m}}H_n)(z)\right]^{(\tau_j-1-\ell)}\left[\frac{(z-z_j)^{\tau_j}f_i(z)H_k(z)}{H_n(z)}\right]^{(\ell)}. \tag{5.29}$$

Define

$$\alpha_n(j,\ell):=\frac{1}{(\tau_j-1)!}\binom{\tau_j-1}{\ell}\lim_{z\to z_j}\left[Q_{n,m}H_n(z)\right]^{(\tau_j-1-\ell)}. \tag{5.30}$$

These quantities do not depend on i or k. By (5.29) and (5.30), we obtain

$$\operatorname{Res}(Q_{n,m}f_iH_k,z_j)=\sum_{\ell=0}^{\tau_j-1}\alpha_n(j,\ell)\left[\frac{(z-z_j)^{\tau_j}f_i(z)H_k(z)}{H_n(z)}\right]^{(\ell)}_{z=z_j}. \tag{5.31}$$

Then, for each $k\geq n-m^*+1$ and $i=1,\ldots,d$, (5.28) and (5.31) give

$$A_{n,k}^{(i)}=\gamma_{n,k}^{(i)}-\sum_{j=1}^{N}\sum_{\ell=0}^{\tau_j-1}\alpha_n(j,\ell)\left[\frac{(z-z_j)^{\tau_j}f_i(z)H_k(z)}{H_n(z)}\right]^{(\ell)}_{z=z_j}.$$

For $k=n-m_i+1,\ldots,n, i=1,\ldots,d$

$$\gamma_{n,k}^{(i)}=\sum_{j=1}^{N}\sum_{\ell=0}^{\tau_j-1}\alpha_n(j,\ell)\left[\frac{(z-z_j)^{\tau_j}f_i(z)H_k(z)}{H_n(z)}\right]^{(\ell)}_{z=z_j}, \tag{5.32}$$

since $A_{n,k}^{(i)}=0$ for these values of k. Thus, we have obtained a system of $|\mathbf{m}|$ equations on $|\mathbf{m}|$ unknowns (the quantities $\alpha_n(j,\ell)$).

The determinant Δ_n of the system has the form

$$\left|\begin{array}{ccc}\left[\frac{(z-z_j)^{\tau_j}f_i(z)H_{n-m_i+1}(z)}{H_n(z)}\right]_{z=z_j} & \cdots & \left[\frac{(z-z_j)^{\tau_j}f_i(z)H_{n-m_i+1}(z)}{H_n(z)}\right]^{(\tau_j-1)}_{z=z_j}\\ \left[\frac{(z-z_j)^{\tau_j}f_i(z)H_{n-m_i+2}(z)}{H_n(z)}\right]_{z=z_j} & \cdots & \left[\frac{(z-z_j)^{\tau_j}f_i(z)H_{n-m_i+2}(z)}{H_n(z)}\right]^{(\tau_j-1)}_{z=z_j}\\ \vdots & \vdots & \vdots\\ \left[\frac{(z-z_j)^{\tau_j}f_i(z)H_n(z)}{H_n(z)}\right]_{z=z_j} & \cdots & \left[\frac{(z-z_j)^{\tau_j}f_i(z)H_n(z)}{H_n(z)}\right]^{(\tau_j-1)}_{z=z_j}\end{array}\right|_{j=1,\ldots,N,i=1,\ldots,d},$$

where the subindex on the determinant means that the indicated groups of columns are successively written for $j=1,2,\ldots,N$ and the rows repeated for $i=1,\ldots,d$. Due to (5.7) we have

