Option pricing with Lévy-Stable processes generated by Lévy-Stable integrated variance

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We show how to calculate European-style option prices when the log-stock price process follows a Lévy-Stable process with index parameter $1 < \alpha \leq 2$ and skewness parameter $-1 < \beta < 1$. Key to our result is to model integrated variance $\int_0^T \sigma_t^2 \, dt$ as an increasing Lévy-Stable process with continuous paths in $T$.

Keywords: Commodity markets; Commodity prices; Lévy process; Hedging techniques

1. Introduction

Up until the early 1990s most of the underlying stochastic processes used in the financial literature were based on Brownian motion, modelling in continuous time a large number of independent 'microscopic' price changes, with finite total variance: and Poisson processes, modelling occasional large changes. These two processes are the canonical models for continuous sample paths and those with a finite number of jumps, respectively. More generally, dropping the assumption of finite variance, the sum of many iid events always has, after appropriate scaling and shifting, a limiting distribution termed a Lévy-Stable law; this is the generalized version of the Central Limit Theorem (CLT) (Samorodnitsky and Taqqu 1994), and the Gaussian distribution is one example. Based on this fundamental result, it is plausible to generalize the assumption of Gaussian price increments by modelling the formation of prices in the market by the sum of many stochastic events with a Lévy-Stable limiting distribution.

An important property of Lévy-Stable distributions is that of stability under addition: when two independent copies of a Lévy-Stable random variable are added then, up to scaling and shift, the resulting random variable is again Lévy-Stable with the same shape. This property is very desirable in models used in finance and particularly in portfolio analysis and risk management; see, for example, Fama (1971), Ziemba (1974) and the more recent work by Tokat and Schwartz (2002), Orteboli et al. (2002) and Mitnik et al. (2002). Only for Lévy-Stable distributed returns do we have the property that linear combinations of different return series, for example portfolios, again have a Lévy-Stable distribution (Feller 1986).

Based on the CLT we have, in general terms, two ways of modelling stock prices or stock returns. If it is believed that stock returns are at least approximately governed by a Lévy-Stable distribution the accumulation of the random events is additive. On the other hand, if it is believed that the logarithms of stock prices is approximately governed by a Lévy-Stable distribution then the accumulation is multiplicative. In the literature, most models have assumed that log-prices, instead of returns, follow a Lévy-Stable process. Mc-Callum (1996) assumes that assets are log Lévy-Stable and prices options using a utility maximization argument; more recently, Carr and Wu (2003) priced European options when the log-stock price follows a maximal skew Lévy-Stable process.

Finally, based on Mandelbrot (1969), Hurst et al. (1999) provide a model to price European options when returns follow a (symmetric) Lévy-Stable process. In their models the Brownian motion that drives the stochastic shocks to the stock process is subordinated to an intrinsic time process that represents 'operational time' on which the market operates. Option pricing can be done within the Black–Scholes framework and one can show that the subordinated Brownian motion is a symmetric Lévy-Stable motion.

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The motivation of this paper is as follows. It is standard to take as a starting point a model for the risk-neutral evolution of the asset price in the form
\[
\frac{dS_t}{S_t} = r dt + \sigma_t dW_t^F,
\]
where $W_t$ is the underlying Brownian motion, $r$ is the constant interest rate and $\sigma_t$ is the volatility process; the case when $\sigma_t$ is constant is the usual Black-Scholes (BS) model. It is then standard to specify a stochastic process for $\sigma_t$, resulting in one of a number of standard stochastic volatility models.

When $\sigma_t$ and $W_t^F$ are independent for all $0 \leq t \leq T$ (as is often approximately the case for FX markets), we have
\[
S_T = S_0 e^{\int_0^T r(t) dt + \int_0^T \sigma(t)^2 \frac{1}{2} dt + \int_0^T \sigma(t) W(t)^F dt},
\]
and then the value of a European vanilla option written on the spot stock price $S_T$ is given by
\[
F(S_T, T, R, \Gamma, \delta, \sigma_T) = \left[ \frac{S_T e^{-\int_0^T \sigma(r)^2 \frac{1}{2} dr} K - E(R)}{\sqrt{T - t}} \right]^+,
\]
where the expected value is with respect to the random variable $\int_0^T \sigma(t)^2 dt$, the integrated variance, under the risk-neutral measure $Q$. and $\Gamma$ is the usual Black-Scholes value for a European option. In general, the distribution or characteristic function of the integrated variance is not known, so evaluating (2) is not straightforward, although when the characteristic function of the integrated variance we can use standard transform methods to evaluate $F(S_T, T, R, \Gamma, \delta, \sigma_T)$ given by equation (3).

