BOOTSTRAP PREDICTION INTERVALS IN STATE–SPACE MODELS

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Abstract. Prediction intervals in state space models can be obtained by assuming Gaussian innovations and using the prediction equations of the Kalman filter, with the true parameters substituted by consistent estimates. This approach has two limitations. First, it does not incorporate the uncertainty caused by parameter estimation. Second, the Gaussianity of future innovations assumption may be inaccurate. To overcome these drawbacks, Wall and Stoffer [*Journal of Time Series Analysis* (2002) Vol. 23, pp. 733 751] propose a bootstrap procedure for evaluating conditional forecast errors that requires the backward representation of the model. Obtaining this representation increases the complexity of the procedure and limits its implementation to models for which it exists. In this article, we propose a bootstrap procedure for constructing prediction intervals directly for the observations, which does not need the backward representation of the model. Consequently, its application is much simpler, without losing the good behaviour of bootstrap prediction intervals. We study its finite sample properties and compare them with those of the standard and the Wall and Stoffer procedures for the local level model. Finally, we illustrate the results by implementing the new procedure to obtain prediction intervals for future values of a real time series.

1. INTRODUCTION

Time series models with unobserved components may be useful in many contexts (see, e.g. Durbin and Koopman, 2001). After casting the model in state–space (SS) form, the Kalman filter allows one to obtain predictions of future values of the series together with their corresponding mean square errors (MSE). Then, prediction intervals are obtained by assuming Gaussian errors. In practice, the parameters of the model are unknown and are substituted by consistent estimates. Therefore, prediction intervals may be inaccurate caused by the estimation uncertainty and/or when the normal distribution is not adequate. In the context of autoregressive integrated moving-average (ARIMA) models, several authors have proposed the use of bootstrap procedures to construct prediction intervals that overcome these limitations (see Thombs and Schucany, 1990, for the original bootstrap procedure for AR(p) models and Pascual *et al.*, 2004, for a simpler procedure not based on the backward representation).

Unlike ARIMA models, models with unobserved components may have several disturbances. Therefore, the bootstrap procedures proposed by Thombs and Schucany (1990) and Pascual *et al.* (2004) cannot be directly applied to them. To overcome this problem, Wall and Stoffer (2002), from now on WS, propose that bootstrap prediction intervals for future observations can be obtained by using the innovation form (IF) of the SS models, which is defined in terms of a unique disturbance. They show that their procedure works well in the context of Gaussian SS models. Moreover, Pfeffermann and Tiller (2005) show that the bootstrap estimator of the underlying unobserved components based on the IF is asymptotically consistent. Following Pascual *et al.* (2004), in this article, we propose a bootstrap procedure to obtain prediction intervals of future observations in SS models that simplifies the WS procedure both from the computational point of view and because it does not require the backward representation. Similar to Wall and Stoffer, our proposed bootstrap procedure is based on the IF. We show that the new procedure has the advantage of being much simpler without losing the good behaviour of bootstrap prediction intervals.

The rest of the article is organized as follows. In section 2, we describe the model, the filters and propose a new bootstrap procedure. We analyse its finite-sample properties, comparing them with those of the standard and the Wall and Stoffer prediction intervals. Section 3 presents an application of the new bootstrap procedure to a real time series. Section 4 concludes the paper with our conclusions and some suggestions for future research.

2. BOOTSTRAP PREDICTION INTERVALS IN SS MODELS

2.1. *SS models and the Kalman filter*

Consider the following time-invariant SS model,

$$y_t = \mathbf{Z}\boldsymbol{\alpha}_t + \mathbf{d} + \varepsilon_t, \tag{1a}$$

$$\boldsymbol{\alpha}_t = \mathbf{T}\boldsymbol{\alpha}_{t-1} + \mathbf{c} + \mathbf{R}\boldsymbol{\eta}_t, \quad t = 1, \ldots, T \tag{1b}$$

where y_t is a univariate time series, $\boldsymbol{\alpha}_t$ is the $m \times 1$ vector of unobservable state variables, ε_t is a serially uncorrelated disturbance with zero mean and variance $\mathbf{H}$ and $\boldsymbol{\eta}_t$ is a $g \times 1$ vector of serially uncorrelated disturbances with zero mean and covariance matrix $\mathbf{Q}$. The disturbances ε_t and $\boldsymbol{\eta}_t$ are uncorrelated with each other in all time periods Finally, the initial state vector, $\boldsymbol{\alpha}_0$, has mean $\mathbf{a}_0$ and covariance matrix $\mathbf{P}_0$.

