COINTEGRATION TESTING using the RANGES

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Abstract

In this paper we propose a method for testing the hypothesis of cointegration in pairs of univariate time series. One of our method's main advantages lies in that it does not impose any restriction on the time series models. Another is that cointegration can be tested regardless of the form of the relationship. Essentially, our test rests on a definition of cointegration which requires the sinchronicity up to a constant delay of the relevant informational events for the series. Thus cointegration can be tested independently on what form of relationship holds between the variables.
We propose three alternative test statistics and obtain, under some assumptions, their asymptotic null distribution. We also propose some graphical techniques consisting in plotting functions of the *range sequences* for the pairs of series. These plots could help in detecting nonlinearities as well as nonstationarities in the cointegrating relationship. Also we show how nonlinearity and/or nonstationarity in the relationship can be detected by analyzing the cross-difference of ranges. We finally report some experiments on financial and monetary time series that compare the performances of our test statistics with more standard ones.

KEY WORDS: *Linear and nonlinear cointegration, Dickey-Fuller test, integrated time series, order statistics, ranks, range, comovement, marked point processes.*

1 Introduction

Processes which exhibit common trends or similar long waves in their sample paths are often called *cointegrated.* The concept of cointegration was inherently linear and originated in macroeconomics and finance (c.f. Granger, 1981[16]; Granger and Engle, 1987[11]), where the theory suggests the presence of economic or institutional forces preventing two or more series to drift too far apart from each other. Take for example, those series as income and expenditure, the prices of a particular good in different markets, the interest rates in different countries, the velocity of circulation of money and short-run interest rates, etc. Cointegration relationships may also appear in engineering applications. For instance, between the outputs signals from different sensing or processing devices having a limited storage capacity or memory, and driven by a common persistent input flow (c.f. Aparicio, 1995[1]).

Underlying the idea of cointegration is that of a *stochastic equilibrium* relationship (i.e. one which, apart from deterministic elements, holds on the average) between two cointegrated variables, y_t, x_t. A strict equilibrium exists when for some $\alpha \neq 0$, one has $y_t = \alpha\, x_t$. This unrealistic situation is replaced, in practice, by that of (linear) cointegration, in which the equilibrium error $z_t = x_t - \alpha\, x_t$ is different from zero but fluctuates around the mean much more frequently than the individual series.

So far, attempts to extend the concept of cointegration beyond the assumption of linearity in the relationship have met with little success. This is essentially due to the fact that a general null hypothesis of cointegration encompassing nonlinear relationships is too wide to be tested. Notwithstanding, the possibility of nonlinear cointegrating relationships is real, and therefore it has prompted some interesting definitions and an ongoing research on the subject. The first of these attempts was due to Hallman (1990)[20] and to Granger and Hallman (1991)[17]. Following this, for a pair of series y_t, x_t to have a cointegrating nonlinear *attractor*, there must be nonlinear measurable functions $f(.), g(.)$ such that $f(y_t)$ and $g(x_t)$ are both $I(d)$, $d > 0$, and $w_t = f(y_t) - g(x_t)$ is $\sim\ I(d_w)$, with $d_w < d$.

Assuming that f and g can be expanded as Taylor series up to some order $p \geq 2$ around the origin, we may write $w_t = c_0 + c_1 z_t + HOT(y_t, x_t)$, where $z_t = y_t - \alpha x_t$, and with $HOT(.,.)$ denoting higher-order terms. It follows that the linear approximation, z_t, to the true cointegration residuals differs from the latter by some higher-order terms which express that the *strengh of attraction* onto the cointegration line $y_t = \alpha x_t$ varies with the levels of both series, y_t and x_t.

As with linear cointegration, the case where $d_x = d_y = 1$, $d_z = 0$ and the cointegration residuals have finite variance is most important in practice, since it allows a straighforward interpretation in terms of equilibrium concepts. Figure 1 illustrates the case of nonlinear cointegration, with simulated nonlinear transformations of random walks.

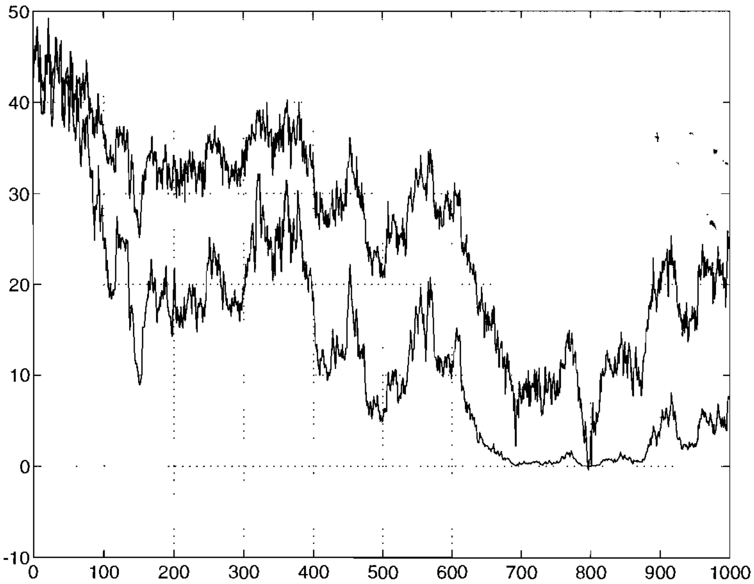

Figure 1: Two simulated nonlinearly cointegrated series. The upper series was obtained as $x_t = w_t + e_{x,t}$, while the lower one corresponds to $y_t = g(w_t) + e_{y,t}$, where $g(.)$ represents a third-order polynomial of its argument random walk variable w_t, and $\epsilon_{x,t}, \epsilon_{y,t}$ are independent$i.i.d.$ sequences.

Further Engle and Granger (1987[11]) proposed a method for testing the hypothesis of non-cointegration against that of linear cointegration. These tests can be decomposed as follows:

- A test for long-memory in the variables, say $y_t \sim I(d_y), x_t \sim I(d_x)$, and estimation of the long-memory parameters d_x, d_y. Then a test of significance for the stochastic difference $\hat{d}_x - \hat{d}_y$. If it is too large as compared to both d_x and d_y, then the variables cannot be cointegrated. Otherwise, we assume $d_x = d_y = d$ and go to next step.

- A test for long-memory in the cointegrating residuals $z_t = y_t - \hat{\alpha}x_t$, and estimation of its long-memory parameter, d_z. Then a test of significance for the difference $d - \hat{d}_z$. Large positive values of this difference as compared to d, can be taken as evidence of the existence of a (maybe fractional) cointegrating relationship between y_t and x_t.

A most investigated case corresponds to when the long-memory features in the variables are only due to unit roots (i.e. $d_x = d_y = 1$). This simplifies the procedure since no estimation of long-memory parameters is required in this case. However, a *test for unit roots* is needed to confirm this hypothesis.
The second part of the test consists in checking for a unit root in the cointegration residuals $\xi_t = x_t - \hat{\alpha}x_t'$ (the hypothesis of a unit root is usually referred to as the null of non-cointegration), for which an estimate, $\hat{\alpha}$, of the cointegrating parameter, is required.

The most general distributional results to unit-root testing were first obtained by Phillips (1987)[25]. Suppose x_t has mean μ_t, and let $\Delta(x_t - \mu_t) = \epsilon_t$, with Δ denoting the first differencing operator. The main assumption imposed to obtain the limit distribution of standard unit-root tests is the following, due to Herrndorf (1984)[22]:
ASSUMPTION AS1 :

1. $E(\epsilon_t) = 0$.

2. $\sup_t E(|\epsilon_t|^\gamma) < \infty$ for some $\gamma > 2$.

3. $0 < \lim_{N\to\infty} E\left(N^{-1}(\sum_{t=1}^N \epsilon_t)^2\right) < \infty$.

4. ϵ_t is strong mixing, with mixing coefficients α_i satisfying $\sum_{i=1}^\infty \alpha_i^{1-2/\gamma} < \infty$.

Under assumption AS1, Herrndorf (1984 [22]) found a Functional Central Limit Theorem (FCLT) for the partial sum of α-mixing sequences.

In a one-sided test for unit roots, the null hypothesis of a unit root in a series x_t, H_0 : $(1-B)(x_t - \mu_t) = \xi_t$, is tested against the alternative H_1 : $(1-\rho B)(x_t - \mu_t) = \xi_t$ with $|\rho| < 1$, where ξ_t denotes a sequence of *i.i.d. r.v.*'s. The decision between H_0 and H_1 is based on the significance of the estimate of $\rho - 1$ in the regression:

$$\Delta(x_t - \mu_t) = (\rho - 1)(x_{t-1} - \mu_{t-1}) + \epsilon_t \tag{1}$$

A commonly used test statistic for testing H_0 is the t-ratio, t_0, of the parameter in the regression equation above. Phillips showed that the distribution of t_0 depends on nuisance parameters from the autocorrelation (ACF) in ϵ_t and its variance. And therefore, he proposed a way of dealing with this problem by estimating those nuisance parameters and correcting accordingly the t-ratio:

$$t_0 = \frac{\hat{\rho} - 1}{\hat{\sigma}_0} \tag{2}$$

where

$$\hat{\sigma}_0 = \sqrt{N^{-1}\sum_{i=1}^{N}\hat{\epsilon}_i^{\,2}}\left(\sum_{i=1}^{N}(x_{i-1} - \mu_{i-1})^2\right)^{-1/2} \tag{3}$$

is the standard deviation of $\hat{\rho} - 1$. The corrected t-ratio, t_1 has the expression:

$$t_1 = \frac{\hat{\rho} - 1}{S_{N,l}\left(\sum_{i=1}^{N}(x_{i-1} - \mu_{i-1})^2\right)^{-1/2}} - \frac{(1/2)(S_{N,l}^2 - S_\epsilon^2)}{S_{N,l}\left(N^{-2}\sum_{i=1}^{N}(x_{i-1} - \mu_{i-1})^2\right)^{1/2}}, \tag{4}$$

and

$$S_{N,l}^2 = \sum_{i=1}^{N}\hat{\epsilon}_i^{\,2} + (2/N)\sum_{j=1}^{l} v_{j,l}\sum_{k=j+1}^{N}\epsilon_j\epsilon_{j-l}, \tag{5}$$

with $v_{j,l} = 1 - j/(l+1)$, is the Newey and West (1987[24]) estimator, and $S_\epsilon^2 = N^{-1}\left(\sum_{i=1}^{N}\Delta(x_i - \mu_i)\right)^2$. The limit distribution of the corrected t-ratio is given in the following theorem by Phillips (1987[25]):

THEOREM TH0:

Under assumption AS1, and if $l = O(N^{1/4})$, then under H_0, t_1 converges to:

$$\frac{(1/2)\left(B_2(1)-1\right)}{\left(\int_0^1 B(r)dr\right)^{l/2}}, \tag{6}$$

where $B(r)$ is the Wiener process on $(0,1)$.

