UNIVERSIDAD CARLOS III DE MADRID working papers

Working Paper 10-18
Statistics and Econometrics Series 07
April 2010

Departamento de Estadística
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 91 624-98-49

THE DECREASING PERCENTILE RESIDUAL LIFE AGING NOTION

Alba M. Franco-Pereira*, Rosa E. Lillo, and Moshe Shaked

Abstract

Earlier researchers have studied some aspects of the classes of distribution functions with decreasing α-percentile residual life (DPRL(α)), $0< \alpha <1$. The purpose of this paper is to note some further properties of these classes, and to initiate a theory of nonparametric statistical estimation of decreasing α-percentile residual life functions. Specifically, the close relationship between the DPRL(α) and the IFR (increasing failure rate) aging notions is studied. Other close relationships, between the DPRL(α) aging notions and the percentile residual life stochastic orders, are described, and further properties of the above classes of distributions are derived. Finally, we introduce an estimator of the percentile residual life function, under the condition that it decreases, and we prove its strongly uniform consistency.

Keywords: Reliability theory, hazard rate, stochastic orders, aging notions, nonparametric estimation, strongly uniform consistency.

* Universidad Carlos III de Madrid, Departament de Statistics, Facultad de Ciencias Sociales y Jurídicas, C/ Madrid 126, 28903 Getafe (Madrid), e-mail: alba.franco@uc3m.es (Alba M. Franco-Pereira).

1 Introduction

Statisticians find it useful to categorize life distributions according to different aging properties. These categories of distributions are useful for modeling situations where items deteriorate with age.

A common approach is to stipulate the decreasingness of the mean residual life function (or of the harmonic mean residual life function) as a model of aging. This approach, however, sometimes has some weaknesses that may prevent its use. For example, the mean residual life function may not exist. Even when it exists it may have some practical shortcomings, especially in situations where the data are censored, or when the underlying distribution is skewed or heavy-tailed. In such cases, either the empirical mean residual life function cannot be calculated, or a single long-term survivor can have a marked effect upon it which will tend to be unstable due to its strong dependence on very long durations. Also, in an experiment it is often impossible or impractical to wait until all items have failed.

An alternative to the mean residual life function is the α-percentile residual life function $q_{X,\alpha}$, where α is some number between 0 and 1. This function is defined as the α-percentile of the residual life at time t. A formal definition of $q_{X,\alpha}(t)$ will be given in Section 2, but here we note that such a function describes, for example, the value that will be survived, by $(1-\alpha)\%$ of items (in reliability theory) or of individuals (in biology), among those that survived up to time t. Questions such as "what proportion of 3-month old cubs will reach the age of two years?" or "what percentage of individuals who have been tumor-free for 3 years will stay tumor-free for 2 more years?" can be answered in terms of the α-percentile residual life function. The α-percentile residual life functions were studied in some detail by Schmittlein and Morrison (1981), Arnold and Brockett (1983), Gupta and Langford (1984), and Joe (1985), and more recently in Lin (2009). Barabás, Csörgö, Horváth, and Yandell (1986), Csörgö and Csörgö (1987), Chung (1989), Feng and Kulasekera (1991), and Csörgö and Viharos (1992) discussed various estimation procedures of the α-percentile residual life function. Raja Rao, Alhumoud, and Damaraju (2006) identified families of distributions for which simple expressions, for the α-percentile residual life functions, can be obtained.

Haines and Singpurwalla (1974) and Joe and Proschan (1984a) studied some aspects of the classes of distribution functions with decreasing α-percentile residual life (DPRL(α)), $0 < \alpha < 1$; the formal definition of these classes will be given in Section 2. The purpose of this paper is to note some further properties of these classes, and to initiate a theory of nonparametric statistical estimation of decreasing α-percentile residual life functions. The close relationship between the DPRL(α) and the IFR (increasing failure rate) aging notions is studied in Section 3, and it is also touched upon in Section 4. Other close relationships, between the DPRL(α) aging notions and the percentile residual life stochastic orders, are described in Section 4. Further properties of the above classes of distributions are derived in Section 5. Finally, in Section 6, we introduce an estimator of the percentile residual life function, under the condition that it decreases, and we prove its strongly uniform consistency. We derive the new properties of these classes by employing recent results involving the α-percentile residual life orders that were obtained in Franco-Pereira, Lillo, Romo, and Shaked (2010).

