Risk-neutral valuation with infinitely many trading dates

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Abstract

The first Fundamental Theorem of Asset Pricing establishes the equivalence between the absence of arbitrage in financial markets and the existence of Equivalent Martingale Measures, if appropriate conditions hold. Since the theorem may fail when dealing with infinitely many trading dates, this paper draws on the A.A. Lyapunov Theorem in order to retrieve the equivalence for complete markets such that the Sharpe Ratio is adequately bounded.

Keywords: A.A. Lyapunov theorem; Asset pricing; Martingale measure; Projective system; Sharpe ratio

1. Introduction

In 1940 Lyapunov proved in [18] that the range of any n-dimensional vector measure is compact, and it is also convex if the measure is atomless. This result has been crucial in control and optimal control theory since, amongst other applications, it allows us to establish the Pontryagin Maximum Principle (see [20]). The A.A. Lyapunov Theorem has been extended in several directions (see [8]) and a recent line of research shows its tie with stopping time linked problems (see [15]), closely related to many important topics in Finance (for example, the problem of pricing and hedging American call or put options, see [11]). The present paper attempts to show some possible relationships between the A.A. Lyapunov Theorem and The First Fundamental Theorem of Asset Pricing, a crucial issue in Mathematical Finance.

Since Harrison and Kreps established in [10] the existence of martingale probability measures for some particular arbitrage-free pricing models their result has been extended in multiple directions, generating the Fundamental Theorem of Asset Pricing. For instance, [6,7] or [12] provide deep characterizations of the existence of martingale measures in different settings.

Nevertheless, a simple version of the Fundamental Theorem cannot be proved, in the sense that the arbitrage absence is not sufficient to construct martingale measures if the set of trading dates is not finite and we are far from a Gaussian world. It was pointed out by Back and Pliska in [1], where a simple dynamic discrete time counter-example

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was provided. To overcome this problem the concept of "free lunch" was introduced in [4] and [5], but this new notion is much weaker than the concept of arbitrage. The absence of free lunch has been the key to yield further extensions of the theorem, even in the imperfect market case (see, for instance, [13]).

Every free lunch can be understood as an "approximated arbitrage". However, it is not an arbitrage, it is not so intuitive and its economic interpretation is not so clear. On the contrary, it is introduced in mathematical terms and solves a mathematical problem, but classical pricing models (binomial model, Black and Scholes model, etc.) usually deal with the concept of arbitrage. Recent studies of efficiency in imperfect markets avoid the use of free lunches and retrieve the concept of arbitrage, but they have to deal with models containing a finite number of trading dates (see [14,22], etc.).

The results of Balbás et al. in [2] have shown that it is possible to solve the counter-example of Back and Pliska without drawing on free lunches. This paper characterizes the arbitrage absence in dynamic discrete time pricing models by building an appropriate projective system of probability measures (see [23]) that are martingale measures for each finite subset of trading dates. Then it is shown that the projective limit may be understood as a martingale measure for the whole set of trading dates. The initial probability measure and the martingale measure cannot be equivalent, as illustrated by using the counter-example of Back and Pliska. However, for any finite subset of trading dates one can find projections of both measures that are equivalent, and there are Radon–Nikodym derivatives in both directions. This property is used in [2] to introduce the concept of "projective equivalence" of probability measures.

We follow the approach of [2] in the sense that the probability space indicating the evolution of prices is given by a projective limit of probability measures. Thus, the existence of a projectively equivalent martingale measure is guaranteed, and our major focus is on the equivalence between this measure and the initial one.

The outline of the paper is as follows. The second section presents the general framework and the financial market model we are going to deal with, as well as those properties of Financial Economics that will apply throughout the article. The third section will use the A.A. Lyapunov Theorem in order to prove Theorem 4, a major result of this paper, since it is shown that the projectively equivalent martingale measure becomes equivalent if the Sharpe Ratio is bounded from above and the market is complete. It will be justified that unbounded Sharpe Ratios hardly make sense in Financial Economics, so our condition is quite intuitive and realistic from the economic point of view.[1] Section 4 will present Theorem 5, where the A.A. Lyapunov Theorem applies again in order to show new conditions guaranteeing bounded Sharpe Ratios. The last section concludes the article.[2]

2. Preliminaries

First of all let us describe our model of a frictionless financial market with a finite number of assets $\{S_0, S_1, \ldots, S_n\}$, $n \in \mathbb{N}$ and a countable set of trading dates

$$T = \{t_0, t_1, \ldots, t_m, \ldots\},$$

$t_0 = 0$ representing the current date. For convenience, we will suppose that there exists a "tree structure" indicating the stochastic evolution of prices (or the arrival of information to the market), i.e., the set of "States of Nature" between consecutive trading dates is finite. Furthermore, the tree of events presents "$n + 1$ branches (or states of nature) per node".

