SEARCHING FOR FRACTIONAL EVIDENCE USING COMBINED UNIT ROOT TESTS

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Key Words

Fractionally integrated processes; *DF* test; *KPSS* test; finite sample analysis; inflation series.

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I. INTRODUCTION

Time series methodology has extensively examined the question of modeling economic series as the sum of a deterministic time trend plus a stochastic term. Within the class of the so-called *difference-stationary* (*DS*) or $I(1)$ models, the stochastic term follows a random walk, which typically implies that the mean and variance increase without bound over time, the precision of the forecast error becomes unbounded as the horizon extends and the effect of any random shock persists. On the other hand, in the so-called *trend-stationary* (*TS*) or $I(0)$ specification, the stochastic term follows a weakly stationary process and hence, the prediction error remains bounded even in the infinite horizon. Moreover, now shocks have only a transitory effect and the model exhibits trend reversion characteristics.

The issue of stochastic versus deterministic trend has considerable implications for our understanding of economic theories. For instance, it has been often argued that the presence or absence of a stochastic trend in the real output decides whether the real business cycle theory or the Keynesian theory should be accepted (see, however, comments by West, 1988). Time series research has not been insensitive to this debate. The seminal study of Nelson and Plosser (1982) which reported strong evidence of unit roots in U.S. historical annual time series led to much subsequent research with both empirical and theoretical dimensions.

At the statistical level, the hypothesis testing for *DS* against *TS* has been formulated in terms of the dominating characteristic root, treating $I(1)$ as the null hypothesis. This in turn has been made possible by the development of new asymptotic statistical theories on the unit root by Fuller (1976), Dickey and Fuller (1979, 1981), Said and Dickey (1984), Phillips (1987) and Phillips and Perron (1988) inter alia, referred to as the *standard unit root tests* along this paper. The usual conclusion that is drawn when these standard unit root tests are applied to the Nelson and Plosser data set is that most aggregate economic time series contain a unit root.

On the other hand, Kwiatkowski et al. (1992) observed that taking the null hypothesis to be $I(1)$ rather than $I(0)$, might itself have led to a bias in favor of the former hypothesis, so that an alternative explanation for the common failure to reject the unit root hypothesis would simply be that standard unit root tests are not very powerful against relevant alternatives. Hence, they proposed testing for *TS* against *DS* and provided a test, the so-called *KPSS* test, of the null hypothesis of stationarity against the $I(1)$ alternative. By proceeding in this way, they concluded that for many of the series of the Nelson and Plosser data set the hypothesis of *TS* could not be rejected.

Consequently, given that the results of the unit root tests are quite sensitive to the formulation of the null hypothesis, it has become a standard testing procedure for the practitioners to perform tests of both the null hypothesis of *DS* as well as tests of the null hypothesis of *TS*. By proceeding in this manner, the combined use of the standard unit root and the *KPSS* tests for a particular series gives rise to one of the following alternatives outcomes:

(*i*) Rejection by the standard unit root tests and failure to reject by the *KPSS* test provides evidence in favor of the *TS* null hypothesis, i.e., the series is $I(0)$.

(*ii*) Failure to reject by the standard unit root tests and rejection by the *KPSS* test supports the *DS* null hypothesis, i.e., the series is $I(1)$.

(*iii*) Failure to reject by both standard and *KPSS* tests shows that the data are not sufficiently informative to distinguish between both hypotheses, and

(*iv*) Rejection by both standard and *KPSS* tests suggests that the series is not well represented as either $I(1)$ or $I(0)$. Others possibilities should be considered.

As regards outcome (*iv*), one of the most explored alternatives in recent years has been to considere the possibility that the underlying series is fractionally integrated. As is well-known, a time series y_t is said to be *fractionally integrated of order d*, denoted $y_t \sim FI(d)$ if it becomes weakly stationary after differentiating d times, and the degree of differentiation or *memory parameter*, d, is a real number. These processes have received an increasing attention because of their ability to provide a natural and flexible characterization of the nonstationary and persistent characteristics of economic time series. See Baillie (1996) for a recent survey.

The aim of this paper is to prove in a rigorous way the empirical rule (*iv*) when the considered alternative is fractional integration, providing some useful modeling guides to practitioners.

For this, and after some preliminary theory included in Section 2, we show in Section 3 that under fractional alternatives the (upper-tailed) *KPSS* and the (lower-tailed) *DF* tests are consistent against fractional alternatives for all $d > 0$ and $d < 1$, respectively. In spite of this finding, however, it should be notice that such results are asymptotic and can differ in samples of finite size and, hence, in real applications. These claims are discussed in Section 4. Finally, conclusions are given in Section 5. Proofs are collected in a mathematical Appendix.

2. PRELIMINARY THEORY

We will say that the zero mean time series $\{\varepsilon_t\}_{t=1}^{\infty}$ is a *short memory* stochastic process if it satisfies (*i*) $T^{-1}E\left(\sum_{j=1}^{T}\varepsilon_j\right)^2 \to \sigma^2$ exists and is non zero and (*ii*) $\forall r \in [0,1]$,

$T^{-1/2}\sum_{j=1}^{[Tr]}\varepsilon_j \Rightarrow \sigma W(r)$, where $W(r)$ is a standard Brownian motion, where the symbol "$\Rightarrow$" denotes weak convergence.

Therefore, according to this definition, a short memory process need not be covariance stationary and some heterogeneity in the process is allowed. As is well-known, when ε_t is stationary the long-run variance σ^2 is proportional to the spectral density at zero frequency, which is required to be neither zero nor infinite. On the other hand, part (*ii*) of the previous definition is just a functional central limit theorem for convergence of partial sums to a Wiener process, where several sets of sufficient conditions for such invariance principle to hold can be found in the literature. See Lee and Schmidt (1996) for an instructive discussion on the suitability of this definition of a short memory process.

On the other hand, we will say that the stochastic process $\{y_t\}_{t=1}^{\infty}$ is a *fractionally integrated process of order d*, denoted $y_t \sim FI(d)$, if it has the representation

$$\Delta^d y_t = \varepsilon_t, \tag{1}$$

where now d is a real number called the *memory parameter* of the y_t series.