Notwithstanding these difficulties, the integrated variance is an important quantity, representing a measure of the total uncertainty in the evolution of the asset price, and we use it as the starting point for our model.

We investigate the properties of a two-factor model in which the integrated variance process follows a Lévy-Stable process, while the shocks to the stock process are conditionally Gaussian, i.e. Brownian motion, with a volatility consistent with the integrated variance process. We then show that the resulting distribution of the log-price process is Lévy-Stable. We also provide a characterization of the most general possible model within our class of integrated variance processes, which is an interesting result in its own right. In addition to pricing options when the integrated variance process and the stock process are independent (as above), we also show how to incorporate a 'leverage' effect, restoring a degree of 'correlation' between the two.

The paper is structured as follows. Section 2 presents definitions and properties of Lévy-Stable processes. In particular, we show how symmetric Lévy-Stable random variables can be 'built' in a combination of two independent Lévy-Stable random variables and define Lévy-Stable processes as in Samorodnitsky and Taqqu (1994). Section 3 discusses the path properties required to model integrated variance as a totally skewed to the right Lévy-Stable process. Section 4 describes the dynamics of the stock process under both the physical and risk-neutral measure and shows how option prices are calculated when the stock returns or log-stock process follows a Lévy-Stable process. Finally, section 5 shows numerical results and section 6 concludes.

2. Lévy-Stable random variables and processes

In this section we show how to obtain any symmetric Lévy-Stable process as a stochastic process whose innovations are the product of two independent Lévy-Stable random variables. The only condition we require, stated precisely in proposition 2.2, is that one of the independent random variables is symmetric and the other is totally skewed to the right. This is a simple, yet very important, result since we can choose a Gaussian random variable as one of the building blocks together with any other totally skewed random variable to 'produce' a symmetric Lévy-Stable random variable. Furthermore, choosing a Gaussian random variable as one of the building blocks of a symmetric random variable will be very convenient since we will be able to formulate any symmetric Lévy-Stable process as a conditional Brownian motion, conditioned on the other building block, the totally skewed Lévy-Stable random variable, which in our case will be the model for the integrated variance:

2.1. Lévy-Stable random variables

The characteristic function of a Lévy-Stable random variable $X$ is given by
\[
\log\mathbb{E}[\exp(i\theta X)] = \begin{cases} \max(\theta \nu, 0) - \frac{\theta^\alpha}{\Gamma(1 - \alpha)} \sin(\pi \alpha) \tan(\pi \alpha/2), & \text{for } \alpha \neq 1, \\ -\nu \log(\theta) + (2\pi)^{\alpha/2} \pi^{\alpha/2} \sin(\pi \alpha/2), & \text{for } \alpha = 1, \end{cases}
\]
where the parameter $\nu \in (0, 2]$ is known as the stability index, $\nu = 2$ if $\alpha = 0$ is a scaling parameter, $\alpha \in (-1, 1)$ is a skewness parameter and $\nu$ is a location parameter (Samorodnitsky and Taqqu 1994). If the random variable $X$ has a Lévy-Stable distribution with parameters $\alpha$, $\nu$, $\beta$ and $\mu$ as we write it $X \sim \nu, \alpha, \beta, \mu$.

It is straightforward to see that, for the case $0 < \alpha < 1$ the random variable $X$ does not have any moments, and for the case $\alpha = 2$ the first moment exists (the case $\alpha = 2$ is Gaussian); however, fractional moments $\mathbb{E}[X^p]$ do exist for $p < \nu$ (Samorodnitsky and Taqqu 1994).

Moreover, given the asymptotic behaviour of the tail of the distribution of a Lévy-Stable random variable it can be shown that the Laplace transform $\mathbb{E}[\exp(-tX)]$ of $X$ exists only when its distribution is totally skewed to the right that is $\beta = 1$, which we state in the following proposition which we use later.

Proposition 2.1: (The Laplace transform; Samorodnitsky and Taqqu (1994)) The Laplace transform $\mathbb{E}[\exp(-t^\alpha X)]$ with $t > 0$ of a Lévy-Stable variable $X \sim \nu, \alpha, \beta, \mu$ with $0 < \alpha < 2$ and scale parameter $\nu = 1$ satisfies
\[
\log\mathbb{E}[\exp(-t^\alpha X)] = \begin{cases} -\alpha \log(t) - \frac{\alpha R}{\nu} \log(t), & \text{for } \alpha \neq 1, \\ \frac{\nu}{\pi} \log(t)^2 + \frac{\nu R}{\pi} \log(t) + \nu \log(t), & \text{for } \alpha = 1, \end{cases}
\]
and
\[
\log\mathbb{E}[\exp(-t^\alpha X)] = \begin{cases} -\alpha \log(t) - \frac{\alpha R}{\nu} \log(t), & \text{for } \alpha \neq 1, \\ \frac{\nu}{\pi} \log(t)^2 + \frac{\nu R}{\pi} \log(t) + \nu \log(t), & \text{for } \alpha = 1, \end{cases}
\]
where
\[
\nu R = \nu \cos(\pi \alpha/2) - \nu \cos(\pi / 2), 
\]
and $R = \nu \sin(\pi \alpha/2) - \nu \sin(\pi / 2)$.