As usual, each sequence of branches on the tree represents an arbitrary "State of the World" or "Trajectory", and the set of states of the world will be denoted by Ω. Moreover, Ω is endowed with the filtration (increasing sequence of σ-algebras) $(\mathcal{F}_m)_{m=0}^{\infty}$, where each σ-algebra $\mathcal{F}_m$ is generated by the subsets of Ω composed of those trajectories with the same branches between $t_0 = 0$ and t_m. Obviously, $\mathcal{F}_m$ is generated by a "canonical partition" of Ω containing $(n + 1)^m$ sets, $m = 0, 1, 2, \ldots$. In particular, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ is the trivial σ-algebra.

Given an arbitrary node on the tree of events, it is the starting point of $n + 1$ branches, and we will assume that the strictly positive probability of any branch is known. Whence, if $m \in \mathbb{N}$, then one can multiply m "single" probabilities

[1] Indeed, it will be shown that unbounded Sharpe ratios will mean that agents can achieve "almost infinite returns" with bounded risk levels which is unrealistic in practice and contradicts the assumptions of many portfolio choice problems (amongst many others, [17,19] or [24] provide modern approaches on portfolio optimization).

[2] In order to simplify the mathematical exposition and some mathematical proofs our financial model will incorporate a finite set of "States of Nature" between consecutive trading dates. This assumption could be relaxed by drawing on the more complex setting of [2].

in order to obtain the probability of an arbitrary event in the canonical partition generating $\mathcal{F}_m$. Consequently, $\mathcal{F}_m$ is endowed with a probability measure that will be represented by μ_m. It may be easily shown that μ_m is a projection of μ_{m+1}, in the sense that both probability measures coincide when applied on measurable elements of $\mathcal{F}_m$, $m = 0, 1, 2, \ldots$.

Fix $m \in \mathbb{N}$ and endow the (finite) set Ω_m of trajectories between 0 and t_m with the discrete topology. Then the set Ω of "whole" trajectories may be endowed with the projective limit topology (see [23]), and becomes a Hausdorff and compact space. If $\mathcal{F}$ represents its Borel σ-algebra, then the Prokhorov Theorem (see [23]) guarantees the existence of a unique Radon probability measure μ on $\mathcal{F}$ such that

$$\mu(F) = \mu_m(F) \tag{1}$$

whenever $F \in \mathcal{F}_m$, $m = 0, 1, 2, \ldots$. Recall that a probability measure is called a Radon measure if it is inner regular (or tight) by compacts [23]. More generally, if ν is a Radon Measure on the Borel σ-algebra $\mathcal{F}$, then ν_m will denote the restriction (or projection) of ν to $\mathcal{F}_m$, i.e.,

$$\nu(F) = \nu_m(F) \tag{2}$$

whenever $F \in \mathcal{F}_m$, $m = 0, 1, \ldots$.

Denote by $S(\omega, t) = (S_0(\omega, t), S_1(\omega, t), \ldots, S_n(\omega, t)) \in \mathbb{R}^{n+1}$ the vector of prices at $t \in T$ under the trajectory $\omega \in \Omega$. Then the price process will be the stochastic process

$$\Omega \times T \ni (\omega, t) \longrightarrow S(\omega, t) \in \mathbb{R}^{n+1}.$$

Following usual conventions, the price process will be adapted to the filtration $(\mathcal{F}_m)_{m=0}^{\infty}$, i.e., it will be adapted to the arrival of information to the market. Besides, the first security will be the riskless asset, and prices will be normalized so that the interest rate vanishes. Thus, $S_0(\omega, t) = 1$ for every trajectory and every trading date.

Portfolios will be represented by adapted stochastic processes

$$\Omega \times T \ni (\omega, t) \longrightarrow x(\omega, t) = (x_0(\omega, t), x_1(\omega, t), \ldots, x_n(\omega, t)) \in \mathbb{R}^{n+1},$$

$x_j(\omega, t)$ reflecting how many units of S_j are being purchased (sold, if $x_j(\omega, t)$ is negative) at t under ω. The price of the portfolio above is given by the adapted process

$$\sum_{j=0}^{n} S_j(\omega, t) x_j(\omega, t).$$

To simplify notations, if there is no confusion, the price process, the portfolio above and its price will be denoted by S, x and Sx, or $S(\omega, t)$, $x(\omega, t)$ and $Sx(\omega, t)$ respectively. For a fixed $t \in T$ we will denote by $S(-, t)$, $x(-, t)$ and $Sx(-, t)$ the $\mathcal{F}_m$-measurable random variables indicating the price process, portfolio x and the price of x at t. For a fixed $\omega \in \Omega$, $S(\omega, -)$, $x(\omega, -)$ and $Sx(\omega, -)$ will be the paths followed by the price process, portfolio x and its price if ω is the finally revealed state of the world.

