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Analytic properties of Krall-type and Sobolev-type orthogonal polynomials

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Ph.D. Thesis

Analytic Properties of Krall-type and Sobolev-type Orthogonal Polynomials

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Propiedades Analíticas de polinomios ortogonales tipo-Krall y tipo-Sobolev

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"What are the base states of the world?"
Richard P. Feynman
The Feynman Lectures on Physics, Volume III, Chapter 8.

Resumen y aportaciones

Esta tesis incluye cinco capítulos, y está dedicada al estudio de familias de polinomios ortogonales de tipo estándar y no estándar. Este Primer Capítulo contiene una breve introducción histórica e información inicial acerca de estos dos tipos de polinomios ortogonales mencionados.

En el Capítulo 2 se presentan algunos conceptos preliminares y notación acerca de las secuencias de polinomios ortogonales estudiadas. Aquí reunimos varios resultados básicos que se usan en capítulos posteriores.

En el Capítulo 3 analizamos el comportamiento de las raíces (o ceros) de las secuencias de polinomios mónicos $\{\widehat{Q}_n(x)\}_{n\geq 0}$, ortogonales con respecto a una medida modificada con una perturbación canónica de tipo Uvarov $d\mu_M(x) = d\mu(x) + M\delta(x-c)$, donde $d\mu(x)$ es una medida de Borel positiva soportada en un conjunto acotado (o no acotado) de la recta real $E = (a,b)$, $\delta(x-c)$ es el funcional delta de Dirac en el punto c, con $c \notin (a,b)$, y M es un número real no negativo, es decir, estudiamos polinomios ortogonales con respecto al producto interno

$$\langle f, g\rangle_M = \int_E f(x)g(x)d\mu(x) + Mf(c)g(c) \tag{1}$$

definido en el espacio vectorial lineal de los polinomios con coeficientes reales $\mathbb{P}$. Aplicamos éstas técnicas al estudio de las propiedades analíticas de secuencias de polinomios ortogonales asociadas con medidas modificadas Jacobi-Koornwinder y Laguerre-Koornwinder. Seguidamente, cuando la medida μ es semiclásica, se proporciona una interpretación electrostática de la distribución de ceros de estas familias. A continuación, extendemos la mencionada interpretación electrostática considerando secuencias de polinomios ortogonales con respecto al producto interno

$$\langle f, g\rangle_m = \int_0^{+\infty} f(x)g(x)d\mu_\alpha(x) + \sum_{j=1}^{m} M_j f(c_j)g(c_j), \tag{2}$$

donde $d\mu_\alpha(x) = x^\alpha e^{-x}dx$ es la medida de Laguerre soportada en $\mathbb{R}_+$, $\alpha > -1$, $c_j < 0$, $M_j > 0$, y f, g son polinomios con coeficientes reales. Obsérvese que este producto interno es una generalización de (1), considerando una cantidad numerable m de iteraciones de la perturbación de Uvarov localizadas fuera del soporte de la medida clásica de Laguerre. Propiedades analíticas de tales secuencias de polinomios cuando E es un intervalo acotado han sido previamente estudiadas en la literatura (ver [22], [25], [16] y las referencias ahí contenidas). Proporcionamos también una interpretación electrostática general de sus ceros en términos de una interacción de tipo potencial logarítmico de cargas unidad sometidas a la acción un campo externo. Esta interpretación electrostática ha sido obtenida mediante técnicas diferentes a las utilizadas para el caso de un solo punto de masa, a partir de los coeficientes de la ecuación diferencial holonómica que satisfacen estas familias de polinomios. La mayor parte de los resultados obtenidos en éste capítulo han sido publicados en [43] , y el resto han sido sometidos para publicación (ver [44]).

En el Capítulo 4 tratamos en primer lugar con secuencias de polinomios con respecto a la mismas perturbaciones que las estudiadas en el Capítulo 3, pero centrándonos en el caso en el que el soporte es no acotado, y la medida perturbada $d\mu(x)$ es la clásica de Laguerre, es decir, $d\mu(x) = x^\alpha e^{-x}dx$, $a = 0$, $b = +\infty$, $\alpha > -1$, $M \in \mathbb{R}_+$, y $c \in \mathbb{R}_-$. Analizamos algunas propiedades asintóticas externas de éstas familias de polinomios ortogonales. También presentamos la representación de estos polinomios en términos de la medida estándar de Laguerre, así como su caracterización como funciones de tipo hipergeométrico. Igualmente, se obtienen los llamados operadores de creación y destrucción asociados a estos polinomios. Finalmente, como en el Capítulo 3, extendemos estos resultados considerando las secuencias de polinomios ortogonales con respecto a (2). Aquí proporcionamos también la asintótica externa relativa general. Todos los resultados mencionados en este capítulo han sido incluidos en los artículos de investigación [23] y [44], éste último actualmente en revisión.

Por último, en el Capítulo 5 consideramos secuencias de polinomios ortogonales con respecto al producto interno de Sobolev discreto

$$\langle f, g\rangle_S = \int_0^{+\infty} f(x)g(x)d\mu(x) + Mf(c)g(c) + Nf'(c)g'(c), \tag{3}$$

donde $d\mu$ es una medida de Borel positiva, $c \leq 0$, y $M, N \geq 0$. Observe que este producto interno es una extensión de los considerados en los Capítulos 3 y 4. Los polinomios ortogonales con respecto a tales productos internos son llamados de tipo-Sobolev discreto.

Seguidamente se obtiene la localización de los ceros de tales polinomios tipo-Sobolev discreto, ortogonales con respecto a la medida de soporte no acotado μ. En particular, para la medida de Laguerre $d\mu(x) = x^{\alpha}e^{-x}dx$, $\alpha > -1$, obtenemos algunas propiedades de los polinomios ortogonales tipo-Sobolev discreto. Por último, presentamos una relación entre la matriz pentadiagonal de Jacobi $\mathbf{H}$, asociada a la relación de recurrencia a cinco términos que satisfacen los polinomios no estándar de tipo-Sobolev $\{s_n^{M,N}(x)\}_{n\geq 0}$, ortonormales con respecto a (3), y la matriz tridiagonal de Jacobi $\mathbf{J}_{[2]}$, asociada a la relación de recurrencia a tres términos que satisfacen los polinomios ortogonales estándar $2-$iterados $\{p_n^{[2]}(x)\}_{n\geq 0}$. La mayoría de los resultados mencionados en este capítulo aparecen publicados en [24] y [32].

A continuación resumimos brevemente las conclusiones aportadas por la presente memoria.

- Se realiza por primera vez un estudio completo del comportamiento de los zeros de familias de polinomios ortogonales con respecto a una medidas modificadas mediante perturbaciones canónicas de tipo Uvarov y Christoffel . El comportamiento de estos ceros se da en términos del parámetro M, el cuál determina como es la intensidad de la perturbación sobre la medida clásica. Hasta el momento, se habían realizado progresos significativos en esta dirección a través de aproximaciones semiclásicas, como en [3] y solamente tratando el comportamiento de propiedades promedio de los ceros, usando el método de WKB.

- Se obtienen resultados asintóticos de las secuencias de polinomios ortogonales mónicos con respecto a la perturbación de Uvarov de la medida clásica de Laguerre, como ejemplo canónico de perturbación fuera del soporte de una medida clásica con soporte no acotado. Hasta la fecha, los puntos de masa se localizaban en la frontera (o fronteras) del soporte de la medida perturbada.

- Se proporciona un modelo electrostático de los ceros de la familia de polinomios ortogonales con respecto a una medida de Laguerre perturbada, con una cantidad numerable m de puntos de masa, fuera del soporte de la medida clásica de Laguerre. Hasta el momento, el único trabajo similar consideraba un solo punto de masa en el origen. Igualmente, describimos el comportamiento de los ceros de los polinomios ortogonales tipo-Krall en términos de los ceros de cierto polinomio de grado $2m$

(siendo m el número de masas de Dirac que aparecen en la medida), y que son las fuentes de un potencial logarítmico de corto alcance que afecta a la localización de los ceros de las secuencias de polinomios ortogonales Krall-Laguerre, considerados como puntos críticos de un problema de equilibio.

- Igualmente se obtienen propiedades asintóticas de secuencias de polinomios ortogonales mónicos de tipo Laguerre-Sobolev, cuando los puntos de masa están situados fuera del soporte de la medida clásica de Laguerre. Hasta la fecha, los puntos de masa se localizaban en la frontera (o fronteras) de los soportes de las medidas perturbadas.

Los resultados originales contenidos en el presente trabajo han sido publicados en distintas revistas de investigación internacionales, todas ellas incluídas en el *Journal of Citation Reports*®, como se detalla a continuación (el número entre corchetes al comienzo señala el orden en que aparece en la bibliografía el correspondiente trabajo)

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[24] H. Dueñas, E. J. Huertas, y F. Marcellán, *Asymptotic properties of Laguerre-Sobolev type orthogonal polynomials*, Numer. Algorithms **60** (1), (2012), 51-73.

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Contents

CHAPTER 1

Introduction

1.1 The standard theory of orthogonal polynomials

After Newton's work, astronomy became much more precise. It was necessary to take into account in calculating the earth's deviation from perfect sphericity and for that, new mathematical functions were necessary. Thus, in the classical memory by *Adrien-Marie Legendre (1752-1813)* on the motion of the planets [61] (1785) the polynomials that now bear his name were introduced, although previously, *Joseph-Louis de Lagrange (1736-1813)* had already used the recurrence relation that defines them [57]. General Jacobi polynomials, which contain as a particular case those of Legendre, appeared in 1859 in the work [49] by *Carl Gustav Jakob Jacobi (1804-1851)*.

In the same way, a first reference to Hermite polynomials appears in the celebrated treatise of celestial mechanics [59] by *Pierre-Simon de Laplace (1749-1827)*, published in 5 volumes for 26 years (1799-1825). These were also studied by the Russian mathematician *Pafnuty Lvovich Chebyshev (1821-1894)* [14] before that *Charles Hermite (1822-1901)* studied them in [42].

As in the previous cases, the Laguerre polynomials of parameter $\alpha = 0$ had already appeared in works of *Niels Henrik Abel (1802-1829)*, J.L. Lagrange and P.L. Chebyshev [14] when *Edmond Nicolas Laguerre (1834-1886)*, studied them in 1879 [58]. The generalization of these Laguerre polynomials was initially made by *Yulian Karol Sokhotski (1842-1927)* and later on by *Nikolay Yakovlevich Sonin (1849-1915)* [93].

Other special classes of orthogonal polynomials (Charlier, Meixner and Pollaczek) were studied by *Carl Vilhelm Ludwig Charlier (1862-1934)*, *Josef Meixner (1908-1994)*, *Ervin Feldheim (1912-1944)*, *Félix Pollaczek (1892-1981)* and *Thomas Joannes Stieltjes (1856-1894)*.

Thus, in the mid-nineteenth century, the families of orthogonal polynomials which belong to the theory of special functions are known as *classical orthogonal polynomials*, and they had been very well studied. It was known that they share similar properties. However there were no comparative studies concerning their similarities that allow us to establish a general theory to characterize all those properties depending on their orthogonality.

The motivation to study systems of orthogonal functions was started under the influence of the invention of the steam engine. The aim was to provide a mathematical framework to the heat conductivity, later developed in thermodynamics, the general science about the laws of thermal motion.

The demands of a mathematical framework in relation to this issue, were highlighted in the contest organized in 1811 by the Academy of Sciences in Paris: *giving a mathematical theory of the laws of heat distribution and compare the results of this theory with experimental data*. The winner turned out to be the Parisian academician *Jean Baptiste Joseph Fourier (1768-1830)*, who in 1807 had addressed the issue by using the trigonometric system of orthogonal functions. In his report of 1811, Fourier developed a powerful method for solving partial differential equations based on the concept of orthogonality. After 11 years, he published the Analytical Theory of Heat, which exerted an enormous influence on the development of mathematics in general. As noted earlier, the first classes of orthogonal polynomials were studied under the refinement of the mathematical apparatus of celestial mechanics.

Later, in the second half of the nineteenth century, again under the influence of the steam engine, were made the first general studies on the subject. Around 1852, P.L. Chebyshev became interested in the study of various articulated mechanisms which transform, in rectilinear movement, the circular motion of the pistons of a steam engine. In many of these mechanisms, the point of contact between the piston rod and the rotating parts is under several forces which modify their rectilinear motion. This causes deviations which have a negative influence on the machine working and leads to the mathematical problem of determining the motion of a certain point M as a function with a minimal deviation from zero inside a given range. This issue gave rise Chebyshev's work on determining polynomials with minimum deviation from zero and the approximation of functions by polynomials, and led him to consider the study of special classes of orthogonal polynomials from a general point of view, abstracting from the peculiarities related to their orthogonality.

About the same time, T. J. Stieltjes, while studying the importance of real numbers, introduced the first notions of approximation of functions by continued fractions and performed with them several studies about families of orthogonal polynomials, which lead him to be considered the co-founder of this theory. Together with P.L. Chebyshev, his student *Andréi Andréyevich Márkov (1856-1922)* and T.J. Stieltjes, the theory of orthogonal polynomials was born as a branch of mathematics, but with a very close link with the theory of approximation of functions.

A second stage in the development of this theory has to do with the rise of the theory of approximation of functions and numerical mathematics, which was imposed by the development of productive forces and world wars of the first half of the 20th century. The most important figures here were the Hungarian mathematicians *Gábor Szegő (1895-1985)* ([99]) and *Géza Freud (1922-1979)* ([33]), the Russians *Sergei Natanovich Bernstein (1880-1968)* and *Yakov Lazarevich Geronimus (1898-1984)* ([37]), toghether with their students. Special mention deserves the G. Szegő's monograph [99], which is considered the bible of orthogonal polynomials. There, he condense all the algebraic, differential and asymptotic results of this theory. Since the mid 70's, with the develop of modern computers and algorithm theory, the improvement in function approximation addressed to the realm of quality, i.e. how much approximant simulates the properties of the approximate function? Can be extracted new information from the approximants?

The answer to the above questions requires a deep knowledge on asymptotic (limit) behavior of orthogonal polynomials. This led to the third stage of development of the standard theory, mainly marked by the asymptotic studies, which extends to the present days. This knowledge has increased considerably the field of extra-mathematical applications, so for example these results are used in signal transmission, data encoding and molecular biology among others.

At this stage of development, it is possible to distinguish at first between the work of rational approximation of *Andrei Aleksandrovich Gonchar (1931)* together with his students in the mid-seventies (which show the need for refinement of the theory of orthogonal polynomials) and the appearance of texts such as [94] in 1992, where are systematized the main results obtained up to date. Nowadays, the location of zeros, recurrence formulas and analytical properties are the focus of interest. Taking into account the mathematical advances in recent decades, new techniques based on the theory of logarithmic potential, geometric theory of functions of complex variables, functional analysis and operator theory are developped. At this time, several new different families of orthogonal polynomials appear, namely, multi-orthogonal polynomials, orthogonal polynomials in several variables, matrix orthogonal polynomials and orthogonal polynomials in Sobolev spaces among others.

In the modern mathematical analysis of abstract spaces are highlighted those linear spaces with metric structure. A particular importance have the spaces with an inner product. Among the basic concepts involved on them are inner products, bases, complete families, best approximation, and so on. They are intrinsically related to the concept of orthogonality.

One of the most interesting structures in mathematics and physics is the space $L^2(\mu)$, formed by square integrable functions with respect to a measure μ supported on a subset of the complex plane. In what follows, we assume by *measure*, a non-negative, finite Borel measure with support E (bounded or unbounded) on the real line $\mathbb{R}$. $L^2(\mu)$ has Hilbert-space structure, with the inner product and the norm given by, respectively, by

$$\left.\begin{array}{c} \langle f, g\rangle_\mu = \int_E f(x)g(x)d\mu(x) \\[2ex] \|f\|_\mu = \sqrt{\langle f, f\rangle} \end{array}\right\} \quad f, g \in L^2(\mu) \tag{1.1}$$

Two functions $f, g \in L^2(\mu)$ are said to be *orthogonal with respect to the measure* μ (or with respect to the inner product $\langle f, g\rangle_\mu$) if $\langle f, g\rangle_\mu = 0$. Complete sets of orthogonal functions, which span linear spaces, are dense in all $L^2(\mu)$ and then they allow to obtain approximate representations of the elements (functions) of such space.

We have many different possibilities to choose systems of orthogonal functions from $L^2(\mu)$. Among the most important, are those formed by *algebraic and trigonometric polynomials*. The advantages of working with polynomials are many. Namely: they provide ease of numerical computation, they are dense in the space of continuous functions on bounded support with uniform norm, and they constitute *Chebyshev systems*, so they are very good interpolants. Additionally algebraic polynomials have the advantage that, if changing the scale of the variable, then a change in the polynomial coefficients holds but not their shape, and their ratios are the more general functions that can be evaluated by a computer (except possibly those other functions involving logical operations or the magnitude of numbers, respectively). Throughout this dissertation, a *polynomial of degree* $n \geq 0$ (in the sequel, $\deg P$ denotes the degree of the polynomial P) will mean a function

$$P_n(x) := k_n x^n + \cdots + k_1 x + k_0 \in \mathbb{P}$$

with real coefficients $k_n, \ldots, k_1, k_0$ and $k_n \neq 0$, where $\mathbb{P}$ will denote the linear space of all polynomials in one real variable and real coefficients. The real number k_n will be said to be the *leading coefficient* of $P_n(x)$. If $k_n = 1$ we say that it is a *monic* polynomial, and we write

$$\widehat{P}_n(x) := x^n + \textit{lower degree terms.}$$

According to (1.1), we define the inner product $\langle \cdot, \cdot \rangle_\mu : \mathbb{P} \times \mathbb{P} \to \mathbb{R}$ associated with μ

$$\langle P, Q \rangle_\mu := \int_E P(x)Q(x)d\mu(x), \quad \text{for every } P,\, Q \in \mathbb{P}, \tag{1.2}$$

and the corresponding norm

$$\|P\|_\mu = \left(\int_E |P(x)|^2 \, d\mu(x) \right)^{1/2}, \quad \text{for every } P \in \mathbb{P}.$$

A *sequence* (*system* or *family*) of polynomials $\{P_n(x)\}_{n\geq 0}$, $n = 0, 1, 2, \ldots$ is said to be *orthogonal with respect to the positive Borel measure* μ on the interval $E \subseteq \mathbb{R}$ (which can be bounded or unbounded) if, for every n

$$\langle P_n, P_m \rangle_\mu \begin{cases} \neq 0, & \text{if } n = m \\ = 0, & \text{if } n \neq m \end{cases}, \tag{1.3}$$

or, analogously

$$\int_E P_n(x) x^m d\mu(x) = \begin{cases} \neq 0, & \text{if } m = n \\ = 0, & \text{if } m < n. \end{cases}$$

Such a sequence $\{P_n(x)\}_{n\geq 0}$ is said to be an *orthogonal polynomial sequence (OPS in short)*. If the polynomials of the sequence are monic, i.e. $\{\widehat{P}_n(x)\}_{n\geq 0}$, then it is customary to say that they constitute a *monic OPS*, or *MOPS* in short. Sometimes we will use a notation relative to the norm of the polynomials. If for every n we have $\|P_n\|_\mu = 1$, then we say that the system of polynomials is *orthonormal*, i.e. the norm of $P_n(x)$ is equal to one. In this case we write $\{p_n(x)\}_{n\geq 0}$ for an *orthonormal polynomial sequence*, and these polynomials will be denoted with lower-case letters. Obviously, for an arbitrary orthonormal polynomial of deg n

$$p_n(x) = \frac{P_n(x)}{\|P_n\|_\mu}$$

and, given an orthonormal polynomial sequence, it holds

$$\langle p_n(x), p_m(x)\rangle_\mu = \delta_{nm}, \quad n, m = 0, 1, 2, \ldots.$$

The most classical example of a system of orthogonal polynomials are the *Chebyshev polynomials* $T_n(x)$ which are orthogonal with respect to the measure $d\mu(x) = \frac{dx}{\sqrt{1-x^2}}$ supported on $[-1, 1]$. More precisely

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = \begin{cases} \pi, & n = m = 0, \\ \pi/2, & n = m > 0, \\ 0, & n \neq m. \end{cases}$$

Such polynomials are expressed by

$$T_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k} x^{n-2k}(x^2-1)^k, \quad n = 0, 1, \cdots, \tag{1.4}$$

where $\left[\frac{n}{2}\right]$ denotes the integer part of $\frac{n}{2}$. For $x \in [-1, 1]$ one has that

$$T_n(x) = \cos(n \arccos(x)).$$

1.2 On Sobolev orthogonality

From the theory of boundary value problems of partial differential equations, it arises the interest in inner products involving not only functions, but also their derivatives up to a given order. These inner products are called *Sobolev inner products*. The definition of *Sobolev spaces* where these products make sense is a delicate matter, so in what follows, we restrict the definition of Sobolev product on the space of polynomials, which allow derivatives of any order and are integrable with respect to any finite Borel measure with finite moments. Let $\{\mu_k\}_{k=0}^d$, $d \in \mathbb{Z}_+$ be a system of $d+1$ positive Borel measures with support $E_k \subset \mathbb{R}$, $k = 0, 1, \cdots, d$, respectively. Assume that at least the support μ_0 contains an infinite number of points and, to avoid trivial cases, which μ_d contains an infinite number of points and which is not negligible. We call

$$\langle P, Q \rangle_S := \sum_{k=0}^{d} \int P^{(k)}(x) Q^{(k)}(x) d\mu_k(x) = \sum_{k=0}^{d} \langle P^{(k)}, Q^{(k)} \rangle_{\mu_k}, \quad P, Q \in \mathbb{P}, \tag{1.5}$$

the *Sobolev inner product* (on the space of polynomials) *associated with the vector of measures* $\{\mu_k\}_{k=0}^d$.

The superscripts in parentheses denote the order of derivation. The associated standard norm (1.5) is called the *Sobolev norm* and is given by the expression

$$\|P\|_S = (\langle P, P \rangle_S)^{\frac{1}{2}} = \left(\sum_{k=0}^{d} \langle P^{(k)}, P^{(k)} \rangle_{\mu_k} \right)^{\frac{1}{2}} = \left(\sum_{k=0}^{d} \|P^{(k)}\|_{\mu_k}^2 \right)^{\frac{1}{2}}. \tag{1.6}$$

It is clear that if $d = 0$ both (1.5) and (1.6) admit extension to the space $L^2(\mu_0)$.

A system $\{Q_n(x)\}_{n \geq 0}$ of polynomials orthogonal with respect to the inner product (1.5) is said to be a *Sobolev OPS*. The study of Sobolev orthogonal polynomials is relatively new (about algebraic properties see [68] and [84], for analytic properties [83]). The work [62] of 1947 is the first publication where norms like (1.6) are studied in the framework of approximation theory. They appear in connection with least squares problems. However, the Sobolev polynomials were first time considered less than forty years ago in [2] and the largest research has been during last the and present decade (see [77]).

In [2] is noted that, although the study of Sobolev orthogonal polynomials may seem similar to the standard case, this is not true at all. Immediately, one finds substantial differences that require other approaches to their study. In the case of orthogonality in the usual sense, the location of the zeros of the family of orthogonal polynomials in the convex hull of the support of the measure of orthogonality is an immediate result of the definition of orthogonality. For a Sobolev product as simple as the case considered by Althamer in [2]

$$\langle f, g \rangle_S := \int_{-1}^{1} f(x) g(x) dx + 10 \int_{-1}^{0} f'(x) g'(x) dx + \int_{0}^{1} f'(x) g'(x) dx, \tag{1.7}$$

one finds that zeros can be outside the convex hull of the support of the measures involved, and they may even be complex. Moreover, no one knows even if the set of zeros of polynomials orthogonal with respect to an arbitrary Sobolev inner product, remains bounded or unbounded in the complex plane. The more general result in this way is the sufficient condition proved in [64] using techniques of bounded operators. So far, only results about Sobolev inner products involving classical weights (or close to them) had been obtained.

One of the main tools in the study of monic orthogonal polynomials (1.3) is the *three term recurrence relation* which they satisfy

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n \geq 1, \tag{1.8}$$

where for every $n \in \mathbb{N}$, γ_n is real and positive, and β_n is real. It is very well known that (1.8) is a consequence of the symmetry of the usual inner product with respect to x. That is, for every pair of polynomials P and Q,

$$\langle xP, Q\rangle_\mu = \langle P, xQ\rangle_\mu$$

and they are called *standard inner products*.

In contrast, this is not true at all for an arbitrary Sobolev inner product, that is, they verify

$$\langle xP, Q\rangle_S \neq \langle P, xQ\rangle_S,$$

for all arbitrary polynomials P and Q, and therefore they are an example of *non-standard inner products*. Because this fact, they lose the nice properties of polynomials orthogonal with respect to a standard inner product. For example, their zeros can be complex or, if real, they can be located outside of the support of the modified measure and they can satisfy recurrence relations with more than three terms. In [64] were the first time when the recurrence relation for Sobolev inner products, and its connection with the corresponding moment matrix and the location of zeros were obtained.

We will split the Sobolev inner products into two different categories. A product like (1.5) is called:

1. *Continuous Sobolev inner product*, when the support of the Borel measures involved in the Sobolev inner product are formed by an infinite number of points. They were already introduced in (1.5).

2. *Sobolev-type or discrete Sobolev-type inner product*, when the support of the measures $\{\mu_k\}_{k=1}^d$ has a finite number of points. The Sobolev-type term is often used, because sometimes we make considerations that go beyond the framework of inner products. A general expression for them is

$$\langle f, g\rangle_S = \int fg d\mu_0 + \sum_{k=0}^{d} \mathcal{F}_k^T \mathcal{M}_k \mathcal{G}_k, \; d \in \mathbb{Z}_+, \tag{1.9}$$

where

$$\mathcal{F}_k = \begin{bmatrix} f(c_k) \\ f^{(1)}(c_k) \\ \vdots \\ f^{(n_k)}(c_k) \end{bmatrix}, \quad \mathcal{G}_k = \begin{bmatrix} g(c_k) \\ g^{(1)}(c_k) \\ \vdots \\ g^{(m_k)}(c_k) \end{bmatrix},$$

$$\mathcal{M}_k = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m_k} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n_k,1} & a_{n_k,2} & \cdots & a_{n_k,m_k} \end{bmatrix},$$

with μ_0 being a finite positive Borel measure, and for $k = 0, 1, \cdots, d$, $d \in \mathbb{Z}_+$, $c_k \in \mathbb{R}$, $a_{i,j} \in \mathbb{R}_+$ and $n_k, m_k > 0$. $\mathcal{F}^T$ denotes the transpose of the vector $\mathcal{F}$. This kind of families have been considered by several authors (see, for instance, [1], [67], [76], [79] and the references therein), and they are known in the literature as *Sobolev-type* or *discrete Sobolev orthogonal polynomials*. Analytic properties of these families of orthogonal polynomials were studied in [76], and some properties concerning the behavior of their zeros were obtained in [73]. A substantial part of this memoir is devoted to the study of OPS with respect to inner products which are particular cases of (1.9).

1.3 Krall-type and Sobolev-type OP: State of the art

Next, we give a brief overview of the state of the art on the two types of polynomials discussed in this work, namely the Krall-type and Sobolev-type orthogonal polynomials.

Krall-type orthogonal polynomials

In the seminal papers by H. L. Krall [56] and A. M. Krall [55] devoted to the spectral analysis of fourth order linear differential operators with polynomial coefficients, for the first time appear some extensions of the classical measures of Laguerre, Legendre and Jacobi. These new (non-classical) orthogonal polynomials were called Laguerre-type, Legendre-type and Jacobi-type orthogonal polynomials. They are orthogonal with respect to modified classical measures by adding one or two Dirac masses, so they are standard inner products. Nowadays, this kind of polynomials are called *Krall-type orthogonal polynomials*.

T. H. Koornwinder [54] analyzed a general situation for Jacobi weights when two masses are added at the end points of the interval $[-1, 1]$. Later on, in [40], Krall-Hermite and Krall-Bessel polynomials are studied in the framework of Darboux transformations.

In [52] analytic properties of orthogonal polynomials with respect to a perturbation of the Laguerre weight when a mass is added at $x = 0$ were considered. Indeed, up to date this case, or when the Dirac masses are added in the boundary of the support of the modified measure, have been extensively studied in the literature, mainly in connection with spectral problems for higher order linear differential operators. In this direction, in [51] and [50] the authors obtain infinite order differential operators such that the Krall-Laguerre and Krall-Jacobi are their respective eigenfunctions. In particular, for some choices of the parameters in Laguerre and Jacobi weights they prove that the differential operator has a finite order.

In recent years, there has been an increasing interest in the so called *spectral transformations of measures*, and the Krall-type orthogonal polynomials have been analyzed from this particular point of view by several authors. Many studies have been done concerning the distribution of their zeros in terms of the mass of the Dirac delta, as well as their interlacing properties, which were analyzed, e.g. in [18]. Moreover, the monotonicity of their zeros in terms of the mass of the perturbation and their asymptotic behavior have been established in some recent works, when the support of the modified measure is either bounded or an unbounded subset of $\mathbb{R}$. (see [18], [19]).

The application of Stieltjes' ideas (see [39] and [101]) to obtain the electrostatic interpretation of the zeros of the Krall-type polynomials as equilibrium points with respect to a logarithmic potential (under the action of an external field) has attracted the attention of the researchers. Pioneer works in this direction are [34], [35], [25], and [26]. The holonomic differential equation that these polynomials satisfy is closely connected with the interpretation of their behavior in terms of a problem of electrostatic equilibrium (see [38], [39], [45], [46] and [47]). In [75], the holonomic equation that these OPS satisfy for such modified measures (with the mass point located in the negative real semi-axis) is deduced for the very first time.

It is worth noting that, when the mass points are located outside the support of the measure, the study of the analytic properties of the Krall-type orthogonal polynomials has not attracted the interest of researchers, up to in the general framework of semiclassical functionals [72]. Indeed, one the main goals of this thesis is to consider the perturbations of the classical measures outside the support of the modified measure. Recent works in this direction are [31], [23], and [43].

Sobolev-type orthogonal polynomials

Concerning the Sobolev-type orthogonal polynomials, as in the Krall-type case, very little is done when the Sobolev-type modification is inside or outside of the support of the modified measure. Nontheless, the case when the Sobolev-type modification is located at the boundary of the support of the modified measure has been studied extensively (see for instance [18], [19], [27], and [28]). For example, when the support of the modified measure is the interval $[0, +\infty)$ and the perturbation is at $x = 0$, Meijer [85] analyzed some analytic properties of the zeros of these Sobolev-type families. Some results of [85] are direct generalizations of the results of [53], where the weight function is the Laguerre classical weight. In [53], the authors established different properties of the discrete Laguerre-Sobolev polynomials such as their representation as a hypergeometric series, an holonomic second order linear differential equation associated with them, properties of the zeros as well as a higher order recurrence relation that such polynomials satisfy. Notice that the asymptotic properties of these discrete Laguerre-Sobolev polynomials have been studied in [4] and [71], while the analysis of convergence of the Fourier expansions in terms of such polynomials was done in [30].

1.4 Outline of this thesis

The thesis includes five chapters and it is focused on the so called standard and non-standard families of orthogonal polynomials. This first Chapter contains a brief historical introduction and some background information.

Chapter 2 presents some preliminary concepts and notations about orthogonal polynomial sequences. There we summarize the results that will be useful in later sections.

In Chapter 3 we analyze the behavior of the zeros of the sequence of monic polynomials $\{\widehat{Q}_n(x)\}_{n\geq 0}$ orthogonal with respect to a Uvarov-perturbed measure $d\mu_M(x) = d\mu(x) + M\delta(x-c)$, where $d\mu(x)$ is a positive Borel measure supported in a finite or infinite interval of the real line $E = (a, b)$, $\delta(x-c)$ is the Dirac delta functional at c, with $c \notin (a, b)$, and M is a nonnegative real number. i.e., we study polynomials orthogonal with respect to the inner product defined in the linear space of polynomials with real coefficients $\mathbb{P}$ by

$$\langle f, g\rangle_M = \int_E f(x)g(x)d\mu(x) + Mf(c)g(c). \tag{1.10}$$

We apply these techniques to the study of analytic properties of orthogonal polynomial sequences associated with the Jacobi and Laguerre-Koornwinder perturbed measures. Next, when the measure μ is semiclassical, an electrostatic interpretation of their zero distribution is given. Finally, we extend the previous electrostatic interpretation by considering the sequences of polynomials orthogonal with respect to the inner product

$$\langle f, g\rangle_m = \int_0^{+\infty} f(x)g(x)d\mu_\alpha(x) + \sum_{j=1}^{m} M_j f(c_j)g(c_j), \tag{1.11}$$

where $d\mu_\alpha(x) = x^\alpha e^{-x}dx$ is the Laguerre measure supported on $\mathbb{R}_+$, $\alpha > -1$, $c_j < 0$, $M_j > 0$, and f, g are polynomials with real coefficients. Notice that this inner product is a generalization of (1.10), by considering m iterations of Uvarov perturbations outside the support of the Laguerre measure. Notice that analytic properties of such polynomial sequences when E is a bounded interval have been studied in the literature (see [22], [25] and the references given therein). We give a general electrostatic interpretation of their zeros in terms of a logarithmic potential interaction of unit charges under an external field. This electrostatic interpretation has been reached by different techniques as those used for just one mass point, from the coefficients of the holonomic equation that these polynomials satisfy. Most of the results obtained in this chapter have been published in [43] , and the other ones have been submitted for publication (see [44]).

In Chapter 4 we first deal with sequences of polynomials orthogonal with respect to the same perturbation as in Chapter 3, but we focus our attention when the support of the measure is unbounded, and the measure $d\mu(x)$ is the Laguerre measure, i.e. $d\mu(x) = x^\alpha e^{-x}dx$, $a = 0$, $b = +\infty$, $\alpha > -1$, $M \in \mathbb{R}_+$, and $c \in \mathbb{R}_-$. We analyze some outer asymptotic properties of such orthogonal polynomials. We also discuss the representation of these polynomials in terms of the standard Laguerre polynomials as well as their characterization as hypergeometric functions. The lowering and raising operators associated with these polynomials are obtained as well. Finally, as in Chapter 3, we extend these results considering the sequences of polynomials orthogonal with respect to (1.11). Here we provide the general outer relative asymptotics. All results mentioned in this chapter yield the paper [23] and the contribution [44], which has been submitted for publication.

Finally, in Chapter 5 we deal with sequences of polynomials orthogonal with respect to the discrete Sobolev inner product

$$\langle f,g\rangle_S=\int_0^{+\infty}f(x)g(x)d\mu(x)+Mf(c)g(c)+Nf'(c)g'(c), \tag{1.12}$$

where $d\mu$ is a positive Borel measure, $c\leq 0$, and $M,N\geq 0$. Notice that this inner product is an extension of the inner product studied in Chapters 3 and 4. Polynomials orthogonal with respect to such an inner product are said to be of *discrete Sobolev-type*. The location of the zeros of such discrete Sobolev-type orthogonal polynomials is given in terms of the zeros of standard polynomials orthogonal with respect to the measure μ with unbounded support. In particular for the Laguerre measure $d\mu(x)=x^\alpha e^{-x}dx$, $\alpha>-1$, we obtain some asymptotics properties for discrete Laguerre Sobolev-type orthogonal polynomials. Finally, we obtain a relation between the pentadiagonal Jacobi matrix $\mathbf{H}$, associated with the five term recurrence relation satisfied by the non-standard sequence of Sobolev-type polynomials $\{s_n^{M,N}(x)\}_{n\geq 0}$, orthonormal with respect to (3), and the tridiagonal Jacobi matrix $\mathbf{J}_{[2]}$, associated with the three term recurrence relation satisfied by the standard $2-$iterated sequence of orthonormal polynomials $\{p_n^{[2]}(x)\}_{n\geq 0}$. The results mentioned in this chapter appear in [24] and [32].

CHAPTER 2

Orthogonal Polynomials on the Real Line

2.1 Basic facts on OPS

In this chapter we summarize some definitions and basic concepts to be used in the sequel.

Let assume that μ is a bounded, and non-decreasing function with an infinite set of points of increase, such that all the integrals

$$m_n = \int_E x^n d\mu(x), \quad E \subseteq \mathbb{R},$$

exist for $n = 0, 1, 2, \ldots$. m_n is said to be the *moment of order n of the function $\mu(x)$*. A point x_0 is said to be of increase of $\mu(x)$ if in every neighborhood $(x_0 - h, x_0 + h)$ of x_0 the inequality $\mu(x_0 + h) - \mu(x_0 - h) > 0$ holds.

We recall that the *mass distribution function* (also *distribution function* or, in short, *m-distribution*) of a positive Borel measure μ is a non-decreasing, right continuous and non-negative function defined by

$$F_\mu(x) := \int_{-\infty}^{x} d\mu(t) = \mu((-\infty, x]).$$

Conversely, any function satisfying these properties is a distribution function for a measure

μ and

$$\int_{\mathbb{R}} f(x)dF_\mu(x) = \int_{\mathbb{R}} f(x)d\mu(x).$$

A finite and positive Borel measure μ is said to be *absolutely continuous* with respect to the Lebesgue measure if there exists a non-negative function $\omega(x)$ such that

$$d\mu(x) = \omega(x)dx.$$

The function $\omega(x)$ is the *density function* of the mass distribution $F_\mu(x)$ (also called the *weight function*) and then sometimes one speaks of orthogonality of a sequence of polynomials with respect to a weight function $\omega(x)$.

From the moments, there exists an explicit expression of the MOPS $\{\widehat{P}_n(x)\}_{n\geq 0}$. Indeed,

$$\widehat{P}_0(x) = 1,$$

$$\widehat{P}_n(x) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} m_0 & m_1 & m_2 & \dots & m_n \\ m_1 & m_2 & m_3 & \dots & m_{n+1} \\ m_2 & m_3 & m_4 & \dots & m_{n+2} \\ . & . & . & \dots & . \\ m_{n-1} & m_n & m_{n+1} & \dots & m_{2n-1} \\ 1 & x & x^2 & \dots & x^n \end{vmatrix}, \quad n \geq 1, \tag{2.1}$$

where

$$\Delta_{n-1} = \begin{vmatrix} m_0 & m_1 & m_2 & \dots & m_{n-1} \\ m_1 & m_2 & m_3 & \dots & m_n \\ m_2 & m_3 & m_4 & \dots & m_{n+1} \\ . & . & . & \dots & . \\ m_{n-1} & m_n & m_{n+1} & \dots & m_{2n-2} \end{vmatrix}, \quad n \geq 1,$$

are the so called *Gram determinants*. Notice that these moment matrices have a Hankel structure.

2.1.1 The moment functional and orthogonality

Let $\{m_n\}_{n\geq 0}$ be a sequence of real numbers and $\mathcal{L}$ a linear functional defined on the linear space $\mathbb{P}$, such that

$$\langle \mathcal{L}, x^n \rangle = m_n, \; n \in \mathbb{N}.$$

$\mathcal{L}$ is said to be a *moment functional* associated with $\{m_n\}_{n\geq 0}$. If $\phi(x)$, $P(x) \in \mathbb{P}$, and D is the usual distributional derivative of $\mathcal{L}$, then we introduce the linear functionals

$$\begin{aligned} \langle \phi(x)\mathcal{L}, P(x)\rangle &= \langle \mathcal{L}, \phi(x)P(x)\rangle\,, \\ \langle D^k\mathcal{L}, P(x)\rangle &= (-1)^k\langle \mathcal{L}, P^{(k)}(x)\rangle, \quad k \geq 0. \end{aligned}$$

Instead of talking about a sequence of polynomials orthogonal with respect to a m-distribution function, (or with respect to a measure μ), in a more general framework one can speak about *sequences of orthogonal polynomials associated with a moment functional*. Given a moment functional $\mathcal{L}$, a sequence of polynomials $\{P_n(x)\}_{n\geq 0}$ is said to be an OPS with respect to $\mathcal{L}$ if:

1. The degree of $P_n(x)$ is n.

2. $\langle \mathcal{L}, P_n(x)P_m(x)\rangle = 0,\ n < m$.

3. $\langle \mathcal{L}, P_n^2(x)\rangle \neq 0,\ n = 0,\, 1,\, 2, \ldots$.

The following theorem, whose proof can be found in [15, Th. 3.1], gives necessary and sufficient conditions for the existence of a MOPS $\{\widehat{P}_n(x)\}_{n\geq 0}$ orthogonal with respect to a moment functional $\mathcal{L}$ associated with $\{m_n\}_{n\geq 0}$.

Theorem 2.1 *Let $\mathcal{L}$ be the moment functional associated with $\{m_n\}_{n\geq 0}$. There exists a MOPS $\{\widehat{P}_n(x)\}_{n\geq 0}$ associated with $\mathcal{L}$ if and only if every leading principal submatrix of the Hankel matrix $[m_{i+j}]_{i,j\in\mathbb{N}}$ is nonsingular.*

In this case, $\mathcal{L}$ is said to be a *regular* or *quasi-definite* moment functional ([15]). If every leading principal submatrix of the Hankel matrix has positive determinant, then $\mathcal{L}$ is said to be a *positive definite moment functional*. Given a polynomial $P(x) \in \mathbb{P}$ and a positive definite moment functional $\mathcal{L}$ then, from the *Riesz representation theorem*, $\mathcal{L}$ has the following integral representation

$$\langle \mathcal{L}, P(x)\rangle = \int_E P(x)d\mu,$$

where μ is a nontrivial positive Borel measure supported on an infinite subset E of the real line.

A remarkable property of any sequence of polynomials $\{\widehat{P}_n(x)\}_{n\geq 0}$, orthogonal with respect to a regular moment functional $\mathcal{L}$ is that any three consecutive polynomials of the sequence are connected by a simple recurrence relation, known as the *Three Term Recurrence Relation (TTRR in short)*. Next we show this TTRR and its converse, the *Favard's Theorem* (see [15])

Theorem 2.2 (Three Term Recurrence Relation) *Let $\mathcal{L}$ be a quasi-definite moment functional and let $\{\widehat{P}_n(x)\}_{n\geq 0}$ be the corresponding MOPS. Then, there exist sequences of real numbers $\{\beta_n\}_{n\geq 1}$ and $\{\gamma_n\}_{n\geq 1}$, with $\gamma_n \neq 0$ for every $n \in \mathbb{N}$, such that*

$$x\widehat{P}_n(x) = \widehat{P}_{n+1}(x) + \beta_n\widehat{P}_n(x) + \gamma_n\widehat{P}_{n-1}(x), \qquad n = 1, 2, \ldots \tag{2.2}$$

with $\widehat{P}_0(x) = 1$, $\widehat{P}_1(x) = x - \beta_0$.

Theorem 2.3 (Favard's Theorem) *Let $\{\beta_n\}_{n\geq 1}$ and $\{\gamma_n\}_{n\geq 1}$ be sequences of real numbers, with $\gamma_n \neq 0$ and $\{\widehat{P}_n(x)\}_{n\geq 0}$ a sequence of monic polynomials such that*

$$x\widehat{P}_n(x) = \widehat{P}_{n+1}(x) + \beta_n\widehat{P}_n(x) + \gamma_n\widehat{P}_{n-1}(x), \qquad n = 1, 2, \ldots$$

where $\widehat{P}_0(x) = 1$, $\widehat{P}_1(x) = x - \beta_0$ Then, there exists a unique moment functional $\mathcal{L}$, such that $\langle \mathcal{L}, 1\rangle = 1$ and $\langle \mathcal{L}, \widehat{P}_n(x)\widehat{P}_m(x)\rangle = 0$ for $n \neq m$, $n, m \in \mathbb{N}$. Under these conditions, $\mathcal{L}$ is quasi-definite and $\{\widehat{P}_n(x)\}_{n\geq 0}$ is the corresponding sequence of monic orthogonal polynomials. Moreover, $\mathcal{L}$ is positive definite if and only if β_n is real and $\gamma_n > 0$ for every $n \geq 1$.

Concerning the zeros of these MOPS, they satisfy (see details in [15] and [99])

Theorem 2.4 *Let $\mathcal{L}$ be a positive definite moment functional supported on an infinite set $E \subseteq \mathbb{R}$ and let $\{\widehat{P}_n(x)\}_{n\geq 0}$ be the corresponding MOPS. Then, for $n \geq 2$, the zeros of $\widehat{P}_n(x)$ are simple, real, interlace with the zeros of $\widehat{P}_{n-1}(x)$ and they lie in the interior of the convex hull of E.*

2.1.2 Kernels and the Christoffel-Darboux summation formula

Given a MOPS with respect to the linear functional $\mathcal{L}$, we define the n-th reproducing kernel as

$$K_n(x, y) = \sum_{k=0}^{n} \frac{\widehat{P}_k(x)\widehat{P}_k(y)}{\langle \mathcal{L}, \widehat{P}_k^2\rangle}. \tag{2.3}$$

Theorem 2.5 (Christoffel-Darboux formula) *Let $\{\widehat{P}_n(x)\}_{n\geq 0}$ be the sequence of polynomials orthogonal with respect to the quasi-definite moment functional $\mathcal{L}$. Then for every $n \in \mathbb{N}$*

$$K_n(x,y) = \sum_{k=0}^{n} \frac{\widehat{P}_k(x)\widehat{P}_k(y)}{\langle \mathcal{L}, \widehat{P}_k^2 \rangle} = \frac{\widehat{P}_{n+1}(x)\widehat{P}_n(y) - \widehat{P}_n(x)\widehat{P}_{n+1}(y)}{\langle \mathcal{L}, \widehat{P}_n^2 \rangle (x-y)}. \tag{2.4}$$

The confluent formula reads as

$$K_n(x,x) = \sum_{k=0}^{n} \frac{(\widehat{P}_k(x))^2}{\langle \mathcal{L}, \widehat{P}_k^2 \rangle} = \frac{1}{\langle \mathcal{L}, \widehat{P}_n^2 \rangle} \left(\widehat{P}'_{n+1}(x)\widehat{P}_n(x) - \widehat{P}'_n(x)\widehat{P}_{n+1}(x) \right). \tag{2.5}$$

Concerning the partial derivatives of (2.3) we will use the following notation

$$\frac{\partial^{j+k}(K_n(x,y))}{\partial x^j \partial y^k} = K_n^{(j,k)}(x,y). \tag{2.6}$$

Let $\{\widehat{P}_n(x)\}_{n\geq 0}$ be a MOPS. From the Christoffel-Darboux formula we have

$$K_{n-1}(x,y) = \frac{1}{\langle \mathcal{L}, \widehat{P}_{n-1}^2 \rangle} \frac{\widehat{P}_n(x)\widehat{P}_{n-1}(y) - \widehat{P}_{n-1}(x)\widehat{P}_n(y)}{(x-y)}.$$

Next, computing the j-th derivative with respect to y we obtain (see [28])

$$K_{n-1}^{(0,j)}(x,y) = \frac{1}{\langle \mathcal{L}, \widehat{P}_{n-1}^2 \rangle} \left(\widehat{P}_n(x) \frac{\partial^j}{\partial y^j} \left(\frac{\widehat{P}_{n-1}(y)}{x-y} \right) - \widehat{P}_{n-1}(x) \frac{\partial^j}{\partial y^j} \left(\frac{\widehat{P}_n(y)}{x-y} \right) \right). \tag{2.7}$$

Using the Leibnitz's rule

$$\frac{\partial^j}{\partial y^j} \left(\frac{\widehat{P}_n(y)}{x-y} \right) = \sum_{k=0}^{j} \frac{j!}{k!} \frac{\widehat{P}_n^{(k)}(y)}{(x-y)^{j-k+1}},$$

and replacing in (2.7) we obtain

$$\begin{aligned} K_{n-1}^{(0,j)}(x,y) &= \frac{1}{\langle \mathcal{L}, \widehat{P}_{n-1}^2 \rangle} \left(\widehat{P}_n(x) \sum_{k=0}^{j} \frac{j!}{k!} \frac{\widehat{P}_{n-1}^{(k)}(y)}{(x-y)^{j-k+1}} - \widehat{P}_{n-1}(x) \sum_{k=0}^{j} \frac{j!}{k!} \frac{\widehat{P}_n^{(k)}(y)}{(x-y)^{j-k+1}} \right) \\ &= \frac{j!}{\langle \mathcal{L}, \widehat{P}_{n-1}^2 \rangle (x-y)^{j+1}} \times \\ &\quad \left(\widehat{P}_n(x) \sum_{k=0}^{j} \frac{1}{k!} \widehat{P}_{n-1}^{(k)}(y)(x-y)^k - \widehat{P}_{n-1}(x) \sum_{k=0}^{j} \frac{1}{k!} \widehat{P}_n^{(k)}(y)(x-y)^k \right). \end{aligned}$$

Thus,

$$K_{n-1}^{(0,j)}(x,c)=\frac{j!}{\langle\mathcal{L},\widehat{P}_{n-1}^2\rangle(x-c)^{j+1}}\left(\widehat{P}_n(x)Q_j(x,c;\widehat{P}_{n-1})-\widehat{P}_{n-1}(x)Q_j(x,c;\widehat{P}_n)\right), \quad (2.8)$$

where $Q_j(x,c;\widehat{P}_{n-1})$ and $Q_j(x,c;\widehat{P}_n)$ denote the Taylor polynomials of degree j, around the point $x=c$, of the polynomials $\widehat{P}_{n-1}(x)$ and $\widehat{P}_n(x)$, respectively.

Next, using the Taylor expansion of $\widehat{P}_n(x)$ and $\widehat{P}_{n-1}(x)$ in (2.8), we can compute $K_{n-1}^{(0,j)}(c,c)$. Indeed,

$$K_{n-1}^{(0,j)}(x,c)=\frac{j!}{\langle\mathcal{L},\widehat{P}_{n-1}^2\rangle(x-c)^{j+1}}\times$$

$$\begin{aligned}&\left[\left(\widehat{P}_n(c)+\widehat{P}_n'(c)(x-c)+\frac{\widehat{P}_n''(c)}{2!}(x-c)^2+\cdots+\frac{\widehat{P}_n^{(n)}(c)}{n!}(x-c)^n\right)\right.\\&\times\left(\widehat{P}_{n-1}(c)+\widehat{P}_{n-1}'(c)(x-c)+\frac{\widehat{P}_{n-1}''(c)}{2!}(x-c)^2+\cdots+\frac{\widehat{P}_{n-1}^{(j)}(c)}{j!}(x-c)^j\right)-\\&\left(\widehat{P}_{n-1}(c)+\widehat{P}_{n-1}'(c)(x-c)+\frac{\widehat{P}_{n-1}''(c)}{2!}(x-c)^2+\cdots+\frac{\widehat{P}_{n-1}^{(n-1)}(c)}{(n-1)!}(x-c)^{n-1}\right)\\&\left.\times\left(\widehat{P}_n(c)+\widehat{P}_n'(c)(x-c)+\frac{\widehat{P}_n''(c)}{2!}(x-c)^2+\cdots+\frac{\widehat{P}_n^{(j)}(c)}{j!}(x-c)^j\right)\right]. \quad (2.9)\end{aligned}$$

Taking into account the coefficient of $(x-c)^{j+1}$ in the right hand side of (2.9), an easy computation shows that, for $x=c$,

$$K_{n-1}^{(0,j)}(c,c)=\frac{1}{(j+1)\langle\mathcal{L},\widehat{P}_{n-1}^2\rangle}(\widehat{P}_{n-1}(c)\widehat{P}_n^{(j+1)}(c)-\widehat{P}_n(c)\widehat{P}_{n-1}^{(j+1)}(c)).$$

To find $K_{n-1}^{(j,j)}(c,c)$, we need to consider the coefficients of $(x-c)^{2j+1}$ in the expression inside the square bracket in (2.9), i.e.

$$\begin{aligned}&\left[\frac{\widehat{P}_{n-1}(c)}{0!}\frac{\widehat{P}_n^{(2j+1)}(c)}{(2j+1)!}+\frac{\widehat{P}_{n-1}'(c)}{1!}\frac{\widehat{P}_n^{(2j)}(c)}{(2j)!}+\cdots+\frac{\widehat{P}_{n-1}^{(j)}(c)}{j!}\frac{\widehat{P}_n^{(j+1)}(c)}{(j+1)!}\right]-\\&\left[\frac{\widehat{P}_n(c)}{0!}\frac{\widehat{P}_{n-1}^{(2j+1)}(c)}{(2j+1)!}+\frac{\widehat{P}_n'(c)}{1!}\frac{\widehat{P}_{n-1}^{(2j)}(c)}{(2j)!}+\cdots+\frac{\widehat{P}_n^{(j)}(c)}{j!}\frac{\widehat{P}_{n-1}^{(j+1)}(c)}{(j+1)!}\right]\\&=\frac{1}{(2j+1)!}\left[\left(\widehat{P}_{n-1}(c)\widehat{P}_n^{(2j+1)}(c)+\widehat{P}_{n-1}'(c)\widehat{P}_n^{(2j)}(c)\binom{2j+1}{1}+\cdots\right.\right.\end{aligned}$$

$$+\binom{2j+1}{j}\widehat{P}_{n-1}^{(j)}(c)\widehat{P}_{n}^{(j+1)}(c)\Big) -$$
$$\Big(\widehat{P}_n(c)\widehat{P}_{n-1}^{(2j+1)}(c) + \widehat{P}'_n(c)\widehat{P}_{n-1}^{(2j)}(c)\binom{2j+1}{1} + \cdots$$
$$+\binom{2j+1}{j}\widehat{P}_{n}^{(j)}(c)\widehat{P}_{n-1}^{(j+1)}(c)\Big)\Big].$$

Hence,

$$K_{n-1}^{(j,j)}(c,c) = \frac{(j!)^2}{(2j+1)!\langle \mathcal{L}, \widehat{P}_{n-1}^2\rangle}\times$$

$$\Big[\Big(\widehat{P}_{n-1}(c)\,\widehat{P}_{n}^{(2j+1)}(c) + \widehat{P}'_{n-1}(c)\,\widehat{P}_{n}^{(2j)}(c)\binom{2j+1}{1} + \cdots$$
$$+\binom{2j+1}{j}\widehat{P}_{n-1}^{(j)}(c)\,\widehat{P}_{n}^{(j+1)}(c)\Big) -$$
$$\Big(\widehat{P}_n(c)\,\widehat{P}_{n-1}^{(2j+1)}(c) + \widehat{P}'_n(c)\,\widehat{P}_{n-1}^{(2j)}(c)\binom{2j+1}{1} + \cdots$$
$$+\binom{2j+1}{j}\widehat{P}_{n}^{(j)}(c)\,\widehat{P}_{n-1}^{(j+1)}(c)\Big)\Big].$$

On the other hand, let $Q(x) \in \mathbb{P}$ be an arbitrary polynomial with $\deg Q(x) \leq n$. It can be written as a linear combination of orthogonal polynomials $\{P_n(x)\}_{n\geq 0}$

$$Q(x) = \sum_{k=0}^{n} \frac{\langle \mathcal{L}, P_k(x)Q(x)\rangle}{\langle \mathcal{L}, P_k^2\rangle} P_k(x).$$

Therefore,

$$[Q]^{(j)}(y) = \sum_{k=0}^{n} \frac{\langle \mathcal{L}, P_k(x)Q(x)\rangle}{\langle \mathcal{L}, P_k^2\rangle} [P_k]^{(j)}(y),$$

and using the fact that

$$\begin{aligned}\langle \mathcal{L}, K_n^{(0,j)}(x,y)Q(x)\rangle &= \left\langle \sum_{k=0}^{n} \frac{P_k(x)[P_k]^{(j)}(y)}{\langle \mathcal{L}, P_k^2\rangle}, Q(x)\right\rangle \\ &= \sum_{k=0}^{n} \frac{\langle \mathcal{L}, P_k(x)Q(x)\rangle}{\langle \mathcal{L}, P_k^2\rangle}[P_k]^{(j)}(y),\end{aligned}$$

then

$$\langle \mathcal{L}, K_n^{(0,j)}(x,y)\,Q\,(x)\rangle = [Q]^{(j)}(y). \tag{2.10}$$

Notice that for $j = 0$ one has the so called *reproducing property* of the kernel (2.3), i.e.

$$\langle \mathcal{L}, K_n(x, y)\, Q(x) \rangle = Q(y). \tag{2.11}$$

2.2 Canonical perturbations of a linear functional

Throughout this dissertation, the notion of quasi-definiteness will be crucial, because this feature ensures that every quasi-definite linear functional has associated an OPS.