The Kalman filter allows one to estimate the state vector, $\boldsymbol{\alpha}_{t+1}$, and its MSE based on the information available at time t. These estimates are given by

$$\mathbf{a}_{t+1|t} = \mathbf{T}\mathbf{a}_{t|t-1} + \mathbf{c} + \mathbf{K}_t F_t^{-1} v_t, \tag{2a}$$

$$\mathbf{P}_{t+1|t} = \mathbf{T}\mathbf{P}_{t|t-1}\mathbf{T}' - \mathbf{K}_t F_t^{-1}\mathbf{K}_t' + \mathbf{R}\mathbf{Q}\mathbf{R}', \tag{2b}$$

where $v_t = y_t - \mathbf{d} - \mathbf{Z}\mathbf{a}_{t|t-1}$ is the innovation, $F_t = \mathbf{Z}\mathbf{P}_{t|t-1}\mathbf{Z}' + \mathbf{H}$ is its variance and $\mathbf{K}_t = \mathbf{T}\mathbf{P}_{t|t-1}\mathbf{Z}'$.

Although the SS model in eqn (1) has several disturbances, the IF has a unique disturbance. The IF is given by eqn (2a) together with

$$y_t = \mathbf{Z}\mathbf{a}_{t|t-1} + \mathbf{d} + v_t. \tag{3}$$

Assuming that future prediction errors are Gaussian, the k-step-ahead prediction intervals for y_{T+k} are given by

$$\left[\tilde{y}_{T+k|T} - z_{1-\alpha/2}\sqrt{F_{T+k|T}}, \tilde{y}_{T+k|T} + z_{1-\alpha/2}\sqrt{F_{T+k|T}}\right], \tag{4}$$

where $z_{1-\alpha/2}$ is the $(1 - (\alpha/2))$-percentile of the standard normal distribution, and $\tilde{y}_{T+k|T}$ and $F_{T+k|T}$ are the k-step-ahead prediction of y_{T+k} and its MSE, given by

$$\tilde{y}_{T+k|T} = \mathbf{Z}\mathbf{T}^k\mathbf{a}_T + \mathbf{Z}\sum_{j=0}^{k-1}\mathbf{T}^j\mathbf{c} + \mathbf{d}, \quad k = 1, 2, \ldots, \tag{5a}$$

$$F_{T+k|T} = \mathbf{Z}(\mathbf{T}^k)\mathbf{P}_T(\mathbf{T}^k)'\mathbf{Z}' + \mathbf{Z}\sum_{j=0}^{k-1}\left[(\mathbf{T}^j)\mathbf{R}\mathbf{Q}\mathbf{R}'(\mathbf{T}^j)'\right]\mathbf{Z}' + \mathbf{H}, \quad k = 1, 2, \ldots. \tag{5b}$$

In practice, the unknown parameters involved in eqn (5) are substituted by consistent estimates, usually quasi-maximum likelihood (QML) estimates because of their well-known asymptotic properties (see, e.g. Durbin and Koopman, 2001). The intervals in eqn (4), where $\tilde{y}_{T+k|T}$ and $F_{T+k|T}$ are obtained by substituting the unknown parameters by their QML estimates, are called standard (ST). Note that the ST prediction intervals underestimate, in general, the variability of the forecast error because they do not take into account the uncertainty caused by parameter estimation. Moreover, these intervals could have inaccurate coverages when the prediction errors are not Gaussian.

To overcome these problems, Wall and Stoffer (2002) propose a bootstrap procedure based on the IF. Following Thombs and Schucany (1990), they propose the use of the backward SS representation to generate bootstrap replicates of the series with fixed last observations. These replicates are used to incorporate the uncertainty caused by parameter estimation in the density of the prediction errors. However, unlike Thombs and Schucany (1990), they propose to obtain bootstrap densities of the prediction errors instead of bootstrapping directly future observations of the series of interest. The bootstrap densities of the prediction errors are obtained in two steps. First, they construct bootstrap prediction errors that incorporate the uncertainty due to the fact that, when predicting, future innovations are equal to zero while in fact they are not. These bootstrap replicates do not incorporate the uncertainty caused by parameter estimation. Therefore, they obtain another set of bootstrap prediction errors that incorporate the variability attributable to parameter estimation through the use of bootstrap-estimated parameters instead of the original estimates. These bootstrap replicates assume that future innovations are zero. Finally, combining both sets of

bootstrap prediction errors, they estimate the density of the conditional forecast errors that can be used for constructing the corresponding bootstrap prediction intervals. They are given by

$$\left[\tilde{y}_{T+k|T} + \mathbf{Q}^*_{\alpha/2,d^*_k}, \tilde{y}_{T+k|T} + \mathbf{Q}^*_{1-\alpha/2,d^*_k}\right] \tag{6}$$

where $\mathbf{Q}^*_{\alpha/2,d^*_k}$ is the $\alpha/2$-percentile of the empirical conditional bootstrap distribution of the k-step-ahead prediction errors of y_{T+k}.