Portfolio x is said to be self-financing if

$$\sum_{j=0}^{n} S_j(\omega, t_i)[x_j(\omega, t_i) - x_j(\omega, t_{i-1})] = 0$$

for $i = 1, 2, \ldots$.

Since the number of branches per node in the tree structure equals the number of available assets, the market will be complete (every pay-off is reachable) as long as the securities are independent. Hereafter we will assume that this property holds, i.e., we have:

Assumption 1. For every $m \in \mathbb{N}$ and every $\mathcal{F}_m$-measurable random variable P there exists a self-financing portfolio x such that $Sx(-, t_m) = P$. □

As usual, an arbitrage strategy provides investors with "money without risk".

Definition 1. The self-financing portfolio x is said to be an arbitrage if:

(a) $Sx(\omega, 0) \leq 0$ (its current price is not positive).
(b) There exists $m \in \mathbb{N}$ such that $Sx(\omega, t_m) \geq 0$ for every $\omega \in \Omega$ (its price is non-negative at a future trading date t_m).
(c) $\mu_m(Sx(\omega, t_m) - Sx(\omega, 0) > 0) > 0$ (the trivial case is excluded).[3] □

We follow usual conventions in order to introduce "Equivalent Risk-Neutral Probabilities", and we adapt the definition of [2] for "Projectively Equivalent Risk-Neutral Probabilities".

Definition 2. The probability measure ν on the σ-algebra $\mathcal{F}$ is said to be an equivalent risk-neutral probability measure (or an equivalent martingale measure) if μ and ν are equivalent ($\mu(F) = 0 \iff \nu(F) = 0$ for $F \in \mathcal{F}$) and the price process is a martingale under ν, i.e.,

$$S(-, t_m) = E_\nu(S(-, t_{m+1}) \mid \mathcal{F}_m) \tag{3}$$

holds for every $m \in \mathbb{N}$, $E_\nu(- \mid \mathcal{F}_m)$ denoting the conditional expectation under ν.

The probability measure ν on the σ-algebra $\mathcal{F}$ is said to be a projectively equivalent risk-neutral probability measure (or a projectively equivalent martingale measure) if μ_m and ν_m are equivalent for every $m \in \mathbb{N}$ and Expression (3) holds. □

Henceforth EMM and PEMM will mean "Equivalent Martingale Measure" and "Projectively Equivalent Martingale Measure". Obviously, every EMM is also a PEMM, but the converse fails in general (see [2]).

The global market above will be represented by $\mathcal{M}$ whereas $\mathcal{M}_m$ will be the restricted model that only involves the finite set of trading dates $\{t_0, t_1, \ldots, t_m\}$, $m = 1, 2, \ldots$.

Under appropriate conditions, the (first) Fundamental Theorem of Asset Pricing establishes the equivalence between the absence of arbitrage and the existence of EMM. For instance, this equivalence would hold if the number of trading dates were finite ([6,12], etc.) and, in particular, if we focused on market $\mathcal{M}_m$, $m = 1, 2, \ldots$. However, when dealing with infinitely many trading dates, the equivalence is not fulfilled, as pointed out by a classical counter-example of Back and Pliska in [1].[4] The lack of equivalent martingale measures for arbitrage-free models was partially solved in [2], where the weaker concept of PEMM was introduced. By readapting some proofs of these authors one can establish the theorem below.

Theorem 3. *Market $\mathcal{M}$ is arbitrage-free if and only if there exists a projectively equivalent martingale measure.* □

Assumption 2. Hereafter we will assume that the market is arbitrage-free. □

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[3] For illustrative reasons, it may be worthwhile to present the notion of "free lunch" of [4] and [5], once adapted to our framework. So, the self-financing portfolio x above is said to be a free lunch if (a) holds and there exist $F \in \mathcal{F}$ and a stopping time τ (i.e. a $\mathcal{F}$-measurable function $\tau : \Omega \longmapsto T$) such that $\mu(F) = 0$, $Sx(\omega, \tau(\omega)) \geq 0$ if $\omega \in \Omega \setminus F$,

$$\{\omega \in \Omega;\, Sx(\omega, \tau(\omega)) - Sx(\omega, 0) > 0\} \in \mathcal{F},$$

and

$$\mu(\{\omega \in \Omega;\, Sx(\omega, \tau(\omega)) - Sx(\omega, 0) > 0\}) > 0.$$

Notice that the arbitrage strategy of Definition 1 is a free lunch since one can take $F = \emptyset$ and the constant stopping time $\tau(\omega) = t_m$ for every $\omega \in \Omega$. Thus, the absence of free lunch is strictly stronger than the absence of arbitrage. Furthermore, if the arbitrage absence and the existence of free lunch simultaneously hold in the model, then agents could purchase portfolios with non-positive price whose liabilities could not be neutralized in a finite period of time, which is hardly compatible with the economic intuition. Consequently, if mathematically possible, it may be worthwhile to characterize the absence of arbitrage rather than the absence of free lunch.