We begin by defining a particular case of linear functional that will appear frequently in next chapters, which is called the *Dirac functional* or the *Dirac delta functional*.

Definition 2.1 (Dirac delta functional) *Let $c \in \mathbb{R}$. The linear functional $\delta(x - c)$ supported at $x = c$, such that*

$$\langle \delta(x - c), P(x) \rangle = P(c), \quad \forall P \in \mathbb{P},$$

is called the Dirac functional at c.

Next, let define three basic canonical transformations of a linear functional $\mathcal{L}$ (see [13] and [103]).

Definition 2.2 *Given a quasi-definite moment functional $\mathcal{L}$ and $p \in \mathbb{P}$*

1. *Christoffel transformation of $\mathcal{L}$ (multiplication by a polynomial)*

$$\mathcal{U}_C = (x - c)\mathcal{L}, \tag{2.12}$$

i.e.

$$\langle \mathcal{U}_C, p \rangle = \langle \mathcal{L}, (x - c)p \rangle.$$

2. *Uvarov transformation of $\mathcal{L}$ (addition of a Dirac mass point)*

$$\mathcal{U}_M = \mathcal{L} + M\delta(x - c), \tag{2.13}$$

i.e.

$$\langle \mathcal{U}_M, p \rangle = \langle \mathcal{L}, p \rangle + Mp(c),$$

where $M \in \mathbb{R}_+$.

3. *Geronimus transformation of $\mathcal{L}$ (division by a polynomial and addition of a Dirac mass point)*

$$\mathcal{U}_G = (x-c)^{-1}\mathcal{L} + M\delta(x-c), \tag{2.14}$$

i.e.

$$\langle \mathcal{U}_G, p\rangle = \left\langle \mathcal{L}, \frac{p(x)-p(c)}{x-c} \right\rangle + Mp(c),$$

where $M \in \mathbb{R}_+$.

We can establish some relationships between them. First, if we make the composition of the Geronimus and Christoffel transformations, we obtain the identity transformation $\mathcal{I}$

$$\mathcal{U}_C \circ \mathcal{U}_G = \mathcal{I},$$

and composing the Christoffel and the Geronimus transformations, we recover the Uvarov transformation

$$\mathcal{U}_G \circ \mathcal{U}_C = \mathcal{U}_M.$$

Next, we characterize the MOPS $\{\widehat{Q}_n(x)\}_{n\geq 0}$ associated with $\mathcal{U}_M$, i.e., the new moment functional (2.13)

Theorem 2.6 *Let $\{\widehat{P}_n(x)\}_{n\geq 0}$ be the MOPS associated with $\mathcal{L}$. If we consider the modified moment functional $\mathcal{U}_M$, the following statements hold:*

1. *$\mathcal{U}_M$ is quasi-definite if and only if $1 + MK_n(c,c) \neq 0$, for every $n \in \mathbb{N}$.*

2. *In the above conditions, if $\{\widehat{Q}_n(x)\}_{n\geq 0}$ is the MOPS associated with $\mathcal{U}_M$, for every $n \in \mathbb{N}$,*

$$\widehat{Q}_n(x) = \widehat{P}_n(x) - \frac{M\,\widehat{P}_n(c)}{1 + MK_{n-1}(c,c)} K_{n-1}(x,c). \tag{2.15}$$

(see [3, (8)] and [100]).

3. *If $\mathcal{L}$ is positive definite, then $\mathcal{U}_M$ is positive definite if and only if $1+MK_n(c,c) > 0$, i.e., $-M < \frac{1}{K_n(c,c)}$ for every $n \geq 0$. In other words, it is enough to consider $-M$ as a lower bound of the sequence $\{\frac{1}{K_n(c,c)}\}_{n\geq 0}$ because $\{K_n(c,c)\}_{n\geq 0}$ is a monotonic non-decreasing sequence.*

2.2.1 k-Iterated Christoffel orthogonal polynomials

Let μ be a positive Borel measure supported on an infinite subset E of the real line, and assume $c \notin E$. Here and subsequently, $\{\widehat{P}_n^{[k]}(x)\}_{n\geq 0}$ denotes the MOPS with respect to the inner product

$$\langle f,g\rangle_{[k]} = \int_E f(x)g(x)d\mu^{[k]} \tag{2.16}$$

where

$$d\mu^{[k]} = (x-c)^k d\mu.$$

Then, the polynomials $\{\widehat{P}_n^{[k]}(x)\}_{n\geq 0}$ are orthogonal with respect to a k-iterated Christoffel perturbation of the measure μ. If $k=1$ we have the Christoffel canonical transformation of the measure (see [103] and [102]). It is well known that $\widehat{P}_n^{[1]}(x)$ is the monic kernel polynomial which can be represented as (see [15, (7.3)])

$$\widehat{P}_n^{[1]}(x) = \frac{1}{(x-c)}\left[\widehat{P}_{n+1}(x) - \frac{\widehat{P}_{n+1}(c)}{\widehat{P}_n(c)}\widehat{P}_n(x)\right] = \frac{\|\widehat{P}_n\|_\mu^2}{\widehat{P}_n(c)}K_n(x,c). \tag{2.17}$$

Since $\widehat{P}_n^{[2]}(x)$ are the polynomials orthogonal with respect to the inner product (2.16) when $k=2$, using (2.17) we can continue in this way to obtain

$$\begin{aligned}\widehat{P}_n^{[2]}(x) &= \frac{1}{(x-c)}\left[\widehat{P}_{n+1}^{[1]}(x) - \frac{\widehat{P}_{n+1}^{[1]}(c)}{\widehat{P}_n^{[1]}(c)}\widehat{P}_n^{[1]}(x)\right] \\ &= \frac{1}{(x-c)^2}\left[\widehat{P}_{n+2}(x) - d_n\widehat{P}_{n+1}(x) + e_n\widehat{P}_n(x)\right],\end{aligned} \tag{2.18}$$

where

$$\begin{aligned} d_n &= \frac{\widehat{P}_{n+2}(c)}{\widehat{P}_{n+1}(c)} + \frac{\widehat{P}_{n+1}^{[1]}(c)}{\widehat{P}_n^{[1]}(c)} = \frac{\widehat{P}_{n+2}(c)+\widehat{P}_n(c)}{\widehat{P}_{n+1}(c)}e_n, \\ e_n &= \frac{\widehat{P}_{n+1}^{[1]}(c)}{\widehat{P}_n^{[1]}(c)}\frac{\widehat{P}_{n+1}(c)}{\widehat{P}_n(c)} = \frac{\|\widehat{P}_{n+1}\|_\mu^2}{\|\widehat{P}_n\|_\mu^2}\frac{K_{n+1}(c,c)}{K_n(c,c)} > 0 \end{aligned}$$

and similar formulas could be obtained for $k>2$.

For $n\geq 1$, $\{\widehat{P}_n^{[k]}(x)\}_{n\geq 0}$ are given by the determinant (2.1) where c_i is replaced by $d_i^{[k]}$, $k\in\mathbb{N}$, where

$$d_n^{[k]} = \int_E x^n(x-c)^k d\mu = d_{n+1}^{[k-1]} - c\,d_n^{[k-1]}, \qquad n=0,1,2,..., \tag{2.19}$$

and $m_n = d_n^{[0]}$.

In the sequel, we will denote

$$||\widehat{P}_n^{[k]}||^2_{\mu,[k]} = \int_E \left(\widehat{P}_n^{[k]}(x)\right)^2 (x-c)^k d\mu$$

and $x_{n,r}^{[k]}$, $r = 1, 2, ..., n$, will denote the zeros of $\widehat{P}_n^{[k]}(x)$ arranged in an increasing order.

2.3 Semiclassical orthogonal polynomials

Next we introduce the concept of semiclassical moment functional and their corresponding sequences of orthogonal polynomials. They were introduced by J. Shohat in the latest thirties [92], in connection with weight functions $\omega(x)$ that satisfy a differential equation known as *Pearson type differential equation* (see, for instance [48]). We first give a definition introduced by Maroni (see [82]).

Definition 2.3 (Admissible pair) *Let $\phi(x)$, $\psi(x)$ be two polynomials in $\mathbb{P}$ such that*

$$\begin{aligned} \deg \phi(x) &= r \geq 0, \\ \deg \psi(x) &= l \geq 1. \end{aligned}$$

A pair of polynomials (ϕ, ψ) is said to be admissible if it satisfies the following conditions:

(i) $\deg \phi' \neq \deg \psi$.

(ii) *If* $\deg \phi' = \deg \psi$, *then* $m\frac{\phi^{(r)}(0)}{r!} + \frac{\phi^{(l)}(0)}{l!} \neq 0, \quad m = 0, 1, \ldots$

Definition 2.4 (Semiclassical moment functional) *The quasi-definite linear functional $\mathcal{L}$ is said to be semiclassical if there exists an admissible pair (ϕ, ψ) such that $\mathcal{L}$ satisfies the following distributional Pearson equation*

$$D[\phi(x)\mathcal{L}] = \psi(x)\mathcal{L}, \tag{2.20}$$

where D denotes the distributional derivative. The corresponding sequence of polynomials orthogonal with respect to $\mathcal{L}$ is said to be semiclassical.

Definition 2.5 (Class of a linear functional) *Let X be the set of admissible pairs (ϕ, ψ) such that (2.20) holds. We define the class of the semiclassical linear functional $\mathcal{L}$ as the non-negative integer*

$$s = \min \{ \max \{ \deg \phi - 2, \deg \psi - 1 \} : (\phi, \psi) \in X \} .$$

Moreover, provided that $\phi(x)$ is a monic polynomial, the pair for which the minimum is attained is unique (see [81] and [82])

Next, we will give some characterizations of semiclassical OPS. They are characterized from their orthogonality and by some special differential-difference equations, known as *structure relations.*

Definition 2.6 (Structure Relation for semi-classical OPS) *Let $\mathcal{L}$ be a regular linear functional and $\{\widehat{P}_n(x)\}_{n\geq 0}$ its corresponding MOPS. The following statements are equivalent:*

1. *$\mathcal{L}$ is semiclassical of class s.*
2. *There exists a polynomial $\phi(x)$ of degree r, $0 \leq r \leq s+2$, such that*

$$\phi(x)\widehat{P}'_{n+1}(x) = \sum_{k=n-s}^{n+r} a_{nk}\widehat{P}_k(x), \quad n \geq s, \tag{2.21}$$

with a_{nk} real numbers such that $a_{n,n-s} \neq 0$, $n \geq s+1$.

Another characterization for the semiclassical OPS (and, in fact, a straightforward consequence of the structure relation), is that they are the polynomial solutions of a particular case of second order linear differential equations known as *holonomic equations.* These differential equations can be obtained from the structure relation, that is associated with the so called *creation and destruction operators.* In physics, these two operators are known as ladder operators, and have an extensive use in quantum mechanics.

Definition 2.7 (The Holonomic Equation) *Let $\mathcal{L}$ be a regular linear functional and $\{\widehat{P}_n(x)\}_{n\geq 0}$ its corresponding OPS. $\mathcal{L}$ is semiclassical of class s if and only if there exist polynomials $\mathcal{A}(x;n)$, $\mathcal{B}(x;n)$ and $\mathcal{C}(x;n)$, whose degrees are independent of n and such that*

$$\mathcal{A}(x;n)[\widehat{P}_n]''(x) + \mathcal{B}(x;n)[\widehat{P}_n]'(x) + \mathcal{C}(x;n)\widehat{P}_n(x) = 0, \quad n \geq 0, \tag{2.22}$$

where deg$\mathcal{A} \leq 2s+2$, deg$\mathcal{B} \leq 2s+1$ and deg$\mathcal{C} \leq 2s$.

2.4 Classical orthogonal polynomials

Next, we study a particular case of the semiclassical sequences of orthogonal polynomials called *classical orthogonal polynomials.* They constitute the most important sequences of orthogonal polynomials (see [5], [6] and [9]). They are widely used in the literature due to their applications in mathematical physics, since they appear when Sturm-Liouville problems for hypergeometric differential equations are studied.

Based on the aforesaid notion of *class of a linear functional*, the classical OPS are those semiclassical of class $s = 0$. Moreover, depending on the polynomial $\phi(x)$ the canonical families are known as

$$\begin{cases} \textit{Hermite:} & \phi(x) = 1, \\ \textit{Laguerre:} & \phi(x) = x, \\ \textit{Jacobi:} & \phi(x) = 1 - x^2, \\ \textit{Bessel:} & \phi(x) = x^2. \end{cases}$$

Throughout this memoir we will use mainly the classical Laguerre and Jacobi OPS. For convenience, next, we summarize some properties of them.

2.4.1 Classical Laguerre orthogonal polynomials

Laguerre orthogonal polynomials are defined as the polynomials orthogonal with respect to the inner product

$$\langle f, g \rangle_\alpha = \int_0^{+\infty} f(x)g(x)\, x^\alpha e^{-x} dx, \quad \alpha > -1, \; f, g \in \mathbb{P}. \tag{2.23}$$

The expression of these polynomials as an ${}_1F_1$ hypergeometric function is very well known in the literature (see [7], [46], [60], [87], [99], among others). The connection between these two facts follows from a characterization of such orthogonal polynomials as eigenfunctions of a second order linear differential operator with polynomial coefficients. Let $\{\widehat{L}_n^\alpha(x)\}_{n\geq 0}$, $\alpha > -1$, denote the family of monic Laguerre orthogonal polynomials, defined by the orthogonality relations

$$\int_0^{+\infty} \widehat{L}_n^\alpha(x)\, x^k d\mu(x) = 0, \quad k = 0, 1, \ldots, n-1,$$

where $\widehat{L}_n^\alpha(x) = x^n +$ *lower degree terms* and $d\mu(x) = x^\alpha e^{-x} dx$, $\alpha > -1$. We are interested in the structural properties of Laguerre polynomials which will be useful in the sequel (see [15] and [9]).

Proposition 2.1 *Let* $\{\widehat{L}_n^\alpha(x)\}_{n\geq 0}$ *be the sequence of monic Laguerre orthogonal polynomials. Then the following statements hold*

1. *Three term recurrence relation. For every* $n \in \mathbb{N}$,

$$x\widehat{L}_n^\alpha(x) = \widehat{L}_{n+1}^\alpha(x) + \beta_n \widehat{L}_n^\alpha(x) + \gamma_n \widehat{L}_{n-1}^\alpha(x), \quad n \geq 1, \tag{2.24}$$

with $\widehat{L}_0^\alpha(x) = 1$, $\widehat{L}_1^\alpha(x) = x - (\alpha + 1)$, $\beta_n = 2n + \alpha + 1$, *and* $\gamma_n = n(n + \alpha)$, $n \geq 1$.

2. *Structure relation. For every* $n \in \mathbb{N}$,

$$\widehat{L}_n^\alpha(x) = \widehat{L}_n^{\alpha+1}(x) + n\widehat{L}_{n-1}^{\alpha+1}(x). \tag{2.25}$$

3. *For every* $n \in \mathbb{N}$,

$$||\widehat{L}_n^\alpha||_\alpha^2 = n!\Gamma(n + \alpha + 1). \tag{2.26}$$

4. *Lowering and raising operators (see [99, 5.1, Formula (5.1.14)]). For every* $n \in \mathbb{N}$,

$$x[\widehat{L}_n^\alpha]'(x) - n\widehat{L}_n^\alpha(x) = n(n + \alpha)\widehat{L}_{n-1}^\alpha(x) \qquad \text{(lowering)} \tag{2.27}$$

and

$$x[\widehat{L}_{n-1}^\alpha]'(x) + (n + \alpha - x)\widehat{L}_{n-1}^\alpha(x) = -\widehat{L}_n^\alpha(x) \qquad \text{(raising).} \tag{2.28}$$

5. *For every* $n \in \mathbb{N}$, $\widehat{L}_n^\alpha(x)$ *is the polynomial eigenfunction of the differential operator*

$$xD^2 + (\alpha + 1 - x)D \tag{2.29}$$

with $-n$ *as the corresponding eigenvalue.*

When another normalization for the Laguerre polynomials is needed, we list other interesting properties of this family (see [99, (5.1.13)])

Proposition 2.2 *Let* $\{L_n^{(\alpha)}(x)\}_{n\geq 0}$ *be the sequence of Laguerre orthogonal polynomials with leading coefficient* $\frac{(-1)^n}{n!}$, *i.e.*

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!}\widehat{L}_n^\alpha(x), \tag{2.30}$$

the following statements hold

1. *For an arbitrary real number α, Laguerre polynomials are defined by*

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}. \tag{2.31}$$

2. *Generating function*

$$(1-t)^{-\alpha-1} e^{\frac{-xt}{(1-t)}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n, \ |t| < 1.$$

3. *Rodrigues formula (see [99, Formula 5.1.5]). For every $n \in \mathbb{N} \cup \{0\}$*

$$L_n^{(\alpha)}(x) = \frac{1}{x^\alpha e^{-x}} \frac{1}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha}).$$

4. *Hahn's condition (see [99, Formula 5.1.14]). For every $n \in \mathbb{N}$,*

$$[L_n^{(\alpha)}]'(x) = -L_{n-1}^{(\alpha+1)}(x). \tag{2.32}$$

5. *Outer strong asymptotics (Perron's asymptotics formula on $\mathbb{C} \setminus \mathbb{R}_+$). Let $\alpha \in \mathbb{R}$. Then*

$$\begin{aligned} L_n^{(\alpha)}(x) &= \frac{1}{2} \pi^{-1/2} e^{x/2} (-x)^{-\alpha/2-1/4} n^{\alpha/2-1/4} \exp\{2(-nx)^{1/2}\} \\ &\quad \times \left\{ \sum_{k=0}^{p-1} C_k(\alpha; x) n^{-k/2} + \mathcal{O}(n^{-p/2}) \right\}. \end{aligned} \tag{2.33}$$

Here $C_k(\alpha; x)$ is independent of n. This relation holds for x in the complex plane with a cut along the positive real semiaxis, and it also holds if x is in the cut plane mentioned. $(-x)^{-\alpha/2-1/4}$ and $(-x)^{1/2}$ must be taken real and positive if $x < 0$. The bound for the remainder holds uniformly in every compact subset of the complex plane with empty intersection with $\mathbb{R}_+$ (see [99], Theorem 8.22.3).

6. *Mehler-Heine type formula. For a fixed j, with $j \in \mathbb{N} \cup \{0\}$, if J_α denotes the Bessel function of the first kind, then*

$$\lim_{n\to\infty} \frac{L_n^{(\alpha)}(x/(n+j))}{n^\alpha} = x^{-\alpha/2} J_\alpha (2\sqrt{x}), \tag{2.34}$$

uniformly on compact subsets of $\mathbb{C}$ (see [99], Theorem 8.1.3)

7. *Plancherel-Rotach type formula. Let* $\varphi(x) = x + \sqrt{x^2-1}$, *with* $\sqrt{x^2-1} > 0$ *if* $|x| > 1$, *be the conformal mapping of* $\mathbb{C} \setminus [-1,1]$ *onto the exterior of the unit circle. Then*

$$\lim_{n\to\infty} \frac{L_{n-1}^{(\alpha)}(nx)}{L_n^{(\alpha)}(nx)} = \frac{-1}{\varphi\left(\left(x-2\right)/2\right)}, \tag{2.35}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0,4]$.

2.4.2 Classical Jacobi orthogonal polynomials

Jacobi orthogonal polynomials are defined as the polynomials orthogonal with respect to the inner product

$$\langle f, g\rangle_{\alpha,\beta} = \int_{-1}^{1} f(x)g(x)(1-x)^{\alpha}(1+x)^{\beta}dx, \quad \alpha,\beta > -1,\ f,g \in \mathbb{P}.$$

The expression of these polynomials as an ${}_2F_1$ hypergeometric function is very well known in the literature (see [7], [46], [60], [87], [99], among others). They satisfy the Pearson equation with $\phi(x) = 1-x^2$ and $\psi_{\alpha,\beta}(x) = -(\alpha+\beta+2)x+(\beta-\alpha)$. Let $\{\widehat{P}_n^{\alpha,\beta}(x)\}_{n\geq 0}$, $\alpha,\beta > -1$, denote the family of classical monic Jacobi orthogonal polynomials. Next we summarize some basic properties of this MOPS.(see [15] and [99]).

Proposition 2.3 *Let* $\{\widehat{P}_n^{\alpha,\beta}(x)\}_{n\geq 0}$ *be the sequence of monic Laguerre orthogonal polynomials. Then the following statements hold*

1. *Three term recurrence relation. For every* $n \in \mathbb{N}$,

$$x\widehat{P}_n^{\alpha,\beta}(x) = \widehat{P}_{n+1}^{\alpha,\beta}(x) + \beta_n^{\alpha,\beta}\widehat{P}_n^{\alpha,\beta}(x) + \gamma_n^{\alpha,\beta}\widehat{P}_{n-1}^{\alpha,\beta}(x), \quad n \geq 1,$$

with $\widehat{P}_0^{\alpha,\beta}(x) = 1$, $\widehat{P}_1^{\alpha,\beta}(x) = x + \frac{\alpha-\beta}{\alpha+\beta+2}$, *and*

$$\begin{aligned} \beta_n^{\alpha,\beta} &= \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)}, \\ \gamma_n^{\alpha,\beta} &= \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}. \end{aligned}$$

2. *Structure relation. For every* $n \in \mathbb{N}$,

$$(1-x^2)[\widehat{P}_n^{\alpha,\beta}]'(x) = a_n^{\alpha,\beta}\widehat{P}_{n+1}^{\alpha,\beta}(x) + b_n^{\alpha,\beta}\widehat{P}_n^{\alpha,\beta}(x) + c_n^{\alpha,\beta}\widehat{P}_{n-1}^{\alpha,\beta}(x),$$

where

$$\begin{aligned}
a_n^{\alpha,\beta} &= -n,\\
b_n^{\alpha,\beta} &= \frac{2(\alpha-\beta)\,n\,(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)},\\
c_n^{\alpha,\beta} &= \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}.
\end{aligned}$$

3. *For every* $n \in \mathbb{N}$,

$$||\widehat{P}_n^{\alpha,\beta}||_{\alpha,\beta}^2 = \frac{2^{2n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)n!}{(2n+\alpha+\beta+1)(\Gamma(2n+\alpha+\beta+1))^2}.$$

4. *For every* $n \in \mathbb{N}$,

$$[\widehat{P}_n^{\alpha,\beta}]'(x) = n\,\widehat{P}_{n-1}^{\alpha+1,\beta+1}(x).$$

5. *For every* $n \in \mathbb{N}$, *there exists a sequence* $\{\lambda_n\}_{n\geq 0}$ *of real numbers such that* $\widehat{P}_n^{\alpha,\beta}(x)$ *satisfies the second order linear differential equation*

$$\phi(x)y'' + \psi_{\alpha,\beta}(x)y' = \lambda_n^{\alpha,\beta} y$$

with $\lambda_n^{\alpha,\beta} = -n(n+1+\alpha+\beta)$.

6. *For every* $n \in \mathbb{N}$,

$$P_n^{\alpha,\beta}(x) = \frac{2^n(\alpha+1)_n}{(n+\alpha+\beta+1)_n}\,{}_2F_1\left(-n,\ n+\alpha+\beta+1;\alpha+1;\frac{1-x}{2}\right),$$

where

$$\begin{aligned}
&{}_2F_1\left(-n,\ n+\alpha+\beta+1;\alpha+1;\frac{1-x}{2}\right)\\
=&\sum_{k=0}^{\infty}\frac{(-n)_k(n+\alpha+\beta+1)_k}{(\alpha+1)_k}\frac{(1-x)^k}{2^k k!}.
\end{aligned}$$

CHAPTER 3

Krall-type Orthogonal Polynomials: Zeros

3.1 Introduction

In this chapter we study the behavior of the zeros of the MOPS $\{\widehat{Q}_n(x)\}_{n\geq 0}$ with respect to the Uvarov perturbed measure (2.13)

$$d\mu_M(x;c) = d\mu(x) + M\delta(x-c)$$

where $d\mu(x)$ is a positive Borel measure supported on a bounded or unbounded interval $E = (a,b) \subseteq \mathbb{R}$, the mass point $c \notin E$, and M is a nonnegative real number. In other words, this sequence of polynomials is orthogonal with respect to the inner product

$$\langle f, g\rangle_M = \int_a^b f(x)g(x)d\mu(x) + Mf(c)g(c),\ M \in \mathbb{R}_+,\ c \notin (a,b). \tag{3.1}$$

The structure of the chapter is as follows. First, we obtain a new connection formula for orthogonal polynomials obtained from Uvarov and Christoffel transformations and several results about their zeros, concerning monotonicity and speed of convergence with respect to M. These results follow from the Christoffel formula, from a connection formula for the perturbed polynomials in terms of the initial ones and from the behavior of the zeros of a linear combination of two polynomials (Lemma B.1 in Appendix B).

In the second part of the chapter, we obtain an electrostatic interpretation of the zeros of $\widehat{Q}_n(x)$ as equilibrium points in a logarithmic potential interaction of positive unit charges under the presence of an external field, following a model by M. E. H. Ismail (see [45] and [47]). Next, we check these results for the Krall-Jacobi and Krall-Laguerre MOPS introduced by T. H. Koornwinder [54].

Finally, we deal with a natural generalization of the above problem. We consider sequences of Krall-Laguerre polynomials $\{\widehat{Q}_n^{\alpha,m}\}_{n\geq 0}$, orthogonal with respect to the iterated Uvarov perturbed measure

$$d\mu_m(x) = d\mu_m(x;\mathbf{c}) = \chi_{\mathbb{R}_+} d\mu_\alpha(x) + \sum_{j=1}^{m} M_j \delta(x - c_j), \quad \mathbf{c} = \{c_1, c_2, \ldots, c_m\} \subset \mathbb{R} \setminus \mathbb{R}_+,$$

In other words, we study iterated-Uvarov perturbed MOPS with respect to the inner product

$$\langle f, g\rangle_m = \int_0^{+\infty} f(x)g(x)d\mu_\alpha(x) + \sum_{j=1}^{m} M_j\, f(c_j)g(c_j), \quad f,\, g \in \mathbb{P}, \tag{3.2}$$

where $d\mu_\alpha(x) = x^\alpha e^{-x}dx$ is the Laguerre measure on $\mathbb{R}_+$, $\alpha > -1$, $\mathbf{c} = \{c_1, c_2, \ldots, c_m\} \subset \mathbb{R} \setminus \mathbb{R}_+$, such that if $i < j$ then $c_j < c_i$, $M_j > 0$, and m is a positive integer. We give the representation of these polynomials in terms of the standard Laguerre polynomials. Next, from the holonomic equation that such polynomials satisfy, we obtain an electrostatic model for their zeros in terms of a logarithmic potential. Here, we use different techniques from those analyzed in the previous section to show some of the different approaches that can be used in order to formulate electrostatic interpretations of zeros of MOPS.

Obviously, for $m = 1$ and $c_1 = c$ we have $Q_n^{(\alpha,1)}(x) = Q_n^{(\alpha)}(x)$ and, then, we recover the results analyzed for one mass point.

3.2 New connection formula

There is a well known connection formula for the MOPS $\{\widehat{Q}_n(x)\}_{n\geq 0}$ taking into account the standard MOPS $\{\widehat{P}_n(x)\}_{n\geq 0}$ and the reproducing kernel (2.15), but in [41] another connection formula was obtained. Using a similar idea as in [41, Proposition 4] we deduce another connection formula for the MOPS $\{\widehat{Q}_n(x)\}_{n\geq 0}$ using the standard MOPS $\{\widehat{P}_n(x)\}_{n\geq 0}$ and the 2-Iterated-Christoffel's MOPS $\{\widehat{P}_{n-1}^{[2]}(x)\}_{n\geq 0}$.

Theorem 3.1 (Connection Formula) *The polynomials* $\widetilde{Q}_n(x)$, *with the normalization* $\widetilde{Q}_n(x) = \kappa_n \widehat{Q}_n(x)$, *can be represented as*

$$\widetilde{Q}_n(x) = \widehat{P}_n(x) + MB_n(x-c)\widehat{P}^{[2]}_{n-1}(x), \tag{3.3}$$

where

$$B_n = \frac{-\widehat{P}_n(c)}{\langle x-c, \widehat{P}^{[2]}_{n-1}\rangle_\mu} = K_{n-1}(c,c) > 0 \tag{3.4}$$

and $\kappa_n = 1 + MK_{n-1}(c,c)$.

Proof. In order to prove the orthogonality of the polynomials given in (3.3), we deal with the basis $1, (x-c), (x-c)^2, \ldots, (x-c)^n$ of the linear space $\mathbb{P}_n$ of polynomials of degree at most n. Then,

$$\begin{aligned}
\langle 1, \widetilde{Q}_n\rangle_M &= \langle 1, \widehat{P}_n\rangle_\mu + MB_n\langle 1, (x-c)\widehat{P}^{[2]}_{n-1}\rangle_\mu + M\widehat{P}_n(c) = 0,\\
\langle (x-c), \widetilde{Q}_n\rangle_M &= \langle (x-c), \widehat{P}_n\rangle_\mu + MB_n\langle 1, \widehat{P}^{[2]}_{n-1}\rangle_{[2]} = 0,\\
\langle (x-c)^{n-1}, \widetilde{Q}_n\rangle_M &= \langle (x-c)^{n-1}, \widehat{P}_n\rangle_\mu + MB_n\langle (x-c)^{n-2}, \widehat{P}^{[2]}_{n-1}\rangle_{[2]} = 0,
\end{aligned}$$

and, finally,

$$\begin{aligned}
\langle (x-c)^n, \widetilde{Q}_n\rangle_M &= \langle (x-c)^n, \widehat{P}_n\rangle_\mu + MB_n\langle (x-c)^{n-1}, \widehat{P}^{[2]}_{n-1}\rangle_{[2]}\\
&= \|\widehat{P}_n\|^2_\mu + MB_n\|\widehat{P}^{[2]}_{n-1}\|^2_{[2]} > 0.
\end{aligned}$$

It remains to prove (3.4). From (2.17) and (2.18) we get

$$\begin{aligned}
\langle x-c, \widehat{P}^{[2]}_{n-1}\rangle_\mu &= \int_E (x-c)\,\widehat{P}^{[2]}_{n-1}(x)d\mu(x)\\
&= \int_E (x-c)\,\frac{1}{x-c}\left[\widehat{P}^{[1]}_n(x) - \frac{\widehat{P}^{[1]}_n(c)}{\widehat{P}^{[1]}_{n-1}(c)}\widehat{P}^{[1]}_{n-1}(x)\right] d\mu(x)\\
&= \int_E \widehat{P}^{[1]}_n(x)d\mu(x) - \frac{\widehat{P}^{[1]}_n(c)}{\widehat{P}^{[1]}_{n-1}(c)}\int_E \widehat{P}^{[1]}_{n-1}(x)d\mu(x)\\
&= \frac{\|\widehat{P}_n\|^2_\mu}{\widehat{P}_n(c)}\int_E K_n(x,c)d\mu(x)\\
&\quad - \frac{\|\widehat{P}_n\|^2_\mu}{\widehat{P}_n(c)}\frac{K_n(c,c)}{K_{n-1}(c,c)}\frac{\widehat{P}_{n-1}(c)}{\|\widehat{P}_{n-1}\|^2_\mu}\frac{\|\widehat{P}_{n-1}\|^2_\mu}{\widehat{P}_{n-1}(c)}\int_E K_{n-1}(x,c)d\mu(x)
\end{aligned}$$

$$= \frac{\|\widehat{P}_n\|_\mu^2}{\widehat{P}_n(c)} - \frac{\|\widehat{P}_n\|_\mu^2}{\widehat{P}_n(c)} \frac{K_n(c,c)}{K_{n-1}(c,c)}$$
$$= \frac{\|\widehat{P}_n\|_\mu^2}{\widehat{P}_n(c)} \left(1 - \frac{K_n(c,c)}{K_{n-1}(c,c)}\right).$$

Thus

$$B_n = \frac{-\widehat{P}_n(c)}{\langle x-c, \widehat{P}^{[2]}_{n-1}\rangle_\mu} = \frac{(\widehat{P}_n(c))^2}{\|\widehat{P}_n\|_\mu^2 \left[K_n(c,c)/K_{n-1}(c,c) - 1\right]} = K_{n-1}(c,c) > 0.$$

■

We also derive a representation of the monic polynomial $\widehat{Q}_n(x)$ as a linear combination of two perturbed Christoffel polynomials that will be useful later on this chapter, assuming that c does not belong to the interior of the convex hull of the support of μ.

Corollary 3.1 *The monic polynomial $\widehat{Q}_n(x)$ can be also represented as*

$$\widehat{Q}_n(x) = \widehat{P}^{[1]}_n(x) + c_n\, \widehat{P}^{[1]}_{n-1}(x), \tag{3.5}$$

where

$$c_n = \frac{1 + MK_n(c,c)}{1 + MK_{n-1}(c,c)} \frac{\widehat{P}_{n-1}(c)}{\widehat{P}_n(c)} \gamma_n \quad \text{and} \quad \gamma_n = \frac{\|\widehat{P}_n\|_\mu^2}{\|\widehat{P}_{n-1}\|_\mu^2}. \tag{3.6}$$

Proof. Using (2.3), we obtain

$$\widehat{P}_n(x) = \frac{\|\widehat{P}_n\|_\mu^2}{\widehat{P}_n(c)} \left[K_n(c,x) - K_{n-1}(c,x)\right]. \tag{3.7}$$

From (2.17) and (3.7), we have

$$\widehat{P}_n(x) = \widehat{P}^{[1]}_n(x) - \gamma_n \frac{\widehat{P}_{n-1}(c)}{\widehat{P}_n(c)} \widehat{P}^{[1]}_{n-1}(x). \tag{3.8}$$

Therefore, replacing (2.17) and (3.8) in (2.15) we get (3.5). ■

3.3 Zeros of Krall-type MOPS

The behavior of the zeros of orthogonal polynomials has been extensively studied because of their applications in many areas of physics and engineering. First, the zeros of orthogonal polynomials are the nodes of the Gaussian quadrature rules and also play an important role in some of their extensions like Gauss-Radau, Gauss- Lobatto, and Gauss-Kronrod rules, among others (see [15], [36] [88]). Second, the zeros of classical orthogonal polynomials are the electrostatic equilibrium points of positive unit charges interacting according to a logarithmic potential under the action of an external field, see Stieltjes' papers ([95], [96], [97] and [98]), Szegő's book ([99, Section 6.7]), and some recent works by D. K. Dimitrov and W. Van Assche ([17]), A. Grünbaum ([38] and [39]), M. E. H. Ismail ([45]), and F. Marcellán, A. Martínez-Finkelshtein and P. Martínez-González ([70]) among others. Third, in a more general framework, the counting measure of zeros weakly converges to the equilibrium measure associated with a logarithmic potential (see [94]). Fourth, zeros of orthogonal polynomials are used in collocation methods for boundary value problems of second order linear differential operators (see [8]). Fifth, global properties of zeros of orthogonal polynomials can be analyzed when they satisfy second order linear differential equations with polynomial coefficients using the WKB method (see [3]). Finally, zeros of orthogonal polynomials are eigenvalues of Jacobi matrices and its role in Numerical Linear Algebra is very well known.

Now, for polynomials orthogonal with respect to (3.1), some natural questions arise: Are there values of the parameter M for which the zeros of $\widehat{Q}_n(x)$ interlace with the zeros of $\widehat{P}_n(x)$? Are the zeros of $\widehat{Q}_n(x)$ monotonic functions in terms of the parameter M? Do these zeros converge when M tends to infinity? If so, what the speed of convergence is?

One of our main contributions in this chapter concerns the questions posed above. We provide an interlacing property as well as the monotonicity and asymptotic behavior of the zeros of the polynomial $\widehat{Q}_n(x)$ with respect to M. All the above questions related to the behavior of the zeros of the polynomials $\widehat{Q}_n(x)$ were answered for two important and particular cases in [18] and [19]. The authors considered the cases when $d\mu(x) = x^\alpha e^{-x}dx$ with $(a,b) = (0,\infty)$ and $c = 0$, and $d\mu(x) = (1-x)^\alpha(1+x)^\beta dx$ with $(a,b) = (-1,1)$ and $c = 1$, respectively.

3.3.1 A description of zero distribution

Theorem 3.2 *Let $\{x_{n,j}\}_{k=1}^n$ and $\{x_{n,j}^{[2]}\}_{k=1}^n$ be the zeros of $\widehat{P}_n(x)$ and $\widehat{P}_n^{[2]}(x)$ respectively. Then, the inequalities*

$$x_{n+1,1} < x_{n,1}^{[2]} < x_{n+1,2} < x_{n,2}^{[2]} < \cdots < x_{n+1,n} < x_{n,n}^{[2]} < x_{n+1,n+1} \tag{3.9}$$

hold for every $n \in \mathbb{N}$.

Proof. Let $\{x_{n,k}^{[1]}\}_{k=1}^{n}$ be the zeros of the monic polynomial $\widehat{P}_n^{[1]}(x)$ of degree n, orthogonal with respect to the inner product (2.16) when $k=1$. In Chihara's book [15, Th. 7.2] the following interlacing property involving the zeros of $\widehat{P}_n^{[1]}(x)$, $\widehat{P}_{n+1}(x)$, and $\widehat{P}_n(x)$ is given.

- If $c \leq a$, then

$$x_{n+1,1} < x_{n,1} < x_{n,1}^{[1]} < x_{n+1,2} < \cdots < x_{n,n} < x_{n,n}^{[1]} < x_{n+1,n+1};$$

- If $c \geq b$, then

$$x_{n+1,1} < x_{n,1}^{[1]} < x_{n,1} < \cdots < x_{n+1,n} < x_{n,n}^{[1]} < x_{n,n} < x_{n+1,n+1}.$$

Since $\widehat{P}_n^{[2]}(x)$ are the MOPS with respect to the inner product (2.16) when $k=2$ it is straightforward to prove that $\widehat{P}_n^{[2]}(c) \neq 0$. Then, using the three term recurrence relation

$$\widehat{P}_{n+1}(x) = (x-\beta_n)\widehat{P}_n(x) - \gamma_n \widehat{P}_{n-1}(x),$$

where

$$\beta_n = \frac{\langle x\widehat{P}_n, \widehat{P}_n\rangle_\mu}{\|\widehat{P}_n\|_\mu^2}, \ n \geq 0, \ \text{ and } \ \gamma_n = \frac{\|\widehat{P}_n\|_\mu^2}{\|\widehat{P}_{n-1}\|_\mu^2} > 0, \ n \geq 1,$$

from (2.17) and (2.18) we obtain

$$\widehat{P}_n^{[2]}(x) = \frac{1}{(x-c)^2}\left[(x-\beta_{n+1}-d_n)\widehat{P}_{n+1}(x) + (e_n - \gamma_{n+1})\widehat{P}_n(x)\right]. \tag{3.10}$$

On the other hand,

$$e_n - \gamma_{n+1} = \frac{\|\widehat{P}_{n+1}\|_\mu^2}{\|\widehat{P}_n\|_\mu^2}\left(\frac{K_{n+1}(c,c)}{K_n(c,c)} - 1\right) > 0. \tag{3.11}$$

Thus, evaluating $\widehat{P}_n^{[2]}(x)$ at the zeros $x_{n+1,k}$, from (3.10) and (3.11),

$$sign\,\widehat{P}_n^{[2]}(x_{n+1,k}) = sign\,\widehat{P}_n(x_{n+1,k}), \ k = 1, \ldots, n+1.$$

Since the zeros of $\widehat{P}_{n+1}(x)$ and $\widehat{P}_n(x)$ interlace, the proof is concluded. ■

Let $\{x_{n,k}^{M}\}_{k=1}^{n}$ be the zeros of $\widehat{Q}_n(x)$. If M is a nonnegative real number then $d\mu_M(x;c)$ is a positive measure, and, as a consequence, the zeros of $\widehat{Q}_n(x)$ are real, simple, and lie in (c,b) (resp. in (a,c)) if $c \leq a$ (resp. if $c \geq b$), that is,

$$\min\{a,c\} < x_{n,1}^{M} < \cdots < x_{n,n}^{M} < \max\{b,c\}.$$

As a consequence of the following theorem, when M tends to infinity notice that the mass point c attracts one zero of $\widehat{Q}_n(x)$, that is, it captures either the least or the largest zero, according to the location of the point c with respect to the interval (a,b). In addition, when either $c<a$ or $c>b$, at most one of the zeros of $\widehat{Q}_n(x)$ is located outside (a,b).

Theorem 3.3 *Let $M>0$ and $\{x_{n,k}^{[2]}\}_{k=1}^{n}$ be the zeros of the polynomial $\widehat{P}_n^{[2]}(x)$ orthogonal with respect to the inner product (2.16) when $k=2$.*

(*i*) *If $c \leq a$, then*

$$c < x_{n,1}^M < x_{n,1} < x_{n-1,1}^{[2]} < x_{n,2}^M < x_{n,2} < \cdots < x_{n-1,n-1}^{[2]} < x_{n,n}^M < x_{n,n}. \tag{3.12}$$

Moreover, each $x_{n,k}^M$ is a decreasing function of M and, for each $k=1,\dots,n-1$, (see the comment in the last sentence 3.3.2)

$$\lim_{M\to\infty} x_{n,1}^M = c, \quad \lim_{M\to\infty} x_{n,k+1}^M = x_{n-1,k}^{[2]},$$

as well as

$$\begin{aligned} \lim_{M\to\infty} M[x_{n,1}^M - c] &= \frac{-\widehat{P}_n(c)}{K_{n-1}(c,c)\widehat{P}_{n-1}^{[2]}(c)}, \\ \lim_{M\to\infty} M[x_{n,k+1}^M - x_{n-1,k}^{[2]}] &= \frac{-\widehat{P}_n(x_{n-1,k}^{[2]})}{K_{n-1}(c,c)(x_{n-1,k}^{[2]}-c)[\widehat{P}_{n-1}^{[2]}]'(x_{n-1,k}^{[2]})}. \end{aligned} \tag{3.13}$$

(*ii*) *If $c \geq b$, then*

$$x_{n,1} < x_{n,1}^M < x_{n-1,1}^{[2]} < \cdots < x_{n,n-1} < x_{n,n-1}^M < x_{n-1,n-1}^{[2]} < x_{n,n} < x_{n,n}^M < c. \tag{3.14}$$

Moreover, each $x_{n,k}^M$ is an increasing function of M and, for each $k=1,\dots,n-1$,

$$\lim_{M\to\infty} x_{n,n}^M = c, \quad \lim_{M\to\infty} x_{n,k}^M = x_{n-1,k}^{[2]},$$

and

$$\begin{aligned} \lim_{M\to\infty} M[c - x_{n,n}^M] &= \frac{\widehat{P}_n(c)}{K_{n-1}(c,c)\widehat{P}_{n-1}^{[2]}(c)}, \\ \lim_{M\to\infty} M[x_{n-1,k}^{[2]} - x_{n,k}^M] &= \frac{\widehat{P}_n(x_{n-1,k}^{[2]})}{K_{n-1}(c,c)(x_{n-1,k}^{[2]}-c)[\widehat{P}_{n-1}^{[2]}]'(x_{n-1,k}^{[2]})}. \end{aligned}$$

Proof. Using the interlacing property (3.9) and the connection formula (3.3), we get

$$\begin{aligned} sign\, \widetilde{Q}_n(x_{n,k}) &= sign\, \left(M(x_{n,k}-c)\widehat{P}_{n-1}^{[2]}(x) \right), \qquad k=1,\ldots,n, \\ sign\, \widetilde{Q}_n(x_{n-1,k}^{[2]}) &= sign\, \widehat{P}_n(x_{n-1,k}^{[2]}), \qquad k=1,\ldots,n-1, \end{aligned}$$

which yield the inequalities stated in (3.12) and (3.14). It remains to show the monotonicity, asymptotics, and the speed of the convergence of the zeros $x_{n,k}^M$ with respect to M. Indeed, it follows from Lemma B.1 concerning the zeros of a linear combination of two polynomials with interlacing zeros. ■

We point out that Theorem 3.3 is general in two aspects and uses new approaches to the analysis of zeros: $d\mu(x)$ is any positive Borel measure and c is outside (a,b).

Next, we deduce the value M_0 of the mass such that for $M > M_0$ one of the zeros of $\widehat{Q}_n(x)$ is located outside (a,b).

Corollary 3.2 *Let $M > 0$.*

(i) If $c < a$, then the least zero $x_{n,1}^M$ satisfies

$$\begin{aligned} x_{n,1}^M &> a, \quad for\ M < M_0, \\ x_{n,1}^M &= a, \quad for\ M = M_0, \\ x_{n,1}^M &< a, \quad for\ M > M_0, \end{aligned}$$

where

$$M_0 = M_0(n,a,c) = \frac{-\widehat{P}_n(a)}{K_{n-1}\,(c,c)\,(a-c)\widehat{P}_{n-1}^{[2]}(a)} > 0.$$

(ii) If $c > b$, then the largest zero $x_{n,n}^M$ satisfies

$$\begin{aligned} x_{n,n}^M &< b, \quad for\ M < M_0, \\ x_{n,n}^M &= b, \quad for\ M = M_0, \\ x_{n,n}^M &> b, \quad for\ M > M_0, \end{aligned}$$

where $M_0 = M_0(n,b,c)$.

The proofs are a direct consequence of the connection formula (3.3).

3.3.2 Convergence of the zeros

We can study to where the zeros of $\widehat{Q}_n(x)$ converge as $M \to \infty$, from another point of view. For this purpose, we express $\widehat{Q}_n(x)$ in terms of the generalized moments using the Gram-Schmidt orthonormalization process for the family of polynomials $\{(x-c)^k\}_{k=0}^n$. Indeed, if $\left\langle (x-c)^k, (x-c)^j \right\rangle = c_{k+j}$ and denoting

$$\Omega_n\left(d\mu\right) = \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-1} \\ c_n & c_{n+1} & \cdots & c_{2n} \end{vmatrix},$$

we get

$$\widehat{P}_n(x) = \frac{1}{\Omega_{n-1}\left(d\mu\right)} \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-1} \\ 1 & x-c & \cdots & (x-c)^n \end{vmatrix}. \tag{3.15}$$

If $d\mu_M(x;c) = d\mu(x) + M\delta(x-c)$, then

$$\widehat{Q}_n(x) = \frac{1}{\Omega_{n-1}\left(d\mu_M\right)} \begin{vmatrix} c_0+M & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-1} \\ 1 & x-c & \cdots & (x-c)^n \end{vmatrix}.$$

Since

$$\Omega_{n-1}\left(d\mu_M\right) = \begin{vmatrix} c_0+M & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} \end{vmatrix} = \Omega_{n-1}\left(d\mu\right) + M\Omega_{n-2}\left((x-c)^2\, d\mu\right)$$

we conclude

$$\widehat{Q}_n(x) = \frac{\Omega_{n-1}\left(d\mu\right)\widehat{P}_n(x) + M\left(x-c\right)\Omega_{n-2}\left((x-c)^2\, d\mu\right)\widehat{P}^{[2]}_{n-1}(x)}{\Omega_{n-1}\left(d\mu\right) + M\Omega_{n-2}\left((x-c)^2\, d\mu\right)}.$$

From the above expression we can easily see that when $M \to \infty$, the mass point c attracts only one zero of $\widehat{Q}_n(x)$, and each zero of $\widehat{P}_{n-1}^{[2]}(x)$ attracts one of the remainder $n-1$ zeros of $\widehat{Q}_n(x)$.

3.4 Application to classical measures

3.4.1 Krall-Jacobi (Jacobi-Koornwinder) MOPS

Let $\{\widehat{P}_n^{\alpha,\beta}(x)\}_{n\geq 0}$ be the monic Jacobi polynomial sequence which is orthogonal with respect to the measure $d\mu_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta dx$, $\alpha, \beta > -1$, supported on $(-1,1)$. We consider the following Uvarov perturbations of $d\mu_{\alpha,\beta}(x)$ where either $c=-1$ or $c=1$, and $M \geq 0$.

$$d\mu_M(x;-1) = d\mu_{\alpha,\beta}(x) + M\delta(x+1), \tag{3.16}$$

$$d\mu_N(x;1) = d\mu_{\alpha,\beta}(x) + N\delta(x-1). \tag{3.17}$$

Such orthogonal polynomials were first studied in 1984 by T. H. Koornwinder (see [54]), in 1984. There, he adds simultaneously two Dirac delta functions at the end points $x=-1$ and $x=1$, that is,

$$d\mu_{M,N}(x) = d\mu_{\alpha,\beta}(x) + M\delta(x+1) + N\delta(x-1).$$

Let denote by $\{\widetilde{Q}_n^{\alpha,\beta}(x;-1)\}_{n\geq 0}$ and $\{\widetilde{Q}_n^{\alpha,\beta}(x;1)\}_{n\geq 0}$ the OPS with respect (3.16) and (3.17), with the normalization introduced in Theorem 3.1, respectively. Then, the connection formulas are

$$\widetilde{Q}_n^{\alpha,\beta}(x;-1) = \widehat{P}_n^{\alpha,\beta}(x) + MK_{n-1}(-1,-1)(x+1)\widehat{P}_{n-1}^{\alpha,\beta+2}(x)$$

and

$$\widetilde{Q}_n^{\alpha,\beta}(x;1) = \widehat{P}_n^{\alpha,\beta}(x) + NK_{n-1}(1,1)(x-1)\widehat{P}_{n-1}^{\alpha+2,\beta}(x).$$

It is straightforward to see that

$$\begin{aligned} K_{n-1}(-1,-1) &= \frac{1}{2^{\alpha+\beta+1}}\frac{\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n)\Gamma(\beta+1)\Gamma(\beta+2)\Gamma(n+\alpha)}, \\ K_{n-1}(1,1) &= \frac{1}{2^{\alpha+\beta+1}}\frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n)\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(n+\beta)}. \end{aligned}$$

Recently, several authors ([3], [19], [25]) have contributed to the analysis of the behavior of the zeros of $\widetilde{Q}_n^{\alpha,\beta}(x;-1)$ and $\widetilde{Q}_n^{\alpha,\beta}(x;1)$.