2.2. A new procedure

We propose to construct bootstrap prediction intervals directly approximating the conditional distribution of y_{T+k} by the distribution of bootstrap replicates that incorporate simultaneously the variability caused by parameter estimation and the uncertainty because of unknown future innovations without using the backward filter. The new procedure consists of the following steps.

Step 1. Estimate the parameters of model (1) by QML, $\hat{\theta}$, and obtain the standardized innovations $\{\hat{v}^s_t; 1 \leq t \leq T\}$.

Step 2. Obtain a sequence of bootstrap standardized innovations $\{\hat{v}^{s*}_t; 1 \leq t \leq T+K\}$ via random draws with replacement from the standardized innovations, $\hat{v}^s_t$.

Step 3. Compute a bootstrap replicate $\{\hat{y}^*_t; 1 \leq t \leq T\}$ by means of the IF in eqns (3) and (2a) using $\hat{v}^{s*}_t$ and the estimated parameters, $\hat{\boldsymbol{\theta}}$. Estimate the corresponding bootstrap parameters, $\hat{\boldsymbol{\theta}}^*$.

Step 4. Run the Kalman filter with $\hat{\boldsymbol{\theta}}^*$ and the original observations and obtain a bootstrap replicate of the state vector at time T which incorporates the uncertainty caused by parameter estimation, $\hat{\mathbf{a}}^*_{T|T-1}$.

Step 5. Obtain conditional bootstrap k-step-ahead predictions, $\{\hat{y}^*_{T+k|T}; 1 \leq k \leq K\}$, by the following expressions

$$\hat{\mathbf{a}}^*_{T+k|T} = \hat{\mathbf{T}}^{*k}\,\hat{\mathbf{a}}^*_{T|T-1} + \sum_{j=0}^{k-1}\hat{\mathbf{T}}^{*k-1-j}\mathbf{c} + \sum_{j=0}^{k-1}\hat{\mathbf{T}}^{*k-1-j}\hat{\mathbf{K}}^*_{T+j}\hat{F}^{*-1}_{T+j}\hat{v}^*_{T+j},$$

$$\hat{y}^*_{T+k|T} = \hat{\mathbf{Z}}^*\,\hat{\mathbf{T}}^{*k}\,\hat{\mathbf{a}}^*_{T|T-1} + \hat{\mathbf{Z}}^*\sum_{j=0}^{k-1}\hat{\mathbf{T}}^{*k-1-j}\mathbf{c} + \mathbf{d}$$

$$+ \hat{\mathbf{Z}}^*\sum_{j=0}^{k-1}\hat{\mathbf{T}}^{*k-1-j}\hat{\mathbf{K}}^*_{T+j}\hat{F}^{*-1}_{T+j}\hat{v}^*_{T+j} + \hat{v}^*_{T+k}, \quad k = 1, \ldots,$$

where $\hat{v}^*_T = y_T - \hat{\mathbf{Z}}^*\hat{\mathbf{a}}^*_{T|T-1}$ and the hat in top of the matrices means that they are obtained by substituting the parameters by their corresponding bootstrap estimates.

Steps 2 to 5 are repeated B times, obtaining B bootstrap replicates of $\hat{y}^*_{T+k|T}$.

Note that the empirical distribution of $\hat{y}^*_{T+k|T}$ incorporates both the variability caused by unknown future innovations and the variability caused by parameter

estimation in just one step. The procedure above, denoted as state-space bootstrap (SSB), has three advantages over the Wall and Stoffer procedure. First, it does not require the use of the backward representation. Second, it is simpler as a unique set of bootstrap replicates of future observations is required instead of two as in the WS procedure. Third, unlike the WS procedure, we do not fix $\hat{\mathbf{a}}^*_{T|T-1} = \hat{\mathbf{a}}_{T|T-1}$ in all bootstrap replicates of future observations. This value depends on the estimated parameters and, consequently, it should be allowed to vary among bootstrap replicates in order to incorporate the uncertainty caused by parameter estimation. However, note that all realizations of $\hat{y}^*_{T+k|T}$ pass through y_T, making the predictions conditional on the available sample.