This distribution has been tabulated in Dickey and Fuller (1979 [8]). The test of unit roots based on the t-ratio of the regression in (1), non-parametrically corrected as above, is therefore commonly referred to as the *Dickey-Fuller* (DF) *test.*

An alternative way to palliate the effect of the autocorrelation in ϵ_t consists in "augmenting" the test by including sufficient lagged first differences $\Delta(x_{t-i} - \mu_{t-i})$, $i \geq 1$, in the left-hand side of (1), so as to remove as much as possible of the serial correlation in ϵ_t. This gives rise to the so called *Augmented Dickey-Fuller* (ADF) *test.* In small samples, the parametric correction implied by the ADF test procedure works better than the previous non-parametric corrections. For an interesting discussion about how to implement the unit-root test in practice, see Hamilton (1994[21]) -chapter 17-.

The results of these unit-root tests applied on the cointegration residuals can be supported by other less conclusive measures of correlation, such as the R^2 of the regression. This quantity, known sometimes as the *coefficient of determination*, measures how well a regression fits -see for example Pindyck and Rubinfeld (1991 [26], page 61)-. In our case, it denotes the proportion of the variance of y_t explained by the regression of y_t on x_t, and the *Durbin-Watson* (*DW*) statistic (c.f. Durbin and Watson, 1951 [10]). The former is expected to be large for a cointegration relationship, but may also be very misleading because of the problem of spurious regressions between $I(1)$ variables. The DW statistic on the cointegration residuals $\hat{\epsilon}_t$ has the form

$$DW = \frac{\sum_{t=2}^{N}\left(\hat{\epsilon}_t - \hat{\epsilon}_{t-1}\right)^2}{\sum_{t=1}^{N}\hat{\epsilon}_t^2} \tag{7}$$

and can be approximated as $DW \approx 2(1-\hat{\rho})$, where $\hat{\rho}$ is an estimate for the parameter in the autoregression $\xi_t = \rho\xi_{t-1} + v_t$, v_t being ideally a zero-mean *i.i.d.* sequence. Thus a low value for DW is usually taken as evidence of no cointegration.

By doing a mean value expansion of a general nonlinear transformation, Granger and Hallman (1991[18]) remarked that the residuals did not satisfy assumption AS1 in several cases -see also Escribano and Mira (1997 [12])-. It followed that those unit-root tests were not invariant to nonlinear transformations of the variables. Therefore they proposed to use a Dickey-Fuller test based on the ranks of x_t[1] , since the ranks are robust to monotonic transformations. Hallman (1990[20]) extended those results to test the null hypothesis of no cointegration on transformed series. However, the limit distribution of the unit-root test based on ranks was unknown. Breitung and Gourieroux (1997[4]) obtained this limit distribution. However, the resulting test has some undesirable properties. First, by means of Monte Carlo experiments, they saw that this limit distribution is a poor approximation in small samples. Second, there are no results for random walks with drifts or more general data generating mechanisms (such as those allowing for autocorrelation in the model errors).

To implement unit-root tests like the ones proposed by Phillips (1987[25]) and Dickey and Fuller (1979 [8]), one needs to use critical values. Most critical values are obtained under the assumption of Normality of the errors. Therefore there is a need to develop tests for cointegration that are robust to non-Normalities in the errors. For example, this is desirable when working with financial data, where non-Normalities appear in the form of fat tails of the error probability densities.
Hallman (1990[20]) proposed to apply a DF test on the ranks of the residuals from linear and nonlinear (transformed) cointegrating relationships. However, as said previously, DF tests based on ranks (RADF) have serious disadvantages.

All the previous techniques can detect at most linear or monotonically nonlinear cointegrating relationships, but more general cointegrating relationships will often pass undetected. This difficulty is inherent to any cointegration testing procedure focussed on the levels of

[1]The rank of x_i in the sample X of size n is defined as $r_{i,n}^{(x)} = \sum_{j=1}^{n} \mathbf{1}(x_i \geq x_j)$ -see for example David (1981[7])-.

the series. At this point, it is important to remark that the two conditions for cointegration are just that:

1. There are informational events that have a permanent effect on the levels of the series (in the linear case, this amounts to saying that the series are integrated).

2. The relevant informational events for both series occur at time instants related by a translation, and their effects on each series are related.

These conditions raise the question of what are those relevant informational events. Of course, it is impossible to identify real informational events from a time series of data. Therefore, the definition that follows is endogeneous to the series and is taken just for convenience. Throughout this paper, we define a *relevant informational event* as one which induces outstanding trending behaviour (either deterministic or stochastic) or a non-reversion to the mean of the series. This definition of relevant informational event cannot be detached from the effects these events have on the series levels. Therefore a cointegration testing procedure only needs checking for the synchronicity (up to a constant delay) of the two sets of jump arrival times.

An appropriate way of analyzing cointegration in the way just described could be by considering the individual series as realizations of *marked point processes* (see Daley and Vere-Jones, 1988[6]) or as *subordinated processes* (see for instance Clark, 1973[5], and Willekens and Teugels, 1988[28]). Thus the *subordinator* processes are point processes (the arrival times of relevant informational events corresponding to each series) which can determine alone whether the pair of series are cointegrated or not. Indeed, the series will be cointegrated if their subordinator processes (as defined previously) are not approximately a delayed replica of each other. On the other hand, the subordinated processes are just the sequence of *marks* associated with these point processes, and will contain the information about the way in which the effects of the relevant events for each series are related, that is, on the form of the cointegrating relationship.

Notice that for linear (nonlinear) cointegration one requires an additional condition: that the effects that these relevant informational events have on each series can be linearly (non-linearly) related. However, the fact that we can always find a function that maps a set of

N points onto another set of N points suggests that it may be impossible to find empirical evidence against nonlinear cointegration, in general. The practical implication of this is that it is only meaningful to test for particular forms of nonlinearity.

In the next section, we follow the previous ideas to derive a cointegration testing framework based on *ranges*, which imposes no restriction on the individual time series models.

2 Cointegration testing using the ranges

The objective of this section is to propose an alternative procedure for testing cointegration, and for testing linearity and/or monotonic nonlinearity in a cointegrating relationship, essentially based on *order statistics*, and which does not rely on any particular model for the series. This methodology also suggests ways of identifying eventual nonlinearities and/or non-stationarities in the cointegrating relationship.

Henceforth, we regard cointegration as the hypothesis consisting in the sinchronicity up to a constant delay of the relevant informational events for these series. We consider as relevant those events which contribute an increase in the trending behaviour of the series. This latter definition can be made operative using some linear functions of order statistics, hereby called *ranges*.

The ranges are defined in terms of the *extremes* -see Galambos (1984 [14])-. For a sample of size n, $x_1, \cdots, x_n$, the order statistics of x_t are given by the sequence $x_{1,n} \leq \cdots \leq x_{n,n}$, obtained after a permutation, of the indexes $\{1, \cdots, n\}$ such that $x_{i,n} \leq x_{i+j,n}$, $\forall j > 0$. The terms $x_{1,n} = \min\{x_1, \cdots, x_n\}$ and $x_{n,n} = \max\{x_1, \cdots, x_n\}$ are called the extremes, and the sequence of ranges for this n-size sample of x_t is defined as $r_n^{(x)} = x_{n,n} - x_{1,n}$. Basically, a process defined by a sequence of ranges is an integrated *jump process*, where each jump corresponds to the arrival of a relevant informational event, which according to our definition is one which contributes either a new maximum or a new minimum level in the series.

2.1 Cointegration testing

Under cointegration, and unless the eventual nonlinearity in the relationship is of a very high order, the jumps in the sequence of ranges of one series tend to occur at a constant time delay from those in the other series. Therefore, a cointegration testing device may consist in checking the constancy of the elements in the sequence obtained by substracting both sets of arrival times. If at least one of the series is not trending (either deterministic or stochastically), the prevalence of noise will erase any trace of relation between these two sets of jump arrival times, and particularly, the translational relation which we are testing for.

Formally, consider the subordinated process represented by the nonzero first differences of ranges, or in other words, the sequence of jumps, $\Delta r^{(v)}_{I_v(t)}$ for $v = x, y$. Here $I_v(t)$ denotes the sequence of arrival times of these jumps for v_t $(v = x, y)$, and it is defined by $I_v(i) = t_i$, where t_i is the time at which the i-th jump appears. Our null hypothesis (H_0) in testing cointegration would be $I_y(t) = I_x(t) + C \; \forall t$, where C is a constant. If the series $I_v(t)$, $(v = x, y)$, could be modelled as integrated processes, then we would only require to test the hypothesis that the cointegration parameter in the regression of $I_y(t)$ on $I_x(t)$ is equal to 1, for which asymptotic results are already available (Dolado and Marmol, 1997[9]).

Another test statistic that may prove useful in cointegration testing is the ratio of the number of jumps with identical arrival times for both series to the minimum number of jumps for either series. Indeed, this statistic will approach the zero value under non-cointegration, whereas it would take values close to (but smaller than) one for linear and nonlinear cointegration.

Finally, if the cointegrating relationship is highly nonlinear or nonstationary (i.e. cointegration only holds for a few time spells) then the statistic will take values in-between.

