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ON THE IDENTIFIABILITY OF THE TWO-STATE *BMAP*

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Abstract

The capability of modeling non-exponentially distributed and dependent inter-arrival times as well as correlated batches makes the Batch Markovian Arrival Processes (*BMAP*) suitable in different real-life settings as teletraffic, queueing theory or actuarial contexts. An issue to be taken into account for estimation purposes is the identifiability of the process. This is an open problem concerning *BMAP*-related processes. This paper explores the identifiability issue of the two-state *BMAP* noted $BMAP_2(k)$, where k is the maximum batch arrival size. It is proven that for $k=2$ the process cannot be identified, under the assumptions that both the interarrival times and batches sizes are observed. Additionally, a method to obtain an equivalent $BMAP_2(k)$, to a given one is provided.

Keywords: Batch Markovian Arrival Process (*BMAP*); Identifiability problems; Hidden Markov models; Redundant representations.

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On the identifiabilify of the two-state $BMAP$*

Joanna Rodríguez[†] Rosa E. Lillo[‡] Pepa Ramírez-Cobo[§]

Abstract

The capability of modeling non-exponentially distributed and dependent inter-arrival times as well as correlated batches makes the Batch Markovian Arrival Processes ($BMAP$) suitable in different real-life settings as teletraffic, queueing theory or actuarial contexts. An issue to be taken into account for estimation purposes is the identifiability of the process. This is an open problem concerning $BMAP$-related processes. This paper explores the identifiability issue of the two-state $BMAP$ noted $BMAP_2(k)$, where k is the maximum batch arrival size. It is proven that for $k = 2$ the process cannot be identified, under the assumptions that both the interarrival times and batches sizes are observed. Additionally, a method to obtain an equivalent $BMAP_2(2)$ to a given one is provided.

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1 Introduction

The Batch Markovian Arrival Process ($BMAP$) is a large class of point processes built as a matrix generalization of the batch Poisson process to allow for non-exponential and dependent times between the arrival of (possibly correlated) batches. $BMAP$s were first introduced by Neuts [17], although the current and more tractable description of a $BMAP$ arises in Lucantoni [14]. The $BMAP$ includes well-known families of processes as the Markovian Arrival Process (MAP), with single arrivals. It is well known ([13]) that the Markov Modulated Poisson Process ($MMPP$) and PH semi-Markov processes are particular cases of the MAP. Their batch counterparts are the Batch PH-renewal

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process and the Batch Modulated Poisson Process, respectively, which obviously are included in the family of the *BMAP*.

Asmussen and Koole [1] show that stationary $(B)MAP$s are capable of approximating any stationary (batch) point process, which suggests the versatility and range of applications of such processes. Actually, many uses of the *BMAP*s have been suggested in real-life contexts, either in queueing or teletraffic, reliability or insurance, where batch dependent arrivals are commonly observed; for example, customers arriving in batches to a queue, packets of bytes of different length in internet, failures occurring at the same time in an electronic device or simultaneous claims in a insurance company. For a recent account of the literature on *BMAP*s applications, we refer the reader to A. Gómez-Corral and A. Economou [5]; Heckmüller and Wolfinger [9]; Klemm et al. [11]; Niyato et al. [18]; Bookbinder et al. [3]. While performance analysis for models incorporating *BMAP*s is a well developed area, less progress has been made on statistical estimation for such models (for estimation of the *MAP*, see Kriege and Buchholz [12] and the references given there).

*BMAP*s are highly-parametrized models where in practice, only interarrival times and sizes of batch arrivals are usually observed and therefore, the observed data can be viewed as being generated from a hidden Markov process (see Ephraim and Merhav [4]). When dealing with inference for hidden Markov processes, it is common to encounter identifiability problems, which happen when different sets of parameters represent the same process (have the same likelihood function). Up to our knowledge, references in the literature devoted to estimation of *BMAP*-related processes did not take into account the issue of identifiability of the model (being Bodrog et al. [2] an exception for the MAP_2). The non-identifiability has serious negative consequences: the likelihood function has infinitely many global maxima and may be highly multimodal, implying that standard methods (as the EM algorithm) will be strongly dependent on the starting values, running the risk of getting stuck at a poor local maximum.

To the best of our knowledge the identifiability of the *BMAP* has not been considered before, although there exist closed results for different subclasses of *MAP*s. For instance, it is known that the Markov Modulated Poisson Process can be identified (up to permutations of states) as shown by Rydén [22]. The PH distributions are known to lack a unique representation [16, 19], and the works by He and Zhang [6–8] study the identifiability problem of PH related distributions as the Coxian distribution, using an algebraic viewpoint. The non-identifiablity of the two-state Markovian arrival process (MAP_2) has been proven by Ramírez-Cobo et al. [21]. Recently, Bodrog et al. [2] provided a canonical representation of the MAP_2, so that the infinitely many equivalent parameterizations are reduced to a unique single one. Ramírez-Cobo and Lillo [20] partially solve the problem for the MAP_3. However, neither of the previous studies considered the inclusion of batch arrivals. In this paper we address the problem of identifiability of the two state *BMAP*, noted $BMAP_2(k)$, where k is the maximum batch arrival size. Under the assumptions that the interarrival times and batch sizes are observed, we prove that the $BMAP_2(k)$ is a nonidentifiable process for $k = 2$.

This paper is organized as follows. Section 2 sets up notation, gives a brief exposition of the $BMAP_2(k)$ and contains some novel results concerning the batch size distribution. In Section 3 our main result is stated and proved, namely, the $BMAP_2(2)$ is a nonidentifiable process. Finally, in Section 4 we provide conclusions and extensions to this work.

2 Background on the two-state *BMAP*

The $BMAP_2(k)$ is a doubly stochastic process $\{J(t), N(t)\}$, where $J(t)$ represents an irreducible, continuous, Markov process with state space $\mathcal{S} = \{1,2\}$ and $N(t)$ is a counting process where the transitions from (i,n) to $(j, n+k_0)$ correspond to batch arrivals of size $k_0 \leq k$, $i,j \in \mathcal{S}$. For a thorough definition of the general $m-$state *BMAP* we refer the reader to Lucantoni [14], [15].

The $BMAP_2(k)$ behaves as follows: the initial state $i_0 \in \mathcal{S}$ is generated according to an initial probability vector $\boldsymbol{\theta} = (\theta, 1-\theta)$ and at the end of an exponentially distributed sojourn time in state i, with mean $1/\lambda_i$, two possible state transitions can occur. First, with probability p_{ij0}, $j \in \mathcal{S}$, no arrival occurs and the $BMAP_2$ enters a different state $j \neq i$. On the other hand, with probability p_{ijl}, $1 \leq l \leq k$, $j \in \mathcal{S}$, there will be a transition to state j with a batch arrival of size l. The transition probabilities satisfy

$$\sum_{j=1, j\neq i}^{2} p_{ij0} + \sum_{l=1}^{k}\sum_{j=1}^{2} p_{ijl} = 1, \quad \text{for all } i \in \mathcal{S}.$$

Figure 1 illustrates by means of a transition diagram the different transitions that can occur in the $BMAP_2(2)$. The values 0, 1 and 2 corresponds to transition with no arrival, a single arrival or two arrivals, respectively.