$$\lim_{n\longrightarrow\infty}\Delta_n=\Delta,$$

where

$$\Delta=\left|\begin{array}{ccc}[(z-z_j)^{\tau_j}\phi^{m_i-1}(z)f_i(z)]_{z=z_j} & \cdots & [(z-z_j)^{\tau_j}\phi^{m_i-1}(z)f_i(z)]^{(\tau_j-1)}_{z=z_j}\\ [(z-z_j)^{\tau_j}\phi^{m_i-2}(z)f_i(z)]_{z=z_j} & \cdots & [(z-z_j)^{\tau_j}\phi^{m_i-2}(z)f_i(z)]^{(\tau_j-1)}_{z=z_j}\\ \vdots & \vdots & \vdots\\ [(z-z_j)^{\tau_j}f_i(z)]_{z=z_j} & \cdots & [(z-z_j)^{\tau_j}f_i(z)]^{(\tau_j-1)}_{z=z_j}.\end{array}\right|_{j=1,2,\ldots,N,i=1,\ldots,d}.$$

Let us show that $\Delta \neq 0$. Assume the contrary. Then there exists a non trivial linear combination of rows giving the zero vector. This mean that there exist polynomials $p_1, \ldots, p_d, \deg p_i \leq m_i - 1$, not all zero such that

$$\sum_{i=1}^{d} [(z - z_j)^{\tau_j} p_i(\phi(z)) f_i(z)]^{(l)}_{z=z_j} = 0, \quad j = 1, \ldots, d, l = 0, \ldots, \tau_j - 1. \tag{5.33}$$

Let us prove that this implies that $\mathbf{f}$ is not polewise independent with respect to $\mathbf{m}$ in $D_{|\mathbf{m}|}(\mathbf{f})$.

In fact, fix $j \in \{1, \ldots, d\}$ and let us rewrite (5.33) as follows

$$[\Phi(z)F(z)]^{(l)}_{z=z_j} = 0, \quad l = 0, \ldots, \tau_j - 1, \tag{5.34}$$

where

$$\Phi(z) := \left(\frac{z - z_j}{\phi(z) - \phi(z_j)}\right)^{\tau_j}, \quad F(z) := (\phi(z) - \phi(z_j))^{\tau_j} \sum_{i=1}^{d} p_i(\phi(z)) f_i(z).$$

Notice that

$$\lim_{z \longrightarrow z_j} \frac{\phi(z) - \phi(z_j)}{z - z_j} = \phi'(z_j) \neq 0,$$

because ϕ is one to one; therefore $\Phi(z_j) := \left(\frac{1}{\phi'(z_j)}\right)^{\tau_j} \neq 0$ is well defined. Also, $(\phi(z) - \phi(z_j))^{\tau_j} f_i(z)$ is holomorphic in a neighborhood of z_j since f_i can have at that point at most a pole of order τ_j. Taking $l = 0$ in (5.34) we get

$$\Phi(z_j)F(z_j) = 0,$$

which implies that $F(z_j) = 0$.

Assume that

$$F^{(l)}(z_j) = 0, \quad l = 0, \ldots, s, \quad s \leq \tau_j - 2, \tag{5.35}$$

and let us show that $F^{(s+1)} = 0$. Indeed, by the Leibnitz formula, (5.34), and (5.35), it follows that

$$0 = [\Phi(z)F(z)]^{s+1}_{z=z_j} = \sum_{k=0}^{s+1} \Phi^{(k)}(z_j) F^{s+1-k}(z_j) = \Phi(z_j)F^{(s+1)}(z_j),$$

which implies what we need since $\Phi(z_j) \neq 0$. Thus, we have proved that

$$\sum_{i=1}^{d} [(\phi(z) - \phi(z_j))^{\tau_j} p_i(\phi(z)) f_i(z)]^{(l)}_{z=z_j} = 0, \quad j = 1, \ldots, d, l = 0, 1, \ldots, \tau_j - 1.$$

Making the change of variables $\phi(z) = \zeta$, we obtain

$$\sum_{i=1}^{d} \left[(\zeta - \zeta_j)^{\tau_j} p_i(\zeta) f_i(\phi^{-1}(\zeta))\right]^{(l)}_{\zeta=\phi(z_j)} = 0, \quad j = 1, \ldots, d, l = 0, \ldots, \tau_j - 1.$$

Thus, $\sum_{i=1}^{d} p_i(\xi) f_i(\phi^{-1}(\xi))$ is holomorphic in the annulus $\{\zeta : 1 < |\zeta| < R_m\}$ against our assumption of polewise independence. Therefore, $\Delta \neq 0$ and $|\Delta_n| \geq C > 0$ for all sufficiently large n. We restrict our attention to such n's.

