External fluctuations in front dynamics with inertia: The overdamped limit

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Abstract. We study the dynamics of fronts when both inertial effects and external fluctuations are taken into account. Stochastic fluctuations are introduced as multiplicative white noise arising from a control parameter of the system. Contrary to the non-inertial (overdamped) case, we find that important features of the system, such as the velocity selection picture, are not modified by the noise. We then compute the overdamped limit of the underdamped dynamics in a more careful way, finding that it does not exhibit any effect of noise either. Our result poses the question as to whether or not external noise sources can be measured in physical systems of this kind.

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1 Introduction

Front propagation is being the subject of very active research in the last few years: Indeed, the problem of the selection of the front velocity is a paradigm of the dynamical selection mechanisms arising in a large number of physical, chemical and biological systems with a certain kind of instability (see [1] and references therein). One of the important questions into which interest has been focused is how the deterministic front scenario is modified by the presence of noise. In this context, the effect of stochastic fluctuations on front dynamics and the modification of its deterministic features have been considered by several authors. A detailed summary of those results, which were mostly devoted to the changes of the front velocity and the spreading of the front position, can be found in [2].

The present work addresses a related problem which, on the other hand, arises naturally from the above line of research: Is the influence of (external) white noise on front propagation the same if inertial effects (as far as we know, neglected in previous work) are taken into account? As is well known, including inertia leads to a description in terms of a damped, hyperbolic partial differential equation. It is important to note that first, a model like this arises when one considers a more realistic jump process for the individuals whose probability density is described by the partial differential equation; and, second, that hyperbolic equations of this kind describe many actual physical phenomena, such as, *e.g.*, population dynamics, nonlinear transmission lines, cell motion, branching random walks, dynamics of ferroelectric domains, and others [3–9]. The rôle of inertia in the scenario of (deterministic) front propagation has been studied recently [10] (see also [6–8], and specially [11] for a detailed study of the underdamped dynamics restricted to the linear regime). In this paper, it was proven that the different dynamical regimes of front propagation using deterministic parabolic models do not change, *i.e.*, the values of the control parameter separating the different regimes do not depend on the inertia parameter ("mass"). However, it was also shown in [10] that the values of the front velocities corresponding to every interval of the control parameter and the spatial shape of the propagating fronts do depend on the inertia parameter.

As stated above, our explicit objective will now be the study of the rôle of multiplicative white noise and its comparison to the results in [12,13], in order to understand the interplay of inertia and external stochastic perturbations. Accordingly, we undertake the study of a hyperbolic partial differential equation with a multiplicative white noise term, used to model front dynamics subjected to both inertia and external fluctuations. Opposite to the deterministic case, in which for very small mass or inertia a naive adiabatic elimination procedure (*i.e.*, leaving the second time derivative term out) gives a parabolic equation which describes very accurately the front dynamics [10], we will prove that this is not at all the case when noise is present. As a matter of fact, the starting hyperbolic equation *with* *noise in the Stratonovič interpretation* transforms, upon adiabatic elimination, into a parabolic equation with an extra term coming from the noise. Furthermore, the so obtained equation turns out to be equivalent to the usual parabolic equation if interpreted in the *Itô sense*. In addition, we include numerical simulation results confirming this unexpected result. We report on these results along the following scheme: We begin by presenting our model and by briefly summarizing what is known about the noise influence in the overdamped case. Subsequently, we concern ourselves with the study of the externally perturbed (stochastic) case. We conclude by summarizing our main findings and discussing their implications.

2 Model definition and notations

Let us begin by introducing the purely deterministic problem. Generally speaking, the situation which we are interested in is generally modeled by the hyperbolic equation

$$\phi_{tt} + \alpha\phi_t = D\phi_{xx} + \tau^{-1} f(\phi, a) \tag{1}$$

where α is the friction (dissipation), D is the diffusion coefficient, τ is the characteristic time of the reaction term and a is the external control parameter of the nonlinear reaction term $f(\phi, a)$. The first step will be the reduction of the number of parameters by introducing the change of variables $t \to \tau\alpha t$, $x \to \sqrt{\tau D}x$; our initial model, equation (1), reduces then to

$$\epsilon\phi_{tt} + \phi_t = \phi_{xx} + f(\phi, a), \tag{2}$$

where a new parameter (the "mass"), $\epsilon = (\tau\alpha^2)^{-1}$, appears. We note that the information regarding both the characteristic reaction time and the dissipation coefficient is contained now in ϵ. With this new notation, the parabolic or overdamped limit is obtained by letting $\epsilon \to 0$ [7], which leads to (note that it is a singular limit)

$$\phi_t = \phi_{xx} + f(\phi, a). \tag{3}$$

We will refer to this procedure along the paper as *naive adiabatic elimination.*