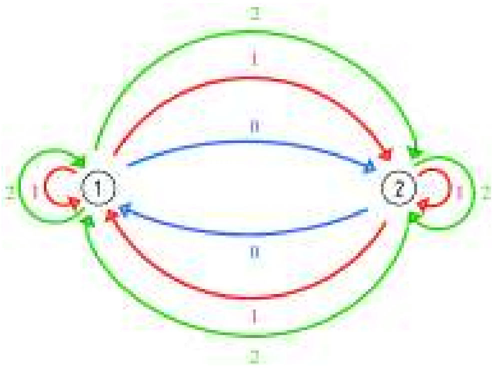

Figure 1: Transition diagram for the $BMAP_2(2)$. 0 denotes moves without arrivals and 1 and 2 denote moves with respective batch arrivals.

A stationary $BMAP_2(k)$ can thus be expressed in terms of the parameters $\{\boldsymbol{\lambda}, P_0, P_1, \ldots, P_k\}$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ and $P_0, \ldots, P_k$ are 2×2 transition probability matrices with elements p_{ij0} $(i \neq j)$, p_{ij1},...,p_{ijk}, respectively. Instead of transition probability matrices, any stationary $BMAP_2(k)$ can be also characterized by rate matrices $\mathcal{B} = \{D_0, D_1, \ldots, D_k\}$ with matrices elements given by

$$\begin{aligned} (D_0)_{ii} &= -\lambda_i, \quad i = 1, 2, \\ (D_0)_{ij} &= \lambda_i p_{ij0}, \quad i, j = 1, 2, \; i \neq j, \\ (D_l)_{ij} &= \lambda_i p_{ijl}, \quad i, j = 1, 2, \; 1 \leq l \leq k. \end{aligned} \tag{1}$$

The matrix D_0 is assumed to be stable, and as a consequence, it is nonsingular and the sojourn times are finite with probability 1. The definition of the rate matrices implies that

$$Q = \sum_{l=0}^{k} D_l$$

is the infinitesimal generator of the underlying Markov process $J(t)$, with stationary probability vector $\boldsymbol{\pi} = (\pi, 1 - \pi)$, computed as $\boldsymbol{\pi} Q = \mathbf{0}$.

An important property of $BMAP$s concerns Markov renewal theory. If X_n is the state of $J(t)$ at the time of the n'th arrival, and T_n the time between the $(n-1)$'th and the n'th (batch) arrival, then $\{X_n, T_n\}_{n=1}^{\infty}$ is a Markov renewal process, which is illustrated by Figure 2 for of a $BMAP_2(2)$. In particular, $\{X_n\}_{n=1}^{\infty}$ is a Markov chain with stationary probability vector $\boldsymbol{\phi}$ computed as

Figure 2: Embedded Markov renewal process in a $BMAP_2(2)$. The b_is denote the batch size of the i-th arrival.

$$\boldsymbol{\phi} = (\boldsymbol{\pi} D \mathbf{e})^{-1} \boldsymbol{\pi} D, \tag{2}$$

where $D = \sum_{l=1}^{k} D_l$ (see [21] for a proof).

In studying MAPs ($BMAP$s with arrivals of size 1) special attention has deserved the analysis of the random variable T, the time between two successive arrivals in the stationary version of the $BMAP_2(k)$. It is well known that T is phase-type distributed with parameters $\{\boldsymbol{\pi}, D_0\}$ (see [10] for more details) and therefore, its moments are computed as

$$E[T^n] = n! \boldsymbol{\phi} (-D_0)^{-n} \mathbf{e},$$

where $\mathbf{e}$ is a vector with all its coordinates equal to one. In addition, if T_n represents the time between the $(n-1)$'th and the n'th arrival, then the autocorrelation function in the stationary version, $\rho(T_1, T_n)$, is given by

$$\rho(T_1, T_n) = \frac{\mu_T \boldsymbol{\pi} \left[(-D_0)^{-1} D\right]^{n-1} (-D_0)^{-1}\mathbf{e} - \mu_T^2}{\sigma_T^2}$$

where $\mu_T = E(T)$ and $\sigma_T^2 = V(T)$.

2.1 Distributional properties of the batch arrival size

This work generalizes previous results on identifiability of *MAP*s to the case where, not only the interarrival times but also batch arrivals are observed. As will be shown, our approach involves looking at the distributional properties of the stationary batch arrival size, B.

First, it is a simple matter to check that

$$P(B = l) = \boldsymbol{\phi}(-D_0)^{-1} D_l \mathbf{e}, \quad \text{for} \quad l = 1, \dots, k,$$

from which the moments of B are obtained as

$$E[B^n] = \boldsymbol{\phi}(-D_0)^{-1} D_n^\star \mathbf{e},$$

where $D_n^\star = \sum_{l=1}^{k} l^n D_l$. Second, the joint Laplace-Stieltjes transform of the interarrival times and batch sizes in the stationary version of the process, $(\boldsymbol{T}, \boldsymbol{B})$ where $\boldsymbol{T} = (T_1, \dots, T_n)$ and $\boldsymbol{B} = (B_1, \dots, B_n)$, is established by the next Lemma.

Lemma 1. *The Laplace-Stieltjes transform of the n first interarrival times and batch sizes of a stationary $BMAP_2(k)$ is given by*

$$f^*_{\boldsymbol{T},\boldsymbol{B}}(\boldsymbol{s}, \boldsymbol{z}) = \boldsymbol{\phi}(s_1 I - D_0)^{-1} \xi(z_1) \dots (s_n I - D_0)^{-1} \xi(z_n) \mathbf{e}, \tag{3}$$

where $\boldsymbol{s} = (s_1, \dots, s_n)$, $\boldsymbol{z} = (z_1, \dots, z_n)$ and $\xi(z_i) = \sum_{l=1}^{k} D_l z_i^l$, for $i = 1, ..., n$.

See Appendix A for the proof. The autocorrelation function of the batch sizes is obtained as an immediate result of the previous Lemma.

Lemma 2. *Let B_n represent the n-th batch arrival size in the stationary version of the $BMAP_2(k)$. Then, the autocorrelation function, $\rho(B_1, B_n)$ is given by*

$$\rho(B_1, B_n) = \frac{\boldsymbol{\phi}(-D_0)^{-1} D_1^\star \left[(-D_0)^{-1} D\right]^{n-2} (-D_0)^{-1} D_1^\star \mathbf{e} - \mu_B^2}{\sigma_B^2} \tag{4}$$

where $\mu_B = E(B)$ and $\sigma_B^2 = Var(B)$.

See Appendix B for the proof.

We should point out here that the lemmas previously stated also hold for the case of the $BMAP_m(k)$, where $m \geq 2$.

3 Identifiability of the $\boldsymbol{BMAP}_2(2)$

As commented in the Introduction, the identifiability of a process is of crucial importance when inference is considered since the lack of a unique representation implies infinite solutions and possibly non-convergence of the typical maximum likelihood approaches. In this section we prove the nonidentifiability of the $BMAP_2(2)$, or in other words, the existence of a differently parametrized representation of the process.

3.1 Preliminary approach

Since only the interarrival times and batch arrival sizes are usually observed, from Ryden [22] and Ramírez-Cobo et al. [21], we provide an analogous definition for the identifiability of the $BMAP_2(k)$, for $k \geq 1$, as follows.

Definition 1. The $BMAP_2(k)$ is a nonidentifiable process if for any fixed $BMAP_2(k)$ with representation $\mathcal{B}$, then there exists another $BMAP_2(k)$ with different representation $\widetilde{\mathcal{B}}$ such that

$$(T_1, \ldots, T_n, B_1, \ldots, B_n) \stackrel{d}{=} (\widetilde{T}_1, \ldots, \widetilde{T}_n, \widetilde{B}_1, \ldots, \widetilde{B}_n) \quad \text{for all } n \geq 1, \tag{5}$$

where T_n and B_n represent the time between the $(n-1)$-th and n-th arrivals, and the batch arrival size of the n-th arrival, respectively, in the $BMAP_2(k)$ defined by $\mathcal{B}$ (similarly define $\widetilde{T}_n$ and $\widetilde{B}_n$).