[4] For illustrative reasons, let us summarize the simple counter-example of Back and Pliska. Imagine the random experiment of rolling a fair die until the first number different from 6 comes out. Denote by $\omega \in \mathbb{N}$ the number of the roll when this occurs. Clearly, the probability of every event ω is $\mu(\omega) = \frac{5}{6}(\frac{1}{6})^{\omega-1}$, and $\mu(\infty) = 0$. Suppose that only two securities can be sold and bought every time $t = 0, 1, 2, \ldots$ that we roll the die. The first one is the riskless bond whose constant price is one dollar. The price process of the second security is

$$S_1(\omega, t) = \begin{cases} 1, & t = 0 \\ \dfrac{(\omega^2 + 2\omega + 2)}{2^t} & 0 < t < \omega \\ \dfrac{1}{2^\omega} & t \geq \omega. \end{cases}$$

Suppose that ν is a PEMM. The completeness of the market allows us to apply the Second Fundamental Theorem of Asset Pricing, which guarantees uniqueness of martingale measures (see [12]). Consequently, the projections (or restrictions) ν_m of ν to $\mathcal{F}_m$ are unique since they are EMM for $\mathcal{M}_m$. Then, the uniqueness of the projective limit of Radon measures (Prokhorov Theorem, see [23]) leads to the uniqueness of ν, PEMM. Thus, the absence of arbitrage, the latter theorem and the ideas above imply that the PEMM exists and is unique. It will be denoted by ν. The major objective of this paper is to find general conditions ensuring that ν is also an EMM.

Since μ_m and ν_m are equivalent, there exists the Radon–Nikodym derivative

$$f_m = \frac{d\nu_m}{d\mu_m} \tag{4}$$

$m = 0, 1, \ldots$, which is strictly positive and is usually called "Stochastic Discount Factor" of Market $\mathcal{M}_n$ (see [3]). If it is constant then Market $\mathcal{M}_m$ is said to be "Risk-Neutral", but the empirical evidence always conclude that real markets are risk adverse. We will impose a strictly weaker assumption.

Assumption 3. $\mathcal{M}_m$ is not risk-neutral, that is, the random variable f_m has positive variance (it is not constant and depends on the state of the world $\omega \in \Omega$), $m = 1, 2, \ldots$. □

Remark 1. If x is a self-financing portfolio with positive current price (at $t = 0$), one can consider its return, expected return and standard deviation at t_m, given by

$$R_x(-, t_m) = \frac{Sx(-, t_m)}{Sx(-, 0)},$$
$$E_x(t_m) = E_\mu(R_x(-, t_m))$$

and

$$\sigma_x(t_m) = \sqrt{E_\mu(R_x(-, t_m)^2) - [E_\mu(R_x(-, t_m))]^2}$$

where $m = 1, 2, \ldots$. If its price at t_m is not constant then its Sharpe Ratio between 0 and t_m is given by

$$\mathcal{S}_x(t_m) = \frac{E_x(t_m) - 1}{\sigma_x(t_m)}. \tag{5}$$

According to [3], Assumption 3 implies that $\mathcal{S}_x(t_m)$ achieves a maximum value $\mathcal{S}(t_m) > 0$, which is attained at those self-financing portfolios x satisfying $Sx(-, 0) > 0$ and

$$Sx(-, t_m) = \alpha_1 - \alpha_2 f_m \tag{6}$$

for some $\alpha_1 > 0$ and $\alpha_2 > 0$.[5]

One can find neither arbitrage opportunities nor equivalent martingale measures. The absence of arbitrage follows from the existence of equivalent martingale measures for every finite subset of trading dates. Indeed, it is sufficient to check that

$$\begin{cases} \tilde{\nu}_t(\omega) = \dfrac{1}{2\omega(\omega+1)} & 1 \le \omega \le t \\ \tilde{\nu}_t[t+1, \infty] = 1 - \displaystyle\sum_{\omega=0}^{t} \dfrac{1}{2\omega(\omega+1)} \end{cases}$$

is a martingale measure for the $\mathcal{M}_t$ market, $t = 1, 2, \ldots$. Back and Pliska showed that there is no martingale measure for the global market $\mathcal{M}$. Moreover, if we adapt the model to our "projective system approach", the projective limit of $(\tilde{\nu}_t)_{t=1}^{\infty}$ is given by

$$\begin{cases} \nu(\omega) = \dfrac{1}{2\omega(\omega+1)} & \omega \ne \infty \\ \nu(\infty) = 1 - \displaystyle\sum_{\omega=0}^{\infty} \dfrac{1}{2\omega(\omega+1)} = \dfrac{1}{2} \end{cases}$$

which is not equivalent to μ because $\mu(\infty) = 0$.