The existence of the Laplace transform of a totally skewed to the right Lévy-Stable random variable will enable us to show how to price options as a weighted average of the classical Black-Scholes prices when the shock to the stock follow a Lévy-Stable process. First we see that any symmetric Lévy-Stable random variable can be represented as the product of a totally skewed with a symmetric Lévy-Stable variable as shown by the following proposition.

2.2. Lévy-Stable processes

A stochastic process $(L_t, t \in T)$ is Lévy-Stable if all its finite-dimensional distributions are Lévy-Stable. A particular case of a Lévy-Stable process, which will be denoted by $(L_t^\alpha, t \geq 0)$, is the Lévy-Stable motion (Samorodnitsky and Taqqu 1994).

Definition 2.3: (Lévy-Stable motion) A Lévy-Stable process $L_t^\alpha$ is called a Lévy-Stable motion if $L_t^\alpha = 0, L_0 = \mu$ has independent increments, and $L_t^\alpha - L_s^\alpha \sim \nu, \alpha, \beta, \mu$ for any $0 < s < t < \infty$ and for some $0 < \alpha < 2$ and $1 \leq \beta < 2$ (time-homogeneity of the increments). Observe that when $\alpha = 2$ and $\beta = 1$ it is Brownian motion, while when $\alpha < 2$ and $\beta = 1$ (respectively $\beta = 2$) the process $L_t^\alpha$ has support on the negative (respectively positive) line.

The log-characteristic function of a Lévy-Stable motion $L_t^\alpha$ is given by (Samorodnitsky and Taqqu 1994)
\[
\log\mathbb{E}[\exp(i\theta X)] = \begin{cases} -\nu \log(\theta) + (2\pi)^{\alpha/2} \pi^{\alpha/2} \sin(\pi \alpha/2), & \text{for } \alpha \neq 1, \\ \frac{\nu}{\pi} \log(t)^2 + \frac{\nu R}{\pi} \log(t) + \nu \log(t), & \text{for } \alpha = 1, \end{cases}
\]
with $R = \nu \cos(\pi \alpha/2) - \nu \cos(\pi / 2)$.

hence the shocks to the process would be symmetric Lévy-Stable by proposition 2.2.

Unfortunately, this model for integrated variance is not able to capture the increments in the integrated variance $\int_0^T \sigma(t)^2 dt$ which is, by construction, a continuous process. However, on the contrary, we have the non-negative Lévy-Stable motion $dL_t^\alpha|_{\geq 0}$ which is, by construction, a purely discontinuous process. Despite these difficulties, we do not abandon the idea of integrating against a Lévy-Stable motion. Instead, we discuss a way of constructing a process for the integrated variance that is Lévy-Stable but with continuous paths in $T$.

If the purely discontinuous process $dL_t^\alpha|_{\geq 0}$ can be modified to
\[
\int_0^T f(t) dL_t^\alpha|_{\geq 0},
\]
for a suitable deterministic function $f(t, X)$, the jumps can be 'damped' and the resulting process made continuous and increasing in $T$. Specifically, we require that $f(t, X)$ be $T$-constant and that $f(t, X) \rightarrow 0$ as $t \rightarrow T$, so that the jump process becomes smooth. For a general discussion of the path behaviour of processes of the type $f(t, X) dL_t^\alpha|_{\geq 0}$, and more general Lévy-Stable
stochastic integrals, see Samorodnitsky and Taqqu (1994). We now give conditions under which the stochastic integral on the right-hand side of equation (6), given by \( \int s(t) \, M(t) \, dW(t) \), is continuous in \( T \), denoting the class of functions \( f(t, T) \) for which this is true by \( \mathbb{F} \).

**Proposition 3.1:** Let \( f(t,T) \) be continuous in \( T \) with \( \mathbb{F}(T) = 0 \), and assume in addition that, for each \( T \), \( f(t,T)|\mathcal{F}(T) = \mathcal{F}(T) \) is continuous on an interval \( 0 < t < T < \infty \). Then the process \( X_T = \int_0^T f(t,T) \, dW(t) \) is continuous in \( T \) for any \( T \) belonging to \( \mathbb{F} \).