Using Cramer's rule, the system of equations (5.32) allows us to express the $\alpha_n(j, l)$ in terms of the $\gamma^{(i)}_{n,k}, i = 1, \ldots, d, k = n - m_i + 1, \ldots, n$. Arguing as in the proof of Theorem 5.3.1 we arrive at the bounds (compare with (5.24))

$$|A^{(i)}_{n,k}| \leq \frac{C}{r^{k-n}(R - \delta)^n}, \quad k \geq n + 1, \quad i = 1, \ldots, d. \tag{5.36}$$

Fix $i \in \{1, 2, \ldots, d\}$. We have

$$Q_{n,\mathbf{m}}(z) Q_{|\mathbf{m}|}(z) f_i(z) - Q_{|\mathbf{m}|}(z) P_{n,\mathbf{m},i}(z) = \sum_{\ell \geq n+1} A^{(i)}_{n,\ell} Q_{|\mathbf{m}|}(z) F_\ell(z) =$$

$$\sum_{\nu=n-|\mathbf{m}|+1}^{n+|\mathbf{m}|-m_i} B^{(i)}_{n,\nu} F_\nu(z) + \sum_{\nu \geq n+|\mathbf{m}|-m_i+1} B^{(i)}_{n,\nu} F_\nu(z), \tag{5.37}$$

where $Q_{|\mathbf{m}|}(z) P_{n,\mathbf{n},i}(z)$ has degree at most $n + |\mathbf{m}| - m_i$.

Fix a compact set $K \subset D_{|\mathbf{m}|}(\mathbf{f})$. In the sequel we assume that R was chosen so that Γ_R also surrounds K. For the series in (5.37), as in the proof of Theorem 5.3.1, it is easy to show that

$$\limsup_{n\to\infty} \| \sum_{\nu \geq n+|\mathbf{m}|-m_i+1} B^{(i)}_{n,\nu} F_\nu(z) \|_K^{1/n} \leq \frac{\|\phi\|_K}{R_{|\mathbf{m}|}(\mathbf{f})}.$$

In order to prove that

$$\limsup_{n\to\infty} \| \sum_{\nu=n-|\mathbf{m}|+1}^{n+|\mathbf{m}|-m_i} B^{(i)}_{n,\nu} F_\nu(z) \|_K^{1/n} \leq \frac{\|\phi\|_K}{R_{|\mathbf{m}|}(\mathbf{f})}$$

one employs (5.36) and the equality

$$B_{n,\nu}^{(i)} = \sum_{\ell \geq n+1} A_{n,\ell}^{(i)} \langle Q_{|\mathbf{m}|}(z) F_\ell(z), F_\nu \rangle,$$

in a similar fashion as in Theorem 5.3.1. Consequently, for each $i = 1, \ldots, d$

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}} Q_{|\mathbf{m}|} f_i - Q_{|\mathbf{m}|} P_{n,\mathbf{n},i}\|_K^{1/n} \leq \frac{\|\phi\|_K}{R_{|\mathbf{m}|}(\mathbf{f})}. \tag{5.38}$$

From here convergence in Hausdorff content readily follows. Therefore, using Gonchar's lemma, each pole of $\mathbf{f}$ in $D_{|\mathbf{m}|}(\mathbf{f})$ attracts as many zeros of $Q_{n,\mathbf{m}}$ as its order.