Let us denote by $x_{n,k}^M(\alpha,\beta)$ and $x_{n,k}(\alpha,\beta)$, $k=1,\ldots,n$, the zeros of $\widetilde{Q}_n^{\alpha,\beta}(x;-1)$ and $\widehat{P}_n^{\alpha,\beta}(x)$, respectively, in an increasing order. Then, applying Theorem 3.3 we obtain

Theorem 3.4 ([19]) *The inequalities*

$$-1 < x_{n,1}^M(\alpha,\beta) < x_{n,1}(\alpha,\beta) < x_{n-1,1}(\alpha,\beta+2) < x_{n,2}^M(\alpha,\beta) < x_{n,2}(\alpha,\beta) < \cdots$$
$$< x_{n-1,n-1}(\alpha,\beta+2) < x_{n,n}^M(\alpha,\beta) < x_{n,n}(\alpha,\beta)$$

hold for every $\alpha,\beta > -1$. Moreover, each $x_{n,k}^M(\alpha,\beta)$ is a decreasing function of M and, for each $k=1,\ldots,n-1$,

$$\lim_{M\to\infty} x_{n,1}^M(\alpha,\beta) = -1, \ \lim_{M\to\infty} x_{n,k+1}^M(\alpha,\beta) = x_{n-1,k}(\alpha,\beta+2),$$

and

$$\lim_{M\to\infty} M[x_{n,1}^M(\alpha,\beta)+1] = h_n(\alpha,\beta),$$
$$\lim_{M\to\infty} M[x_{n,k+1}^M(\alpha,\beta) - x_{n-1,k}(\alpha,\beta+2)] = \frac{[1-x_{n-1,k}(\alpha,\beta+2)]\,h_n(\alpha,\beta)}{2(\beta+2)},$$

where

$$h_n(\alpha,\beta) = \frac{2^{\alpha+\beta+2}\Gamma(n)\Gamma(\beta+2)\Gamma(\beta+3)\Gamma(n+\alpha)}{\Gamma(n+\beta+2)\Gamma(n+\alpha+\beta+2)}.$$

Proof. From (3.13)

$$\lim_{M\to\infty} M[x_{n,1}^M(\alpha,\beta)+1] = \frac{-\widehat{P}_n^{\alpha,\beta}(-1)}{K_{n-1}(-1,-1)\,\widehat{P}_{n-1}^{\alpha,\beta+2}(-1)}.$$

Since

$$\widehat{P}_n^{\alpha,\beta}(-1) = \frac{(-1)^n 2^n \Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(\beta+1)\Gamma(2n+\alpha+\beta+1)}$$

and

$$K_{n-1}(-1,-1) = \frac{1}{2^{\alpha+\beta+1}}\frac{\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n)\Gamma(\beta+1)\Gamma(\beta+2)\Gamma(n+\alpha)}$$

we obtain

$$\frac{-\widehat{P}_n^{\alpha,\beta}(-1)}{K_{n-1}(-1,-1)\,\widehat{P}_{n-1}^{\alpha,\beta+2}(-1)} = \frac{2^{\alpha+\beta+2}\Gamma(n)\Gamma(\beta+2)\Gamma(\beta+3)\Gamma(n+\alpha)}{\Gamma(n+\beta+2)\Gamma(n+\alpha+\beta+2)} = h_n(\alpha,\beta).$$

It also follows from (3.13) that

$$\begin{aligned}&\lim_{M\to\infty} M[x^M_{n,k+1}(\alpha,\beta)-x_{n-1,k}(\alpha,\beta+2)]\\ =\;&\frac{-\widehat{P}^{\alpha,\beta}_n(x_{n-1,k}(\alpha,\beta+2))}{K_{n-1}(-1,-1)(x_{n-1,k}(\alpha,\beta+2)+1)[\widehat{P}^{\alpha,\beta+2}_{n-1}]'(x_{n-1,k}(\alpha,\beta+2))}.\end{aligned}$$

On the other hand, from

$$n(n+\alpha)(1+x)\widehat{P}^{\alpha,\beta+2}_{n-1}(x)=n(n+\alpha+\beta+1)\widehat{P}^{\alpha,\beta}_n(x)+(\beta+1)(1-x)[\widehat{P}^{\alpha,\beta}_n]'(x)$$

we get

$$\begin{aligned}&n(n+\alpha+\beta+1)\,\widehat{P}^{\alpha,\beta}_n(x_{n-1,k}(\alpha,\beta+2))\\ =\;&-(\beta+1)(1-x_{n-1,k}(\alpha,\beta+2))[\widehat{P}^{\alpha,\beta}_n]'(x_{n-1,k}(\alpha,\beta+2))\end{aligned}$$

as well as

$$\begin{aligned}&n(n+\alpha)(1+x_{n-1,k}(\alpha,\beta+2))[\widehat{P}^{\alpha,\beta}_n]'(x_{n-1,k}(\alpha,\beta+2))\\ =\;&[n(n+\alpha+\beta+1)-(\beta+1)][\widehat{P}^{\alpha,\beta}_n]'(x_{n-1,k}(\alpha,\beta+2))\\ &+(\beta+1)(1-x_{n-1,k}(\alpha,\beta+2))[\widehat{P}^{\alpha,\beta}_n]''(x_{n-1,k}(\alpha,\beta+2)).\end{aligned}$$

Using the last two equalities and the second order linear differential equation for the Jacobi polynomials

$$(1-x^2)[\widehat{P}^{\alpha,\beta}_n]''(x)+[\beta-\alpha-(\alpha+\beta+1)x][\widehat{P}^{\alpha,\beta}_n]'(x)+n(n+\alpha+\beta+1)\widehat{P}^{\alpha,\beta}_n(x)=0$$

we obtain

$$\begin{aligned}&(1+x_{n-1,k}(\alpha,\beta+2))[\widehat{P}^{\alpha,\beta+2}_n]'(x_{n-1,k}(\alpha,\beta+2))\\ =\;&\frac{-(n+\beta+1)(n+\alpha+\beta+1)}{(\beta+1)(1-x_{n-1,k}(\alpha,\beta+2))}\widehat{P}^{\alpha,\beta}_n(x_{n-1,k}(\alpha,\beta+2)).\end{aligned}$$

Therefore,

$$\begin{aligned}&\lim_{M\to\infty} M[x^M_{n,k+1}(\alpha,\beta)-x_{n-1,k}(\alpha,\beta+2)]\\ =\;&\frac{-\widehat{P}^{\alpha,\beta}_n(x_{n-1,k}(\alpha,\beta+2))}{K_{n-1}(-1,-1)(x_{n-1,k}(\alpha,\beta+2)+1)[\widehat{P}^{\alpha,\beta+2}_{n-1}]'(x_{n-1,k}(\alpha,\beta+2))}\end{aligned}$$

$$= \frac{[1 - x_{n-1,k}(\alpha, \beta + 2)]\, h_n(\alpha, \beta)}{2(\beta + 2)}$$

■

Let $x^N_{n,k}(\alpha, \beta)$ be the zeros of $\widetilde{Q}^{\alpha,\beta}_n(x; 1)$. Then

Theorem 3.5 ([19]) *The inequalities*

$$x_{n,1}(\alpha, \beta) < x^N_{n,1}(\alpha, \beta) < x_{n-1,1}(\alpha + 2, \beta) < \cdots <$$
$$x_{n,n-1}(\alpha, \beta) < x^N_{n,n-1}(\alpha, \beta) < x_{n-1,n-1}(\alpha + 2, \beta) < x_{n,n}(\alpha, \beta) < x^N_{n,n}(\alpha, \beta) < 1$$

hold for every $\alpha, \beta > -1$. Moreover, each $x^N_{n,k}(\alpha, \beta)$ is an increasing function of N and, for each $k = 1, \ldots, n-1$,

$$\lim_{N\to\infty} x^N_{n,n}(\alpha, \beta) = 1, \ \lim_{N\to\infty} x^N_{n,k}(\alpha, \beta) = x_{n-1,k}(\alpha + 2, \beta),$$

and

$$\lim_{N\to\infty} N[1 - x^N_{n,n}(\alpha, \beta)] = g_n(\alpha, \beta),$$
$$\lim_{N\to\infty} N[x_{n-1,k}(\alpha + 2, \beta) - x^N_{n,k}(\alpha, \beta)] = \frac{[1 + x_{n-1,k}(\alpha + 2, \beta)]\, g_n(\alpha, \beta)}{2(\alpha + 2)},$$

where

$$g_n(\alpha, \beta) = \frac{2^{\alpha+\beta+2}\Gamma(n)\Gamma(\alpha + 2)\Gamma(\alpha + 3)\Gamma(n + \beta)}{\Gamma(n + \alpha + 2)\Gamma(n + \alpha + \beta + 2)}.$$

Proof. We can proceed as in the proof of Theorem 3.4. We only observe that

$$\frac{n + \beta}{2n}(x - 1)\widehat{P}^{\alpha+2,\beta}_{n-1}(x) = \widehat{P}^{\alpha,\beta}_n(x) - \frac{\alpha + 1}{n(n + \alpha + \beta + 1)}(1 + x)[\widehat{P}^{\alpha,\beta}_n]'(x).$$

■

To illustrate the results of Theorem 3.5, the graphs of $\widetilde{Q}^{\alpha,\beta}_3(x; 1)$, for $\alpha = \beta = 0$ and $N + \varepsilon$, for some values of $\varepsilon > 0$ appear in Figure 3.1 where the monotonicity of the zeros of $\widetilde{Q}^{\alpha,\beta}_3(x; 1)$ as a function of the mass N is shown.

In Table 3.1 we describe the zeros of $\widetilde{Q}^{\alpha,\beta}_3(x; 1)$, with $\alpha = \beta = 0$, for several choices of N.

Notice that the largest zero converges to 1 and the other two zeros converge to the zeros of the Jacobi polynomial $\widehat{P}^{(2,0)}_2(x)$, that is, they converge to $x_{2,1}(2, 0) = -0.75497$ and $x_{2,2}(4) = 0.0883037$. Also note that all the zeros increase when N increase.

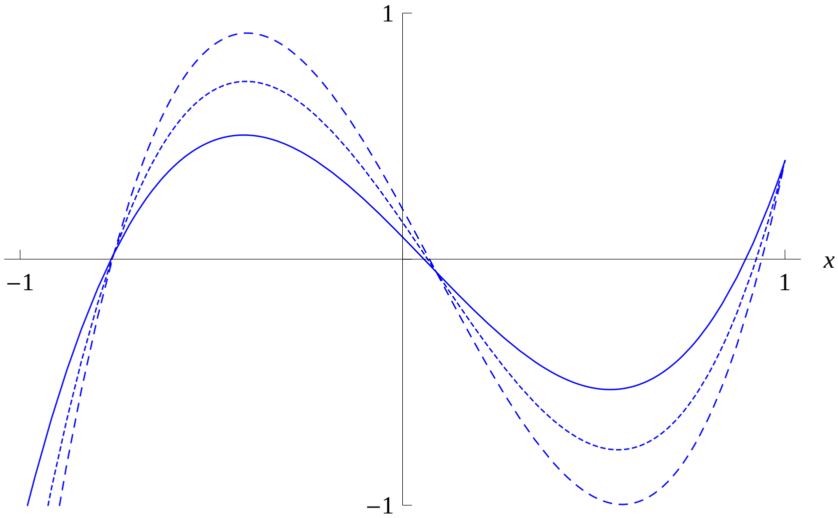


Figure 3.1: The graphs of $\widetilde{Q}_3^{\alpha,\beta}(x;1)$ with $N+\varepsilon$, for some values of ε.

Table 3.1: Zeros of $\widetilde{Q}_3^{\alpha,\beta}(x;1)$ for some values of N.

λ	$x_{3,1}(0,0;\lambda)$	$x_{3,2}(0,0;\lambda)$	$x_{3,3}(0,0;\lambda)$
0	−0.774597	0	0.774597
1	−0.757872	0.0753429	0.955257
10	−0.755305	0.0868168	0.994575
100	−0.755004	0.0881528	0.999446
1000	−0.754974	0.0882886	0.999944

3.4.2 Krall-Laguerre (Laguerre-Koornwinder) MOPS

Let $\{\widehat{L}_n^{\alpha}(x)\}_{n\geq 0}$ be the monic Laguerre polynomials which are orthogonal with respect to the classical Laguerre measure $d\mu_\alpha(x) = x^\alpha e^{-x}dx$, $\alpha > -1$, supported on $[0,+\infty)$. Let us denote by $\{\widehat{Q}_n^{\alpha}(x)\}_{n\geq 0}$ the sequence of polynomials orthogonal with respect to

$$d\mu_M(x;c) = d\mu_\alpha(x) + M\delta(x-c), \qquad M \geq 0. \tag{3.18}$$

When $c = 0$, this family was also obtained by T. H. Koornwinder [54] as a special limit case of the Jacobi-Koornwinder (Jacobi type) orthogonal polynomial. Analytic properties of these polynomials have been studied in the last years (see [3], [18], [26], [52], among others). The connection formula (3.3) reads

$$\widetilde{Q}_n^{\alpha}(x) = \widehat{L}_n^{\alpha}(x) + M K_{n-1}(0,0)\, x\, \widehat{L}_{n-1}^{\alpha+2}(x),$$

where

$$K_{n-1}(0,0) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n)\Gamma(\alpha+1)\Gamma(\alpha+2)}.$$

Now, we will analyze the behavior of their zeros. Let denote by $x_{n,k}^M(\alpha)$ and $x_{n,k}(\alpha)$, $k = 1, \ldots, n$, the zeros of the Krall-Laguerre polynomial $\widetilde{Q}_n^{\alpha}(x)$ and the classical Laguerre orthogonal polynomial $\widehat{L}_n^{\alpha}(x)$, respectively. Applying the results of Theorem 1 we obtain

Theorem 3.6 ([18]) *The inequalities*

$$0 < x_{n,1}^M(\alpha) < x_{n,1}(\alpha) < x_{n-1,1}(\alpha+2) < x_{n,2}^M(\alpha) < \\ x_{n,2}(\alpha) < \cdots < x_{n-1,n-1}(\alpha+2) < x_{n,n}^M(\alpha) < x_{n,n}(\alpha)$$

hold for every $\alpha > -1$*. Moreover, each* $x_{n,k}^M(\alpha)$ *is a decreasing function of* M *and, for each* $k = 1, \ldots, n-1$,

$$\lim_{M\to\infty} x_{n,1}^M(\alpha) = 0, \quad \lim_{M\to\infty} x_{n,k+1}^M(\alpha) = x_{n-1,k}(\alpha+2),$$

as well as

$$\lim_{M\to\infty} M x_{n,1}^M(\alpha) = g_n(\alpha),$$
$$\lim_{M\to\infty} M[x_{n,k+1}^M(\alpha) - x_{n-1,k}(\alpha+2)] = \frac{g_n(\alpha)}{\alpha+2},$$

where

$$g_n(\alpha) = \frac{\Gamma(n)\Gamma(\alpha+2)\Gamma(\alpha+3)}{\Gamma(n+\alpha+2)}.$$

Proof. From (3.13)

$$\lim_{M\to\infty} M x_{n,1}^M(\alpha) = \frac{-\widehat{L}_n^{\alpha}(0)}{K_{n-1}(0,0)\, \widehat{L}_{n-1}^{\alpha+2}(0)}.$$

Since

$$\widehat{L}_n^{\alpha}(0) = \frac{(-1)^n\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \quad \text{and} \quad K_{n-1}(0,0) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n)\Gamma(\alpha+1)\Gamma(\alpha+2)},$$

we obtain

$$\frac{-\widehat{L}_n^{\alpha}(0)}{K_{n-1}(0,0)\, \widehat{L}_{n-1}^{\alpha+2}(0)} = \frac{\Gamma(n)\Gamma(\alpha+2)\Gamma(\alpha+3)}{\Gamma(n+\alpha+2)} = g_n(\alpha).$$

From (3.13)

$$\lim_{M\to\infty} M[x^M_{n,k+1}(\alpha) - x_{n-1,k}(\alpha+2)]$$
$$= \frac{-\widehat{L}^{\alpha}_n(x_{n-1,k}(\alpha+2))}{K_{n-1}(0,0)x_{n-1,k}(\alpha+2)[\widehat{L}^{\alpha+2}_{n-1}]'(x_{n-1,k}(\alpha+2))}.$$

On the other hand, it is easy to verify that

$$x\widehat{L}^{\alpha+2}_{n-1}(x) = \widehat{L}^{\alpha}_n(x) + \frac{\alpha+1}{n}[\widehat{L}^{\alpha}_n]'(x).$$

Thus,

$$[\widehat{L}^{\alpha}_n]'(x_{n-1,k}(\alpha+2)) = -\frac{n}{\alpha+1}\widehat{L}^{\alpha}_n(x_{n-1,k}(\alpha+2))$$

and

$$x_{n-1,k}(\alpha+2)[\widehat{L}^{\alpha}_n]'(x_{n-1,k}(\alpha+2))$$
$$= [\widehat{L}^{\alpha}_n]'(x_{n-1,k}(\alpha+2)) + \frac{\alpha+1}{n}[\widehat{L}^{\alpha}_n]''(x_{n-1,k}(\alpha+2)).$$

Using the last two equalities and the second order linear differential equation for the Laguerre polynomials

$$x[\widehat{L}^{\alpha}_n]''(x) + (\alpha+1-x)[\widehat{L}^{\alpha}_n]'(x) + n\widehat{L}^{\alpha}_n(x) = 0$$

we obtain

$$x_{n-1,k}(\alpha+2)[\widehat{L}^{\alpha}_n]'(x_{n-1,k}(\alpha+2)) = \frac{-(n+\alpha+1)}{\alpha+1}\widehat{L}^{\alpha}_n(x_{n-1,k}(\alpha+2)).$$

Therefore,

$$\lim_{M\to\infty} M[x^M_{n,k+1}(\alpha) - x_{n-1,k}(\alpha+2)]$$
$$= \frac{-\widehat{L}^{\alpha}_n(x_{n-1,k}(\alpha+2))}{K_{n-1}(0,0)x_{n-1,k}(\alpha+2)[\widehat{L}^{\alpha+2}_{n-1}]'(x_{n-1,k}(\alpha+2))}$$
$$= \frac{\Gamma(n)\Gamma(\alpha+2)\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)} = \frac{g_n(\alpha)}{\alpha+2}.$$

■

To illustrate the results of Theorem 3.6 we enclose the graphs of $\widehat{Q}^{\alpha}_3(x)$ for $\alpha = 2$ and $M+\varepsilon$, for some values of $\varepsilon > 0$. Figure 3.2 shows the monotonicity of the zeros of $\widehat{Q}^{\alpha}_3(x)$ as a function of the mass M.

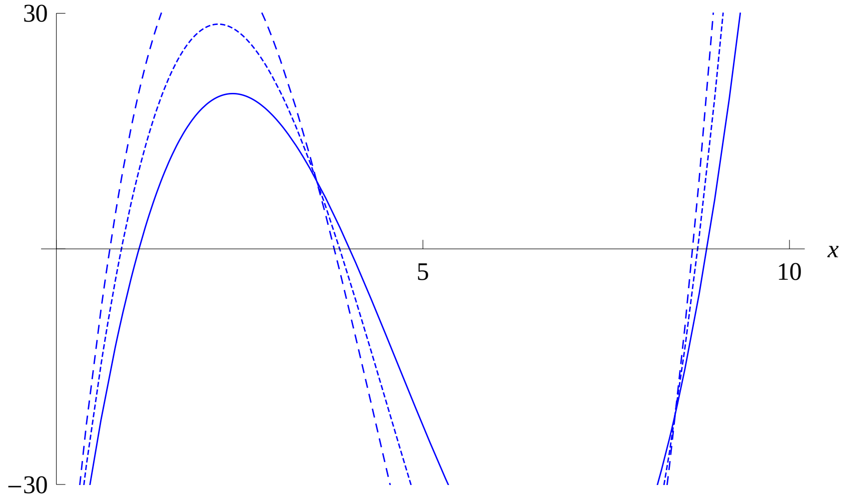


Figure 3.2: The graphs of $\widehat{Q}_3^\alpha(x)$ for $c = 0$ and $M + \varepsilon$, for some values of ε.

Table 3.2: Zeros of $\widehat{Q}_3^\alpha(x)$ for $c = 0$ and some values of M.

λ	$x_{3,1}(2;\lambda)$	$x_{3,2}(2;\lambda)$	$x_{3,3}(2;\lambda)$
0	1.51739	4.31158	9.17103
1	0.321731	3.64053	8.53774
10	0.0390611	3.5604	8.45936
100	0.00399042	3.55151	8.45049
1000	0.00039990	3.55061	8.44959

Table 3.2 describes the zeros of $\widehat{Q}_3^\alpha(x)$, with $\alpha = 2$, for several choices of M. Observe that the least zero converges to 0 and the other two zeros converge to the zeros of the Laguerre polynomial $\widehat{L}_2^4(x)$, that is, they converge to $x_{2,1}(4) = 3.55051$ and $x_{2,2}(4) = 8.44949$. Note that all the zeros decrease when M increases

3.5 Electrostatic model for one mass point

3.5.1 Ismail's electrostatic model for semiclassical measures

We assume that $d\mu(x) = \omega(x)dx$, where $\omega(x)$ is a weight function supported on the real line. We can associate with $\omega(x)$ an external potential $\upsilon(x)$ such that $\omega(x) =$

$\exp\left(-\upsilon\left(x\right)\right)$. Notice that if $\upsilon(x)$ is assumed to be differentiable on the support of $d\mu(x) = \omega\left(x\right)dx$ then

$$\frac{\omega'\left(x\right)}{\omega\left(x\right)} = -\upsilon'\left(x\right).$$

If $\upsilon'(x)$ is a rational function on (a, b), then the weight function $\omega(x)$ is said to be semiclassical (see [82], [92]). The linear functional $\mathcal{L}$ associated with $\omega(x)$,

$$\langle \mathcal{L}, f(x) \rangle = \int_a^b f(x)\omega\left(x\right)dx,$$

satisfies the Pearson's equation

$$D[\sigma(x)\mathcal{L}] = \tau(x)\mathcal{L},$$

where D is the distributional derivative of $\mathcal{L}$ introduced in Section 2.1.1, and $\sigma(x)$, $\tau(x)$ are non-zero polynomials such that $\sigma(x)$ is monic and $\deg(\tau(x)) \geq 1$. Notice that, in terms of the weight function, the above relation reads

$$\frac{\omega'(x)}{\omega(x)} = \frac{\tau(x) - \sigma'(x)}{\sigma(x)}, \text{ or, equivalently } \upsilon'(x) = -\frac{\tau(x) - \sigma'(x)}{\sigma(x)}.$$

Let consider the linear functional $\mathcal{U}_C$ associated with the measure $d\mu^{[1]}(c, x) = (x-c)d\mu(x)$ (see (2.12)). In order to find the Pearson equation that $\mathcal{U}_C$ satisfies, we analyze two situations:

(i) If $\sigma(c) \neq 0$, then

$$\begin{aligned} D[(x-c)\sigma(x)\mathcal{U}_C] &= D[(x-c)^2\sigma(x)\mathcal{L}] = 2(x-c)\sigma(x)\mathcal{L} + (x-c)^2 D[(\sigma(x)\mathcal{L}] \\ &= 2\sigma(x)\mathcal{U}_C + (x-c)^2\tau(x)\mathcal{L} = \left[2\sigma(x) + (x-c)\tau(x)\right]\mathcal{U}_C. \end{aligned}$$

Thus,

$$D[\phi(x)\mathcal{U}_C] = \psi(x)\mathcal{U}_C,$$

where

$$\begin{cases} \phi(x) = (x-c)\sigma(x) \\ \psi(x) = 2\sigma(x) + (x-c)\tau(x). \end{cases} \tag{3.19}$$

(ii) If $\sigma(c) = 0$, i.e., $\sigma(x) = (x-c)\bar{\sigma}(x)$, then

$$\begin{aligned} D[\sigma(x)\mathcal{U}_C] &= D[(x-c)\bar{\sigma}(x)\mathcal{U}_C] = D[(x-c)^2\bar{\sigma}(x)\mathcal{L}] = D[(x-c)\sigma(x)\mathcal{L}] \\ &= \sigma(x)\mathcal{L} + (x-c)D[\sigma(x)\mathcal{L}] = \sigma(x)\mathcal{L} + (x-c)\tau(x)\mathcal{L} = (\bar{\sigma}(x)+\tau(x))\,\mathcal{U}_C. \end{aligned}$$

In this case,

$$D[\phi(x)\mathcal{U}_C] = \psi(x)\mathcal{U}_C,$$

with

$$\begin{cases} \phi(x) = \sigma(x) \\ \psi(x) = \bar{\sigma}(x) + \tau(x). \end{cases} \tag{3.20}$$

It is well known that the sequence of monic polynomials $\{\widehat{P}_n^{[1]}(x)\}_{n\geq 0}$, orthogonal with respect to $d\mu^{[1]}(x) = (x-c)d\mu$, satisfies the structure relation (see [20] and [82])

$$\phi(x)[\widehat{P}_n^{[1]}]'(x) = A(x,n)\widehat{P}_n^{[1]}(x) + B(x,n)\widehat{P}_{n-1}^{[1]}(x), \tag{3.21}$$

where $A(x,n)$ and $B(x,n)$ are polynomials of a fixed degree, and the three term recurrence relation (see [13])

$$x\,\widehat{P}_n^{[1]}(x) = \widehat{P}_{n+1}^{[1]}(x) + \tilde{\beta}_n\,\widehat{P}_n^{[1]}(x) + \tilde{\gamma}_n\,\widehat{P}_{n-1}^{[1]}(x), \quad n \geq 0, \tag{3.22}$$

with initial conditions $\widehat{P}_0^{[1]}(x) = 1$ and $\widehat{P}_{-1}^{[1]}(x) = 0$, and

$$\tilde{\beta}_n = \beta_{n+1} + \frac{\widehat{P}_{n+2}(c)}{\widehat{P}_{n+1}(c)} - \frac{\widehat{P}_{n+1}(c)}{\widehat{P}_n(c)}, \qquad n \geq 0, \tag{3.23}$$

and

$$\tilde{\gamma}_n = \frac{\widehat{P}_{n+1}(c)\widehat{P}_{n-1}(c)}{[\widehat{P}_n(c)]^2}\gamma_n > 0, \qquad n \geq 1. \tag{3.24}$$

Lemma 3.1 *[45] We have, for $n \geq 2$*

$$A(x,n) + A(x,n-1) + \frac{(x-\tilde{\beta}_{n-1})}{\tilde{\gamma}_{n-1}}B(x,n-1) = \phi'(x) - \psi(x). \tag{3.25}$$

Proof. According to a result by Ismail ([45], (1.12)) which must be adapted to our situation since we use monic polynomials, we get

$$A(x,n) + A(x,n-1) + \frac{(x-\tilde{\beta}_{n-1})}{\tilde{\gamma}_{n-1}}B(x,n-1)$$

$$= -\phi(x)\frac{\left[\omega_{[1]}(x)\right]'}{\omega_{[1]}(x)} - \phi(x)\frac{\psi(x)-\phi'(x)}{\phi(x)}\phi'(x) - \psi(x),$$

where $\omega_{[1]}(x) = (x-c)\omega(x)$. ■

Next, in order to find the holonomic equation that $\widehat{Q}_n$ satisfy (following the Ismail's model). Applying the derivative operator in (3.5) and multiplying it by $\phi(x)$, we obtain

$$\phi(x)[\widehat{Q}_n]'(x) = \phi(x)[\widehat{P}_n^{[1]}]'(x) + c_n\phi(x)[\widehat{P}_{n-1}^{[1]}]'(x). \tag{3.26}$$

Thus, substituting (3.21) in (3.26), yields

$$\begin{aligned}\phi(x)[\widehat{Q}_n]'(x) &= A(x,n)\widehat{P}_n^{[1]}(x) \\ &\quad + \left[B(x,n) + c_nA(x,n-1)\right]\widehat{P}_{n-1}^{[1]}(x) + c_nB(x,n-1)\widehat{P}_{n-2}^{[1]}(x).\end{aligned}$$

Using (3.22) in the above expression, we obtain

$$\phi(x)[\widehat{Q}_n]'(x) = \tilde{A}(x,n)\widehat{P}_n^{[1]}(x) + \tilde{B}(x,n)\widehat{P}_{n-1}^{[1]}(x), \tag{3.27}$$

where

$$\tilde{A}(x,n) = A(x,n) - \frac{c_n}{\tilde{\gamma}_{n-1}}B(x,n-1) \quad \text{and} \tag{3.28}$$

$$\tilde{B}(x,n) = B(x,n) + c_nA(x,n-1) + \frac{c_n}{\tilde{\gamma}_{n-1}}(x-\tilde{\beta}_{n-1})B(x,n-1). \tag{3.29}$$

Therefore, from (3.5) and (3.27), it follows that

$$\begin{bmatrix} 1 & c_n \\ \tilde{A}(x,n) & \tilde{B}(x,n) \end{bmatrix}\begin{bmatrix} \widehat{P}_n^{[1]}(x) \\ \widehat{P}_{n-1}^{[1]}(x) \end{bmatrix} = \begin{bmatrix} \widehat{Q}_n(x) \\ \phi(x)[\widehat{Q}_n]'(x) \end{bmatrix},$$

that is,

$$\widehat{P}_n^{[1]}(x) = \frac{\tilde{B}(x,n)}{\tilde{B}(x,n) - c_n\tilde{A}(x,n)}\widehat{Q}_n(x) - \frac{c_n\phi(x)}{\tilde{B}(x,n) - c_n\tilde{A}(x,n)}[\widehat{Q}_n]'(x)$$

and

$$\widehat{P}_{n-1}^{[1]}(x) = \frac{-\tilde{A}(x,n)}{\tilde{B}(x,n) - c_n\tilde{A}(x,n)}\widehat{Q}_n(x) + \frac{\phi(x)}{\tilde{B}(x,n) - c_n\tilde{A}(x,n)}[\widehat{Q}_n]'(x).$$

Substituting the above two expressions in (3.21), we deduce

$$\phi(x)\left[\frac{\tilde{B}(x,n)\widehat{Q}_n(x)}{\tilde{B}(x,n) - c_n\tilde{A}(x,n)} - \frac{c_n\phi(x)[\widehat{Q}_n]'(x)}{\tilde{B}(x,n) - c_n\tilde{A}(x,n)}\right]'$$

$$= A(x,n)\left(\frac{\tilde{B}(x,n)\widehat{Q}_n(x)}{\tilde{B}(x,n)-c_n\tilde{A}(x,n)} - \frac{c_n\phi(x)[\widehat{Q}_n]'(x)}{\tilde{B}(x,n)-c_n\tilde{A}(x,n)}\right)$$
$$+ B(x,n)\left(\frac{-\tilde{A}(x,n)\widehat{Q}_n(x)}{\tilde{B}(x,n)-c_n\tilde{A}(x,n)} + \frac{\phi(x)[\widehat{Q}_n]'(x)}{\tilde{B}(x,n)-c_n\tilde{A}(x,n)}\right).$$

Then the following statement follows

Theorem 3.7 (Holonomic eq. for one mass point) *The MOPS* $\{\widehat{Q}_n(x)\}_{n\geq 0}$ *satisfies the holonomic equation (second order linear differential equation)*

$$\mathcal{A}(x;n)[\widehat{Q}_n]''(x) + \mathcal{B}(x;n)[\widehat{Q}_n]'(x) + \mathcal{C}(x;n)\widehat{Q}_n(x) = 0, \tag{3.30}$$

where

$$\begin{aligned}
\mathcal{A}(x;n) &= \frac{c_n\,[\phi(x)]^2}{\tilde{B}(x,n)-c_n\tilde{A}(x,n)},\\
\mathcal{B}(x;n) &= \frac{\phi(x)\left[B(x,n)-\tilde{B}(x,n)+c_n(\phi'(x)-A(x,n))\right]}{\tilde{B}(x,n)-c_n\tilde{A}(x,n)}\\
&\qquad -\frac{c_n\phi(x)^2[\tilde{B}(x,n)-c_n\tilde{A}(x,n)]'}{\left(\tilde{B}(x,n)-c_n\tilde{A}(x,n)\right)^2},\\
\mathcal{C}(x;n) &= \frac{A(x,n)\tilde{B}(x,n)-B(x,n)\tilde{A}(x,n)}{\tilde{B}(x,n)-c_n\tilde{A}(x,n)} - \phi(x)\left[\frac{\tilde{B}(x,n)}{\tilde{B}(x,n)-c_n\tilde{A}(x,n)}\right]'.
\end{aligned}$$

If we denote $u(x;n) := \tilde{B}(x,n) - c_n\tilde{A}(x,n)$ and using (3.25), (3.28), and (3.29), then

$$u(x;n) = B(x,n) + c_n\left[-2A(x,n) + \phi'(x) - \psi(x) + \frac{c_n}{\tilde{\gamma}_{n-1}}B(x,n-1)\right]. \tag{3.31}$$

Notice that, according to the fact that the zeros of $\widehat{Q}_n$ are simple, then $u(x_{n,k};n) \neq 0$.

On the other hand, from (3.31) and (3.28), we obtain

$$\tilde{B}(x,n) - B(x,n) + c_nA(x,n) - c_n\phi'(x) = -c_n\psi(x).$$

If we evaluate this second order linear differential equation (3.30) at $x^M_{n,k}$, $k = 1, 2, \ldots n$, then we obtain

$$\mathcal{A}(x^M_{n,k};n)[\widehat{Q}_n]''(x^M_{n,k}) + \mathcal{B}(x^M_{n,k};n)[\widehat{Q}_n]'(x^M_{n,k}) = 0.$$

Hence,

$$\frac{[\widehat{Q}_n]''(x_{n,k}^M)}{[\widehat{Q}_n]'(x_{n,k}^M)} = -\frac{\mathcal{B}(x_{n,k}^M;n)}{\mathcal{A}(x_{n,k}^M;n)}, \quad k=1,2,\dots n. \tag{3.32}$$

Substituting $\mathcal{A}(x_{n,k}^M;n)$ and $\mathcal{B}(x_{n,k}^M;n)$ in the right hand side of (3.32), we get

$$\frac{[\widehat{Q}_n]''(x_{n,k}^M)}{[\widehat{Q}_n]'(x_{n,k}^M)} = \frac{[\tilde{B}(x_{n,k}^M,n) - c_n\tilde{A}(x_{n,k}^M,n)]'}{\tilde{B}(x_{n,k}^M,n) - c_n\tilde{A}(x_{n,k}^M,n)} + \frac{\tilde{B}(x_{n,k}^M,n) - B(x_{n,k}^M,n) + c_nA(x_{n,k}^M,n) - c_n\phi'(x_{n,k}^M)}{c_n\phi(x_{n,k}^M)}.$$

Thus

$$\frac{[\widehat{Q}_n]''(x_{n,k}^M)}{[\widehat{Q}_n]'(x_{n,k}^M)} = [\ln u]'(x_{n,k}^M,n) - \frac{\psi(x_{n,k}^M)}{\phi(x_{n,k}^M)}, \qquad k=1,2,\dots n.$$

We consider two external fields

$$-\int\frac{\psi(x)}{\phi(x)}dx \quad \text{and} \quad \ln u(x;n).$$

In this model, the (double) total external potential $\mathcal{V}(x)$ is given by

$$\mathcal{V}(x) = -\int\frac{\psi(x)}{\phi(x)}dx + \ln u(x;n). \tag{3.33}$$

Let us consider a system of n movable positive unit charges in (c,b) or (a,c), depending on the location of the point c with respect to (a,b), in the presence of the external potential $\mathcal{V}(x)$ given in (3.33). Let $\mathbf{x} := (x_1,\dots,x_n)$, where $x_1,\dots,x_n$ denote the location of the charges. Following Ismail ([46, Ch. 3], [45] and [47]), in this model the potential energy at x of a point charge e located at certain arbitrary point t is $-2e\ln|x-t|$. Thus,

$$\mathcal{E}(\mathbf{x}) = \sum_{k=1}^{n}\mathcal{V}(x_k) - 2\sum_{1\le j<k\le n}\ln|x_j - x_k|$$

can be interpreted as the (double, see [80, Ch. 3]) total energy of the system.

To standardize all the electrostatic interpretations provided in this memoir, we will consider that the total electrostatic energy stored because the position of the charges in the global system is actually given by the half of that considered in Ismail's model. Therefore, the actual total electrostatic potential considered is

$$V(x_k) = \frac{1}{2}\mathcal{V}(x_k)$$

and the total electrostatic energy is

$$E(\mathbf{x}) = \frac{1}{2}\mathcal{E}(\mathbf{x}) = \sum_{k=1}^{n} V(x_k) - \sum_{1\le j<k\le n} \ln|x_j - x_k| . \tag{3.34}$$

In order to find the critical points of $E(\mathbf{x})$ we set

$$-\frac{\partial}{\partial x_j}E(\mathbf{x}) = 0 \Leftrightarrow \frac{\psi(x_j)}{\phi(x_j)} - \frac{u'(x_j;n)}{u(x_j;n)} + 2\sum_{1\le k\le n, k\neq j} \frac{1}{x_j - x_k} = 0, \quad j = 1,\dots,n. \tag{3.35}$$

Let $f(y) := (y - x_1)\cdots(y - x_n)$. Thus,

$$\frac{\psi(x_j)}{\phi(x_j)} - \frac{u'(x_j;n)}{u(x_j;n)} + \frac{f''(x_j)}{f'(x_j)} = 0, \quad j = 1,\dots,n,$$

or, equivalently,

$$f''(y) + \frac{\mathcal{B}(y;n)}{\mathcal{A}(y;n)}f'(y) = 0, \quad y = x_1,\dots,x_n.$$

Therefore

$$f''(y) + \frac{\mathcal{B}(y;n)}{\mathcal{A}(y;n)}f'(y) + \frac{\mathcal{C}(y;n)}{\mathcal{A}(y;n)}f(y) = 0, \quad y = x_1,\dots,x_n. \tag{3.36}$$

On the other hand, from (3.36) we obtain $f(y) = \widehat{Q}_n(y)$, which means that the zeros of $\widehat{Q}_n(x)$ satisfy (3.35).

3.5.2 Example with the zeros of Krall-Laguerre MOPS

We give an electrostatic interpretation for the zeros of Krall-Laguerre polynomials $\widehat{Q}_n^{\alpha}(x)$ orthogonal with respect to the measure (3.18).

We analyze two cases:

1. First, we consider $c = 0$. Thus, the polynomials $\widehat{Q}_n^{\alpha}(x;0)$ are orthogonal with respect to

$$d\mu_M(x;0) = x^{\alpha}e^{-x}dx + M\delta(x), \quad \alpha > -1,$$

The measure

$$d\mu^{[1]}(x) = x^{\alpha+1}e^{-x}dx$$

satisfies a Pearson equation with (see (3.20))

$$\phi(x) = \sigma(x) = x, \quad \psi(x) = \bar{\sigma}(x) + \tau(x) = \alpha + 2 - x.$$

On the other hand, the structure relation (3.21) reads (see [99])

$$\phi(x)[\widehat{L}_n^{\alpha+1}]'(x) = A(x,n)\widehat{L}_n^{\alpha+1}(x) + B(x,n)\widehat{L}_{n-1}^{\alpha+1}(x),$$

where

$$\phi(x) = x, \quad A(x,n) = n, \quad B(x,n) = n+\alpha+1.$$

The coefficients (3.6) and (3.22) are

$$\begin{aligned} \tilde{\gamma}_n &= n(n+\alpha+1), \\ c_n &= -\frac{1+MK_n(0,0)}{1+MK_{n-1}(0,0)}\frac{\widehat{L}_{n-1}^{\alpha}(0)}{\widehat{L}_n^{\alpha}(0)}n(n+\alpha) \\ &= \frac{n!\Gamma(\alpha+1)\Gamma(\alpha+2)+M\Gamma(n+\alpha+2)}{(n-1)!\Gamma(\alpha+1)\Gamma(\alpha+2)+M\Gamma(n+\alpha+1)} \end{aligned}$$

As a conclusion, $u(x;n)$ in (3.31) becomes

$$u\,(x;n) = n(n+\alpha+1) - c_n\,(2n+1+\alpha-c_n) + c_n x$$

with a zero at

$$z_n = (2n+1+\alpha-c_n) - \frac{n(n+\alpha+1)}{c_n}.$$

It is easy to see that $0 < c_n < n+\alpha+1$. Thus, $u(0,n) < 0$ and it implies that $z_n > 0$.

The electrostatic interpretation of the distribution of zeros means that we have an equilibrium position under the presence of a total external field

$$V(x) = \frac{1}{2}\ln u(x;n) - \frac{1}{2}\ln x^{\alpha+2}e^{-x},$$

where the first term represents a short range potential corresponding to a unit charge located at z_n and the second one is a long range potential associated with the weight function (see also [47] and [46]).

Next we show in (3.5.2) the position of the least two zeros of the Krall-Laguerre polynomial $\widehat{Q}_5^{\alpha}(x)$ (at $x_{5,1} = 0.0892553$ and $x_{5,2} = 1.27635$) and the unique zero $z_n = -0.0357143$ of $u\,(x,n)$. Parameters considered were $n = 5$, $\alpha = 0$, and $M = 0.5$.

2. Now, we take $c < 0$. In this case $d\mu^{[1]}(x) = (x-c)x^{\alpha}e^{-x}dx$. Thus, the structure relation (3.21) for $d\mu^{[1]}(x)$ is

$$\phi(x)[\widehat{L}_n^{\alpha,[1]}]'(x) = A(x,n)\widehat{L}_n^{\alpha,[1]}(x) + B(x,n)\widehat{L}_{n-1}^{\alpha,[1]}(x),$$

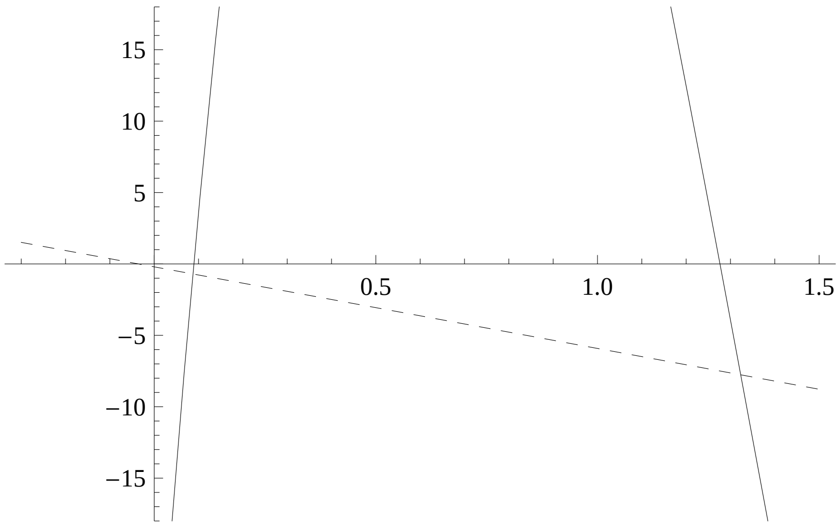


Figure 3.3: The least two zeros of $\widehat{Q}_5^{\alpha}(x)$ and the polynomial $u(x;n)$ (dashed line).

where

$$\begin{aligned}\phi(x) &= (x-c)x,\\ A(x,n) &= n\left[x-(n+1+a_n)\left(1+\frac{n+\alpha}{a_{n-1}}\right)\right],\\ B(x,n) &= \frac{n\,(n+\alpha)}{a_{n-1}}\left[a_n x-(n+1+a_n)\,(n+1+a_n+\alpha)\right].\end{aligned}$$

Proof. From (2.17) we obtain

$$(x-c)\,\widehat{L}_n^{\alpha,[1]}(x)=\widehat{L}_{n+1}^{\alpha}(x)-\frac{\widehat{L}_{n+1}^{\alpha}(c)}{\widehat{L}_n^{\alpha}(c)}\widehat{L}_n^{\alpha}(x).$$

Taking derivatives with respect to x in both hand sides of the above expression, and multiplying the resulting expression by x, we derive

$$x\widehat{L}_n^{\alpha,[1]}(x)+x\,(x-c)\,[\widehat{L}_n^{\alpha,[1]}]'(x)=x[\widehat{L}_{n+1}^{\alpha}]'(x)-\frac{\widehat{L}_{n+1}^{\alpha}(c)}{\widehat{L}_n^{\alpha}(c)}x[\widehat{L}_n^{\alpha}]'(x).$$

Using the structure relation and the three term recurrence relation for Laguerre polynomials we obtain

$$x\widehat{L}_n^{\alpha,[1]}(x)+x\,(x-c)\,[\widehat{L}_n^{\alpha,[1]}]'(x)=$$

$$(n+1)\,\widehat{L}_{n+1}^{\alpha}(x)+(n+1)\,(n+1+\alpha)\,\widehat{L}_{n}^{\alpha}(x)-\frac{\widehat{L}_{n+1}^{\alpha}(c)}{\widehat{L}_{n}^{\alpha}(c)}\left[n\widehat{L}_{n}^{\alpha}(x)+n\,(n+\alpha)\,\widehat{L}_{n-1}^{\alpha}(x)\right],$$

or

$$x\widehat{L}_{n}^{\alpha,[1]}(x)+x\,(x-c)\,[\widehat{L}_{n}^{\alpha,[1]}]'(x)=$$

$$\left[(n+1)\,(x-n)-\frac{n\widehat{L}_{n+1}^{\alpha}(c)}{\widehat{L}_{n}^{\alpha}(c)}\right]\widehat{L}_{n}^{\alpha}(x)-\left(n\,(n+1)\,(n+\alpha)+n\,(n+\alpha)\,\frac{\widehat{L}_{n+1}^{\alpha}(c)}{\widehat{L}_{n}^{\alpha}(c)}\right)\widehat{L}_{n-1}^{\alpha}(x).$$

Let

$$a_n=\frac{\widehat{L}_{n+1}^{\alpha}\,(c)}{\widehat{L}_{n}^{\alpha}\,(c)}$$

and, from (3.8),

$$\widehat{L}_{n}^{\alpha}(x)=\widehat{L}_{n}^{\alpha,[1]}(x)-\frac{\gamma_n}{a_{n-1}}\widehat{L}_{n-1}^{\alpha,[1]}(x),$$

we have

$$x\widehat{L}_{n}^{\alpha,[1]}(x)+x\,(x-c)\,[\widehat{L}_{n}^{\alpha,[1]}]'(x)=$$

$$\begin{aligned}&((x-n)\,(n+1)-na_n)\,\widehat{L}_{n}^{\alpha,[1]}(x)\\&-\left[((n+1)\,(x-n)-na_n)\,\frac{n\,(n+\alpha)}{a_{n-1}}+n\,(n+\alpha)\,(n+1+a_n)\right]\widehat{L}_{n-1}^{\alpha,[1]}(x)\\&\quad+n\,(n-1)\,(n+\alpha)\,(n-1+\alpha)\,(n+1+a_n)\,\frac{\widehat{L}_{n-2}^{\alpha,[1]}(x)}{a_{n-2}}.\end{aligned}$$

Using (3.22) and (3.23)

$$\frac{\tilde{\gamma}_{n-1}}{\gamma_{n-1}}=\frac{\widehat{L}_{n}^{\alpha}(c)}{\widehat{L}_{n-1}^{\alpha}(c)}\frac{\widehat{L}_{n-2}^{\alpha}(c)}{\widehat{L}_{n-1}^{\alpha}(c)}$$

we obtain

$$\frac{a_{n-1}}{a_{n-2}}=\frac{\tilde{\gamma}_{n-1}}{(n-1)\,(n-1+\alpha)}\frac{(n-1)\,(n-1+\alpha)}{a_{n-2}}=\frac{\tilde{\gamma}_{n-1}}{a_{n-1}}$$

and, then,

$$x\widehat{L}_{n}^{\alpha,[1]}(x)+x\,(x-c)\,[\widehat{L}_{n}^{\alpha,[1]}]'(x)=$$

$$\begin{aligned}&\left[((n+1)\,(x-n)-na_n)-\frac{n\,(n+\alpha)\,(n+1+a_n)}{a_{n-1}}\right]\widehat{L}_{n}^{\alpha,[1]}(x)\\&+n\,(n+\alpha)\left[\frac{1}{a_{n-1}}\,(n+1+a_n)\left(x-\tilde{\beta}_{n-1}\right)\right.\end{aligned}$$

$$- \frac{1}{a_{n-1}} \left((n+1)(x-n) - na_n \right) - (n+1+a_n) \Big] \widehat{L}_{n-1}^{\alpha,[1]}(x)$$

Therefore,

$$\phi(x)[\widehat{L}_n^{\alpha,[1]}]'(x) = A(x,n)\widehat{L}_n^{\alpha,[1]}(x) + B(x,n)\widehat{L}_{n-1}^{\alpha,[1]}(x),$$

where

$$\begin{aligned} \phi(x) &= x(x-c), \\ A(x,n) &= ((n+1)(x-n) - na_n) - \frac{n(n+\alpha)(n+1+a_n)}{a_{n-1}} - x \\ B(x,n) &= n(n+\alpha) \left[\frac{1}{a_{n-1}} (n+1+a_n) \left(x - \tilde{\beta}_{n-1} \right) \right. \\ &\quad \left. - \frac{1}{a_{n-1}} ((n+1)(x-n) - na_n) - (n+1+a_n) \right]. \end{aligned}$$

Simplifying these expressions we get

$$\begin{aligned} A(x,n) &= n \left[x - (n+1+a_n) \left(1 + \frac{n+\alpha}{a_{n-1}} \right) \right], \\ B(x,n) &= n(n+\alpha) \left[\frac{a_n}{a_{n-1}} x + \frac{n+1+a_n}{a_{n-1}} \left(n - \tilde{\beta}_{n-1} \right) - (n+1+a_n) \right]. \end{aligned}$$

In the last expression, using again (3.23)

$$\tilde{\beta}_n = \beta_{n+1} + a_{n+1} - a_n = 2n + \alpha + 3 + a_{n+1} - a_n$$

we obtain

$$B(x,n) = \frac{n(n+\alpha)}{a_{n-1}} \left[a_n x - (n+1+a_n)(n+1+a_n+\alpha) \right].$$

■

This is an alternative approach to the method described in [75]. Notice that the Pearson equation for the linear functional associated with the measure $d\mu^{[1]} = (x-c)d\mu$ becomes

$$D[\phi \mathcal{U}_C] = \psi \mathcal{U}_C,$$

where (see (3.19))

$$\begin{aligned} \phi(x) &= (x-c)\sigma(x) = (x-c)x, \\ \psi(x) &= 2\sigma(x) + (x-c)\tau(x) = 2x + (x-c)(\alpha+1-x). \end{aligned}$$

According to (3.23) and (3.24),

$$\tilde{\beta}_{n-1} = \beta_n + a_n - a_{n-1}, \text{ and } \tilde{\gamma}_{n-1} = \frac{a_{n-1}}{a_{n-2}}\gamma_{n-1}.$$

This means that $u(x;n)$ in (3.31) is the following quadratic polynomial

$$u(x,n) = c_n x^2 + r_n x + s_n$$

with

$$\begin{aligned} r_n &= n(n+\alpha)\frac{a_n}{a_{n-1}} + c_n^2 - c_n(c+\alpha+1+2n) \\ &= (c_n+a_n)(c_n-a_n) - (c_n-a_n)c - (c_n+a_n)(2n+\alpha+1) \end{aligned}$$

and

$$\begin{aligned} s_n = {} & (n+1+a_n)\left[(n+1+a_n+\alpha)(2n+1+a_n+\alpha-c-2c_n) + 2cc_n\right] \\ & + c\alpha c_n + c_n^2(a_n - a_{n-1} + 1 - c). \end{aligned}$$

The zeros of this polynomial are

$$\begin{aligned} z_{1,n} &= -\frac{1}{2c_n}\left(r_n + \sqrt{r_n^2 - 4s_n c_n}\right), \\ z_{2,n} &= -\frac{1}{2c_n}\left(r_n - \sqrt{r_n^2 - 4s_n c_n}\right). \end{aligned}$$

Taking into account

$$\frac{\psi(x)}{\phi(x)} = \frac{2}{x-c} + \frac{\alpha+1}{x} - 1,$$

the electrostatic interpretation means the equilibrium position for the zeros under the presence of a total external field

$$V(x) = \frac{1}{2}\ln u(x,n) - \frac{1}{2}\ln(x-c)^2 x^{\alpha+1}e^{-x}, \tag{3.37}$$

where the first one is a short range potential corresponding to two unit charges located at $z_{1,n}$ and $z_{2,n}$ and the second one is a long range potential associated with a polynomial perturbation of the weight function.

In figure 3.4 we show the position of the least two zeros of the Krall-Laguerre polynomial $\widehat{Q}_6^{\alpha}(x)$ (at $x_{6,1} = -0.145632$ and $x_{6,2} = 0.714756$) and the two zeros of $u(x;n)$ (at

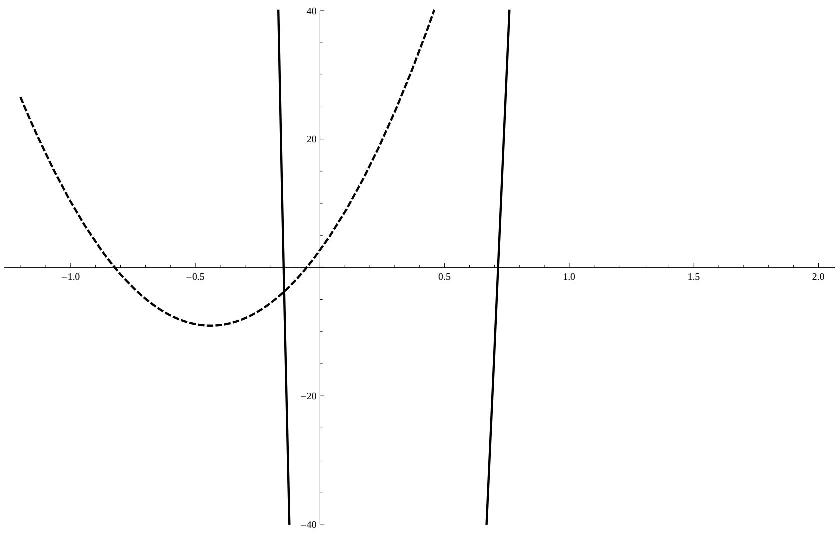


Figure 3.4: The least two zeros of $\widehat{Q}_6^{\alpha}(x)$ and polynomial $u(x;n)$ (dashed line).