Finally, the SSB bootstrap prediction intervals are constructed directly by the percentile method[1] as follows

$$\left[\mathbf{Q}^*_{\alpha/2,\hat{y}^*_{T+k|T}}, \mathbf{Q}^*_{1-\alpha/2,\hat{y}^*_{T+k|T}} \right] \tag{7}$$

where $\mathbf{Q}^*_{\alpha/2,\hat{y}^*_{T+k|T}}$ is the $\alpha/2$-percentile of the empirical distribution $\hat{y}^*_{T+k|T}$.

2.3. *Finite-sample properties*

To analyse the finite-sample properties of the SSB prediction intervals, we consider the local level model given by

$$y_t = \mu_t + \varepsilon_t, \quad \varepsilon_t \sim \text{i.i.d.}(0, \sigma^2_\varepsilon), \tag{8a}$$

$$\mu_t = \mu_{t-1} + \eta_t, \quad \eta_t \sim \text{i.i.d.}(0, q\sigma^2_\varepsilon), \tag{8b}$$

where μ_t is the level of the series that evolves over time following a random walk and q is known as the signal-to-noise ratio.

Simulation results are based on $R = 1000$ replicates of series of sizes $T = 50$, 100 and 500. The parameters of the model have been chosen to cover a wide range of situations from cases in which the noise is large relative to the signal, i.e. q is small, to cases in which q is large. In particular, we consider $q = \{0.1, 1, 2\}$. With respect to the disturbances, we consider two distributions, Gaussian and a centred and re-scaled chi-square with 1 degree of freedom,[2] $\chi^2_{(1)}$. For each simulated series, $\{y^{(r)}_1, \ldots, y^{(r)}_T\}$, $r = 1, 2, \ldots, R$, we first generate $B = 1000$ observations of $y^{(r)}_{T+k}$ for prediction horizons $k = 1$, 5 and 15, and then obtain 95% prediction intervals computed using the ST intervals in eqn (4), the Wall and Stoffer intervals in eqn (6) and the SSB intervals in eqn (7). Finally, we compute the coverage of each of these intervals as well as the length and the percentage of observations left out on the right and on the left of the limits of the prediction intervals.[3]

Table I reports the Monte Carlo averages of these quantities when both disturbances are Gaussian. Although, in this case, the three procedures have very similar properties, the average coverage of the ST intervals is smaller than the coverage of the SSB intervals when the sample size is small ($T = 50$) and the prediction horizon is 5 and 15. This result can be explained by the fact that ST