The following conventions are used in this paper. By "increasing" and "decreasing" we mean "nondecreasing" and "nonincreasing", respectively. For any distribution function F we let the function F^{-1} be the left continuous version of the inverse of F, that is,

$$F^{-1}(p) = \inf\{x : F(x) \geq p\}, \quad p \in (0, 1).$$

For the corresponding survival function $\overline{F} \equiv 1 - F$ we let the function $\overline{F}^{-1}$ be defined as $\overline{F}^{-1}(p) = F^{-1}(1-p)$, $p \in (0, 1)$; this, in fact, gives the right continuous version of the inverse of $\overline{F}$.

2 Definitions and basic properties

Let X be a random variable, and let u_X be the right endpoint of its support. For any $t < u_X$, the *residual life* at time t, that is associated with X, is any random variable that has the conditional distribution of $X - t$ given that $X > t$. We denote it by

$$X_t = [X - t \big| X > t], \quad t < u_X. \tag{2.1}$$

If F_X denotes the distribution function of X, and $\overline{F}_X = 1 - F_X$ denotes the corresponding survival function, then the survival function of X_t is given by

$$\overline{F}_{X_t}(x) = \frac{\overline{F}_X(t + x)}{\overline{F}_X(t)}, \quad x \geq 0. \tag{2.2}$$

The residual life is of interest in many areas of applied probability and statistics such as actuarial studies, biometry, survivorship analysis, and reliability — see, for example, Lillo (2005) for a list of references.

The α-percentile residual life function $q_{X,\alpha}$, where α is some number between 0 and 1, is defined by

$$q_{X,\alpha}(t) = \begin{cases} F_{X_t}^{-1}(\alpha), & t < u_X; \\ 0, & t \geq u_X. \end{cases}$$

It is useful to note that $q_{X,\alpha}$ satisfies

$$\overline{F}_X(t + q_{X,\alpha}(t)) = \overline{\alpha}\overline{F}_X(t) \quad \text{for all } t, \tag{2.3}$$

where $\overline{\alpha} = 1 - \alpha$. Also,

$$q_{X,\alpha}(t) = \overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t)) - t, \quad t < u_X. \tag{2.4}$$

or, alternatively,

$$q_{X,\alpha}(t) = F_X^{-1}(\alpha + (1 - \alpha)F_X(t)) - t, \quad t < u_X. \tag{2.5}$$

Let $0 < \alpha < 1$. A random variable X is said to have (or to be) DPRL(α) if $q_{X,\alpha}(t)$ is decreasing in t. It is also possible to similarly define the notion of increasing α-percentile residual life (IPRL(α)). However, note that with our definition of $q_{X,\alpha}$, in order for a random variable to be IPRL(α) it is necessary that $u_X = \infty$.

Some useful equivalent conditions for the DPRL(α) notion are given in the following proposition for absolutely continuous random variables with interval support (which may be finite or infinite). For such random variable X we denote by f_X its density function, and by $r_X \equiv f_X/\overline{F}_X$ its hazard rate function.

Proposition 2.1. *Let X be an absolutely continuous random variable with interval support (l_X, u_X). The following conditions are equivalent:*

(i) *X is* DPRL(α)*;*

(ii) *$\overline{\alpha} f_X(t) \le f_X(\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t)))$ for all $t \in (l_X, u_X)$;*

(iii) *$\overline{\alpha} f_X(\overline{F}_X^{-1}(p)) \le f_X(\overline{F}_X^{-1}(\overline{\alpha}p))$ for all $p \in (0,1)$;*

(iv) *$r_X(t) \le r_X(t + q_{X,\alpha}(t))$ for all $t \in (l_X, u_X)$.*

Proof. Assume (i). Then $q_{X,\alpha}(t)$ is decreasing in $t \in (l_X, u_X)$. Therefore, by differentiating (2.4) we see that

$$0 \ge \frac{d}{dt} q_{X,\alpha}(t) = \frac{\overline{\alpha} f_X(t)}{f_X(\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t)))} - 1,$$

and (ii) follows. In fact, the proof shows that (i)$\Longleftrightarrow$(ii).