For the sake of definiteness, we take as a representative example of nonlinear reaction term

$$f(\phi, a) = \phi(1 - \phi)(a + \phi). \tag{4}$$

Such a term can be obtained from a local (bistable) potential, $f(\phi, a) = -V'(\phi)$, with

$$V(\phi) = -\frac{a}{2}\phi^2 - \frac{1-a}{3}\phi^3 + \frac{1}{4}\phi^4. \tag{5}$$

It is then straightforward to show that the steady states are, $\phi_1 = 0, \phi_2 = 1$ and $\phi_3 = -a$. We are interested in those solutions which are front-like (kinks) connecting the (unstable if $a > 0$ and metastable otherwise) state $\phi_1 = 0$ with the globally stable state $\phi_2 = 1$. Consequently, we supplement equations (2) and (3) with boundary conditions $\phi(-\infty, t) = \phi_2$, $\phi(\infty, t) = \phi_1$.

Let us now move on to the stochastic version of the problem. As is well known, thermal (additive) noise is *not* expected to be relevant in actual, experimental contexts, as its ratio to other terms in the governing equations can be estimated to be 10^{-9} [1]. However, in addition to thermal noise we must expect [1] multiplicative noise sources arising from fluctuating control parameters [14]. Examples of this case are recent experiments on the Belousov-Zabotinsky reaction in a light-sensitive medium [15–17]. The fluctuating light intensity enters as a multiplicative noise in the theoretical modelization of this chemical reaction. According to this, noise is introduced in the system described so far through the parameter a, which fluctuates according to

$$a \to a + \xi(x, t). \tag{6}$$

The noise ξ is Gaussian, with zero mean and correlation given by

$$\langle \xi(x, t)\xi(x', t') \rangle = 2\,\sigma^2 C(x - x')\delta(t - t'), \tag{7}$$

with $C(x-x')$ being the spatial correlation function, normalized by imposing $\int C(x)\mathrm{d}x = 1$. The noisy parabolic case considered in [12,13] corresponds to the following stochastic partial differential equation:

$$\phi_t = \phi_{xx} + f(\phi, a) + g(\phi)\xi(x, t), \tag{8}$$

with $g(\phi) = \phi(1 - \phi)$ in case $f(\phi, a)$ is given by equation (4). Equation (8) with noise statistical properties (7), being well known, will be taken as our reference scenario; its main features are summarized below. Nevertheless, before going into those results, it is important to note that, prior to any other consideration, it is necessary to prescribe a mathematical interpretation of the noise in equation (8). Based on physical and mathematical grounds we will follow the Stratonovič interpretation (see [18,19] for in-depth discussions of the interpretation of stochastic differential equations). This interpretation fulfills two important properties from a physical point of view: First, it corresponds to the white noise limit of a real (nonwhite) noise, and second, when manipulating stochastic terms, usual rules of calculus apply. We note that in an actual experimental situation the noise has necessarily a finite (non zero) characteristic time. However, this time is indeed very small compared with any other characteristic time of the system, and therefore the assumptions of white noise and Stratonovič interpretation seem physically well founded.

3 Stochastic results on the overdamped model

As already mentioned in the introduction, the overdamped limit with noise, equation (8), has been studied recently [2,12,13]. It is worth summarizing here the main points in order to compare with our results. In addition, we will refer to the quantities introduced in these section later along the text.

In the parabolic model (8), when $-1/2 < a < 1/2$ (metastable and nonlinear regimes of the deterministic equation, see [20–22]), starting from a sufficiently localized initial front evolves to the nonlinear solution, and correspondingly the selected velocity of the front is

$$v_{nl}(a) = \frac{2a+1}{\sqrt{2\,(1-2\sigma^2 C(0))}}\,. \qquad (9)$$

On the other hand, in the linear regime, $1/2 < a < 1$, the velocity is given by

$$v_l = 2\sqrt{a + \sigma^2 C(0)}. \qquad (10)$$

We note that, in the three regimes, this velocity increases monotonously as a function of the noise intensity; we will come back to this result below.

4 Stochastic perturbation

We now proceed to analyze the case we are interested in, namely the stochastic version of the hyperbolic partial differential equation (2). For convenience, by introducing a new field variable ψ, we cast it in the form

$$\begin{aligned}\phi_t &= \psi,\\ \epsilon\psi_t &= -\psi + \phi_{xx} + f(\phi, a) + g(\phi)\xi(x,t).\end{aligned} \qquad (11)$$

At this point, it is important to note that naive adiabatic elimination leads again to equation (8).