Note that the equality in distribution (5) is equivalent to the equality of the Laplace-Stieltjes transforms defined in (3),

$$f^*_{\boldsymbol{T},\boldsymbol{B}}(\boldsymbol{s}, \boldsymbol{z}) = f^*_{\widetilde{\boldsymbol{T}};\widetilde{\boldsymbol{B}}}(\boldsymbol{s}, \boldsymbol{z}) \tag{6}$$

for all $\boldsymbol{s} = (s_1, \ldots, s_n)$, $\boldsymbol{z} = (z_1, \ldots, z_n)$ and all $n \geq 1$. As will be seen, the proof for the nonidentifiability of the $BMAP_2(2)$ consists in the existence of infinite solutions to the system of equations given by (5) in terms of the Laplace transforms.

A first approach to study the identifiability of the $BMAP_2(2)$ follows Ramírez-Cobo et al. [21], where the corresponding system of equations to (6) in the MAP_2 case (that is, setting $\boldsymbol{z} = 1$) is solved for $n = 1$ and $n = 2$. It is proven there that the (infinite) solutions found in these cases also satisfy the equations for $n \geq 3$ and therefore the MAP_2 is not identifiable. We show next how this procedure cannot be exactly implemented in the case of the $BMAP_2(2)$, since it ends up with a system of equations impossible to solve in practice when the cases $n = 1, 2$ are jointly considered for $\boldsymbol{s}$ and $\boldsymbol{z}$. However, the results found for the construction of an equivalent MAP_2 are the starting point for our second approach, which will be given later on.

From now on, a stationary $BMAP_2(2)$ will be represented by $\{D_0, D_1, D_2\}$ where

$$D_0 = \begin{pmatrix} x & y \\ r & u \end{pmatrix}, \quad D_1 = \begin{pmatrix} w & m \\ v & q \end{pmatrix}, \quad D_2 = \begin{pmatrix} n & -x-y-w-m-n \\ t & -r-u-v-q-t \end{pmatrix}, \tag{7}$$

where without loss of generality it is assumed that $x \le u$. According to (1),

$$x=-\lambda_1,\quad y=\lambda_1 p_{120},\quad w=\lambda_1 p_{111},\quad m=\lambda_1 p_{121},\quad n=\lambda_1 p_{112},$$

$$r=\lambda_2 p_{210},\quad u=-\lambda_2,\quad v=\lambda_2 p_{211},\quad q=\lambda_2 p_{221},\quad t=\lambda_2 p_{212}.$$

The stationary probability distribution $\boldsymbol{\phi}=(\phi,1-\phi)$ is computed from (2) in terms of the model parameters as

$$\phi=\frac{rn+rw-xt-xv}{ux-yr+rn+rw-xt-xv-yt-yv+un+uw}.$$

Take $n=1$ in (6), which from Lemma 1 becomes

$$\boldsymbol{\phi}(sI-D_0)^{-1}\xi(z)\mathbf{e}=\widetilde{\boldsymbol{\phi}}(sI-\widetilde{D}_0)^{-1}\widetilde{\xi}(z)\mathbf{e}. \tag{8}$$

From (7) it can be seen that (8) is equivalent to

$$\frac{z(s\alpha+\beta)+z^2(s\gamma-\beta+\eta)}{s^2+sv+\eta}=\frac{z(s\widetilde{\alpha}+\widetilde{\beta})+z^2(s\widetilde{\gamma}-\widetilde{\beta}+\widetilde{\eta})}{s^2+s\widetilde{v}+\widetilde{\eta}},$$

where α, β, γ, η, v (respectively $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\gamma}$, $\widetilde{\eta}$, $\widetilde{v}$) are given by

$$\begin{aligned}
\alpha &= \phi(m+w-v-q)+(v+q),\\
\beta &= \phi(-uw+yv-um+yq-rw+xv-rm+xq)+(rw-xv+rm-xq),\\
\gamma &= \phi(-m-w-x-y+q+u+v+r)-(q+u+v+r),\\
\eta &= xu-yr,\\
v &= -u-x.
\end{aligned} \tag{9}$$

Note that since D_0 is invertible, then necessarily $\eta\neq 0$. Consider now (6) with $n=2$. We proceed analogously to the case $n=1$ to find that (6) becomes

$$\phi(s_1I-D_0)^{-1}\xi(z_1)(s_2I-D_0)^{-1}\xi(z_2)\mathbf{e}=\widetilde{\phi}(s_1I-\widetilde{D}_0)^{-1}\widetilde{\xi}(z_1)(s_2I-\widetilde{D}_0)^{-1}\widetilde{\xi}(z_2)\mathbf{e}, \tag{10}$$

where

$$\begin{aligned}
&\phi(s_1I-D_0)^{-1}\xi(z_1)(s_2I-D_0)^{-1}\xi(z_2)\mathbf{e}=\\
&\frac{z_1z_2(s_1\delta_1+s_2\delta_2+s_1s_2\delta_3+\delta_4)+z_1z_2^2(s_1(\alpha\eta-\delta_1)+s_2\delta_5+s_1s_2\delta_6)}{(s_1^2+s_2^2)\eta+s_1^2s_2^2+(s_1^2s_2+s_1s_2^2)v+(s_1+s_2)\eta v+s_1s_2v^2+\eta^2}+\\
&\frac{z_1z_2^2(\beta\eta-\delta_4)+z_1^2z_2(s_1\delta_7+s_2\delta_8+s_1s_2\delta_9+\delta_{10})}{(s_1^2+s_2^2)\eta+s_1^2s_2^2+(s_1^2s_2+s_1s_2^2)v+(s_1+s_2)\eta v+s_1s_2v^2+\eta^2}+\\
&\frac{z_1^2z_2^2(s_1(\eta\gamma-\delta_7)+s_2\delta_{11}+s_1s_2\delta_{12}+\eta^2-\eta\beta-\delta_{10})}{(s_1^2+s_2^2)\eta+s_1^2s_2^2+(s_1^2s_2+s_1s_2^2)v+(s_1+s_2)\eta v+s_1s_2v^2+\eta^2},
\end{aligned}$$

(respectively the right-hand side of (10)) where δ_i for $i=1,\ldots,12$ are given by

$$\delta_1 = \beta(q-m)+(\phi(w-v-q+m)+v)(vy-mu+qy-uw)$$

$$
\begin{aligned}
& \quad -m(qx-mr-rw+vx), \\
\delta_2 &= \beta(q+w)+(\phi(x+y-r-u)+(r-x))(mv-qw), \\
\delta_3 &= \alpha(q+w)+mv-qw, \\
\delta_4 &= \beta(mr-qx-uw+vy)+\eta(mv-qw), \\
\delta_5 &= \phi((r+u)(mr+mu-qx-qy)+(x+y)(-vx-vy+wr+wu)) \\
& \quad +(r+u)(qx-mr)+(x+y)(vx-wr)-\delta_2, \\
\delta_6 &= \phi((r+u)(q-m)+(x+y)(v-w))-q(r+u)-v(x+y)-\delta_3, \qquad (11)\\
\delta_7 &= (\gamma-n)(rm+rw-xq-xv)-\beta(t-n)+t(-mu+qy-uw+vy), \\
\delta_8 &= (\eta-\beta)(q+v)+(\phi(nr+nu-tx-ty)+(tx-nr))(q+v-m-w), \\
\delta_9 &= \gamma(q+v)+(\phi(t-n)-t)(q+v-m-w), \\
\delta_{10} &= \eta(mr-qx-vx+rw+mt+tw-nq-nv) \\
& \quad -\beta(mr-qx-vx+rw+nr+nu-tx-ty), \\
\delta_{11} &= (\phi(-nr-nu+tx+ty)+(rn-tx))(r+u-x-y) \\
& \quad +(r+u)(\beta-\eta)-\delta_8, \\
\delta_{12} &= -\gamma(r+u)+(r+u-x-y)(\phi(n-t)+t)-\delta_9,
\end{aligned}
$$