Since $\deg Q_{n,\mathbf{m}} \leq |\mathbf{m}|$ and the total number of poles of $\mathbf{f}$ in $D_{|\mathbf{m}|}(\mathbf{f})$ equals $|\mathbf{m}|$ we have $\deg Q_{n,\mathbf{m}} = |\mathbf{m}|$ for all sufficiently large n. This implies that for such n's $\mathbf{R}_{n,\mathbf{m}}$ is unique. In fact, if this were not the case we could find an infinite subsequence of indices for which Definition 5.1.1 has solutions with $\deg Q_{n,\mathbf{m}} < |\mathbf{m}|$, which contradicts what was proved. In the sequel, we only consider such n's.

As the poles of $\mathbf{f}$ attract all the zeros of $Q_{n,|\mathbf{m}|}$, we obtain

$$Q_{n,\mathbf{m}}(z) = \prod_{j=1}^{|\mathbf{m}|} \left(1 - \frac{z}{z_{n,j}}\right), \qquad n \geq n_0,$$

and

$$\lim_{n\to\infty} Q_{n,\mathbf{m}}(z) = \prod_{k=1}^{N} \left(1 - \frac{z}{z_k}\right)^{\tau_k} = Q_{|\mathbf{m}|}(z).$$

Therefore, (5.2) follows at once from (5.38).

Let us prove (5.3). To this end we start by showing that for $k = 1, \ldots, N$

$$\limsup_{n\to\infty} |Q_{n,\mathbf{m}}^{(j)}(z_k)|^{1/n} \leq |\phi(z_k)|/R_{|\mathbf{m}|}(\mathbf{f}), \qquad j = 0, \ldots, \tau_k - 1. \tag{5.39}$$

Suppose that the pole z_k attains its order with the function f_i. Let $\varepsilon > 0$ be sufficiently small so that the closed disk $C_{k,\varepsilon} = \{z : |z - z_k| \leq \varepsilon\}$ is contained in $D_{|\mathbf{m}|}(\mathbf{f})$ and contains no other pole of $\mathbf{f}$. On account of (5.38)

$$\limsup_{n\to\infty} \|(z - z_k)^{\tau_k} f_i Q_{n,\mathbf{m}} - (z - z_k)^{\tau_k} P_{n,\mathbf{m},i}\|_{C_{k,\varepsilon}}^{1/n} \leq \frac{\|\phi\|_{C_{k,\varepsilon}}}{R_{|\mathbf{m}|}(\mathbf{f})},$$

and using Cauchy's integral formula for the derivative, we have

$$\limsup_{n\to\infty} \|[(z-z_k)^{\tau_k} f_i Q_{n,\mathbf{m}} - (z-z_k)^{\tau_k} P_{n,\mathbf{m},i}]^{(j)}\|_{C_{k,\varepsilon}}^{1/n} \leq \frac{\|\phi\|_{C_{k,\varepsilon}}}{R_{|\mathbf{m}|}(\mathbf{f})}, \tag{5.40}$$

for all $j \geq 0$. In particular, taking $z = z_k$ and $j = 0$, we obtain

$$\limsup_{n\to\infty} |AQ_{n,\mathbf{m}}(z_k)|^{1/n} \leq \frac{|\phi(z_k)|}{R_{|\mathbf{m}|}(\mathbf{f})},$$

where $A = \lim_{z\to z_k}(z-z_k)^{\tau_k} f_i(z) \neq 0$ since z_k is a pole of f_i of order τ_k. Therefore

$$\limsup_{n\to\infty} |Q_{n,\mathbf{m}}(z_k)|^{1/n} \leq \frac{|\phi(z_k)|}{R_{|\mathbf{m}|}(\mathbf{f})}.$$