EXAMPLES :

Figure 2 show the range sequences $r_i^{(y)}$ and $r_i^{(x)}$ for pairs of linearly, nonlinearly (cubic nonlinearity), non-cointegrated, and $I(0)$ comoving series. It can be seen that, for cointegrated series (either linear or nonlinear), the jumps occur at approximately the same instants, even though their amplitudes may not be related by a linear relationship (see figures 3 and 4). On the contrary, the jump sequences corresponding to the non-cointegrated series show no apparent relation between the arrival times of the two sets of jumps, and a similar behaviour is obtained when the series are $I(0)$ but comoving (figures 5 and 6).

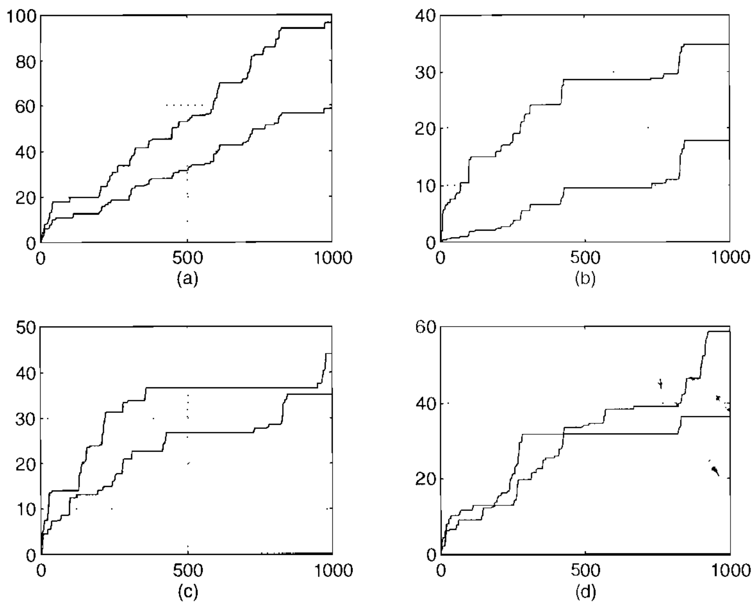

Figure 2: Cross-plots of the sequences of ranges for a pair of linearly, nonlinearly, non-cointegrated, and $I(0)$ comoving series: (a) linear cointegration, (b) nonlinear cointegration, (c) independent random walks, and (d) $I(0)$ comoving series. Under cointegration, either linear or nonlinear, the jumps in the range sequences take place at approximately the same instants. For (non)linearly cointegrated series, the amplitude of the jumps are (non)linearly related. On the contrary, the jumps in a pair of independent random walks or of $I(0)$ comoving series occur at remarkably different instants.

Figure 7 shows the cross-plots of the range sequences for the four pair of series. It is apparent that cointegration implies a sort of continuity in these plots. For the pairs of independent random walks and the pairs of $I(0)$ comoving series, the sequences of ranges evolve differently, which explain the discontinuities in the corresponding cross-plots. These discontinuities are

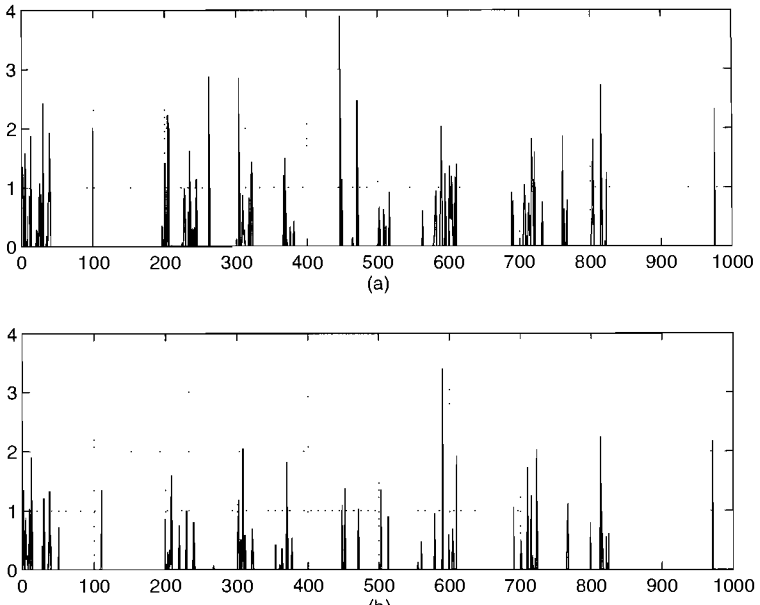

Figure 3: Sequences of jumps $\Delta r_i^{(y)}$ and $\Delta r_i^{(x)}$ for the linearly cointegrated series used in figure 2.

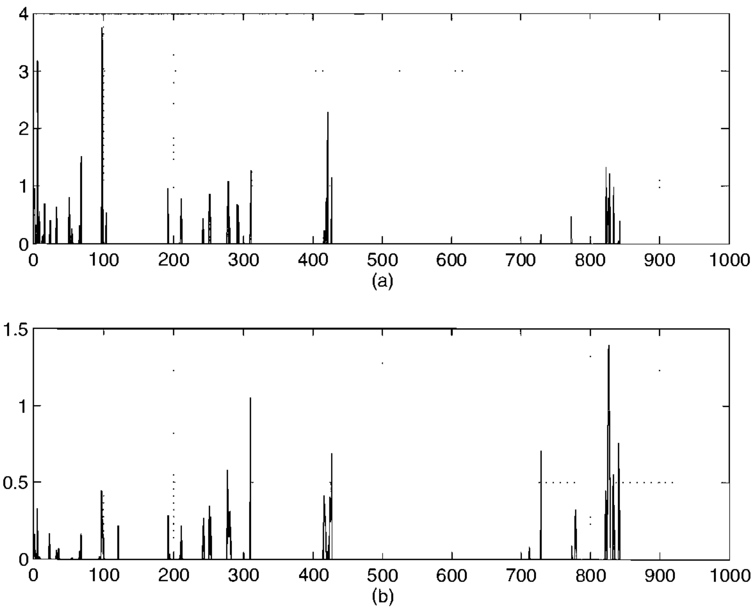

Figure 4: Sequences of jumps $\Delta r_i^{(y)}$ and $\Delta r_i^{(x)}$ for the nonlinearly cointegrated series used in figure 2.

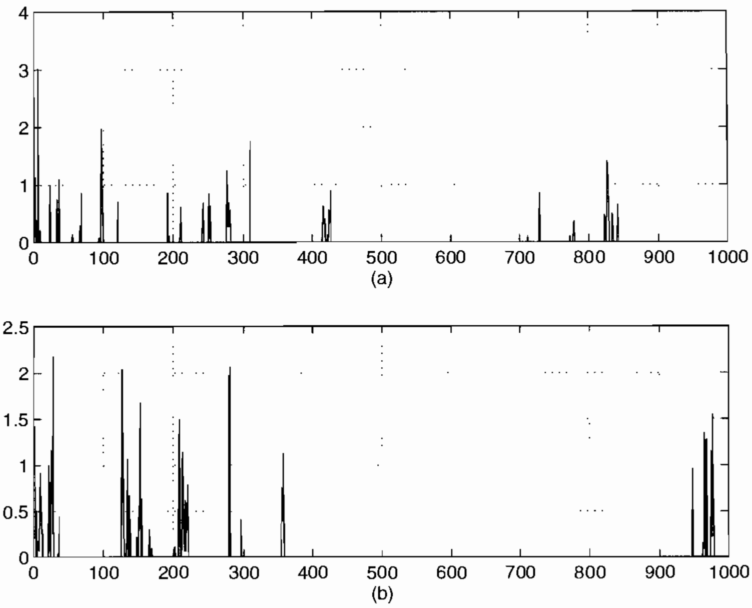

Figure 5: Sequences of jumps $\Delta r_i^{(y)}$ and $\Delta r_i^{(x)}$ for the independent random walks used in figure 2.

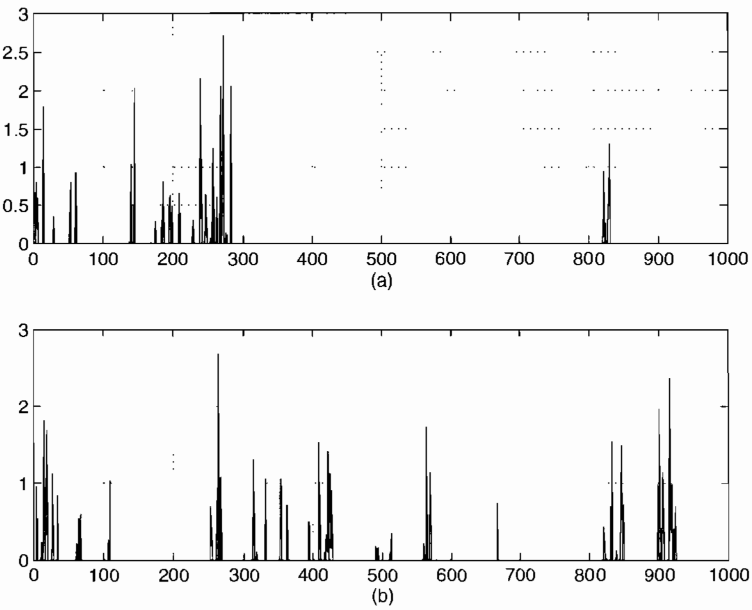

Figure 6: Sequences of jumps $\Delta r_i^{(y)}$ and $\Delta r_i^{(x)}$ for the pair of $I(0)$ comoving series used in figure 2.

more pronounced for pairs of independent random walks since the sample paths of the series consist of long strides, while for the $I(0)$ comoving series, the the different ways in which the range series evolve are hidden by high-frequency fluctuations. As a consequence, the discontinuities in the last cross-plot are not so outstanding.