(respectively, $\widetilde{\delta}_i$ for $i=1,\ldots,12$). Next, define the three following conditions

$$
\begin{aligned}
C1: &\quad \alpha\eta-\beta(\alpha+\gamma)\neq 0, \\
C2: &\quad \beta+\alpha(\alpha+\gamma-\upsilon)\neq 0, \\
C3: &\quad \eta+(\gamma-\upsilon+\alpha)(\alpha+\gamma)\neq 0.
\end{aligned}
$$

Then, it is tedious but straightforward to prove that if $C1$, $C2$ and $C3$ holds, then the solution to both equations (8) and (10) is

$$
\widetilde{\alpha}=\alpha, \quad \widetilde{\beta}=\beta, \quad \widetilde{\gamma}=\gamma, \quad \widetilde{\eta}=\eta, \quad \widetilde{\upsilon}=\upsilon, \quad \widetilde{\delta}_i=\delta_i, \quad \text{for } i=1,\ldots,12. \qquad (12)
$$

Given a known $BMAP_2(2)$ defined by $\{D_0,D_1,D_2\}$ (or alternatively by $x,y,\ldots,t$) as in (7), then the set of equations given by (12) provides the solutions $\{\widetilde{D}_0,\widetilde{D}_1,\widetilde{D}_1\}$ (alternatively $\tilde{x},\tilde{y},\ldots,\tilde{t}$) of differently parametrized $BMAP_2(2)$s such that (6) holds for $n=1$ and $n=2$. However, due to the complexity of the set of equations (9) and (11), it was not possible to (symbolically) obtain the values of $\{\widetilde{D}_0,\widetilde{D}_1,\widetilde{D}_1\}$ $(\tilde{x},\tilde{y},\ldots,\tilde{t})$ that solve the system (12) via standard symbolic packages as Maple or Matlab. In consequence, a different approach for solving the identifiability problem of the $BMAP_2(2)$ needs to be considered. The proof of the main result in next Section shows such a procedure.

3.2 Main result

Next result provides the solutions to (12).

Proposition 1. *Consider a $BMAP_2(2)$ as in (7). For all $\tilde{u} < 0$ and all $\tilde{r} > 0$, let $\tilde{x}(\tilde{u},\tilde{r})$, $\tilde{y}(\tilde{u},\tilde{r})$, $\tilde{v}(\tilde{u},\tilde{r})$, $\tilde{w}(\tilde{u},\tilde{r})$, $\tilde{m}(\tilde{u},\tilde{r})$, $\tilde{q}(\tilde{u},\tilde{r})$, $\tilde{n}(\tilde{u},\tilde{r})$ and $\tilde{t}(\tilde{u},\tilde{r})$ be defined as*

$$
\begin{aligned}
\tilde{x}(\tilde{u},\tilde{r}) &= -\tilde{u} + x + u,\\
\tilde{y}(\tilde{u},\tilde{r}) &= -\frac{(\tilde{u}^2 - \tilde{u}x - \tilde{u}u + xu - ry)}{\tilde{r}},\\
\tilde{q}(\tilde{u},\tilde{r}) &= h_1(\widetilde{\phi}, \tilde{x}, \tilde{y}, D_0, D_1, D_2),\\
\tilde{m}(\tilde{u},\tilde{r}) &= h_2(\widetilde{\phi}, \tilde{x}, \tilde{y}, \tilde{q}, D_0, D_1, D_2),\\
\tilde{n}(\tilde{u},\tilde{r}) &= h_3(\widetilde{\phi}, \tilde{x}, \tilde{y}, \tilde{q}, D_0, D_1, D_2),\\
\tilde{t}(\tilde{u},\tilde{r}) &= h_4(\widetilde{\phi}, \tilde{x}, \tilde{y}, \tilde{q}, D_0, D_1, D_2),\\
\tilde{w}(\tilde{u},\tilde{r}) &= \left(\frac{xun + ryt - \tilde{u}xw + ryv + \tilde{u}xv + \tilde{u}uv - ryn + u\tilde{r}t - \tilde{r}yv + \tilde{u}\tilde{r}n}{-\tilde{u}u - u\tilde{r} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{r} + \tilde{r}^2 - \tilde{u}x - \tilde{r}x - ry}\right.\\
&\quad + \frac{-\tilde{u}\tilde{r}v - \tilde{r}yt + 2\tilde{r}xu - \tilde{r}xw - \tilde{u}\tilde{r}t + u\tilde{r}v - ryw + \tilde{u}xt - \tilde{u}uw}{-\tilde{u}u - u\tilde{r} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{r} + \tilde{r}^2 - \tilde{u}x - \tilde{r}x - ry}\\
&\quad + \frac{-xu^2 - \tilde{u}xn - r\tilde{r}y + rx\tilde{r} + ruy - rxu + \tilde{u}^2w + \tilde{u}^2n - r\tilde{u}^2}{-\tilde{u}u - u\tilde{r} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{r} + \tilde{r}^2 - \tilde{u}x - \tilde{r}x - ry}\\
&\quad + \frac{-\tilde{u}^2t - \tilde{u}^2v + \tilde{u}^3 + \tilde{r}u^2 - \tilde{r}^2x - 2\tilde{u}^2u + r^2y + \tilde{u}u^2 - 2\tilde{u}\tilde{r}x}{-\tilde{u}u - u\tilde{r} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{r} + \tilde{r}^2 - \tilde{u}x - \tilde{r}x - ry}\\
&\quad + \frac{-\tilde{r}^2u - 3\tilde{u}\tilde{r}u + 2\tilde{u}xu + 2\tilde{u}^2\tilde{r} + r\tilde{u}u - \tilde{u}\tilde{r}r - \tilde{u}^2x + \tilde{r}^2\tilde{u}}{-\tilde{u}u - u\tilde{r} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{r} + \tilde{r}^2 - \tilde{u}x - \tilde{r}x - ry}\\
&\quad + \frac{\tilde{u}ut + r\tilde{u}x - \tilde{r}xn + xuw - \tilde{u}un + ru\tilde{r} + r\tilde{r}w + r\tilde{r}n}{-\tilde{u}u - u\tilde{r} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{r} + \tilde{r}^2 - \tilde{u}x - \tilde{r}x - ry}\\
&\quad \left. + \frac{-xuv - xut + \tilde{u}\tilde{r}w - ry\tilde{u}}{-\tilde{u}u - u\tilde{r} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{r} + \tilde{r}^2 - \tilde{u}x - \tilde{r}x - ry}\right) - \tilde{n}\\
\tilde{v}(\tilde{u},\tilde{r}) &= \left(\frac{\tilde{r}(-xv - xt - yv - yt + rw + rn + uw + un - \tilde{r}w - \tilde{r}n + r\tilde{r})}{-\tilde{u}u - u\tilde{r} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{r} + \tilde{r}^2 - \tilde{u}x - \tilde{r}x - ry}\right.\\
&\quad + \frac{\tilde{r}(r\tilde{u} + \tilde{u}u - \tilde{r}^2 - 2\tilde{u}\tilde{r} + u\tilde{r} + \tilde{r}v)}{-\tilde{u}u - u\tilde{r} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{r} + \tilde{r}^2 - \tilde{u}x - \tilde{r}x - ry}\\
&\quad \left. + \frac{\tilde{r}(\tilde{r}t + \tilde{u}v + \tilde{u}t - \tilde{u}w - \tilde{u}n - \tilde{u}^2)}{-\tilde{u}u - u\tilde{r} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{r} + \tilde{r}^2 - \tilde{u}x - \tilde{r}x - ry}\right) - \tilde{t},
\end{aligned}
\tag{13}
$$