[5] As a consequence, if one maximizes the Sharpe Ratio then the resulting portfolio is quite close to the "Market Portfolio" or the "Stochastic Discount Factor", crucial strategies when introducing the Classical Equilibrium Financial Models, "Capital Asset Pricing Model (CAPM)" and

If the price $Sx(-, t_m)$ above is constant, i.e., if $\sigma_x(t_m) = 0$, then the absence of arbitrage implies that Expression (5) leads to $0/0$ but we will accept that the Sharpe ratio also attains the value $\mathcal{S}(t_m)$ in this case.

Finally, since prices have been normalized so that the risk-free rate vanishes, it is easy to show that the sequence of optimal Sharpe ratios is increasing, that is, $0 < \mathcal{S}(t_1) \leq \mathcal{S}(t_2) \leq, \ldots$. $\square$

3. The first fundamental theorem of asset pricing

According to the A.A. Lyapunov Theorem, $\mu(\mathcal{F})$ and $\nu(\mathcal{F})$ are compact subsets of $\mathbb{R}$, and

$$(\mu, \nu)(\mathcal{F}) = \{(\mu(F), \nu(F)) \in \mathbb{R}^2; F \in \mathcal{F}\}$$

is a compact subset of $[0, 1]^2$. Therefore, there exist $\tilde{\mu}, \tilde{\nu} \in \mathbb{R}$ and $F_\mu, F_\nu \in \mathcal{F}$ such that

$$\tilde{\mu} = \mu(F_\mu) \geq \mu(F)$$

for every $F \in \mathcal{F}$ with $\nu(F) = 0$, and

$$\tilde{\nu} = \nu(F_\nu) \geq \nu(F)$$

for every $F \in \mathcal{F}$ with $\mu(F) = 0$. Obviously, ν is μ-continuous (respect. μ is ν-continuous) if and only if $\tilde{\nu} = 0$ (respect. $\tilde{\mu} = 0$), and the equivalence between μ and ν holds if and only if

$$\tilde{\mu} = \tilde{\nu} = 0. \tag{7}$$

Suppose that U is an upper bound for $\mathcal{S}(t_m)$, i.e.,

$$\mathcal{S}(t_m) \leq U \tag{8}$$

holds for $m = 1, 2, \ldots$. Then

$$E_x(t_m) \leq 1 + U\sigma_x(t_m), \tag{9}$$

for every self-financing portfolio with positive initial price. Expression (9) means that expected returns $E_x(t_m)$ cannot be "too large" unless risk levels $\sigma_x(t_m)$ become "too large" as well. This is a meaningful idea from the economic viewpoint. Indeed, if (8) failed then agents could reach "infinite expected returns" in the long term, despite prices being normalized and the risk-free rate becoming zero. Thus, agents could borrow one dollar and invest this money in a self-financing strategy with a Sharpe Ratio as high as desired. It is almost an arbitrage, though the exact definition of arbitrage is not fulfilled. Actually, under appropriate assumptions, it might be proved that this strategy would be a free lunch, in the sense of [4] and [5], although it has been introduced by using economic arguments rather than technical and mathematical conditions.

According to the statement below, from a mathematical point of view, the economically meaningful Expression (8) also provides an adequate condition to solve the lack of equivalence between μ and ν. In particular, the counter-example of [1] will reflect unbounded Sharpe Ratios.

Theorem 4. *Suppose that there exists $U > 0$ such that $\mathcal{S}(t_m) \leq U$ holds for every $m = 1, 2, \ldots$. Then, ν and μ are equivalent (or the PEMM ν becomes a EMM).*

Proof. Let us prove that ν is μ-continuous, i.e., according to (7), $\tilde{\nu} = 0$. If $\tilde{\nu} > 0$, since μ and ν are Radon measures, there exists a compact set $K \subset \Omega$ such that

$$\nu(K) > 0 \tag{10}$$

and

$$\mu(K) = 0. \tag{11}$$

"Arbitrage Pricing Theory (APT)" (see [3], for further details). In particular, if risk levels are given by standard deviations, then the efficient frontier can be easily computed if one combines the stochastic discount factor and the riskless asset. If risk levels are not given by standard deviations then the efficient frontier must be computed by solving a vector optimization problem. If so, both classical analyses or balance point linked methods (see [9]) may apply.