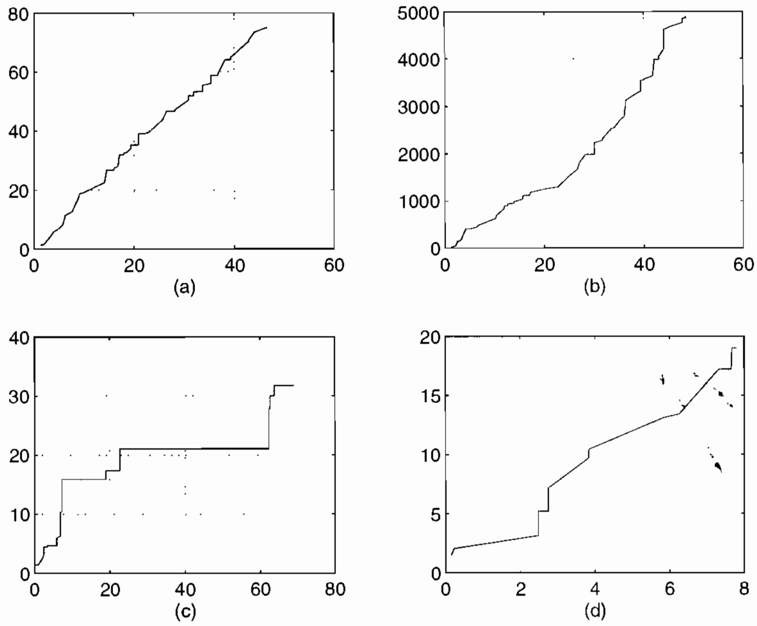

Figure 7: Cross-plots of the sequences of ranges for a pair of linearly, nonlinearly, non-cointegrated, and $I(0)$ comoving series: (a) linear cointegration, (b) nonlinear cointegration, (c) independent random walks, and (d) $I(0)$ comoving series.

2.2 Linear and monotonically nonlinear cointegration testing

As we have said, if the arrival times of these jumps representing the arrival of the relevant informational events, are translationally related then the series will be cointegrated. Otherwise, if the clusters in the plots of the first-differences of ranges have orthogonal supports then the series will be non-cointegrated. Further, to distinguish between linear and nonlinear cointegration, it would be enough to remark that while for linear cointegration, informational events having identical arrival times in both series will have approximately the same impact on their levels, for nonlinear or non-stationary cointegration these shocks may have quite different effects on each series.

Assuming that no series lags behind the other, then under cointegration, a cross-plot of the first differences of ranges for both series would show many points in the first quadrant, while for independent random walks or for $I(0)$ comoving series, the points in these plots would tend to lie very close to or along the non-negative half-axis. Therefore, one would be inclined to believe that the quality of fit of a regression line from the origin to the points in these plots would necessarily be less bad under cointegration than under non-cointegration. Figure 8 shows these cross-plots obtained from 100 replications of each pair of series.

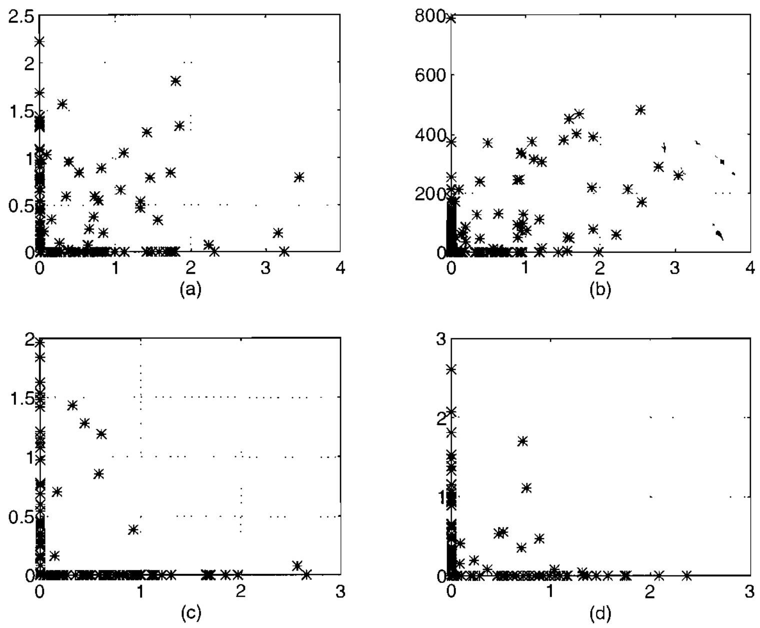

Figure 8: $\Delta r_i^{(y)}$ versus $\Delta r_i^{(x)}$ for pairs of: (a) linearly cointegrated series, (b) nonlinearly cointegrated series (quadratic transformation), (c) non-cointegrated series (independent random walks), and (d) $I(0)$ linearly comoving series.

In order to summarize the information collected by the cross-plots of $\Delta r_t^{(y)}$ versus $\Delta r_t^{(x)}$ in a statistic, we remark that in the presence of a linear or a monotonic nonlinear cointegrating component in x_t, y_t the sequences of ranges, $r_1^{(x)}, \cdots, r_n^{(x)}$ and $r_1^{(y)}, \cdots, r_n^{(y)}$, will be approximately proportional. We expect a similar behaviour from the sequences of jumps, $\Delta r_1^{(x)}, \cdots, \Delta r_n^{(x)}$ and $\Delta r_1^{(y)}, \cdots, \Delta r_n^{(y)}$. Thus, a non-parametric measure of linear cointegration could be provided by the following statistic, which provides a measure of the quality of fit of a regression line from the origin to the points in the cross-plots for the jump sequences $\Delta r_t^{(x)}$ and $\Delta r_t^{(y)}$:

$$\rho_{x,y}^{(n)} = \frac{\sum_{i=2}^{n}(r_i^{(x)} - r_{i-1}^{(x)})(r_i^{(y)} - r_{i-1}^{(y)})}{\left(\sum_{i=2}^{n}\left(r_i^{(x)} - r_{i-1}^{(x)}\right)^2\right)^{1/2}\left(\sum_{i=2}^{n}\left(r_i^{(y)} - r_{i-1}^{(y)}\right)^2\right)^{1/2}}, \tag{8}$$

Alternatively, we could consider the statistic:

$$\bar{\rho}_{x,y}^{(n)} = \frac{\sum_{i=2}^{n}(r_i^{(x)} - r_{i-1}^{(x)} - \mu_{\Delta r}^{(x)})(r_i^{(y)} - r_{i-1}^{(y)} - \mu_{\Delta r}^{(y)})}{\left(\sum_{i=2}^{n}(r_i^{(x)} - r_{i-1}^{(x)} - \mu_{\Delta r}^{(x)})^2\right)}, \tag{9}$$

where $\mu_{\Delta r}^{(x)}$ and $\mu_{\Delta r}^{(y)}$ represent the means of the sequences $\Delta r_i^{(x)}$ and $\Delta r_i^{(y)}$, respectively. This reminds a sample cross-correlation, and can be estimated directly from the regression equation:

$$r_i^{(y)} - r_{i-1}^{(y)} = \bar{\rho}_{x,y}^{(n)}(r_i^{(x)} - r_{i-1}^{(x)}) + u_t \tag{10}$$

In table 1, the mean values and their standard deviations (given in brackets) for the range statistic $\rho_{x,y}^{(n)}$ is given for an experiment involving 100 replications of cointegrated (linearly and nonlinearly -quadratic-) and non-cointegrated series of length $n = 1000$. The nonlinearly cointegrated series were obtained as in figure 1, using a quadratic transformation of a common random walk component, plus an added independent white Gaussian noise. We also estimated the mean and standard deviation of the range statistic on a pair of linearly comoving $I(0)$ series generated with the following model:

$$x_t = 0.6x_{t-1} + e_{t,1} \tag{11}$$

$$y_t = 2.0x_t + e_{t,2}, \tag{12}$$

where $e_{t,1}, e_{t,2}$ are independent $i.i.d.$ sequences of Gaussian $r.v.'s$.
In the sequel LC will stand for linear cointegration, NLC for nonlinear cointegration, NC for non-cointegration (independent randon walks), and SMC for $I(0)$ comoving.

Clearly, the case of independent randon walks (NC) can be easily discriminated using this statistic in a unilateral test. Similar results were obtained using $\bar{\rho}_{x,y}^{(n)}$ as test statistic. However, the values taken by this statistic are not bounded to the interval $[0, 1]$ as in the case of

Test statistic	*LC*	*NLC (quadratic)*	*NC*	*SMLC*
$\rho_{x,y}^{(n)}$	0.33 (0.15)	0.52 (0.12)	0.04 (0.04)	0.7184 (0.1494)

Table 1: Mean values and standard deviations of the range statistic $\rho_{x,y}^{(n)}$, evaluated on 100 replications of linearly (LC) and nonlinearly (NLC) cointegrated, of independent random walks (NC), and on a pair of $I(0)$ linearly comoving series (SMLC), for a sample size of $n = 1000$.

$\rho_{x,y}^{(n)}$.

Figure 9 shows the histogram plots of $\rho_{x,y}^{(n)}$.

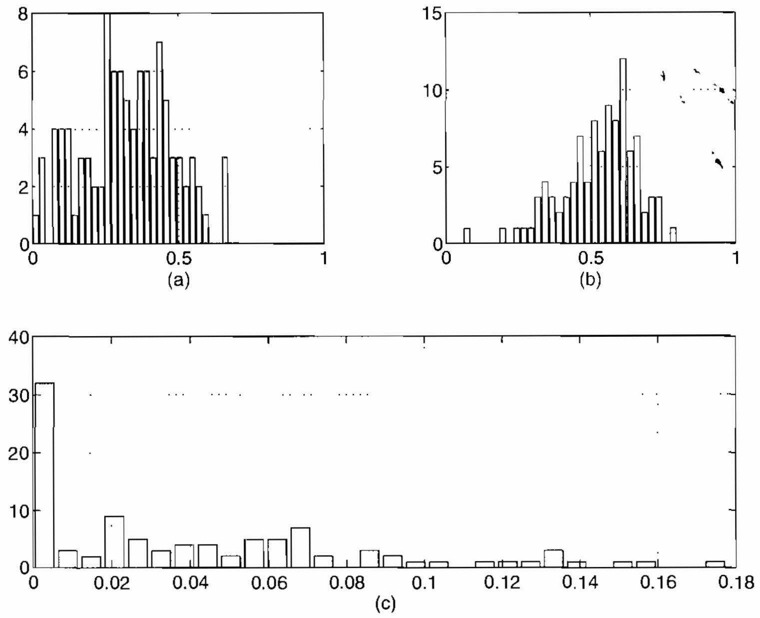

Figure 9: Histogram plots for $\rho_{x,y}^{(n)}$ where the frequencies are estimated from 100 replications of: (a) linearly cointegrated series, (b) nonlinearly (quadratic) cointegrated series, and (c) non-cointegrated series (independent random walks).