**Proof:** Integrating by parts (Protter 1992), and using \( (T,T,N_2) = 0 \),

\[
\int_0^T \int s(t) \, M(t) \, dW(t) = \int_0^T s(t) \, M(t) \, dW(t) - \int_0^T s(t) \, M(t) \, dW(t)
\]

The first term is continuous in \( T \) by assumption on \( f(t,T) \), as \( s(t) \) is fixed. Evaluating the second term at \( T + \epsilon \) and \( T \) and subtracting gives

\[
\int_0^T \frac{d}{dt} [s(t) \, M(t) \, dW(t)] = \int_0^T s(t) \, M(t) \, dW(t)
\]

Both terms on the right clearly tend to zero with \( \epsilon \). □

Since we are interested in pricing options where the underlying stochastic component is driven by a symmetric Levy-Stable process we would like to specify a kernel \( f(t,T) \) on the finite-dimensional distribution of integrated variance is totally skewed to the right Levy-Stable. We propose a model for integrated variance

\[
Y_T = \int_0^T s(t) \, dW(t) + \int_0^T f(t,T) \, dW(t)
\]

for suitable positive functions \( h(t,T) \) and \( f(t,T) \). We assume that \( \mathbb{F}(T) = 0 \) for all \( T \) to dump the Levy-Stable jumps, and that \( h(t,T) = 0 \) for consistency when \( t = T \). For the same reason we also need to take \( f(t,T) = 0 \), \( 0 < t < T \); this is shown below. For \( \mathbb{F}(T) \) (respectively \( f(t,T) \)) we require that \( h(T,T) > 0 \) (respectively \( f(T,T) > 0 \)) to ensure that \( Y_T \) is strictly positive and properly random. Further conditions on \( h(t,T) \) and \( f(t,T) \) which specify their general form are given in proposition 3.2. For example, in our model we may choose

\[
h(t,T) = \left( 1 - e^{-\gamma(t-T)} \right) \quad \text{and} \quad f(t,T) = \frac{1}{\gamma} \quad e^{-\gamma(t-T)}
\]

for \( y > 0 \) in (9) to obtain, as a particular case, the OU-type model for integrated variance first introduced by Barndorff-Nielsen and Shephard (2001) where the increments in (9) are driven by a general non-negative Levy process \( L(t) \). (Note, however, that, in general, the functions \( h(t,T) \) and \( f(t,T) \) do not depend only on the lag \( T - t \) (respectively \( T - t \)) as one might expect. Their most general form is given below.)

Before proceeding, we note an important point concerning units. The integrated variance is dimensionless (that is, as a pure number it has no units). Hence the function \( h(t,T) \) must have dimensions of time, and since the Levy process \( L^{(1)} \) scales as time to the power \( 2 \), the function \( f(t,T) \) must have dimensions of time to the power \( -2 \). This distinction only matters, of course, if we change the unit of time: in \((\theta, f(t,T)) \) contains an implicit dimension constant, equal to 1 in the units of the model to make the dimensions correct.

**Proposition 3.2:** Suppose that the functions \( f(t,T) \) and \( h(t,T) \) are twice differentiable in their second argument and once differentiable in their first argument, with \( f(t,T) > 0 \) for all \( t < T \), while \( f(T,T) = 0 \), and \( h(t,T) > 0 \) for all \( t < T \), while \( h(t,t) > 0 \). Then the process

\[
Y_T = \int_0^T s(t) \, dW(t) + \int_0^T f(t,T) \, dW(t)
\]

is non-negative, continuous and increasing in \( T \), and satisfies the consistency condition \( Y_T = Y_T \) for \( \mathbb{F}(T) = 0 \). We also note that \( h(t,T) \) and \( f(t,T) \) are non-negative and take the form

\[
h(t,T) = \frac{h(t,T) - h(T)}{\mathbb{F}(T)} \quad \text{and} \quad f(t,T) = f(t,T) - \mathbb{F}(T)
\]

where \( h(t,T) \) is a strictly monotonic, differentiable function with derivative \( H(t) \), and \( F(t) \) is continuous and positive (respectively negative) if \( h(t,T) \) is increasing (respectively decreasing).

**Proof:** We use subscripts (1 respectively 2) on \( h_1 \) and \( f_2 \) to denote differentiation with respect to the first (respectively second) argument, with an obvious extension to higher derivatives.

Suppose that, for \( \gamma > 0 \),

\[
\int_s^T \frac{d}{dt} [h(t,T) \, dW(t)] + \int_s^T f(t,T) \, dW(t)
\]

where \( L^{(1)} \) is a special case of this. This is clearly a positive process with respect to some filtered measure \( \mathbb{G} \).