			Mean coverage			Mean coverage in tails			Mean length		
						ST	WS	SSB			
		k	ST	WS	SSB	Below/Above	Below/Above	Below/Above	ST	WS	SSB
T	50										
q	0.1	1	0.927	0.935	0.936	0.036/0.037	0.030/0.035	0.031/0.033	4.530	4.597	4.774
		5	0.927	0.940	0.943	0.036/0.037	0.029/0.031	0.028/0.029	5.182	5.285	5.539
		15	0.915	0.928	0.940	0.042/0.042	0.035/0.037	0.030/0.031	6.460	6.633	7.052
q	1	1	0.936	0.923	0.928	0.029/0.035	0.036/0.041	0.036/0.035	6.157	6.250	6.280
		5	0.927	0.921	0.938	0.035/0.039	0.037/0.042	0.032/0.031	9.722	9.718	10.274
		15	0.914	0.909	0.934	0.041/0.045	0.043/0.047	0.033/0.033	15.258	15.194	16.469
q	2	1	0.938	0.930	0.930	0.032/0.029	0.036/0.034	0.036/0.034	7.424	7.56	7.433
		5	0.926	0.924	0.931	0.037/0.036	0.038/0.038	0.034/0.034	12.849	12.880	13.088
		15	0.918	0.915	0.930	0.041/0.041	0.042/0.042	0.035/0.035	20.889	20.830	21.632
T	100										
q	0.1	1	0.945	0.941	0.943	0.025/0.030	0.031/0.028	0.026/0.031	4.569	4.576	4.618
		5	0.945	0.942	0.948	0.025/0.030	0.030/0.028	0.024/0.029	5.206	5.238	5.334
		15	0.938	0.938	0.945	0.029/0.033	0.032/0.030	0.026/0.030	6.498	6.575	6.743
q	1	1	0.944	0.940	0.939	0.028/0.028	0.030/0.029	0.030/0.031	6.271	6.314	6.278
		5	0.939	0.937	0.942	0.031/0.030	0.032/0.031	0.029/0.029	9.874	9.873	10.120
		15	0.934	0.932	0.940	0.033/0.033	0.034/0.034	0.030/0.030	15.547	15.521	16.165
q	2	1	0.945	0.937	0.939	0.028/0.027	0.032/0.030	0.031/0.030	7.476	7.537	7.460
		5	0.939	0.938	0.939	0.030/0.030	0.031/0.031	0.031/0.031	13.137	13.155	13.210
		15	0.935	0.935	0.937	0.032/0.032	0.032/0.033	0.031/0.031	21.509	21.539	21.758
T	500										
q	0.1	1	0.946	0.948	0.945	0.027/0.027	0.025/0.028	0.028/0.027	4.592	4.577	4.582
		5	0.946	0.947	0.946	0.026/0.028	0.025/0.028	0.027/0.027	5.217	5.206	5.223
		15	0.946	0.945	0.945	0.026/0.029	0.026/0.029	0.027/0.028	6.515	6.477	6.511
q	1	1	0.948	0.948	0.947	0.029/0.023	0.027/0.025	0.027/0.025	6.339	6.335	6.314
		5	0.948	0.947	0.947	0.027/0.025	0.028/0.025	0.027/0.026	10.075	10.049	10.073
		15	0.947	0.946	0.947	0.027/0.026	0.027/0.027	0.027/0.026	15.956	15.919	15.944
q	2	1	0.947	0.945	0.947	0.027/0.026	0.029/0.026	0.027/0.026	7.563	7.546	7.540
		5	0.948	0.948	0.948	0.027/0.027	0.027/0.025	0.027/0.027	13.418	13.446	13.387
		15	0.947	0.948	0.947	0.027/0.026	0.026/0.026	0.026/0.027	22.066	22.112	22.051

intervals do not incorporate the uncertainty caused by parameter estimation. Comparing the two bootstrap intervals, we can observe that both are similar, with the average coverages of SSB being closer to the nominal in small samples and/or when the prediction horizon increases. This result is illustrated in Figure 1 that plots kernel estimates of the ST, Wall and Stoffer and SSB densities for 15-step-ahead predictions for one particular series generated by each of the three models considered together with the empirical density. Note that when the signal-to-noise ratio is small, i.e. $q = 0.1$ and $T = 50$, the SSB density seems to be more similar to the empirical density than the other densities.

Table II, that reports the results when ε_t is $\chi^2_{(1)}$ and η_t is Gaussian, shows that, although the mean coverage of the ST intervals is close to the nominal, they are not capable of dealing with the asymmetry in the distribution of ε_t. The average coverage in the left tail is smaller than that in the right tail. The difference between
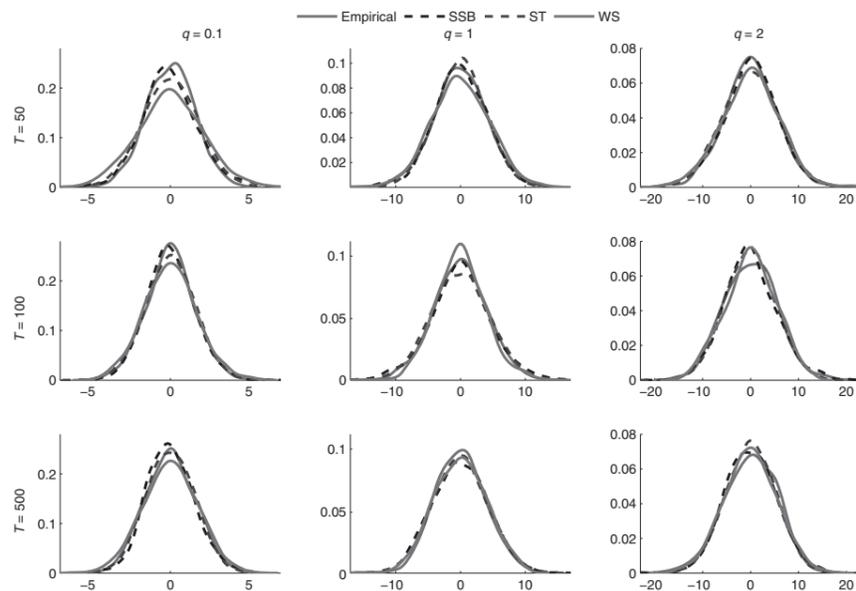
FIGURE 1. Kernel estimates densities of y_{T+k} for k 15. Normal case.