We computed the 5% right critical values of the empirical distribution of $\rho_{x,y}^{(n)}$ for different sample sizes ($n = 100, 500, 1000$), and for 1000 simulated pairs of independent random walks with *i.i.d.* Normally distributed errors (*model 0*). The results are summarized in table 2 below.

n=100	n=500	n=1000
0.39251	0.22647	0.16781

Table 2: Simulated 5% right critical values of the empirical distribution of the test statistic $\rho_{x,y}^{(n)}$ under the hypothesis NC.

Next, we computed the power of a unilateral test that uses these critical values against different alternatives. We considered the following data generating mechanisms:

1. *Model 1*

$$w_t = w_{t-1} + e_{t,0} \quad (13)$$

$$x_t = w_t + e_{t,1} \quad (14)$$

$$y_t = aw_t + e_{t,2}, \quad (15)$$

2. *Model 2*

   (a) *Model 2a*

$$w_t = w_{t-1} + e_{t,0} \quad (16)$$

$$x_t = w_t + e_{t,1} \quad (17)$$

$$y_t = aw_t^2 + e_{t,2}, \quad (18)$$

   (b) *Model 2b*

$$w_t = w_{t-1} + e_{t,0} \quad (19)$$

$$x_t = w_t + e_{t,1} \quad (20)$$

$$y_t = alog(w_t + 1000) + e_{t,2}, \quad (21)$$

(c) *Model 2c*

$$w_t = w_{t-1} + e_{t,0} \tag{22}$$

$$x_t = w_t + e_{t,1} \tag{23}$$

$$y_t = aexp(w_t/100) + e_{t,2}, \tag{24}$$

3. *Model 3*

$$x_t = 0.6x_{t-1} + e_{t,1} \tag{25}$$

$$y_t = 2.0x_t + e_{t,2}, \tag{26}$$

4. *Model 4*

$$x_t = 0.6x_{t-1} + e_{t,1} \tag{27}$$

$$y_t = 0.8y_{t-1} + e_{t,2}, \tag{28}$$

5. *Model 5*

$$x_t = 0.6x_{t-1} + e_{t,1} \tag{29}$$

$$y_t = y_{t-1} + e_{t,2}, \tag{30}$$

where a is a nonzero real number generated at random and $e_{t,0}, e_{t,1}, e_{t,2}$ are independent *i.i.d.* sequences of Gaussian *r.v.'s*.

Table 3 shows the estimated power of the test on the previous models.

The power figures which we found when using $\bar{\rho}_{x,y}^{(n)}$ as test statistic pointed to the same direction, in spite of a slightly smaller power of this statistic against the case of linear cointegration (*model 1*). For economy of space, we omit the details.

Model	*n=100*	*n=500*	*n=1000*
1	0.481	0.774	0.890
2a	0.899	0.991	0.998
2b	0.069	0.098	-
2c	0.553	0.074	-
3	0.979	0.992	0.999
4	0.114	0.255	-
5	0.076	0.080	-

Table 3: Estimated power for the test based on $\rho_{x,y}^{(n)}$ against different alternatives.

The results in table 3 show that our testing device cannot discriminate properly between models 1 (linear cointegration) and 3 ($I(0)$ linearly comoving series). Moreover, it may reject the null hypothesis of non-cointegration more often than desired when applied to pairs of independent $I(0)$ series. Finally, it is generally unable to detect cointegration when nonlinearities appear in the relationship.

An inspection of the cross-plots for the jump series reveals that the points in these plots tend to cluster at the origin for pairs of $I(0)$ comoving series, meaning that there are large time spells in which no relevant informational event appears for either series. Indeed, the very nature of the sample paths of $I(0)$ series entails that all relevant features of the series are captured in a comparatively small time interval, whereas the long strides of the sample paths of integrated series preclude this possibility. This explains why the quality of fit of a regression line from the origin to the points in the cross-plots cannot be distinguished from that obtained for pairs of linearly cointegrated series. This calls for a complementary test statistic that takes into account these features in order to discriminate between pairs of cointegrated series and pairs of $I(0)$ series. Therefore we propose a second test statistic $R_{x,y}^{(n)}$, which we define as

$$R_{x,y}^{(n)} = \frac{J^+}{NJ}, \tag{31}$$

where J^+ denotes the number of points in the plots which occur on the positive half axes, and NJ the number of points at the origin of these plots. In other words, $R^{(n)}_{x,y}$ measures the proportion of informational events that are only relevant to one series with respect to those which are not for either. The variable selected for the numerator in this ratio ensures that for pairs of independent random walks $R^{(n)}_{x,y}$ will take large values as compared to the cases of $I(0)$ series and LC.

For different sample sizes ($n = 100$, $n = 500$ and $n = 1000$) we simulated the 5% critical values of the empirical distribution of $R^{(n)}_{x,y}$ from 1000 replications of linear cointegration with *i.i.d.* Normally distributed errors. Table 4 shows the estimated left (c_l) and right (c_r) 5% critical values.

critical value	*n=100*	*n=500*	*n=1000*
left (c_l)	0.10714	0.05161	0.03575
right (c_r)	0.39063	0.15854	0.10544

Table 4: Simulated 5% left critical values of the empirical distribution of the test statistic $R^{(n)}_{x,y}$ under the hypothesis LC.

The power of a test based on $R^{(n)}_{x,y}$ was estimated for the different models considered above and from 1000 replications of each. For *model 3*, we analyzed the power behaviour for different values of the AR(1) coefficient, b, going from 0.6 to 0.99. We also studied the power of $R^{(n)}_{x,y}$ against pairs of $I(0)$ monotonically nonlinearly comoving series using *model 6* below.

1. *Model 6a*

$$x_t = 0.6x_{t-1} + e_{t,1} \tag{32}$$

$$y_t = 0.5x_t^2 + e_{t,2}, \tag{33}$$

2. *Model 6b*

$$x_t = 0.6x_{t-1} + e_{t,1} \tag{34}$$

$$y_t = log(x_t + 1000) + e_{t,2}, \tag{35}$$

3. *Model 6c*

$$x_t = 0.6x_{t-1} + e_{t,1} \tag{36}$$

$$y_t = exp(x_t/100) + e_{t,2}, \tag{37}$$

The results are shown in table 5 below.

Model	*n=100*	*n=500*	*n=1000*
2a ($< c_l$)	0.113	0.123	-
2b ($< c_l$)	0.018	0.032	-
2c ($< c_l$)	0.024	0.023	-
3 ($b = 0.6$) ($< c_l$)	0.681	0.993	-
3 ($b = 0.9$) ($< c_l$)	-	0.982	-
3 ($b = 0.95$) ($< c_l$)	-	0.938	-
3 ($b = 0.99$) ($< c_l$)	0.359	0.676	0.857
4 ($< c_l$)	0.098	0.805	-
6a ($< c_l$)	0.477	0.994	-
6b ($< c_l$)	0.164	0.909	-
6c ($< c_l$)	0.300	0.896	-
5 ($< c_l$)	0.015	0.013	-
0 ($> c_r$)	0.637	0.736	0.838
0 ($< c_l$)	0.000	0.000	-

Table 5: Estimated power for the test based on $R_{x,y}^{(n)}$ against different alternatives.

The power results obtained for *model 2* (NLC) and *model 6* (pairs of $I(0)$ monotonically nonlinearly comoving series) reveal a remarkable robustness of the unilateral test that uses the simulated left critical values against monotonic nonlinearities in the relationship. Therefore if we use $R_{x,y}^{(n)}$ in combination with $\rho_{x,y}^{(n)}$ (second and first stage of our test, respectively), it suggests the possibility of discriminating the hypothesis of cointegration (either linear or monotonically nonlinear) from that of pairs of $I(0)$ series.

It can be seen that by computing the statistics $\rho_{x,y}^{(n)}$ and $R_{x,y}^{(n)}$ on the pair of series, the associated unilateral tests discussed previously lead to the diagnostics shown in table 6. Notice that when neither test rejects the combined procedure cannot decided between a pair of nonlinearly cointegrated series and a pair $I(0)/I(1)$ of series. However, it is not totally unreasonable to have both cases under the same category, since a nonlinearity of a very high-order can constrain the sample paths of the common $I(1)$ component in such a way that one of the series looks like $I(0)$.

1st. test	*2nd.test*	*diagnostic*
rejects H_0	rejects H_0	SMC
rejects H_0	holds H_0	cointegration
holds H_0	rejects H_0	independence
holds H_0	holds H_0	NLC or pair of $I(0)/I(1)$

Table 6: Table of possible diagnostics when combining the testing procedures based on $\rho_{x,y}^{(n)}$ (1st. test) and on $R_{x,y}^{(n)}$ (2nd. test).

We finally studied the robustness of the test against deviations from the assumption of independence in the errors. For example, when the errors $e_{t,2}$ in *model 1* (LC) are correlated. For our study, we considered an AR(1) structure for these errors where the autoregressive parameter d was allowed to take the values $0.6, 0.9, 0.95$ and 0.99. Formally, the model analyzed was:

$$w_t = w_{t-1} + e_{t,0} \tag{38}$$

$$x_t = w_t + e_{t,1} \tag{39}$$

$$y_t = aw_t + e_{t,2} \quad (40)$$

$$e_{t,2} = de_{t-1,2} + \epsilon_t, \quad (41)$$

where $e_{t,0}, e_{t,1}, \epsilon_t$ are independent $i.i.d.$ sequences of Gaussian $r.v.'s$. We obtained the results shown in table 7.

Model	*n=100*	*n=500*	*n=1000*
1 $(d = 0.6)$ $(< c_l)$	0.12195	0.05353	-
1 $(d = 0.9)$ $(< c_l)$	0.14286	0.06318	-
1 $(d = 0.95)$ $(< c_l)$	0.15000	0.06652	-
1 $(d = 0.99)$ $(< c_l)$	0.16883	0.07860	-

Table 7: Power of the test based on $R_{x,y}^{(n)}$ against linear correlation in the model error structure under the null hypothesis (LC).