Next assume (ii). Putting there $t = \overline{F}_X^{-1}(p)$ we obtain (iii). In fact, the proof shows that (ii)$\Longleftrightarrow$(iii).

Finally, assume (ii) again. For $t \in (l_X, u_X)$ divide the left hand side of the inequality in (ii) by $\overline{\alpha}\overline{F}_X(t)$, and divide the right hand side of the inequality in (ii) by $\overline{F}_X(t + q_{X,\alpha}(t))$, which are equal by (2.3). We obtain

$$r_X(t) \le \frac{f_X(\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t)))}{\overline{F}_X(t + q_{X,\alpha}(t))} = \frac{f_X(t + q_{X,\alpha}(t))}{\overline{F}_X(t + q_{X,\alpha}(t))},$$

where the last equality follows from (2.4). This gives (iv). In fact, the proof shows that (ii)$\Longleftrightarrow$(iv). $\square$

The equivalence (i)$\Longleftrightarrow$(iv) can be found already in Haines and Singpurwalla (1974) and in Joe and Proschan (1984a).

3 Relations between the DPRL and IFR aging notions

From Proposition 2.1(iv) it is seen that if r_X is increasing (that is, if X has an increasing failure rate (IFR)) then X is DPRL(α) for any $\alpha \in (0,1)$. On the other hand, if X is DPRL(α) for some $\alpha \in (0,1)$ then it is not necessary that X be IFR. In fact, the following example shows that, given any $\varepsilon > 0$, then, even if X is DPRL(α) for every $\alpha \ge \varepsilon$, it is not necessary that X is IFR. A related result, with a more positive flavor, will be given later in Proposition 4.2.

Example 3.1. Fix an $\varepsilon \in (0,1)$ and denote

$$a = (-\log \varepsilon)^{1/2}.$$

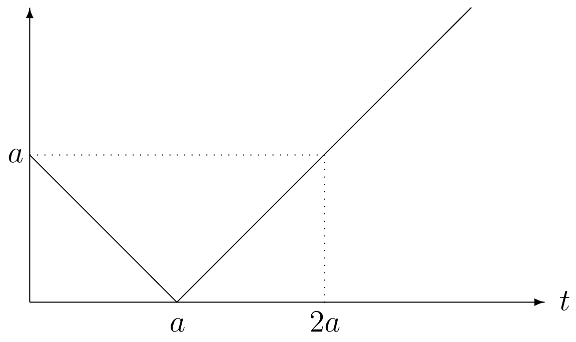

Figure 1: The hazard rate function r_X in Example 3.1

Consider a random variable X with the hazard rate function (see Figure 1)

$$r_X(t) = \begin{cases} a - t, & 0 \leq t \leq a; \\ t - a, & t > a. \end{cases}$$

A straightforward computation yields the survival function of X:

$$\overline{F}_X(t) = \begin{cases} 1, & t \leq 0; \\ \exp\{-at + \frac{t^2}{2}\}, & 0 < t \leq a; \\ \exp\{-\frac{a^2}{2} - \frac{(t-a)^2}{2}\}, & t > a. \end{cases}$$

Note that

$$\overline{F}_X(2a) = \exp\{-a^2\} = \varepsilon.$$

From a Remark in page 672 of Joe and Proschan (1984a) it follows that X is DPRL(α) for every $\alpha \geq \varepsilon$. However, obviously, X is not IFR. ◀

From Example 3.1 it is seen that, given any $\alpha \in (0, 1)$, it is possible to find a random variable that is DPRL(α), but that is not DPRL(β) for $\beta < \alpha$. Thus, a natural question to ask now is whether X being DPRL(α) implies that X is also DPRL(β) for $\beta > \alpha$. In the next example we show that the answer to this question is negative. That is, the following example shows that, given $\alpha \in (0, 1)$, it is possible to find a random variable X, and a $\beta \in (\alpha, 1)$, such that X is DPRL(α) but it is not DPRL(β).