Being (μ, ν) the projective limit of $(\mu_n, \nu_n)_{n=1}^{\infty}$, one has that

$$(\mu, \nu)(K) = \mathrm{Lim}(\mu_m, \nu_m)(K_m), \tag{12}$$

K_m being the set of $\mathcal{F}_m$ obtained as the union of those sets of the canonical generator of $\mathcal{F}_m$ whose intersection with K is non-void. Since the market is complete (Assumption 1) there exists a self-financing portfolio x_m such that

$$Sx_m(-, t_m) = \nu_m(K_m)\mathcal{X}_{\Omega} - \mathcal{X}_{K_\mathrm{m}}, \tag{13}$$

where, as usual, the characteristic function of any $V \subset \Omega$ is given by

$$\mathcal{X}_V(\omega) = \begin{cases} 1, & \omega \in V \\ 0, & \omega \in \Omega \setminus V. \end{cases}$$

Consider $y_m = x_m + (10^{-m}, 0, \dots, 0)$, i.e., Strategy y_m "is obtained by adding x_m plus 10^{-m} dollars invested in the riskless asset". According to Expression (3) and Theorem 3,

$$\begin{aligned} Sy_m(-, 0) &= \int_{\Omega} ((\nu_m(K_m) + 10^{-m})\mathcal{X}_{\Omega} - \mathcal{X}_{K_\mathrm{m}})\mathrm{d}\nu_m \\ &= (\nu_m(K_m) + 10^{-m}) - \nu_m(K_m) \\ &= 10^{-m}. \end{aligned}$$

Besides, (13) leads to

$$Sy_m(-, t_m) = (\nu_m(K_m) + 10^{-m})\mathcal{X}_{\Omega} - \mathcal{X}_{K_\mathrm{m}},$$

and

$$\begin{cases} E_{y_\mathrm{m}}(t_m) = 10^m[(\nu_m(K_m) + 10^{-m}) - \mu_m(K_m)], \\ \sigma_{y_\mathrm{m}}(t_m) = 10^m\sqrt{\mu_m(K_m) - \mu_m(K_m)^2}. \end{cases}$$

Therefore,

$$\begin{aligned} \mathcal{S}(t_m) \geq \mathcal{S}_{y_\mathrm{m}}(t_m) &= \frac{10^m[\nu_m(K_m) + 10^{-m} - \mu_m(K_m)] - 1}{10^m\sqrt{\mu_m(K_m) - \mu_m(K_m)^2}} \\ &= \frac{[\nu_m(K_m) + 10^{-m} - \mu_m(K_m)] - 10^{-m}}{\sqrt{\mu_m(K_m) - \mu_m(K_m)^2}} \\ &= \frac{\nu_m(K_m) - \mu_m(K_m)}{\sqrt{\mu_m(K_m) - \mu_m(K_m)^2}}, \end{aligned}$$

which, according to (10)–(12), tends to ∞, against the existence of the upper bound U.[6]

In order to prove that μ is ν-continuous, suppose the existence of a compact set $K^* \subset \Omega$ satisfying $\mu(K^*) > 0$ and $\nu(K^*) = 0$, and repeat the arguments above by taking Strategy x_m^* such that

$$Sx_m^*(-, t_m) = \mathcal{X}_{K_m^*} - \nu_m(K_m^*)\mathcal{X}_{\Omega}$$

instead of (13). □

4. Atomless measures and the converse theorem

Throughout this section let us assume that μ is an atomless measure. Then, if δ is a μ-continuous probability measure on $\mathcal{F}$ it is atomless too. Indeed, suppose that F is δ-atom with $\delta(F) > 0$. Obviously $\mu(F) > 0$ and, according to the Saks Theorem (see [21]), given $\varepsilon > 0$ there exists a partition of F such that $\mu(F_\varepsilon) < \varepsilon$ if F_ε is in the partition. Fix F_ε so as to guarantee that $\delta(F) = \delta(F_\varepsilon)$. Then we get the contradiction $\mathrm{Lim}\,\mu(F_\varepsilon) = 0$ and $\mathrm{Lim}\,\delta(F_\varepsilon) = \delta(F) > 0$.

[6] Notice that $\mu_m(K_m) - \mu_m(K_m)^2 = 0$ does not make any sense for m large enough, since this equality would imply that $\mu_m(K_m) = 0$, in contradiction with the absence of arbitrage, or $\mu_m(K_m) = 1$, in contradiction with

$$\mathrm{Lim}\,\mu_m(K_m) = \mu(K) = 0.$$

According to the A.A. Lyapunov Theorem

$$(\mu, \delta)(\mathcal{F}) = \{(\mu(F), \delta(F)) \in \mathbb{R}^2; F \in \mathcal{F}\}$$

is a convex and compact subset of $[0, 1]^2$ and, therefore, the set

$$(\mu, \delta)(\mathcal{F}) \cap (\{u\} \times [0, 1]) = \{(r, s) \in (\mu, \delta)(\mathcal{F}); r = u\}$$

is compact and non-void for every $u \in [0, 1]$. Then one can define the function

$$\delta_\mu : [0, 1] \longrightarrow [0, 1]$$

given by

$$\delta_\mu(u) = \mathrm{Max}\{s \in [0, 1]; (u, s) \in (\mu, \delta)(\mathcal{F})\}.$$