It can be proved (see, e.g. Baillie, 1996) that a $FI(d)$ process is stationary and invertible if and only if $d \in \left(-\frac{1}{2},\frac{1}{2}\right)$ and nonstationary if $d \geq \frac{1}{2}$. The memory parameter d can always be decomposed into the sum of an integer number, q, plus a real number $\delta \in \left(-\frac{1}{2},\frac{1}{2}\right)$. For instance, if $d = 1.3$, then $q = 1$, $\delta = .3$. If $d = 1$, then $q = 1$, $\delta = 0$ and if $d = .9$ then $q = 1, \delta = -.1$. Hence, a nonstationary fractionally integrated process (*NFI*) of order d, can always be reparameterized in a suitable manner as the sum of an integrated process of order q, $I(q)$, process plus a stationary fractionally integrated process (*SFI*) of order δ :

$$\Delta^d y_t = \varepsilon_t \Leftrightarrow \Delta^{q+\delta} y_t = \varepsilon_t \Leftrightarrow \Delta^q y_t = \Delta^{-\delta} \varepsilon_t \Leftrightarrow \Delta^q y_t = \Im_t, \Delta^\delta \Im_t = \varepsilon_t .$$

Throughout this paper we shall assume that the true data generating mechanism of the relevant y_t series is well represented by expression (1). With respect to the short memory term ε_t, we will proceed under the following assumption:

Assumption 1. $\varepsilon_t \sim iid(0, \sigma^2)$, *with* $E|\varepsilon_t|^r < \infty$ *for* $r \geq max\{4, -8\delta/(1+2\delta)\}$.

This assumption follows from Sowell (1990) and is slightly weaker than other assumptions made in the literature. For example, Lee and Schmidt (1996) assume that the ε_t are i.i.d. $N(0, \sigma^2)$ and Lo (1991) assumes normality and stationarity of ε_t.

Under the previous assumption, Sowell (1990) shows that

$$T^{-1-2\delta} \sigma^2_{\Im T} \xrightarrow{p} \frac{\sigma^2 \Gamma(1-2\delta)}{(1+2\delta)\Gamma(1+\delta)\Gamma(1-\delta)} \equiv \theta^2_\delta , \quad (2)$$

and

$$\sigma^{-1}_{\Im T} S_{[Tr]} \Rightarrow \frac{1}{\Gamma(1+\delta)} \int_0^r (r-s)^\delta dW(s) \equiv W_\delta(r), \quad (3)$$

where $\sigma^2_{\Im T} = \text{var}\left(\sum_{t=1}^T \Im_t\right)$ and $W_\delta(r)$ is a standard fractional Brownian motion as defined, e.g., by Beran (1994, p. 56).

3. DICKEY-FULLER AND *KPSS* TESTS

The most commonly used tests of the null hypothesis of a unit root in an observed time series are derivatives of the Dickey-Fuller (*DF*) tests. These tests are based on the regression of the observed series on its one-period lagged value, with the regression sometimes including an intercept and time trend. Thus, they are based on regressions of the form:

$$y_t = \rho y_{t-1} + \varepsilon_t, \tag{4}$$

$$y_t = \alpha + \rho y_{t-1} + \varepsilon_t, \tag{5}$$

$$y_t = \alpha + \rho y_{t-1} + \beta t + \varepsilon_t, \tag{6}$$

for $t = 1,2,\ldots,T$. In the presence of *NFI* processes, the natural analogs of regressions (4)-(6) are

$$y_t = \rho y_{t-1} + \Im_t, \tag{4'}$$

$$y_t = \alpha + \rho y_{t-1} + \Im_t, \tag{5'}$$

$$y_t = \alpha + \rho y_{t-1} + \beta t + \Im_t. \tag{6'}$$

Regression (4') was analyzed by Sowell (1990). He showed that under (1) and Assumption 1, the lower-tailed *DF* *t*-test diverges to $-\infty$ when $d \in \left(\frac{1}{2},1\right)$ being, therefore, a consistent test. Conversely, when $d \in \left(1,\frac{3}{2}\right)$, it diverges to $+\infty$ asymptotically, thus having zero power against fractional alternatives. Indeed, note that an upper-tailed *DF* test is consistent against $d \in \left(\frac{1}{2},\frac{3}{2}\right)$. On the other hand, regressions (5') and (6') have been recently studied by Haldrup and Marmol (1998) obtaining similar qualitative results. For the sake of completeness, we report below the following theorem proved by these authors concerning model (5').

Theorem 1. *Under Assumption* 1, *with* $y_t \sim NFI(d), d \in (0.5, 1.5)$ *and* $H_0: \alpha = 0, \rho = 1$, *the DF tests in regression* (5') *have the following asymptotic distributions*:

(*i*) *When* $d = 1$,

$$T(\hat{\rho}-1) \Rightarrow \frac{\frac{1}{2}\left[W^2(1)-1\right]-W(1)\int W}{\int W^2-\left(\int W\right)^2},$$

$$t_\rho \Rightarrow \frac{\frac{1}{2}\left[W^2(1)-1\right]-W(1)\int W}{\left[\int W^2-\left(\int W\right)^2\right]^{1/2}},$$

(ii) when $d \in \left(1, \frac{3}{2}\right)$, *i.e.,* $\delta \in \left(0, \frac{1}{2}\right)$,

$$T(\hat{\rho}-1) \Rightarrow \frac{\frac{1}{2}W_\delta^2(1)-W_\delta(1)\int W_\delta}{\int W_\delta^2-\left(\int W_\delta\right)^2},$$

$$T^{-\delta}t_\rho \Rightarrow \frac{\psi_\delta\left[\frac{1}{2}W_\delta^2(1)-W_\delta(1)\int W_\delta\right]}{\left[\int W_\delta^2-\left(\int W_\delta\right)^2\right]^{1/2}},$$

so that $t_\rho \xrightarrow{p} \infty$, *and*

(iii) when $d \in \left(\frac{1}{2}, 1\right)$, *i.e.,* $\delta \in \left(-\frac{1}{2}, 0\right)$,

$$T^{1+2\delta}(\hat{\rho}-1) \Rightarrow \frac{-1}{2\psi_\delta^2\left[\int W_\delta^2-\left(\int W_\delta\right)^2\right]},$$

$$T^{\delta}t_\rho \Rightarrow \frac{-1}{2\psi_\delta\left[\int W_\delta^2-\left(\int W_\delta\right)^2\right]^{1/2}},$$

so that $T(\hat{\rho}-1) \xrightarrow{p} -\infty$ *and* $t_\rho \xrightarrow{p} -\infty$, *where* $\psi_\delta^2 = \theta_\delta^2\left[E\left(\Im_t^2\right)\right]^{-1}$.