The robustness of our test against correlation in the model errors' structure seems remarkable in the light of these results.

2.3 Linear cointegration testing

A possible way of testing for linear cointegration is by remarking that $r_t^{(y)} - r_t^{(x)}$ is monotonic under linear cointegration, whereas it is not in other cases. This possibility of discriminating between linear and nonlinear cointegration is established with proposition PR1 in the next section.

3 Some inference results for ranges

In this section we provide some inferential results for some of the tests' statistics defined above.

3.1 Test for linear and monotonically nonlinear cointegration

In what follows we will see that, under alternative regularity conditions, it is possible to approximate the form of the asymptotic distributions of $\rho_{x,y}^{(n)}$ and of $\bar{\rho}_{x,y}^{(n)}$ under linear cointegration.

Let the sequences of jumps be defined as

$$\Delta r_t^{(x)} = r_t^{(x)} - r_{t-1}^{(x)} \quad (42)$$

$$\Delta r_t^{(y)} = r_t^{(y)} - r_{t-1}^{(y)} \quad (43)$$

for the series x_t, y_t generated with *model 1*, and consider the following assumptions:

ASSUMPTION AS2 :

- a) There exists stationary Gaussian processes $u_t^{(x)}, u_t^{(y)}$ such that
  1. $u_t^{(v)} = \sum_{j=0}^{\infty} \alpha_j^{(v)} \varepsilon_{t-j}^{(v)}$ ($v = x, y$), with $\varepsilon_t^{(v)}$ denoting a zero mean *i.i.d.* Gaussian sequence with bounded third-order moment.
  2. It is possible to write $\Delta r_t^{(v)} = \left(u_t^{(v)}\right)^2$.
- b) The coefficients $\alpha_j^{(v)}$ above satisfy:

$$\sum_{j=1}^{\infty} j^2 \left(\alpha_j^{(v)}\right)^2 < \infty, \quad v = x, y \quad (44)$$

- c) There is no lag behaviour between the series y_t and x_t.
- d) The errors $e_{t,1}, e_{t,2}$ in *model 1* have finite variance.

THEOREM TH1 :

Under assumption AS2 one has $n^{1/2}\, \rho_{x,y}^{(n)} \Rightarrow \chi_{1,\delta}^2$ up to a scaling factor. Under linear cointegration $\delta > 0$, whereas $\delta = 0$ under non-cointegration.

To prove this theorem we invoke the following lemmas:

LEMMA LE0:

Under linear cointegration and if the model errors for the individual series have finite variance then $\lim_{n\to\infty} \mu_{\Delta r}^{(z)} = 0$ *in probability, where* $z = x, y$.

PROOF OF LEMMA LE0:

This is a straightforward consequence of the boundness of the jump series under linear cointegration with finite variance model errors.

LEMMA LE1 :

If v_t, w_t *are stationary time series satisfying AS2(a-1) and AS2(b), then a central limit result applies to their cross-correlation estimate* $\hat{\gamma}_{v,w}^{(n)}$. *More precisely,* $n^{1/2}(\hat{\gamma}_{v,w}^{(n)} - \gamma_{v,w})$ *converges to a central normal distribution as* $n \to \infty$.

PROOF OF LEMMA LE1 :

For linearly cointegrated series, this follows from a modest extension of Theorem 6.7 in Hall and Heyde (1980 [19], page 188), while for x_t, y_t independent, it follows from Theorem 6.5.2 in Fuller (1976[13]) -page 267-.

LEMMA LE2 :

Let $y_t = x_t^2$, $y_t' = (x_t')^2$, *with* x_t, x_t' *two Gaussian zero-mean and unit variance sequences. Then the cross-correlation coefficients of the sequences verify* $\gamma_{y,y'} = 2\,\gamma_{x,x'}^2$.

PROOF OF LEMMA LE2 :

Let $H_n(x)$ be the Hermite polynomials. These are defined in terms of the standard normal *pdf*, $\varphi(x)$, as $H_n(x) = (-1)^n \varphi^{(n)}(x)/\varphi(x)$. Then it is easily shown that

$$E[H_n(x)H_k(y)] = \begin{cases} n!\,\gamma_{x,y}^n, & \text{if } n = k \\ 0, & \text{otherwise} \end{cases}$$

Noting that

$$Cov(y_t, y_t') = E\left(\sum_{j=1}^{\infty} a_j H_j(x_t) \sum_{i=1}^{\infty} a_i' H_i(x_t')\right) \tag{45}$$

with $a_i, a_i' = 0 \; \forall i \neq 2$ but that $a_2 = a_2' = 1$, the result follows. □

LEMMA LE3 :

Let S_n be a r.v. defined on an interval I of the real line, and satisfying $S_n \Rightarrow \mathcal{N}(\mu, b_n\sigma^2)$, with $b_n \to 0$ as $n \to \infty$, and let $g(x)$ be a real-valued function defined on I and possesing continuous derivatives of order $m > 1$ in a neighbourhood of $x = \mu$, with all derivatives of order j $(1 \leq j \leq m-1)$ vanishing at $x = \mu$, but with the $m-th$ order derivative not vanishing at this point. Then

$$b_n^{-1/2}[g(S_n) - g(\mu)] \Rightarrow (m!)^{-1} g^{(m)}(x)\rfloor_{x=\mu} \prod_{j=1}^{m} Z_j. \tag{46}$$

with $Z_j \sim \mathcal{N}(0, \sigma^2)$, and $g^{(m)}(.)$ denoting the m-th derivative of $g(.)$.

PROOF OF LEMMA LE3 :

This follows from theorem B in Serfling (1980 [27], page 124).□

PROOF OF THEOREM TH1 :

From lemma LE0, we can neglect the effect of $\mu_{\Delta r}^{(z)}$ $(z = x, y)$ on $\rho_{x,y}^{(n)}$ as n grows without bound. From lemma LE1, and under assumptions AS2(a-b), it follows that $n^{1/2}\hat{\gamma}_{u^{(x)},u^{(y)}}^{(n)} \Rightarrow \mathcal{N}(\gamma_{u^{(x)},u^{(y)}}, \sigma_1^2)$, with $\sigma_1^2 < \infty$. Now, from lemmas LE2 and LE3, it follows that $n^{1/2}\,\rho_{x,y}^{(n)} \Rightarrow \chi_{1,\delta}^2$, up to a scaling factor, and with $\delta = \rho_{x,y}$. Finally, under linear and monotonically non-linear cointegration, and under assumption AS2(c), the sequences of jumps are correlated, and thus $\rho_{x,y} > 0$. On the contrary, under non-cointegration one has $\rho_{x,y} = 0$. □

REMARKS :

1. Under perfect (linear) cointegration $y_t = ax_t + \epsilon_t$, $r_j^{(y)} \approx ar_j^{(x)}$. Therefore $\rho_{x,y}^{(n)}$ will tend to be substantially larger than zero, for large n. Under monotonically nonlinear

cointegration, we may write $r_j^{(y)} \approx c_j r_j^{(x)}$, where c_j defines a slowly varying sequence with $c_j \neq 0$. In this case, we may expect again $0 < \rho_{x,y}^{(n)} < 1$. Under non-cointegration, the sequences of extremes will be uncorrelated and so will be the sequences of ranges. Thus $\rho_{x,y}^{(n)} \to 0$ as $n \to \infty$.

2. The centrality parameter δ encodes the "degree" of cointegration.

3. Assumption AS2(c) precludes the possibility of lag behaviour between the series. Because of its sensitivity to lag effects, the range correlation statistic may reject the hypothesis of cointegration in pairs of series with apparent comovements in the mean.

4. The assumption that ε_t is a Gaussian identically distributed sequence in AS2(a) is not required for lemma LE1, but is needed for lemma LE2. However, this assumption cannot be justified for sequences containing trails of zeros, such as $\Delta r_t^{(x)}, \Delta r_t^{(y)}$. However, it would be more plausible as an assumption to consider the jumps associated with the T-step range sequences, that is $\Delta^T r_t^{(z)} = r_{t+T}^{(z)} - r_t^{(z)}$, $z = x, y$, where T can be chosen so as to keep the probability that $\Delta^T r_t^{(z)} = 0$ arbitrarily small.

To approximate the asymptotic distribution of the test statistic $\bar{\rho}_{x,y}^{(n)}$, let $\Delta r_t^{(v)}$, $v = x, y$ have an $ARMA(p^*, p)$ representation such as:

$$\Delta r_t^{(v)} = c_0^{(v)} + c_1^{(v)} \Delta r_{t-1}^{(v)} + \cdots + c_{p^*}^{(v)} \Delta r_{t-p^*}^{(v)} + u_t^{(v)} + b_1^{(v)} u_{t-1}^{(v)} + \cdots + b_p^{(v)} u_{t-p}^{(v)}, \tag{47}$$

where some constraints are applied to the coefficients $c_i^{(v)}$ and $b_j^{(v)}$, $i = 0, \cdots, p^*$, $j = 1, \cdots, p$, as well as on the support of distribution of the errors $u_t^{(v)}$, so that $\Delta r_t^{(v)}$ is always non-negative -see for example Gourieroux (1997[15]), chapter 3-.
We will assume that $u_t^{(v)}$ satisfies $E(u_t^{(v)} \mid \mathbf{u}_{t-1}^{(v)}) = 0$, where $\mathbf{u}_{t-1}^{(v)} = (u_{t-1}^{(v)}, u_{t-2}^{(v)}, \cdots)'$, that is, $u_t^{(v)}$ is a martingale sequence (but possibly heteroskedastic).