Differentiating \( \mathbb{G} \) and using the Ito formula,

\[
\frac{\partial}{\partial t} h(t,T) = \frac{1}{\gamma} \quad e^{-\gamma(t-T)}
\]

for \( y > 0 \) in (9) to obtain, as a particular case, the OU-type model for integrated variance first introduced by Barndorff-Nielsen and Shephard (2001) where the increments in (9) are driven by a general non-negative Levy process \( L(t) \). (Note, however, that, in general, the functions \( h(t,T) \) and \( f(t,T) \) do not depend only on the lag \( T - t \) (respectively \( T - t \)) as one might expect. Their most general form is given below.)

\[
\frac{\partial}{\partial t} f(t,T) = \frac{1}{\gamma} \quad e^{-\gamma(t-T)}
\]

Note that this immediately implies that

\[
h(t,T) = 0
\]

as stated above.

\[
\int_0^T \frac{d}{dt} h(t,T) \, dW(t) + \int_0^T f(t,T) \, dW(t)
\]

we have

\[
\int_0^T \frac{d}{dt} h(t,T) \, dW(t) + \int_0^T f(t,T) \, dW(t)
\]

from which both sides are equal to an arbitrary function \( \phi \) of the resulting ordinary differential equation in \( t \) for \( f(t,T) \) with the condition \( f_2(t,T) = 0 \), shows that

\[
\frac{d}{dt} f(t,T) = \frac{1}{\gamma} \quad e^{-\gamma(t-T)}
\]

the latter being differentiable. Substitution back into (16) shows that \( G(t) = \frac{1}{\gamma} \) as required. The sign of \( G(t) \) clearly follows from (13) given that \( h(t,T) \) is monotonic. The converse is shown by direct substitution.

Two possible choices for \( f(t,T) \) and \( h(t,T) \) are

\[
f(t,T) = \frac{1}{h(t,T)} \quad e^{-\gamma(t-T)}
\]

\[
h(t,T) = \frac{1}{\gamma} \quad e^{-\gamma(t-T)}
\]

for \( s < t \leq T \) and \( \gamma > 0 \) is a positive constant that can be seen as a damping factor which we can use to smooth the process, and \( \gamma \\n\geq 0 \) is constant. Both choices satisfy the additivity condition (6); for example, \( h(t,T) = \frac{1}{\gamma} \) in proposition 3.2.

Henceforth, we take \( \gamma > 0 \) without loss of generality, and we further assume that

\[
\int_0^T h(t,T) \, dW(t) = \infty \quad 0 < T < \infty
\]

which is a condition we will require below to price instruments under the risk-neutral measure. It simply amounts to saying that \( F(T) = 0 \), namely that the time \( T \) is not special (recall that \( h(t,T) \) cannot have turning points for \( t > T \)).

**3.1 Illustration**

We now illustrate the different building blocks needed to obtain the integrated variance process described above. First we simulate a slightly skewed to the right Levy-Stable motion; then we get the spot variance process, by choosing an appropriate kernel; then we produce the integrated variance process. We focus on kernels of the integrated variance of the form (19). The solid line in the two bottom graphs of figure 1 represents the case where \( n = 1, r = 0, 0 < T < 1, \alpha = 0, \gamma = 25 \), which is a standard OU-type process as in Barndorff-Nielsen and Shephard (2001) with a two-week mean-reversion period. In the same figure the dotted lines represent the case \( n = 1, r = 0, 0 < T < 1, \alpha = 0, \gamma = 25 \).

\[\text{Although these functions are apparently the same, as remarked above, there is a dimensionless constant multiplying these which would change if the units were changed.}\]
4. Model dynamics and option prices

We now turn to models of the asset price evolution and the pricing of vanilla options. Section 4.1 looks at a basic model where the shocks to the returns or log-stock process are symmetric; section 4.2 extends it to a model where shocks can also be asymmetric. Finally, section 4.3 shows how to price vanilla options when the shocks to the underlying stock process follow a Lévy-Stable process for α > 1 and −1 < α < 1.

Given the nature of the model, there is no unique equivalent martingale measure (EMM). In line with most of the Lévy process literature we choose an EMM that is structure-preserving (see, among other features (Cont and Tankov 2004)), transform methods for pricing are straightforward to implement; this is discussed at the end of section 4.2.