Proceeding by induction, take $s \leq \tau_k$ and assume that

$$\limsup_{n\to\infty} |(Q_{n,\mathbf{m}}^{(j)}(z_k)|^{1/n} \leq \frac{|\phi(z_k)|}{R_{|\mathbf{m}|}(\mathbf{f})}, \qquad j = 0, \ldots, s-2, \tag{5.41}$$

and let us show that (5.41) holds for $j = s-1$. As $s - 1 < \tau_k$, using (5.40) we deduce that

$$\limsup_{n\to\infty} |[(z-z_k)^{\tau_k} f_i Q_{n,\mathbf{m}}]^{(s-1)}(z_k)| \leq \frac{|\phi(z_k)|}{R_{|\mathbf{m}|}(\mathbf{f})}. \tag{5.42}$$

Applying the Leibnitz formula it follows that

$$[(z-z_k)^{\tau_k} f_i Q_{n,\mathbf{m}}]^{(s-1)}(z_k) = \sum_{\ell=0}^{s-1} \binom{s-1}{\ell} Q_{n,\mathbf{m}}^{(s-1-\ell)}(z_k)[(z-z_k)^{\tau_k} f_i]^{(\ell)}(z_k).$$

Using (5.41), (5.42), and that $A \neq 0$, we conclude that

$$\limsup_{n\to\infty} |(Q_{n,\mathbf{m}}^{(s-1)}(z_k)|^{1/n} \leq \frac{|\phi(z_k)|}{R_{|\mathbf{m}|}(\mathbf{f})}.$$

Consider a basis of polynomials $\{q_{k,s} : k = 1, \ldots, N, s = 0, \ldots, \tau_k - 1\}$ such that $\deg q_{k,s} \leq |\mathbf{m}| - 1$ for all k, s and

$$q_{k,s}^{(j)}(z_i) = \delta_{i,k}\delta_{j,s}, \quad 1 \leq i \leq N, \quad 0 \leq j \leq \tau_i - 1.$$

Then

$$Q_{n,\mathbf{m}}(z) = \sum_{k=1}^{N} \sum_{s=0}^{\tau_k - 1} Q_{n,\mathbf{m}}^{(s)}(z_k) q_{k,s}(z) + C_n Q_{|\mathbf{m}|}(z),$$

where $C_n = \prod_{k=1}^{N} z_k^{\tau_k} / \prod_{j=1}^{|\mathbf{m}|} z_{n,j}$. From (5.39) it readily follows that

$$\limsup_{n\to\infty} \|Q_{n,\mathbf{m}} - C_n Q_{|\mathbf{m}|}\|^{1/n} \leq \frac{\max\{|\phi(\zeta)| : \zeta \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{f})\}}{R_{|\mathbf{m}|}(\mathbf{f})}.$$

Evaluating at zero, we obtain

$$\limsup_{n\to\infty} |1 - C_n|^{1/n} \leq \frac{\max\{|\phi(\zeta)| : \zeta \in \mathcal{P}_{|\mathbf{m}|}(\mathbf{f})\}}{R_{|\mathbf{m}|}(\mathbf{f})},$$

which combined with the previous estimate gives us (5.3). We are done. $\square$

We wish to underline that as far as the measure is concerned, the proof of Corollary 5.3.1 and Theorem 5.1.3 only relies on the properties (5.5)-(5.8) of the system of orthonormal polynomials. Therefore, for any measure μ for which the corresponding sequence of orthonormal polynomials satisfies these relations the same results hold. For example, such is the case when the measure μ is supported on $I \cup e$ where e is a denumerable set of points contained in the real line whose accumulation points are contained in I and $\mu' > 0$ a.e. on I (see [9]). Corollary 5.3.1 complements Theorem 5.1.3 as occurs with similar results of Chapter 4.

Chapter 6

Concluding remarks

We have proved the convergence of row sequences of simultaneous rational approximations of systems of meromorphic functions in different settings. These include the case of interpolating rational functions or Hermite-Padé approximation, and rational functions constructed from Fourier expansions in terms of orthonormal systems of polynomials with respect to measures supported on the unit circle or on a segment of the real line.