the coverage in both tails is larger in the model with $q = 0.1$, where the signal is relatively small with respect to the non-Gaussian noise. Note that the inability of the ST intervals to deal with the asymmetry in the distribution of ε_t is larger the larger the sample size. On the other hand, the coverages of the Wall and Stoffer and SSB intervals are rather similar, with SSB being again slightly closer to the nominal for almost all models and sample sizes considered. Both bootstrap intervals are capable of coping with the asymmetry of the distribution of ε_t. Consequently, according to the results reported in Table II, using the much simpler SSB method does not imply a worse performance of the prediction intervals. Figure 2 illustrates these results by plotting the kernel density of the simulated y_{T+1} together with the ST, Wall and Stoffer and SSB densities obtained with a particular series generated by each of the models and sample sizes considered. This figure also illustrates the lack of fit of the ST density when $q = 0.1$ and 1. On the other hand, the shapes of the Wall and Stoffer and SSB densities are similar, with SSB being always closer to the empirical.

3. EMPIRICAL APPLICATION

We illustrate the performance of the proposed procedure to construct bootstrap prediction intervals by implementing it on the standardized quarterly mortgages

		Mean coverage			Mean coverage in tails			Mean length		
					ST	WS	SSB			
Case	k	ST	WS	SSB	Below/Above	Below/Above	Below/Above	ST	WS	SSB
T	50									
q	0.1									
	1	0.941	0.940	0.942	0.010/0.049	0.030/0.030	0.027/0.031	4.513	4.909	4.734
	5	0.943	0.934	0.946	0.013/0.044	0.039/0.027	0.027/0.026	5.221	5.507	5.596
	15	0.930	0.919	0.950	0.025/0.045	0.053/0.029	0.027/0.023	6.572	6.665	7.329
q	1									
	1	0.935	0.932	0.934	0.026/0.039	0.034/0.034	0.034/0.032	6.200	6.514	6.459
	5	0.926	0.926	0.930	0.034/0.040	0.040/0.034	0.038/0.032	9.682	9.803	9.919
	15	0.913	0.914	0.923	0.042/0.045	0.041/0.036	0.045/0.041	15.126	15.176	15.597
q	2									
	1	0.937	0.933	0.932	0.028/0.035	0.034/0.034	0.035/0.033	7.378	7.714	7.575
	5	0.927	0.924	0.927	0.035/0.038	0.040/0.036	0.038/0.035	12.805	12.897	12.957
	15	0.919	0.917	0.923	0.040/0.041	0.043/0.040	0.040/0.037	20.839	20.880	21.236
T	100									
q	0.1									
	1	0.947	0.939	0.943	0.006/0.048	0.033/0.028	0.027/0.029	4.552	4.773	4.710
	5	0.946	0.937	0.942	0.010/0.043	0.037/0.026	0.031/0.027	5.196	5.356	5.414
	15	0.939	0.929	0.944	0.021/0.040	0.045/0.026	0.032/0.024	6.491	6.597	6.912
q	1									
	1	0.942	0.939	0.943	0.021/0.037	0.030/0.030	0.030/0.027	6.244	6.483	6.501
	5	0.937	0.936	0.939	0.028/0.034	0.035/0.029	0.022/0.035	9.813	9.919	10.017
	15	0.932	0.930	0.935	0.033/0.035	0.037/0.033	0.028/0.032	15.438	15.445	15.742
q	2									
	1	0.947	0.944	0.945	0.022/0.031	0.028/0.028	0.027/0.028	7.507	7.685	7.686
	5	0.942	0.941	0.943	0.028/0.030	0.031/0.028	0.030/0.027	13.220	13.307	13.399
	15	0.938	0.937	0.940	0.030/0.031	0.033/0.030	0.032/0.028	21.659	21.752	21.941
T	500									
q	0.1									
	1	0.948	0.940	0.950	0.006/0.045	0.033/0.026	0.023/0.027	4.575	4.697	4.707
	5	0.948	0.939	0.947	0.011/0.041	0.036/0.025	0.028/0.025	5.184	5.272	5.314
	15	0.946	0.937	0.946	0.019/0.035	0.039/0.024	0.031/0.024	6.455	6.507	6.631
q	1									
	1	0.947	0.947	0.948	0.020/0.033	0.020/0.033	0.027/0.026	6.338	6.492	6.472
	5	0.948	0.948	0.948	0.024/0.028	0.029/0.023	0.027/0.025	10.073	10.181	10.137
	15	0.947	0.946	0.947	0.025/0.027	0.029/0.024	0.028/0.025	15.952	15.983	15.957
q	2									
	1	0.944	0.945	0.944	0.026/0.030	0.027/0.028	0.029/0.026	7.554	7.648	7.636
	5	0.947	0.947	0.948	0.026/0.027	0.028/0.025	0.028/0.024	13.369	13.447	13.466
	15	0.947	0.947	0.948	0.026/0.027	0.028/0.025	0.027/0.025	21.968	21.992	22.085