As a starting point, let us recall that in [10] we proved that the velocity of the deterministic hyperbolic model can be obtained from the parabolic model by using the transformation

$$v_{nl}(\epsilon, a) = \frac{1}{\sqrt{\epsilon + v_{nl}(a)^{-2}}}\,. \qquad (12)$$

Therefore, our first aim here is to check whether or not this result applies in the presence of multiplicative noise, *i.e.*, whether or not we can still use the expression (12) substituting for the deterministic velocities the stochastic ones found in [12,13]. From those papers, we know that the important noise effects come from the fact that the multiplicative noise term has *a non zero mean value*, provided the Stratonovič interpretation is used. Within that interpretation, it can be found by using only usual stochastic calculus and a little algebra that the mean value of the noisy term in equation (11) is [2]

$$\langle g(\phi)\xi(x,t)\rangle = \sigma^2 C(0)\langle \frac{\partial g(\phi)}{\partial\phi}\frac{\delta\phi}{\delta\xi(x',t')}\rangle\Big|_{x'=x,\,t'=t} = 0. \qquad (13)$$

This result allows us to *conjecture* that in the hyperbolic case the noise is not as relevant as in the parabolic case. In order to verify this conjecture, we carried out numerical simulations of equation (11) (with noise white in space, implying $C(0) = 1/\Delta x$ with Δx the spatial discretization step) for different choices of the parameters, finding that the velocity is not affected by the perturbation even for very small values of ϵ. In Figure 1 we present some of these numerical results. It can be clearly seen the dramatic difference between the effects of noise in the two models. It is most important to stress that the results in Figure 1 correspond to *individual realizations* of the noise; of course, we have verified that this is the typical behavior by repeating the simulations very many times. This means that our conjecture that the noise effects and in particular those of the front velocity are not important, is in fact true beyond mean values, *i.e.*, individual fronts do not change their deterministic velocity in the presence of noise. For the parabolic case we see that the velocity does depend on the intensity of the noise, being independent in the hyperbolic case. This is an unexpected result but, as we have just seen, it is in complete agreement with our theoretical analysis, that leads to equation (13).

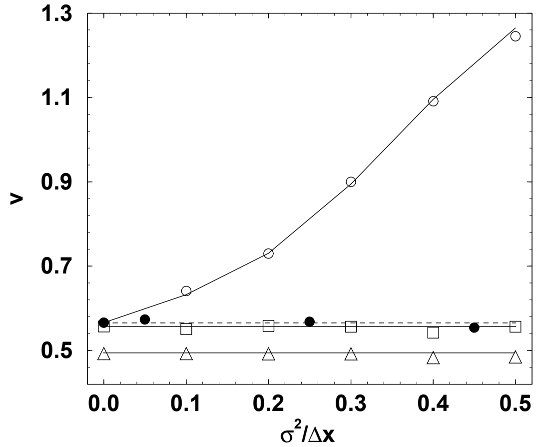

Fig. 1. Front velocity *versus* the effective noise intensity for $\epsilon = 0$ (circles, parabolic case (8)), $\epsilon = 0.1$ (squares) and $\epsilon = 1$ (triangles). Filled circles correspond to the dynamics given by equation (17). In all cases, $a = -0.1$; similar results are obtained for other values of a. Lines correspond to equations (9, 10), and (12), the three straight lower lines with $\sigma = 0$; among these, the dashed line is the theoretical prediction for equation (17).

Let us now discuss the reasons for this result. It is clear from equation (13) that such a null result comes from the fact that the response of the field ϕ to the noise is zero at $t' = t$. This is not the case in the parabolic case (8), for which one finds [12,13]

$$\begin{aligned}\langle g(\phi)\xi(x,t)\rangle &= \sigma^2 C(0)\langle \frac{\partial g(\phi)}{\partial\phi}\frac{\delta\phi}{\delta\xi(x',t')}\rangle\Big|_{x'=x,t'=t}\\ &= \sigma^2 C(0)\langle \frac{\partial g(\phi)}{\partial\phi} g(\phi)\rangle.\end{aligned} \qquad (14)$$

This non zero mean average is responsible for the strong effects of noise in the parabolic model (8) which were studied and quantified in [12,13]. As it turns out, this is not the case in the hyperbolic model (11); therefore, it seems likely that the naive adiabatic elimination procedure, which led to equation (8), is not the proper way to take the limit $\epsilon \to 0$ in equation (11), in view of the different behavior of the two models.