Let $p^* = \max(p, q)$. Then following Bollerslev (1986[3]) we can write an equivalent $GARCH(p, q)$ representation for the process given by the previous $ARMA(p^*, p)$ model:

$$h_t^{(v)} = c_0^{(v)} + \sum_{j=1}^{q} a_j^{(v)} \Delta r_{t-j}^{(v)} + \sum_{j=1}^{p} b_j^{(v)} h_{t-j}^{(v)}, \tag{48}$$

where $c_j^{(v)} = a_j^{(v)} + b_j^{(v)}$, $j = 1, \cdots, p^*$, with $b_j^{(v)} = 0$ for $j > p$ and $a_i^{(v)} = 0$ for $i > q$.
Letting $\delta_t^{(v)}$ be an *i.i.d.* sequence with $P(\delta_t^{(v)} = 1) = P(\delta_t^{(v)} = -1) = 1/2$, we have -see Gourieroux (1997[15])- that:

$$E(u_t^{(v)} \mid u_{t-1}^{(v)}) = 0 \tag{49}$$

$$var(u_t^{(v)} \mid u_{t-1}^{(v)}) = h_t^{(v)}, \tag{50}$$

where $u_t^{(v)} = \delta_t^{(v)} \sqrt{\Delta r_t^{(v)}}$ and $h_t^{(v)}$ follows the $GARCH(p, q)$ model above.
Now, if all the coefficients in this representation are non-negative, and if $\sum_{i=1}^{q} a_i^{(v)} + \sum_{j=1}^{p} b_j^{(v)} < 1$, then it can be shown that $u_t^{(v)}$ is asymptotically second-order stationary. Therefore we can safely test for cointegration by using the t-ratio from the OLS parameter estimates of the following regression equation:

$$\Delta r_t^{(y)} = \alpha_0 + \alpha_1 \Delta r_t^{(x)} + \epsilon_t. \tag{51}$$

Under non-cointegration (H_0) we have $\alpha_1 = 0$, whereas under cointegration (H_1), $\alpha_1 \neq 0$. Since ϵ_t might have heteroskedasticity as well as non-zero autocorrelation, we could use the autocorrelation and heteroskedasticity consistent covariance matrix estimator to form robust t-ratios -see Newey and West (1987[24])-.

3.2 Test for linear cointegration

Now let x_t, y_t be two I(1) time series, and define the sequence $z_t = r_t^{(y)} - r_t^{(x)}$. Without loss of generality, we may write $r_t^{(y)} = c_t r_t^{(x)}$, for a given sequence c_t.

ASSUMPTION AS3 :
The following limits exist $\forall\, j \geq 0$:

1. $\lim_{n \to \infty} n^{-1} \sum_{t=1}^{n} c_t = \bar{c}$
2. $\lim_{n \to \infty} (n - j)^{-1} \sum_{t=j+1}^{n} c_t c_{t-j} = \eta_j^{(c)}$

where $\bar{c}, \eta_j^{(c)}$ are (possibly degenerate) $r.v.$'s.

REMARK :

Notice that the process defining c_t needs not be ergodic.

PROPOSITION PR1 :

Under assumption AS3 above:

1. $P[\lim_{n\to\infty}(n-j)^{-1}\sum_{t=j+1}^{n} z_t z_{t-j} \geq 0] = 1,\ \forall j > 0$, *under linear cointegration.*

2. $P[\lim_{n\to\infty}(n-j)^{-1}\sum_{t=j+1}^{n} z_t z_{t-j} \geq 0] < 1,\ \forall j > 0$, *under nonlinear and non-cointegration.*

PROOF :

Suppose that there exists a nonzero real number a such that $\epsilon_t = y_t - ax_t \sim I(0)$. Therefore both x_t and y_t will dominate over ϵ_t after a transient $t > t_0$. Thus $r_t^{(y)} = ar_t^{(x)} + o(r_t^{(x)}, r_t^{(y)})$ and $r_t^{(y)} - r_t^{(x)} = (a-1)\, r_t^{(x)} + o(r_t^{(x)}, r_t^{(y)})$. The latter is equal to $(a-1)\, r_t^{(x)}$, after $t > t_0$ for some finite t_0, and the result in PR1(1) follows.

Under nonlinear cointegration and non-cointegration one could write $r_t^{(y)} \approx c_t r_t^{(x)}$, where c_t defines a sequence possibly dependent on the original series. Then $z_t = (c_t - 1)\, r_t^{(x)}$ and

$$\begin{aligned} P[\lim_{n\to\infty}(n-j)^{-1}\sum_{t=j+1}^{n} z_t z_{t-j} \geq 0] &= P[\lim_{n\to\infty}(n-j)^{-1}\sum_{t=j+1}^{n}(c_t-\bar{c})(c_{t-j}-\bar{c}) \geq -(\bar{c}-1)^2] \\ &< P[\lim_{n\to\infty}(n-j)^{-1}\sum_{t=j+1}^{n}(c_t-\bar{c})^2 \geq -(\bar{c}-1)^2] = 1 \quad (52) \end{aligned}$$

and the result in PR1(2) follows. □

REMARKS :

AS3 could be replaced by the weaker assumption that $P(u_t > u) > P(u_{t,j} > u)$, $\forall u$ and $\forall j > 0$, where $u_t = (c_t - 1)^2$ and $u_{t,j} = (c_t - 1)(c_{t-j} - 1)$. With this condition, the sample averages in AS3 need not converge at all.

EXRPY/EXRPM	*EXRPY/EXRPD*	*EXRPM/EXRPD*	*STR1/STR2*
-1.196	-5.249	-4.889	-13.486

Table 8: Values taken by the Dickey-Fuller test statistic $\tau_{df}(x,y) = N(\hat{\rho}-1)$ on the two pairs of foreign exchange rate series and the pair of stock return series. Here $\hat{\rho}$ is the OLS estimator of the parameter in the regression of y_t on x_t.

EXRPY/EXRPM	*EXRPY/EXRPD*	*EXRPM/EXRPD*	*STR1/STR2*
-4.649	-7.005	-7.460	-34.522

Table 9: Values taken by the Dickey-Fuller test statistic $\tau_{df}(x,y) = N(\hat{\rho}-1)$ on the two pairs of foreign exchange rate series and the pair of stock return series. Here $\hat{\rho}$ is the OLS estimator of the parameter in the regression of the series of ranks for y_t on the series of ranks for x_t.

4 Experiment on monetary and financial data

Some of the statistics proposed in the previous sections for testing cointegration were evaluated on two pairs of exchange rate series (figure 11), and on a pair of stock return series (STR1,STR2) from a Japanese food company (figure 10). The former group of series were the rates of exchange of the US Dollar (EXRPD), the Deutsch Mark (EXRPM) and the Japanese Yen (EXRPY) (in units of 100 yens) against the Spanish Peseta. We took the first $n = 1000$ daily observations from series starting at January the 1st. 1987. For the exchange-rate data, EXRPD was taken as the reference series.

First of all, we run an *Augmented Dickey-Fuller* (ADF) test (the conventional DF test was augmented with one lag in the first differences of the series) on the regression residuals of the three pairs of data sets considered above. If we denote by $\hat{\rho}$ the OLS estimator of the parameter in the regression of y_t on x_t, then the ADF test statistic is $\tau = N(\hat{\rho}-1)$ and its values are shown in the tables 8 and 9, for the levels and for the ranks of the series, respectively.

Using the critical values given by Mackinnon (1990) [23] (-2.57, -1.94 and -1.62 at the 1%, 5% and 10% levels, respectively), the hypothesis of (linear) cointegration (i.e. that τ takes values smaller than the tabulated critical values) is accepted in all cases except for the pair (EXRPY,EXRPM) when the test statistic is computed on the levels of the series, and in all cases when it is computed on their ranks.

Plots of the jump series (first differences of the range series) are shown in figures 12,15,14 and 13 for the pairs (STR1,STR2), (EXRPY,EXRPM), (EXRPY,EXRPD) and (EXRPM,EXRPD) respectively.

As we pointed in a previous section, the relative way in which jumps cluster along the horizontal axis in these plots is related to the likelihood of the cointegration hypothesis. Accordingly, the evidence of cointegration is comparatively weak for the pair (EXRPY,EXRPM), since no translational relation between the two sets of arrival times transpires from the figures (the two jump series have almost no overlapping support). For the pairs (EXRPY,EXRPD) and (EXRPM,EXRPD), most of the jumps of one series are synchronous or close to synchronous with those of the other. However, at some time spells, jumps appear for one series and not for the other, thus suggesting that there may be cointegration in a nonstationary or in a nonlinear way. Finally, for the pair (STR1,STR2) the jump arrival times for the series are more aligned, thus supporting the evidence of linear cointegration.

The values obtained with the range statistic $\rho_{x,y}^{(n)}$ for $n = 1000$ are given in table 10 below. It shows that only for the pair (EXRPY,EXRPM) the hypothesis of cointegration could be easily rejected, while linear cointegration seems to be the most likely outcome for the remaining pairs of series, especially for the pair (STR1,STR2).

Finally, table 11 shows the values taken by our test statistic $R_{x,y}^{(n)}$ on the four pairs of series.

EXRPY/EXRPM	EXRPY/EXRPD	EXRPM/EXRPD	STR1/STR2
0.037	0.152	0.231	0.589

Table 10: Values taken by $\rho_{x,y}^{(n)}$ on the four pairs of financial time series, for $n = 1000$.

All these values are consistent with the hypothesis of cointegration at the 10% significance level.

EXRPY/EXRPM	EXRPY/EXRPD	EXRPM/EXRPD	STR1/STR2
0.08009	0.07427	0.07427	0.09429

Table 11: Values taken by $R_{x,y}^{(n)}$ on the four pairs of financial time series, for $n = 1000$.

Thus the results obtained with the standard and the proposed testing procedures point to the same conclusion.

5 Conclusion

In this paper we have proposed using first differences of ranges for testing the hypothesis of cointegration in the bivariate time series case. The method is fully nonparametric and postulates no model at all for the individual series.
The plots of the sequences of first differences of ranges suggest a new definition of cointegration where the relevant feature is the simultaneity of the arrival times of significant informational events. That is, one could say that a pair of series are non-cointegrated when the sequences of first differences of ranges have orthogonal supports (i.e. they do not overlap).
Comparison of the behaviour of the jump sequences obtained for each series led us to propose two complementary testing procedures that when used in combination allow to discriminate between the alternatives of cointegration, independence, and comovements. The first testing

stage, based on a correlation measure for the jump series, is unable to find whether the series are integrated or not, but rejects the null hypothesis of independent random walks when there is a linear relationship between the series. The second testing stage, based on a ratio of counts, can solve most of the ambiguities of the former stage, and has a role similar to a unit-root testing device, with the advantage of not being bound to any particular model (in this case, a unit-root time series model).
We have shown that the proposed statistics behave similarly to standard measures such as the ADF and RADF test statistics.