for specific values of functions h_1, h_2, h_3 and h_4. Then, the set of values $\{\tilde{u},\tilde{r},\tilde{x}(\tilde{u},\tilde{r}),\tilde{y}(\tilde{u},\tilde{r}),\tilde{v}(\tilde{u},\tilde{r}),\tilde{w}(\tilde{u},\tilde{r}),\tilde{m}(\tilde{u},\tilde{r}),\tilde{q}(\tilde{u},\tilde{r}),\tilde{n}(\tilde{u},\tilde{r}),\tilde{t}(\tilde{u},\tilde{r})\}$ solves the system of equations given by (12).

Remark 1. Closed-form expressions for h_i, $i = 1, 2, 3, 4$ can be found at

`http://www.joavrweb.3owl.com/Software.html`

They have been included neither in Proposition 1. nor in the Appendix due to their large extension (around 43 pages).

Proof. Since it is not possible to symbolically solve the set of equations (12), the next alternative procedure was applied. Consider a fixed $BMAP_2(2)$ with representation $\mathcal{B} = \{D_0, D_1, D_2\}$ as in (7). Then, it is clear that $\mathcal{M} = \{R_0 = D_0, R_1 = D_1 + D_2\}$ defines a MAP_2, where arrivals of batch 1 or 2 are considered as being equal. Assume we look for a $BMAP_2(2)$ defined by $\widetilde{\mathcal{B}} = \{\widetilde{D}_0, \widetilde{D}_1, \widetilde{D}_2\}$ that is equivalent to $\mathcal{B}$ according to Definition 1. In particular the equality

$$(T_1, \ldots, T_n) \stackrel{d}{=} (\widetilde{T}_1, \ldots, \widetilde{T}_n)$$

must hold, and therefore the underlying MAP_2 defined by $\mathcal{M} = \{\widetilde{R}_0 = \widetilde{D}_0, \widetilde{R}_1 = \widetilde{D}_1 + \widetilde{D}_2\}$ must be equivalent to $\mathcal{M}$, according to Definition 1. in Ramírez-Cobo et al [21]. This approach can be seen to reduce the number of unknown variables from 10 to 4: given the underlying MAP_2 defined by

$$R_0 = D_0 = \begin{pmatrix} x & y \\ r & u \end{pmatrix}, \quad R_1 = D_1 + D_2 = \begin{pmatrix} w+n & -x-y-w-n \\ v+t & -r-u-v-t \end{pmatrix}, \tag{14}$$

then the method provided by Theorem 4.1. of Ramírez-Cobo et al [21] allows one to calculate an equivalent MAP_2 to (14),

$$\widetilde{R}_0 = \begin{pmatrix} \tilde{x} & \tilde{y} \\ \tilde{r} & \tilde{u} \end{pmatrix}, \quad \widetilde{R}_1 = \begin{pmatrix} \tilde{d}_{111} & -\tilde{x}-\tilde{y}-\tilde{d}_{111} \\ \tilde{d}_{211} & -\tilde{r}-\tilde{u}-\tilde{d}_{211} \end{pmatrix}, \tag{15}$$

where $\tilde{r}$ and $\tilde{u}$ are free parameters, and $\tilde{x}$, $\tilde{y}$ $\tilde{d}_{111}$ and $\tilde{d}_{211}$ are obtained as

$$\begin{aligned}
\tilde{x}(\tilde{u},\tilde{r}) &= -\tilde{u} + x + u, \\
\tilde{y}(\tilde{u},\tilde{r}) &= -\frac{(\tilde{u}^2 - \tilde{u}x - \tilde{u}u + xu - ry)}{\tilde{r}}, \\
\tilde{d}_{111} &= \Bigg(\frac{xun + ryt - \tilde{u}xw + ryv + \tilde{u}xv + \tilde{u}uv - ryn + u\tilde{r}t - \tilde{r}yv + \tilde{u}\tilde{r}n}{-\tilde{u}u - u\tilde{r} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{r} + \tilde{r}^2 - \tilde{u}x - \tilde{r}x - ry} \\
&\quad + \frac{-\tilde{u}\tilde{r}v - \tilde{r}yt + 2\tilde{r}xu - \tilde{r}xw - \tilde{u}\tilde{r}t + u\tilde{r}v - ryw + \tilde{u}xt - \tilde{u}uw}{-\tilde{u}u - u\tilde{r} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{r} + \tilde{r}^2 - \tilde{u}x - \tilde{r}x - ry} \\
&\quad + \frac{-xu^2 - \tilde{u}xn - r\tilde{r}y + rx\tilde{r} + ruy - rxu + \tilde{u}^2w + \tilde{u}^2n - r\tilde{u}^2}{-\tilde{u}u - u\tilde{r} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{r} + \tilde{r}^2 - \tilde{u}x - \tilde{r}x - ry} \\
&\quad + \frac{-\tilde{u}^2t - \tilde{u}^2v + \tilde{u}^3 + \tilde{r}u^2 - \tilde{r}^2x - 2\tilde{u}^2u + r^2y + \tilde{u}u^2 - 2\tilde{u}\tilde{r}x}{-\tilde{u}u - u\tilde{r} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{r} + \tilde{r}^2 - \tilde{u}x - \tilde{r}x - ry} \\
&\quad + \frac{-\tilde{r}^2u - 3\tilde{u}\tilde{r}u + 2\tilde{u}xu + 2\tilde{u}^2\tilde{r} + r\tilde{u}u - \tilde{u}\tilde{r}r - \tilde{u}^2x + \tilde{r}^2\tilde{u}}{-\tilde{u}u - u\tilde{r} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{r} + \tilde{r}^2 - \tilde{u}x - \tilde{r}x - ry} \\
&\quad + \frac{\tilde{u}ut + r\tilde{u}x - \tilde{r}xn + xuw - \tilde{u}un + ru\tilde{r} + r\tilde{r}w + r\tilde{r}n}{-\tilde{u}u - u\tilde{r} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{r} + \tilde{r}^2 - \tilde{u}x - \tilde{r}x - ry} \\
&\quad + \frac{-xuv - xut + \tilde{u}\tilde{r}w - ry\tilde{u}}{-\tilde{u}u - u\tilde{r} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{r} + \tilde{r}^2 - \tilde{u}x - \tilde{r}x - ry} \Bigg)
\end{aligned}$$

$$\tilde{d}_{211} = \left(\frac{\tilde{r}(-xv - xt - yv - yt + rw + rn + uw + un - \tilde{r}w - \tilde{r}n + r\tilde{r})}{-\tilde{u}u - u\tilde{r} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{r} + \tilde{r}^2 - \tilde{u}x - \tilde{r}x - ry} \right.$$
$$+ \frac{\tilde{r}(r\tilde{u} + \tilde{u}u - \tilde{r}^2 - 2\tilde{u}\tilde{r} + u\tilde{r} + \tilde{r}v)}{-\tilde{u}u - u\tilde{r} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{r} + \tilde{r}^2 - \tilde{u}x - \tilde{r}x - ry}$$
$$\left. + \frac{\tilde{r}(\tilde{r}t + \tilde{u}v + \tilde{u}t - \tilde{u}w - \tilde{u}n - \tilde{u}^2)}{-\tilde{u}u - u\tilde{r} + xu + \tilde{u}^2 + 2\tilde{u}\tilde{r} + \tilde{r}^2 - \tilde{u}x - \tilde{r}x - ry} \right).$$