$z_{1,6} = -0.82332$ and $z_{2,6} = -0.0530003$). The parameters are considered when $n = 6$, $\alpha = 0$, $c = -1$ and $M = 5.0 \cdot 10^{-3}$.

Next we also show some numerical experiments dealing with the least zero of Krall-Laguerre orthogonal polynomial $\widehat{Q}_n^{\alpha}(x)$. We are interested to analyze when such a zero is negative as well as how fast converges to the point c. To obtain these results we have implemented a Matlab® software using the classical Chebyshev algorithm, since these polynomials satisfy a three term recurrence relation and therefore, this technique can be used.

In the first two tables, for $M = 0$ obviously we recover the least zero and the second zero of the classical Laguerre polynomials (in bold). The following table shows this effect for the first and second zeros of Krall-Laguerre polynomial of degree $n = 15$ and $\alpha = 0$, for some choices of the mass M, when $c = -10$

$x_{15,k}^M(0)$	$M = 0$	$M = 5.0 \cdot 10^{-12}$	$M = 5.0 \cdot 10^{-8}$	$M = 5.0 \cdot 10^{-4}$	$M = 5.0 \cdot 10^{-2}$
$k = 1$	**0.0933078**	−9.9962381	−9.9999996	−10.0	−10.0
$k = 2$	**0.4926917**	0.1117925	0.1117908	0.1117908	0.1117908

as well as when the masspoint is located at $c = -1$

$x^M_{15,k}(0)$	$M = 0$	$M = 5.0 \cdot 10^{-12}$	$M = 5.0 \cdot 10^{-8}$	$M = 5.0 \cdot 10^{-4}$	$M = 5.0 \cdot 10^{-2}$
$k = 1$	**0.0933078**	0.0933077	0.09307224	−0.9279098	−0.9992515
$k = 2$	**0.4926917**	0.4926916	0.4917823	0.1649767	0.1632534

.

In the next table we give the least zero for polynomials of degree $n = 7, \alpha = 0$, as well as we point out the fact that there exists $M = M_0$ such that this zero is negative. In this particular example, with the mass point located at $c = -10$, this value is roughly $1,0 \cdot 10^{-9} < M_0 < 2 \cdot 10^{-9}$

$x^M_{7,k}(0)$	$M = 0$	$M = 1.0 \cdot 10^{-9}$	$M = 2.0 \cdot 10^{-9}$	$M = 5.0 \cdot 10^{-9}$	$M = 5.0 \cdot 10^{-5}$
$k = 1$	**0.193044**	0.048634	−**0.775950**	−3.598918	−9.998880

and with the mass point located at $c = -1$, we need larger values of M_0 to get the least zero as a negative real number. Now the estimate is $1.0 \cdot 10^{-3} < M_0 < 2 \cdot 10^{-3}$.

$x^M_{7,k}(0)$	$M = 0$	$M = 5.0 \cdot 10^{-9}$	$M = 1.0 \cdot 10^{-3}$	$M = 2.0 \cdot 10^{-3}$	$M = 5.0 \cdot 10^{-2}$
$k = 1$	**0.193044**	0.193043	0.059013	−**0.094368**	−0.915571

Another interesting question is to analyze, for a fixed value M, the behavior of zeros of Krall-Laguerre polynomials in terms of the parameter α. Notice that, for a fixed value of α we can lose its negative zero. Again we show the behavior of the first two zeros to give more information about their relative spacing.

For instance, let us show the first two zeros of the Krall-Laguerre polynomials of degree $n = 6$, when $M = 5.0 \cdot 10^{-8}$ and the mass point is located at $c = -10$,

$x^M_{6,k}(\alpha)$	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 3$	$\alpha = 10$
$k = 1$	−3.498898	−1.606946	−0.173020	**1.271640**	**4.890738**
$k = 2$	0.333321	0.592795	1.031807	3.044173	8.143534

and again, the first two zeros when $M = 5.0 \cdot 10^{-3}$ and $c = -1$.

$x^M_{6,k}(\alpha)$	$\alpha = 0$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 3$	$\alpha = 10$
$k = 1$	−0.145632	−0.146124	−0.083058	**0.835712**	**4.890712**
$k = 2$	0.714756	0.982986	1.270833	2.677161	8.143506

Finally, another interesting numerical approach is to consider a different choice of the fixed parameters. For instance, for a fixed M we would find values of the degree n for which

$\widehat{Q}_n^{\alpha}(x)$ has a negative zero. Locating the mass point at $c = -10$, we have

$M = 5.0 \cdot 10^{-8}$	$n \geq n_0$
$\alpha = -0.99$	$n \geq 5$
$\alpha = 0$	$n \geq 6$
$\alpha = 1$	$n \geq 6$
$\alpha = 2.5$	$n \geq 7$
$\alpha = 5$	$n \geq 10$
$\alpha = 8$	$n \geq 14$

$M = 5.0 \cdot 10^{-6}$	$n \geq n_0$
$\alpha = -0.99$	$n \geq 4$
$\alpha = 0$	$n \geq 4$
$\alpha = 1$	$n \geq 5$
$\alpha = 2.5$	$n \geq 5$
$\alpha = 5$	$n \geq 7$
$\alpha = 8$	$n \geq 11$

$M = 5.0 \cdot 10^{-4}$	$n \geq n_0$
$\alpha = -0.99$	$n \geq 3$
$\alpha = 0$	$n \geq 3$
$\alpha = 1$	$n \geq 3$
$\alpha = 2.5$	$n \geq 4$
$\alpha = 5$	$n \geq 5$
$\alpha = 8$	$n \geq 8$

3.5.3 Example with the zeros of Krall-Jacobi MOPS

We give an electrostatic interpretation for the zeros of Krall-Jacobi polynomials $\widehat{Q}_n^{\alpha,\beta}(x;c)$ which are orthogonal with respect to the measure

$$d\mu_M(x;c) = (1-x)^{\alpha}(1+x)^{\beta}dx + M\delta(x-c),$$

with $c \notin (-1,1)$ and $M \geq 0$.

We analyze two cases:

1. First, we consider $c = -1$. Thus, the polynomials $\widehat{Q}_n^{\alpha,\beta}(x;-1)$ are orthogonal with respect to

$$d\mu_M(x;-1) = (1-x)^{\alpha}(1+x)^{\beta}dx + M\delta(x+1).$$

The measure

$$d\mu^{[1]}(x) = (x-(-1))d\mu(x) = (1-x)^{\alpha}(1+x)^{\beta+1}dx$$

satisfies a Pearson equation with (see (3.20))

$$\phi(x) = \sigma(x) = 1-x^2, \quad \psi(x) = \bar{\sigma}(x) + \tau(x) = (\beta-\alpha+1) - (\alpha+\beta+3)\,x.$$

On the other hand, the structure relation (3.21) reads

$$\phi(x)[\widehat{P}_n^{\alpha,\beta+1}]'(x) = A(x,n)\widehat{P}_n^{\alpha,\beta+1}(x) + B(x,n)\widehat{P}_{n-1}^{\alpha,\beta+1}(x),$$

where

$$A(x,n) \quad = \quad \frac{-n[\beta-\alpha+1+(2n+\alpha+\beta+1)x]}{2n+\alpha+\beta+1},$$

$$B(x,n) = \frac{4n(n+\alpha)(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)^2(2n+\alpha+\beta)}.$$

The coefficient $\tilde{\gamma}_n$ in (3.22) when $c=-1$ and $\widehat{P}_n^{\alpha,\beta,[1]}(x) = \widehat{P}_n^{\alpha,\beta+1}(x)$ is

$$\tilde{\gamma}_n = \gamma_n^{\alpha,\beta+1} = \frac{4n(n+\alpha)(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)^2(2n+\alpha+\beta+2)}$$

and

$$c_n = \frac{1+MK_n(-1,-1)}{1+MK_{n-1}(-1,-1)} \frac{2n(n+\alpha)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} > 0,$$

Thus,

$$u(x;n) = B(x,n)$$
$$+c_n\left[(2n+\alpha+\beta)c_n - \frac{(\alpha+\beta+1)(\beta-\alpha+1)}{2n+\alpha+\beta+1}\right] + (2n+\alpha+\beta+1)c_n x.$$

Observe that the zero of $u(x,n)$ belongs to $(-1,1)$. In fact, after some tedious calculations we see that

$$u(1;n) = B(1,n)$$
$$+c_n\left[(2n+\alpha+\beta)c_n + \frac{2(2n(n+\alpha+\beta+1)+\alpha(\alpha+\beta+1))}{2n+\alpha+\beta+1}\right] > 0,$$

and

$$u(-1,n)$$
$$\begin{aligned} &= \frac{-2^{\alpha+\beta+3}(\beta+1)\Gamma(n)\Gamma(n+\alpha)\Gamma(\beta+2)^2\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)M}{2n+\alpha+\beta} \\ &\quad \times \frac{1}{2^{\alpha+\beta+1}\Gamma(n)\Gamma(n+\alpha)\Gamma(\beta+1)\Gamma(\beta+2)+M\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)} \\ &< 0. \end{aligned}$$

Using some known properties of the Jacobi polynomials we conclude that

$$\begin{aligned} z_n &= -1+2M\frac{n(n+\alpha)\left(\widehat{P}_n^{\alpha,\beta}(-1)\right)^2/||\widehat{P}_n^{\alpha,\beta}||_\mu^2}{(2n+\alpha+\beta+1)^2\left(1+MK_n(-1,-1)\right)} \\ &\quad \times\frac{\left(\widehat{P}_n^{\alpha,\beta}(-1)\right)^2/||\widehat{P}_n^{\alpha,\beta}||_\mu^2}{K_{n-1}(-1,-1)\left(1+MK_{n-1}(-1,-1)\right)}. \end{aligned}$$

The electrostatic interpretation means that the equilibrium position for the zeros under the presence of a total external field

$$V(x) = \frac{1}{2}\ln u(x,n) - \frac{1}{2}\ln(1-x)^{\alpha+1}(1+x)^{\beta+2},$$

where the first one is a short range potential corresponding to a unit charge located at the zero of $u(x,n)$ and the other one is a long range potential associated with the weight function.

2. We take $c < -1$. Then,

$$d\mu^{[1]}(x) = (x-c)(1-x)^{\alpha}(1+x)^{\beta}dx$$

and the structure relation *(3.21)* for the above measure is

$$\phi(x)[\widehat{P}_n^{\alpha,\beta,[1]}]'(x) = A(x,n)\widehat{P}_n^{\alpha,\beta,[1]}(x) + B(x,n)\widehat{P}_{n-1}^{\alpha,\beta,[1]}(x),$$

where

$$\begin{aligned} \phi(x) &= (x-c)(1-x^2), \\ A(x,n) &= a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n - (a_{n+1}\gamma_n + \lambda_n b_n)\frac{1}{\lambda_{n-1}} - 1 + x^2, \\ B(x,n) &= (a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n)\frac{\gamma_n}{\lambda_{n-1}} + a_{n+1}\gamma_n + \lambda_n b_n \\ &\quad -(a_{n+1}\gamma_n + \lambda_n b_n)\frac{x-\tilde{\beta}_{n-1}}{\lambda_{n-1}}. \end{aligned}$$

Let

$$\lambda_n = \lambda_n^{\alpha,\beta}(c) = \frac{\widehat{P}_{n+1}^{\alpha,\beta}(c)}{\widehat{P}_n^{\alpha,\beta}(c)}. \tag{3.38}$$

From (2.17) we obtain

$$(x-c)\,\widehat{P}_n^{\alpha,\beta,[1]}(x) = \widehat{P}_{n+1}^{\alpha,\beta}(x) - \lambda_n\widehat{P}_n^{\alpha,\beta}(x).$$

Taking derivatives with respect to x in both hand sides of the above expression, and multiplying them by $(1-x^2)$, we see that

$$\begin{aligned} &(1-x^2)\widehat{P}_n^{\alpha,\beta,[1]}(x) + (x-c)\,(1-x^2)[\widehat{P}_n^{\alpha,\beta,[1]}]'(x) \\ = \;& (1-x^2)[\widehat{P}_{n+1}^{\alpha,\beta}]'(x) - \lambda_n(1-x^2)[\widehat{P}_n^{\alpha,\beta}]'(x). \end{aligned}$$

Since

$$(1-x^2)[\widehat{P}_n^{\alpha,\beta}]'(x) = a_n \widehat{P}_n^{\alpha,\beta}(x) + b_n \widehat{P}_{n-1}^{\alpha,\beta}(x),$$

where

$$\begin{aligned} a_n &= a_n^{\alpha,\beta}(x) = \frac{-n[\beta-\alpha+(2n+\alpha+\beta)x]}{2n+\alpha+\beta}, \\ b_n &= b_n^{\alpha,\beta} = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2(2n+\alpha+\beta-1)}, \end{aligned}$$

we obtain

$$\begin{aligned} & (1-x^2)\widehat{P}_n^{\alpha,\beta,[1]}(x) + (x-c)\,(1-x^2)[\widehat{P}_n^{\alpha,\beta,[1]}]'(x) \\ = \; & a_{n+1}\widehat{P}_{n+1}^{\alpha,\beta}(x) + (b_{n+1}-\lambda_n a_n)\,\widehat{P}_n^{\alpha,\beta}(x) - \lambda_n b_n \widehat{P}_{n-1}^{\alpha,\beta}(x). \end{aligned}$$

The three term recurrence relation of monic Jacobi polynomials yields

$$\begin{aligned} & (1-x^2)\widehat{P}_n^{\alpha,\beta,[1]}(x) + (x-c)\,(1-x^2)[\widehat{P}_n^{\alpha,\beta,[1]}]'(x) \\ = \; & [a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n]\,\widehat{P}_n^{\alpha,\beta}(x) - (a_{n+1}\gamma_n + \lambda_n b_n)\widehat{P}_{n-1}^{\alpha,\beta}(x). \end{aligned}$$

From (3.8) and (3.38),

$$\widehat{P}_n^{\alpha,\beta}(x) = \widehat{P}_n^{\alpha,\beta,[1]}(x) - \frac{\gamma_n}{\lambda_{n-1}}\widehat{P}_{n-1}^{\alpha,\beta,[1]}(x).$$

Then

$$\begin{aligned} & (1-x^2)\widehat{P}_n^{\alpha,\beta,[1]}(x) + (x-c)\,(1-x^2)[\widehat{P}_n^{\alpha,\beta,[1]}]'(x) \\ = \; & [a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n]\,\widehat{P}_n^{\alpha,\beta,[1]}(x) \\ & - \left[(a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n)\,\frac{\gamma_n}{\lambda_{n-1}} + a_{n+1}\gamma_n + \lambda_n b_n\right]\widehat{P}_{n-1}^{\alpha,\beta,[1]}(x) \\ & \quad (a_{n+1}\gamma_n + \lambda_n b_n)\frac{\gamma_{n-1}}{\lambda_{n-2}}\widehat{P}_{n-2}^{\alpha,\beta,[1]}(x). \end{aligned}$$

From (3.22) for monic kernels,

$$\begin{aligned} & (1-x^2)\widehat{P}_n^{\alpha,\beta,[1]}(x) + (x-c)\,(1-x^2)[\widehat{P}_n^{\alpha,\beta,[1]}]'(x) \\ = \; & [a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n]\,\widehat{P}_n^{\alpha,\beta,[1]}(x) \\ & - \left[(a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n)\,\frac{\gamma_n}{\lambda_{n-1}} + a_{n+1}\gamma_n + \lambda_n b_n\right]\widehat{P}_{n-1}^{\alpha,\beta,[1]}(x) \end{aligned}$$

$$(a_{n+1}\gamma_n + \lambda_n b_n)\frac{\gamma_{n-1}}{\lambda_{n-2}}\frac{(x-\tilde{\beta}_{n-1})\widehat{P}_{n-1}^{\alpha,\beta,[1]}(x) - \widehat{P}_n^{\alpha,\beta,[1]}(x)}{\tilde{\gamma}_{n-1}}.$$

According to (3.23) and (3.38), we obtain

$$\frac{\tilde{\gamma}_{n-1}}{\gamma_{n-1}} = \frac{\lambda_{n-1}}{\lambda_{n-2}}.$$

Therefore

$$\begin{aligned} &(1-x^2)\widehat{P}_n^{\alpha,\beta,[1]}(x) + (x-c)\,(1-x^2)[\widehat{P}_n^{\alpha,\beta,[1]}]'(x) \\ = \;& \left[a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n - (a_{n+1}\gamma_n + \lambda_n b_n)\frac{1}{\lambda_{n-1}}\right]\widehat{P}_n^{\alpha,\beta,[1]}(x) \\ & - \left[(a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n)\,\frac{\gamma_n}{\lambda_{n-1}} + a_{n+1}\gamma_n + \lambda_n b_n \right. \\ & \left. \quad -(a_{n+1}\gamma_n + \lambda_n b_n)\frac{x-\tilde{\beta}_{n-1}}{\lambda_{n-1}}\right]\widehat{P}_{n-1}^{\alpha,\beta,[1]}(x). \end{aligned}$$

Thus

$$\phi(x)[\widehat{P}_n^{\alpha,\beta,[1]}]'(x) = A(x,n)\widehat{P}_n^{\alpha,\beta,[1]}(x) + B(x,n)\widehat{P}_{n-1}^{\alpha,\beta,[1]}(x),$$

where

$$\begin{aligned} \phi(x) &= (x-c)(1-x^2), \\ A(x,n) &= a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n - (a_{n+1}\gamma_n + \lambda_n b_n)\frac{1}{\lambda_{n-1}} - 1 + x^2, \\ B(x,n) &= (a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n)\,\frac{\gamma_n}{\lambda_{n-1}} + a_{n+1}\gamma_n + \lambda_n b_n \\ & \quad -(a_{n+1}\gamma_n + \lambda_n b_n)\frac{x-\tilde{\beta}_{n-1}}{\lambda_{n-1}}. \end{aligned}$$

Simplifying these expressions we have

$$\begin{aligned} A(x,n) &= A_{n,0} + A_{n,1}x + A_{n,2}x^2, \\ B(x,n) &= B_{n,0} + B_{n,1}x + B_{n,2}x^2. \end{aligned}$$

Notice that the Pearson equation for the linear functional associated with the measure

$$d\mu^{[1]}(x) = (x-c)(1-x)^{\alpha}(1+x)^{\beta}dx$$

becomes

$$D[\phi \mathfrak{U}_C] = \psi \mathfrak{U}_C,$$

with (see (3.19))

$$\begin{aligned}
\phi(x) &= (x-c)\sigma(x) = (x-c)(1-x^2), \\
\psi(x) &= 2\sigma(x) + (x-c)\tau(x) = 2(1-x^2) + (x-c)(\beta-\alpha-(\alpha+\beta+2)x),
\end{aligned}$$

which means that $u\,(x;n)$ is the following quadratic polynomial

$$\begin{aligned}
u\,(x;n) &= B(x,n) + c_n\left[-2A(x,n) + \phi'(x) - \psi(x) + \frac{c_n}{\tilde{\gamma}_{n-1}}B(x,n-1)\right] \\
&= \left[B_{n,2} + \left(\alpha+\beta+1-2A_{n,2}+\frac{c_n B_{n-1,2}}{\tilde{\gamma}_{n-1}}\right)c_n\right]x^2 \\
&\quad + \left\{\frac{c_n^2 B_{n-1,1}}{\tilde{\gamma}_{n-1}} + B_{n,1} - [\alpha(c-1)+\beta(c+1)+2A_{n,1}]c_n\right\}x \\
&\qquad + B_{n,0} - [2A_{n,0}+1+c(\alpha-\beta)]c_n + \frac{c_n^2 B_{n-1,0}}{\tilde{\gamma}_{n-1}}.
\end{aligned}$$

Taking into account

$$\frac{\psi(x)}{\phi(x)} = \frac{2}{x-c} - \frac{\alpha+1}{1-x} + \frac{\beta+1}{1+x},$$

the electrostatic interpretation means that the equilibrium position for the zeros under the presence of a total external field

$$V(x) = \frac{1}{2}\ln u(x;n) - \frac{1}{2}\ln{(x-c)^2}\,(1-x)^{\alpha+1}(1+x)^{\beta+1},$$

where the first one is a short range potential corresponding to two unit charges located at the zeros of $u(x;n)$ and the second one is a long range potential associated with a polynomial perturbation of the weight function.

3.6 General electrostatic model for Krall-Laguerre OPS

In this section, we apply again the Stieltjes' ideas (see [39] and [101]) to study the electrostatic interpretation of zeros of perturbed MOPS from an alternative point of view to that followed in the preceding sections. We consider only the case of orthogonality with respect to a standard Laguerre weight, perturbed with a finite number m of positive mass points on the negative real axis. Note that this is a generalization of the problem discussed in the preceding sections, but taking into account only a canonical example of a measure with an unbounded support.

Next, we provide the holonomic equation that the family $\{\widehat{Q}_n^{\alpha,m}\}_{n\geq 0}$ satisfy. This differential equation will be useful to get a new electrostatic model in this Section. We begin by proving a lemma, concerning two connection formulas that will be needed later.

Lemma 3.2 *For the sequences of polynomials* $\{\widehat{Q}_n^{\alpha,m}\}_{n\geq 0}$ *and* $\{\widehat{L}_n^{\alpha}\}_{n\geq 0}$ *we get*

$$\begin{aligned} R_m(x)\widehat{Q}_n^{\alpha,m}(x) &= A_1(x;n)\widehat{L}_n^{\alpha}(x) + B_1(x;n)\widehat{L}_{n-1}^{\alpha}(x), && (3.39)\\ x[R_m(x)\widehat{Q}_n^{\alpha,m}(x)]' &= C_1(x;n)\widehat{L}_n^{\alpha}(x) + D_1(x;n)\widehat{L}_{n-1}^{\alpha}(x), && (3.40)\end{aligned}$$

where

$$\begin{aligned} A_1(x;n) &= R_m(x) - \sum_{j=1}^{m}\left(\frac{M_j\,\widehat{L}_{n-1}^{\alpha}(c_j)\,\widehat{Q}_n^{\alpha,m}(c_j)}{(n-1)!\Gamma(n+\alpha)}\right)R_{m,j}(x),\\ B_1(x;n) &= \sum_{j=1}^{m}\left(\frac{M_j\,\widehat{L}_n(c_j)\,\widehat{Q}_n^{\alpha,m}(c_j)}{(n-1)!\Gamma(n+\alpha)}\right)R_{m,j}(x), && (3.41)\\ R_{m,k}(x) &= \prod_{\substack{j=1\\ j\neq k}}^{m}(x-c_j)\,,\\ C_1(x;n) &= nA_1(x;n) - B_1(x;n) + xA_1'(x;n),\\ D_1(x;n) &= n(n+\alpha)A_1(x;n) + (x-(n+\alpha))B_1(x;n) + xB_1'(x;n).\end{aligned}$$

Proof. Since $K_{n-1}(x,y)$ is a polynomial of degree $n-1$ in the variable y, we have

$$\begin{aligned} &\langle K_{n-1}(x,y), \widehat{Q}_n^{\alpha,m}(y)\rangle_m = 0,\\ &\langle K_{n-1}(x,y), \widehat{Q}_n^{\alpha,m}(y)\rangle_\alpha = -\sum_{j=1}^{m} M_j\,K_{n-1}(x,c_j)\,\widehat{Q}_n^{\alpha,m}(c_j). \end{aligned} \qquad (3.42)$$

Using in (3.42) the Christoffel-Darboux formula, we have

$$\begin{aligned} \langle K_{n-1}(x,y), \widehat{Q}_n^{\alpha,m}(y)\rangle_\alpha &= -\left(\sum_{j=1}^{m}\frac{M_j\,\widehat{L}_{n-1}^{\alpha}(c_j)\,\widehat{Q}_n^{\alpha,m}(c_j)}{(n-1)!\Gamma(n+\alpha)(x-c_j)}\right)\widehat{L}_n^{\alpha}(x)\\ &\quad -\left(\sum_{j=1}^{m}\frac{M_j\,\widehat{L}_n^{\alpha}(c_j)\,\widehat{Q}_n^{\alpha,m}(c_j)}{(n-1)!\Gamma(n+\alpha)(x-c_j)}\right)\widehat{L}_{n-1}^{\alpha}(x). && (3.43)\end{aligned}$$

Replacing (3.43) in (3.42) and multiplying by $R_m(x)$, we deduce (3.39) for $x \in \mathbb{C} \setminus \{\mathbb{R}_+ \cup \{c_1, \dots, c_m\}\}$. To prove (3.40), we can take derivatives in both hand sides of (3.39)

$$\begin{aligned} [R_m(x)\widehat{Q}_n(x)]' &= A_1'(x;n)\widehat{L}_n^\alpha(x) + A_1(x;n)[\widehat{L}_n^\alpha]'(x) + \\ &\quad + B_1'(x;n)\widehat{L}_{n-1}^\alpha(x) + B_1(x;n)\,[\widehat{L}_{n-1}^\alpha]'(x). \end{aligned} \tag{3.44}$$

Now, multiplying (3.44) by x and using (2.27)–(2.28), we obtain (3.40). ■

Lemma 3.3 *The sequences of monic orthogonal polynomials $\{\widehat{Q}_n^{\alpha,m}\}_{n\geq 0}$ and $\{\widehat{L}_n^\alpha\}_{n\geq 0}$ are also related by*

$$R_m(x)\widehat{Q}_{n-1}^{\alpha,m}(x) = A_2(x;n)\widehat{L}_n^\alpha(x) + B_2(x;n)\widehat{L}_{n-1}^\alpha(x), \tag{3.45}$$

$$x[R_m(x)\widehat{Q}_{n-1}^{\alpha,m}(x)]' = C_2(x;n)\widehat{L}_n^\alpha(x) + D_2(x;n)\widehat{L}_{n-1}^\alpha(x), \tag{3.46}$$

where

$$\begin{aligned} A_2(x;n) &= \frac{-1}{(n-1+\alpha)(n-1)}B_1(x;n-1), \\ B_2(x;n) &= A_1(x;n-1) + \frac{(x+1-2n-\alpha)}{(n-1+\alpha)(n-1)}B_1(x;n-1), \\ C_2(x;n) &= \frac{-1}{(n-1+\alpha)(n-1)}D_1(x;n-1), \\ D_2(x;n) &= C_1(x;n-1) + \frac{(x+1-2n-\alpha)}{(n-1+\alpha)(n-1)}D_1(x;n-1). \end{aligned} \tag{3.47}$$

Proof. The proof of (3.45)-(3.46) is a straightforward consequence of (3.39)–(3.41) and the three term recurrence relation (2.24) for monic Laguerre polynomials. ■

The following lemma shows the converse relation of (3.39)–(3.45) for the polynomials $\widehat{L}_n^\alpha(x)$ and $\widehat{L}_{n-1}^\alpha(x)$

Lemma 3.4

$$\widehat{L}_n^\alpha(x) = \frac{R_m(x)}{\Delta(x;n)}\left(B_2(x;n)\widehat{Q}_n^{\alpha,m}(x) - B_1(x;n)\widehat{Q}_{n-1}^{\alpha,m}(x)\right), \tag{3.48}$$

$$\widehat{L}_{n-1}^\alpha(x) = \frac{R_m(x)}{\Delta(x;n)}\left(-A_2(x;n)\widehat{Q}_n^{\alpha,m}(x) + A_1(x;n)\widehat{Q}_{n-1}^{\alpha,m}(x)\right). \tag{3.49}$$

where

$$\Delta(x;n) = A_1(x;n)B_2(x;n) - B_1(x;n)A_2(x;n), \quad \deg \Delta(x;n) = 2m.$$

Proof. Note that (3.39)–(3.45) is a system of two linear equations with two unknowns $\widehat{L}_n^{\alpha}(x)$ and $\widehat{L}_{n-1}^{\alpha}(x)$ and from the Cramer's rule the lemma follows. ■

Lemma 3.5

$$\begin{aligned} G(x;n)\widehat{Q}_n^{\alpha,m}(x) + F(x;n)[\widehat{Q}_n^{\alpha,m}]'(x) &= H(x;n)\widehat{Q}_{n-1}^{\alpha,m}(x), \quad (3.50)\\ J(x;n)\widehat{Q}_{n-1}^{\alpha,m}(x) + F(x;n)[\widehat{Q}_{n-1}^{\alpha,m}]'(x) &= K(x;n)\widehat{Q}_n^{\alpha,m}(x), \quad (3.51) \end{aligned}$$

where

$$\begin{aligned} F(x;n) &= x\Delta(x;n)R_m(x),\\ G(x;n) &= x\Delta(x;n)R'_m(x) + R_m(x)\left[D_1(x;n)A_2(x;n)\right.\\ &\quad \left. -C_1(x;n)B_2(x;n)\right],\\ H(x;n) &= R_m(x)[D_1(x;n)A_1(x;n) - C_1(x;n)B_1(x;n)], \quad (3.52)\\ J(x;n) &= x\Delta(x;n)R'_m(x) + R_m(x)\left[C_2(x;n)B_1(x;n)\right.\\ &\quad \left. -D_2(x;n)A_1(x;n)\right],\\ K(x;n) &= R_m(x)[C_2(x;n)B_2(x;n) - D_2(x;n)A_2(x;n)]. \end{aligned}$$

Proof. Replacing (3.48)–(3.49) in (3.40) and (3.46), (3.50) and (3.51), holds. ■

From (3.50)

$$\widehat{Q}_{n-1}^{\alpha,m}(x) = \frac{1}{H(x;n)}(G(x;n)\widehat{Q}_n^{\alpha,m}(x) + F(x;n)[\widehat{Q}_n^{\alpha,m}]'(x)),$$

and replacing this polynomial in (3.51), after some cumbersome computations, we obtain

Theorem 3.8 (Holonomic Equation) *The $n-th$ monic orthogonal polynomial with respect to the inner product (3.2) is a polynomial solution of the second order linear differential equation with rational functions as coefficients*

$$[\widehat{Q}_n^{\alpha,m}]''(x) + \mathcal{R}_1(x;n)[\widehat{Q}_n^{\alpha,m}]'(x) + \mathcal{R}_0(x;n)\widehat{Q}_n^{\alpha,m}(x) = 0, \quad (3.53)$$

where

$$\begin{aligned} \mathcal{R}_1(x;n) &= -\frac{u'_{2m}(x;n)}{u_{2m}(x;n)} + 2\frac{R'_m(x)}{R_m(x)} + \frac{\alpha+1}{x} - 1,\\ \mathcal{R}_0(x;n) &= \frac{H(x;n)G'(x;n) - G(x;n)H'(x;n)}{H(x;n)F(x;n)} \end{aligned}$$

$$
\begin{aligned}
& +\frac{J(x;n)G(x;n) - K(x;n)H(x;n)}{F^2(x;n)}, \\
u_{2m}(x;n) &= D_1(x;n)A_1(x;n) - C_1(x;n)B_1(x;n). \qquad (3.54)
\end{aligned}
$$

Note that $u_{2m}(x;n)$ is a polynomial of degree $2m$.

Remark 3.1 *Notice that the polynomial $u_{2m}(x;n)$ plays the same role that the polynomial $u(x;n)$ (see (3.31)), In the Ismail's model for just one mass point. Notice that $\deg u(x;n) = 1$ if $c = 0$ and $\deg u(x;n) = 2$ if $c < 0$.*

Next, we apply again the Stieltjes' ideas (see [39] and [101]) to study the electrostatic interpretation of zeros of perturbed MOPS from an alternative point of view to that followed in the preceding sections, for the specific case of the Laguerre measure. This result generalizes that of (3.37) for an iterated Uvarov perturbation on the Laguerre measure $d\mu_\alpha(x) = x^\alpha e^{-x}dx$, $\alpha > -1$.

Let $\widehat{Q}_n^{\alpha,m}(x)$ be the n-th monic orthogonal polynomial with respect to the inner product (3.2), i.e.

$$
\langle \widehat{Q}_n^{\alpha,m}(x), x^k \rangle_m =
$$

$$
\int_0^{+\infty} \widehat{Q}_n^{\alpha,m}(x)\, x^k \, d\mu_\alpha(x) + \sum_{j=1}^{m} M_j \, \widehat{Q}_n^{\alpha,m}(c_j)\, c_j^k = 0, \quad k = 0, 1, 2, \ldots, n-1. \qquad (3.55)
$$

If

$$
R_m(z) = \prod_{j=1}^{m} (z - c_j),
$$

then is straightforward to see that $\widehat{Q}_n^{\alpha,m}(x)$ is quasi-orthogonal of order m (see [11, Definition 1]) with respect to $R_m(x)d\mu_\alpha(x)$, i.e.

$$
\int_0^{+\infty} x^k \widehat{Q}_n^{\alpha,m}(x) R_m(x) d\mu_\alpha(x) = 0, \quad k = 0, 1, 2, \ldots, n-m-1. \qquad (3.56)
$$

As a well known consequence (see [99, §3.3]), the polynomial $\widehat{Q}_n^{\alpha,m}(x)$ has at least $n-m$ changes of sign on $[0, +\infty)$. Hence, $\widehat{Q}_n^{\alpha,m}(x)$ has at least $n-m$ zeros of odd multiplicity on $[0, +\infty)$.

Furthermore, there is at most one zero of $\widehat{Q}_n$ in each gap between c_k's, assuming $c_0 = 0$. This can be proved by contradiction. Suppose that the polynomial $\widehat{Q}_n(x)$, orthogonal with respect to the inner product (3.2), has two simple zeros x_1 and x_2 both inside the interval (c_k, c_{k-1}). We can write $\widehat{Q}_n(x)$ in the form

$$\widehat{Q}_n(x) = (x - x_1)(x - x_2)q_{n-2}(x), \tag{3.57}$$

where $q_{n-2}(x)$ is certain polynomial of degree $n-2$. Obviously, if $d\mu_m(x) = \chi_{\mathbb{R}_+} x^\alpha e^{-x} dx + \sum_{j=1}^m M_j \delta(x - c_j)$, then

$$I_n = \int_{\mathbb{R}} \widehat{Q}_n(x) q_{n-2}(x) d\mu_m(x) = 0 \tag{3.58}$$

because the orthogonality of $\widehat{Q}_n(x)$ to polynomials of lower degree.

On the other hand, $(x - x_1)(x - x_2) > 0$ whenever $x \notin (c_k, c_{k-1})$, hence from (3.57)

$$\begin{aligned} I_n &= \int_{\mathbb{R}} (x - x_1)(x - x_2) q_{n-2}^2(x) d\mu_m(x) \\ &= \int_{\mathbb{R} \setminus (c_k, c_{k-1})} (x - x_1)(x - x_2) q_{n-2}^2(x) d\mu_m(x) > 0, \end{aligned}$$

contrary to (3.58). This implies that $\widehat{Q}_n(x)$ cannot have two zeros in (c_k, c_{k-1}).

From the Fourier expansion of the polynomials $\{\widehat{Q}_n^{\alpha,m}(x)\}_{n\geq 0}$ in terms of the monic polynomials $\{\widehat{L}_n^\alpha(x)\}_{n\geq 0}$ and the definition (2.3) of kernel polynomial associated with Laguerre polynomials, it is straightforward to prove that $\{\widehat{Q}_n^{\alpha,m}\}_{n\geq 0}$ and $\{\widehat{L}_n^\alpha\}_{n\geq 0}$ are related by

$$\widehat{Q}_n^{\alpha,m}(x) = \widehat{L}_n^\alpha(x) - \sum_{j=1}^m M_j \widehat{Q}_n^{\alpha,m}(c_j) K_{n-1}(x, c_j). \tag{3.59}$$

Evaluating (3.59) in $x = c_k$, with $k = 1, 2, \ldots, m$, we obtain the following system of m linear equations $(1 \leq k \leq m)$ with m unknowns $\widehat{Q}_n^{\alpha,m}(c_j)$ $(1 \leq j \leq m)$

$$\widehat{L}_n^\alpha(c_k) = (1 + a_k K_{n-1}(c_k, c_k))\, \widehat{Q}_n^{\alpha,m}(c_k) + \sum_{\substack{j=1 \\ j\neq k}}^m M_j K_{n-1}(c_j, c_k) \widehat{Q}_n^{\alpha,m}(c_j).$$

Next, we present the electrostatic interpretation of the distribution of the zeros of $\{\widehat{Q}_n^{\alpha,m}(x)\}_{n\geq 0}$ as the logarithmic potential interaction of positive unit charges in the presence of an external field, for $\{c_1, c_2, \ldots, c_m\} \notin [0, +\infty)$. We use the fact that this family of monic polynomials satisfies the second order linear differential equation (3.53). Notice that the zeros of $\widehat{Q}_n^{\alpha,m}(x)$ are real, simple and belong to the interior of the convex hull of

$\mathbb{R}_+ \cup \{c_1, c_2, \dots, c_m\}$, because $d\mu_m$ is a positive Borel measure. Now we evaluate (3.53) at $x^m_{n,k}$, where $\left\{x^m_{n,k}\right\}_{k=1}^{n}$ are the zeros of $\widehat{Q}_n^{\alpha,m}(x)$ arranged in an increasing order, yielding

$$\frac{[\widehat{Q}_n^{\alpha,m}]''(x^m_{n,k})}{[\widehat{Q}_n^{\alpha,m}]'(x^m_{n,k})} = -\mathcal{R}_1(x^m_{n,k}; n).$$

Using the explicit expressions $\mathcal{R}_1(x^m_{n,k}; n)$ we get, for $1 \leq k \leq n$,

$$\frac{[\widehat{Q}_n^{\alpha,m}]''(x^m_{n,k})}{[\widehat{Q}_n^{\alpha,m}]'(x^m_{n,k})} = \frac{u'_{2m}(x^m_{n,k}; n)}{u_{2m}(x^m_{n,k}; n)} - 2\frac{R'_m(x^m_{n,k})}{R_m(x^m_{n,k})} - \frac{\alpha+1}{x^m_{n,k}} + 1. \tag{3.60}$$

Taking into account the fact that the zeros of $\widehat{Q}_n^{\alpha,m}(x)$ are simple, then

$$[\widehat{Q}_n^{\alpha,m}]'(x) = \sum_{i=1}^{n} \prod_{\substack{j=1,\\ j\neq i}}^{n} (x - x^m_{n,j}), \qquad [\widehat{Q}_n^{\alpha,m}]''(x) = \sum_{i=1}^{n} \sum_{\substack{j=1,\\ j\neq i}}^{n} \prod_{\substack{l=1,\\ i\neq l\neq j}}^{n} (x - x^m_{n,l}),$$

$$[\widehat{Q}_n^{\alpha,m}]'(x^m_{n,k}) = \prod_{\substack{j=1,\\ j\neq k}}^{n} (x^m_{n,k} - x^m_{n,j}), \qquad [\widehat{Q}_n^{\alpha,m}]''(x^m_{n,k}) = 2\sum_{\substack{i=1,\\ i\neq k}}^{n} \prod_{\substack{j=1,\\ i\neq j\neq k}}^{n} (x^m_{n,k} - x^m_{n,j}).$$

Consequently, (3.60) reads as the "electrostatic equilibrium condition" (see [39] and [47] for other examples). Indeed, with $1 \leq k \leq n$,

$$\sum_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{x^m_{n,j} - x^m_{n,k}} + \frac{1}{2}\frac{u'_{2m}(x^m_{n,k}; n)}{u_{2m}(x^m_{n,k}; n)} - \frac{R'_m(x^m_{n,k})}{R_m(x^m_{n,k})} - \frac{\alpha+1}{2x^m_{n,k}} + \frac{1}{2} = 0. \tag{3.61}$$

We should notice that according to the Lemma 3.2 and the fact that the zeros of $\widehat{Q}_n^{\alpha,m}(x)$ are simple, then $u_{2m}(x^m_{n,k}; n) \neq 0$. The above equation means that have an unstable (at least in principle) equilibrium on the zeros of the family $\{\widehat{Q}_n^{\alpha,m}(x)\}_{n\geq 1}$, that is, the gradient of the total energy is zero at the points $\{x^m_{n,k}\}_{1\leq k\leq n}$. In other words, the energy functional has a critical point (either a relative extreme or a saddle point) therein.

We now consider n unit positive charges located in the real line, with a logarithmic interaction under an external field $V(x)$. For $x \in R \setminus \{c_j\}$ the total potential is

$$V(x) = \frac{1}{2}\ln u_{2m}(x; n) - \ln R_m(x) - \frac{\alpha+1}{2}\ln x + \frac{1}{2}x. \tag{3.62}$$

The term $-\ln R_m(x)$ is the potential field due to the mass points of our measure. Thus, we have in (3.62)

$$V(x) = \frac{1}{2}\ln u_{2m}(x;n) - \frac{1}{2}\ln\left(R_m^2(x)x^{\alpha+1}e^{-x}\right).$$

Following [45], the term

$$\upsilon_{long}(x) = \frac{-1}{2}\ln\left(R_m^2(x)x^{\alpha+1}e^{-x}\right),$$

is said to be a long range potential, which is associated with a polynomial perturbation of the Laguerre weight function. Similarly,

$$\upsilon_{short}(x) = \frac{1}{2}\ln u_{2m}(x;n)$$

represents a short range potential (or varying external potential) corresponding to $2m$ unit charges located at the zeros of $u_{2m}(x;n)$.

Next we give some numerical experiments using Mathematica® software, dealing with the least zeros of Krall-Laguerre polynomials. We are interested to show the location of their zeros outside the interval $[0,+\infty)$ and the position of the source-charges of the short range potential $\upsilon_{short}(x)$, which are the roots of the polynomial $u_4(x;n)$. In these experiments we consider in the inner product (3.2) two fixed mass points (that is, $m=2$) at points $c_1=-1$ and $c_1=-2$. The parameter $\alpha=0$ and the masses are always $a_1=a_2=1$. Notice that in the examples shown, the zeros of the Krall-Laguerre polynomials never match with the zeros of u_4 given in (3.54), i.e. the polynomial u_4 never vanishes at the zeros of any Krall-Laguerre polynomial. The negative zeros appear in bold.

Next, we show the position of the zeros of the Krall-Laguerre polynomial of degree $n=4$ and the four real zeros of the polynomial $u_4(x;n)$. Notice that the polynomial $u_4(x;n)$ have four negative real roots, but there is only one zero of the Krall-Laguerre polynomial on $\mathbb{R}_-$.

zero	*1st*	*2nd*	*3rd*	*4th*
$\widehat{Q}_4(x)$	$\mathbf{-1.84565}$	0.0122706	2.65152	7.49184
$u_4(x;4)$	$\mathbf{-1.93302}$	$\mathbf{-1.48646}$	$\mathbf{-0.60338}$	$\mathbf{-0.000119291}$

As n increases, the situation changes as expected according to the Hurwitz's Theorem, and the mass points attract exactly one zero of the Krall-Laguerre polynomial in each gap between them

zero	*1st*	*2nd*	*3rd*	*4th*	*5th*
$\widehat{Q}_5(x)$	$\mathbf{-1.9219}$	$\mathbf{-0.439622}$	1.73422	5.20588	10.7544
$u_4(x;5)$	$\mathbf{-1.96394}$	$\mathbf{-1.56249}$	$\mathbf{-0.767607}$	$\mathbf{-0.11943}$	–

Next two tables show the behavior of the zeros of Krall-Laguerre polynomials and u_4 for degrees $n = 6$ and $n = 10$ respectively. Notice that the two negative zeros of $\widehat{Q}_n(x)$ and the four zeros of u_4 become more negative approaching to the position of the mass points

zero	*1st*	*2nd*	*3rd*	*4th*	*5th*	*6th*
$\widehat{Q}_6(x)$	$\mathbf{-1.96485}$	$\mathbf{-0.711952}$	1.23489	3.98228	8.03313	14.1729
$u_4(x;6)$	$\mathbf{-1.9831}$	$\mathbf{-1.61526}$	$\mathbf{-0.871511}$	$\mathbf{-0.275212}$	–	–

zero	*1st*	*2nd*	*3rd*	*4th*	*5th*	*6th*
$\widehat{Q}_{10}(x)$	$\mathbf{-1.99898}$	$\mathbf{-0.979076}$	0.515223	2.00183	4.11731	6.87812
$u_4(x;10)$	$\mathbf{-1.99949}$	$\mathbf{-1.69674}$	$\mathbf{-0.989683}$	$\mathbf{-0.54116}$	–	–

In Figure 3.5 we present an example of the electrostatic behavior of the zeros of $\widehat{Q}_3^{\alpha,m}(x)$, the perturbed Laguerre polynomial of degree $n = 3$ with parameter $\alpha = 0$. The dotted line shows the monic standard Laguerre polynomial of degree $n = 3$ with parameter $\alpha = 0$, $\widehat{L}_3^{\alpha}(x)$ and the thick black line shows the graph of $\widehat{Q}_3^{\alpha,m}(x)$ in the particular case of two mass points a_1 and a_2 located at $x = c_1$ and $x = c_2$, respectively. The thick blue line shows the total potential $V(x)$ that rules the behavior of the zeros of $\widehat{Q}_3^{\alpha,m}(x)$, and the green and orange thin lines show the short range $v_{short}(x)$ and long range $v_{long}(x)$ potentials, respectively. Notice that $v_{short}(x)$ has its source points in the four zeros of the polynomial $u_{2m}(x;n)$ (thick red line).

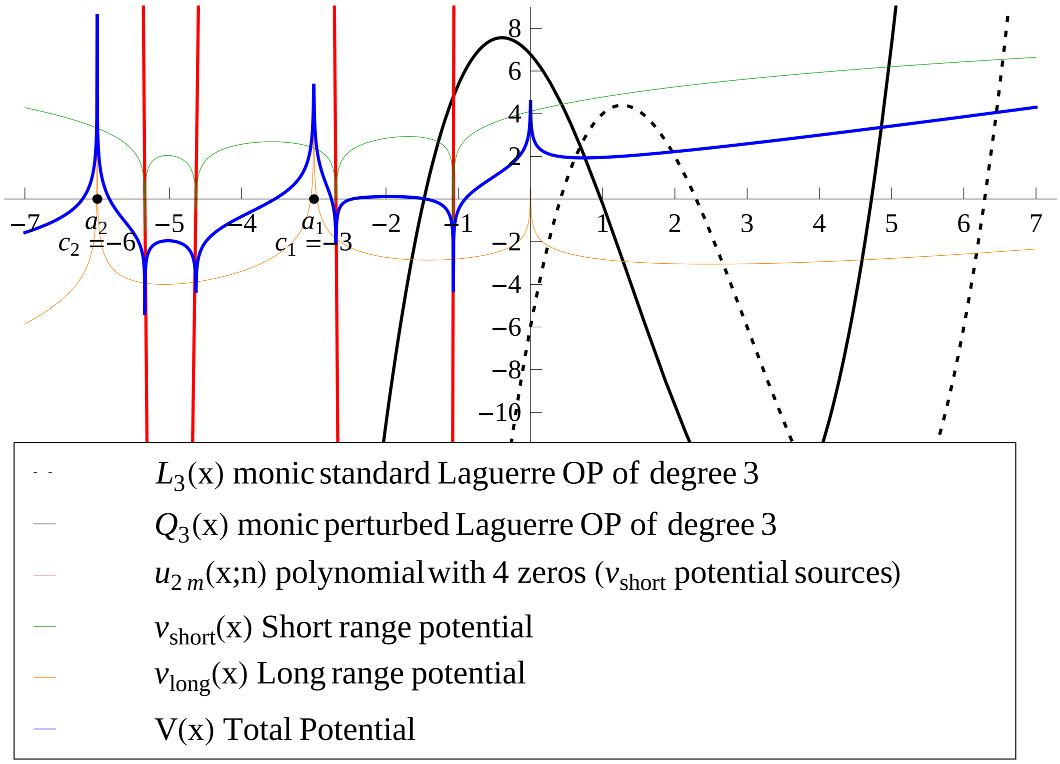


Figure 3.5: Potential graphs for two mass points.

CHAPTER 4

Krall-Laguerre OPS: Asymptotics

4.1 Introduction

In this chapter we consider the MOPS $\{\widehat{Q}_n^{\alpha}(x)\}_{n\geq 0}$ with respect to the inner product

$$\langle f,g\rangle_M = \int_0^{+\infty} f(x)g(x)\,d\mu_\alpha(x) + Mf(c)g(c), \quad f,\, g \in \mathbb{P}, \tag{4.1}$$

where $d\mu_\alpha(x) = x^\alpha e^{-x}dx$, $\alpha > -1$, $M \in \mathbb{R}^+$, and $c \in \mathbb{R}_-$. This is the Uvarov canonical transformation of the measure $d\mu_\alpha$ introduced in (2.13). The polynomials $\{\widehat{Q}_n^{\alpha}(x)\}_{n\geq 0}$ are said to be either Laguerre-type or Krall-Laguerre OPS.

In [52] analytic properties of orthogonal polynomials with respect to a perturbation of the Laguerre weight when a mass is added at $x = 0$ are considered. When the mass point is located at $c = 0$ in (4.1), an electrostatic interpretation of the zeros as equilibrium points with respect to a logarithmic potential, under the action of an external field, has been done in [26].

First, we consider some algebraic and analytic properties of such polynomials and to present a comparison with those of Laguerre polynomials. We obtain the representation of these polynomials in terms of the standard Laguerre polynomials as well as hypergeometric functions. The lowering and raising operators associated with these polynomials are also obtained . Second, we analyze the outer relative asymptotics as well as the Mehler-Heine formula for these polynomials.

Finally, we consider again Laguerre-type MOPS $\{\widehat{Q}_n^{\alpha,m}\}_{n\geq 0}$ with respect to the inner product (3.2) in order to generalize the previour results for m mass points. We find the explicit formula for their outer relative asymptotics that such polynomials satisfy. Of course, for $m=1$ and $c_1=c$ we recover the same results for only one mass point.

From now on, the notation $u_n \cong v_n$ means that the sequence u_n/v_{n_n} converges to 1 as $n\to\infty$.

4.2 Analytic properties of Krall-Laguerre OPS

4.2.1 The connection formula

Applying Theorem 2.6 to the sequence of monic Laguerre-type orthogonal polynomials $\{\widehat{Q}_n^{\alpha}(x)\}_{n\geq 0}$, we get

$$\widehat{Q}_n^{\alpha}(x) = \widehat{L}_n^{\alpha}(x) - M\widehat{Q}_n^{\alpha}(c)\,K_{n-1}(x,c)\,. \tag{4.2}$$

In order to find $\widehat{Q}_n^{\alpha}(c)$, we evaluate (4.2) in $x=c$. Thus

$$\widehat{Q}_n^{\alpha}(c) = \frac{\widehat{L}_n^{\alpha}(c)}{1+MK_{n-1}(c,c)} \tag{4.3}$$

and replacing this value in (4.2), we have

$$\widehat{Q}_n^{\alpha}(x) = \widehat{L}_n^{\alpha}(x) - \frac{M\,\widehat{L}_n^{\alpha}(c)}{1+MK_{n-1}(c,c)}K_{n-1}(x,c). \tag{4.4}$$

On the other hand, from (2.4)

$$(x-c)K_{n-1}(x,c) = \frac{1}{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2}\left(\widehat{L}_n^{\alpha}(x)\widehat{L}_{n-1}^{\alpha}(c) - \widehat{L}_{n-1}^{\alpha}(x)\widehat{L}_n^{\alpha}(c)\right),$$

and (4.4) becomes

$$\begin{aligned}(x-c)\widehat{Q}_n^{\alpha}(x) &= (x-c)\widehat{L}_n^{\alpha}(x) - \\ &- \frac{M\,\widehat{L}_n^{\alpha}(c)}{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2\,(1+MK_{n-1}(c,c))}\left(\widehat{L}_n^{\alpha}(x)\widehat{L}_{n-1}^{\alpha}(c) - \widehat{L}_{n-1}^{\alpha}(x)\widehat{L}_n^{\alpha}(c)\right) \\ &= \widehat{L}_{n+1}^{\alpha}(x) + A_n\widehat{L}_n^{\alpha}(x) + B_n\widehat{L}_{n-1}^{\alpha}(x),\end{aligned}$$

where

$$\begin{aligned} A_n &= \beta_n - c - \frac{M\,\widehat{L}_n^{\alpha}(c)\widehat{L}_{n-1}^{\alpha}(c)}{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2\,(1+MK_{n-1}(c,c))}, \\ B_n &= \gamma_n + \frac{M\left(\widehat{L}_n^{\alpha}(c)\right)^2}{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2\,(1+MK_{n-1}(c,c))} = \gamma_n\left(\frac{1+MK_n(c,c)}{1+MK_{n-1}(c,c)}\right). \end{aligned}$$

Introducing the notation

$$a_n = \frac{\widehat{L}_{n+1}^{\alpha}(c)}{\widehat{L}_n^{\alpha}(c)}, \quad b_n = \frac{1+MK_n(c,c)}{1+MK_{n-1}(c,c)} \tag{4.5}$$

we get

$$\begin{aligned} A_n &= \beta_n - c - M\frac{\gamma_n}{||\widehat{L}_n^{\alpha}||_{\alpha}^2}\frac{\widehat{L}_{n-1}^{\alpha}(c)}{\widehat{L}_n^{\alpha}(c)}\frac{\left(\widehat{L}_n^{\alpha}(c)\right)^2}{(1+MK_{n-1}(c,c))} \\ &= \beta_n - c - M\,\gamma_n\frac{\widehat{L}_{n-1}^{\alpha}(c)}{\widehat{L}_n^{\alpha}(c)}\frac{((1+K_n(c,c)) - K_{n-1}(c,c) - 1)}{||\widehat{L}_n^{\alpha}||_{\alpha}^2\,(1+MK_{n-1}(c,c))} \\ &= \beta_n - c - \gamma_n\frac{\widehat{L}_{n-1}^{\alpha}(c)}{\widehat{L}_n^{\alpha}(c)}\left(\frac{1+MK_n(c,c)}{1+MK_{n-1}(c,c)} - 1\right) \\ &= \frac{(\beta_n - c)\,\widehat{L}_n^{\alpha}(c) + \gamma_n\widehat{L}_{n-1}^{\alpha}(c)}{\widehat{L}_n^{\alpha}(c)} - \gamma_n\frac{\widehat{L}_{n-1}^{\alpha}(c)}{\widehat{L}_n^{\alpha}(c)}\frac{1+MK_n(c,c)}{1+MK_{n-1}(c,c)} \\ &= \frac{-\widehat{L}_{n+1}^{\alpha}(c)}{\widehat{L}_n^{\alpha}(c)} - \gamma_n\frac{\widehat{L}_{n-1}^{\alpha}(c)}{\widehat{L}_n^{\alpha}(c)}\frac{1+MK_n(c,c)}{1+MK_{n-1}(c,c)} \\ &= -a_n - \gamma_n\frac{b_n}{a_{n-1}} = -a_n - \frac{B_n}{a_{n-1}} \end{aligned} \tag{4.6}$$

and

$$B_n = \gamma_n b_n. \tag{4.7}$$

Notice that from (4.5)

$$\begin{aligned} B_n &= \frac{\langle (x-c)\,\widehat{Q}_n^{\alpha}(x), \widehat{L}_{n-1}^{\alpha}(x)\rangle_{\alpha}}{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2} \\ &= \frac{\langle \widehat{Q}_n^{\alpha}(x), (x-c)\,\widehat{L}_{n-1}^{\alpha}(x)\rangle_{\alpha}}{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2} \end{aligned}$$

$$= \frac{||\widehat{Q}_n^\alpha||_M^2}{||\widehat{L}_{n-1}^\alpha||_\alpha^2} = \gamma_n \frac{||\widehat{Q}_n^\alpha||_M^2}{||\widehat{L}_n^\alpha||_\alpha^2}.$$

This yields an expression for the ratio of the energy of polynomials (in the sense of signal theory) $\widehat{Q}_n^\alpha$ and $\widehat{L}_n^\alpha$ with respect to the norms associated with their corresponding inner products.