Example 3.2. Fix an $\alpha \in (0, 1)$ and let θ be such that

$$\theta > \frac{3 \log(1 - \alpha)}{2 \log\left(\frac{-\log(1-\alpha)}{4\pi - \log(1-\alpha)}\right)}; \tag{3.1}$$

it is not hard to verify that the right hand side of (3.1) is positive. Furthermore, let ε be such that

$$\frac{-\log(1 - \alpha)(1 - (1 - \alpha)^{\frac{3}{2\theta}})}{2\pi(1 + (1 - \alpha)^{\frac{3}{2\theta}})} < \varepsilon \leq \frac{-\log(1 - \alpha)}{-\log(1 - \alpha) + 2\pi}; \tag{3.2}$$

it is not hard to verify that, when (3.1) holds, then the left hand side of (3.2) is smaller than the right hand side of (3.2).

Now define

$$k(x) = 1 + \varepsilon \sin\Big(\frac{2\pi x}{\log(1-\alpha)}\Big), \quad x \in \mathbb{R};$$

and

$$H(t) = (1-t)^\theta \cdot k\big(\log\left[(1-t)^\theta\right]\big), \quad 0 \le t \le 1.$$

Below, first we show that H is a survival function. Second, we show that a random variable X that has the survival function H is DPRL(α). Finally, we show that there exists a $\beta > \alpha$ such that X is not DPRL(β).

Obviously, $H(0) = 1$ and $H(1) = 0$. If we can find an $\varepsilon > 0$ such that $H(t)$ is decreasing in $0 \le t \le 1$, then it would follow that H is a survival function. In order to identify such an ε, we note that the derivative of k is given by

$$k'(x) = \varepsilon \cos\Big(\frac{2\pi x}{\log(1-\alpha)}\Big) \cdot \frac{2\pi}{\log(1-\alpha)}, \quad x \in \mathbb{R},$$

and thus the derivative of H is given by

$$H'(t) = -\theta(1-t)^{\theta-1}\Big[1 + \varepsilon \sin\Big(\frac{2\pi\theta\log(1-t)}{\log(1-\alpha)}\Big) - \frac{2\pi\varepsilon}{-\log(1-\alpha)}\cos\Big(\frac{2\pi\theta\log(1-t)}{\log(1-\alpha)}\Big)\Big], \quad 0 \le t \le 1.$$

Therefore H is decreasing if, and only if,

$$\varepsilon\Big[\log(1-\alpha)\sin\Big(\frac{2\pi\theta\log(1-t)}{\log(1-\alpha)}\Big) + 2\pi\cos\Big(\frac{2\pi\theta\log(1-t)}{\log(1-\alpha)}\Big)\Big] \le -\log(1-\alpha), \quad 0 \le t \le 1. \tag{3.3}$$

Since

$$\varepsilon\Big[\log(1-\alpha)\sin\Big(\frac{2\pi\theta\log(1-t)}{\log(1-\alpha)}\Big) + 2\pi\cos\Big(\frac{2\pi\theta\log(1-t)}{\log(1-\alpha)}\Big)\Big] \le \varepsilon(-\log(1-\alpha) + 2\pi), \ 0 \le t \le 1,$$

we see that if

$$\varepsilon \le \frac{-\log(1-\alpha)}{-\log(1-\alpha) + 2\pi} \tag{3.4}$$

then (3.3) holds. But (3.4) is the right hand side of (3.2), and therefore H is a survival function.

Now, let X have the survival function H, and let Y be a random variable with survival function $\overline{F}_Y$ given by

$$\overline{F}_Y(t) = (1-t)^\theta, \quad 0 \le t \le 1.$$

From Gupta and Langford (1984) we know that $q_{X,\alpha}(t) = q_{Y,\alpha}(t)$ for all t. Computing $q_{Y,\alpha}$, and using the equality $q_{X,\alpha} = q_{Y,\alpha}$, we obtain

$$q_{X,\alpha}(t) = \begin{cases} 1 - \overline{\alpha}^{1/\theta} - t, & t < 0, \\ (1 - \overline{\alpha}^{1/\theta})(1-t), & 0 \le t < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, X is DPRL(α).