The convexity of $(\mu, \delta)(\mathcal{F})$ trivially implies that δ_μ is concave. Moreover, since $(\mu, \delta)(\mathcal{F})$ and

$$\mathcal{A} = \{(r, s) \in \mathbb{R}^2; r \le 0, s \ge 0\} \tag{14}$$

are convex sets and $(\mu, \delta)(\mathcal{F})$ does not contain any interior point of $\mathcal{A}$, there exists a separating hyperplane (see [16], the Hahn–Banach Theorem and consequences). It is easy to see that the separating hyperplane takes the form

$$-\theta r + \rho s = 0,$$

with $\theta > 0$, $\rho \ge 0$. As usual, if one can choose $\rho > 0$ then we will say that the separating hyperplane is non-vertical, $\lambda = \theta/\rho$ will be called finite and positive super-gradient of δ_μ at 0 and we will denote $\lambda \in \partial\delta_\mu(0)$.

Next we will prove that the A.A. Lyapunov Theorem permits us to characterize those models with bounded Sharpe Ratio, i.e., if the "Real Probability Measure" μ has no atoms then some kind of converse of Theorem 4 may be stated.

Theorem 5. *The following assertions are equivalent and they imply the existence of* $U > 0$ *such that* $\mathcal{S}(t_m) \le U$ *holds for every* $m = 1, 2, \ldots$.

(a) *Measures* ν *and* μ *are equivalent and there exists a finite and positive element in* $\partial\nu_\mu(0)$.
(b) *The Radon–Nikodym derivatives* $(f_m)_{m=1}^\infty$ *are bounded from above, i.e., there exists* $U^* > 0$ *such that* $f_m \le U^*$ *holds for every* $m = 1, 2, \ldots$.

Proof. First of all let us prove that the fulfillment of (b) implies the existence of U. Expression (6) shows that $\mathcal{S}(t_m)$ is achieved at $2 - \alpha_m f_m$ if $\alpha_m \ge 0$, is such that the price of this payoff equals one. It is easy to compute α_m since we have to impose

$$\begin{aligned} 1 &= 2\int_\Omega f_m \mathrm{d}\mu - \alpha_m \int_\Omega f_m^2 \mathrm{d}\mu \\ &= 2\nu_m(\Omega) - \alpha_m \int_\Omega f_m^2 \mathrm{d}\mu \\ &= 2 - \alpha_m \int_\Omega f_m^2 \mathrm{d}\mu, \end{aligned}$$

and consequently

$$\alpha_m = \frac{1}{\int_\Omega f_m^2 \mathrm{d}\mu}. \tag{15}$$

One has that

$$\begin{aligned} E_\mu(2 - \alpha_m f_m) &= \int_\Omega (2 - \alpha_m f_m) \mathrm{d}\mu \\ &= 2 - \alpha_m \int_\Omega f_m \mathrm{d}\mu \\ &= 2 - \alpha_m \nu_m(\Omega) \\ &= 2 - \alpha_m. \end{aligned} \tag{16}$$

Besides, if $\sigma(t_m)$ denotes the standard deviation of the payoff $2 - \alpha_m f_m$ we have

$$\sigma(t_m) = \sqrt{E_\mu((2-\alpha_m f_m)^2) - ((2-\alpha_m)^2)}.$$

Since (15) leads to

$$\begin{aligned} E_\mu((2-\alpha_m f_m)^2) &= \int_\Omega (2-\alpha_m f_m)^2 \mathrm{d}\mu \\ &= 4 - 3\alpha_m, \end{aligned}$$

we get

$$\sigma(t_m) = \sqrt{4 - 3\alpha_m - (2-\alpha_m)^2} = \sqrt{\alpha_m - \alpha_m^2}.$$

The last expression and (16) give

$$\mathcal{S}(t_m) = \sqrt{\frac{1}{\alpha_m} - 1}. \tag{17}$$

Therefore, if (b) holds we have that

$$\int_\Omega f_m^2 \mathrm{d}\mu \le (U^*)^2$$

and (15) and (17) lead to

$$\mathcal{S}(t_m) \le \sqrt{(U^*)^2 - 1}.$$

Next we will prove that (b) $\Longrightarrow$ (a). Indeed, if (b) holds then the existence of U and Theorem 4 show that ν and μ are equivalent. Furthermore, consider the probability measure $\delta_m : \mathcal{F} \longrightarrow [0, 1]$ such that