Proposition 4.1 *Let* $||\widehat{Q}_n^\alpha||_M^2$ *be the norm of Laguerre-type monic polynomials with respect to (4.1). Then*

$$\frac{||\widehat{Q}_n^\alpha||_M^2}{||\widehat{L}_n^\alpha||_\alpha^2} = \frac{1 + MK_n(c,c)}{1 + MK_{n-1}(c,c)}$$

Proof. Taking in account

$$\begin{aligned} ||\widehat{Q}_n^\alpha||_M^2 &= \langle \widehat{Q}_n^\alpha(x), x^n \rangle_M \\ &= \langle \widehat{Q}_n^\alpha(x), \widehat{L}_n^\alpha(x) \rangle_M \\ &= \langle \widehat{Q}_n^\alpha(x), \widehat{L}_n^\alpha(x) \rangle_\alpha + M\, \widehat{Q}_n^\alpha(c) \widehat{L}_n^\alpha(c) \end{aligned}$$

and using (4.3), we get

$$\begin{aligned} ||\widehat{Q}_n^\alpha||_M^2 &= ||\widehat{L}_n^\alpha||_\alpha^2 + \frac{M\left(\widehat{L}_n^\alpha(c)\right)^2}{1 + MK_{n-1}(c,c)} \\ &= ||\widehat{L}_n^\alpha||_\alpha^2 \frac{(1 + MK_{n-1}(c,c)) + \frac{M\left(\widehat{L}_n^\alpha(c)\right)^2}{||\widehat{L}_n^\alpha||_\alpha^2}}{1 + MK_{n-1}(c,c)} \\ &= ||\widehat{L}_n^\alpha||_\alpha^2 \frac{1 + MK_n(c,c)}{1 + MK_{n-1}(c,c)}. \end{aligned}$$

■

Remark. From the expressions (4.6) and (4.7) we observe that two basic parameters (a_n, b_n) are needed in the connection formula.

As a conclusion,

Theorem 4.1 *Let* $\{\widehat{Q}_n^\alpha(x)\}_{n\geq 0}$ *be the sequence of monic Laguerre-type polynomials orthogonal with respect to (4.1). Then*

$$(x - c)\widehat{Q}_n^\alpha(x) = \widehat{L}_{n+1}^\alpha(x) + A_n \widehat{L}_n^\alpha(x) + B_n \widehat{L}_{n-1}^\alpha(x), \tag{4.8}$$

where

$$A_n = (2n+1+\alpha-c) - \frac{M\,\widehat{L}_n^{\alpha}(c)\widehat{L}_{n-1}^{\alpha}(c)}{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2\,(1+MK_{n-1}(c,c))} = -a_n - \gamma_n \frac{b_n}{a_{n-1}}, \quad (4.9)$$

$$B_n = n\,(n+\alpha)\,\frac{1+MK_n(c,c)}{1+MK_{n-1}(c,c)} = \gamma_n b_n, \quad (4.10)$$

with

$$a_n = \frac{\widehat{L}_{n+1}^{\alpha}(c)}{\widehat{L}_n^{\alpha}(c)}, \quad b_n = \frac{1+MK_n(c,c)}{1+MK_{n-1}(c,c)}.$$

4.2.2 Hypergeometric representation

Next we will focus our attention in the representation of this new family of orthogonal polynomials as hypergeometric functions. From (4.8) and the hypergeometric representation of the classical Laguerre polynomials

$$\widehat{L}_n^{\alpha}(x) = \frac{(-1)^n\,\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\,{}_1F_1(-n;\alpha+1;x),$$

we can write

$$\begin{aligned}
(x-c)\,\widehat{Q}_n^{\alpha}(x) &= -(-1)^n(\alpha+n+1)(\alpha+1)_n \sum_{k=0}^{\infty} \frac{(-n-1)}{(-n-1+k)} \left(\frac{(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!}\right) \\
&\quad - \left(a_n + \gamma_n \frac{b_n}{a_{n-1}}\right)(-1)^n(\alpha+1)_n \sum_{k=0}^{\infty} \left(\frac{(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!}\right) \\
&\quad - \gamma_n b_n (-1)^n \frac{(\alpha+1)_n}{(\alpha+n)} \sum_{k=0}^{\infty} \frac{(-n+k)}{(-n)} \left(\frac{(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!}\right) \\
&= (-1)^n(\alpha+1)_n \sum_{k=0}^{\infty} \left(\frac{(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!}\right) \\
&\quad \times \left[\frac{-(\alpha+n+1)(-n-1)}{(-n-1+k)} - \left(a_n + \gamma_n \frac{b_n}{a_{n-1}}\right) - \frac{\gamma_n b_n}{(\alpha+n)}\frac{(-n+k)}{(-n)}\right].
\end{aligned}$$

Now let's write the expression in brackets as a rational function in the variable k. A careful computation of the elements of the sum inside these brackets, yields

$$\left[\frac{-(\alpha+n+1)(-n-1)}{(-n-1+k)} - \left(a_n + \gamma_n \frac{b_n}{a_{n-1}}\right) - \frac{\gamma_n b_n}{(\alpha+n)}\frac{(-n+k)}{(-n)}\right]$$

$$= \quad b_n \frac{(k-e_1)(k-e_2)}{(k-(n+1))}$$

where

$$\begin{aligned}
e_1 &= \frac{-1}{2b_n a_{n-1}}\left(d_1 + \sqrt{d_1^2 - 4d_0 b_n a_{n-1}}\right), \\
e_2 &= \frac{-1}{2b_n a_{n-1}}\left(d_1 - \sqrt{d_1^2 - 4d_0 b_n a_{n-1}}\right), \\
d_0 &= (n+1)\left(\gamma_n b_n + a_{n-1}\left(a_n + 1 + \alpha + n\left(1+b_n\right)\right)\right), \\
d_1 &= -\gamma_n b_n - a_{n-1}\left(b_n\left(2n+1\right) + a_n\right).
\end{aligned}$$

Thus

$$(x-c)\,\widehat{Q}_n^{\alpha}(x) = \sum_{k=0}^{\infty}\left(\frac{(-1)^n(\alpha+1)_n(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!}\right) \times \left[b_n \frac{(k-e_1)(k-e_2)}{(k-(n+1))}\right]. \tag{4.11}$$

Notice that although the sum is up to infinity, this is a terminating hypergeometric series, because the Pochhammer symbol $(-n)_k$ becomes zero if $k > n+1$. Now, using some properties of the Pochhammer symbol, we can write

$$\begin{aligned}
\frac{(k-e_1)(k-e_2)}{(k-(n+1))} &= \frac{(-e_1)(1-e_1)_k}{(-e_1)_k}\frac{(-e_2)(1-e_2)_k}{(-e_2)_k}\left(\frac{-(-n-1)_k}{(n+1)(-n)_k}\right) \\
&= \left(\frac{e_1 e_2}{(n+1)}\right)\frac{(1-e_1)_k(1-e_2)_k(-n-1)_k}{(-e_1)_k(-e_2)_k(-n)_k}
\end{aligned}$$

and, therefore,

$$\begin{aligned}
(x-c)\,\widehat{Q}_n^{\alpha}(x) &= (-1)^n b_n\left(\frac{e_1 e_2}{n+1}\right)(\alpha+1)_n \sum_{k=0}^{\infty}\frac{(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!} \times \frac{(1-e_1)_k(1-e_2)_k(-n-1)_k}{(-e_1)_k(-e_2)_k(-n)_k} \\
&= (-1)^n b_n\left(\frac{e_1 e_2}{n+1}\right)(\alpha+1)_n \sum_{k=0}^{\infty}\frac{(1-e_1)_k(1-e_2)_k(-n-1)_k}{(-e_1)_k(-e_2)_k(\alpha+1)_k}\frac{x^k}{k!} \\
&= C_{n,\alpha}\,{}_3F_3\left(1-e_1, 1-e_2, -n-1; -e_1, -e_2, \alpha+1; x\right).
\end{aligned}$$

Finally, the hypergeometric representation is

$$\widehat{Q}_n^{\alpha}(x) = \left(\frac{C_{n,\alpha}}{x-c}\right){}_3F_3\left(1-e_1, 1-e_2, -n-1; -e_1, -e_2, \alpha+1; x\right). \tag{4.12}$$

4.2.3 The three term recurrence formula

Taking into account

$$x\widehat{Q}_n^\alpha(x) = \widehat{Q}_{n+1}^\alpha(x) + \widetilde{\beta}_n \widehat{Q}_n^\alpha(x) + \widetilde{\gamma}_n \widehat{Q}_{n-1}^\alpha(x) \tag{4.13}$$

we get

$$\begin{aligned}
\widetilde{\beta}_n &= \frac{\langle x\widehat{Q}_n^\alpha(x), \widehat{Q}_n^\alpha(x)\rangle_M}{||\widehat{Q}_n^\alpha||_M^2} = c + \frac{\langle (x-c)\,\widehat{Q}_n^\alpha(x), \widehat{Q}_n^\alpha(x)\rangle_M}{||\widehat{Q}_n^\alpha||_M^2} \\
&= c + \frac{\langle (x-c)\,\widehat{Q}_n^\alpha(x), \widehat{Q}_n^\alpha(x)\rangle_\alpha}{||\widehat{Q}_n^\alpha||_M^2}.
\end{aligned}$$

But, from the connection formula, and after some tedious computations, the previous expression becomes

$$\begin{aligned}
\widetilde{\beta}_n &= c + \frac{\left\langle \widehat{L}_{n+1}^\alpha(x) - \left(a_n + \gamma_n \frac{b_n}{a_{n-1}}\right) \widehat{L}_n^\alpha(x) + \gamma_n b_n \widehat{L}_{n-1}^\alpha(x), \widehat{Q}_n^\alpha(x) \right\rangle_\alpha}{||\widehat{Q}_n^\alpha||_M^2} \\
&= c - \left(\frac{a_n}{b_n} + \frac{\gamma_n}{a_{n-1}}\right) - \frac{M}{||\widehat{L}_{n-1}^\alpha||_\alpha^2} \frac{\widehat{L}_n^\alpha(c)\widehat{L}_{n-1}^\alpha(c)}{1 + MK_{n-1}(c,c)} \\
&= \beta_n + a_n\left(1 - \frac{1}{b_n}\right) - a_{n-1}\left(1 - \frac{1}{b_{n-1}}\right).
\end{aligned}$$

On the other hand

$$\widetilde{\gamma}_n = \frac{b_n}{b_{n-1}}\gamma_n.$$

Thus, as a conclusion

Proposition 4.2 *The coefficients of the three term recurrence relation for the sequence of monic orthogonal polynomials* $\{\widehat{Q}_n^\alpha(x)\}_{n\geq 0}$ *are*

$$\widetilde{\beta}_n = \beta_n + a_n\left(1 - \frac{1}{b_n}\right) - a_{n-1}\left(1 - \frac{1}{b_{n-1}}\right), \tag{4.14}$$

$$\widetilde{\gamma}_n = \frac{b_n}{b_{n-1}}\gamma_n. \tag{4.15}$$

Indeed,

$$\frac{\widetilde{\beta}_n}{\beta_n} = 1 + \frac{a_n}{\beta_n}\left(1 - \frac{1}{b_n}\right) - \frac{a_{n-1}}{\beta_n}\left(1 - \frac{1}{b_{n-1}}\right)$$

$$= 1 + \frac{1}{\beta_n} \frac{\widehat{L}^{\alpha}_{n+1}(c)}{\widehat{L}^{\alpha}_{n}(c)} \left(M \frac{\frac{\left(\widehat{L}^{\alpha}_{n}(c)\right)^2}{\|\widehat{L}^{\alpha}_{n}\|^2_{\alpha}}}{1 + MK_n(c,c)} \right) - \frac{1}{\beta_n} \frac{\widehat{L}^{\alpha}_{n}(c)}{\widehat{L}^{\alpha}_{n-1}(c)} \left(M \frac{\frac{\left(\widehat{L}^{\alpha}_{n-1}(c)\right)^2}{\|\widehat{L}^{\alpha}_{n-1}\|^2_{\alpha}}}{1 + MK_{n-1}(c,c)} \right).$$

But

$$\frac{M\widehat{L}^{\alpha}_{n+1}(c)\,\widehat{L}^{\alpha}_{n}(c)}{\|\widehat{L}^{\alpha}_{n}\|^2_{\alpha} + M(n+1)!n!\left(\widehat{L}^{\alpha+1}_{n}(c)\right)^2 \left(1 - \frac{\widehat{L}^{\alpha+1}_{n+1}(c)}{\widehat{L}^{\alpha+1}_{n}(c)} \frac{\widehat{L}^{\alpha+1}_{n-1}(c)}{\widehat{L}^{\alpha+1}_{n}(c)}\right)}$$

$$= \frac{\left(1 - \frac{\widehat{L}^{\alpha+1}_{n+1}(c)}{\widehat{L}^{\alpha+1}_{n}(c)}\right)\left(1 - \frac{\widehat{L}^{\alpha+1}_{n-1}(c)}{\widehat{L}^{\alpha+1}_{n}(c)}\right)}{\frac{\|\widehat{L}^{\alpha}_{n}\|^2_{\alpha}}{M(n+1)!n!\left(\widehat{L}^{\alpha+1}_{n}(c)\right)^2} + \left(1 - \frac{\widehat{L}^{\alpha+1}_{n+1}(c)}{\widehat{L}^{\alpha+1}_{n}(c)} \frac{\widehat{L}^{\alpha+1}_{n-1}(c)}{\widehat{L}^{\alpha+1}_{n}(c)}\right)}. \tag{4.16}$$

According to Perron's formula (see also Appendix C)

$$\frac{\widehat{L}^{\alpha+1}_{n}(c)}{\widehat{L}^{\alpha+1}_{n-1}(c)} = 1 + \frac{\sqrt{|c|}}{\sqrt{n}} + \mathcal{O}\left(n^{-1}\right)$$

the expression (4.16) becomes

$$\frac{\left(-\frac{\sqrt{|c|}}{\sqrt{n+1}} + \mathcal{O}\left(n^{-1}\right)\right)\left(\frac{\sqrt{|c|}}{\sqrt{n}} + \mathcal{O}\left(n^{-1}\right)\right)}{1 - \left(1 + \frac{\sqrt{|c|}}{\sqrt{n+1}}\right)\left(1 - \frac{\sqrt{|c|}}{\sqrt{n}}\right)}$$

$$= \frac{\frac{-|c|}{\sqrt{n}\sqrt{n+1}} + \mathcal{O}\left(n^{-3/2}\right)}{\frac{\sqrt{|c|}}{\sqrt{n}} - \frac{\sqrt{|c|}}{\sqrt{n+1}} + \mathcal{O}\left(n^{-1/2}\right)} = -2\sqrt{|c|}n^{1/2} + \mathcal{O}\left(n^{-1/2}\right).$$

Thus

$$\begin{aligned} \frac{\widetilde{\beta}_n}{\beta_n} &= 1 + \frac{1}{2n+1+\alpha}\left(-2\sqrt{|c|}n^{1/2} + \mathcal{O}\left(n^{-1/2}\right)\right) \\ &\quad - \frac{1}{2n+1+\alpha}\left(-2\sqrt{|c|}\,(n-1)^{1/2} + \mathcal{O}\left((n-1)^{-1/2}\right)\right) \\ &= 1 - \frac{\sqrt{|c|}}{2}n^{-3/2} + \mathcal{O}\left(n^{-5/2}\right). \end{aligned}$$

On the other hand, from (4.14) and (4.5)

$$\frac{\widetilde{\gamma}_n}{\gamma_n} = \frac{b_n}{b_{n-1}},$$

again we take into account

$$\frac{1 + MK_n(c,c)}{1 + MK_{n-1}(c,c)}$$

$$\begin{aligned}
&= 1 + \frac{\frac{M\left(\widehat{L}_n^{\alpha}(c)\right)^2}{\|\widehat{L}_n^{\alpha}\|_{\alpha}^2}}{1 + MK_{n-1}(c,c)} \\
&= 1 + \frac{\left(\widehat{L}_n^{\alpha}(c)\right)^2}{\frac{\|\widehat{L}_n^{\alpha}\|_{\alpha}^2}{M\,(n!)^2} 1 + (n+\alpha)\left[\left(\widehat{L}_{n-1}^{\alpha+1}(c)\right)^2 - \widehat{L}_n^{\alpha+1}(c)\,\widehat{L}_{n-2}^{\alpha+1}(c)\right]} \\
&= 1 + \frac{\left(1 - \frac{\widehat{L}_n^{\alpha+1}(c)}{\widehat{L}_{n-1}^{\alpha+1}(c)}\right)^2}{\frac{\|\widehat{L}_n^{\alpha}\|_{\alpha}^2}{M\,(n!)^2\left(\widehat{L}_{n-1}^{\alpha+1}(c)\right)^2} + (n+\alpha)\left[1 - \frac{\widehat{L}_n^{\alpha+1}(c)}{\widehat{L}_{n-1}^{\alpha+1}(c)}\frac{\widehat{L}_{n-2}^{\alpha+1}(c)}{\widehat{L}_{n-1}^{\alpha+1}(c)}\right]} \\
&\cong 1 + \frac{\frac{|c|}{n} + \mathcal{O}\left(n^{-3/2}\right)}{(n+\alpha)\left[1 - \left(\frac{\sqrt{|c|}}{\sqrt{n}} + 1 + \mathcal{O}\left(n^{-1}\right)\right)\left(-\frac{\sqrt{|c|}}{\sqrt{n-1}} + 1 + \mathcal{O}\left(n^{-1}\right)\right)\right]} \\
&= 1 + \frac{\frac{|c|}{n} + \mathcal{O}\left(n^{-3/2}\right)}{(n+\alpha)\,\frac{\sqrt{|c|}}{2n^{3/2}}} = 1 + 2\sqrt{|c|}n^{-1/2} + \mathcal{O}\left(n^{-1}\right).
\end{aligned}$$

As a conclusion, we have

Proposition 4.3

$$\begin{aligned}
\frac{\widetilde{\beta}_n}{\beta_n} &= 1 - \frac{\sqrt{|c|}}{2}n^{-3/2} + \mathcal{O}(n^{-5/2}), \\
\frac{\widetilde{\gamma}_n}{\gamma_n} &= 1 + 2\sqrt{|c|}n^{-1/2} + \mathcal{O}(n^{-1}).
\end{aligned}$$

4.2.4 Lowering and raising operators

From the connection formula (4.8) we get

$$(x-c)\widehat{Q}_n^{\alpha}(x) = (x + A_n - \beta_n)\,\widehat{L}_n^{\alpha}(x) + (B_n - \gamma_n)\,\widehat{L}_{n-1}^{\alpha}(x). \tag{4.17}$$

Notice that for $x = c$

$$(c + A_n - \beta_n)\,\widehat{L}_n^{\alpha}(c) = (\gamma_n - B_n)\,\widehat{L}_{n-1}^{\alpha}(c).$$

Thus

$$c + A_n - \beta_n = (\gamma_n - B_n)\,\frac{1}{a_{n-1}}$$

and (4.17) becomes

$$\begin{aligned}(x-c)\widehat{Q}_n^{\alpha}(x) &= \left(x - c + (\gamma_n - B_n)\,\frac{1}{a_{n-1}}\right)\widehat{L}_n^{\alpha}(x) + (B_n - \gamma_n)\,\widehat{L}_{n-1}^{\alpha}(x)\\ &= \left(x - c - a_{n-1}\left(1 - \frac{1}{b_{n-1}}\right)\right)\widehat{L}_n^{\alpha}(x) + \gamma_n\,(b_n - 1)\,\widehat{L}_{n-1}^{\alpha}(x). \end{aligned} \tag{4.18}$$

On the other hand, introducing the shift $n \to n-1$, the above expression becomes

$$(x-c)\widehat{Q}_{n-1}^{\alpha}(x) = \left(x - c - a_{n-2}\left(1 - \frac{1}{b_{n-2}}\right)\right)\widehat{L}_{n-1}^{\alpha}(x) + \gamma_{n-1}\,(b_{n-1} - 1)\,\widehat{L}_{n-2}^{\alpha}(x)$$

and, using the three term recurrence relation, we get

$$(x-c)\widehat{Q}_{n-1}^{\alpha}(x) =$$
$$\left(x - c - a_{n-2}\left(1 - \frac{1}{b_{n-2}}\right) + (b_{n-1} - 1)\,(x - \beta_{n-1})\right)\widehat{L}_{n-1}^{\alpha}\,(x) - (b_{n-1} - 1)\,\widehat{L}_n^{\alpha}(x). \tag{4.19}$$

Thus, taking $x = c$, we obtain

$$\left(-a_{n-2}\left(1 - \frac{1}{b_{n-2}}\right) + (b_{n-1} - 1)\,(c - \beta_{n-1})\right)\widehat{L}_{n-1}^{\alpha}(c) = (b_{n-1} - 1)\,\widehat{L}_n^{\alpha}(c)$$

i.e.

$$\left(-a_{n-2}\left(1 - \frac{1}{b_{n-2}}\right) + (b_{n-1} - 1)\,(c - \beta_{n-1})\right) = (b_{n-1} - 1)\,a_{n-1}.$$

Replacing it in (4.19) we get

$$(x-c)\widehat{Q}_{n-1}^{\alpha}(x) = (b_{n-1}\,(x - c) + a_{n-1}\,(b_{n-1} - 1))\,\widehat{L}_{n-1}^{\alpha}(x) + (1 - b_{n-1})\,\widehat{L}_n^{\alpha}(x). \tag{4.20}$$

As a conclusion, from (4.18) and (4.20) deduce the representation of Laguerre polynomials in terms of Laguerre-type polynomials which will be very useful in the sequel.

$$\widehat{L}_n^\alpha(x) =$$

$$\frac{\begin{vmatrix} (x-c)\,\widehat{Q}_n^\alpha(x) & \gamma_n(b_n-1) \\ (x-c)\,\widehat{Q}_{n-1}^\alpha(x) & b_{n-1}(x-c)+a_{n-1}\,(b_{n-1}-1) \end{vmatrix}}{\begin{vmatrix} x-c-a_{n-1}(1-\dfrac{1}{b_{n-1}}) & \gamma_n(b_n-1) \\ 1-b_{n-1} & b_{n-1}(x-c)+a_{n-1}\,(b_{n-1}-1) \end{vmatrix}}$$

$$= \frac{\left[(x-c)+a_{n-1}\left(1-\dfrac{1}{b_{n-1}}\right)\right]\widehat{Q}_n^\alpha(x)-\dfrac{a_{n-1}^2}{b_{n-1}}\left(1-\dfrac{1}{b_{n-1}}\right)\widehat{Q}_{n-1}^\alpha(x)}{(x-c)}, \tag{4.21}$$

$$\widehat{L}_{n-1}^\alpha(x) =$$

$$\frac{\begin{vmatrix} x-c-a_{n-1}\left(1-\dfrac{1}{b_{n-1}}\right) & \widehat{Q}_n^\alpha(x) \\ (1-b_{n-1}) & \widehat{Q}_{n-1}^\alpha(x) \end{vmatrix}}{b_{n-1}\,(x-c)}$$

$$= \frac{\left(1-\dfrac{1}{b_{n-1}}\right)\widehat{Q}_n^\alpha(x)+\dfrac{1}{b_{n-1}}\left[(x-c)-a_{n-1}\left(1-\dfrac{1}{b_{n-1}}\right)\right]\widehat{Q}_{n-1}^\alpha(x)}{(x-c)}. \tag{4.22}$$

Next, taking derivatives in (4.18) and multiplying by x

$$\begin{aligned} x\widehat{Q}_n^\alpha(x)+x\,(x-c)\,[\widehat{Q}_n^\alpha]'(x) &= x\widehat{L}_n^\alpha(x)+\left(x-c-a_{n-1}\left(1-\frac{1}{b_{n-1}}\right)\right)x\,[\widehat{L}_n^\alpha]'(x) \\ &\quad +\gamma_n(b_n-1)\,x\,[\widehat{L}_{n-1}^\alpha]'(x), \end{aligned}$$

but, according to the lowering operator for Laguerre polynomials (2.27) and the three term recurrence relation that they satisfy, the above expression becomes

$$\begin{aligned} & x\widehat{Q}_n^\alpha(x)+x\,(x-c)\,[\widehat{Q}_n^\alpha]'(x) \\ = \;& x\widehat{L}_n^\alpha(x)+\left(x-c-a_{n-1}\left(1-\frac{1}{b_{n-1}}\right)\right)\left[n\widehat{L}_n^\alpha(x)+\gamma_n\widehat{L}_{n-1}^\alpha(x)\right] \end{aligned}$$

$$
\begin{aligned}
& +\gamma_n(b_n-1)\left[(n-1)\,\widehat{L}_{n-1}^{\alpha}(x)+\gamma_{n-1}\widehat{L}_{n-2}^{\alpha}(x)\right] \\
= \;& x\widehat{L}_n^{\alpha}(x)+\left(x-c-a_{n-1}\left(1-\frac{1}{b_{n-1}}\right)\right)\left[n\widehat{L}_n^{\alpha}(x)+\gamma_n\widehat{L}_{n-1}^{\alpha}(x)\right] \\
& +\gamma_n(b_n-1)\left[(n-1+x-\beta_{n-1})\,\widehat{L}_{n-1}^{\alpha}(x)-\widehat{L}_n^{\alpha}(x)\right] \\
= \;& \left[(n+1)\,x-nc-na_{n-1}\left(1-\frac{1}{b_{n-1}}\right)-n\,(n+\alpha)\,(b_n-1)\right]\widehat{L}_n^{\alpha}(x) \\
& +\gamma_n\left[b_nx-c-a_{n-1}\left(1-\frac{1}{b_{n-1}}\right)-(n+\alpha)\,(b_n-1)\right]\widehat{L}_{n-1}^{\alpha}(x).
\end{aligned}
$$

Multiplying both hand sides by $(x-c)$ and using (4.21) and (4.22) we get

$$
x\,(x-c)\,\widehat{Q}_n^{\alpha}(x)+x\,(x-c)^2\,[\widehat{Q}_n^{\alpha}]'(x)=
$$

$$
\begin{aligned}
& \left[(n+1)\,x-nc-na_{n-1}\left(1-\frac{1}{b_{n-1}}\right)-n\,(n+\alpha)\,(b_n-1)\right](x-c)\,\widehat{L}_n^{\alpha}(x) \\
& +\gamma_n\left[b_nx-c-a_{n-1}\left(1-\frac{1}{b_{n-1}}\right)-(n+\alpha)\,(b_n-1)\right](x-c)\,\widehat{L}_{n-1}^{\alpha}(x)\,.
\end{aligned}
$$

Introducing the parameter

$$
c_n=c+a_{n-1}\left(1-\frac{1}{b_{n-1}}\right)-(n+\alpha)\,(b_n-1)
$$

the above expression reads as

$$
x\,(x-c)\,\widehat{Q}_n^{\alpha}(x)+x\,(x-c)^2\,[\widehat{Q}_n^{\alpha}]'(x)=
$$

$$
\begin{aligned}
& [(n+1)\,x-nc_n]\left\{\left[(x-c)+a_{n-1}\left(1-\frac{1}{b_{n-1}}\right)\right]\widehat{Q}_n^{\alpha}(x)-\frac{a_{n-1}^2}{b_{n-1}}\left(1-\frac{1}{b_{n-1}}\right)\widehat{Q}_{n-1}^{\alpha}(x)\right\} \\
& +\gamma_n\,[b_nx-c_n]\left\{\left(1-\frac{1}{b_{n-1}}\right)\widehat{Q}_n^{\alpha}(x)+\frac{1}{b_{n-1}}\left[(x-c)-a_{n-1}\left(1-\frac{1}{b_{n-1}}\right)\right]\widehat{Q}_{n-1}^{\alpha}(x)\right\}.
\end{aligned}
$$

As a conclusion,

$$
\begin{aligned}
& \left[x\,(x-c)^2\,\frac{d}{dx}+x\,(x-c)-[(n+1)\,x-nc_n]\left(x-c+a_{n-1}\left(1-\frac{1}{b_{n-1}}\right)\right)\right. \\
& \left.-\,\gamma_n\,[b_nx-c_n]\left(1-\frac{1}{b_{n-1}}\right)\right]\widehat{Q}_n^{\alpha}(x)
\end{aligned}
$$

$$
\begin{aligned}
= & \left[-\left[(n+1)x - nc_n\right]\frac{a_{n-1}^2}{b_{n-1}}\left(1-\frac{1}{b_{n-1}}\right)\right. \\
& \left.+\frac{\gamma_n}{b_{n-1}}\left[b_n x - c_n\right]\left(x-c-a_{n-1}\left(1-\frac{1}{b_{n-1}}\right)\right)\right]\widehat{Q}_{n-1}^{\alpha}(x).
\end{aligned}
$$

Thus, we get the expression for the lowering operator $\mathcal{L}_n$

$$\mathcal{L}_n\widehat{Q}_n^{\alpha}(x) = u_n(x)\,\widehat{Q}_{n-1}^{\alpha}(x)$$

where

$$\mathcal{L}_n = x(x-c)^2\frac{d}{dx} - nx^2 + D_n x + E_n$$

and

$$
\begin{aligned}
D_n &= \frac{1}{b_{n-1}}\left(a_{n-1}+\gamma_n b_n\right)+\gamma_n - a_{n-1} - 2\left(\gamma_n b_n - cn\right), \\
E_n &= c_n\left(\left(na_{n-1}+\gamma_n\right)\left(1-\frac{1}{b_{n-1}}\right)-cn\right).
\end{aligned}
$$

Notice that $u_n(x)$ is a quadratic polynomial

$$u_n(x) = F_n x^2 + G_n x + H_n,$$

with

$$
\begin{aligned}
F_n &= \gamma_n\frac{b_n}{b_{n-1}}, \\
G_n &= \left(\frac{1}{b_{n-1}}-1\right)\frac{a_{n-1}^2}{b_{n-1}}(n+1)-\frac{\gamma_n}{b_{n-1}}\left(c_n+b_n\left(c-\left(\frac{1}{b_{n-1}}-1\right)a_{n-1}\right)\right), \\
H_n &= \gamma_n c_n\frac{c-\left(\frac{1}{b_{n-1}}-1\right)a_{n-1}}{b_{n-1}} - nc_n\left(\frac{1}{b_{n-1}}-1\right)\frac{a_{n-1}^2}{b_{n-1}}.
\end{aligned}
$$

Taking into account the three term recurrence relation (4.13)

$$\widetilde{\gamma}_n\mathcal{L}_n\widehat{Q}_n^{\alpha}(x) = u_n(x)\left[\left(x-\widetilde{\beta}_n\right)\widehat{Q}_n^{\alpha}(x)-\widehat{Q}_{n+1}^{\alpha}(x)\right].$$

Thus, we get the raising operator

$$\mathcal{R}_n = -\widetilde{\gamma}_n\mathcal{L}_n + \left(x-\widetilde{\beta}_n\right)u_n(x)$$

i.e

$$\mathcal{R}_n\widehat{Q}_n^{\alpha}(x) = u_n(x)\,\widehat{Q}_{n+1}^{\alpha}(x).$$

Notice that, combining the former raising and lowering operators, we can obtain the holonomic equation (3.36) from a different point of view. Let

$$\frac{1}{u_n(x)}\mathcal{R}_n\widehat{Q}_n^\alpha(x)=\widehat{Q}_{n+1}^\alpha(x).$$

Applying the lowering operator in the above expression yields

$$\mathcal{L}_{n+1}\left[\frac{1}{u_n(x)}\mathcal{R}_n\right]\left(\widehat{Q}_n^\alpha(x)\right)=u_{n+1}(x)\,\widehat{Q}_n^\alpha(x)$$

and thus, the holonomic equation satisfied by the Krall-Laguerre OPS follows directly.

4.3 Krall-Laguerre OPS: Asymptotics for one mass point

4.3.1 Outer relative asymptotics

In this section, we will obtain some results concerning the asymptotic behavior of Laguerre-type orthogonal polynomials in the exterior of the positive real semi axis. Taking into account (4.4),

$$\widehat{Q}_n^\alpha(x)=\widehat{L}_n^\alpha(x)-\frac{M\widehat{L}_n^\alpha(c)}{1+MK_{n-1}(c,c)}K_{n-1}(x,c),$$

by using the standard normalization for Laguerre polynomials (2.30), and dividing by $L_n^{(\alpha)}(x)$ in both hand sides of the above expression, we get

$$\frac{Q_n^{(\alpha)}(x)}{L_n^{(\alpha)}(x)}=1-\frac{ML_n^{(\alpha)}(c)}{1+MK_{n-1}(c,c)}\frac{K_{n-1}(x,c)}{L_n^{(\alpha)}(x)}.$$

Using the Christoffel-Darboux formula, this expression becomes

$$\frac{Q_n^{(\alpha)}(x)}{L_n^{(\alpha)}(x)}=1-$$

$$\frac{ML_n^{(\alpha)}(c)}{\|\widehat{L}_{n-1}^\alpha\|_\alpha^2+M\left[n\widehat{L}_{n-1}^{\alpha+1}(c)\,\widehat{L}_{n-1}^\alpha(c)-(n-1)\widehat{L}_{n-2}^{\alpha+1}(c)\,\widehat{L}_n^\alpha(c)\right]}$$
$$\times\frac{\widehat{L}_n^\alpha(x)\,\widehat{L}_{n-1}^\alpha(c)-\widehat{L}_{n-1}^\alpha(x)\,\widehat{L}_n^\alpha(c)}{(x-c)\,L_n^{(\alpha)}(x)}$$

$$= 1 + \frac{Mn!\,(n-1)!\left(L_n^{(\alpha)}(c)\right)^2\left(\dfrac{L_{n-1}^{(\alpha)}(c)}{L_n^{(\alpha)}(c)} - \dfrac{L_{n-1}^{(\alpha)}(x)}{L_n^{(\alpha)}(x)}\right)\dfrac{1}{x-c}}{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2 + Mn!\,(n-1)!\left(\left[L_{n-1}^{(\alpha+1)}(c)\right]^2 - L_{n-2}^{(\alpha+1)}(c)L_n^{(\alpha+1)}(c)\right)}$$

$$= 1 + \frac{M\left[1 - \dfrac{L_n^{(\alpha+1)}(c)}{L_{n-1}^{(\alpha+1)}(c)}\right]^2\left(\dfrac{L_{n-1}^{(\alpha)}(c)}{L_n^{(\alpha)}(c)} - \dfrac{L_{n-1}^{(\alpha)}(x)}{L_n^{(\alpha)}(x)}\right)\dfrac{1}{x-c}}{\dfrac{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2}{n!\,(n-1)!\left(L_{n-1}^{(\alpha+1)}(c)\right)^2} + M\dfrac{L_{n-2}^{(\alpha+1)}(c)}{L_{n-1}^{(\alpha+1)}(c)}\left[\dfrac{L_{n-1}^{(\alpha+1)}(c)}{L_{n-2}^{(\alpha+1)}(c)} - \dfrac{L_n^{(\alpha+1)}(c)}{L_{n-1}^{(\alpha+1)}(c)}\right]}. \quad (4.23)$$

Next, applying Lemmas C.1 and C.2 in (4.23) we get

$$\begin{aligned}
\frac{Q_n^{(\alpha)}(x)}{L_n^{(\alpha)}(x)} &\cong 1 + \frac{M\left(\dfrac{\sqrt{|c|}}{\sqrt{n}}\right)^2\left(1 - \dfrac{\sqrt{|c|}}{\sqrt{n}} - 1 + \dfrac{\sqrt{-x}}{\sqrt{n}}\right)\dfrac{1}{x-c}}{M\left(-1 - \dfrac{\sqrt{|c|}}{\sqrt{n}} + 1 + \dfrac{\sqrt{|c|}}{\sqrt{n-1}}\right)} \\
&\cong 1 + 2\frac{\sqrt{|c|}\left(\sqrt{(-x)} - \sqrt{|c|}\right)}{x-c} \\
&= 1 - 2\sqrt{|c|}\frac{1}{\sqrt{(-x)} + \sqrt{|c|}} \\
&= \frac{\sqrt{-x} - \sqrt{|c|}}{\sqrt{-x} + \sqrt{|c|}}
\end{aligned}$$

and the convergence is locally uniformly on $\mathbb{C}\setminus\mathbb{R}_+$. Notice that, according to the Hurwitz's theorem, the above result shows that the point $x = c$ attracts one zero of $Q_n^{(\alpha)}(x)$ for n large enough.

4.3.2 Mehler-Heine type formula

Concerning the Mehler-Heine formula, notice that from (4.23)

$$\frac{Q_n^{(\alpha)}(x/n)}{L_n^{(\alpha)}(x/n)} = 1 +$$

$$\frac{M\left[1-\dfrac{L_n^{(\alpha+1)}(c)}{L_{n-1}^{(\alpha+1)}(c)}\right]^2\left(\dfrac{L_{n-1}^{(\alpha)}(c)}{L_n^{(\alpha)}(c)}-\dfrac{L_{n-1}^{(\alpha)}(x/n)}{L_n^{(\alpha)}(x/n)}\right)\dfrac{1}{\dfrac{x}{n}-c}}{\dfrac{||\widehat{L}_n^{\alpha}||_\alpha^2}{n!\,(n-1)!\left(L_{n-1}^{(\alpha+1)}(c)\right)^2}+M\dfrac{L_{n-2}^{(\alpha+1)}(c)}{L_{n-1}^{(\alpha+1)}(c)}\left[\dfrac{L_{n-1}^{(\alpha+1)}(c)}{L_{n-2}^{(\alpha+1)}(c)}-\dfrac{L_n^{(\alpha+1)}(c)}{L_{n-1}^{(\alpha+1)}(c)}\right]}.$$

Proceeding as above

$$\frac{Q_n^{(\alpha)}(x/n)}{n^\alpha}\cong\frac{L_n^{(\alpha)}(x/n)}{n^\alpha}+$$

$$\begin{aligned}
&\frac{M\dfrac{|c|}{n}+\mathcal{O}\left(n^{3/2}\right)}{\dfrac{||\widehat{L}_n^{\alpha}||_\alpha^2}{n!\,(n-1)!\left(L_{n-1}^{(\alpha+1)}(c)\right)^2}+M\dfrac{L_{n-2}^{(\alpha+1)}(c)}{L_{n-1}^{(\alpha+1)}(c)}\left[\dfrac{L_{n-1}^{(\alpha+1)}(c)}{L_{n-2}^{(\alpha+1)}(c)}-\dfrac{L_n^{(\alpha+1)}(c)}{L_{n-1}^{(\alpha+1)}(c)}\right]}\\
&\times\left(\frac{-1}{c}\right)\left(\frac{L_n^{(\alpha)}(x/n)}{n^\alpha}-\frac{L_n^{(\alpha)}(c)}{L_{n-1}^{(\alpha)}(c)}\frac{L_{n-1}^{(\alpha)}(x/n)}{n^\alpha}\right)\\
\cong\;&\frac{L_n^{(\alpha)}(x/n)}{n^\alpha}-\frac{M\dfrac{|c|}{n}\dfrac{1}{c}L_n^{(\alpha)}(c)\left(\dfrac{L_n^{(\alpha)}(x/n)}{n^\alpha}-\left(1+\dfrac{\sqrt{|c|}}{\sqrt{n}}\right)\dfrac{L_n^{(\alpha)}(x/n)}{n^\alpha}\right)}{M\sqrt{|c|}\left(\dfrac{1}{\sqrt{n-1}}-\dfrac{1}{\sqrt{n}}\right)}\\
\cong\;&\frac{L_n^{(\alpha)}(x/n)}{n^\alpha}-2\frac{L_n^{(\alpha)}(x/n)}{n^\alpha}=-x^{-\alpha/2}J_\alpha\left(2\sqrt{x}\right).
\end{aligned}$$

Notice that the addition of a mass point changes the sign in the Mehler-Heine formula that the standard Laguerre polynomials satisfy.

In Figure 4.1 we present the convergence predicted by the above Mehler-Heine type formulas. The first one shows the ratio $L_n^{(\alpha)}(x/n)/n^\alpha$ for three increasing values of n: $n=60$ (red solid), $n=100$ (red dashed), and $n=170$ (red dotted), with parameter $\alpha=1$. The convergence is towards the thick black graph of the function $x^{-\alpha/2}J_\alpha\left(2\sqrt{x}\right)$. The second one shows the ratio $Q_n^{(\alpha)}(x/n)/n^\alpha$ for $\alpha=1$ and other three different increasing values of n: $n=50$ (blue solid), $n=100$ (red dashed), and $n=150$ (red dotted).

Notice that, in this case, the convergence is towards the thick black graph of the function $-x^{-\alpha/2}J_\alpha\left(2\sqrt{x}\right)$, showing how the addition of a mass point changes the sign in the Mehler-Heine formula.

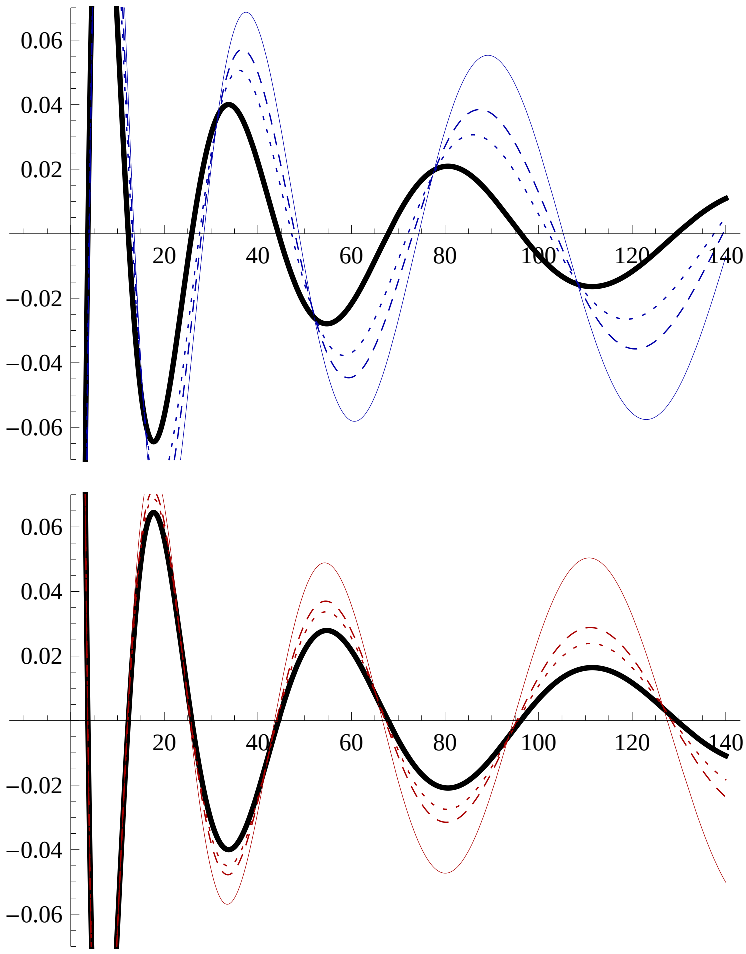


Figure 4.1: Comparison between Mehler-Heine type formulas

4.4 Asymptotics for Krall-Laguerre OPS with m mass points

4.4.1 Outer relative asymptotics

In this section we need two useful lemmas concerning the rate of convergence of the ratio of two classical Laguerre polynomials of different parameter and degree (see Appendix C) outside the support of the measure. Using these two lemmas, we deduce

$$\begin{aligned}
&\frac{L_{n-1}^{(\alpha)}(c_i)}{L_n^{(\alpha)}(c_i)} - \frac{L_{n-1}^{(\alpha)}(c_j)}{L_n^{(\alpha)}(c_j)} = \frac{\sqrt{|c_j|}-\sqrt{|c_i|}}{\sqrt{n}} + \frac{|c_i|-|c_j|}{2n} + \mathcal{O}(n^{-3/2}),\\
&\frac{L_{n-1}^{(\alpha+1)}(c_i)}{L_{n-2}^{(\alpha+1)}(c_i)} - \frac{L_n^{(\alpha)}(c_i)}{L_{n-1}^{(\alpha)}(c_i)} = \frac{1}{2n} + \mathcal{O}\left(n^{-3/2}\right).
\end{aligned} \tag{4.24}$$

On the other hand, from (3.59) we have

$$Q_n^{(\alpha,m)}(x) = L_n^{(\alpha)}(x) - \sum_{j=1}^{m} M_j Q_n^{(\alpha,m)}(c_j) K_{n-1}(c_j,x), \tag{4.25}$$

where $Q_n^{(\alpha,m)}(x) = \frac{(-1)^n}{n!}\widehat{Q}_n^{\alpha,m}(x)$. Dividing by $L_n^{(\alpha)}(x)$ in both hand sides of (4.25), we get

$$\frac{Q_n^{(\alpha,m)}(x)}{L_n^{(\alpha)}(x)} = 1 - \sum_{j=1}^{m} M_j Q_n^{(\alpha)}(c_j)\frac{K_{n-1}(c_j,x)}{L_n^{(\alpha)}(x)}. \tag{4.26}$$

Next, we will analyze

$$\lim_{n\to\infty} \frac{Q_n^{(\alpha,m)}(x)}{L_n^{(\alpha)}(x)} \tag{4.27}$$

when $x \in \mathbb{C}\diagdown\mathbb{R}_+$. In order to prove the existence of such a limit, we will find the solutions of the following linear system

$$\left[1 + a_k K_{n-1}(c_k,c_k)\right] Q_n^{(\alpha,m)}(c_k) + \sum_{\substack{j=1\\ j\neq k}}^{m} M_j K_{n-1}(c_j,c_k) Q_n^{(\alpha,m)}(c_j) = L_n^{(\alpha)}(c_k) \tag{4.28}$$

with $k = 1,2,\dots,m$, obtained from (4.25) where x is evaluated at $c_1, c_2, \dots, c_m$. Let us define

$$P_n^{(\alpha)}(c_j,x) = -M_j Q_n^{(\alpha,m)}(c_j)\frac{K_{n-1}(c_j,x)}{L_n^{(\alpha)}(x)} \tag{4.29}$$

and

$$\lim_{n\to\infty} P_n^{(\alpha)}(c_j,x) = \bar{p}^{(\alpha)}(c_j,x). \tag{4.30}$$

From (4.26) and (4.27) we need to figure out the values of $\bar{p}^{(\alpha)}(c_1,x),\dots,\bar{p}^{(\alpha)}(c_m,x)$ to obtain the outer relative asymptotic for $Q_n^{(\alpha,m)}(x)$. From Cramer's rule, we see that $\{Q_n^{(\alpha,m)}(c_j)\}_{j=1}^m$ are affected by the location of all the m mass points. From (4.29) we have

$$Q_n^{(\alpha,m)}(c_j) = \frac{-L_n^{(\alpha)}(x) P_n^{(\alpha)}(c_j,x)}{M_j K_{n-1}(c_j,x)}$$

and then we replace for all $j = 1, \dots, m$, these expressions in (4.28) to obtain the next linear system in the unknowns $P_n^{(\alpha)}(c_1, x), \dots, P_n^{(\alpha)}(c_m, x)$

$$\left.\begin{array}{ccccccccc} \Phi_n(1,x)P_n^{(\alpha)}(c_1,x) & + & \cdots & + & \Psi_n(1,m,x)P_n^{(\alpha)}(c_m,x) & = & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_n(m,1,x)P_n^{(\alpha)}(c_1,x) & + & \cdots & + & \Phi_n(m,x)P_n^{(\alpha)}(c_m,x) & = & -1 \end{array}\right\}, \tag{4.31}$$

where

$$\Phi_n(i,x) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(c_i)} \frac{1/a_i + K_{n-1}(c_i,c_i)}{K_{n-1}(c_i,x)} \tag{4.32}$$

$$\Psi_n(i,j,x) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(c_j)} \frac{K_{n-1}(c_i,c_j)}{K_{n-1}(c_i,x)}. \tag{4.33}$$

No matter the number of equations of the previous system, in each of the m previous equations we will have always only two different quantities. Only one (in each equation) of the type $\Phi_n(i,x)$ and $m-1$ of the type $\Psi_n(i,j,x)$.

Next we estimate the rate of convergence of (4.32) and (4.33) as $n \to \infty$. Taking into account

$$u - v = (\sqrt{|v|} + \sqrt{|u|})(\sqrt{|v|} - \sqrt{|u|}), \quad \forall u, v \in \mathbb{R}_-,$$

in (4.33) we obtain for $x \in \mathbb{C} \setminus \mathbb{R}_+$

$$\Psi_n(i,j,x) = \frac{(\sqrt{-x} + \sqrt{|c_i|})}{(\sqrt{|c_j|} + \sqrt{|c_i|})} \frac{(\sqrt{-x} - \sqrt{|c_i|})}{(\sqrt{|c_j|} - \sqrt{|c_i|})} \frac{\left(\frac{L_{n-1}^{(\alpha)}(c_i)}{L_n^{(\alpha)}(c_i)} - \frac{L_{n-1}^{(\alpha)}(c_j)}{L_n^{(\alpha)}(c_j)} \right)}{\left(\frac{L_{n-1}^{(\alpha)}(c_i)}{L_n^{(\alpha)}(c_i)} - \frac{L_{n-1}^{(\alpha)}(x)}{L_n^{(\alpha)}(x)} \right)}, \quad i \neq j.$$

From (4.24) we deduce $\Psi_n(i,j,x) = \dfrac{(\sqrt{-x} + \sqrt{|c_i|}) + \mathcal{O}(n^{-1/2})}{(\sqrt{|c_j|} + \sqrt{|c_i|}) + \mathcal{O}(n^{-1/2})}$, and, as a consequence,

$$\lim_{n\to\infty} \Psi_n(i,j,x) = \frac{\sqrt{-x} + \sqrt{|c_i|}}{\sqrt{|c_j|} + \sqrt{|c_i|}}. \tag{4.34}$$

On the other hand,

$$\Phi_n(i,x) = (c_i - x) \frac{\frac{||L_n^{(\alpha)}||_\alpha^2}{n \cdot a_i \left(L_n^{(\alpha)}(c_i)\right)^2} + \frac{L_{n-2}^{(\alpha+1)}(c_i)}{L_n^{(\alpha)}(c_i)} \frac{L_{n-1}^{(\alpha)}(c_i)}{L_n^{(\alpha)}(c_i)} \left(\frac{L_{n-1}^{(\alpha+1)}(c_i)}{L_{n-2}^{(\alpha+1)}(c_i)} - \frac{L_n^{(\alpha)}(c_i)}{L_{n-1}^{(\alpha)}(c_i)} \right)}{\left(\frac{L_{n-1}^{(\alpha)}(c_i)}{L_n^{(\alpha)}(c_i)} - \frac{L_{n-1}^{(\alpha)}(x)}{L_n^{(\alpha)}(x)} \right)}$$

and using (4.24) we get

$$\Phi_n(i,x) = \frac{(\sqrt{-x}+\sqrt{|c_i|})(\sqrt{-x}-\sqrt{|c_i|})\frac{1}{2\sqrt{|c_i|}} + \mathcal{O}(n^{-1/2})}{(\sqrt{-x}-\sqrt{|c_i|}) + \mathcal{O}(n^{-1/2})}.$$

Thus,

$$\lim_{n\to\infty} \Phi_n(i,x) = \frac{\sqrt{-x}+\sqrt{|c_i|}}{2\sqrt{|c_i|}}. \tag{4.35}$$

Next, from (4.30), (4.34), and (4.35), and taking limits, when $n \to \infty$, in both hand sides of (4.31)

$$\left.\begin{array}{rcl} \frac{\sqrt{-x}+\sqrt{|c_1|}}{2\sqrt{|c_1|}}\bar{p}^{(\alpha)}(c_1,x) + \cdots + \frac{\sqrt{-x}+\sqrt{|c_m|}}{\sqrt{|c_1|}+\sqrt{|c_m|}}\bar{p}^{(\alpha)}(c_m,x) & = & -1 \\ \vdots & \vdots & \vdots \\ \frac{\sqrt{-x}+\sqrt{|c_1|}}{\sqrt{|c_m|}+\sqrt{|c_1|}}\bar{p}^{(\alpha)}(c_1,x) + \cdots + \frac{\sqrt{-x}+\sqrt{|c_m|}}{2\sqrt{|c_m|}}\bar{p}^{(\alpha)}(c_m,x) & = & -1 \end{array}\right\}$$

It is not difficult to prove that the m solutions of the above linear system are

$$\bar{p}^{(\alpha)}(c_i,x) = \frac{-2\sqrt{|c_i|}}{\sqrt{-x}+\sqrt{|c_i|}} \prod_{\substack{j=1\\ j\neq i}}^{m} \left(\frac{\sqrt{|c_i|}+\sqrt{|c_j|}}{\sqrt{|c_i|}-\sqrt{|c_j|}}\right), \quad \forall i = 1,\ldots,m.$$

Now, from (4.29) and (4.30) we conclude that, if $x \in \mathbb{C}\setminus\mathbb{R}_+$, then

$$\begin{aligned} \lim_{n\to\infty} \frac{Q_n^{(\alpha,m)}(x)}{L_n^{(\alpha)}(x)} &= 1 + \sum_{i=1}^{m} \lim_{n\to\infty}\left(-a_i Q_n^{(\alpha,m)}(c_i)\frac{K_{n-1}(c_i,x)}{L_n^{(\alpha)}(x)}\right) \\ &= 1 + \sum_{i=1}^{m} \bar{p}^{(\alpha)}(c_i,x) \\ &= 1 + \sum_{i=1}^{m}\left(\frac{-2\sqrt{|c_i|}}{\sqrt{-x}+\sqrt{|c_i|}} \prod_{\substack{j=1\\ j\neq i}}^{m} \frac{\sqrt{|c_i|}+\sqrt{|c_j|}}{\sqrt{|c_i|}-\sqrt{|c_j|}}\right). \end{aligned}$$

From the expression above, we obtain

Theorem 4.2

$$\lim_{n\to\infty} \frac{Q_n^{(\alpha,m)}(x)}{L_n^{(\alpha)}(x)} = \prod_{k=1}^{m}\left(\frac{\sqrt{-x}-\sqrt{|c_k|}}{\sqrt{-x}+\sqrt{|c_k|}}\right) \tag{4.36}$$

uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}_+$.