Our results extend in new directions the classical Montessus de Ballore theorem as well as results of Graves-Morris/Saff, A.A. Gonchar, and S.P. Suetin. Theorems 1.2.4 and 3.3.3 constitute our most elaborated contributions where we manage to give necessary and sufficient conditions for the uniform convergence of the simultaneous approximants inside the largest possible region, indicating the exact rate of convergence of the approximants as well as of their poles. These results emulate a celebrated one of A.A. Gonchar given for classical Padé approximation. For the case of Fourier-Padé approximations we only give sufficient conditions for their convergence providing an analog of a theorem by Graves-Morris/Saff proved for Hermite-Padé approximation.

The main instrument we use is that of incomplete Padé and Fourier-Padé approximation. This construction is new and interesting in its own right. Its flexibility allows it to be used in other contexts like Padé-type approximation and least-squares rational approximation. These applications need to be further explored.

For the future there are some questions we would like to address.

- Suppose that $\lim_{n\to\infty} Q_{n,\mathbf{m}} = Q_{|\mathbf{m}|}$, where $\deg Q_{|\mathbf{m}|} = |\mathbf{m}|, Q_{|\mathbf{m}|}(0) \neq 0$. No assumption is made on the rate of convergence. In the scalar case, when $d = 1$ and $\mathbf{m} = m \in \mathbb{N}$, if the zeros of Q_m are ordered increasingly

$$0 < |z_1| \leq \cdots \leq |z_N| < |z_{N+1}| = \cdots = |z_{|\mathbf{m}|}|,$$

S.P. Suetin proved in [40] that $z_1, \ldots, z_N$ are the poles of f in $D_{m-1}(f)$ and $z_{N+1}, \ldots, z_m$ are singularities of f. What can be said in the vector case?

- Assume that there exists a sequence $\{z_n\}_{n\geq|\mathbf{m}|}$ such that $Q_{n,\mathbf{m}}(z_n) = 0$ and $\lim_{n\to\infty} z_n = a \neq 0$. For the scalar case, A. A. Gonchar posed several conjectures concerning this situation. They were solved by S.P. Suetin in [39]. What can be proved in the vector case under these assumptions. In particular, when is a a pole of $\mathbf{f}$ and of what order? Is a at least a singularity of $\mathbf{f}$?

- We would like to achieve results similar to Theorems 1.2.4 and 3.3.3 for the case of simultaneous Fourier-Padé approximation and study for these approximants questions similar to those proposed in the two previous items.

- We would also like to explore our approach using other criteria for the constructions of the simultaneous approximants. For example:

  - Interpolation along a table of points, also called multipoint approximation.
  - Using least squares vector-valued approximation as presented in several papers of A. Sidi (see reference list).
  - Using Fourier expansions in terms of orthonormal systems with respect to area measure or measures supported on an arc of the complex plane.
  - Expansions in terms of Faber polynomials.

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Appendix A

VECTOR PADÉ APPROXIMATION: VRMPade06

We implement a code in Maple to compute simultaneous Padé approximants.

A.1 Procedure to build the system whose solution gives the coefficients of $Q_{n,\mathbf{m}}$

```
> bloque:=proc(fv1,nn,mm,mmi,d::evaln,b::evaln)
> local l,j,c,k,i;
>  l := 0; for j from 0 to mmi-1 do
>        l := l+1; c := 0;
>       for k from 0 to mm-1 do c := c+1;
>      if 0 <= nn-mmi-k+j then
>       d[l, c] := fv1[nn-mmi-k+j+1]
>         else
>       d[l, c] := 0
>     end if
>     end do
```

```
> end do;
>
> for i to mmi do
> b[i] := -fv1[nn-mmi+1+i];
> end do;
>
> end proc:
```