Consider now the limiting behavior of the *DF* tests against *SFI* alternatives, i.e., when $y_t = \Im_t \sim SFI(\delta)$, $\delta \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. For simplicity, we shall only consider the case where the observed series is regressed on its lagged value.

Theorem 2. *Under Assumption* 1, *with* $y_t \sim SFI(\delta)$, $\delta \in \left(-\frac{1}{2},\frac{1}{2}\right)$ *and* $H_0\!:\rho = 1$ *in regression* (4'), *then,*

$$\left(\hat{\rho}-1\right) \xrightarrow{p} \frac{2\delta-1}{1-\delta} < 0,$$

$$T^{-1/2} t_{\rho} \xrightarrow{p} -\left(1-2\delta\right)^{1/2} < 0,$$

so that $T\left(\hat{\rho}-1\right) \xrightarrow{p} -\infty$ *and* $t_{\rho} \xrightarrow{p} -\infty$.

On the other hand, Kwiatkowski et al. (1992) suggested switching from a unit root null to one of $I(0)$ or *TS*. More precisely, they proposed to test the hypothesis that deviations of a series from deterministic trend are short memory.

In this section we shall consider the following version of this test: Let ξ_t be the residuals from a regression of y_t on intercept and let S_t the partial sum process of the ξ_t so that $S_t = \sum_{j=1}^{t} \xi_t = \sum_{j=1}^{t} \left(y_t - \bar{y}\right)$, $t = 1,2,\ldots,T$. Hence, in this case, the so-called *KPSS* statistic for testing the null of stationarity can be expresses as the following LM statistic:

$$\hat{\eta}_{\mu} = \frac{T^{-2}\sum S_t^2}{s^2\left(\ell\right)}, \tag{7}$$

where $s^2(\ell)$ is the Newey-West estimator

$$s^2(\ell) = T^{-1}\sum_{t=1}^{T} \xi_t^2 + 2T^{-1}\sum_{\wp=1}^{\ell} \omega\left(\wp,\ell\right) \sum_{t=\wp+1}^{T} \xi_t \xi_{t-\wp}, \tag{8}$$

with the kernel weight $\omega\left(\wp,\ell\right)$ defined as

$$\omega\left(\wp,\ell\right) = 1 - \frac{\wp}{1+\ell}, \tag{9}$$

(Barlett spectral window) and with the bandwidth parameter ℓ being a function of the sample size satisfying that $\ell, T \to \infty$ but $\ell/T \to 0$. Qualitatively similar results would be obtained for the *KPSS* statistic $\hat{\eta}_\tau$, where now ξ_t denote the residuals from a regression of y_t on intercept and trend. See Lee and Amsler (1997) for further comments.

Under the alternative that y_t is an $I(1)$ process, Kwiatkowski et al. (1992) and Shin and Schmidt (1992) showed that $\hat{\eta}_\mu = O_p(T/\ell)$. Since $T/\ell \to \infty$ and the *KPSS* statistic is an upper-tailed test, it is consistent. In the same manner, Lee and Schmidt (1996) prove that the *KPSS* test is also consistent against *SFI* alternatives. Specifically, they show that if $y_t = \Im_t \sim SFI(\delta)$ then $T^{-2\delta}\hat{\eta}_\mu \Rightarrow \psi_\delta^2 \int_0^1 W_\delta^+(\varphi)d\varphi$ if $\ell = 0$ and $(\ell/T)^{2\delta}\hat{\eta}_\mu \Rightarrow \int_0^1 W_\delta^+(\varphi)d\varphi$ if $\ell \neq 0$, where $W_\delta^+(\varphi)$ is a standard fractional Brownian bridge, $W_\delta^+(\varphi) = W_\delta(\varphi) - \varphi W_\varphi(1)$, $\varphi \in [0,1]$. Consequently, for $\delta > 0$, $\hat{\eta}_\mu \xrightarrow{p} \infty$ and the *KPSS* test is consistent, but for $\delta < 0$, $\hat{\eta}_\mu \xrightarrow{p} 0$ and the *KPSS* test has zero power asymptotically. Indeed, as the authors note, a two-tailed test is consistent against $\delta > 0$ and against $\delta < 0$ for all value of the truncation parameter ℓ.

Consider now the asymptotic behavior of the *KPSS* test under the alternative that the series is a *NFI* process, i.e., against the alternative that $\Delta y_t = \Im_t \sim SFI(\delta)$.[1]

Theorem 3. *Suppose that* $\ell, T \to \infty$ *but* $\ell/T \to 0$. *Then, under Assumption* 1, *with* $y_t \sim NFI(d)$, $d \in (\frac{1}{2}, \frac{3}{2})$, *the KPSS test has the following asymptotic distributions:*

[1] An alternative proof of Theorem 3 for the case $d \in (1/2, 1)$ and ε_t a Gaussian white noise has been recently provided by Lee and Amsler (1997) in independent work.

$$T^{-1}\hat{\eta}_\mu \Rightarrow \frac{\int_0^1 \left[\int_0^\tau W_\delta^*(\varphi)\right]^2 d\tau}{\int_0^1 \left[W_\delta^*(\varphi)\right]^2 d\varphi}, \tag{10}$$

if $\ell = 0$, *and*

$$(\ell/T)\hat{\eta}_\mu \Rightarrow \frac{\int_0^1 \left[\int_0^\tau W_\delta^*(\varphi)\right]^2 d\tau}{\int_0^1 \left[W_\delta^*(\varphi)\right]^2 d\varphi}, \tag{11}$$

if $\ell \neq 0$, *where* $W_\delta^*(\varphi) = W_\delta(\varphi) - \int_0^1 W_\delta(v)dv$ *is a demeaned fractional Brownian motion. Hence,* $\hat{\eta}_\mu \xrightarrow{p} \infty$ *and the KPSS test is consistent against the class of NFI alternatives.*

Theorem 3 implies that the *KPSS* test has the same orders in probability for $d \in \left(\frac{1}{2}, \frac{3}{2}\right)$. In fact, notice that these orders of probability are independent of *d*, in contrast to the *SFI* case. On the other hand, the *KPSS* test with different critical values was suggested by Shin and Schmidt (1992) as the basis for a unit root test. In this sense, Lee and Amsler (1997) showed that the *KPSS* statistic cannot distinguish consistently between the $d \in \left(\frac{1}{2}, 1\right)$ *and* $d = 1$ cases. From our Theorem 3, however, we obtain that this statistic can distinguish consistently between *stationary and nonstationary long memory*.