4.1. Modelling returns

As pointed out in the Introduction we can model either returns or log-stock prices; when shocks are symmetric we can take either route. For example, if we believe that the shocks to the returns process follow a Lévy-Stable distribution, we assume that, in the physical measure P,

$$dS_t = \mu dt + \sigma_t dW_t,$$

and by independence of $\sigma_t$ and $dW_t$, $\int_0^T \sigma_t dW_t$ is a zero-mean Normal variable whose variance is the random variable $\int_0^T \sigma_t^2 dt$. Thus the characteristic function of $\int_0^T \sigma_t dW_t$ is given by

$$E[e^{i\theta \int_0^T \sigma_t dW_t}] = \exp \left[ -\frac{1}{2} \int_0^T \sigma_t^2 dt \right].$$

Further, using (5) we have that

$$\int_0^T f(x, \text{d}L_t^\alpha) \text{d}t = \mathcal{S}_0 \left( \int_0^T f(x, t \alpha^{\frac{1}{\alpha}}) \text{d}t \right)^{\alpha^{-1}} 1,0,$$

and using proposition 2.1 we write

$$E[e^{i\theta \int_0^T \sigma_t dW_t}] = \mathcal{S}_0 \left( \int_0^T f(x, t \alpha^{\frac{1}{\alpha}}) \text{d}t \right)^{\alpha^{-1}} 1,0.$$

This is clearly the characteristic function of the sum of a Gaussian process and an independent symmetric Lévy-Stable process with index $\alpha$.

Note that we might also stipulate that our departure point is the risk-neutral dynamics for the stock process and that our model is as above with $\mu$ replaced by $r$.

$$dS_t = \mu dt + \sigma_t dW_t^p,$$

where $dW_t^p$ is the increment of the standard Brownian motion, $\mu$, and $\sigma_t$ satisfy the conditions in proposition 3.2, $\sigma_t$ is $\mathcal{F}$ and $\mu$ is a constant. In the following proposition we show the distribution of the stock process.

Proposition 4.1: Let the stock process follow (21) and the integrated stock process follow (22). Assume further that $W_t$ and $L_t^{\alpha^{-1}}$ are independent, then the log-stock process (21) is the sum of two independent processes: a symmetric Lévy-Stable process and a Gaussian process.

Proof: First note that the stochastic component of the log-stock process is given by

$$U_t = \int_0^t \sigma_t dW_t,$$

Now we calculate the characteristic function of the random process $U_t$. We have

$$E[e^{i\theta \int_0^T \sigma_t dW_t}] = E[e^{i\theta \int_0^T \sigma_t \text{d}W_t}],$$

and by independence of $\sigma_t$ and $W_t$, $\int_0^T \sigma_t dW_t$ is a zero-mean Normal variable whose variance is the random variable $\int_0^T \sigma_t^2 dt$. Thus the characteristic function of $\int_0^T \sigma_t dW_t$ is given by

$$E[e^{i\theta \int_0^T \sigma_t dW_t}] = \exp \left[ -\frac{1}{2} \int_0^T \sigma_t^2 dt \right].$$

and further using (5) we have that

$$\int_0^T f(x, \text{d}L_t^\alpha) \text{d}t = \mathcal{S}_0 \left( \int_0^T f(x, t \alpha^{\frac{1}{\alpha}}) \text{d}t \right)^{\alpha^{-1}} 1,0,$$

and using proposition 2.1 we write

$$E[e^{i\theta \int_0^T \sigma_t dW_t}] = \mathcal{S}_0 \left( \int_0^T f(x, t \alpha^{\frac{1}{\alpha}}) \text{d}t \right)^{\alpha^{-1}} 1,0.$$
The inclusion of the leverage is straightforward in this setting, hence the risk-neutral dynamics of the model (25) and (26) become:

\[ \log(S_T)/S_0 = (r - \delta) t + \frac{1}{2} \sigma^2_S t + \sigma^2_D \text{sec} \theta \left( \frac{T - t}{T} \right) - \delta \frac{T - t}{T} \]

and second, that \( Z_t \) is a martingale and

\[ \mathbb{E}[Z_T] = 1. \]  

Since \( r - \delta \) is a constant the first condition is satisfied if \( P \left[ 0 < \frac{\sigma^2_S}{\sigma^2_D} < \infty \right] = 1 \) and \( P \left[ \frac{\sigma^2_D}{\sigma^2_S} < \infty \right] = 1 \) for \( 0 \leq T < \infty \). To show the first, note that \( X_{t,T} := \int_0^T f(t, T) \mathcal{S}_{t,T} \text{dr} = \sigma_D \left( \int_0^T f(t, T) \mathcal{S}_{t,T} \text{dr} \right)^{\frac{1}{2}} \), therefore, \( P \left[ \frac{\sigma^2_S}{\sigma^2_D} < \infty \right] = 1 \) for all \( T \in [0, \infty) \). To show the second, we use (13) and (20) to show that \( \frac{\sigma^2_S}{\sigma^2_D} \text{dr} \) is bounded above:

\[ \int_0^T \frac{1}{\sigma_S^2} \text{dr} \leq \frac{1}{\sigma_D^2} \left( \int_0^T f(t, T) \mathcal{S}_{t,T} \text{dr} \right)^{-\frac{1}{2}}< \infty \]

and the variance of \( \frac{\sigma^2_S}{\sigma^2_D} \text{dr} \) is bounded above:

\[ \mathbb{E}[X_{t,T}^2] = \mathbb{E}[\int_0^T f(t, T) \mathcal{S}_{t,T} \text{dr}]^2 = \sigma_D^2 \left( \int_0^T f(t, T) \mathcal{S}_{t,T} \text{dr} \right)^{-1} \]

Therefore, \( \mathbb{E}[Z_T] = 1 \) and the variance of \( \frac{\sigma^2_S}{\sigma^2_D} \text{dr} \) is bounded above:

\[ \mathbb{E}[Z_T^2] = \mathbb{E}[\frac{1}{\sigma_D^2} \left( \int_0^T f(t, T) \mathcal{S}_{t,T} \text{dr} \right)^{-\frac{1}{2}}] < \infty \]

Moreover, it is simple to calculate \( \mathbb{E}[Z_T:Z_T = S_T] \) for \( 0 \leq t < T \). According to Girsanov’s theorem, the Radon-Nikodym derivative

\[ Z_t = \exp \left[ \int_0^t \left( \delta - r \right) \text{d}s + \sigma^2_D \left( \int_0^t \text{d}s \right)^{\frac{1}{2}} \right] \]

is: 

\[ \mathbb{E}[Z_T:Z_T = S_T] = \exp \left[ \int_0^T \left( \delta - r \right) \text{d}s + \sigma^2_D \left( T - t \right)^{\frac{1}{2}} \right] \]

5. Numerical illustrations: Levy-Schieber option prices

In this section we show how vanilla option prices can be calculated according to the above derivations. One route is to calculate option prices using the Black-Scholes model weighted by the stochastic volatility component and the leverage effect. Another route is to use put option prices on stock prices that follow a geometric Levy process to compute the option value as an integral in Fourier space, using Complex Fourier Transform techniques (Carr and Madan 2001, Lewis 2001). We use the Black-Scholes model as a benchmark to compare the expected option prices obtained when the returns follow a Levy-Schieber process. Our results are consistent with the findings of Hull and White (1987) where the Black-Scholes model underperforms in- and out-of-the-money call option prices and overprices at-the-money options.

5.1. Option prices for symmetrical Levy-Schieber log-price prices

We first obtain option prices and implied volatilities when the log-returns follow a symmetric Levy-Schieber process. Recall that, under the risk-neutral measure \( \mathbb{Q} \), and assuming, for simplicity, that \( \sigma_D = 0 \), the stock price and variance process are given by

\[ S_T = \exp \left[ \left( \delta - r \right) T + \sigma^2_S T + \sigma^2_D \left( T - t \right)^{\frac{1}{2}} \right] \]

and option prices are calculated as:

\[ P \left[ \frac{\sigma^2_S}{\sigma^2_D} < \infty \right] = 1 \]

and second, that \( Z_t \) is a martingale and

\[ \mathbb{E}[Z_T] = 1. \] 

Therefore, the Radon-Nikodym derivative

\[ Z_t = \exp \left[ \int_0^t \left( \delta - r \right) \text{d}s + \sigma^2_D \left( \int_0^t \text{d}s \right)^{\frac{1}{2}} \right] \]

is:

\[ \mathbb{E}[Z_T:Z_T = S_T] = \exp \left[ \int_0^T \left( \delta - r \right) \text{d}s + \sigma^2_D \left( T - t \right)^{\frac{1}{2}} \right] \]

which is the Finite Moment Log-Stock (FMLS) model of Carr and Wu (2003).

Proposition 4.2: It is possible to extend the results above to European call and put options when the skewness coefficient \( \beta \neq 0 \).

Proof: Using put call inversion (McCauley 1996), we have by no-arbitrage that European call and put options are related by:

\[ \text{C}(S,t;K,T;\alpha,\beta) = \text{BS}(S,t;K,T;\alpha) \times \text{SK}(S,t;K,T) \]

where \( S_t \) and \( \text{SK}(S,t;K,T) \) are the Black-Scholes model option prices and the skewness function, respectively. Using this relationship, we can show that the option prices for the Levy-Schieber model are consistent with the findings of Hull and White (1987) where the Black-Scholes model underperforms in- and out-of-the-money call option prices and overprices at-the-money options.
kernel \( f_x(T) = \frac{1}{\sqrt{T}} \left( 1 - e^{-x^2 T} \right) \), which is as in (19) with \( \alpha = 1 \), where for illustrative purposes we have assumed mean-reversion over a two-week period, i.e. \( \gamma = 25 \).