We should point out here that given any MAP_2 as in (14), then there exist infinite equivalent MAP_2s as in (15), each one constructed from a specific choice of a certain parameter ε (see Theorem 4.1. in [21]). Therefore, $\widetilde{R}_0$ and $\widetilde{R}_1$ are indeed, $\widetilde{R}_0(\varepsilon)$ and $\widetilde{R}_1(\varepsilon)$. Imposing the condition that equivalent $BMAP_2$s must have equivalent underlying MAP_2s leads to

$$\widetilde{D}_0 = \widetilde{R}_0 = \begin{pmatrix} \tilde{x} & \tilde{y} \\ \tilde{r} & \tilde{u} \end{pmatrix}, \quad \widetilde{D}_1 = \begin{pmatrix} \tilde{w} & \tilde{m} \\ \tilde{v} & \tilde{q} \end{pmatrix}, \quad \widetilde{D}_2 = \begin{pmatrix} \tilde{n} & -\tilde{x} - \tilde{y} - \tilde{w} - \tilde{m} - \tilde{n} \\ \tilde{t} & -\tilde{r} - \tilde{u} - \tilde{v} - \tilde{q} - \tilde{t} \end{pmatrix}, \tag{16}$$

where necessarily $\widetilde{D}_1 + \widetilde{D}_2 = \widetilde{R}_1$ or in other words, $\tilde{w} = \tilde{d}_{111} - \tilde{n}$ and $\tilde{v} = \tilde{d}_{211} - \tilde{t}$. In order to find the remaining unknowns $\tilde{n}$, $\tilde{t}$, $\tilde{m}$ and $\tilde{q}$, the known values are substituted into the next subset of equations of (12),

$$\widetilde{\beta} = \beta, \quad \widetilde{\gamma} = \gamma, \quad \widetilde{\delta}_3 = \delta_3, \quad \widetilde{\delta}_4 = \delta_4,$$

to yield the expressions given by (13). Finally, it is cumbersome but straightforward to check that the solutions in (13) also satisfy the rest of equations in (12). □

Remark 2. The set of values in (13) solves the equality of Laplace transforms (6) for $n = 1, 2$. Or equivalently, given a $BMAP_2(2)$ as in (7) it allows to compute the values of $\{\widetilde{D}_0, \widetilde{D}_1, \widetilde{D}_2\}$ such that (6) holds for $n = 1, 2$. However, we should point out here an interesting result, which is that not all the infinite solutions in (13) define real $BMAP_2(2)$s, in the sense that it may be the case that any of the parameters of $\widetilde{D}_1$ or $\widetilde{D}_2$ take a negative value. See the next numerical example for an illustration of this fact.

Example 1. Consider the $BMAP_2(2)$ defined by

$$D_0 = \begin{pmatrix} -7.0666 & 0.0779 \\ 0.0047 & -6.9116 \end{pmatrix}, D_1 = \begin{pmatrix} 4.5829 & 0.4523 \\ 1.3993 & 1.5595 \end{pmatrix}, D_2 = \begin{pmatrix} 0.0354 & 1.9181 \\ 3.3994 & 0.5488 \end{pmatrix},$$

whose underlying MAP_2 is

$$R_0 = \begin{pmatrix} -7.0666 & 0.0779 \\ 0.0047 & -6.9116 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 4.6183 & 2.3704 \\ 4.7986 & 2.1083 \end{pmatrix}.$$

If the value of $\varepsilon = 0.0018$ is set in the method derived in ([21]) to find an equivalent MAP_2 then

$$\widetilde{R}_0 = \begin{pmatrix} -7.0649 & 0.0985 \\ 0.0064 & -6.9134 \end{pmatrix}, \quad \widetilde{R}_1 = \begin{pmatrix} 6.4265 & 0.5398 \\ 6.6068 & 0.3001 \end{pmatrix}$$

is obtained. Computing the values of $\tilde{n}$, $\tilde{t}$, $\tilde{m}$ and $\tilde{q}$ as in (13), finally leads to

$$\widetilde{D}_0 = \begin{pmatrix} -7.0649 & 0.0985 \\ 0.0064 & -6.9134 \end{pmatrix}, \widetilde{D}_1 = \begin{pmatrix} 5.1102 & -0.6433 \\ 1.9265 & 1.0323 \end{pmatrix}, \widetilde{D}_2 = \begin{pmatrix} 5.1102 & -0.6433 \\ 1.9265 & 1.0323 \end{pmatrix},$$

which is not a well-defined $BMAP_2(2)$. Note that despite obtaining a non-real $BMAP_2(2)$, the equations (12) still hold.

The previous example motivates the seek for *optimal* values for the free parameters, $\tilde{x}$ and $\tilde{r}$ such that (13) correctly defines the parameters of a $BMAP_2(2)$. Next Proposition provides these values.

Proposition 2. *Consider a $BMAP_2(2)$ with representationa $\mathcal{B}$ as in (16), and define*

$$\begin{aligned}
\kappa_1 &= -x, \\
\kappa_2 &= \frac{u-x}{2}, \\
\kappa_3 &= \frac{r(1-\phi)}{\phi}, \\
\kappa_4 &= \frac{rq}{v}, \\
\kappa_5 &= -\frac{r}{t}(r+u+v+q+t), \\
\kappa_6 &= \frac{(u-x)+\sqrt{(x-u)^2+4ry}}{2}, \\
\kappa_7 &= \frac{r}{2v}\left[(q-w)+\sqrt{(q-w)^2+4vm}\right], \\
\kappa_8 &= -\frac{r}{2t}\left[u+v+q+n+t+r-\right. \\
&\qquad \left.\sqrt{(u+v+q+n+t+r)^2+4t(-y-x-w-m-n)}\right],
\end{aligned}$$

Let κ be chosen from

$$0 < \kappa < \min\{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_6, \kappa_7, \kappa_8\}, \quad \text{if } x < u, \tag{17}$$

$$0 < \kappa < \min\{\kappa_1, \kappa_3, \kappa_4, \kappa_5, \kappa_6, \kappa_7, \kappa_8\}, \quad \text{if } x = u, \tag{18}$$

and set $\tilde{u} \equiv u - \kappa$ and $\tilde{r} \equiv r + \kappa$.

Then there exist an infinite number of $BMAP_2(2)$s, $\widetilde{\mathcal{B}}$, given by $\mathcal{F} = \{\tilde{u}, \tilde{r}, \tilde{x}(\tilde{u},\tilde{r}), \tilde{y}(\tilde{u},\tilde{r}), \tilde{v}(\tilde{u},\tilde{r}), \tilde{w}(\tilde{u},\tilde{r}), \tilde{m}(\tilde{u},\tilde{r}), \tilde{q}(\tilde{u},\tilde{r}), \tilde{n}(\tilde{u},\tilde{r}), \tilde{t}(\tilde{u},\tilde{r})\}$, where $\tilde{x}(\tilde{u},\tilde{r})$, $\tilde{y}(\tilde{u},\tilde{r})$, $\tilde{v}(\tilde{u},\tilde{r})$, $\tilde{w}(\tilde{u},\tilde{r})$, $\tilde{m}(\tilde{u},\tilde{r})$, $\tilde{q}(\tilde{u},\tilde{r})$, $\tilde{n}(\tilde{u},\tilde{r})$ and $\tilde{t}(\tilde{u},\tilde{r})$ are defined by (13), such that (6) holds.