$$f_m = \frac{\mathrm{d}\delta_m}{\mathrm{d}\mu},$$

$m = 1, 2, \ldots$. Expressions (1) and (4) imply that δ_m extends ν_m from $\mathcal{F}_m$ to $\mathcal{F}$, and, as stated at the beginning of this section, the μ-continuity of δ_m guarantees that this measure is atomless. Besides, we have

$$\frac{\delta_m(F)}{\mu(F)} = \frac{\int_F f_m \mathrm{d}\mu}{\mu(F)} \le \frac{U^* \mu(F)}{\mu(F)} = U^*,$$

for $F \in \mathcal{F}$, $\mu(F) \neq 0$, $m = 1, 2, \ldots$. Whence,

$$-U^* \mu(F) + \delta_m(F) \le 0,$$

for $F \in \mathcal{F}$, $m = 1, 2, \ldots$. Therefore, the closed convex half-space

$$-U^* r + s \le 0 \tag{18}$$

contains the convex closed set $(\mu, \delta_m)(\mathcal{F})$, $m = 1, 2, \ldots$. Suppose that we prove the inclusion

$$(\mu, \nu)(\mathcal{F}) \subset \mathrm{Ad}\left[\bigcup_{m=1}^{\infty} ((\mu, \delta_m)(\mathcal{F}))\right] \tag{19}$$

where the symbol Ad represents the adherence. Then $(\mu, \nu)(\mathcal{F})$ will be included in the half-space (18) and the hyperplane $-U^* r + s = 0$ will separate $(\mu, \nu)(\mathcal{F})$ and the set $\mathcal{A}$ of (14), from where $\lambda = U^*$ will be a positive and finite element in $\partial \nu_\mu(0)$. Thus, let us prove (19). Since μ and ν are Radon measures, for every $F \in \mathcal{F}$ and every $\varepsilon > 0$ there exists a compact set $K \subset F$ with

$$(\mu, \nu)(K) \le (\mu, \nu)(F) \le (\mu, \nu)(K) + (\varepsilon, \varepsilon).$$

Since K is compact Expression (12) applies, and

$$(\mu, \nu)(F) - (\varepsilon, \varepsilon) \le (\mu_m, \nu_m)(K_m) \le (\mu, \nu)(F) + (\varepsilon, \varepsilon)$$

if m is large enough.[7] Bearing in mind that μ extends μ_m and $\nu_m(K_m) = \delta_m(K_m)$ (notice that $K_m \in \mathcal{F}_m$)

$$(\mu, \nu)(F) - (\varepsilon, \varepsilon) \leq (\mu, \delta_m)(K_m) \leq (\mu, \nu)(F) + (\varepsilon, \varepsilon)$$

if m is large enough. Now take the sequences

$$\varepsilon = 1, 1/2, 1/3, \ldots$$

and

$$m_1 < m_2 < \cdots$$

such that

$$(\mu, \nu)(F) - \left(\frac{1}{s}, \frac{1}{s}\right) \leq (\mu, \delta_{m_s})(K_{m_s}) \leq (\mu, \nu)(F) + \left(\frac{1}{s}, \frac{1}{s}\right).$$

We have

$$(\mu, \nu)(F) = \mathrm{Lim}(\mu, \delta_{m_s})(K_{m_s}).$$

Finally, let us prove that (a) $\Longrightarrow$ (b). Indeed, take $\lambda \in (0, \infty) \cap \partial \nu_\mu(0)$ and one has that

$$-\lambda r + s = 0$$

is a separating hyperplane, so

$$-\lambda \mu(F) + \nu(F) \leq 0$$

holds for every $F \in \mathcal{F}$. In particular,

$$\frac{\nu(F)}{\mu(F)} \leq \lambda$$

for every $F \in \mathcal{F}$ such that $\mu(F) > 0$. Hence, (1) and (2) imply that

$$\frac{\nu_m(F)}{\mu_m(F)} \leq \lambda$$

whenever $F \in \mathcal{F}_m$ such that $\mu_m(F) > 0$, $m = 1, 2, \ldots$. Thus, Expression (4), along with the existence of a finite partition of Ω generating $\mathcal{F}_m$, give

$$f_m \leq \lambda,$$

$m = 1, 2, \ldots$. □

5. Conclusions

The first Fundamental Theorem of Asset Pricing establishes the equivalence between the existence of Martingale Measures and the Absence of Arbitrage in a Financial Market satisfying appropriate conditions. However, if the set of trading dates is not finite and the "real world" is not Gaussian then the equivalence may fail, as pointed out by several counter-examples. We have used the A.A. Lyapunov Theorem and projective systems of Radon measures in order to retrieve the equivalence for complete markets with a countable family of trading dates and bounded Sharpe Ratios and/or Discount Factors. The interest of these results seems to be clear since we are dealing with a central topic in Mathematical Finance.

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[7] Notice that $(\mu_m, \nu_m)(K_m) \geq (\mu, \nu)(K)$.

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