Table 1 summarizes the asymptotic results obtained in Theorems 1-3, whereas their combined use gives rise to the set of possible outcomes collected in Table 2 according to alternatives (*i*)-(*iv*) in the introductory section. From these tables, it appears that we should perform two-sided *DF* and *KPSS* tests to avoid erroneous decisions. For instance, when the true series has memory parameter $d \in \left(1, \frac{3}{2}\right)$, then from the last row of Table 2 we have that the combined use of the standard *DF* and *KPSS* tests will lead asymptotically to the non rejection of the unit root hypothesis.

TABLE 1 ABOUT HERE

TABLE 2 ABOUT HERE

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4. FINITE SAMPLE REMARKS

Altogether, the results obtained in the preceding section are asymptotic and can be different in finite samples. More specifically, Sowell (1990) conjectured that the *DF* test might be severely misleading in all but very large sample. This is so because its distribution depends on two underlying random variables with a very slow rate of convergence to its limiting distribution for a very plausible range of d values, resulting in a finite sample similarity of the $I(1)$ distribution and the fractional unit root distribution is spite of their sharp asymptotic differences as presented in Theorem 1.

Sowell's conjecture was supported by Diebold and Rudebusch (1991) in Monte Carlo experiments. They showed that, for a fixed memory parameter value d, power increases monotonically with T, as expected, and that, for fixed sample size, power increases monotonically with the Euclidean distance $|d-1|$. Moreover, they also reported that, for fixed sample size, power is always asymmetric around the unit root null hypothesis and that the power of the $T(\hat{\rho}-1)$ and t_ρ tests is always approximately equal for $d<1$, whereas the $T(\hat{\rho}-1)$ test is less powerful than the t_ρ test for $d>1$.

In this sense, it is worth noting how Theorem 1 helps to explain their experimental findings. In effect, the power of the DF tests is asymmetric around the $d = 1$ null due to the fact that they have different limiting distributions whether we consider the alternative $d < 1$ or the alternative $d > 1$. Equally, the power of the DF tests is equal for $d < 1$ and different for $d > 1$ with the $T(\hat{\rho} - 1)$ test being less powerful than the t_ρ test because for $d < 1$ both tests diverge to $-\infty$ whereas for $d > 1$ the t_ρ test continues diverging to $+\infty$ but $T(\hat{\rho} - 1) = O_p(1)$.

Overall, their research leads to the conclusion that the power of the DF test against fractionally integrated alternatives is quite low. Moreover, Hassler and Wolters (1994) provide both analytical as well as Monte Carlo evidence that other standard unit root tests such as the Augmented Dickey-Fuller (*ADF*) or the Phillips-Perron (*PP*) tests also perform poorly when the alternative is fractionally integrated.

For instance, from their Table 1, page 4, we have that when $d = \frac{3}{4}$ the DF tests (lower-tailed performed with level 0.05) rejects the unit root hypothesis in about 50% if $T = 100$, whereas the ADF test rejects this null hypothesis in about 21% with 2 lags and only in about 6% if 12 lags are included in the augmented Dickey-Fuller regression. Likewise, for $d = 0.9$ the DF tests rejects the DS null in about 14% if $T = 100$ and in about 20% if $T = 200$. Added to that, and in agreement with our Theorem 2, with parameter values of *SFI* processes, the wrong null hypothesis is always rejected.

As regards the *KPSS* test, Lee and Schmidt (1996) provide some evidence on the power against fractional alternatives of the $\hat{\eta}_\mu$ test. On the one hand, they obtain power increases with the sample size, which is a reflection of the fact that $\hat{\eta}_\mu \xrightarrow{p} \infty$, i.e., of the consistency of the test. On the other hand, they also report that power is higher when the memory parameter d is larger, i.e., as the alternative hypothesis becomes farther from the null of *TS*. Finally, they conclude that power is lower when the lag truncation parameter ℓ is higher in accordance with the asymptotics of Theorem 3 which indicate that power depends on (ℓ/T) even asymptotically.

In finite samples, for $T = 100$ and $\ell = 4$, and accordingly to Lee and Schmidt (1996), the power of the $\hat{\eta}_\mu$ test (upper-tailed performed with level 0.05) rejects is around 83% for $d = 1$, 65% for $d = 0.7$, 48% for $d = \frac{1}{2}$, 27% for $d = 0.3$ and only around 10% for $d = 0.1$. Similarly, for $T = 250$ (and $\ell = 4$), the $\hat{\eta}_\mu$ test rejects the *TS* null the 95 percent of times for $d = 1$ and only in around 13% if $d = 0.1$. More experimental evidence in this direction has been recently also provided by Lee and Amsler (1997).

Summing up, it appears that the power of both the standard unit root and the *KPSS* tests against fractionally integrated alternatives is quite low except for rather large samples. This implies that, for small to moderate samples, the asymptotic results obtained at the end of Section 3 and collected in Table 2 should be modified in the following way: For values of d near to 1, the power of the customary *KPSS* test is high, rejecting the *TS* null, but the *DF* test has low power in this range, failing to reject the *DS* null. Consequently, the series would be classified as $I(1)$. Conversely, for values of d near to 0, the high power of the *DF* test in this case can be compensated with the low power of the *KPSS* in this range, leading to the conclusion that the underlying series is $I(0)$.