For the reminder of this example let H and $-H'$ be denoted by $\overline{F}_X$ and f_X, respectively. We will now show that there exists a $\beta \in (\alpha, 1)$ such that X is not DPRL(β). Specifically, let $\beta = 1-(1-\alpha)^{\frac{3}{2}}$ ($>\alpha$). We will show that for $t_0 = 1-(1-\alpha)^{\frac{3}{2\theta}}$ we have

$$\overline{\beta} f_X(t_0) > f_X(\overline{F}_X^{-1}(\overline{\beta}\,\overline{F}_X(t_0))), \tag{3.5}$$

and then use Proposition 2.1(ii).

We compute

$$\begin{aligned}\overline{F}_X(t_0) = \overline{F}_X(1-(1-\alpha)^{\frac{3}{2\theta}}) &= (1-\alpha)^{\frac{3}{2}} k(\theta \log\left[(1-\alpha)^{\frac{3}{2\theta}}\right]) \\ &= (1-\alpha)^{\frac{3}{2}} k\big(\tfrac{3}{2}\log(1-\alpha)\big) = (1-\alpha)^{\frac{3}{2}}(1+\varepsilon\sin(3\pi)) = (1-\alpha)^{\frac{3}{2}}.\end{aligned}$$

So

$$\overline{\beta}\,\overline{F}_X(t_0) = (1-\alpha)^3 = (1-\alpha)^3[1+\varepsilon\sin(6\pi)] = \overline{F}_X(1-(1-\alpha)^{\frac{3}{\theta}}).$$

Hence

$$\overline{F}_X^{-1}(\overline{\beta}\,\overline{F}_X(t_0)) = 1-(1-\alpha)^{\frac{3}{\theta}}.$$

So (3.5) is equivalent to

$$\overline{\beta} f_X(1-(1-\alpha)^{\frac{3}{2\theta}}) > f_X(1-(1-\alpha)^{\frac{3}{\theta}}),$$

which is equivalent to

$$\begin{aligned}(1-\alpha)^{\frac{3}{2}}\theta(1-\alpha)^{\frac{3(\theta-1)}{2\theta}}&\left[1+\varepsilon\sin(3\pi)+\tfrac{2\pi\varepsilon}{\log(1-\alpha)}\cos(3\pi)\right] \\ &> \theta(1-\alpha)^{\frac{3(\theta-1)}{\theta}}\left[1+\varepsilon\sin(6\pi)+\tfrac{2\pi\varepsilon}{\log(1-\alpha)}\cos(6\pi)\right],\end{aligned}$$

which is equivalent to

$$(1-\alpha)^{\frac{3}{2\theta}}\left[1+\tfrac{2\pi\varepsilon}{-\log(1-\alpha)}\right] > \left[1-\tfrac{2\pi\varepsilon}{-\log(1-\alpha)}\right],$$

which is equivalent to

$$\varepsilon > \frac{-\log(1-\alpha)(1-(1-\alpha)^{\frac{3}{2\theta}})}{2\pi(1+(1-\alpha)^{\frac{3}{2\theta}})}.$$

The last inequality is the left hand side of (3.2). So (3.5) holds, and therefore X is not DPRL(β). ◀

The previous example shows that if X is DPRL(α) it does not necessarily follow that X is DPRL(β) for $\beta > \alpha$. In the next proposition we notice that if the density function of X is decreasing on a specific region of its support, then, if X is DPRL(α), it does follow that X is DPRL(β) for $\beta > \alpha$.