Proof. The proof is based on the decomposition in partial fractions and the Residue Theorem. To simplify the notation, we write $t_i = \sqrt{|c_i|}, \quad z = \sqrt{-x}$. Thus (4.36) becomes a rational function

$$\lim_{n\to\infty} \frac{Q_n^{(\alpha,m)}(x)}{L_n^{(\alpha)}(x)} = r(z) = \frac{q_m(z)}{p_m(z)},$$

where $q_m(z)$ and $p_m(z)$ are monic polynomials of degree m, i.e.

$$q_m(z) = \prod_{j=1}^{m}(z - t_j), \quad p_m(z) = \prod_{j=1}^{m}(z + t_j).$$

Notice that

$$\frac{q_m(z)}{p_m(z)} = 1 + \frac{[q_m(z) - p_m(z)]}{p_m(z)} \tag{4.37}$$

and the numerator in the above expression is a polynomial of degree at most $m - 1$. In these conditions, $r(z)-1$ is a *proper rational function*, i.e. a ratio between two polynomials such that the degree of the numerator is less than the degree of the denominator. Under the above assumptions, when $-t_i$ are simple zeros of the polynomial $p_m(z)$, it is well known that always exists a decomposition in partial fractions of (4.37) as

$$\frac{[q_m(z) - p_m(z)]}{p_m(z)} = \sum_{i=1}^{m} \frac{A_i}{z + t_i}, \quad \text{where} \quad A_i = \lim_{z\to -t_i} (z + t_i)\frac{[q_m(z) - p_m(z)]}{p_m(z)}.$$

Applying l'Hôpital's rule we have

$$A_i = \frac{[q_m(-t_i) - p_m(-t_i)]}{p'_m(-t_i)} = \frac{\displaystyle\prod_{j=1}^{m}(-t_j - t_i) - \prod_{j=1}^{m}(t_j - t_i)}{\displaystyle\prod_{\substack{j=1\\ j\neq i}}^{m}(t_j - t_i)} = -2t_i \prod_{\substack{j=1\\ j\neq i}}^{m} \frac{t_i + t_j}{t_i - t_j},$$

for all $i = 1, \ldots, m$. Thus, the proof is completed. ∎

Remark 4.1 *Notice that outer relative asymptotics for orthogonal polynomials with respect to perturbations of measures supported on $\mathbb{R}_+$ or $\mathbb{R}$ (non–rescale case) have been studied in connection with rational approximation. In [65], for orthogonal polynomials normalized with the non-standard condition $Q_n(-1) = L_n(-1) = (-1)^n$, the author finds the relative asymptotic behavior, that is an analog of our Theorem 4.2. Notwithstanding* *that the conditions on the measures are general, we infer the relative asymptotic behavior for either monic orthogonal polynomials or orthonormal from [65, Th. 1] is an unsolved problem. The same problem is studied in [66, Th. 3 and Th. 4], but under more restrictive conditions on the modified weight functions. These conditions are not satisfied in our case*

Remark 4.2 *Notice that that according to Hurwitz's theorem, each* c_k *attracts exactly one zero of the polynomial* $Q_n^{(\alpha)}(x)$ *for* n *large enough. In other words, we have exactly one zero in each gap.*

CHAPTER 5

Discrete Sobolev Orthogonal Polynomials

5.1 Introduction

In this chapter we deal with sequences of polynomials orthogonal with respect to the discrete Sobolev inner product

$$\langle f, g\rangle_S = \int_0^{\infty} f(x)g(x)d\mu(x) + Mf(c)g(c) + Nf'(c)g'(c), \tag{5.1}$$

where $\mu(x)$ is a general positive Borel measure supported on $E = [0, +\infty)$, $c \in \mathbb{R}_-$, and $M, N \in \mathbb{R}_+$. Let $\{\widehat{S}_n^{M,N}(x)\}_{n\geq 0}$ denote the MOPS with respect to (5.1). They are known in the literature as *Sobolev-type* or *discrete Sobolev* orthogonal polynomials. It is worth to point out that many properties of the standard orthogonal polynomials are lost when an inner product (5.1) is considered. In a more general framework, for measures supported on the interval $[0, +\infty)$ the zeros can be complex or, if real, they can be located outside $[c, +\infty)$.

First, we obtain the representation of these polynomials in terms of the MOPS with respect to $d\mu(x)$ and $(x-c)^k d\mu(x)$ (when $k = 2, 4$), and we analyze the distribution of the zeros of the MOPS with respect to (5.1). Second, we study the asymptotic properties and the zeros of the MOPS $\{\widehat{S}_n^{M,N}(x)\}_{n\geq 0}$, when $d\mu(x)$ is the standard Laguerre measure.

Third, we provide a matrix interpretation of the Sobolev-type orthonormal polynomial sequence.

In the second half of this chapter, we find several results when the modified measure is the Laguerre classical measure. We obtain some asymptotic results and the zero behavior for the Laguerre Sobolev-type OPS. Moreover, we consider an extremal characterization of certain interesting polynomials which arise from the computations, and that it is quasi-orthogonal of order 2 with respect to the modified Laguerre measure $(x-c)^2x^\alpha e^{-x}dx$. The zeros of this polynomial are the limit of the zeros of $\{\widehat{S}_n^{M,N}(x)\}_{n\geq 0}$, when $M=0$ and $N\to\infty$ in (5.1).

5.1.1 Auxiliary results

Next, we prove some useful results concerning k-iterated orthogonal polynomials to be used in the sequel. We remind that these polynomials were introduced in Section 2.2.1.

Proposition 5.1 *Let* $D_{n-1}^{[k]}=\det[a_{ij}^{[k]}]_{0\leq i,j\leq n-1}$, *where* $a_{ij}^{[k]}=d_{i+j}^{[k]}$, $k\in\mathbb{N}$. *Then, the following relation holds*

$$D_{n-1}^{[k]}=(-1)^nD_{n-1}^{[k-1]}\widehat{P}_n^{[k-1]}(c), \tag{5.2}$$

with $D_{n-1}^{[0]}=\Delta_{n-1}$.

Proof. For $n\geq 1$ and $k\in\mathbb{N}$,

$$\widehat{P}_n^{[k-1]}(x)=\frac{1}{D_{n-1}^{[k-1]}}\begin{vmatrix} d_0^{[k-1]} & d_1^{[k-1]} & \cdots & d_n^{[k-1]}\\ d_1^{[k-1]} & d_2^{[k-1]} & \cdots & d_{n+1}^{[k-1]}\\ \vdots & \vdots & \ddots & \vdots\\ d_{n-1}^{[k-1]} & d_n^{[k-1]} & \cdots & d_{2n-1}^{[k-1]}\\ 1 & x & \cdots & x^n \end{vmatrix}, \tag{5.3}$$

with $\widehat{P}_n=\widehat{P}_n^{[0]}$. The determinant in (5.3) becomes (see [99, formula (2.2.9)])

$$\widehat{P}_n^{[k-1]}(x)=\frac{(-1)^n}{D_{n-1}^{[k-1]}}\begin{vmatrix} d_1^{[k-1]}-d_0^{[k-1]}x & d_2^{[k-1]}-d_1^{[k-1]}x & \cdots & d_n^{[k-1]}-d_{n-1}^{[k-1]}x\\ d_2^{[k-1]}-d_1^{[k-1]}x & d_3^{[k-1]}-d_2^{[k-1]}x & \cdots & d_{n+1}^{[k-1]}-d_n^{[k-1]}x\\ \vdots & \vdots & \ddots & \vdots\\ d_n^{[k-1]}-d_{n-1}^{[k-1]}x & d_{n+1}^{[k-1]}-d_n^{[k-1]}x & \cdots & d_{2n-1}^{[k-1]}-d_{2n-2}^{[k-1]}x \end{vmatrix}.$$

Now, by using (2.19), (5.2) follows. ■

Next we will compute some integrals involving $\widehat{P}_n^{[k]}$.

Proposition 5.2 *The following relations hold*

(i)

$$\int_0^\infty \widehat{P}_n^{[k]}(x)(x-c)^{k-1}d\mu(x) = \frac{||\widehat{P}_n^{[k-1]}||^2_{\mu,[k-1]}}{\widehat{P}_n^{[k-1]}(c)} = \begin{cases} \dfrac{||\widehat{P}_n||^2_\mu}{\widehat{P}_n(c)}, & k=1, \\ \dfrac{(-1)^{k-1}}{\widehat{P}_n^{[k-1]}(c)} \displaystyle\prod_{i=1}^{k-1} \frac{\widehat{P}_{n+1}^{[i-1]}(c)}{\widehat{P}_n^{[i-1]}(c)} ||\widehat{P}_n||^2_\mu, & k \geq 2; \end{cases}$$

(ii)

$$\int_0^\infty \widehat{P}_n^{[k]}(x)(x-c)^{k-2}d\mu(x) = \frac{[\widehat{P}_{n+1}^{[k-2]}]'(c)||\widehat{P}_n^{[k-2]}||^2_{\mu,[k-2]}}{\widehat{P}_n^{[k-1]}(c)\widehat{P}_n^{[k-2]}(c)} = \begin{cases} \dfrac{[\widehat{P}_{n+1}]'(c)||\widehat{P}_n||^2_\mu}{\widehat{P}_n(c)\widehat{P}_n^{[1]}(c)}, & k=2, \\ \dfrac{(-1)^k[\widehat{P}_{n+1}^{[k-2]}]'(c)}{\widehat{P}_n^{[k-1]}(c)\widehat{P}_n^{[k-2]}(c)} \prod_{i=1}^{k-2} \dfrac{\widehat{P}_{n+1}^{[i-1]}(c)}{\widehat{P}_n^{[i-1]}(c)} ||\widehat{P}_n||^2_\mu, & k \geq 3. \end{cases}$$

Proof. (i) Using (2.19) in a recursive way, as well as some properties of determinants, we have

$$D_{n-1}^{[k]} \int_0^\infty \widehat{P}_n^{[k]}(x)(x-c)^{k-1}d\mu(x) = \begin{vmatrix} d_0^{[k]} & d_1^{[k]} & d_2^{[k]} & \cdots & d_n^{[k]} \\ d_1^{[k]} & d_2^{[k]} & d_3^{[k]} & \cdots & d_{n+1}^{[k]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{n-1}^{[k]} & d_n^{[k]} & d_{n+1}^{[k]} & \cdots & d_{2n-1}^{[k]} \\ d_0^{[k-1]} & d_1^{[k-1]} & d_2^{[k-1]} & \cdots & d_n^{[k-1]} \end{vmatrix}$$

$$= \begin{vmatrix} d_1^{[k-1]} & d_2^{[k-1]} & d_3^{[k-1]} & \cdots & d_{n+1}^{[k-1]} \\ d_2^{[k-1]} & d_3^{[k-1]} & d_4^{[k-1]} & \cdots & d_{n+2}^{[k-1]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_n^{[k-1]} & d_{n+1}^{[k-1]} & d_{n+2}^{[k-1]} & \cdots & d_{2n}^{[k-1]} \\ d_0^{[k-1]} & d_1^{[k-1]} & d_2^{[k-1]} & \cdots & d_n^{[k-1]} \end{vmatrix} = (-1)^n D_n^{[k-1]}.$$

On the other hand,

$$||\widehat{P}_n^{[k-1]}||^2_{\mu,[k-1]} = \int_0^\infty x^n \widehat{P}_n^{[k-1]}(x)(x-c)^{k-1} d\mu(x) = \frac{D_n^{[k-1]}}{D_{n-1}^{[k-1]}}$$

and by using (5.2)

$$\int_0^\infty \widehat{P}_n^{[k]}(x)(x-c)^{k-1} d\mu(x) = \frac{(-1)^n D_{n-1}^{[k-1]} ||\widehat{P}_n^{[k-1]}||^2_{\mu,[k-1]}}{D_{n-1}^{[k]}} = \frac{||\widehat{P}_n^{[k-1]}||^2_{\mu,[k-1]}}{\widehat{P}_n^{[k-1]}(c)}. \tag{5.4}$$

On the other hand, we get (see [99, Theorem 2.5])

$$(x-c)\widehat{P}_n^{[k]}(x) = \widehat{P}_{n+1}^{[k-1]}(x) - \frac{\widehat{P}_{n+1}^{[k-1]}(c)}{\widehat{P}_n^{[k-1]}(c)} \widehat{P}_n^{[k-1]}(x). \tag{5.5}$$

Therefore,

$$||\widehat{P}_n^{[k]}||^2_{\mu,[k]} = -\frac{\widehat{P}_{n+1}^{[k-1]}(c)}{\widehat{P}_n^{[k-1]}(c)} ||\widehat{P}_n^{[k-1]}||^2_{\mu,[k-1]}. \tag{5.6}$$

By using the above relation in a recursive way we obtain

$$||\widehat{P}_n^{[k]}||^2_{\mu,[k]} = (-1)^k \prod_{i=1}^{k} \frac{\widehat{P}_{n+1}^{[i-1]}(c)}{\widehat{P}_n^{[i-1]}(c)} ||\widehat{P}_n||^2_{\mu}, \quad k \geq 2. \tag{5.7}$$

Combining (5.4) and (5.7), our statement follows.

(ii) We have

$$[\widehat{P}_{n+1}^{[k-2]}]'(x) = \frac{1}{D_n^{[k-2]}} \begin{vmatrix} d_0^{[k-2]} & d_1^{[k-2]} & d_2^{[k-2]} & \cdots & d_{n+1}^{[k-2]} \\ d_1^{[k-2]} & d_2^{[k-2]} & d_3^{[k-2]} & \cdots & d_{n+2}^{[k-2]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_n^{[k-2]} & d_{n+1}^{[k-2]} & d_{n+2}^{[k-2]} & \cdots & d_{2n+1}^{[k-2]} \\ 0 & 1 & 2x & \cdots & nx^{n-1} \end{vmatrix}, \; n \geq 0. \tag{5.8}$$

Now, adding to the last column the n-th and $(n-1)$-th columns multiplied by $-2x$ and x^2, respectively, and repeating this operation for each of the preceding columns, we obtain

$$[\widehat{P}_{n+1}^{[k-2]}]'(x) = \frac{1}{D_n^{[k-2]}} \times$$

$$\begin{vmatrix} d_0^{[k-2]} & d_1^{[k-2]} & d_2^{[k-2]} - 2xd_1^{[k-2]} + x^2 d_0^{[k-2]} & \cdots & d_{n+1}^{[k-2]} - 2xd_n^{[k-2]} + x^2 d_{n-1}^{[k-2]} \\ d_1^{[k-2]} & d_2^{[k-2]} & d_3^{[k-2]} - 2xd_2^{[k-2]} + x^2 d_1^{[k-2]} & \cdots & d_{n+2}^{[k-2]} - 2xd_{n+1}^{[k-2]} + x^2 d_n^{[k-2]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_n^{[k-2]} & d_{n+1}^{[k-2]} & d_{n+2}^{[k-2]} - 2xd_{n+1}^{[k-2]} + x^2 d_n^{[k-2]} & \cdots & d_{2n+1}^{[k-2]} - 2xd_{2n}^{[k-2]} + x^2 d_{2n-1}^{[k-2]} \\ 0 & 1 & 0 & \cdots & 0 \end{vmatrix}$$

$$= \frac{1}{D_n^{[k-2]}} \times$$

$$\begin{vmatrix} d_2^{[k-2]} - 2xd_1^{[k-2]} + x^2 d_0^{[k-2]} & d_3^{[k-2]} - 2xd_2^{[k-2]} + x^2 d_1^{[k-2]} & \cdots & d_{n+2}^{[k-2]} - 2xd_{n+1}^{[k-2]} + x^2 d_n^{[k-2]} \\ d_3^{[k-2]} - 2xd_2^{[k-2]} + x^2 d_1^{[k-2]} & d_4^{[k-2]} - 2xd_3^{[k-2]} + x^2 d_2^{[k-2]} & \cdots & d_{n+3}^{[k-2]} - 2xd_{n+2}^{[k-2]} + x^2 d_{n+1}^{[k-2]} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n+1}^{[k-2]} - 2xd_n^{[k-2]} + x^2 d_{n-1}^{[k-2]} & d_{n+2}^{[k-2]} - 2xd_{n+1}^{[k-2]} + x^2 d_n^{[k-2]} & \cdots & d_{2n+1}^{[k-2]} - 2xd_{2n}^{[k-2]} + x^2 d_{2n-1}^{[k-2]} \\ d_0^{[k-2]} & d_1^{[k-2]} & \cdots & d_n^{[k-2]} \end{vmatrix} \tag{5.9}$$

On the other hand,

$$D_{n-1}^{[k]} \int_0^\infty \widehat{P}_n^{[k]}(x)(x-c)^{k-2} d\mu(x) = \begin{vmatrix} d_0^{[k]} & d_1^{[k]} & \cdots & d_n^{[k]} \\ d_1^{[k]} & d_2^{[k]} & \cdots & d_{n+1}^{[k]} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n-1}^{[k]} & d_n^{[k]} & \cdots & d_{2n-1}^{[k]} \\ d_0^{[k-2]} & d_1^{[k-2]} & \cdots & d_n^{[k-2]} \end{vmatrix},$$

and by using (5.2), (5.7), and (5.9) we get

$$\begin{aligned} \int_0^\infty \widehat{P}_n^{[k]}(x)(x-c)^{k-2} d\mu(x) &= \frac{D_n^{[k-2]}\,[\widehat{P}_{n+1}^{[k-2]}]'(c)}{D_{n-1}^{[k-2]}\,\widehat{P}_n^{[k-1]}(c)\widehat{P}_n^{[k-2]}(c)} \\ &= \frac{[\widehat{P}_{n+1}^{[k-2]}]'(c)\,||\widehat{P}_n^{[k-2]}||^2_{\mu,[k-2]}}{\widehat{P}_n^{[k-1]}(c)\widehat{P}_n^{[k-2]}(c)} \end{aligned}$$

$$= \frac{(-1)^k[\widehat{P}_{n+1}^{[k-2]}]'(c)}{\widehat{P}_n^{[k-1]}(c)\widehat{P}_n^{[k-2]}(c)} \prod_{i=1}^{k-2} \frac{\widehat{P}_{n+1}^{[i-1]}(c)}{\widehat{P}_n^{[i-1]}(c)} ||\widehat{P}_n||_\mu^2.$$

■

Finally, we state some results concerning the zeros $x_{n,r}^{[k]}$, $r = 1, \dots, n$ of $\widehat{P}_n^{[k]}(x)$.

Proposition 5.3 *(i) The zeros of $\widehat{P}_n^{[k]}(x)$ interlace with both the zeros of $\widehat{P}_{n+1}^{[k-1]}(x)$ and $\widehat{P}_n^{[k-1]}(x)$, i.e.*

$$x_{n,r}^{[k-1]} < x_{n,r}^{[k]} < x_{n+1,r+1}^{[k-1]}, \qquad r = 1, 2, ..., n.$$

(ii) Between two consecutive zeros of $\widehat{P}_{n+1}^{[k-2]}$, $k \geq 2$, there is exactly one zero of $\widehat{P}_n^{[k]}$.

(iii) It holds

$$sign\, \widehat{P}_n^{[k-2]}(x_{n-1,r}^{[k]}) = (-1)^{n-r} = -sign\, \widehat{P}_{n-2}^{[k+2]}(x_{n-1,r}^{[k]}), \qquad r = 1, 2, ..., n-1.$$

Proof. (i) Here we will use the same argument as in [15, p. 65] (see also Appendix B). It is well known that the zeros of $\widehat{P}_{n+1}^{[k-1]}(x)$ interlace with the zeros of $\widehat{P}_n^{[k-1]}(x)$, i.e.

$$0 < x_{n+1,1}^{[k-1]} < x_{n,1}^{[k-1]} < x_{n+1,2}^{[k-1]} < \cdots < x_{n,n}^{[k-1]} < x_{n+1,n+1}^{[k-1]} < \infty.$$

From (5.2) $\frac{\widehat{P}_{n+1}^{[k-1]}(c)}{\widehat{P}_n^{[k-1]}(c)} < 0$ and taking into account (5.5)

$$\begin{aligned} sign\, \widehat{P}_n^{[k]}(x_{n+1,r}^{[k-1]}) &= sign\, \widehat{P}_n^{[k-1]}(x_{n+1,r}^{[k-1]}) = (-1)^{n-r+1},\ r = 1, 2, ..., n+1, \\ sign\, \widehat{P}_n^{[k]}(x_{n,r}^{[k-1]}) &= sign\, \widehat{P}_{n+1}^{[k-1]}(x_{n,r}^{[k-1]}) = (-1)^{n-r+1},\ r = 1, 2, ..., n, \end{aligned}$$

Thus, there exist zeros $x_{n,r}^{[k]}$, $r = 2, 3, ..., n$, of $\widehat{P}_n^{[k]}(x)$ satisfying

$$x_{n,r}^{[k-1]} < x_{n,r}^{[k]} < x_{n+1,r+1}^{[k-1]},\ r = 1, 2, ..., n.$$

(ii) Notice that this result was proved in Theorem 3.2 for $k = 2$. By using (5.5) and the recurrence relation, we obtain

$$(x-c)^2 \widehat{P}_n^{[k]}(x) = (d_{1,n}x + d_{2,n})\widehat{P}_{n+1}^{[k-2]}(x) + d_{3,n}\widehat{P}_n^{[k-2]}(x).$$

Since $\widehat{P}_{n+1}^{[k-2]}(c) \neq 0$ we have $d_{3,n} \neq 0$. Now, the rest of the proof can be done in a similar way as in [85, Lemma 6.1] (see also [86, Lemma 4.1]).

(iii) From *(ii)*

$$x_{n,r}^{[k-2]} < x_{n-1,r}^{[k]} < x_{n,r+1}^{[k-2]}, \ r = 1, 2, ..., n-1.$$

Therefore,

$$sign\,\widehat{P}_n^{[k-2]}(x_{n-1,r}^{[k]}) = (-1)^{n-r}.$$

Again, according to *(ii)*

$$x_{n-2,r-1}^{[k+2]} < x_{n-1,r}^{[k]} < x_{n-2,r}^{[k+2]}, \ r = 1, 2, ..., n-2,$$

$$x_{n-2,n-2}^{[k+2]} < x_{n-1,n-1}^{[k]}.$$

Therefore,

$$sign\,\widehat{P}_{n-2}^{[k+2]}(x_{n-1,r}^{[k]}) = (-1)^{n-r-1}$$

and

$$sign\,\widehat{P}_{n-2}^{[k+2]}(x_{n-1,n-1}^{[k]}) = 1.$$

As a conclusion,

$$sign\,\widehat{P}_n^{[k-2]}(x_{n-1,r}^{[k]}) = -sign\,\widehat{P}_{n-2}^{[k+2]}(x_{n-1,r}^{[k]}), \ r = 1, 2, ..., n-1.$$

■

5.2 Sobolev-type orthogonal polynomials

In this section we deal with general measures with unbounded support $[0, +\infty)$. The Laguerre measure is a particular case of this type of measures, and later on, we will get some results when μ is the Laguerre measure.

5.2.1 Some connection formulas

As we have seen in previous chapters, the connection formulas are the main tool to study the analytical properties of new families of OPS, in terms of other families of OPS with well-known analytical properties. Indeed, the problem of finding such expressions is called "the connection problem", and it is of great importance in this context.

In this subsection we will give three connection formulas which will be useful later.

Connection formula (I)

The most common way to represent the Sobolev-type orthogonal polynomials is using the kernel and its derivatives. We will prove the following

Theorem 5.1 (Connection formula (I)) *Let* $\{\widehat{S}_n^{M,N}(x)\}_{n\geq 0}$ *be the MOPS with respect to (5.1). Then the polynomials* $\widehat{S}_n^{M,N}(x)$ *can be represented by*

$$(x-a)^2\widehat{S}_n^{M,N}(x) = A(x;n)\widehat{P}_n(x) + B(x;n)\widehat{P}_{n-1}(x),$$

where

$$\begin{aligned} A(x;n) &= (x-c)^2 - \frac{N[\widehat{S}_n^{M,N}]'(c)[\widehat{P}_{n-1}]'(c) + M\widehat{S}_n^{M,N}(c)\widehat{P}_{n-1}(c)}{||\widehat{P}_{n-1}||_\mu^2}(x-c) \\ &\quad - \frac{N[\widehat{S}_n^{M,N}]'(c)\widehat{P}_{n-1}(c)}{||\widehat{P}_{n-1}||_\mu^2}, \\ B(x;n) &= \frac{M\widehat{S}_n^{M,N}(c)\widehat{P}_n(c) + N[\widehat{S}_n^{M,N}]'(c)[\widehat{P}_n]'(c)}{||\widehat{P}_{n-1}||_\mu^2}(x-c) + \frac{N[\widehat{S}_n^{M,N}]'(c)\widehat{P}_n(c)}{||\widehat{P}_{n-1}||_\mu^2}, \end{aligned}$$

and

$$\begin{aligned} \widehat{S}_n^{M,N}(c) &= \frac{\begin{vmatrix} \widehat{P}_n(c) & NK_{n-1}^{(0,1)}(c,c) \\ [\widehat{P}_n]'(c) & 1+NK_{n-1}^{(1,1)}(c,c) \end{vmatrix}}{\begin{vmatrix} 1+MK_{n-1}(c,c) & NK_{n-1}^{(0,1)}(c,c) \\ MK_{n-1}^{(1,0)}(c,c) & 1+NK_{n-1}^{(1,1)}(c,c) \end{vmatrix}}, \\ [\widehat{S}_n^{M,N}]'(c) &= \frac{\begin{vmatrix} 1+MK_{n-1}(c,c) & \widehat{P}_n(c) \\ MK_{n-1}^{(1,0)}(c,c) & [\widehat{P}_n]'(c) \end{vmatrix}}{\begin{vmatrix} 1+MK_{n-1}(c,c) & NK_{n-1}^{(0,1)}(c,c) \\ MK_{n-1}^{(1,0)}(c,c) & 1+NK_{n-1}^{(1,1)}(c,c) \end{vmatrix}}. \end{aligned}$$

Proof. Let $\{\widehat{S}_n^{M,N}(x)\}_{n\geq 0}$ denote the SMOP with respect to the discrete Sobolev inner product (5.1), then we can expand the polynomial $\widehat{S}_n^{M,N}(x)$ as follows

$$\widehat{S}_n^{M,N}(x) = \widehat{P}_n(x) + \sum_{i=0}^{n-1} a_{n,i}\widehat{P}_i(x), \tag{5.10}$$

where

$$a_{n,j} = \frac{\langle \widehat{P}_j(x), \widehat{S}_n^{M,N}(x)\rangle_\mu}{||\widehat{P}_j||_\mu^2}, \quad 0 \leq j \leq n-1. \tag{5.11}$$

Thus, using (2.3) and (2.9), (5.10) becomes

$$\widehat{S}_n^{M,N}(x) = \widehat{P}_n(x) - M\widehat{S}_n^{M,N}(c)K_{n-1}(x,c) - N[\widehat{S}_n^{M,N}]'(c)K_{n-1}^{(0,1)}(x,c). \tag{5.12}$$

Next, we need to find expressions for $\widehat{S}_n^{M,N}(c)$ and $[\widehat{S}_n^{M,N}]'(c)$. We first derive (5.12) with respect to x

$$[\widehat{S}_n^{M,N}]'(x) = [\widehat{P}_n(x)]' - M\widehat{S}_n^{M,N}(c)K_{n-1}^{(1,0)}(x,c) - N[\widehat{S}_n^{M,N}]'(c)K_{n-1}^{(1,1)}(x,c) \tag{5.13}$$

and $x = c$ in (5.12) and (5.13) a linear system

$$\begin{bmatrix} 1 + MK_{n-1}(c,c) & NK_{n-1}^{(0,1)}(c,c) \\ MK_{n-1}^{(1,0)}(c,c) & 1 + NK_{n-1}^{(1,1)}(c,c) \end{bmatrix} \begin{bmatrix} \widehat{S}_n^{M,N}(c) \\ [\widehat{S}_n^{M,N}]'(c) \end{bmatrix} = \begin{bmatrix} \widehat{P}_n(c) \\ [\widehat{P}_n]'(c) \end{bmatrix}$$

in the unknowns $\widehat{S}_n^{M,N}(c)$ and $[\widehat{S}_n^{M,N}]'(c)$, whose solutions are

$$\widehat{S}_n^{M,N}(c) = \frac{\begin{vmatrix} \widehat{P}_n(c) & NK_{n-1}^{(0,1)}(c,c) \\ [\widehat{P}_n]'(c) & 1 + NK_{n-1}^{(1,1)}(c,c) \end{vmatrix}}{\begin{vmatrix} 1 + MK_{n-1}(c,c) & NK_{n-1}^{(0,1)}(c,c) \\ MK_{n-1}^{(1,0)}(c,c) & 1 + NK_{n-1}^{(1,1)}(c,c) \end{vmatrix}},$$

$$[\widehat{S}_n^{M,N}]'(c) = \frac{\begin{vmatrix} 1 + MK_{n-1}(c,c) & \widehat{P}_n(c) \\ MK_{n-1}^{(1,0)}(c,c) & [\widehat{P}_n]'(c) \end{vmatrix}}{\begin{vmatrix} 1 + MK_{n-1}(c,c) & NK_{n-1}^{(0,1)}(c,c) \\ MK_{n-1}^{(1,0)}(c,c) & 1 + NK_{n-1}^{(1,1)}(c,c) \end{vmatrix}}.$$

On the other hand, we multiply both sides of (5.13) by $(x-c)^2$

$$(x-a)^2\widehat{S}_n^{M,N}(x) =$$

$$(x-c)^2\widehat{P}_n(x) - M\widehat{S}_n^{M,N}(c)(x-c)^2K_{n-1}(x,c) - N[\widehat{S}_n^{M,N}]'(c)(x-c)^2K_{n-1}^{(0,1)}(x,c).$$

Since (2.4) and (2.7), we have

$$(x-c)^2K_{n-1}^{(0,1)}(x,c) =$$

$$\frac{\widehat{P}_n(x)\widehat{P}_{n-1}(c)}{||\widehat{P}_{n-1}||_\mu^2} - \frac{\widehat{P}_{n-1}(x)\widehat{P}_n(c)}{||\widehat{P}_{n-1}||_\mu^2} + (x-c)\frac{\widehat{P}_n(x)[\widehat{P}_{n-1}]'(c)}{||\widehat{P}_{n-1}||_\mu^2} - (x-c)\frac{\widehat{P}_{n-1}(x)[\widehat{P}_n]'(c)}{||\widehat{P}_{n-1}||_\mu^2}$$

and

$$(x-c)K_{n-1}(x,c) = \frac{\widehat{P}_n(x)\widehat{P}_{n-1}(c)}{||\widehat{P}_n||_\mu^2} - \frac{\widehat{P}_{n-1}(x)\widehat{P}_n(c)}{||\widehat{P}_n||_\mu^2}.$$

Hence

$$(x-a)^2\widehat{S}_n^{M,N}(x) = A(x;n)\widehat{P}_n(x) + B(x;n)\widehat{P}_{n-1}(x),$$

where $A(x;n)$ and $B(x;n)$ are the following two polynomials

$$\begin{aligned} A(x;n) &= (x-c)^2 - \frac{N[\widehat{S}_n^{M,N}]'(c)[\widehat{P}_{n-1}]'(c) + M\widehat{S}_n^{M,N}(c)\widehat{P}_{n-1}(c)}{||\widehat{P}_{n-1}||_\mu^2}(x-c) \\ &\quad - \frac{N[\widehat{S}_n^{M,N}]'(c)\widehat{P}_{n-1}(c)}{||\widehat{P}_{n-1}||_\mu^2}, \\ B(x;n) &= \frac{M\widehat{S}_n^{M,N}(c)\widehat{P}_n(c) + N[\widehat{S}_n^{M,N}]'(c)[\widehat{P}_n]'(c)}{||\widehat{P}_{n-1}||_\mu^2}(x-c) + \frac{N[\widehat{S}_n^{M,N}]'(c)\widehat{P}_n(c)}{||\widehat{P}_{n-1}||_\mu^2}, \end{aligned}$$

of degree 2 and 1 respectively. ■

Connection formula (II)

Next we prove that the Sobolev-type orthogonal polynomials $\{\widehat{S}_n^{M,N}(x)\}_{n\geq 0}$ can be expressed in terms of polynomials orthogonal with respect to the measures $d\mu(x)$ and $(x-c)^k d\mu(x)$. Moreover, the behavior of the coefficients $A_{n,1}$ and $A_{n,2}$ is studied with more detail.

Theorem 5.2 (Connection Formula (II)) *Let $M \geq 0$ and $N \geq 0$. There are real constants $A_{n,1}$ and $A_{n,2}$ such that*

$$\widehat{S}_n^{M,N}(x) = \widehat{P}_n(x) + A_{n,1}(x-c)\widehat{P}_{n-1}^{[2]}(x) + A_{n,2}(x-c)^2\widehat{P}_{n-2}^{[4]}(x), \tag{5.14}$$

where

$$\begin{aligned} A_{n,1} &= \frac{NI_{2,n}(c)[\widehat{P}_n]'(c) - MI_{3,n}(c)\widehat{P}_n(c)}{I_{1,n}(c)I_{3,n}(c) - NI_{2,n}(c)\widehat{P}_{n-1}^{[2]}(c)}, \\ A_{n,2} &= \frac{MN\widehat{P}_n(c)\widehat{P}_{n-1}^{[2]}(c) - NI_{1,n}(c)[\widehat{P}_n]'(c)}{I_{1,n}(c)I_{3,n}(c) - NI_{2,n}(c)\widehat{P}_{n-1}^{[2]}(c)}, \end{aligned}$$

$$\begin{aligned} I_{1,n}(c) &= -\frac{\widehat{P}_n(c)}{K_{n-1}(c,c)}, \\ I_{2,n}(c) &= \frac{\widehat{P}_{n-1}(c)\widehat{P}^{[1]}_{n-1}(c)[\widehat{P}^{[2]}_{n-1}]'(c)}{\widehat{P}_{n-2}(c)\widehat{P}^{[1]}_{n-2}(c)\widehat{P}^{[2]}_{n-2}(c)\widehat{P}^{[3]}_{n-2}(c)}||\widehat{P}_{n-2}||^2_\mu, \\ I_{3,n}(c) &= -\frac{\widehat{P}_{n-1}(c)\widehat{P}^{[1]}_{n-1}(c)\widehat{P}^{[2]}_{n-1}(c)}{\widehat{P}_{n-2}(c)\widehat{P}^{[1]}_{n-2}(c)\widehat{P}^{[2]}_{n-2}(c)\widehat{P}^{[3]}_{n-2}(c)}||\widehat{P}_{n-2}||^2_\mu. \end{aligned}$$

Proof. We will prove that

$$\langle \widehat{S}^{M,N}_n, (x-c)^k\rangle_S = 0$$

for $k = 0, 1, \ldots, n-1$.For $k \geq 2$ and $n > k$,

$$\begin{aligned} \langle \widehat{S}^{M,N}_n, (x-c)^k\rangle_S &= \int_0^\infty \widehat{S}^{M,N}_n(x)(x-c)^k d\mu(x) \\ &= \int_0^\infty \widehat{P}_n(x)(x-c)^k d\mu(x) + A_{n,1}\int_0^\infty (x-c)^2\widehat{P}^{[2]}_{n-1}(x)(x-c)^{k-1}d\mu(x) \\ &+ A_{n,2}\int_0^\infty (x-c)^4\widehat{P}^{[4]}_{n-2}(x)(x-c)^{k-2}d\mu(x) \\ &= 0. \end{aligned}$$

Now, let us consider $k = 0$ and $n \geq 1$. We have

$$\begin{aligned} \langle \widehat{S}^{M,N}_n, 1\rangle_S &= \int_0^\infty \widehat{S}^{M,N}_n(x)d\mu(x) + M\widehat{S}^{M,N}_n(c) \\ &= A_{n,1}\int_0^\infty (x-c)\widehat{P}^{[2]}_{n-1}(x)d\mu(x) + A_{n,2}\int_0^\infty (x-c)^2\widehat{P}^{[4]}_{n-2}(x)d\mu(x) + M\widehat{P}_n(c). \end{aligned}$$

On the other hand, by using Proposition 5.2(i)

$$I_{1,n}(c) = \int_0^\infty \widehat{P}^{[2]}_{n-1}(x)(x-c)d\mu(x) = -\frac{\widehat{P}_n(c)}{\widehat{P}_{n-1}(c)\widehat{P}^{[1]}_{n-1}(c)}||\widehat{P}_{n-1}||^2_\mu, \tag{5.15}$$

taking derivatives in (5.5) and then substitute $x = c$, we get

$$\widehat{P}^{[k]}_{n-1}(c) = [\widehat{P}^{[k-1]}_n]'(c) - \frac{\widehat{P}^{[k-1]}_n(c)}{\widehat{P}^{[k-1]}_{n-1}(c)}[\widehat{P}^{[k-1]}_{n-1}]'(c). \tag{5.16}$$

Combining (2.5), (5.15), and (5.16), we get

$$I_{1,n}(c) = \frac{-\widehat{P}_n(c)}{K_{n-1}(c,c)}.$$

Using Proposition 5.2(ii)

$$\begin{aligned} I_{2,n}(c) &= \int_0^\infty \widehat{P}^{[4]}_{n-2}(x)(x-c)^2 d\mu(x) = \frac{[\widehat{P}^{[2]}_{n-1}]'(c)||\widehat{P}^{[2]}_{n-2}||^2_{\mu,[2]}}{\widehat{P}^{[2]}_{n-2}(c)\widehat{P}^{[3]}_{n-2}(c)} \\ &= \frac{\widehat{P}_{n-1}(c)\widehat{P}^{[1]}_{n-1}(c)\,[\widehat{P}^{[2]}_{n-1}]'(c)}{\widehat{P}_{n-2}(c)\widehat{P}^{[1]}_{n-2}(c)\widehat{P}^{[2]}_{n-2}(c)\widehat{P}^{[3]}_{n-2}(c)}||\widehat{P}_{n-2}||^2_{\mu}. \end{aligned} \tag{5.17}$$

Therefore,

$$\langle \widehat{S}^{M,N}_n, 1\rangle_S = A_{n,1}I_{1,n}(c) + A_{n,2}I_{2,n}(c) + M\widehat{P}_n(c).$$

In the same way, for $k=1$ and $n\geq 2$, we have

$$\begin{aligned} \langle \widehat{S}^{M,N}_n, (x-c)\rangle_S &= \int_0^\infty \widehat{S}^{M,N}_n(x)(x-c)d\mu(x) + N\,[\widehat{S}^{M,N}_n]'(c) \\ &= A_{n,2}I_{3,n}(c) + NA_{n,1}\widehat{P}^{[2]}_{n-1}(c) + N\,[\widehat{P}_n]'(c), \end{aligned}$$

where

$$\begin{aligned} I_{3,n}(c) = \int_0^\infty (x-c)^3\widehat{P}^{[4]}_{n-2}(x)d\mu(x) &= \frac{||\widehat{P}^{[3]}_{n-2}||^2_{\mu,[3]}}{\widehat{P}^{[3]}_n(c)} \\ &= -\frac{\widehat{P}_{n-1}(c)\widehat{P}^{[1]}_{n-1}(c)\widehat{P}^{[2]}_{n-1}(c)}{\widehat{P}_{n-2}(c)\widehat{P}^{[1]}_{n-2}(c)\widehat{P}^{[2]}_{n-2}(c)\widehat{P}^{[3]}_{n-2}(c)}||\widehat{P}_{n-2}||^2_{\mu}. \end{aligned}$$

Finally, using the expressions of $A_{n,1}$ and $A_{n,2}$, our statement follows. ∎

Next, we will study the behavior of the coefficients $A_{n,1}$ and $A_{n,2}$.

Proposition 5.4 (i)

$$I_{1,n}(c)I_{3,n}(c) - NI_{2,n}(c)\widehat{P}^{[2]}_{n-1}(c) = -I_{2,n}(c)\widehat{P}^{[2]}_{n-1}(c)\left(N+\alpha_n\beta_n\right),$$

where $0<\alpha_n = \dfrac{I_{1,n}(c)}{\widehat{P}^{[2]}_{n-1}(c)} < d^{[1]}_0,\ \dfrac{d^{[3]}_0}{d^{[2]}_0} < \dfrac{-[\widehat{P}^{[2]}_{n-1}]'(c)}{\widehat{P}^{[2]}_{n-1}(c)} = \dfrac{I_{2,n}(c)}{I_{3,n}(c)} = \dfrac{1}{\beta_n} < -\dfrac{n}{c},$

(ii)

$$NI_{2,n}(c)[\widehat{P}_n]'(c) - MI_{3,n}(c)\widehat{P}_n(c) = I_{2,n}(c)[\widehat{P}_n]'(c)\left(N+M\beta_n\gamma_n\right),$$

where $\dfrac{d^{[1]}_0}{m_0} < \dfrac{-[\widehat{P}_n]'(c)}{\widehat{P}_n(c)} = \dfrac{1}{\gamma_n} < -\dfrac{n}{c},$

(iii)
$$MN\widehat{P}_n(c)\widehat{P}^{[2]}_{n-1}(c) - NI_{1,n}(c)[\widehat{P}_n]'(c) = N\widehat{P}_n(c)\widehat{P}^{[2]}_{n-1}(c)\left(M + \frac{\alpha_n}{\gamma_n}\right).$$

Proof. (i) From Christoffel-Darboux formula for polynomials $\{\widehat{P}^{[2]}_n\}_{n\geq 0}$ we have

$$\begin{aligned} &(x-c)\sum_{k=0}^{n}\frac{\widehat{P}^{[2]}_k(x)\widehat{P}^{[2]}_k(y)}{||\widehat{P}^{[2]}_k||^2_{\mu,[2]}} - \sum_{k=0}^{n}\frac{\widehat{P}^{[2]}_k(x)}{||\widehat{P}^{[2]}_k||^2_{\mu,[2]}}(y-c)\widehat{P}^{[2]}_k(y) \\ = &\frac{1}{||\widehat{P}^{[2]}_n||^2_{\mu,[2]}}\left(\widehat{P}^{[2]}_{n+1}(x)\widehat{P}^{[2]}_n(y) - \widehat{P}^{[2]}_n(x)\widehat{P}^{[2]}_{n+1}(y)\right). \end{aligned} \tag{5.18}$$

If we multiply (5.18) by $(y-c)$ and integrate on $(0,\infty)$ with respect to $d\mu$, the evaluation at $x = c$ yields

$$\begin{aligned} &-\sum_{k=0}^{n}\frac{\widehat{P}^{[2]}_k(c)}{||\widehat{P}^{[2]}_k||^2_{\mu,[2]}}\int_0^\infty \widehat{P}^{[2]}_k(y)(y-c)^2 d\mu(y) \\ = &\frac{1}{||\widehat{P}^{[2]}_n||^2_{\mu,[2]}}\left(\widehat{P}^{[2]}_{n+1}(c)I_{1,n+1}(c) - \widehat{P}^{[2]}_n(c)I_{1,n+2}(c)\right). \end{aligned}$$

Since
$$\int_0^\infty \widehat{P}^{[2]}_k(y)(y-c)^2 d\mu(y) = 0,\ k = 1,\ 2,\ 3, \ldots,\ n,$$
and $\widehat{P}^{[2]}_0 = 1$, the left hand side is negative. Therefore,

$$\widehat{P}^{[2]}_{n+1}(c)I_{1,n+1}(c) - \widehat{P}^{[2]}_n(c)I_{1,n+2}(c) < 0.$$

From (5.2)

$$\begin{aligned} sign\,\widehat{P}^{[2]}_{n+1}(c) &= (-1)^{n+1}, \\ sign\,\widehat{P}^{[2]}_n(c) &= (-1)^n. \end{aligned}$$

Thus, $\widehat{P}^{[2]}_{n+1}(c)\widehat{P}^{[2]}_n(c)$ is negative and, as a consequence,

$$\frac{I_{1,n+2}(c)}{\widehat{P}^{[2]}_{n+1}(c)} < \frac{I_{1,n+1}(c)}{\widehat{P}^{[2]}_n(c)}.$$

By using the above relation in a recursive way, we get

$$\frac{I_{1,n}(c)}{\widehat{P}^{[2]}_{n-1}(c)} < I_{1,1}(c) = d^{[1]}_0.$$

On the other hand, from (5.2) and (5.15)

$$sign\, I_{1,n}(c) = (-1)^{n+1}.$$

Therefore,

$$0 < \frac{I_{1,n}(c)}{\widehat{P}^{[2]}_{n-1}(c)} < d_0^{[1]}.$$

From (5.18)

$$0 < \sum_{k=0}^{n} \frac{(\widehat{P}^{[2]}_k(c))^2}{||\widehat{P}^{[2]}_k||^2_{\mu,[2]}} = \frac{1}{||\widehat{P}^{[2]}_n||^2_{\mu,[2]}} \left([\widehat{P}^{[2]}_{n+1}]'(c)\widehat{P}^{[2]}_n(c) - [\widehat{P}^{[2]}_n]'(c)\widehat{P}^{[2]}_{n+1}(c) \right).$$

Since $\widehat{P}^{[2]}_{n+1}(c)\widehat{P}^{[2]}_n(c)$ is negative this yields

$$\frac{[\widehat{P}^{[2]}_{n+1}]'(c)}{\widehat{P}^{[2]}_{n+1}(c)} < \frac{[\widehat{P}^{[2]}_n]'(c)}{\widehat{P}^{[2]}_n(c)}.$$

By using the above relation in a recursive way, we have

$$\frac{[\widehat{P}^{[2]}_{n+1}]'(c)}{\widehat{P}^{[2]}_{n+1}(c)} < \frac{[\widehat{P}^{[2]}_1]'(c)}{\widehat{P}^{[2]}_1(c)} = -\frac{d_0^{[3]}}{d_0^{[2]}}.$$

Let $0 < x^{[2]}_{n,1} < x^{[2]}_{n,2} < \cdots < x^{[2]}_{n,n}$ denote the zeros of $\widehat{P}^{[2]}_n$. Then

$$-\frac{[\widehat{P}^{[2]}_n]'(c)}{\widehat{P}^{[2]}_n(c)} = \frac{1}{x^{[2]}_{n,1} - c} + \frac{1}{x^{[2]}_{n,2} - c} + \cdots + \frac{1}{x^{[2]}_{n,n} - c} < -\frac{n}{c}.$$

Statements (*ii*) and (*iii*) can be proved in the similar way as we did in (*i*). ■

Proposition 5.5 *Let $M, N \geq 0$ and not both zero. Then,*

$$\begin{aligned} sign\, A_{n,1} &= -1, \\ sign\, A_{n,2} &= -1. \end{aligned}$$

Proof. From (5.2) and Proposition 5.4

$$sign\, A_{n,1} = -sign\, \frac{[\widehat{P}_n]'(c)}{\widehat{P}^{[2]}_{n-1}(c)} = sign \left(-\frac{[\widehat{P}_n]'(c)}{\widehat{P}_n(c)} \right) sign\, \frac{\widehat{P}_n(c)}{\widehat{P}^{[2]}_{n-1}(c)} = -1.$$

In a similar way,

$$\begin{aligned} sign\, A_{n,2} &= -sign\left(\frac{\widehat{P}_n(c)}{I_{2,n}}\right) = sign\left(-\frac{\widehat{P}^{[2]}_{n-1}(c)}{[\widehat{P}^{[2]}_{n-1}]'(c)}\right) \\ &\times sign\,\frac{\widehat{P}_n(c)\widehat{P}_{n-2}(c)\widehat{P}^{[1]}_{n-2}(c)\widehat{P}^{[2]}_{n-2}(c)\widehat{P}^{[3]}_{n-2}(c)}{\widehat{P}_{n-1}(c)\widehat{P}^{[1]}_{n-1}(c)\widehat{P}^{[2]}_{n-1}(c)} \\ &= -1. \end{aligned}$$

■

Connection formula (III)

Next, let us obtain a third representation for the discrete Sobolev OPS in terms of the polynomials orthonormal with respect to $(x-c)^2 d\mu(x)$. This expression will be very useful to find a connection of these polynomials with the matrix orthogonal polynomials. Let us denote $\{s_n^{M,N}\}_{n\geq 0}$, $\{p_n\}_{n\geq 0}$ the sequences of polynomials orthonormal with respect to (5.1) and (1.2) respectively. Throughout the proof

$$\begin{aligned} s_n^{M,N} &= t_n x^n + \text{ lower degree terms}, \quad t_n > 0, \\ p_n &= r_n x^n + \text{ lower degree terms}, \quad r_n > 0, \\ p_n^{[k]} &= r_n^{[k]} x^n + \text{ lower degree terms}, \quad r_n^{[k]} > 0. \end{aligned}$$

We first prove a couple of auxiliary results

Proposition 5.6 *The sequence of discrete Sobolev orthonormal polynomials* $\{s_n^{M,N}(x)\}_{n\geq 0}$ *can be expressed as*

$$\begin{aligned} s_n^{M,N}(x) &= \eta_1(c,n)p_{n+1}(x) + \eta_2(c,n)p_n(x) \\ &\quad -Ms_n^{M,N}(c)K_{n+1}(x,c) - N[s_n^{M,N}]'(c)K^{(0,1)}_{n+1}(x,c) \end{aligned} \tag{5.19}$$

where

$$\begin{aligned} \eta_1(c,n) &= N[s_n^{M,N}]'(c)[p_{n+1}]'(c) + Ms_n^{M,N}(c)p_{n+1}(c), \\ \eta_2(c,n) &= \frac{t_n}{r_n} + Ms_n^{M,N}(c)p_n(c) + N[s_n^{M,N}]'(c)[p_n]'(c). \end{aligned}$$

Proof. Using orthonormal polynomials, the formula (5.12) reads

$$s_n^{M,N}(x) = \frac{t_n}{r_n} p_n(x) - M s_n^{M,N}(c) K_{n-1}(x,c) - N [s_n^{M,N}]'(c) K_{n-1}^{(0,1)}(x,c) \tag{5.20}$$

and from the orthonormal version of the kernel (2.3), we have

$$\begin{aligned} K_{n-1}(x,c) &= K_{n+1}(x,c) - p_{n+1}(x)p_{n+1}(c) - p_n(x)p_n(c), \\ K_{n-1}^{(0,1)}(x,c) &= K_{n+1}^{(0,1)}(x,c) - p_{n+1}(x)[p_{n+1}]'(c) - p_n(x)[p_n]'(c). \end{aligned}$$