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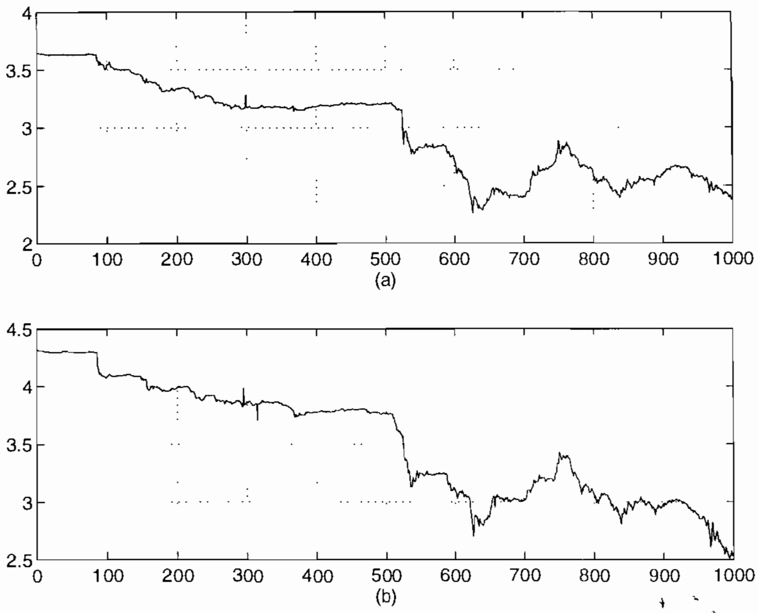

Figure 10: Two stock return series from the Japanese food company *Ajinomoto*.

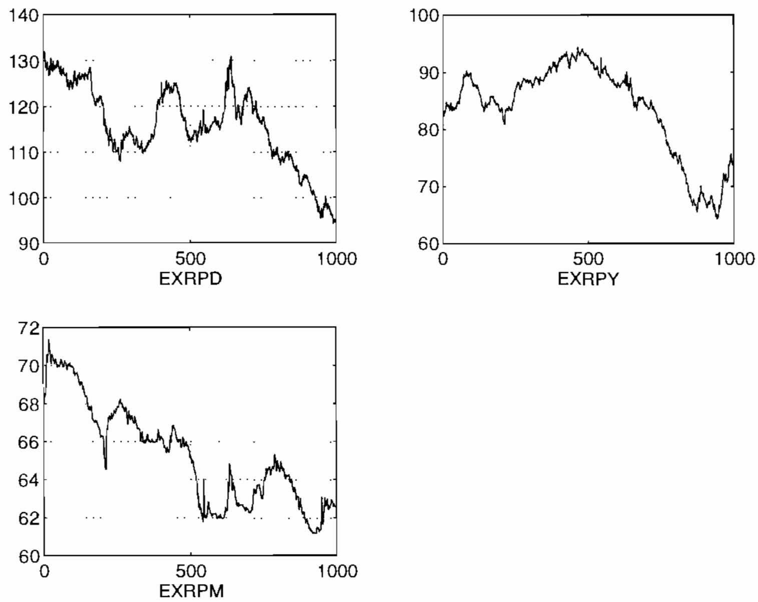

Figure 11: Daily foreign exchange rate series from January 1987: EXRPD (Peseta/US Dollar), EXRPY (Peseta/100 Yens), EXRPM (Peseta/Deutsch Mark).

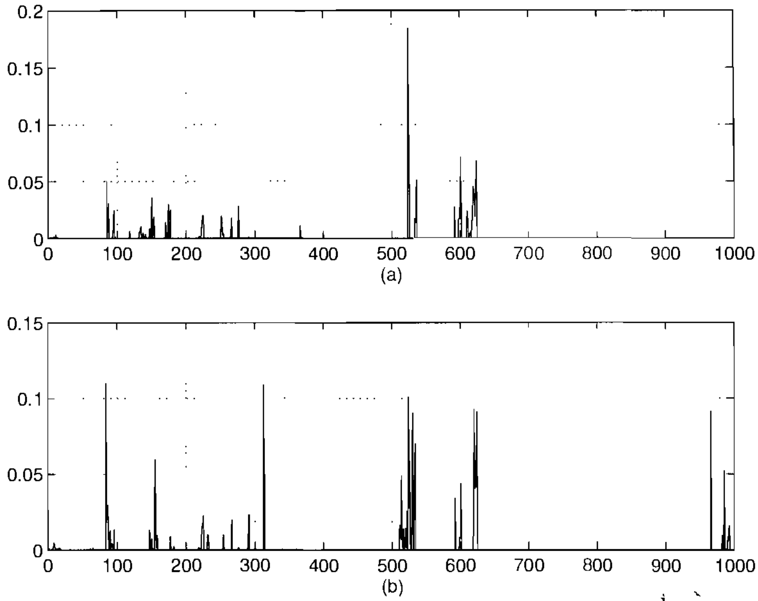

Figure 12: Jump series for the pair (STR1,STR2).

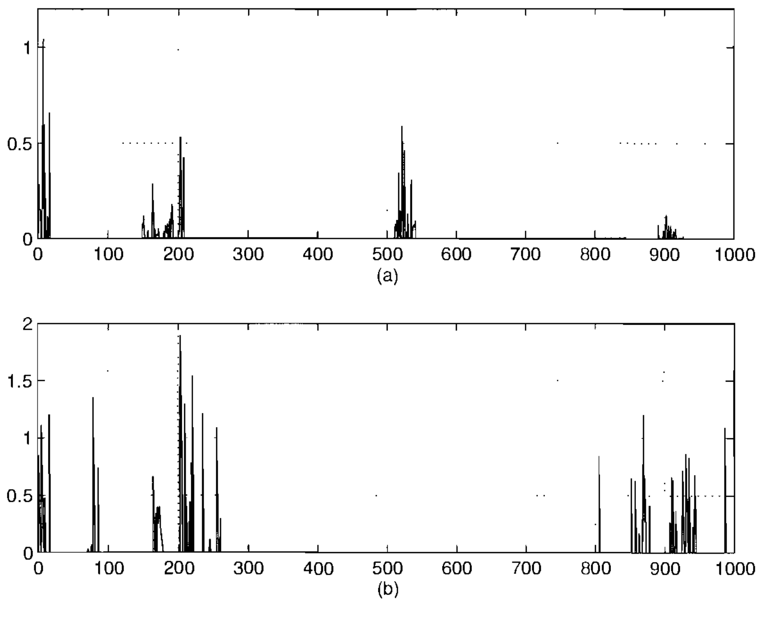

Figure 13: Jump series for the pair (EXRPM,EXRPD).

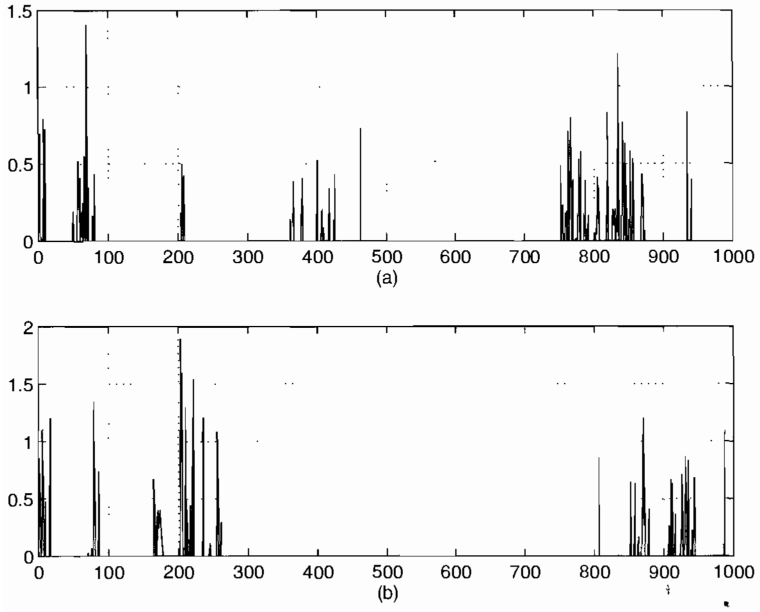

Figure 14: Jump series for the pair (EXRPY,EXRPD).

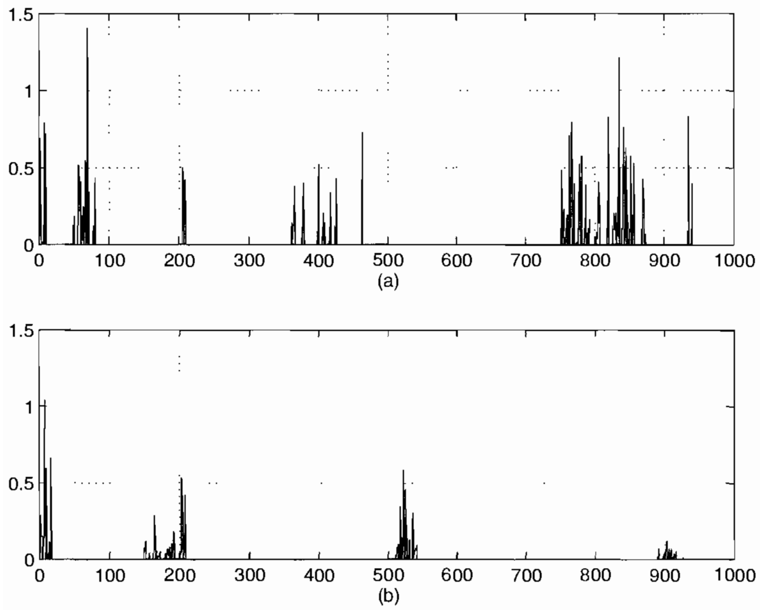

Figure 15: Jump series for the pair (EXRPY,EXRPM).