Moreover, and according to the above mentioned results, all those problems will be exacerbated either if we use other standard unit root tests such as the *ADF* test with moderate to large number of lags included in the Dickey-Fuller regression or if we increase the number of lags included in the Newey-West spectral estimator in the *KPSS* test.

5. A DISCUSSION OF EMPIRICAL EVIDENCE

Recently, Baillie et al. (1996) considered the application of fractionally integrated processes with conditionally heteroscedastic innovations to describing monthly CPI (Consumer Price Index) inflation from 1948 to 1990 for the G7 countries and also for three high-inflation economies (Argentina, Brazil and Israel). In their Table IV, page 30, they present the results of applying the *PP* and *KPSS* tests to the inflation series of these countries. They find that, for eight countries (except Germany and Japan) it is possible to reject both a unit root and stationarity, suggesting the possibility of fractional integration. For Germany and Japan, rejection by the *PP* test and failure to reject by the *KPSS* statistic is indicative of inflation being $I(0)$ in both countries.

In order to test for the possibility that the ten inflation series are fractionally integrated, Baillie et al. (1996) propose minimizing the conditional sum of squares (*CSS*) function of an $ARFIMA(0,d,1)\times(0,0,2)_{12}-GARCH(1,1)\sim Student\ t$ model. For fixed initial conditions, the *CSS* estimator is asymptotically equivalent to maximum likelihood estimation. See Chung and Baillie (1993) for further details. Using the *CSS* estimation procedure, Baillie et al. (Table VII, page 33) obtain the following estimates of d for the inflation series of the following ten countries (in parenthesis, the corresponding standard errors): Argentina, 0.598 (0.086); Brazil, 0.595 (0.061); Canada, 0.386 (0.083); France, 0.452 (0.058); Germany, 0.181 (0.051); Israel, 0.591 (0.080); Italy, 0.449 (0.056); Japan, 0.084 (0.056); U.K., 0.202 (0.048) and U.S.A., 0.472 (0.065).

Therefore, for Argentina, Brazil and Israel, the estimated value of d is approximately 0.59 so that the inflation series for these three countries is considered to be nonstationary with infinite variance but mean reverting, i.e., returning to its equilibrium or long-run behavior after any random shock. For the G7 low-inflation economies, the estimated d is less than $\frac{1}{2}$, implying that for these countries the inflation series is covariance stationary with long-memory properties, i.e., with autocorrelations decaying at the hyperbolic rate. Only for Japan can the hypothesis that $d = 0$ not be rejected. Hence, only the inflation series of Japan appears to be a covariance stationary process with short-memory properties, i.e., with autocorrelations that die out at an exponential rate.

All these results are consistent with the analysis made above in Section 3, except in the case of Germany. In effect, rejection by the *PP* statistic and failure to reject by the *KPSS* statistic suggest the possibility that the inflation series is $I(0)$ in this country. However, the estimated value of d for Germany is 0.18 which is significantly different from zero. This result, notwithstanding, is not surprising in view of the comments in Section 4: the power of the *KPSS* test is very low against local fractional alternatives. Indeed, Lee and Schmidt (1996) and Lee and Amsler (1997) show as an important practical conclusion that, in spite of the consistency of the customary *KPSS* test against fractional alternatives, it requires a rather large sample size, such as $T = 1000$, to distinguish a long-memory from a short-memory process with any reasonable degree of reliability.

6. CONCLUSIONS

In this paper we tried to answer from an analytical point of view the question of the suitability of the conventional *DF* and *KPSS* tests to detect that a series is best characterized by a *FI* process by rejecting the null hypotheses of $d = 0$ and $d = 1$, respectively. We proved that these tests are consistent against fractional alternatives but that, taken together, they (*asymptotically*) lead to correct conclusions for all $d > -\frac{1}{2}$ *only if* two-tailed tests are performed.

In finite samples, however, the power of these tests is very low against local *FI* alternatives except for rather large samples. That could lead to erroneous inferences, as we have illustrated with an empirical example.

Therefore, in spite of the consistency property, when working with small to moderate samples it appears necessary to explicitly investigate the possibility that the underlying time series be fractionally integrated.

TABLE 1

Asymptotic behavior of the *DF* and *KPSS* tests against *FI*(*d*) alternatives

Value of d	DF_t	DF_{2T}	$KPSS_u$	$KPSS_{2T}$
$-\frac{1}{2} < d < 0$	*Rejection* $H_0{:}I(1)$	*Rejection* $H_0{:}I(1)$	*No Rejection* $H_0{:}I(1)$	*Rejection* $H_0{:}I(0)$
$d = 0$	*Rejection* $H_0{:}I(1)$	*Rejection* $H_0{:}I(1)$	*Correct* $H_0{:}I(0)$	*Correct* $H_0{:}I(0)$
$0 < d < \frac{1}{2}$	*Rejection* $H_0{:}I(1)$	*Rejection* $H_0{:}I(1)$	*Rejection* $H_0{:}I(0)$	*Rejection* $H_0{:}I(0)$
$\frac{1}{2} < d < 1$	*Rejection* $H_0{:}I(1)$	*Rejection* $H_0{:}I(1)$	*Rejection* $H_0{:}I(0)$	*Rejection* $H_0{:}I(0)$
$d = 1$	*Correct* $H_0{:}I(1)$	*Correct* $H_0{:}I(1)$	*Rejection* $H_0{:}I(0)$	*Rejection* $H_0{:}I(0)$
$1 < d < \frac{3}{2}$	*No Rejection* $H_0{:}I(1)$	*Rejection* $H_0{:}I(1)$	*Rejection* $H_0{:}I(0)$	*Rejection* $H_0{:}I(0)$

DF_t: lower-tailed *DF* test. $KPSS_u$: upper-tailed *KPSS* test. DF_{2T}: two-tailed *DF* test. $KPSS_{2T}$: two-tailed *KPSS* test.