change in the United States' home equity debt outstanding, unscheduled payments, observed from the first quarter of 1991 to the second quarter of 2007 (Mortgages) and measured in USD Billions.[4] We use the observations up to the first quarter of 2001, $T = 61$, to estimate the local level model, leaving the rest to evaluate the out-of-sample forecast performance of the procedure. The QML estimates of the parameters are $\hat{\sigma}^2_\varepsilon = 0.126$ and $\hat{q} = 0.671$. These estimates are used in the Kalman filter to obtain estimates of the innovations and their variances. Figure 3 plots the correlogram and a kernel estimate of the density of the within-sample standardized one-step-ahead errors. The correlations and partial correlations are not significant. However, the density of the errors suggests that they are obviously far from normality. Therefore, although the local level model seems appropriate to represent the dependencies in the conditional mean of the Mortgages series, it is convenient to implement a prediction procedure that takes into account the non-normality of the
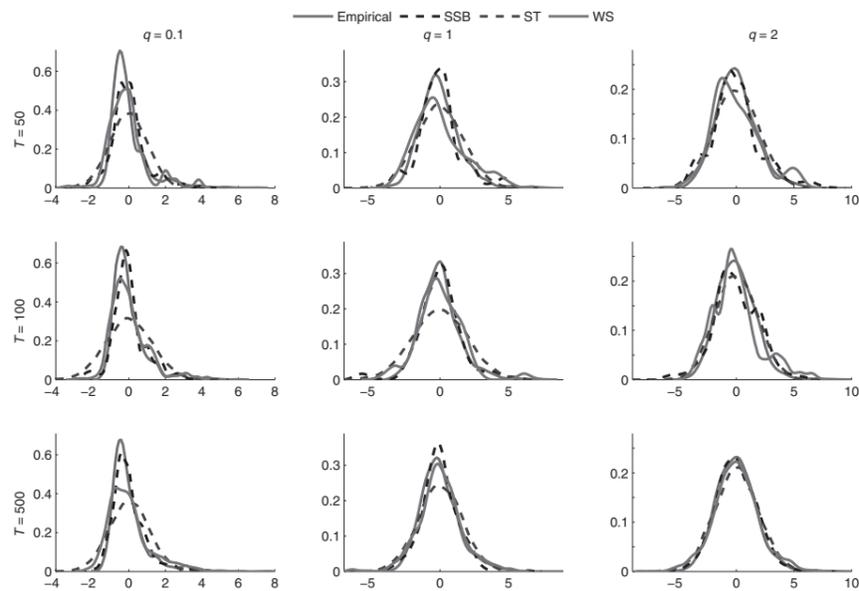
FIGURE 2. Kernel estimates densities of y_{T+k} for k 1. $\chi^2_{(1)}$ case.

errors. We construct prediction intervals up to five-steps ahead using the ST, WS and SSB procedures. The resulting intervals are plotted in Figure 4 together with the observed values of the Mortgages series. First, observe that the two bootstrap procedures generate very similar intervals which are wider than the ST intervals as expected, given that they incorporate the uncertainty caused by parameter estimation. Second, note that the observations corresponding to the second quarter of 2006 and the first quarter of 2007, fall outside the ST prediction intervals. However, both bootstrap intervals still contain these two values. It is important to note that although bootstrap procedures are computationally intensive, in this application with $B = 2000$ bootstrap replicates, the SSB procedure requires 110 seconds using a MATLAB algorithm in an AMD Athlon 2.00 GHz processor of a PC desktop with 2.00 Gb of RAM. The WS procedure requires 160 seconds. There is a reduction of 31% in the computer time required by the new procedure proposed in this article.