Figure 2 shows the difference between European call options when the stock returns are distributed according to a symmetric Lévy-Stable motion with \( \alpha = 1.7 \) and when returns follow a Brownian motion with annual volatility \( \sigma_{BS} = 0.20 \). For out-the-money call options the Lévy-Stable call prices are higher than the Black-Scholes and for at-the-money options Black-Scholes delivers higher prices. These results are a direct consequence of the heavier tails under the Lévy-Stable case.

**Proposition 5.2:** The characteristic function of \( Z_x^T \) is given by

\[
\mathbb{E} [ e^{itZ_x^T} ] = \exp \left( -\frac{1}{2\sigma^2} \left( e^{\frac{1}{2}|t|^\alpha} + e^{-\frac{1}{2}|t|^\alpha} \right) \right) \times \int f_x(T)^{\frac{1}{2}} \, ds + \left( T - \theta(T) \right) \left( \frac{1}{2} \mathbb{E} [ Z_x^T ] \right)
\]

where \(-1 \leq t \leq 0\), \( \theta = e^{\frac{1}{2}|t|^\alpha} \), and \( Z_x^T \) is analytic in the strip \(-1 < t < 0\).

**Proof:** The proof is very similar to the one above. It suffices to note that, for \( x \leq 0 \),

\[
\mathbb{E} \left[ e^{itZ_x^T} \right] \leq \mathbb{E} \left[ \left| e^{itZ_x^T} \right| \right] = \mathbb{E} \left[ e^{\frac{1}{2}|t|^\alpha} e^{itZ_x^T} \right] \leq \mathbb{E} \left[ e\frac{1}{2}|t|^\alpha \right] < \infty.
\]

Moreover, for \( x < 0 \), we have that \( \mathbb{E} \left[ e^{itZ_x^T} \right] \) is analytic, i.e.

\[
\left| \frac{d}{dt} \mathbb{E} \left[ e^{itZ_x^T} \right] \right| = \mathbb{E} \left[ \left| e^{itZ_x^T} \right| \right] < \infty.
\]

Putting these results together with the results from proposition 5.1 we get the desired result. The requirement \(-1 < t < 0\) arises because \( dZ_x^T \) is totally skewed to the left, so we need \( -\xi \geq 0 \).

We use the same \( f_x(T) \) as above and include a leverage parameter \( \lambda = 1 \) and \( \phi = 0.15 \) so that returns follow a negatively skewed process with \( \phi(t,T) = -0.5 \) when there is 3 months to expiry. Figure 4 shows the difference between Lévy-Stable and Black-Scholes call option prices for different expiry dates. In the Black-Scholes case, annual volatility is \( \sigma_{BS} = 0.20 \). Figure 5 shows the corresponding implied volatility. The negative skewness introduced produces a 'hump' for call prices with strike below 100. This is financially intuitive since relative to the Black-Scholes the risk-neutral probability of the call option ending out-of-the-money is substantially higher in the Lévy-Stable case.

**6. Conclusion**

The GCLT provides a very strong theoretical foundation to argue that the limiting distribution of stock returns or log-stock prices follows a Lévy-Stable process. We have shown how to model stock returns and log-stock prices where the stochastic component is Lévy-Stable distributed covering the whole range of skewness \( \beta \in [-1,1] \). We showed that European-style
option prices are straightforward to calculate using transform methods and we compare them to Black-Scholes prices where we obtain the expected volatility smile.

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References


Appendix A

Suppose that the stock process, as assumed above in section 4.2, follows

$$\frac{\log S_T}{S_0} = \mu (T-t) + \int_t^T \sigma dW_r + \int_t^T \int_r^T \sigma dZ_r dZ_u,$$

under $P$ where $dW_r$ denotes the increment of the standard Brownian motion independent of both $dR_r$ and $dZ_r$. Then it is straightforward to verify that the shocks to the above log-stock process under the measure $P$ are the sum of two independent processes: those of a Gaussian component and those of a Lévy-Stable process with negative skewness $\beta \in (-1, 0)$. Let $G(t, T) = \int_t^T g(t, r) dr$ and, for simplicity in the calculations, assume that $\sigma^2 = 0$ (so we focus only on the asymmetric Lévy process).

Now consider the process

$$U_{t, \alpha} = \int_t^T \sigma^2 dW_r + \alpha \int_t^T \int_r^T \sigma dZ_r dZ_u.$$