See Appendix C for the proof of Proposition 2.

Note that Proposition 2 is analogous to Theorem 4.1. in [21] in the case of the MAP_2, where certain value of ε is obtained in order to define feasible values for the parameters of a equivalent

MAP_2 to a fixed one. Note too that the range of values for such an ε contains the interval from where κ is chosen.

Consider Example 1. From Proposition 2 a value of κ needs to be selected from

$$0 < \kappa < min\{7.0666, 0.0775, 0.0023, 0.1573, 0.0007, 0.0052, 0.0008, 0.0039\}$$

Take for example, $\kappa = 5 \times 10^{-4}$, then the obtained (feasible) equivalent $BMAP_2(2)$ is

$$\widetilde{D}_0 = \begin{pmatrix} -7.0661 & 0.0856 \\ 0.0052 & -6.9121 \end{pmatrix}, \widetilde{D}_1 = \begin{pmatrix} 4.7399 & 0.0859 \\ 1.5562 & 1.4026 \end{pmatrix}, \widetilde{D}_2 = \begin{pmatrix} 0.4167 & 1.7380 \\ 3.7807 & 0.1675 \end{pmatrix}.$$

Finally, it can be seen that the feasible solutions obtained in Proposition 2. also satisfy the equality of Laplace transform (6) for all $n \geq 3$, leading to our main result.

Theorem 1. *The $BMAP_2(2)$ is not an identifiable process.*

Proof. The proof of Theorem 1 is parallel to the proof of Theorem 4.2. in [21] (therefore the details are omitted), where in this case $\Delta(s)$ is replaced by $\Delta(s, z)$ obtained as

$$\Delta(s, z) = (sI - D_0)^{-1}\xi(z),$$

with parametrization

$$\Delta(s, z) = \begin{pmatrix} a(s, z) & b(s, z) \\ c(s, z) & d(s, z) \end{pmatrix}. \tag{19}$$

□

3.3 Numerical examples

In this section we illustrate our approach for finding equivalent $BMAP_2(2)$ to a fixed given one by some more examples.

Example 2. Consider a $BMAP_2(2)$ defined by

$$D_0 = \begin{pmatrix} -9.9271 & 2.1354 \\ 0.3991 & -9.8909 \end{pmatrix}, D_1 = \begin{pmatrix} 1.3419 & 0.1245 \\ 1.2953 & 1.6686 \end{pmatrix}, D_2 = \begin{pmatrix} 2.6025 & 3.7228 \\ 0.2105 & 6.3174 \end{pmatrix}.$$

Set $\kappa = 0.0159$, chosen from

$$0 < \kappa < min\{9.9271, 0.0181, 1.3310, 0.9414, 0.1839, 0.5141, 11.9751, 7.4213\}.$$

Then, it can be easily seen that from applying Proposition 2. the obtained $BMAP_2(2)$

$$\widetilde{D}_0 = \begin{pmatrix} -9.9111 & 2.0542 \\ 0.4150 & -9.9068 \end{pmatrix}, \widetilde{D}_1 = \begin{pmatrix} 1.3936 & 0.1303 \\ 1.3470 & 1.6169 \end{pmatrix}, \widetilde{D}_2 = \begin{pmatrix} 2.6109 & 3.7221 \\ 0.2189 & 6.3090 \end{pmatrix}$$

makes (6) hold for all s, z and n.

Example 3. Consider a $BMAP_2(2)$ for which $x = u$,

$$D_0 = \begin{pmatrix} -4 & 1.3 \\ 1 & -4 \end{pmatrix}, D_1 = \begin{pmatrix} 0.1 & 0.3 \\ 0.28 & 0.74 \end{pmatrix}, D_2 = \begin{pmatrix} 1.6 & 0.7 \\ 1 & 0.98 \end{pmatrix}.$$

Set $\kappa = 0.1105$, chosen from

$$0 < \kappa < min\{4, 0.9144, 1.1402, 2.6848, 2.6429, 0.98, 0.5822\},$$

then, the equivalent $BMAP_2(2)$ is

$$\widetilde{D}_0 = \begin{pmatrix} -3.8895 & 1.1596 \\ 1.1105 & -4.1105 \end{pmatrix}, \widetilde{D}_1 = \begin{pmatrix} 0.1309 & 0.3308 \\ 0.3109 & 0.7091 \end{pmatrix}, \widetilde{D}_2 = \begin{pmatrix} 1.7105 & 0.5576 \\ 1.1105 & 0.8695 \end{pmatrix}.$$

4 Conclusions

This paper deepens the understanding of the identifiability of *BMAP*-related process, a relevant aspect not only from the theoretical viewpoint, but also when inference for the process is to be undertaken. Specifically, it proves that the two-state *BMAP* or $BMAP_2(k)$, with maximum batch size $k = 2$ is nonidentifiable, extending previous works focused on the case $k = 1$.

All calculations to prove the nonidentifiability of the $BMAP_2(2)$ have been carried out using MATLAB® version 7.1.0.246 (R14). In the spirit of a reproducible research the codes utilized in this paper are available at

`http://www.joavrweb.3owl.com/Software.html`

Prospects regarding this work may concern both the estimation of the $BMAP_2(k)$ and the study of the nonidentifiability for higher order $BMAP_m(k)$, for $m \geq 3$ and for greater batch sizes $k \geq 3$ which are expected to show more versatility for modeling purposes. Concerning the second point, we are aware of the complexity of such a problem due to the increasing number of parameters. These complications define challenging tasks that we hope to address in the future.

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Appendix A: Proof of Lemma 1

Let $\boldsymbol{T} = (T_1, \ldots, T_n)$, $\boldsymbol{B} = (B_1, \ldots, B_n)$, $\boldsymbol{s} = (s_1, \ldots, s_n)$, $\boldsymbol{z} = (z_1, \ldots, z_n)$, $\boldsymbol{t} = (t_1, \ldots, t_n)$, $\boldsymbol{b} = (b_1, \ldots, b_n)$. Then, expression (3) easily follows from the definition of the Laplace-Stieltjes transform of $(\boldsymbol{T}, \boldsymbol{B})$,

$$f^*_{\boldsymbol{T},\boldsymbol{B}}(\boldsymbol{s}, \boldsymbol{z}) = \int_0^\infty e^{-s_1 t_1} \ldots e^{-s_n t_n} \prod_{j=1}^{n} \left(\sum_{b_j=1}^{\infty} z_j^{b_j} \right) f_{\boldsymbol{T},\boldsymbol{B}}(\boldsymbol{t}, \boldsymbol{b})\ d\boldsymbol{t},$$

and the joint density function, $f_{\boldsymbol{T},\boldsymbol{B}}(\boldsymbol{t}, \boldsymbol{b})$, given by (see [11])

$$f_{\boldsymbol{T},\boldsymbol{B}}(\boldsymbol{t}, \boldsymbol{b}) = \boldsymbol{\phi} e^{D_0 t_1} D_{b_1} \ldots e^{D_0 t_n} D_{b_n} \mathbf{e}.$$

□

Appendix B: Proof of Lemma 2

Let $\boldsymbol{T} = (T_1, \ldots, T_n)$, $\boldsymbol{B} = (B_1, \ldots, B_n)$, $\boldsymbol{s} = (s_1, \ldots, s_n)$ and $\boldsymbol{z} = (z_1, \ldots, z_n)$. Since

$$E[B_1 B_n] = \frac{\partial f^*_{(\boldsymbol{T},\boldsymbol{B})}(\boldsymbol{s}, \boldsymbol{z})}{\partial z_1 \partial z_n}\Big|_{\boldsymbol{s}=\boldsymbol{0};\boldsymbol{z}=\boldsymbol{1}},$$

then, it follows from (3) that

$$\begin{aligned} E[B_1 B_n] &= \boldsymbol{\phi}(-D_0)^{-1} \left(\sum_{l=1}^{k} l D_l z_1^{l-1} \right) \kappa (-D_0)^{-1} \left(\sum_{l=1}^{k} l D_l z_n^{l-1} \right) \mathbf{e} \Big|_{\boldsymbol{s}=\boldsymbol{0};\boldsymbol{z}=\boldsymbol{1}} \\ &= \boldsymbol{\phi}(-D_0)^{-1} D_1^\star [(-D_0)^{-1} D]^{n-2} (-D_0)^{-1} D_1^\star \mathbf{e}, \end{aligned}$$

where $\kappa = \sum_{m=2}^{n-1} (s_m I - D_0)^{-1} \left(\sum_{l=1}^{k} D_l z_m^l \right)$. Therefore (4) is obtained.