4. CONCLUSIONS

This article proposes a new procedure to obtain bootstrap prediction intervals in the context of state–space models. It is based on obtaining the density of future observations in a single step that incorporates simultaneously the uncertainties
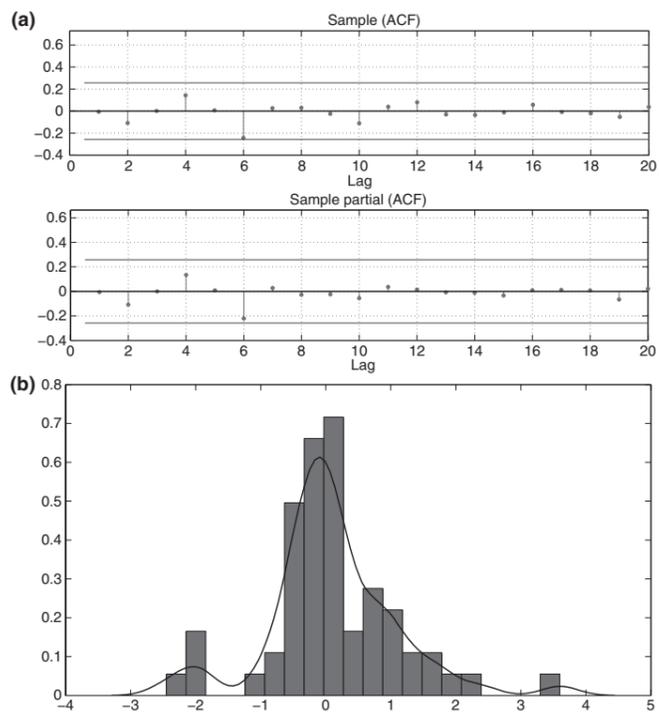
FIGURE 3. (a) Sample autocorrelations and partial autocorrelations of standardized one step ahead errors. (b) Empirical density and histogram of the standardized one step ahead error.
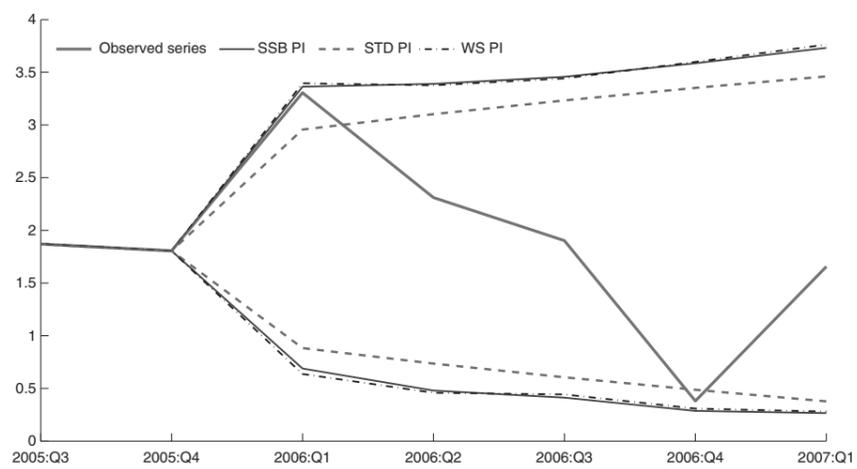

FIGURE 4. Prediction intervals for the out of sample forecasts of the Mortgage series.

caused by parameter estimation and error distribution. More importantly, our bootstrap procedure does not rely on the backward representation. Consequently, it is computationally very simple and can be extended to models without such representation.

We show that our procedure, although much simpler, has slightly better finite-sample properties than the bootstrap intervals of Wall and Stoffer (2002). As expected, we also show that bootstrap intervals are more adequate than standard intervals mainly in the presence of non-normal errors.

Finally, our proposed bootstrap procedure is implemented to obtain intervals for future values of a series of Mortgages. In this case, the two bootstrap intervals considered in this article are very similar. However, there is an important improvement in terms of computer time when implementing our proposed procedure.

When fitting state–space models to represent the dynamic evolution of a time series, it is often of interest to obtain predictions not only of future values of the series but also of future unobserved states. We are working on the adequacy of the proposed bootstrap prediction intervals when implemented with this goal. An issue left for further research is the implementation of the proposed procedure when the system matrices are time-varying.

NOTES

1. We try alternative methods such as the bias-corrected and the acceleration bias-corrected with similar results; see Efron (1987) for a definition of these intervals.
2. We are particularly interested in dealing with this distribution because of its relation with the linear transformation of the autoregressive stochastic volatility model; see, for instance, Harvey *et al.* (1994). Results for other distributions are similar and are not reported to save space. They are available from the authors upon request.
3. All the results have been obtained using MATLAB, version 7.2, programs developed by the first author.
4. The data have been downloaded from EcoWin.

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