TABLE 2

Final decisions combining the *DF* and *KPSS* tests

Value of d	$DF_{\ell} + KPSS_u$	$DF_{\ell} + KPSS_{2T}$	$DF_{2T} + KPSS_u$	$DF_{2T} + KPSS_{2T}$
$-\frac{1}{2} < d < 0$	$y_t \sim I(0)$ *	$y_t \sim FI(d)$	$y_t \sim I(0)$ *	$y_t \sim FI(d)$
$d = 0$	$y_t \sim I(0)$	$y_t \sim I(0)$	$y_t \sim I(0)$	$y_t \sim I(0)$
$0 < d < \frac{1}{2}$	$y_t \sim FI(d)$	$y_t \sim FI(d)$	$y_t \sim FI(d)$	$y_t \sim FI(d)$
$\frac{1}{2} < d < 1$	$y_t \sim FI(d)$	$y_t \sim FI(d)$	$y_t \sim FI(d)$	$y_t \sim FI(d)$
$d = 1$	$y_t \sim I(1)$	$y_t \sim I(1)$	$y_t \sim I(1)$	$y_t \sim I(1)$
$1 < d < \frac{3}{2}$	$y_t \sim I(1)$ *	$y_t \sim I(1)$ *	$y_t \sim FI(d)$	$y_t \sim FI(d)$

Possibilities: (*i*) Rejection by the *DF* test and failure to reject by the *KPSS* test: $y_t \sim I(0)$. (*ii*) Failure to reject by the *DF* test and rejection by the *KPSS* test: $y_t \sim I(1)$. (*iii*) Failure to reject by both tests: the data are not sufficiently informative. (*iv*) Rejection by both tests: $y_t \sim FI(d)$ (among other alternatives).

*: Erroneous decisions.

MATHEMATICAL APPENDIX

PROOF OF THEOREM 2. Given that $y_t = \Im_t \sim SFI(\delta)$, $\delta \in \left(-\frac{1}{2},\frac{1}{2}\right)$, then the manipulation of regression (4') yields

$$\hat{\rho} = \frac{\sum_{t=1}^{T} y_t y_{t-1}}{\sum_{t=1}^{T} y_{t-1}^2} = \frac{\sum_{t=1}^{T} \Im_t \Im_{t-1}}{\sum_{t=1}^{T} \Im_{t-1}^2} = \frac{T^{-1}\sum_{t=1}^{T} \Im_t \Im_{t-1}}{T^{-1}\sum_{t=1}^{T} \Im_{t-1}^2} \xrightarrow{p} \frac{E(\Im_t \Im_{t-1})}{E(\Im_t^2)},$$

where the weak consistency result follows from the fact that $\Im_t$ is a stationary and ergodic process.

Therefore, given that

$$E(\Im_t \Im_{t-j}) = \sigma^2 \frac{\Gamma(j+\delta)\Gamma(1-2\delta)}{\Gamma(j+1-\delta)\Gamma(1-\delta)\Gamma(\delta)}, j \geq 0,$$

(see, e.g., Baillie, 1996, Table 2, page 19) and the well-know recursive identity $\Gamma(1+z) = z\Gamma(z)$, it follows that

$$\hat{\rho} \xrightarrow{p} \frac{E(\Im_t \Im_{t-1})}{E(\Im_t^2)} = \frac{\delta}{1-\delta},$$

implying that

$$(\hat{\rho}-1) \xrightarrow{p} \frac{2\delta-1}{1-\delta} = \rho_\delta, \text{ say,}$$

which, in turn, given that $\delta \in \left(-\frac{1}{2},\frac{1}{2}\right)$, entails that $\rho_\delta \in (-1.33, 0)$. Consequently, $T(\hat{\rho}-1) \xrightarrow{p} -\infty$ as claimed.

With respect to the t-test,

$$t_\rho = \frac{(\hat{\rho}-1)}{\hat{\sigma}_\rho},$$

where $\hat{\sigma}_\rho^2 = \hat{\sigma}^2\left(\sum_{t=1}^T y_{t-1}^2\right)^{-1}$ and $\hat{\sigma}^2 = T^{-1}\sum_{t=1}^T\left(y_t - \hat{\rho}y_{t-1}\right)^2$, it is straightforward to prove that

$$\hat{\sigma}^2 = T^{-1}\sum_{t=1}^{T}\left(y_t - \hat{\rho}y_{t-1}\right)^2 = T^{-1}\sum_{t=1}^{T}\left(\Im_t - \hat{\rho}\Im_{t-1}\right)^2 \xrightarrow{p} \frac{E^2\left(\Im_t^2\right) - E^2\left(\Im_t\Im_{t-1}\right)}{E\left(\Im_t^2\right)}$$

and

$$T\hat{\sigma}_\rho^2 = \frac{\hat{\sigma}^2}{T^{-1}\sum_{t=1}^{T} y_{t-1}^2} \xrightarrow{p} 1 - \left[\frac{E\left(\Im_t\Im_{t-1}\right)}{E\left(\Im_t^2\right)}\right]^2 = \frac{\left(1-2\delta\right)}{\left(1-\delta\right)^2},$$

entailing

$$T^{-1/2}t_\rho = \frac{\left(\hat{\rho}-1\right)}{T^{1/2}\hat{\sigma}_\rho} \xrightarrow{p} -\left(1-2\delta\right)^{1/2} = t_\delta \text{, say.}$$

Finally, since $t_\delta \in \left(-1.41, 0\right)$, it follows that $t_\rho \xrightarrow{p} -\infty$.■

PROOF OF THEOREM 3. As regards the numerator of the $\hat{\eta}_\mu$ test, notice that

$$S_{[T\tau]} = \sum_{t=1}^{[T\tau]}\xi_t = \sum_{t=1}^{[T\tau]} y_t - [T\tau]\bar{y} = \sum_{t=1}^{[T\tau]} y_t - [T\tau]T^{-1}\sum_{t=1}^{T} y_t .$$

Hence, using expressions (2) and (3) in the main text and the continuous mapping theorem (*CMT*) yields

$$T^{-3/2-\delta}S_{[T\tau]} \Rightarrow \theta_\delta\int_0^\tau W_\delta\left(\varphi\right)d\varphi - \tau\theta_\delta\int_0^1 W_\delta\left(\varphi\right)d\varphi ,$$

so that

$$T^{-4-2\delta}\sum_{t=1}^{T} S_t^2 = T^{-1}\sum_{t=1}^{T}\left(T^{-3/2-\delta}S_t\right)^2 \Rightarrow \theta_\delta^2\int_0^1\left[\int_0^\tau W_\delta^*\left(\varphi\right)d\varphi\right]^2 d\tau . \qquad \text{(A1)}$$

where $W_\delta^*\left(\varphi\right)$ is a demeaned fractional Brownian motion and $\varphi, \tau \in \left[0,1\right]$.