Replacing in (5.20), we get

$$\begin{aligned} s_n^{M,N}(x) &= \left(N[s_n^{M,N}]'(c)[p_{n+1}]'(c) + M s_n^{M,N}(c) p_{n+1}(c)\right) p_{n+1}(x) \\ &+ \left(\frac{t_n}{r_n} + M s_n^{M,N}(c) p_n(c) + N[s_n^{M,N}]'(c)[p_n]'(c)\right) p_n(x) \\ &- M s_n^{M,N}(c) K_{n+1}(x,c) - N[s_n^{M,N}]'(c) K_{n+1}^{(0,1)}(x,c), \end{aligned}$$

which proves our statement. ■

Proposition 5.7 *The sequence of polynomials $\{p_n(x)\}_{n\geq 0}$, orthonormal with respect to $d\mu(x)$, can be expressed in terms of the 2−iterated orthonormal polynomials $\{p_n^{[2]}(x)\}_{n\geq 0}$ as follows*

$$p_n(x) = \phi_1(c,n) p_n^{[2]}(x) + \phi_2(c,n) p_{n-1}^{[2]}(x) + \phi_3(c,n) p_{n-2}^{[2]}(x), \tag{5.21}$$

where

$$\begin{aligned} \phi_1(c,n) &= \frac{r_n^{[1]}}{r_{n+1}^{[1]}} \frac{r_n}{r_{n+1}} \frac{p_{n+1}^{[1]}(c)}{p_n^{[1]}(c)} \frac{p_{n+1}(c)}{p_n(c)}, \\ \phi_2(c,n) &= \frac{-r_{n-1}^{[1]}}{r_n^{[1]}} \left(\frac{p_n^{[1]}(c)}{p_{n-1}^{[1]}(c)} \frac{r_{n-1}}{r_n} + \frac{r_n}{r_{n+1}} \frac{p_{n+1}(c)}{p_n(c)} \right), \\ \phi_3(c,n) &= \frac{r_{n-2}^{[1]}}{r_{n-1}^{[1]}} \frac{r_{n-1}}{r_n}. \end{aligned}$$

Proof. Using orthonormal polynomials, the Christoffel-Darboux formula (2.4) reads

$$K_n(x,y) = \sum_{k=0}^{n} p_k(x)p_k(y) = \frac{r_n}{r_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{(x-y)}. \tag{5.22}$$

Hence, taking into account (2.17), it follows that, for $k = 1, 2$ we get

$$p_n^{[1]}(x) = \frac{1}{(x-c)}\left[p_n(x) - \frac{p_n(c)}{p_{n+1}(c)}p_{n+1}(x)\right], \tag{5.23}$$

$$p_n^{[2]}(x) = \frac{1}{(x-c)}\left[p_n^{[1]}(x) - \frac{p_n^{[1]}(c)}{p_{n+1}^{[1]}(c)}p_{n+1}^{[1]}(x)\right]. \tag{5.24}$$

Next, from (5.22)

$$\begin{aligned} p_n(x) &= \frac{1}{p_n(c)}\left(K_n(x,c) - K_{n-1}(x,c)\right) \\ &= \frac{r_{n-1}}{r_n}p_{n-1}^{[1]}(x) - \frac{r_n}{r_{n+1}}\frac{p_{n+1}(c)}{p_n(c)}p_n^{[1]}(x). \end{aligned} \tag{5.25}$$

Therefore, from (5.25) we get

$$\begin{aligned} p_n(x) &= \frac{r_{n-1}}{r_n}p_{n-1}^{[1]}(x) - \frac{r_n}{r_{n+1}}\frac{p_{n+1}(c)}{p_n(c)}p_n^{[1]}(x) \\ &= \frac{r_n^{[1]}}{r_{n+1}^{[1]}}\frac{r_n}{r_{n+1}}\frac{p_{n+1}^{[1]}(c)}{p_n^{[1]}(c)}\frac{p_{n+1}(c)}{p_n(c)}p_n^{[2]}(x) \\ &\quad -\frac{r_{n-1}^{[1]}}{r_n^{[1]}}\left(\frac{p_n^{[1]}(c)}{p_{n-1}^{[1]}(c)}\frac{r_{n-1}}{r_n} + \frac{r_n}{r_{n+1}}\frac{p_{n+1}(c)}{p_n(c)}\right)p_{n-1}^{[2]}(x) + \frac{r_{n-2}^{[1]}}{r_{n-1}^{[1]}}\frac{r_{n-1}}{r_n}p_{n-2}^{[2]}(x), \end{aligned}$$

which is our statement. ■

Proposition 5.8 *Let $\{s_n^{M,N}(x)\}_{n\geq 0}$ be the sequence Sobolev-type polynomials orthonormal with respect to (5.1), and let $\{p_n^{[2]}(x)\}_{n\geq 0}$ be the sequence of polynomials orthonormal with respect to the inner product (2.16) with $k = 2$. Then, the following expression holds*

$$s_n^{M,N}(x) = \alpha(n,c)p_n^{[2]}(x) + \beta(n,c)p_{n-1}^{[2]}(x) + \gamma(n,c)p_{n-2}^{[2]}(x), \tag{5.26}$$

where

$$\begin{aligned} \alpha(n,c) &= \frac{t_n}{r_n^{[2]}}, \\ \beta(n,c) &= \eta_1(c,n)\phi_3(c,n+1) + \eta_2(c,n)\phi_2(c,n), \\ \gamma(n,c) &= \frac{t_n}{r_{n-2}^{[2]}} \end{aligned}$$

and

$$\begin{aligned}
\eta_1(c,n) &= N[s_n^{M,N}]'(c)[p_{n+1}]'(c) + Ms_n^{M,N}(c)p_{n+1}(c),\\
\eta_2(c,n) &= \frac{t_n}{r_n} + Ms_n^{M,N}(c)p_n(c) + N[s_n^{M,N}]'(c)[p_n]'(c),\\
\phi_2(c,n) &= \frac{-r_{n-1}^{[1]}}{r_n^{[1]}}\left(\frac{p_n^{[1]}(c)}{p_{n-1}^{[1]}(c)}\frac{r_{n-1}}{r_n} + \frac{r_n}{r_{n+1}}\frac{p_{n+1}(c)}{p_n(c)}\right),\\
\phi_3(c,n+1) &= \frac{r_{n-1}^{[1]}}{r_n^{[1]}}\frac{r_n}{r_{n+1}}.
\end{aligned}$$

Proof. For $\alpha(n,c)$, matching the leading coefficients of $s_n^{M,N}(x)$ and $p_n^{[2]}(x)$, it is trivial to see that

$$\alpha(n,c) = \langle s_n^{M,N}(x), p_n^{[2]}(x)\rangle_{[2]} = \frac{t_n}{r_n^{[2]}}.$$

For $\beta(n,c)$ we need some more work. From (5.19)we have

$$\begin{aligned}
\beta(n,c) &= \langle s_n^{M,N}(x), p_{n-1}^{[2]}(x)\rangle_{[2]}\\
&= \eta_1(c,n)\langle p_{n+1}(x), p_{n-1}^{[2]}(x)\rangle_{[2]}\\
&\quad +\eta_2(c,n)\langle p_n(x), p_{n-1}^{[2]}(x)\rangle_{[2]}\\
&\quad -Ms_n^{M,N}(c)\langle K_{n+1}(x,c), p_{n-1}^{[2]}(x)\rangle_{[2]}\\
&\quad -N[s_n^{M,N}]'(c)\langle K_{n+1}^{(0,1)}(x,c), p_{n-1}^{[2]}(x)\rangle_{[2]},
\end{aligned}$$

where, taking into account the reproductive property of the kernel

$$\begin{aligned}
\langle K_{n+1}(x,c), p_{n-1}^{[2]}(x)\rangle_{[2]} &= 0,\\
\langle K_{n+1}^{(0,1)}(x,c), p_{n-1}^{[2]}(x)\rangle_{[2]} &= 0,
\end{aligned}$$

and using (5.21), we get

$$\beta(n,c) = \eta_1(c,n)\phi_3(c,n+1) + \eta_2(c,n)\phi_2(c,n).$$

For the last coefficient, as a straigtforward consequence of (5.1), we have

$$\begin{aligned}
\gamma(n,c) &= \langle s_n^{M,N}(x), p_{n-2}^{[2]}(x)\rangle_{[2]}\\
&= \langle s_n^{M,N}(x), (x-c)^2 p_{n-2}^{[2]}(x)\rangle_S = \frac{t_n}{r_{n-2}^{[2]}}.
\end{aligned}$$

Finally, for $0 \leq j \leq n-3$, we have

$$\langle s_n^{M,N}(x), p_j^{[2]}(x)\rangle_{[2]} = \langle s_n^{M,N}(x), (x-c)^2 p_j^{[2]}(x)\rangle_S$$

which vahishses for $j \leq n-3$. Hence, there is no more coefficients in (5.26), and it completes the proof. ■

5.2.2 The five term recurrence relation

In this section, we will obtain the five term recurrence relation that the sequence of Sobolev-type orthonormal polynomials $\{s_n^{M,N}(x)\}_{n\geq 0}$ satisfies. We are interested in using orthonormal polynomials because with this normalization, all the matrices that we next obtain are symmetric. Later on, we will derive an interesting relation between the pentadiagonal matrix $\mathbf{H}$ associated with the multiplication operator by $(x-c)^2$, and the tridiagonal Jacobi matrix $\mathbf{J}_{[2]}$, associated with the three term recurrence relation satisfied by the 2−iterated orthonormal polynomials $\{p_n^{[2]}(x)\}_{n\geq 0}$.

To do that, we will use the remarkable fact, which is a straightforward consequence of (5.1), that the multiplication operator by $(x-c)^2$ is a symmetric operator with respect to the discrete Sobolev inner product (5.1). In other words, if p, $q \in \mathbb{P}$

$$\langle (x-c)^2 p, q\rangle_S = \langle p, (x-c)^2 q\rangle_S. \tag{5.27}$$

First we will obtain the aforementioned five term recurrence relation. Let consider the Fourier expansion of $(x-c)^2 s_n^{M,N}(x)$ in terms of $\{s_n^{M,N}(x)\}_{n\geq 0}$

$$(x-c)^2 s_n^{M,N}(x) = \sum_{k=0}^{n+2} \rho_{n,k} s_k^{M,N}(x), \tag{5.28}$$

where

$$\rho_{n,k} = \left\langle (x-c)^2 s_n^{M,N}(x), s_k^{M,N}(x)\right\rangle_S, \quad k = 0, \ldots, n+2.$$

From (5.27)

$$\rho_{n,k} = \left\langle s_n^{M,N}(x), (x-c)^2 s_k^{M,N}(x)\right\rangle_S, \quad k = 0, \ldots, n+2.$$

Hence, $\rho_{n,k} = 0$ for $k = 0, \ldots, n-3$. Taking into account that

$$\left[(x-c)^2 s_n^{M,N}(x)\right]|_{x=c} = \left[(x-c)^2 s_n^{M,N}(x)\right]'|_{x=c} = 0,$$

and using theorem (5.2) we get

$$\left\langle (x-c)^2 s_n^{M,N}(x), s_k^{M,N}(x) \right\rangle_S = \left\langle s_n^{M,N}(x), s_k^{M,N}(x) \right\rangle_{[2]}$$

Next, using connection formula (5.26) we have

$$\begin{aligned}
\rho_{n,n+2} &= \left\langle (x-c)^2 s_n^{M,N}(x), s_{n+2}^{M,N}(x) \right\rangle_S = \left\langle s_n^{M,N}(x), s_k^{M,N}(x) \right\rangle_{[2]} \\
&= \alpha(n,c)\gamma(n+2,c) \left\langle p_n^{[2]}(x), p_n^{[2]}(x) \right\rangle_{[2]} \\
&= \alpha(n,c)\gamma(n+2,c),
\end{aligned}$$

$$\begin{aligned}
\rho_{n,n+1} &= \left\langle (x-c)^2 s_n^{M,N}(x), s_{n+1}^{M,N}(x) \right\rangle_S = \left\langle s_n^{M,N}(x), s_{n+1}^{M,N}(x) \right\rangle_{[2]} \\
&= \alpha(n,c)\beta(n+1,c) \left\langle p_n^{[2]}(x), p_n^{[2]}(x) \right\rangle_{[2]} \\
&\quad +\beta(n,c)\gamma(n+1,c) \left\langle p_{n-1}^{[2]}(x), p_{n-1}^{[2]}(x) \right\rangle_{[2]} \\
&= \alpha(n,c)\beta(n+1,c) + \beta(n,c)\gamma(n+1,c),
\end{aligned}$$

$$\begin{aligned}
\rho_{n,n} &= \left\langle (x-c)^2 s_n^{M,N}(x), s_n^{M,N}(x) \right\rangle_S = \left\langle s_n^{M,N}(x), s_n^{M,N}(x) \right\rangle_{[2]} \\
&= \alpha^2(n,c) \left\langle p_n^{[2]}(x), p_n^{[2]}(x) \right\rangle_{[2]} + \beta^2(n,c) \left\langle p_{n-1}^{[2]}(x), p_{n-1}^{[2]}(x) \right\rangle_{[2]} \\
&\quad +\gamma^2(n,c) \left\langle p_{n-2}^{[2]}(x), p_{n-2}^{[2]}(x) \right\rangle_{[2]} \\
&= \alpha^2(n,c) + \beta^2(n,c) + \gamma^2(n,c),
\end{aligned}$$

$$\begin{aligned}
\rho_{n,n-1} &= \left\langle (x-c)^2 s_n^{M,N}(x), s_{n-1}^{M,N}(x) \right\rangle_S = \left\langle s_n^{M,N}(x), s_{n-1}^{M,N}(x) \right\rangle_{[2]} \\
&= \alpha(n-1,c)\beta(n,c) \left\langle p_{n-1}^{[2]}(x), p_{n-1}^{[2]}(x) \right\rangle_{[2]} + \beta(n-1,c)\gamma(n,c) \left\langle p_{n-2}^{[2]}(x), p_{n-2}^{[2]}(x) \right\rangle_{[2]} \\
&= \alpha(n-1,c)\beta(n,c) + \beta(n-1,c)\gamma(n,c),
\end{aligned}$$

$$\begin{aligned}
\rho_{n,n-2} &= \left\langle (x-c)^2 s_n^{M,N}(x), s_{n-2}^{M,N}(x) \right\rangle_S = \left\langle s_n^{M,N}(x), s_{n-2}^{M,N}(x) \right\rangle_{[2]} \\
&= \alpha(n-2,c)\gamma(n,c) \left\langle p_{n-2}^{[2]}(x), p_{n-2}^{[2]}(x) \right\rangle_{[2]} \\
&= \alpha(n-2,c)\gamma(n,c).
\end{aligned}$$

Rearranging index, we can rewrite the above coefficients as

$$\begin{aligned}\rho_{n,n-2} &= a_n, \quad \rho_{n,n+2} = a_{n+2},\\ \rho_{n,n-1} &= b_n, \quad \rho_{n,n+1} = b_{n+1},\\ \rho_{n,n} &= c_n.\end{aligned}$$

Hence, (5.28) becomes

$$(x-c)^2 s_n^{M,N}(x) = \\ a_{n+2}s_{n+2}^{M,N}(x) + b_{n+1}s_{n+1}^{M,N}(x) + c_n s_n^{M,N}(x) + b_n s_{n-1}^{M,N}(x) + a_n s_{n-2}^{M,N}(x), \quad (5.29)$$

where, by convention

$$s_{-2}^{M,N}(x) = s_{-1}^{M,N}(x) = 0.$$

The matrix representation of (5.29) is

$$(x-c)^2 \bar{\mathbf{s}}^{M,N} = \mathbf{H}\,\bar{\mathbf{s}}^{M,N}, \quad (5.30)$$

where $\mathbf{H}$ is the pentadiagonal semi-infinite symmetric matrix

$$\mathbf{H} = \begin{bmatrix} c_0 & b_1 & a_2 & 0 & \cdots \\ b_1 & c_1 & b_2 & a_3 & \cdots \\ a_2 & b_2 & c_2 & b_3 & \ddots \\ 0 & a_3 & b_3 & c_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \quad (5.31)$$

and

$$\bar{\mathbf{s}}^{M,N} = \begin{bmatrix} s_0^{M,N}(x) & s_1^{M,N}(x) & s_2^{M,N}(x) & \cdots \end{bmatrix}^T.$$

On the other hand, the matrix analog to (5.26) can be stated as

$$\bar{\mathbf{s}}^{M,N} = \mathbf{T}\,\bar{\mathbf{p}}^{[2]} \quad (5.32)$$

where $\mathbf{T}$ is the lower tridiagonal, semi-infinite, and invertible matrix

$$\mathbf{T} = \begin{bmatrix} \alpha(0,c) & 0 & 0 & 0 \\ \beta(1,c) & \alpha(1,c) & 0 & 0 \\ \gamma(2,c) & \beta(2,c) & \alpha(2,c) & 0 \\ 0 & \ddots & \ddots & \ddots \end{bmatrix}$$

and

$$\bar{\mathbf{p}}^{[2]} = \left[p_0^{[2]}(x) \quad p_1^{[2]}(x) \quad p_2^{[2]}(x) \quad \cdots \right]^T.$$

Next, we show that the pentadiagonal matrix $\mathbf{H}$ associated with the five term recurrence relation (5.29) can be given in terms of the pentadiagonal matrix $\mathbf{J}_{[2]} - c\mathbf{I}$. The matrix $\mathbf{J}_{[2]}$ is the tridiagonal Jacobi matrix associated with the three term recurrence relation which verify the family of $2-$iterated OPS $\{p_n^{[2]}(x)\}_{n\geq 0}$. Note that this is a standard OPS, so therefore

$$x\bar{\mathbf{p}}^{[2]} = \mathbf{J}_{[2]}\,\bar{\mathbf{p}}^{[2]},$$

which in turn implies

$$(x-c)^2\,\bar{\mathbf{p}}^{[2]} = \left(\mathbf{J}_{[2]} - c\mathbf{I}\right)^2 \bar{\mathbf{p}}^{[2]}. \tag{5.33}$$

Combining (5.30) with (5.32), we get

$$\mathbf{T}(x-c)^2\bar{\mathbf{p}}^{[2]} = \mathbf{H}\mathbf{T}\,\bar{\mathbf{p}}^{[2]}. \tag{5.34}$$

Substituting (5.33) into (5.34)

$$\mathbf{T}\left(\mathbf{J}_{[2]} - c\mathbf{I}\right)^2\,\bar{\mathbf{p}}^{[2]} = \mathbf{H}\mathbf{T}\,\bar{\mathbf{p}}^{[2]}.$$

Hence,

Proposition 5.9 *The semi-infinite pentadiagonal matrix* $\mathbf{H}$*, associated with the operator* $(x-c)^2$ *can be obtained from the matrix* $\left(\mathbf{J}_{[2]} - c\mathbf{I}\right)^2$ *as follows*

$$\mathbf{H} = \mathbf{T}\left(\mathbf{J}_{[2]} - c\mathbf{I}\right)^2 \mathbf{T}^{-1}. \tag{5.35}$$

5.2.3 Zeros of Sobolev-type orthogonal polynomials

In this subsection, we will analyze the zeros of the polynomials $\widehat{S}_n^{M,N}$. The techniques are the same as those used by Meijer in [85] and [86].

Theorem 5.3 *The discrete Sobolev orthogonal polynomial* $\widehat{S}_n^{M,N}$ *has* n *real simple zeros and at most one of them is outside* $[c,\infty)$*.*

Proof. For $N=0$, $\widehat{S}_n^{M,N}$ is a standard orthogonal polynomial. In the sequel we will consider the cases when $N>0$ and $M\geq 0$. Let denote by $\nu_{n,r}$, $r=1,2,...,n$, the zeros of $\widehat{S}_n^{M,N}(x)$ on (c,∞) with odd multiplicity in an increasing order. Let us introduce the polynomial

$$\phi(x) = (x-\nu_{n,1})(x-\nu_{n,2})\cdots(x-\nu_{n,k}).$$

Notice that $\phi(c)$ and $\phi'(c)$ have opposite sign and $\phi(x)\widehat{S}_n^{M,N}(x)$ does not change sign on $[c,\infty)$. If $\deg\phi \leq n-2$, then

$$0 = \langle \phi, \widehat{S}_n^{M,N}\rangle_S = \int_0^\infty \phi(x)\widehat{S}_n^{M,N}(x)d\mu(x) + M\,\phi(c)\widehat{S}_n^{M,N}(c) + N\,\phi'(c)[\widehat{S}_n^{M,N}]'(c),$$

$$0 = \langle (x-c)\phi, \widehat{S}_n^{M,N}\rangle_S = \int_0^\infty (x-c)\phi(x)\widehat{S}_n^{M,N}(x)d\mu(x) + N\,\phi(c)[\widehat{S}_n^{M,N}]'(c).$$

This means that $\phi'(c)[\widehat{S}_n^{M,N}]'(c)$ and $\phi(c)[\widehat{S}_n^{M,N}]'(c)$ have the same sign, and therefore $\phi'(c)$ and $\phi(c)$ have the same sign. This yields a contradiction.

As a conclusion, either $\deg\phi = n-1$ or $\deg\phi = n$, which proves our statement. ∎

Next, we prove that the zeros of $\widehat{S}_n^{M,N}(x)$ interlace with the zeros of $\widehat{P}_{n-1}^{[2]}(x)$ if $\widehat{S}_n^{M,N}(x)$ has a zero outside $[c,\infty)$. Notice that, by Theorem 5.2, $\widehat{S}_n^{M,N}(c) \neq 0$.

Theorem 5.4 *Let denote by $\nu_{n,r}$, $r = 1,2,...,n$, the zeros of $\widehat{S}_n^{M,N}(x)$ in an increasing order. Suppose that $\nu_{n,1} < c$. Then, $2c - x_{n-1,1}^{[2]} < \nu_{n,1} < c$ and*

$$c < \nu_{n,2} < x_{n-1,1}^{[2]} < \cdots < \nu_{n,n} < x_{n-1,n-1}^{[2]}.$$

Proof. From Theorem 5.2

$$\widehat{S}_n^{M,N}(x_{n-1,r}^{[2]}) = \widehat{P}_n(x_{n-1,r}^{[2]}) + A_{n,2}\left(x_{n-1,r}^{[2]} - c\right)^2 \widehat{P}_{n-2}^{[4]}(x_{n-1,r}^{[2]}),\ r = 1,\,2,\,\ldots,\,n-1.$$

Then from Proposition 5.3(iii) and Proposition 5.5 we get

$$sign\,\widehat{S}_n^{M,N}(x_{n-1,r}^{[2]}) = (-1)^{n-r},\ r = 1,\,2,\ldots,\,n-1,$$

On the other hand, from (5.2) and Theorem 5.2

$$sign\,\widehat{S}_n^{M,N}(c) = sign\,\widehat{P}_n(c) = (-1)^n.$$

Therefore, every interval $(c, x_{n-1,1}^{[2]})$ and $\left(x_{n-1,r}^{[2]}, x_{n-1,r+1}^{[2]}\right)$, $r = 1,\,\ldots,\,n-2$, contains an odd number of zeros of $\widehat{S}_n^{M,N}(x)$. Since $\widehat{S}_n^{M,N}$ has n real zeros and at most one of them is outside of (c,∞), then

$$c < \nu_{n,2} < x_{n-1,1}^{[2]} < \cdots < \nu_{n,n} < x_{n-1,n-1}^{[2]}.$$

Now, we will prove that $2c - x^{[2]}_{n-1,r+1} < \nu_{n,1} < c$. Let

$$\widehat{S}^{M,N}_n(x) = (x - \nu_{n,1})(x - \nu_{n,2}) \cdots (x - \nu_{n,n}).$$

By Theorem 5.2 and Proposition 5.4

$$[\widehat{S}^{M,N}_n]'(c) = [\widehat{P}_n]'(c) + A_{n,1}\widehat{P}^{[2]}_{n-2}(c) = \frac{\beta_n \widehat{P}_n(c)(M + \frac{\alpha_n}{\gamma_n})}{N + \alpha_n\beta_n}.$$

Therefore,

$$sign\,[\widehat{S}^{M,N}_n]'(c) = sign\,\widehat{P}_n(c) = sign\,\widehat{S}^{M,N}_n(c)$$

and

$$0 < \frac{[\widehat{S}^{M,N}_n]'(c)}{\widehat{S}^{M,N}_n(c)} = \frac{1}{c - \nu_{n,1}} - \frac{1}{\nu_{n,2} - c} - \cdots - \frac{1}{\nu_{n,n} - c}.$$

Hence

$$\frac{1}{c - \nu_{n,1}} > \frac{1}{\nu_{n,2} - c} \Rightarrow x^{[2]}_{n-1,1} - c > \nu_{n,2} - c > c - \nu_{n,1} \Rightarrow 2c - x^{[2]}_{n-1,1} < \nu_{n,1}.$$

Our statement follows. ■

5.3 Connection with Matrix Orthogonality

Next, we will establish a connection between the Sobolev-type orthonormal polynomials $\{s^{M,N}_n\}_{n\geq 0}$ and orthogonal matrix and vector polynomials. Concerning the matrix orthogonality, we will show that the five term recurrence relation (5.29) can be expressed in terms of orthonormal matrix polynomials for which the coefficients are 2×2 matrices. This construction is due to A. Durán and W. Van Assche and is a nice picture of the Sobolev-type orthogonality from a different point of view. Here we only apply this theory to our particular case of Sobolev-type polynomials. For a treatment of a more general case we refer to the reader to [21].

Set $h(x) = (x - c)^2$. We will use the following basis in the linear space of polynomials $\mathbb{P}$

$$\left\{1, x, (x - c)^2, x(x - c)^2, (x - c)^4, x(x - c)^4, \ldots\right\}.$$

Let $\Sigma = \{0, 1\}$. The polynomial $s^{M,N} \in \mathbb{P}$ of degree $2k+l$, $0 \leq l < 2$ can then be expanded in this basis as

$$s^{M,N}(x) = \sum_{\sigma\in\Sigma}\sum_{m=0}^{k} a_{\sigma,m} x^{\sigma} h^m(x).$$

Next, let

$$R_{h,\sigma}(p)(x) = \sum_{m=0}^{k} a_{\sigma,m} x^m, \quad p \in \mathbb{P},\ \sigma \in \Sigma,$$

such that it takes from p those terms of the form $a_{\sigma,m}x^{\sigma}h^m$ and then it removes the common factor x^{σ} and changes $h(x)$ to x. With $\{s_n^{M,N}\}_{n\geq 0}$ satisfying the five term recurrence relation (5.29), the polynomial $s_n^{M,N}$ is equivalent (and we will write $s_n^{M,N}(x) \equiv \mathbf{s}_n^{M,N}$), modulo $h(x)$, with the vector polynomial given by

$$\mathbf{s}_n^{M,N} = \begin{bmatrix} R_{h,0}(s^{M,N})(h(x)) & R_{h,1}(s^{M,N})(h(x)) \end{bmatrix}.$$

Furthermore, under these assumptions, we can write the polynomials $s_{2k+l}^{M,N}(x)$, of degree $2k+l$, $0 \leq l < 2$ as

$$s_{2k+l}^{M,N}(x) = R_{h,0}(s_{2k+l}^{M,N}(h(x))) + xR_{h,1}(s_{2k+l}^{M,N}(h(x))).$$

The main Theorem in [21, Section 2] states that the inner product (5.1) can be easily rewritten as

$$\begin{aligned} \langle f, g\rangle_S &= \int_0^{\infty} \begin{bmatrix} R_{h,0}(f)(h(x)) & R_{h,1}(g)(h(x)) \end{bmatrix} d\mathbf{M}(x) \begin{bmatrix} R_{h,0}(g)(h(x)) \\ R_{h,1}(g)(h(x)) \end{bmatrix} \\ &\quad + \begin{bmatrix} R_{h,0}(f)(0) & R_{h,1}(g)(0) \end{bmatrix} \mathbf{L} \begin{bmatrix} R_{h,0}(g)(0) \\ R_{h,1}(g)(0) \end{bmatrix}, \end{aligned}$$

where $\mathbf{M}$ is the 2×2 matrix of measures

$$d\mathbf{M}(x) = \begin{bmatrix} d\mu(x) & xd\mu(x) \\ xd\mu(x) & x^2 d\mu(x) \end{bmatrix},$$

and $\mathbf{L}$ is the 2×2 matrix

$$\begin{aligned} \mathbf{L} &= M \begin{bmatrix} 1 \\ c \end{bmatrix} \begin{bmatrix} 1 & c \end{bmatrix} + N \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} M+N & Mc+N \\ Mc+N & Mc^2+N \end{bmatrix}. \end{aligned}$$

Under the conditions stated above, the Sobolev-type polynomials $s_n^{M,N}(x)$ orthogonal with respect to (5.1) yield the matrix polynomials

$$\mathcal{S}_n^{M,N}(x) = \begin{bmatrix} R_{h,0}(s_{2n}^{M,N})(x) & R_{h,1}(s_{2n}^{M,N})(x) \\ R_{h,0}(s_{2n+1}^{M,N})(x) & R_{h,1}(s_{2n+1}^{M,N})(x) \end{bmatrix}$$

orthonormal with respect to the matrix of measures $\mathbf{M}(h^{-1})$, perturbed with a "matrix mass point" given by the matrix $\mathbf{L}$ at $x = 0$.

The matrix of measures satisfies

$$\int F(x)d\mathbf{M}(h^{-1}(x)) = \int F(h(x))d\mathbf{M}(x),$$

where $F : \mathbb{R} \to \mathbb{R}^2$ is a vector function such that $F(h(x)) \in L_1(\mathbf{M})$.

Moreover, the matrix orthogonal polynomials $\mathcal{S}_n^{M,N}(x)$ satisfy the three term recurrence relation

$$x\mathcal{S}_n^{M,N}(x) = D_{n+1}\mathcal{S}_n^{M,N}(x) + E_n\mathcal{S}_n^{M,N}(x) + D_n\mathcal{S}_{n-1}^{M,N}(x),$$

where D_n and E_n are the 2×2 matrices (D_n being a lower triangular matrix different from the zero matrix $O_{2\times 2}$)

$$D_n = \begin{bmatrix} a_{2n} & 0 \\ b_{2n} & a_{2n+1} \end{bmatrix}, \quad E_n = \begin{bmatrix} c_{2n} & b_{2n+1} \\ b_{2n+1} & c_{2n+1} \end{bmatrix},$$

whose entries are in terms of the coeffients a_n, b_n and c_n in the five term recurrence relation (5.29) for the Sobolev-type OPS $\{s^{M,N}(x)\}_{n\geq 0}$.

5.4 Laguerre Sobolev-type orthogonal polynomials

5.4.1 k-iterated Laguerre polynomials

Let $\{L_n^{(\alpha),[k]}\}_{n\geq 0}$, $k \in \mathbb{N}$, denote the OPS with respect to the modified Laguerre measure $(x - c)^k d\mu(x)$, $c < 0$, normalized by the condition that $L_n^{(\alpha),[k]}$ has the same leading coefficient $\frac{(-1)^n}{n!}$ as the classical Laguerre orthogonal polynomial $L_n^{(\alpha)} = L_n^{(\alpha),[0]}$. With the notation $\{\widehat{L}_n^{\alpha,[k]}\}_{n\geq 0}$ we have the same monic modified Laguerre orthogonal polynomials, i.e.

$$L_n^{(\alpha),[k]}(x) = \frac{(-1)^n}{n!}\widehat{L}_n^{\alpha,[k]}(x).$$

Next we summarize some asymptotic properties of polynomials $L_n^{(\alpha),[k]}(x)$, which will be used in the sequel.

Proposition 5.10 *([29])*

1. *Normalization*
$$L_n^{(\alpha),[k]}(x) = \frac{(-1)^n}{n!}\widehat{L}_n^{\alpha,[k]}(x).$$

2. *(Outer relative asymptotics) From the Perron's formula (2.33), we get*
$$\lim_{n\to\infty} \frac{L_n^{(\alpha),[k]}(x)}{n^{k/2}L_n^{(\alpha)}(x)} = \frac{1}{(\sqrt{-x}+\sqrt{|c|})^k}, \tag{5.36}$$
uniformly on compact subsets of $\mathbb{C}\backslash[0,\infty)$.

3. *(Mehler-Heine type formula) Uniformly on compact subsets of* $\mathbb{C}$
$$\lim_{n\to\infty} \frac{L_n^{(\alpha),[k]}(x/(n+j))}{n^{\alpha+k/2}} = \frac{1}{\left(\sqrt{|c|}\right)^k} x^{-\alpha/2} J_\alpha(2\sqrt{x}) \tag{5.37}$$
where $j \in \mathbb{N}\cup 0$ *and* J_α *is the Bessel function of the first kind.*

4. *Plancherel-Rotach type outer asymptotics for* $L_n^{(\alpha),[k]}$
$$\lim_{n\to\infty} \frac{L_n^{(\alpha),[k]}((n+j)x)}{L_n^{(\alpha)}((n+j)x)} = \left(\frac{\phi((x-2)/2)+1}{x}\right)^k, \tag{5.38}$$
holds uniformly on compact subsets of $\mathbb{C}\backslash[0,4]$ *and uniformly on* $j\in\mathbb{N}\cup\{0\}$, *where* ϕ *is the conformal mapping of* $\mathbb{C}\backslash[-1,1]$ *onto the exterior of the unit circle*
$$\phi(x) = x+\sqrt{x^2-1},\ x\in\mathbb{C}\backslash[-1,1],$$
with $\sqrt{x^2-1}>0$ *when* $x>1$.

Proposition 5.11 *It holds*
$$[L_n^{(\alpha),[2]}]'(c) \cong \frac{n}{4c}L_n^{(\alpha+1)}(c).$$

Proof. Using integration by parts we have
$$\int_0^\infty [L_n^{(\alpha),[2]}]'(x)L_k^{(\alpha+1),[3]}(x)(x-c)^3x^{\alpha+1}e^{-x}dx = \begin{cases} 0, & \text{if } k\le n-3,\\ n(n-1)||\widehat{L}_n^{\alpha,[2]}||^2_{\alpha,[2]}, & \text{if } k=n-2.\end{cases}$$
Therefore,
$$[L_n^{(\alpha),[2]}]'(x) = -L_{n-1}^{(\alpha+1),[3]}(x)+H_nL_{n-2}^{(\alpha+1),[3]}(x),$$

where

$$H_n = \frac{n(n-1)||\widehat{L}_n^{\alpha,[2]}||^2_{\alpha,[2]}}{||\widehat{L}_{n-2}^{\alpha+1,[3]}||^2_{\alpha+1,[3]}}.$$

Using (5.5), (2.33) and (5.36)

$$\begin{aligned} H_n &= \frac{(n+1)^2(n+\alpha)}{(n-1)^3}\frac{L_{n-2}^{(\alpha+1),[2]}(c)}{L_{n-1}^{(\alpha+1),[2]}(c)}\prod_{i=1}^{2}\frac{L_{n-2}^{(\alpha+1),[i-1]}(c)}{L_{n-1}^{(\alpha+1),[i-1]}(c)}\frac{L_{n+1}^{(\alpha),[i-1]}(c)}{L_{n}^{(\alpha),[i-1]}(c)} \\ &= \frac{L_{n-2}^{(\alpha+1),[2]}(c)}{L_{n-1}^{(\alpha+1),[2]}(c)}\prod_{i=1}^{2}\frac{L_{n-2}^{(\alpha+1),[i-1]}(c)}{L_{n-1}^{(\alpha+1),[i-1]}(c)}\frac{L_{n+1}^{(\alpha),[i-1]}(c)}{L_{n}^{(\alpha),[i-1]}(c)} + \mathcal{O}\left(n^{-1}\right). \end{aligned}$$

On the other hand, from [29, Proposition 2.2]

$$[L_n^{(\alpha),[2]}]'(x) = -L_{n-1}^{(\alpha),[3]}(x) + F_n L_{n-2}^{(\alpha+1),[3]}(x), \tag{5.39}$$

where

$$\begin{aligned} F_n &= H_n - \frac{n^3}{(n-1)^3}\prod_{i=1}^{3}\frac{L_{n-2}^{(\alpha+1),[i-1]}(c)}{L_{n-1}^{(\alpha+1),[i-1]}(c)}\frac{L_{n}^{(\alpha),[i-1]}(c)}{L_{n-1}^{(\alpha),[i-1]}(c)} \\ &= \prod_{i=1}^{3}\frac{L_{n-2}^{(\alpha+1),[i-1]}(c)}{L_{n-1}^{(\alpha+1),[i-1]}(c)}\left(\frac{L_{n+1}^{(\alpha)}(c)L_{n+1}^{(\alpha),[1]}(c)}{L_{n}^{(\alpha)}(c)L_{n}^{(\alpha),[1]}(c)} - \frac{L_{n}^{(\alpha)}(c)L_{n}^{(\alpha),[1]}(c)L_{n}^{(\alpha),[2]}(c)}{L_{n-1}^{(\alpha)}(c)L_{n-1}^{(\alpha),[1]}(c)L_{n-1}^{(\alpha),[2]}(c)}\right) + \mathcal{O}\left(n^{-1}\right). \end{aligned}$$

Again, from [29, Proposition 2.2]

$$\frac{L_{n+1}^{(\alpha)}(c)L_{n+1}^{(\alpha),[1]}(c)}{L_{n}^{(\alpha)}(c)L_{n}^{(\alpha),[1]}(c)} = \frac{L_{n+1}^{(\alpha)}(c)L_{n+1}^{(\alpha-1),[1]}(c)}{L_{n}^{(\alpha)}(c)L_{n}^{(\alpha),[1]}(c)} + \frac{L_{n+2}^{(\alpha-1)}(c)}{L_{n+1}^{(\alpha-1)}(c)} + \mathcal{O}\left(n^{-1}\right),$$

$$\begin{aligned} \frac{L_{n}^{(\alpha)}(c)L_{n}^{(\alpha),[1]}(c)L_{n}^{(\alpha),[2]}(c)}{L_{n-1}^{(\alpha)}(c)L_{n-1}^{(\alpha),[1]}(c)L_{n-1}^{(\alpha),[2]}(c)} &= \frac{L_{n}^{(\alpha)}(c)L_{n}^{(\alpha),[1]}(c)L_{n}^{(\alpha-1),[2]}(c)}{L_{n-1}^{(\alpha)}(c)L_{n-1}^{(\alpha),[1]}(c)L_{n-1}^{(\alpha),[2]}(c)} \\ &\quad + \frac{L_{n+1}^{(\alpha-1)}(c)L_{n+1}^{(\alpha-1),[1]}(c)}{L_{n}^{(\alpha-1)}(c)L_{n}^{(\alpha-1),[1]}(c)} + \mathcal{O}\left(n^{-1}\right), \end{aligned}$$

and

$$\frac{L_{n+2}^{(\alpha-1)}(c)}{L_{n+1}^{(\alpha-1)}(c)} - \frac{L_{n+1}^{(\alpha-1)}(c)L_{n+1}^{(\alpha-1),[1]}(c)}{L_{n}^{(\alpha-1)}(c)L_{n}^{(\alpha-1),[1]}(c)} =$$

$$\frac{L_{n+2}^{(\alpha-2)}(c)}{L_{n+1}^{(\alpha-1)}(c)} + 1 - \frac{L_{n+1}^{(\alpha-1)}(c)L_{n+1}^{(\alpha-2),[1]}(c)}{L_{n}^{(\alpha-1)}(c)L_{n}^{(\alpha-1),[1]}(c)} - \frac{L_{n+2}^{(\alpha-2)}(c)}{L_{n+1}^{(\alpha-2)}(c)} + \mathcal{O}\left(n^{-1}\right)$$
$$= \frac{L_{n+2}^{(\alpha-2)}(c)}{L_{n+1}^{(\alpha-1)}(c)} - \frac{L_{n+1}^{(\alpha-1)}(c)L_{n+1}^{(\alpha-2),[1]}(c)}{L_{n}^{(\alpha-1)}(c)L_{n}^{(\alpha-1),[1]}(c)} - \frac{L_{n+2}^{(\alpha-3)}(c)}{L_{n+1}^{(\alpha-2)}(c)} + \mathcal{O}\left(n^{-1}\right)$$

Therefore, by (2.33) and (5.36)

$$\sqrt{n}F_n \cong -\sqrt{|c|}.$$

and, taking into account, (5.39) the result follows. ■

5.4.2 Laguerre Sobolev-type OPS: Asymptotics

Let $\{S_n^{(\alpha,M,N)}(x)\}_{n\geq 0}$ denote the sequence of polynomials orthogonal with respect to the discrete Sobolev inner product (5.1), where $d\mu(x) = x^\alpha e^{-x}dx$ and $c < 0$, normalized by the condition that $S_n^{(\alpha,M,N)}(x)$ has the same leading coefficient $\frac{(-1)^n}{n!}$ as the classical Laguerre orthogonal polynomial $L_n^{(\alpha)}(x)$. With this normalization we get

Theorem 5.5 *Let $M \geq 0$ and $N \geq 0$. There are real constants $B_{n,0}$, $B_{n,1}$, and $B_{n,2}$ such that*

$$S_n^{(\alpha,M,N)}(x) = B_{n,0}L_n^{(\alpha)}(x) + B_{n,1}(x-c)L_{n-1}^{(\alpha),[2]}(x) + B_{n,2}(x-c)^2 L_{n-2}^{(\alpha),[4]}(x), \qquad (5.40)$$

where $B_{n,0} = \dfrac{1}{1+A_{n,1}+A_{n,2}}$, $B_{n,1} = -\dfrac{A_{n,1}}{n(1+A_{n,1}+A_{n,2})}$, $B_{n,2} = \dfrac{A_{n,2}}{n(n-1)(1+A_{n,1}+A_{n,2})}$.

Moreover,

(*i*) *If $M > 0$ and $N > 0$, then*

$$B_{n,0} \cong \frac{8cn^\alpha}{M\left(L_n^{(\alpha)}(c)\right)^2}, \qquad B_{n,1} \cong -\frac{32c\sqrt{|c|}\,n^{\alpha-1/2}}{M\left(L_n^{(\alpha)}(c)\right)^2}, \qquad B_{n,2} \cong \frac{1}{n^2}. \qquad (5.41)$$

(*ii*) *If $M = 0$ and $N > 0$, then*

$$B_{n,0} \cong \frac{1}{4\sqrt{|c|n}}, \qquad B_{n,1} \cong -\frac{1}{n}, \qquad B_{n,2} \cong \frac{1}{4n^2\sqrt{|c|n}}.$$

(*iii*) *If $M > 0$ and $N = 0$, then*

$$B_{n,0} \cong \frac{\sqrt{|c|}}{Mn^{1/2-\alpha}\left(L_{n-1}^{(\alpha)}(c)\right)^2}, \qquad B_{n,1} \cong -\frac{1}{n}, \qquad B_{n,2} = 0.$$

Proof. From Theorem 5.2

$$S_n^{(\alpha,M,N)}(x) = \frac{(-1)^n \widehat{S}_n^{\alpha,M,N}(x)}{n!(1+A_{n,1}+A_{n,2})}$$

and, as a consequence,

$$S_n^{(\alpha,M,N)}(x) = B_{n,0}L_n^{(\alpha)}(x) + B_{n,1}(x-c)L_{n-1}^{(\alpha),[2]}(x) + B_{n,2}(x-c)^2 L_{n-2}^{(\alpha),[4]}(x),$$

where $B_{n,0} = \frac{1}{1+A_{n,1}+A_{n,2}}$, $B_{n,1} = -\frac{A_{n,1}}{n(1+A_{n,1}+A_{n,2})}$, $B_{n,2} = \frac{A_{n,2}}{n(n-1)(1+A_{n,1}+A_{n,2})}$.

Now, from Proposition 5.4 we can obtain the behavior of the coefficients $B_{n,0}$, $B_{n,1}$ and $B_{n,2}$ for n large enough. In order to estimate $A_{n,1}$ and $A_{n,2}$, first we compute $\alpha_n\beta_n$, α_n/γ_n, $\beta_n\gamma_n$ and $I_{2,n}(c)$. From (5.15) and Proposition 5.10

$$\begin{aligned}
\alpha_n\beta_n &= -\frac{I_{1,n}(c)}{[\widehat{L}_{n-1}^{\alpha,[2]}]'(c)} = \frac{\widehat{L}_n^{\alpha}(c)}{\widehat{L}_{n-1}^{\alpha}(c)\widehat{L}_{n-1}^{\alpha,[1]}(c)[\widehat{L}_{n-1}^{\alpha,[2]}]'(c)} ||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2 \\
&= -\frac{\Gamma(n+\alpha)}{\Gamma(n)} \frac{nL_n^{(\alpha)}(c)}{L_{n-1}^{(\alpha)}(c)L_{n-1}^{(\alpha),[1]}(c)\left(L_{n-1}^{(\alpha),[2]}\right)(c)} \cong \frac{8(-c)^{3/2}n^{\alpha-1/2}}{L_n^{(\alpha)}(c)L_n^{(\alpha+1)}(c)},
\end{aligned}$$

$$\begin{aligned}
\frac{\alpha_n}{\gamma_n} &= -\frac{I_{1,n}(c)[\widehat{L}_n^{\alpha}]'(c)}{\widehat{L}_n^{\alpha}(c)\widehat{L}_{n-1}^{\alpha,[2]}(c)} \\
&= \frac{[\widehat{L}_n^{\alpha}]'(c)}{\widehat{L}_{n-1}^{\alpha}(c)\widehat{L}_{n-1}^{\alpha,[1]}(c)\widehat{L}_{n-1}^{\alpha,[2]}(c)} ||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2 \\
&= \frac{\Gamma(n+\alpha)}{\Gamma(n)} \frac{nL_{n-1}^{(\alpha+1)}(c)}{L_{n-1}^{(\alpha)}(c)L_{n-1}^{(\alpha),[1]}(c)L_{n-1}^{(\alpha),[2]}(c)} \cong \frac{8(-c)^{3/2}n^{\alpha-1/2}L_n^{(\alpha+1)}(c)}{\left(L_n^{(\alpha)}(c)\right)^3},
\end{aligned}$$

$$\beta_n\gamma_n = \alpha_n\beta_n\frac{\gamma_n}{\alpha_n} \cong \left(\frac{L_n^{(\alpha)}(c)}{L_n^{(\alpha+1)}(c)}\right)^2 \cong -\frac{c}{n},$$

$$\begin{aligned}
I_{2,n}(c) &\cong (-1)^{n-1}(n-2)!\, n^{\alpha+3} \frac{L_{n-1}^{(\alpha)}(c)L_{n-1}^{(\alpha),[1]}(c)[L_{n-1}^{(\alpha),[2]}]'(c)}{L_{n-2}^{(\alpha)}(c)L_{n-2}^{(\alpha),[1]}(c)L_{n-2}^{(\alpha),[2]}(c)L_{n-3}^{(\alpha),[3]}(c)} \\
&\cong \frac{8c(-1)^{n-1}(n-2)!\, n^{\alpha+2}}{L_n^{(\alpha)}(c)}.
\end{aligned}$$

Next, we will analyze the following three situations.

(i) Let $M>0$ and $N>0$. Then,

$$A_{n,1}\cong -\frac{[\widehat{L}_n^{\alpha}]'(c)}{\widehat{L}_{n-1}^{\alpha,[2]}(c)}=\frac{n[L_n^{(\alpha)}]'(c)}{L_{n-1}^{(\alpha),[2]}(c)}=-\frac{nL_{n-1}^{(\alpha+1)}(c)}{L_{n-1}^{(\alpha),[2]}(c)}\cong -4\sqrt{|c|n},$$

$$A_{n,2}\cong -\frac{M\widehat{L}_n^{\alpha}(c)}{I_{n,2}(c)}\cong \frac{M\left(L_n^{(\alpha)}(c)\right)^2}{8cn^{\alpha}}.$$

Therefore,

$$B_{n,0}\cong \frac{8cn^{\alpha}}{M\left(L_n^{(\alpha)}(c)\right)^2},\qquad B_{n,1}\cong \frac{32c\sqrt{|c|}\,n^{\alpha-1/2}}{M\left(L_n^{(\alpha)}(c)\right)^2},\qquad B_{n,2}\cong \frac{1}{n^2}.$$

(ii) Let $M=0$ and $N>0$. Then,

$$\begin{aligned} A_{n,1} &\cong -4\sqrt{|c|n},\\ A_{n,2} &= -\frac{\widehat{L}_n^{\alpha}(c)}{I_{n,2}(c)}\frac{\alpha_n}{\gamma_n}\cong -1. \end{aligned}$$

Therefore,

$$B_{n,0}\cong -\frac{1}{4\sqrt{|c|n}},\qquad B_{n,1}\cong -\frac{1}{n},\qquad B_{n,2}\cong \frac{1}{4n^2\sqrt{|c|n}}.$$

(iii) Let $M>0$ and $N=0$. Then,

$$\begin{aligned} A_{n,1} &= \frac{M\widehat{L}_n^{\alpha}(c)}{I_{n,1}(c)}=-\frac{M\widehat{L}_{n-1}^{\alpha}(c)\widehat{L}_{n-1}^{\alpha,[1]}(c)}{||L_{n-1}^{(\alpha)}||_{\alpha}^2}\cong -\frac{Mn^{1/2-\alpha}}{\sqrt{|c|}}\left(L_{n-1}^{(\alpha)}(c)\right)^2,\\ A_{n,2} &= 0. \end{aligned}$$

Therefore,

$$B_{n,0}\cong -\frac{\sqrt{|c|}}{Mn^{1/2-\alpha}\left(L_{n-1}^{(\alpha)}(c)\right)^2},\qquad B_{n,1}\cong -\frac{1}{n},\qquad B_{n,2}=0.$$

■

Outer relative asymptotics

Finally, we deduce several asymptotic properties for discrete Laguerre-Sobolev polynomials when $M, N \geq 0$.

Theorem 5.6 *Uniformly on compact subsets of* $\mathbb{C}\backslash[0,\infty)$ *we get*

(a) *If* $M > 0$ *and* $N > 0$, *then*

$$\lim_{n\to\infty} \frac{S_n^{(\alpha,M,N)}(x)}{L_n^{(\alpha)}(x)} = \left(\frac{\sqrt{-x}-\sqrt{|c|}}{\sqrt{-x}+\sqrt{|c|}}\right)^2.$$

Notice that, according to the Hurwitz's theorem, the point c *attracts two negative zeros of* $S_n^{(\alpha,M,N)}(x)$ *for* n *large enough.*

(b) *If* $M = 0$ *and* $N > 0$ *or* $M > 0$ *and* $N = 0$, *then*

$$\lim_{n\to\infty} \frac{S_n^{(\alpha,M,N)}(x)}{L_n^{(\alpha)}(x)} = \frac{\sqrt{-x}-\sqrt{|c|}}{\sqrt{-x}+\sqrt{|c|}}.$$

Notice that, according to the Hurwitz's theorem, the point c *attracts one negative zero of* $S_n^{(\alpha,M,N)}(x)$ *for* n *large enough.*

Proof. We will prove the theorem when $M > 0$ and $N > 0$. The proofs of the other cases can be done in a similar way.

From (5.40)

$$\frac{S_n^{(\alpha,M,N)}(x)}{L_n^{(\alpha)}(x)} = B_{n,0} + nB_{n,1}(x-c)\frac{L_{n-1}^{(\alpha),[2]}(x)}{nL_n^{(\alpha)}(x)} + n^2B_{n,2}(x-c)^2\frac{L_{n-2}^{(\alpha),[4]}(x)}{n^2L_n^{(\alpha)}(x)}.$$

Now, (5.36) and (5.41) yield

$$\lim_{n\to\infty} \frac{S_n^{(\alpha,M,N)}(x)}{L_n^{(\alpha)}(x)} = (x-c)^2 \lim_{n\to\infty} \frac{L_{n-2}^{(\alpha),[4]}(x)}{n^2L_n^{(\alpha)}(x)} = \left(\frac{\sqrt{-x}-\sqrt{|c|}}{\sqrt{-x}+\sqrt{|c|}}\right)^2.$$

■

Mehler-Heine type formula

Theorem 5.7 *Mehler-Heine formula*

(a) *If* $M > 0$ *and* $N > 0$

$$\lim_{n\to\infty} \frac{S_n^{(\alpha,M,N)}(x/n)}{n^\alpha} = x^{-\alpha/2} J_\alpha(2\sqrt{x}),$$

(b) *If* $M = 0$ *and* $N > 0$ *or* $M > 0$ *and* $N = 0$

$$\lim_{n\to\infty} \frac{S_n^{(\alpha,M,N)}(x/n)}{n^\alpha} = -x^{-\alpha/2} J_\alpha(2\sqrt{x}),$$

uniformly on compact subsets of $\mathbb{C}$.

Proof. We will prove the theorem when $M > 0$ and $N > 0$. The proofs of the other cases can be done in a similar way.

Scaling the variable as $x \to x/n$ in (5.40) then dividing by n^α we get

$$\frac{S_n^{(\alpha,M,N)}(x/n)}{n^\alpha} = B_{n,0}\frac{L_n^{(\alpha)}(x/n)}{n^\alpha} + nB_{n,1}(x/n-c)\frac{L_{n-1}^{(\alpha),[2]}(x/n)}{n^{\alpha+1}} + n^2 B_{n,2}(x/n-c)^2\frac{L_{n-2}^{(\alpha),[4]}(x/n)}{n^{\alpha+2}}.$$

Now, (5.37) and (5.41) yield

$$\lim_{n\to\infty} \frac{S_n^{(\alpha,M,N)}(x/n)}{n^\alpha} = (-c)^2 \lim_{n\to\infty} \frac{L_{n-2}^{(\alpha),[4]}(x)}{n^{\alpha+2}} = x^{-\alpha/2} J_\alpha(2\sqrt{x}).$$

■

Plancherel-Rotach type outer asymptotics

Theorem 5.8 • *If* $M \geq 0$ *and* $N \geq 0$, *then*

$$\lim_{n\to\infty} \frac{S_n^{(\alpha,M,N)}(nx)}{L_n^{(\alpha)}(nx)} = 1,$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,4]$.