In order to check this idea, we have followed an alternative, non naive adiabatic elimination procedure [23] to see whether an equation different from (8) arises for the overdamped limit. We will outline here the main steps of the calculation following [23] (see [19] for an alternative presentation). For the sake of simplicity, we rewrite equation (11) in a more compact form:

$$\phi_t = \psi; \quad \psi_t = \frac{1}{\epsilon}\left(F(\phi) + g(\phi)\xi(x,t)\right), \qquad (15)$$

where $F(\phi) = f(\phi, a) + \phi_{xx}$. Formally integrating the second expression in (15) and using the first one, we find an integro-differential equation for the variable ϕ:

$$\psi = \phi_t = \int_0^t \mathrm{d}t \, \frac{\mathrm{e}^{-(t-t')/\epsilon}}{\epsilon}\left(F(\phi) + g(\phi)\xi(x,t)\right), \qquad (16)$$

with the initial condition $\psi(0) = 0$. Subsequently, formal integrations by parts are performed in order to obtain a series of terms in powers of ϵ, which is the situation we are interested in (ϵ small). From this formal expansion a Fokker-Planck equation, whose first order term does not depend on ϵ, is obtained. The calculation is involved, but it does not require any further physical assumptions, it is only (lengthy) algebra. We then skip the details and refer the reader interested in them to [23]. The final result is that the corresponding Langevin equation in the Stratonovič interpretation is

$$\phi_t = \phi_{xx} + f(\phi, a) - \sigma^2 C(0)\frac{\partial g(\phi)}{\partial \phi} g(\phi) + g(\phi)\xi(x,t). \qquad (17)$$

We have thus obtained a different overdamped limit, for which one can check that, according to equation (14), the mean value of the noisy term compensates the new term in equation (17), hence rendering the noise contribution null, as in the $\epsilon \neq 0$ case. Therefore, equation (17) is the physically consistent overdamped limit of equation (11), insofar it exhibits the same behavior as this last one does for any value of the "mass" ϵ.

5 Discussion and conclusions

In this work, we have shown analytically and numerically that inertial effects of any magnitude suppress the external white noise influence on the velocity of fronts. Whereas the theoretical result has been conjectured by taking averages, our numerical simulations show that the velocity of *individual* fronts is unchanged by noise. This means that the overdamped (parabolic) equation usually employed to describe front propagation in reaction diffusion model systems is not simply the limit of an underdamped (hyperbolic) version, as in that case it is known that the velocity of the front does depend on the noise strength. In other words, the naive prediction based on deterministic results, equation (12), is not correct in the presence of multiplicative noise. We have also shown that, by means of a more involved adiabatic elimination procedure, it is possible to obtain an equation for the overdamped limit which does not show noise effects. However, this equation differs from the usual one by an extra term, arising from elimination, which exactly cancels the noise contribution to the mean velocity of the fronts.

From the physical viewpoint, these results are very relevant. Indeed, we have seen that any amount of inertia present in the system will lead to front propagation at the deterministic velocity even in the presence of external white noise. *Although the calculation has been done for a specific model, the reasons for the vanishing of the noise contribution are generic and do not depend on the specific choice of the reaction term.* In this context, it is then clear that, even if a system is considered overdamped, generally speaking there will be some degree of inertia in its dynamics. In that case, the predicted changes in the velocity [12,13] will not apply in this case. Therefore, our results are in fact a criterion to establish whether a system (where front propagation arises) is truly overdamped and correspondingly described by a parabolic equation, or in turn, it is an inertial system with very large damping: As we have seen, the response of the system to external white noise would be fundamentally different in both cases. This result is of particular importance in the study of excitable media in noisy environments [15,17]. An additional implication of our findings is that, if one is interested in identifying or measuring possible noise effects in hyperbolic problems in the class of equation (11), the front velocity is not a good observable and it is necessary to resort to other indicators, such as, *e.g.*, the fluctuations of the velocity.

Finally, another remarkable fact is that the naive parabolic limit (8) if interpreted in the Itô sense, it is stochastically equivalent to the limit (17) valid for the Stratonovič interpretation. This is but a further indication that the overdamped limit of equation (11) is problematic and has to be carefully performed, as in any other instance where, for physical reasons, multiplicative noise has to be considered. On the other hand, the fact that equation (8) is the consistent overdamped limit in the Itô interpretation has to do as well with the known result that, in general, perturbative expansions such as the one needed in the adiabatic elimination procedure have many identically zero terms when carried out in Itô sense [19]. This result poses questions of a more mathematical character that would be interesting to address in a general framework for stochastic partial differential equations.

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Note added in proof

After completion of this work, it was pointed out to us the possibility of considering colored noise (finite correlation time τ). We emphasize that the present work corresponds to taking first the limit $\tau \to 0$ (white noise limit) and then $\epsilon \to 0$. Reversing these two limits leads to equation (8) in the Stratonovič interpretation, *i.e.*, the two limits do not commute. Preliminary numerical simulations confirm this result. We thank Jaume Casademunt for this remark.

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