A.2 Procedure to build the system whose solution gives the coefficients of $P_{n,\mathbf{m},j}$, $j = 1, \ldots, d$

```
> Pnmj:=proc(fv1,qt,mi,m,n,qw::evaln)
> local n3,mm,j,s,t,k,r1,poli,i,poli1;
> t:=0;
> mm:=m;
> r1:=array(1..n-mi+1);
> for j from 0 to n-mi do
> s:=0;
> poli:=0;
>
> t:=t+1;
> for k from 0 to j do
> if j-k>=0 and j-k<=mm then
> s:=s+fv1[k+1]*qt[j-k+1];
> end:
> end:
> r1[t]:=s;
> end:
> for i from 0 to n-mi do
```

```
>    poli:=poli+r1[i+1]*(z^(i));
>    end:
>
>    poli1:=convert(poli, polynom);
>    #print(poli1):
>    qw:=poli1;
>    end proc:
```

A.3 The principal procedure

```
>    VRMPade:=proc(V,M,nn)
>    local qd,sg,qt,qs,r0,q,d1,b1,r,c1,f1,d0,b0,tr,f,sk,t,
>    mmi,i,nv,fv1, s1,ss,k,n3,mm,d,b,qw,dt,TW,j,TW1,qx,
>    qxx,caux;
>    fv1 := array(1 .. nn+1);
>    with(Matlab):
>    nv:=Matlab[dimensions](M);
>    nv:=convert(nv,Vector);
>    sk:=0;
>    for t to nv[1] do
>    sk:=sk+M[t];
>    end do:
>    mm:=sk;
>    d:=array(1..mm,1..mm);
>    d0:=array(1..mm,1..mm);
>    b:=array(1..mm);
>    dt:=array(1..mm+1,1..mm+1);
>    b0:=array(1..mm);
>    tr:=0;
```

```
>  for i from 1 to nv[1] do
>  s1 := series(V[i], z = 0, nn+1):
>  ss := convert(s1, polynom);
>  for k from 0 to nn do
> fv1[k+1] := coeff(ss, z, k);
> end do:
> mmi:=M[i];
> bloque(fv1,nn,mm,mmi,d::evaln,b::evaln);
> for f1 to mmi do
> for c1 to mm do
> d0[f1+tr,c1]:=d[f1,c1];
>  end do;
>  end do;
>  for f from 1 to mmi do
>  b0[f+tr]:=b[f];
>  end do:
>  tr:=tr+mmi;
>  end do:
>  with(LinearAlgebra):
>  d1:=Matrix(mm,d0);
>  b1:=Vector(mm,b0);
>  print(d1);
>  q := LinearSolve(d1, Transpose(b1));
>  print(q);
>  sg := 1;
>  qt[1]:=sg;
>  for i to mm do
>  qt[i+1]:=q[i];
>  sg := sg+q[i]*z^i;
```

```
> end do;
> qs:=convert(s, polynom);
> #print(factor(sg));
> r0 := solve(sg = 0, z);
> printf("Raices de Qnm=");
> print(evalf(r0));
> for i from 1 to nv[1] do
> s1 := series(V[i], z = 0, nn+1):
> ss := convert(s1, polynom);
> for k from 0 to nn do
>         fv1[k+1] := coeff(ss, z, k);
> end do:
> Pnmj(fv1,qt,mmi,mm,nn,qw::evaln);
> print(factor(qw)/factor(sg));
> end do:
> for i from 1 to mm do
> TW[i,1]:=-b1[i];
> for j from 2 to mm+1 do
> TW[i,j]:=d1[i,j-1];
> end do:
> TW[mm+1,i]:=z^(i-1);
> end do:
> TW[mm+1,mm+1]:=z^mm;
> TW1:=Matrix(mm+1,TW);
> #print(TW1);
> print(Determinant(TW1)/Determinant(d1));
> qx:=Determinant(TW1);
> qxx:=convert(qx,polynom);
> print(evalf(solve(qxx=0,z)));
```

```
> end proc:
```