Proof. Dividing (5.40) by $L_n^{(\alpha)}(x)$ and scaling the variable as $x \to nx$ we get

$$\frac{S_n^{(\alpha,M,N)}(nx)}{L_n^{(\alpha)}(nx)} = B_{n,0} + nB_{n,1}\frac{nx-c}{n}\frac{L_{n-1}^{(\alpha),[2]}(nx)}{L_{n-1}^{(\alpha)}(nx)}\frac{L_{n-1}^{(\alpha)}(nx)}{L_n^{(\alpha)}(nx)} + n^2 B_{n,2}\frac{(nx-c)^2}{n^2}\frac{L_{n-2}^{(\alpha),[4]}(nx)}{L_{n-2}^{(\alpha)}(nx)}\frac{L_{n-2}^{(\alpha)}(nx)}{L_n^{(\alpha)}(nx)}.$$

From (2.35), (5.38) and (5.41)

$$\lim_{n\to\infty}\frac{S_n^{(\alpha,M,N)}(nx)}{L_n^{(\alpha)}(nx)} = x^2\left(\frac{\phi((x-2)/2)+1}{x}\right)^4 \frac{1}{(\phi((x-2)/2))^2}.$$

Now, using the fact that

$$(\phi(z)+1)^2 = 2(z+1)\phi(z),\ |z|>1,$$

we get our result. ■

5.4.3 Laguerre Sobolev-type OPS: Zeros

In this subsection, we analyze the behavior of the zeros of the discrete Laguerre-Sobolev polynomials $S_n^{(\alpha,M,N)}(x)$ when $M = 0$, i.e. $S_n^{(\alpha,0,N)}(x) = S_n^{(\alpha,N)}(x)$. We are interested to find results concerning the monotonicity and speed of convergence of the zeros of $S_n^{(\alpha,N)}(x)$ in terms of their dependence on N. We also get the interesting fact that the mass point c does not attract any zero of $\widehat{S}_n^{\alpha,N}(x)$ when $N \to \infty$ and the exact value for which the least zero of $S_n^{(\alpha,N)}(x)$ is located outside $[0, +\infty)$.

For this purpose we will need the useful Lemma B.1 (see Appendix B) concerning the behavior and the asymptotics of the zeros of a polynomial that is a linear combination of two polynomials of the same degree with interlacing zeros. Let us introduce the nth-degree monic polynomial,

$$\begin{aligned}\widehat{G}_{n,c}^{\alpha}(x) &= \lim_{N\to\infty}\left(\widehat{L}_n^{\alpha}(x) - N\frac{[\widehat{L}_n^{\alpha}]'(c)}{1+NK_{n-1}^{(1,1)}(c,c)}K_{n-1}^{(0,1)}(x,c)\right)\\ &= \widehat{L}_n^{\alpha}(x) - \frac{[\widehat{L}_n^{\alpha}]'(c)}{K_{n-1}^{(1,1)}(c,c)}K_{n-1}^{(0,1)}(x,c), \qquad (5.42)\end{aligned}$$

To characterize these polynomials, first we will observe that they are quasi-orthogonal of order 2 (see [11, Definition 1]) with respect to the modified measure $(x-c)^2x^\alpha e^{-x}dx$, which is the 2-iterated Christoffel perturbation of the Laguerre measure. It means that $\widehat{G}^{\alpha}_{n,c}(x)$ is a linear combination of three consecutive polynomials of the sequence $\{\widehat{L}_n^{\alpha,[2]}(x)\}_{n\geq 0}$ of monic polynomials orthogonal with respect to $(x-c)^2x^{\alpha}e^{-x}$. Indeed, for $n \geq 2$,

$$\widehat{G}^{\alpha}_{n,c}(x) = \widehat{L}_n^{\alpha,[2]}(x) + B_n\widehat{L}_{n-1}^{\alpha,[2]}(x) + C_n\widehat{L}_{n-2}^{\alpha,[2]}(x),$$

where B_n and $C_n \neq 0$ are real numbers. Since

$$C_n = \frac{\int_0^{\infty} \widehat{G}^{\alpha}_{n,c}(x)\widehat{L}_{n-2}^{\alpha,[2]}(x)(x-c)^2x^{\alpha}e^{-x}dx}{||\widehat{L}_{n-2}^{\alpha,[2]}||^2_{\alpha,[2]}}$$

we have

Proposition 5.12 *If c is a real negative number, then C_n is positive for every $n \geq 2$.*

Proof. We only need to study the sign of the numerator. According to (5.42)

$$\int_0^{\infty} \widehat{G}^{\alpha}_{n,c}(x)\widehat{L}_{n-2}^{\alpha,[2]}(x)(x-c)^2x^{\alpha}e^{-x}dx =$$

$$\int_0^{\infty} \widehat{L}_n^{\alpha}(x)\widehat{L}_{n-2}^{\alpha,[2]}(x)(x-c)^2x^{\alpha}e^{-x}dx - \frac{[\widehat{L}_n^{\alpha}]'(c)}{K_{n-1}^{(1,1)}(c,c)}\int_0^{\infty} K_{n-1}^{(0,1)}(x,c)\widehat{L}_{n-2}^{\alpha,[2]}(x)(x-c)^2x^{\alpha}e^{-x}dx$$

$$= \quad ||\widehat{L}_n^{\alpha}||^2_{\alpha} - \frac{[\widehat{L}_n^{\alpha}]'(c)}{K_{n-1}^{(1,1)}(c,c)}\int_0^{\infty} K_{n-1}^{(0,1)}(x,c)\widehat{L}_{n-2}^{\alpha,[2]}(x)(x-c)^2x^{\alpha}e^{-x}dx.$$

From (2.3) and (2.6) it follows immediately that

$$K_{n-1}^{(0,1)}(x,c) = K_n^{(0,1)}(x,c) - \frac{\widehat{L}_n^{\alpha}(x)[\widehat{L}_n^{\alpha}]'(c)}{||\widehat{L}_n^{\alpha}||^2_{\alpha}},$$

hence

$$\int_0^{\infty} K_{n-1}^{(0,1)}(x,c)\widehat{L}_{n-2}^{\alpha,[2]}(x)(x-c)^2x^{\alpha}e^{-x}dx =$$

$$\int_0^{\infty} K_n^{(0,1)}(x,c)\widehat{L}_{n-2}^{\alpha,[2]}(x)(x-c)^2x^{\alpha}e^{-x}dx - \frac{[\widehat{L}_n^{\alpha}]'(c)}{||\widehat{L}_n^{\alpha}||^2_{\alpha}}\int_0^{\infty} \widehat{L}_n^{\alpha}(x)\widehat{L}_{n-2}^{\alpha,[2]}(x)(x-c)^2x^{\alpha}e^{-x}dx.$$

The second integral in the right-hand side is

$$\int_0^{\infty} \widehat{L}_n^{\alpha}(x)\widehat{L}_{n-2}^{\alpha,[2]}(x)(x-c)^2x^{\alpha}e^{-x}dx = ||\widehat{L}_n^{\alpha}||^2_{\alpha},$$

while the first one vanishes because deg $K_n^{(0,1)}(x,c)$ = deg $\widehat{L}_{n-2}^{\alpha,[2]}(x)(x-c)^2 = n$, and therefore we can apply the property (2.10)

$$\int_0^\infty K_n^{(0,1)}(x,c)\, q(x)\, d\mu_\alpha = q'(c)\,.$$

If we denote $q(x) = \widehat{L}_{n-2}^{\alpha,[2]}(x)(x-c)^2$, then $q'(x) = [\widehat{L}_{n-2}^{\alpha,[2]}]'(x)\,(x-c)^2 + 2(x-c)\widehat{L}_{n-2}^{\alpha,[2]}(x)$ and consequently $q'(c) = 0$. Therefore

$$\int_0^\infty \widehat{G}_{n,c}^{\alpha}(x)\widehat{L}_{n-2}^{\alpha,[2]}(x)(x-c)^2 x^\alpha e^{-x}dx = ||\widehat{L}_n^\alpha(x)||_\alpha^2 + \frac{\left([\widehat{L}_n^\alpha]'(c)\right)^2}{K_{n-1}^{(1,1)}(c,c)}.$$

Thus,

$$C_n = \frac{\int_0^\infty \widehat{G}_{n,c}^{\alpha}(x)\widehat{L}_{n-2}^{\alpha,[2]}(x)(x-c)^2 x^\alpha e^{-x}dx}{||\widehat{L}_{n-2}^{\alpha,[2]}||_{\alpha,[k]}^2} > 0, \text{ for every } n \geq 2.$$

■

On the other hand, let $\{\eta_{n,k}\}_{k=1}^n \equiv \eta_{n,1} < \eta_{n,2} < ... < \eta_{n,n}$ be the zeros of $\widehat{S}_n^{\alpha,N}(x)$ and $\{x_{n,k}\}_{k=1}^n \equiv x_{n,1} < x_{n,2} < ... < x_{n,n}$ be the zeros of $\widehat{L}_n^\alpha(x)$. Notice that these zeros are real and simple (see [73], Proposition 3.2). Thus

Proposition 5.13 *([73], Proposition 6.2) The polynomial $\widehat{G}_{n,c}^{\alpha}(x)$ has n real and simple zeros $\{y_{n,k}\}_{k=1}^n \equiv y_{n,1} < y_{n,2} < ... < y_{n,n}$. The inequalities*

$$y_{n,1} < c < x_{n,1} < y_{n,2} < x_{n,2} < \cdots < y_{n,n} < x_{n,n} \tag{5.43}$$

hold for every $n \geq 2,\ n \in \mathbb{N}$.

Notice that $\widehat{S}_1^{\alpha,N}(x) = \widehat{L}_1^\alpha(x)$.

Next, we express $\widehat{S}_n^{\alpha,N}(x)$ in a proper way, in order to use Lemma B.1 to study the behavior of their zeros in terms of the mass N.

Proposition 5.14 *The polynomials $\{\widetilde{S}_n^{\alpha,N}(x)\}_{n\geq 0}$, with $\widetilde{S}_n^{\alpha,N}(x) = \lambda_{n-1}\widehat{S}_n^{\alpha,N}(x)$, can be represented as*

$$\widetilde{S}_n^{\alpha,N}(x) = \widehat{L}_n^\alpha(x) + NK_{n-1}^{(1,1)}(c,c)\,\widehat{G}_{n,c}^{\alpha}(x)\,, \tag{5.44}$$

where

$$\lambda_{n-1} = 1 + NK_{n-1}^{(1,1)}(c,c)\,.$$

Proof. First, we express the Laguerre Sobolev-type polynomials in terms of the standard Laguerre orthogonal polynomials $\widehat{L}_n^{\alpha}(x)$ and the Kernel polynomial (2.8). Taking into account the Fourier expansion

$$\widehat{S}_n^{\alpha,N}(x) = \widehat{L}_n^{\alpha}(x) + \sum_{k=0}^{n-1} a_{n,k}\widehat{L}_k^{\alpha}(x),$$

we obtain, for $k = 0, 1, 2, ..., n-1$,

$$a_{n,k} = \frac{-N\,[\widehat{S}_n^{\alpha,N}]'(c)[\widehat{L}_k^{\alpha}]'(c)}{||\widehat{L}_k^{\alpha}||_{\alpha}^2}.$$

Thus

$$\widehat{S}_n^{\alpha,N}(x) = \widehat{L}_n^{\alpha}(x) - N[\widehat{S}_n^{\alpha,N}]'(c)\sum_{k=0}^{n-1}\frac{[\widehat{L}_k^{\alpha}]'(c)\widehat{L}_k^{\alpha}(x)}{||\widehat{L}_k^{\alpha}||_{\alpha}^2}$$

and, from (2.6),

$$\widehat{S}_n^{\alpha,N}(x) = \widehat{L}_n^{\alpha}(x) - N\,[\widehat{S}_n^{\alpha,N}]'(c)K_{n-1}^{(0,1)}(x,c). \tag{5.45}$$

Our next step is to find $[\widehat{S}_n^{\alpha,N}]'(c)$. In order to do that, we take the derivative in the former expression and evaluate it at $x = c$, so

$$[\widehat{S}_n^{\alpha,N}]'(c) = \frac{[\widehat{L}_n^{\alpha}]'(c)}{1 + NK_{n-1}^{(1,1)}(c,c)},$$

and therefore

$$\widehat{S}_n^{\alpha,N}(x) = \widehat{L}_n^{\alpha}(x) - N\frac{[\widehat{L}_n^{\alpha}]'(c)}{1 + NK_{n-1}^{(1,1)}(c,c)}K_{n-1}^{(0,1)}(x,c). \tag{5.46}$$

Next, Replacing (5.42) in (5.44)

$$\begin{aligned}
\lambda_{n-1}\widehat{S}_n^{\alpha,N}(x) &= \widehat{L}_n^{\alpha}(x) + N\,K_{n-1}^{(1,1)}(c,c)\left(\widehat{L}_n^{\alpha}(x) - \frac{[\widehat{L}_n^{\alpha}]'(c)}{K_{n-1}^{(1,1)}(c,c)}K_{n-1}^{(0,1)}(x,c)\right),\\
\left(1 + NK_{n-1}^{(1,1)}(c,c)\right)\widehat{S}_n^{\alpha,N}(x) &= \left(1 + NK_{n-1}^{(1,1)}(c,c)\right)\widehat{L}_n^{\alpha}(x) - N\,[\widehat{L}_n^{\alpha}]'(c)K_{n-1}^{(0,1)}(x,c),\\
\widehat{S}_n^{\alpha,N}(x) &= \widehat{L}_n^{\alpha}(x) - N\frac{[\widehat{L}_n^{\alpha}]'(c)}{1 + NK_{n-1}^{(1,1)}(c,c)}K_{n-1}^{(0,1)}(x,c),
\end{aligned}$$

which is the connection formula (5.46). ■

We point out the fact the Laguerre Sobolev-type polynomial $\widehat{S}_n^{\alpha,N}(x)$ appears as a linear combination of two polynomials of degree n. Thus, from (5.44), (5.43), and Lemma B.1, we immediately conclude

Theorem 5.9 *If* $c \in \mathbb{R}_-$, *then*

$$y_{n,1} < \eta_{n,1} < x_{n,1} < y_{n,2} < \eta_{n,2} < x_{n,2} < \cdots < y_{n,n} < \eta_{n,n} < x_{n,n}.$$

Moreover, each $\eta_{n,k}$ *is a decreasing function of* N *and, for each* $k = 1, \ldots, n$,

$$\lim_{N\to\infty} \eta_{n,k} = y_{n,k},$$

as well as

$$\lim_{N\to\infty} N\,[\eta_{n,k} - y_{n,k}] = \frac{-\widehat{L}_n^{\alpha}(y_{n,k})}{[\widehat{G}_{n,c}^{\alpha}]'(y_{n,k})}.$$

Notice that the mass point c does not attract any zero of $\widehat{S}_n^{\alpha,N}(x)$, when $N \to \infty$, as in the standard case (i.e. when $M = N = 0$, see [43]). The least zero of $\widehat{S}_n^{\alpha,N}(x)$ is attracted by the least zero of the polynomial $\widehat{G}_{n,c}^{\alpha}(x)$. Next, we are going to look closely at this polynomial. Notice that $[\widehat{G}_{n,c}^{\alpha}]'(c) = 0$. We are interested to give an interesting enough extremal characterization of $\widehat{G}_{n,c}^{\alpha}(x)$.

Let

$$\widehat{R}_n(x) = x^n + \textit{lower degree terms}$$

be a monic polynomial and consider the optimization problem with constrains

$$\text{minimize} \quad \begin{cases} & ||\widehat{R}_n||_\alpha^2 = \int_0^\infty |\widehat{R}_n(x)|^2 x^\alpha e^{-x} dx \\ \text{with} & \widehat{R}_n(x) = x^n + \textit{ lower degree terms} \\ \text{and} & [\widehat{R}_n]'(c) = 0. \end{cases} \tag{5.47}$$

It can be solved in the following way. Let

$$\begin{aligned} \widehat{R}_n(x) &= \widehat{L}_n^{\alpha}(x) + \sum_{k=0}^{n-1} a_{n,k}\,\ell_k^{\alpha}(x), \\ [\widehat{R}_n]'(x) &= [\widehat{L}_n^{\alpha}]'(x) + \sum_{k=0}^{n-1} a_{n,k}\,[\ell_k^{\alpha}]'(x), \end{aligned} \tag{5.48}$$

where $\{\ell_k^{\alpha}(x)\}_{k\geq 0}$ is the orthonormal Laguerre polynomial sequence. Then,

$$||\widehat{R}_n||_\alpha^2 = ||\widehat{L}_n^{\alpha}||_\alpha^2 + \sum_{k=0}^{n-1} |a_{n,k}|^2. \tag{5.49}$$

On the other hand, taking $x = c$ in (5.48),

$$\begin{aligned}
[\widehat{R}_n]'(c) &= [\widehat{L}_n^{\alpha}]'(c) + \sum_{k=0}^{n-1} a_{n,k}\,[\ell_k^{\alpha}]'(c) = 0,\\
-[\widehat{L}_n^{\alpha}]'(c) &= \sum_{k=0}^{n-1} a_{n,k}\,[\ell_k^{\alpha}]'(c).
\end{aligned}$$

Next, using the Cauchy-Schwarz inequality we get

$$\left|[\widehat{L}_n^{\alpha}]'(c)\right|^2 \leq \sum_{k=0}^{n-1} |a_{n,k}|^2 \sum_{k=0}^{n-1} \left|[\ell_k]'(c)\right|^2$$

or, equivalently,

$$\frac{\left|[\widehat{L}_n^{\alpha}]'(c)\right|^2}{K_{n-1}^{(1,1)}(c,c)} \leq \sum_{k=0}^{n-1} |a_{n,k}|^2.$$

Thus, taking into account (5.49), the infimum of (5.47) is

$$||\widehat{L}_n^{\alpha}||_{\alpha}^2 + \frac{\left|[\widehat{L}_n^{\alpha}]'(c)\right|^2}{K_{n-1}^{(1,1)}(c,c)}$$

that is attained by the polynomial $\widehat{G}_{n,c}^{\alpha}(x)$.

The Minimum Mass

When $c \in \mathbb{R}_-$, at most one of the zeros of $\widehat{S}_n^{\alpha,N}(x)$ is located outside $[0,+\infty)$. Next we provide the explicit value N_0 of the mass such that for $N > N_0$ this situation appears, i.e, one of the zeros is located outside $[0,+\infty)$.

Corollary 5.1 *If $c \in \mathbb{R}_-$, then the least zero $\eta_{n,1} = \eta_{n,1}(c)$ satisfies*

$$\begin{aligned}
\eta_{n,1} > 0, &\quad \textit{for } \ N < N_0,\\
\eta_{n,1} = 0, &\quad \textit{for } \ N = N_0,\\
\eta_{n,1} < 0, &\quad \textit{for } \ N > N_0,
\end{aligned}$$

where

$$N_0 = N_0(n,\alpha,c) = \left(\frac{[\widehat{L}_n^{\alpha}]'(c)}{\widehat{L}_n^{\alpha}(0)} K_{n-1}^{(0,1)}(0,c) - K_{n-1}^{(1,1)}(c,c)\right)^{-1} > 0. \qquad (5.50)$$

Proof. It suffices to use (5.46) together with the fact that $\widehat{S}_n^{\alpha,N}(0) = 0$ if and only if $N = N_0$

$$\begin{aligned} \widehat{S}_n^{\alpha,N}(0) &= \widehat{L}_n^{\alpha}(0) - N_0 \frac{[\widehat{L}_n^{\alpha}]'(c)}{1 + N_0 K_{n-1}^{(1,1)}(c,c)} K_{n-1}^{(0,1)}(0,c) = 0, \\ \widehat{L}_n^{\alpha}(0) &= N_0 \frac{[\widehat{L}_n^{\alpha}]'(c)}{1 + N_0 K_{n-1}^{(1,1)}(c,c)} K_{n-1}^{(0,1)}(0,c), \\ N_0 &= N_0(n,\alpha,c) = \left(\frac{[\widehat{L}_n^{\alpha}]'(c)}{\widehat{L}_n^{\alpha}(0)} K_{n-1}^{(0,1)}(0,c) - K_{n-1}^{(1,1)}(c,c) \right)^{-1}. \end{aligned}$$

■

Notice that, according to the Hurwitz's theorem, for n large enough, only one zero of $\widehat{S}_n^{\alpha,N}$ is located outside of $[0,+\infty)$ and it is attracted by c.

Next we show some numerical experiments using Mathematica® software, dealing with the least zero of Laguerre Sobolev-type polynomials. We are interested to show the location and behavior of this least zero. In the first two tables we recover the results in [89] when the mass point is located at $x = 0$, for $n = 2,\ 3$ and $\alpha = -1/2,\ 1,\ 5$. (Notice that in this work the authors label the zeros in a reverse order).

N	$\eta_{2,1}(-1/2)$	N	$\eta_{2,1}(1)$	N	$\eta_{2,1}(5)$
1/2	0.115964	3/2	0.271499	710	0.0419159
$N_0 = \sqrt{\pi}/2$	0	$N_0 = 2$	0	$N_0 = 720$	0
1	−0.0313955	5/2	−0.230139	730	−0.0414199

N	$\eta_{3,1}(-1/2)$	N	$\eta_{3,1}(1)$	N	$\eta_{3,1}(5)$
1/4	0.00211646	1/5	0.407703	79	0.0251697
$N_0 = \sqrt{\pi}/7$	0	$N_0 = 2/5$	0	$N_0 = 80$	0
1/2	−0.133233	3/2	−0.275762	81	−0.0248324

In the next two tables, we show the position for the first and second zeros of Laguerre Sobolev-type polynomial of degree $n = 15$ and $\alpha = 0$, for some choices of the mass N. For $N = 0$ obviously we recover the least zero and the second zero of the classical Laguerre polynomials (in bold). When the mass point is located at $c = 0$ we get

$\eta_{15,k}$	$N = 0$	$N = 5.0 \cdot 10^{-12}$	$N = 5.0 \cdot 10^{-8}$	$N = 5.0 \cdot 10^{-4}$	$N = 5.0 \cdot 10^{-2}$
$k = 1$	**0.0933078**	0.0933078	0.0933046	0.0620821	−0.146205
$k = 2$	**0.492692**	0.492692	0.492682	0.417657	0.263754

as well as when the mass point is located at $c = -1$.

$\eta_{15,k}$	$N = 0$	$N = 5.0 \cdot 10^{-12}$	$N = 5.0 \cdot 10^{-8}$	$N = 5.0 \cdot 10^{-4}$	$N = 5.0 \cdot 10^{-2}$
$k = 1$	**0.0933078**	0.0933076	0.0915341	-1.35377	-1.36544
$k = 2$	**0.492692**	0.492691	0.485200	0.148587	0.148434

In the next two tables we provide numerical evidences in support of Corollary 5.1, where the exact values of N_0 are calculated for two specific cases. For this purpose we begin by analyzing the least zero of the Laguerre Sobolev-type polynomials of degree $n = 7$, $\alpha = 2$ and with the mass point located at $c = -2$. Calculations show that for the values of N_0 given by (5.50), we have $N_0 = 3.21582 \cdot 10^{-4} \in (3.0 \cdot 10^{-4}, 4.0 \cdot 10^{-4})$.

$\eta_{7,k}:$	$N = 0$	$N = 5.0 \cdot 10^{-5}$	$N = 3.0 \cdot 10^{-4}$	$N = 4.0 \cdot 10^{-4}$	$N = 5.0 \cdot 10^{-3}$
$k = 1$	**0.783096**	0.705892	0.0636699	**−0.775950**	-2.70450

The table below shows that, with the mass point located at $c = -1$, we need larger values of N_0 to get the least zero as a negative real number. Now the estimate is $1.0 \cdot 10^{-3} < N_0 < 2.0 \cdot 10^{-3}$, according to the exact value $N_0(7, 2, -1) = 1.88442 \cdot 10^{-3}$.

$\eta_{7,k}:$	$N = 0$	$N = 5.0 \cdot 10^{-4}$	$N = 1.0 \cdot 10^{-3}$	$N = 2.0 \cdot 10^{-3}$	$N = 5.0 \cdot 10^{-2}$
$k = 1$	**0.783096**	0.603763	0.384610	**−0.0452617**	-1.81059

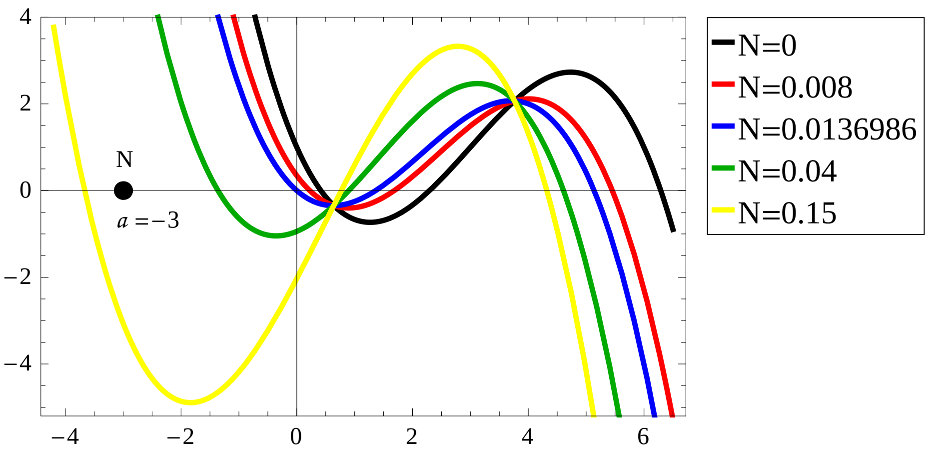
Another interesting question is to study, for a fixed value N, the behavior of zeros of Laguerre Sobolev-type polynomials in terms of the parameter α. Notice that, for a fixed value of α we can loose its negative zero, as it occurs in the standard case (see [23]). We show the behavior of the first two zeros to give more information about their relative spacing. For instance, let us show the first two zeros of the Laguerre Sobolev-type polynomials of degree $n = 12$, when $N = 1.5 \cdot 10^{-7}$ and the mass point is located at $c = -3$

$\eta_{12,k}:$	$\alpha = -1/2$	$\alpha = 0$	$\alpha = 2$	$\alpha = 3$	$\alpha = 5$
$k = 1$	-2.81937	-2.52014	-0.0397219	**0.625246**	**1.29029**
$k = 2$	0.0716143	0.164964	0.855437	1.54668	2.57453

and again, the first two zeros when $N = 5.0 \cdot 10^{-5}$ and $c = -1$.

$\eta_{12,k}:$	$\alpha = -1/2$	$\alpha = 0$	$\alpha = 2$	$\alpha = 3$	$\alpha = 5$
$k = 1$	-0.16977	-0.185167	**0.242738**	**0.600667**	**1.27787**
$k = 2$	0.137987	0.272018	1.0244	1.53932	2.55799

Finally, we show the graph of the Laguerre Sobolev-type polynomial of degree 3, $S_3^{(0,N)}(x)$ for several values of the mass N, located at $c = -3$. Obviously, when $N = 0$ we recover the standard Laguerre polynomial of degree 3, $L_3^{(0)}(x)$, showed in the picture with black graph. Using the formula (5.50) we get the value $N_0(3, 0, -3) = 0.0136986$. For $N < N_0$, the least zero of $S_3^{(0,N)}(x)$ is still positive (red graph). For the exact value $N = N_0(3, 0, -3)$, we have $\tilde{x}_{3,1} = 0$ (blue graph) and with slightly higher values of $N > N_0$, we get $\tilde{x}_{3,1} < 0$ (green graph). Further increasing the value of N we get $\tilde{x}_{3,1}$ out of the interval $[c, +\infty)$ (yellow graph).



APPENDIX A

Conclusions and Future Research

A.1 Main contributions

Here we summarize the contributions of this dissertation.

- For the first time a comprehensive study of the behavior of zeros of polynomials orthogonal with respect to Uvarov and Christoffel perturbed measures is done. The behavior of the zeros is given in terms of the parameter M, which determines how important the perturbation on the classical measure is. So far, significant progress in this direction have been done through semiclassical approximations, as in [3] and only concerning the behavior of the mean average properties of zeros using the WKB method.

- Asymptotic results for the MOPS with respect to the Uvarov transformation of the Laguerre measure, as a canonical example of unbounded supported measure, are deduced when the mass point is located outside the support of the Laguerre measure. Up to date, the mass points were located at the boundary/boundaries of the support of the measure.

- In case of a finitely many mass points outside the support of the Laguerre measure, an electrostatic model is provided. To date, the only similar work is considering a single mass point at the origin. We describe the behavior of zeros of Krall-Laguerre polynomials in terms of the zeros of a polynomial of degree $2m$ (being m the number of Dirac masses on the measure), which are sources of a short range potential field in the location of the zeros of the Krall-Laguerre OPS as critical points in the equilibrium problem.

- Asymptotic properties of Laguerre Sobolev-type MOPS, when the mass point is located outside the support of the Laguerre measure are deduced. Up to date, the mass points were located at the boundary/boundaries of the support of the measure.

The original results contained in this memoir have been published in several international research Journals, all of them indexed in the *Journal of Citation Reports*®, as follows (the numbers in square brackets means the order in which the corresponding work appear in the bibliography)

[23] H. Dueñas, E. J. Huertas, and F. Marcellán, *Analytic properties of Laguerre-type orthogonal polynomials*, Integral Transforms Spec. Funct. **22** (2011), 107-122.

[24] H. Dueñas, E. J. Huertas, and F. Marcellán, *Asymptotic properties of Laguerre-Sobolev type orthogonal polynomials*, Numer. Algorithms **60** (1), (2012), 51-73.

[32] F. Marcellán, R. Xh. Zejnullahu, B. Xh. Fejzullahu, and E. J. Huertas, *On orthogonal polynomials with respect to certain discrete Sobolev inner product*, Pacific J. Math. **257** (1), (2012), 167-188.

[43] E. J. Huertas, F. Marcellán and F. R. Rafaeli, *Zeros of Orthogonal Polynomials Generated by Canonical Perturbations of Measures*, Appl. Math. Comput. **218**, (2012), 7109-7127.

[44] E. J. Huertas, F. Marcellán, and H. Pijeira, *An Electrostatic Model for Zeros of Laguerre Polynomials*. Submitted to Proceedings of the American Mathematical Society, December 2011. Under review.

A.2 Open problems

Finally, we discuss some related work as well as we propose a set of open problems for a future research.

P1. Infinitely many mass points

Let $\{Q_n^{(\alpha,m)}(x)\}_{n\geq 0}$ be the sequence of Krall-Laguerre orthogonal polynomials considered in Chapter 3. The problem is to describe the behavior of $Q_n^{(\alpha,m)}(x)$, relative to the Laguerre OPS $\{L_n^{(\alpha)}(x)\}_{n\geq 0}$, when an infinite number of mass points is added in the negative real semiaxis $\mathbb{R}_-$, i.e. $m \to +\infty$. Namely, we would like to evaluate the limit as $m \to \infty$ in (4.36)

$$\lim_{n\to\infty} \frac{Q_n^{(\alpha,m)}(x)}{L_n^{(\alpha)}(x)} = \prod_{k=1}^{m} \frac{\sqrt{-x} - \sqrt{|c_k|}}{\sqrt{-x} + \sqrt{|c_k|}}. \tag{A.1}$$

Notice that we deal with a nonincreasing sequence $\{c_k\}_{k=1}^m$ as $k = 1, \ldots, m$.

P2. Mass points inside the support of the Laguerre measure

The results of Chapters 4 and 5 for Krall-Laguerre and Laguerre-Sobolev-type orthogonal polynomials were obtained when the mass points are located outside the support of the Laguerre measure. It will be of interest to study the zeros and the asymptotic behavior for this measure when the mass points belong to $(0, +\infty)$.

P3. Electrostatic model in more general Sobolev cases

Find electrostatic models for the zeros of polynomials orthogonal with respect to the following discrete Sobolev inner products

$$\langle f, g\rangle_S = \int_0^\infty f(x)g(x)d\mu + Mf^{(j)}(c)g^{(j)}(c), \quad c \notin [0, +\infty),\, M \in \mathbb{R}_+,\, j \in \mathbb{N}, \tag{A.2}$$

and

$$\langle f, g\rangle_{S,m} = \int_0^\infty f(x)g(x)d\mu + \sum_{k=0}^{m} M_k\, f^{(j)}(c_k)g^{(j)}(c_k), \quad \{c_k\}_{k=0}^m \notin [0, +\infty),\, M_k \in \mathbb{R}_+. \tag{A.3}$$

P4. General positive Borel measures with unbounded support

Let study the above questions for general positive Borel measures μ supported on $\mathbb{R}$.

P5. Geronimus transformation of measures

Given the Geronimus canonical transformation of a positive Borel measure (2.14), it would be interesting to explore the behavior of the zeros of polynomials orthogonal with respect to the inner product

$$\langle f, g\rangle_G = \int_E f(x)g(x)\frac{1}{x-c}d\mu + Mf(c)g(c), \quad c \notin E,\, M \in \mathbb{R}_+$$

in terms of M.

P6. Connection with vector orthogonality

Motivated by the recent work [10], the analysis of the matrix orthogonality provided in section (5.3) would be a nice approach to study analytic properties of such matrix orthogonal polynomials generated from the Sobolev type orthogonality. In particular, an interesting problem would be to analyze if they satisfy ordinary linear differential equations with matrix polynomials as coefficients.

APPENDIX B

Interlacing Lemma

The following Lemma deals with the behavior of the zeros of a linear combination of two polynomials with interlacing zeros. For the convenience of the reader, since this is central to understand some results of this thesis, we reproduce here the complete proof. The best general references here are [90, p. 31-33], [12, Lemma 1] or [19, Lemma 3]. This proof is due to C. Bracciali, D. Dimitrov and S. Ranga.

Lemma B.1 *Let $h_n(x) = a(x - x_1)\cdots(x - x_n)$ and $g_n(x) = b(x - y_1)\cdots(x - y_n)$ be two polynomials with real zeros, where a and b are positive constants.*

(*i*) *If*

$$y_1 < x_1 < \cdots < y_n < x_n,$$

then, for any real constant $c > 0$, the polynomial

$$f(x) = h_n(x) + cg_n(x)$$

has n real zeros $\eta_1 < \cdots < \eta_n$ which interlace with the zeros of $h_n(x)$ and $g_n(x)$ as follows

$$y_1 < \eta_1 < x_1 < \cdots < y_n < \eta_n < x_n.$$

Moreover, each $\eta_k = \eta_k(c)$ *is a decreasing function of* c*, and for every* $k = 1, \ldots, n$,

$$\lim_{c\to\infty} \eta_k = y_k \qquad \text{and}$$
$$\lim_{c\to\infty} c[\eta_k - y_k] = \frac{-h_n(y_k)}{g_n'(y_k)}$$

(*ii*) *If*

$$x_1 < y_1 < \cdots < x_n < y_n,$$

then, for any real constant $c > 0$*, the polynomial*

$$f(x) = h_n(x) + cg_n(x)$$

has n *real zeros* $\eta_1 < \cdots < \eta_n$ *which interlace with the zeros of* $h_n(x)$ *and* $g_n(x)$ *as follows*

$$x_1 < \eta_1 < y_1 < \cdots < x_n < \eta_n < y_n. \tag{B.1}$$

Moreover, each $\eta_k = \eta_k(c)$ *is an increasing function of* c*, and for every* $k = 1, \ldots, n$,

$$\begin{aligned} &\lim_{c\to\infty} \eta_k = y_k \qquad \text{and} \\ &\lim_{c\to\infty} c[y_k - \eta_k] = \frac{h_n(y_k)}{g_n'(y_k)} \end{aligned} \tag{B.2}$$

Proof. It is enough to prove (*ii*) because the case (*i*) is similar. Since $h_n(x)$ and $g_n(x)$ are monic polynomials with

$$x_1 < y_1 < \cdots < x_n < y_n,$$

we have

$$sign\, f(x_k) = sign\, g_n(x_k) = (-1)^{n-k+1}, \qquad k = 1, \ldots, n.$$

Hence, there exist $n-1$ zeros $\eta_1, \ldots, \eta_{n-1}$ of $f(x)$ such that

$$x_1 < \eta_1 < x_2 < \ldots < x_{n-1} < \eta_{n-1} < x_n.$$

The existence of $\eta_n > x_n$ follows from $f(x_n) < 0$ and the fact that $f(x) \to +\infty$ as $x \to +\infty$. Since similar arguments apply to the case

$$\eta_1 < y_1 < \cdots < \eta_n < y_n,$$

we conclude that (B.1) is true.

To prove the monotonicity of zeros of in terms of c, we define the polynomial

$$f_\varepsilon(x) = h_n(x) + (c+\varepsilon)g_n(x),$$

with $\varepsilon \geq 0$, and let denote its zeros by $\eta_1(\varepsilon) < \cdots < \eta_n(\varepsilon)$. It is easy to check that $\eta_k = \eta_k(0)$ and

$$f_\varepsilon(x) = f(x) + \varepsilon g_n(x).$$

Therefore, $f_\varepsilon(\eta_k) = \varepsilon g_n(\eta_k)$ and then, for $\varepsilon > 0$,

$$sign\, f_\varepsilon(\eta_k) = sign\, g_n(\eta_k) = (-1)^{n-k+1}$$

because the interlacing property (B.1). Thus, $\eta_k < \eta_k(\varepsilon)$, i.e. every zero η_k is an increasing function of c.

For the proof of the limits (B.2), we define the polynomial $q(x)$ by

$$q(x) = \frac{1}{c}h_n(x) + g_n(x).$$

Notice that the zeros of $f(x)$ and $q(x)$ coincide for each c. Since

$$\lim_{c\to\infty} q(x) = g_n(x),$$

by the Hurwitz's theorem (see [99]), the zeros η_k of $f(x)$ converge to the zeros y_k of $g_n(x)$ when c tends to infinity.

Next, by the Mean Value Theorem, there exist real real numbers $\theta_k \in (\eta_k, y_k)$, $k = 1, \ldots, n$, such that

$$\frac{cg_n(y_k) - cg_n(\eta_k)}{y_k - \eta_k} = cg_n'(\theta_k)$$

or, equivalently,

$$c[y_k - \eta_k] = \frac{h_n(\eta_k)}{g_n'(\theta_k)}.$$

On the other hand, $h_n(x)$ and $g_n(x)$ are polynomials with simple zeros. Since $g'(y_k) \neq 0$, then there exists $\delta_1 > 0$ such that

$$m_k = \min\{|g_n'(x)| : x \in [y_k - \delta_1, y_k]\} \neq 0.$$

Let $\varepsilon > 0$. Hence, there exists $\delta_2 > 0$, with $\delta_2 < \delta_1$, such that

$$|h_n(x) - h_n(y_k)| < \frac{\varepsilon m_k}{2}, \qquad |g_n(x) - g_n(y_k)| < \frac{\varepsilon m_k^2}{2|h_n(y_k)|},$$

when $x \in [y_k - \delta_2, y_k]$. Since $\eta_k, \theta_k \to y_k$ as $c \to \infty$, then there exists $c_0 > 0$ such that, for all $c > c_0$, $\eta_k,\ \theta_k \in [y_k - \delta_2, y_k]$. Thus, for $c > c_0$

$$\begin{aligned}
\left|\frac{h_n(\eta_k)}{g_n'(\theta_k)} - \frac{h_n(y_k)}{g_n'(y_k)}\right| &= \left|\frac{h_n(\eta_k)g_n'(y_k) - h_n(y_k)g_n'(\theta_k)}{g_n'(\theta_k)g_n'(y_k)}\right| \\
&\leq |h_n(\eta_k) - h_n(y_k)|\frac{1}{|g_n'(\theta_k)|} + |g_n'(\theta_k) - g_n'(y_k)|\frac{|h_n(y_k)|}{|g_n'(\theta_k)||g_n'(y_k)|} \\
&\leq |h_n(\eta_k) - h_n(y_k)|\frac{1}{m_k} + |g_n'(\theta_k) - g_n'(y_k)|\frac{|h_n(y_k)|}{m_k^2} \\
&< \varepsilon.
\end{aligned}$$

Hence,

$$\lim_{c\to\infty} c[y_k - \eta_k] = \lim_{c\to\infty} \frac{h_n(\eta_k)}{g_n'(\theta_k)} = \frac{h_n(y_k)}{g_n'(y_k)},$$

and the proof is complete. ■

APPENDIX C

Rate of Convergence

In this appendix, from the Perron's formula (2.33) we prove two lemmas concerning the rate of convergence of the ratio of two standard Laguerre polynomials as $n \to \infty$. Throughout this dissertation, we have needed to compute several ratios of Laguerre polynomials with different parameters and degrees, and this is the reason why these Lemmas will be useful. Taking $p = 3$ in (2.33) we have

$$\begin{aligned} L_n^{(\alpha)}(x) &= \frac{1}{2}\pi^{-1/2}e^{x/2}(-x)^{-\alpha/2-1/4}\, n^{\alpha/2-1/4}e^{2(-nx)^{1/2}} \\ &\quad \times \left\{ C_0(\alpha;x) + C_1(\alpha;x)n^{-1/2} + C_2(\alpha;x)n^{-1} + \mathcal{O}\left(n^{-3/2}\right) \right\}, \end{aligned}$$

where every $C_\nu(\alpha;x)$ is independent of n, but depends on α. Thus,

$$\begin{aligned} \frac{L_{n+j}^{(\alpha)}(x)}{L_{n+k}^{(\beta)}(x)} &= (-x)^{-\alpha/2+\beta/2}\left[\frac{(n+j)^{\alpha/2-1/4}}{(n+k)^{\beta/2-1/4}}\right] e^{\left[2(-(n+j)x)^{1/2}-\left(2(-(n+k)x)^{1/2}\right)\right]} \\ &\quad \times \frac{C_0(\alpha;x)+C_1(\alpha;x)(n+j)^{-1/2}+C_2(\alpha;x)\,(n+j)^{-1}+\mathcal{O}\left((n+j)^{-3/2}\right)}{C_0(\beta;x)+C_1(\beta;x)(n+k)^{-1/2}+C_2(\beta;x)\,(n+k)^{-1}+\mathcal{O}\left((n+k)^{-3/2}\right)}. \end{aligned} \tag{C.1}$$

First, we study the term inside the square brackets. From $(n+j)^{\frac{\alpha}{2}-\frac{1}{4}} = n^{\frac{\alpha}{2}-\frac{1}{4}}\left(1+\frac{j}{n}\right)^{\frac{\alpha}{2}-\frac{1}{4}}$

and $(n+k)^{\frac{\beta}{2}-\frac{1}{4}} = n^{\frac{\beta}{2}-\frac{1}{4}}\left(1+\frac{k}{n}\right)^{\frac{\beta}{2}-\frac{1}{4}}$, we obtain

$$\left[\frac{(n+j)^{\alpha/2-1/4}}{(n+k)^{\beta/2-1/4}}\right] = \frac{n^{\frac{\alpha}{2}-\frac{1}{4}}\left(1+\frac{j}{n}\right)^{\frac{\alpha}{2}-\frac{1}{4}}}{n^{\frac{\beta}{2}-\frac{1}{4}}\left(1+\frac{k}{n}\right)^{\frac{\beta}{2}-\frac{1}{4}}} = n^{\frac{\alpha}{2}-\frac{\beta}{2}}\frac{\left(1+\frac{j}{n}\right)^{\frac{\alpha}{2}-\frac{1}{4}}}{\left(1+\frac{k}{n}\right)^{\frac{\beta}{2}-\frac{1}{4}}}. \tag{C.2}$$

Next, using the expansion

$$(1+z)^y = 1 + yz + \mathcal{O}\left(z^2\right), \ \ |z|<1,$$

for both $y = \frac{\alpha}{2}-\frac{1}{4}$ with $z=\frac{j}{n}$ and $y = -\left(\frac{\beta}{2}-\frac{1}{4}\right)$ with $z = \frac{k}{n}$, we obtain

$$\frac{(n+j)^{\alpha/2-1/4}}{(n+k)^{\beta/2-1/4}} = n^{\frac{\alpha}{2}-\frac{\beta}{2}}\left[1+\frac{j}{n}\left(\frac{\alpha}{2}-\frac{1}{4}\right)+\mathcal{O}\left(\frac{1}{n^2}\right)\right]\times\left[1-\frac{k}{n}\left(\frac{\beta}{2}-\frac{1}{4}\right)+\mathcal{O}\left(\frac{1}{n^2}\right)\right].$$

The product of the last two square brackets yields

$$1+\frac{j}{n}\left(\frac{\alpha}{2}-\frac{1}{4}\right)-\frac{k}{n}\left(\frac{\beta}{2}-\frac{1}{4}\right)+\mathcal{O}\left(\frac{1}{n^2}\right).$$

As a conclusion, (C.2) becomes

$$\frac{(n+j)^{\alpha/2-1/4}}{(n+k)^{\beta/2-1/4}} = n^{\frac{\alpha}{2}-\frac{\beta}{2}}\left(1+\left[j\left(\frac{\alpha}{2}-\frac{1}{4}\right)-k\left(\frac{\beta}{2}-\frac{1}{4}\right)\right]\frac{1}{n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right). \tag{C.3}$$

On the other hand, in (C.1)

$$e^{\left[2(-(n+j)x)^{1/2}-\left(2(-(n+k)x)^{1/2}\right)\right]} = e^{2\sqrt{-x}\left((n+j)^{1/2}-(n+k)^{1/2}\right)}. \tag{C.4}$$

But

$$\begin{aligned}(n+j)^{1/2}-(n+k)^{1/2} &= n^{1/2}\left(\left(1+\frac{j}{n}\right)^{1/2}-\left(1+\frac{k}{n}\right)^{1/2}\right)\\ &= \frac{1}{2}(j-k)\,n^{-1/2}+\mathcal{O}(n^{-3/2}).\end{aligned}$$

Therefore, (C.4) becomes

$$e^{2\sqrt{-x}\left((n+j)^{1/2}-(n+k)^{1/2}\right)} = \exp\left[\frac{\sqrt{-x}}{\sqrt{n}}(j-k)+\mathcal{O}(n^{-3/2})\right],$$

and using

$$\exp z = 1 + z + \frac{1}{2}z^2 + \mathcal{O}\left(z^3\right),$$

we can write

$$e^{2\sqrt{-x}\left((n+j)^{1/2}-(n+k)^{1/2}\right)} = 1 + \frac{\sqrt{-x}}{\sqrt{n}}(j-k) - \frac{x}{2n}(j-k)^2 + \mathcal{O}(n^{-3/2}). \tag{C.5}$$

Multiplying (C.3) and (C.5) we can rewrite (C.1) as

$$\frac{L_{n+j}^{(\alpha)}(x)}{L_{n+k}^{(\beta)}(x)} = (-x)^{-\frac{\alpha}{2}+\frac{\beta}{2}} n^{\frac{\alpha}{2}-\frac{\beta}{2}} \times$$

$$\left(1 + \frac{\sqrt{-x}}{\sqrt{n}}(j-k) + \left[\left(\frac{\alpha}{2}-\frac{1}{4}\right)j - \left(\frac{\beta}{2}-\frac{1}{4}\right)k - \frac{x}{2}(j-k)^2\right]\frac{1}{n} + \mathcal{O}(n^{-3/2})\right) \tag{C.6}$$
$$\times \frac{C_0(\alpha;x) + C_1(\alpha;x)(n+j)^{-1/2} + C_2(\alpha;x)(n+j)^{-1} + \mathcal{O}\left((n+j)^{-3/2}\right)}{C_0(\beta;x) + C_1(\beta;x)(n+k)^{-1/2} + C_2(\beta;x)(n+k)^{-1} + \mathcal{O}\left((n+k)^{-3/2}\right)}.$$

Next we will state two useful lemmas considering some particular values of the parameters α, β, j and k.

Lemma C.1 *Given two standard Laguerre polynomials of the same parameter α and different degree, the following statement holds. For $x \in \mathbb{C}\setminus\mathbb{R}_+$*

$$\frac{L_{n+j}^{(\alpha)}(x)}{L_n^{(\alpha)}(x)} = 1 + \frac{\sqrt{-x}}{\sqrt{n}}j + \left[\left(\frac{\alpha}{2}-\frac{1}{4}\right)j - \frac{x}{2}j^2\right]\frac{1}{n} + \mathcal{O}(n^{-3/2})$$

where $\sqrt{-x}$ must be taken real and positive if $x < 0$.

Proof. Letting $\alpha = \beta$ and $k = 0$ in (C.6) yields

$$\frac{L_{n+j}^{(\alpha)}(x)}{L_n^{(\alpha)}(x)} = \left(1 + \frac{\sqrt{-x}}{\sqrt{n}}j + \left[\left(\frac{\alpha}{2}-\frac{1}{4}\right)j - \frac{x}{2}j^2\right]\frac{1}{n} + \mathcal{O}(n^{-3/2})\right) \tag{C.7}$$
$$\times \frac{C_0(\alpha;x) + C_1(\alpha;x)(n+j)^{-1/2} + C_2(\alpha;x)(n+j)^{-1} + \mathcal{O}\left((n+j)^{-3/2}\right)}{C_0(\alpha;x) + C_1(\alpha;x)n^{-1/2} + C_2(\alpha;x)n^{-1} + \mathcal{O}\left(n^{-3/2}\right)}.$$

In the numerator of (C.7) we have

$$C_0(\alpha;x) + C_1(\alpha;x)(n+j)^{-1/2} + C_2(\alpha;x)(n+j)^{-1} + \mathcal{O}((n+j)^{-3/2})$$

$$= \quad C_0(\alpha;x) + C_1(\alpha;x)n^{-1/2}\left(1+\frac{j}{n}\right)^{-1/2} + C_2(\alpha;x)n^{-1}\left(1+\frac{j}{n}\right)^{-1} + \mathcal{O}\left(n^{-3/2}\right)$$

Thus

$$\frac{C_0(\alpha;x) + C_1(\alpha;x)(n+j)^{-1/2} + C_2(\alpha;x)\,(n+j)^{-1} + \mathcal{O}\left((n+j)^{-3/2}\right)}{C_0(\alpha;x) + C_1(\alpha;x)n^{-1/2} + C_2(\alpha;x)n^{-1} + \mathcal{O}\left(n^{-3/2}\right)} = 1 + \mathcal{O}(n^{-3/2}) \tag{C.8}$$

Under these conditions, this shows that there are no terms of order either $\mathcal{O}\left(n^{-1/2}\right)$ or $\mathcal{O}\left(n^{-1}\right)$ in the expansion (C.8). Thus, we can rewrite (C.7) as

$$\begin{aligned}\frac{L_{n+j}^{(\alpha)}(x)}{L_n^{(\alpha)}(x)} &= \left(1+\frac{\sqrt{-x}}{\sqrt{n}}j + \left[\left(\frac{\alpha}{2}-\frac{1}{4}\right)j - \frac{x}{2}j^2\right]\frac{1}{n} + \mathcal{O}(n^{-3/2})\right) \times \left(1+\mathcal{O}\left(n^{-3/2}\right)\right) \\ &= 1+\frac{\sqrt{-x}}{\sqrt{n}}j + \left[\left(\frac{\alpha}{2}-\frac{1}{4}\right)j - \frac{x}{2}j^2\right]\frac{1}{n} + \mathcal{O}(n^{-3/2}).\end{aligned}$$

■

Lemma C.2 *Given two standard Laguerre polynomials of equal degree n and different parameter, the following statements hold. For $x \in \mathbb{C}\setminus\mathbb{R}_+$*

$$\begin{aligned}\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha+1)}(x)} &= \frac{\sqrt{-x}}{\sqrt{n}} + \left[\left(\frac{\alpha}{2}+\frac{1}{4}\right)+\frac{x}{2}\right]\frac{1}{n} + \mathcal{O}(n^{-3/2}), \\ \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha+2)}(x)} &= \frac{-x}{n} + \mathcal{O}(n^{-3/2}),\end{aligned}$$

where $\sqrt{-x}$ must be taken real and positive if $x < 0$.

Proof. Using (2.25) and proceeding by induction, it is easy to see that

$$L_n^{(\alpha)}(x) = \sum_{\nu=0}^{\ell}(-1)^\nu\binom{\ell}{\nu}L_{n-\nu}^{(\alpha+\ell)}(x), \;\; \ell = 1,2,\ldots$$

and therefore

$$\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha+\ell)}(x)} = \sum_{\nu=0}^{\ell}(-1)^\nu\binom{\ell}{\nu}\frac{L_{n-\nu}^{(\alpha+\ell)}(x)}{L_n^{(\alpha+\ell)}(x)}.$$

Next we can use Lemma C.1 with parameter $\alpha+\ell$ and $j=-\nu$ in order to evaluate the last ratios as follows

$$\frac{L_{n-\nu}^{(\alpha+\ell)}(x)}{L_n^{(\alpha+\ell)}(x)} = 1 - \frac{\sqrt{-x}}{\sqrt{n}}\nu - \left[\left(\frac{\alpha+\ell}{2} - \frac{1}{4}\right)\nu + \frac{x}{2}\nu^2\right]\frac{1}{n} + \mathcal{O}(n^{-3/2}).$$

Hence

$$\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha+\ell)}(x)} = \tag{C.9}$$

$$\sum_{\nu=0}^{\ell}(-1)^{\nu}\binom{\ell}{\nu}\left(1 - \frac{\sqrt{-x}}{\sqrt{n}}\nu - \left[\left(\frac{\alpha+\ell}{2} - \frac{1}{4}\right)\nu + \frac{x}{2}\nu^2\right]\frac{1}{n}\right) + \mathcal{O}(n^{-3/2}).$$

Taking $\ell = 1$ in (C.9)

$$\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha+1)}(x)} = \frac{\sqrt{-x}}{\sqrt{n}} + \left[\left(\frac{\alpha}{2} + \frac{1}{4}\right) + \frac{x}{2}\right]\frac{1}{n} + \mathcal{O}(n^{-3/2}).$$

If $\ell = 2$ we have

$$\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha+2)}(x)} = \frac{-x}{n} + \mathcal{O}(n^{-3/2}).$$

■

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