□

Appendix C: Proof of Proposition 2

We prove here that the set $\mathcal{F}$ in Proposition 2 provides feasible solutions to the problem of equivalent $BMAP_2(2)$s. Assume first that $x < u$. Let κ be defined as in (17), that is,

$$0 < \kappa < min \{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_6, \kappa_7, \kappa_8\}.$$

First, we prove that $\min \{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_6, \kappa_7, \kappa_8\} > 0$. It is straightforward to check that

$$min \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\} > 0.$$

Also, $\kappa_5 = -\frac{r}{t}(r+u+v+q+t) > 0$, since $r, t > 0$ and $-r-u-v-q-t > 0$. Next, $\kappa_6 > 0$ since $u - x > 0$, and $\kappa_7 > 0$ since $(q-w) < \sqrt{(q-w)^2 + 4vm}$, because $v, m > 0$. Finally, $\kappa_8 > 0$ since $-\dfrac{r}{2t} < 0$, $-y-x-w-m-n > 0$ and therefore,

$$(u+v+q+n+t+r)^2 < (u+v+q+n+t+r)^2 + 4t(-y-x-w-m-n).$$

Then,

$$\tilde{u} = u - \kappa < 0 \quad \text{and} \quad \tilde{r} = r + \kappa > 0.$$

Moreover, since $\kappa < \kappa_2 = \dfrac{u-x}{2}$, this assures that $\tilde{x} < \tilde{u}$, and thus the parameterization of $\widetilde{\mathcal{M}}$ is different from that of $\mathcal{M}$ with permuted states.

We have that $\kappa < \kappa_3 = \frac{r(1-\phi)}{\phi}$,

$$\widetilde{\phi} \equiv \frac{(r+\kappa)\phi}{\phi} \in [0,1],$$

And that $\kappa < \kappa_4 = \frac{rq}{v}$,

$$\tilde{q} \equiv \frac{qr - v\kappa}{r} > 0.$$

Next,

$$\frac{(u-x) - \sqrt{(x-u)^2 + 4ry}}{2} < 0 < \kappa < \frac{(u-x) + \sqrt{(x-u)^2 + 4ry}}{2},$$

implies that

$$\tilde{y}(\tilde{u}, \tilde{r}) \equiv \frac{-(\kappa^2 + (x-u)\kappa - ry)}{r+\kappa} > 0,$$

and

$$\frac{r}{2v}\left[(q-w) - \sqrt{(q-w)^2 + 4vm}\right] < 0 < \kappa < \frac{r}{2v}\left[(q-w) + \sqrt{(q-w)^2 + 4vm}\right],$$

implies that

$$\tilde{m}(\tilde{u}, \tilde{r}) \equiv \frac{-(v\kappa^2 - r(q-w)\kappa - mr^2)}{r(r+\kappa)} > 0,$$

In addition,

$$\tilde{w}(\tilde{u}, \tilde{r}) \equiv \frac{wr + v\kappa}{r} > 0, \qquad \tilde{v}(\tilde{u}, \tilde{r}) \equiv \frac{v(r+\kappa)}{r} > 0,$$
$$\tilde{t}(\tilde{u}, \tilde{r}) \equiv \frac{t(r+\kappa)}{r} > 0, \qquad \tilde{n}(\tilde{u}, \tilde{r}) \equiv \frac{nr + t\kappa}{r} > 0.$$

It remains to prove that $-\tilde{r} - \tilde{u} - \tilde{v} - \tilde{q} - \tilde{t} > 0$ and $-\tilde{x} - \tilde{y} - \tilde{w} - \tilde{m} - \tilde{n} > 0$. It is easy to check that

$$\begin{aligned} -\tilde{r} - \tilde{u} - \tilde{v} - \tilde{q} - \tilde{t} &= -r - u - \frac{v(r+\kappa)}{r} - \frac{qr - v\kappa}{r} - \frac{t(r+\kappa)}{r} \\ & -r - u - \frac{r(v+q) + t(r+\kappa)}{r}, \end{aligned}$$

which is positive if and only if $\kappa < \kappa_5 = -\frac{t}{t}(r + u + v + q + t)$. Finally, an easy computation shows that $-\tilde{x} - \tilde{y} - \tilde{w} - \tilde{m} - \tilde{n} > 0$ is equivalent to

$$-\kappa - x > \frac{-(\kappa^2 + (x - u)\kappa - ry)}{r + \kappa} + \frac{-(v\kappa^2 - r(q - w)\kappa - mr^2)}{r(r + \kappa)} + \frac{(nr + t\kappa) + (wr + v\kappa)}{r}$$

which holds if and only if $\kappa \in (r_1, r_2)$ where

$$\begin{aligned} r_1 &= -\frac{r}{2t}\Big[(u + v + q + n + t + r) \\ &\quad + \sqrt{(u + v + q + n + t + r)^2 + 4t(-y - w - m - n - x)}\Big] < 0, \\ r_2 &= -\frac{r}{2t}\Big[(u + v + q + n + t + r) \\ &\quad - \sqrt{(u + v + q + n + t + r)^2 + 4t(-y - w - m - n - x)}\Big] = \kappa_8 > 0. \end{aligned}$$

Now, assume that $x = u$. Then, let κ be defined as in (18), where in this case, $\kappa_6 \equiv \sqrt{ry}$. Then,

$$\begin{aligned} \tilde{u} &= u - \kappa < 0, \\ \tilde{r} &= r + \kappa > 0, \\ \tilde{x} &= x + \kappa < 0 \quad (\text{since } \kappa < -x), \\ \tilde{y} &= \frac{ry - \kappa^2}{r + \kappa} > 0 \quad (\text{since } \kappa < \sqrt{ry}), \\ \tilde{w} &= \frac{wz + v\kappa}{v} > 0, \\ \tilde{v} &= \frac{v(z + \kappa)}{z} > 0, \\ \tilde{n} &= \frac{nr + t\kappa}{r} > 0, \\ \tilde{t} &= \frac{t(r + \kappa)}{r} > 0, \end{aligned}$$

and $\tilde{q}$, $\tilde{m}$, $\tilde{\phi} \in [0, 1]$, $-\tilde{r} - \tilde{u} - \tilde{v} - \tilde{q} - \tilde{>}0$ and $-\tilde{x} - \tilde{y} - \tilde{w} - \tilde{m} - \tilde{n} > 0$ follow from the assumptions $\kappa < \kappa_4$, $\kappa < \kappa_7$, $\kappa < \kappa_3$, $\kappa < \kappa_5$ and $\kappa < \kappa_8$, respectively. □

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