With respect to the denominator, assume first that $\ell = 0$, so that

$$s^2(0) = T^{-1}\sum_{t=1}^{T}\xi_t^2 + 2T^{-1}\sum_{\wp=1}^{\ell}\omega(\wp,0)\sum_{t=\wp+1}^{T}\xi_t\xi_{t-\wp} = T^{-1}\sum_{t=1}^{T}\xi_t^2$$

$$= T^{-1}\sum_{t=1}^{T}(y_t - \bar{y})^2 = T^{-1}\sum_{t=1}^{T}y_t^2 - \bar{y}^2 .$$

Now, since

$$T^{-1/2-\delta}\bar{y} = \left(T^{-1/2-\delta}\sigma_{\Im T}\right)T^{-1}\sum_{t=1}^{T}\left(\sigma_{\Im T}^{-1}S_t\right) \Rightarrow \theta_\delta\int_0^1 W_\delta(\varphi)d\varphi$$

and

$$T^{-2-2\delta}\sum_{t=1}^{T}y_t^2 = \left(T^{-1-2\delta}\sigma_{\Im T}^2\right)T^{-1}\sum_{t=1}^{T}\left(\sigma_{\Im T}^{-1}S_t\right)^2 \Rightarrow \theta_\delta^2\int_0^1 W_\delta^2(\varphi)d\varphi ,$$

it follows that

$$T^{-1-2\delta}s^2(0) = T^{-2-2\delta}\sum_{t=1}^{T}y_t^2 - \left(T^{-1/2-\delta}\bar{y}\right)^2 \Rightarrow \theta_\delta^2\left[\int_0^1 W_\delta^2(\varphi)d\varphi - \left(\int_0^1 W_\delta(\varphi)d\varphi\right)^2\right]$$

$$= \theta_\delta^2\int_0^1\left[W_\delta^*(\varphi)\right]^2 d\varphi . \qquad \text{(A2)}$$

Consequently, from (A1), (A2) and the *CMT* we obtain

$$T^{-1}\hat{\eta}_\mu = \frac{T^{-2-2\delta}T^{-2}\sum S_t^2}{T^{-1-2\delta}s^2(0)} \Rightarrow \frac{\int_0^1\left[\int_0^\tau W_\delta^*(\varphi)\right]^2 d\tau}{\int_0^1\left[W_\delta^*(\varphi)\right]^2 d\varphi} .$$

Consider now the case where $\ell \neq 0$. For this, assume first that the lag truncation parameter ℓ is fixed and denote the sample cross moments of the residuals as

$$\sum_{t=\wp+1}^{T}\xi_t\xi_{t-\wp} = \sum_{t=\wp+1}^{T}(y_t - \bar{y})(y_{t-\wp} - \bar{y}) = \sum_{t=\wp+1}^{T}y_t^*y_{t-\wp}^* ,$$

say. In this case, given that $y_t = y_{t-\wp} + \sum_{j=0}^{\wp-1}\Im_{t-j}$, it follows

$$\sum_{t=\wp+1}^{T} y_t^* y_{t-\wp}^* = \sum_{t=\wp+1}^{T} \left(y_{t-\wp}^*\right)^2 + \sum_{t=\wp+1}^{T} y_{t-\wp}^* \left(\sum_{j=0}^{\wp-1} \Im_{t-j}\right).$$

Moreover, since

$$\sum_{t=\wp+1}^{T} \left(y_{t-\wp}^*\right)^2 = \sum_{t=1}^{T-\wp} \left(y_t^*\right)^2,$$

then

$$T^{-2-2\delta} \sum_{t=\wp+1}^{T} \left(y_{t-\wp}^*\right)^2 = T^{-2-2\delta} \sum_{t=1}^{T-\wp} \left(y_t^*\right)^2 \Rightarrow \theta_\delta^2 \int_0^\lambda \left[W_\delta^*(\varphi)\right]^2 d\varphi \to \theta_\delta^2 \int_0^1 \left[W_\delta^*(\varphi)\right]^2 d\varphi$$

where $\lambda = (T-\wp)/T \to 1$ provided that $\wp/T \to 0$.

On the other hand, notice that

$$\sum_{t=\wp+1}^{T} y_{t-\wp}^* \left(\sum_{j=0}^{\wp-1} \Im_{t-j}\right) = \sum_{t=\wp+1}^{T} y_{t-\wp}^* \Im_t + \sum_{t=\wp+1}^{T} y_{t-\wp}^* \Im_{t-1} + \cdots + \sum_{t=\wp+1}^{T} y_{t-\wp}^* \Im_{t-\wp+1},$$

$$= \sum_{t=\wp+1}^{T} y_{t-\wp}^* \left(\sum_{j=0}^{\wp-1} \Im_{t-j}\right) = \sum_{i=1}^{\wp} \nabla_{T,\wp}(i),$$

where $\nabla_{T,\wp}(i) = \sum_{t=\wp+1}^{T} y_{t-\wp}^* \Im_{t-\wp+i}$.

With regard the $\nabla_{T,\wp}(i) = \sum_{t=\wp+1}^{T} y_{t-\wp}^* \Im_{t-\wp+i}$ terms, notice that

$$\nabla_{T,\wp}(i) = \sum_{t=\wp+1}^{T} y_{t-\wp}^* \Im_{t-\wp+1} = \sum_{t=1}^{T-\wp} y_t^* \Im_{t+i}, \quad i = 1, \ldots, \wp,$$

and hence, when $i=1$, from the identity $\left(y_{t+1}^*\right)^2 = \left(y_t^*\right)^2 + \Im_{t+1}^2 + 2y_t^* \Im_{t+1}$, we obtain

$$\nabla_{T,\wp}(1) = \sum_{t=1}^{T-\wp} y_t^* \Im_{t+1} = \frac{1}{2} \sum_{t=1}^{T-\wp} \left[\left(y_{t+1}^*\right)^2 - \left(y_t^*\right)^2\right] - \frac{1}{2} \sum_{t=1}^{T-\wp} \Im_{t+1}^2 = -\frac{1}{2}\left(y_1^*\right)^2 - \frac{1}{2} \sum_{t=1}^{T-\wp} \Im_{t+1}^2.$$

In the same manner,

$$\nabla_{T,\wp}(2) = -\frac{1}{2}\left(y_2^*\right)^2 - \frac{1}{2} \sum_{t=1}^{T-\wp} \Im_{t+2}^2 - \sum_{t=1}^{T-\wp} \Im_{t+1} \Im_{t+2},$$

$$\nabla_{T,\wp}(3) = -\frac{1}{2}\left(y_3^*\right)^2 - \frac{1}{2}\sum_{t=1}^{T-\wp}\Im_{t+3}^2 - \sum_{t=1}^{T-\wp}\Im_{t+1}\Im_{t+3} - \sum_{t=1}^{T-\wp}\Im_{t+2}\Im_{t+3},$$

$$\vdots$$

$$\nabla_{T,\wp}(\wp) = -\frac{1}{2}\left(y_\wp^*\right)^2 - \frac{1}{2}\sum_{t=1}^{T-\wp}\Im_{t+\wp}^2 - \sum_{j=1}^{\wp-1}\sum_{t=1}^{T-\wp}\Im_{t+j}\Im_{t+\wp},$$

meaning that

$$\sum_{t=\wp+1}^{T} y_{t-\wp}^*\left(\sum_{j=0}^{\wp-1}\Im_{t-j}\right) = \sum_{i=1}^{\wp}\left\{-\frac{1}{2}\left(y_i^*\right)^2 - \frac{1}{2}\sum_{t=1}^{T-\wp}\Im_{t+i}^2 - \sum_{j=1}^{i-1}\sum_{t=1}^{T-\wp}\Im_{t+j}\Im_{t+i}\right\}$$

$$= -\frac{1}{2}\sum_{i=1}^{\wp}\left(y_i^*\right)^2 - \frac{1}{2}\sum_{i=1}^{\wp}\sum_{t=1}^{T-\wp}\Im_{t+i}^2 - \sum_{i=1}^{\wp}\sum_{j=1}^{i-1}\sum_{t=1}^{T-\wp}\Im_{t+j}\Im_{t+i}.$$

Moreover, since

$$\sum_{i=1}^{\wp}\left(y_i^*\right)^2 = \sum_{i=1}^{\wp}\left(y_i - \bar{y}\right)^2 = \sum_{i=1}^{\wp}\left(\sum_{j=1}^{i}\Im_j\right)^2 + \wp\bar{y}^2 - 2\bar{y}\left(\sum_{i=1}^{\wp}S_i\right),$$

then $\sum_{i=1}^{\wp}\left(y_i^*\right)^2 = O_p\left(T^{1+2\delta}\right)$. Consequently, given that

$$\frac{1}{2}\sum_{i=1}^{\wp}\sum_{t=1}^{T-\wp}\Im_{t+i}^2 - \sum_{i=1}^{\wp}\sum_{j=1}^{i-1}\sum_{t=1}^{T-\wp}\Im_{t+j}\Im_{t+i} = O_p(T),$$

it can be deduced that

$$\sum_{t=\wp+1}^{T} y_{t-\wp}^*\left(\sum_{j=0}^{\wp-1}\Im_{t-j}\right) = \begin{cases} O_p(T) & if\ \delta \le 0 \\ O_p\left(T^{1+2\delta}\right) & if\ \delta > 0. \end{cases}$$

Therefore,

$$T^{-2-2\delta}\sum_{t=\wp+1}^{T}\xi_t\xi_{t-\wp} = T^{-2-2\delta}\sum_{t=\wp+1}^{T}\left(y_{t-\wp}^*\right)^2 + o_p(1) \Rightarrow \theta_\delta^2\int_0^1\left[W_\delta^*(\varphi)\right]^2 d\varphi.$$

entailing

$$T^{-1-2\delta}s^2(\ell) = T^{-2-2\delta}\sum_{t=1}^{T}\xi_t^2 + 2\sum_{\wp=1}^{\ell}\omega(\wp,\ell)T^{-2-2\delta}\sum_{t=\wp+1}^{T}\xi_t\xi_{t-\wp}$$

$$\Rightarrow \theta_\delta^2 \int_0^1 \left[W_\delta^*(\varphi)\right]^2 d\varphi \left\{1 + 2\sum_{\wp=1}^{\ell} \omega(\wp,\ell)\right\}.$$

Now, using the properties of the Barlett spectral window, letting $\ell \to \infty$, it follows that

$$\ell^{-1}\left[T^{-1-2\delta} s^2(\ell)\right] = \theta_\delta^2 \int_0^1 \left[W_\delta^*(\varphi)\right]^2 d\varphi \left\{\ell^{-1} + 2\ell^{-1}\sum_{\wp=1}^{\ell} \omega(\wp,\ell)\right\}$$

$$= \theta_\delta^2 \int_0^1 \left[W_\delta^*(\varphi)\right]^2 d\varphi \left\{\ell^{-1} + \ell^{-1}\sum_{\wp=-\ell}^{\ell} \omega(\wp,\ell)\right\}$$

$$\to \theta_\delta^2 \int_0^1 \left[W_\delta^*(\varphi)\right]^2 d\varphi \left\{\int_{-1}^{1} (1-|\wp|) d\wp\right\} = \theta_\delta^2 \int_0^1 \left[W_\delta^*(\varphi)\right]^2 d\varphi .$$

Finally, collecting all the above results, yields

$$\left(\frac{\ell}{T}\right)\hat{\eta}_\mu = \frac{T^{-2-2\delta}T^{-2}\sum S_t^2}{\ell^{-1}T^{-1-2\delta}s^2(\ell)} \Rightarrow \frac{\int_0^1 \left[\int_0^\tau W_\delta^*(\varphi)\right]^2 d\tau}{\int_0^1 \left[W_\delta^*(\varphi)\right]^2 d\varphi},$$

which completes the proof of the theorem. ■

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