Proposition 3.3. *Let X be an absolutely continuous random variable with interval support (l_X, u_X), such that $u_X < \infty$, and with density and survival functions f_X and $\overline{F}_X$, respectively. Let $\alpha \in (0,1)$. If X is* DPRL(α) *and if f_X is increasing on $[\overline{F}_X^{-1}(\overline{\alpha}), u_X]$, then X is* DPRL(β) *for all $\beta > \alpha$.*

Proof. Let $\beta > \alpha$. Then, for all $p \in (0,1)$ we have

$$\begin{aligned}
\overline{\beta} f_X(\overline{F}_X^{-1}(p)) &\le \overline{\alpha} f_X(\overline{F}_X^{-1}(p)) \\
&\le f_X(\overline{F}_X^{-1}(\overline{\alpha} p)) \\
&\le f_X(\overline{F}_X^{-1}(\overline{\beta} p)),
\end{aligned}$$

where the second inequality follows from Proposition 2.1(iii), and the last inequality follows from the increasingness of f_X. The stated result now follows from Proposition 2.1(iii). $\square$

Note that if f_X is increasing on its support, then the monotonicity condition on f_X in Proposition 3.3 obviously holds. However, this observation does not tell us anything new because if f_X is increasing on its support then X is IFR, and, as we noted after Proposition 2.1, this implies that X is DPRL(α) for all $\alpha \in (0,1)$.

It is worthwhile to mention that Launer (1993) has shown that a nonnegative random variable X, with a bathtub-shaped hazard rate function r_X, is DPRL(α) for all $\alpha \in (\alpha_0, 1)$ for some $\alpha_0 > 0$, provided there exists a $t_0 \ge 0$ such that $r_X(t_0) \ge r_X(0)$.

4 Relationships with the PRL stochastic orders

We recall the following family of stochastic orders that was recently studied in Franco-Pereira, Lillo, Romo, and Shaked (2010). Let $0 < \alpha < 1$. Let X and Y be two random variables with percentile residual life functions $q_{X,\alpha}$ and $q_{Y,\alpha}$, respectively. If

$$q_{X,\alpha}(t) \le q_{Y,\alpha}(t) \quad \text{for all } t$$

then X is said to be smaller than Y in the α-percentile residual life order (denoted as $X \le_{\alpha\text{-rl}} Y$).

In the following result we provide some characterizations of the DPRL(α) aging notion in terms of the α-percentile residual life order. Recall the definition of X_t from (2.1).

Theorem 4.1. *Let X be an absolutely continuous random variable with interval support. Then X is* DPRL(α) *if, and only if, any of the following equivalent conditions holds:*

(i) $X_t \ge_{\alpha\text{-rl}} X_{t'}$ *whenever* $t \le t' < u_X$;

(ii) $X \ge_{\alpha\text{-rl}} X_t$ *whenever* $0 \le t < u_X$ *(when X is a nonnegative random variable);*

(iii) $X + t \le_{\alpha\text{-rl}} X + t'$ *whenever* $t \le t'$.

Proof. From (2.2) it is easy to verify that

$$q_{X_t,\alpha}(x) = \overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t+x)) - (t+x) \quad \text{for all } 0 < x < u_X - t.$$

Now, let $t \le t' < u_X$. Then $X_t \ge_{\alpha\text{-rl}} X_{t'}$ if, and only if,

$$\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t+x)) - (t+x) \ge \overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t'+x)) - (t'+x) \quad \text{for all } x < u_X - t';$$

that is (by (2.4)), $q_{X,\alpha}(t+x) \geq q_{X,\alpha}(t'+x)$ whenever $t+x \leq t'+x < u_X$; that is, $q_{X,\alpha}$ is decreasing. This proves the equivalence of DPRL(α) and (i).

Next, let $0 \leq t < u_X$. Then $X \geq_{\alpha\text{-rl}} X_t$ if, and only if,

$$\overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(x)) - x \geq \overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t+x)) - (t+x) \quad \text{for all } x < u_X - t;$$

that is (by (2.4)), $q_{X,\alpha}(x) \geq q_{X,\alpha}(t+x)$ whenever $t+x \leq u_X$; that is, $q_{X,\alpha}$ is decreasing. This proves the equivalence of DPRL(α) and (ii).

In order to prove the equivalence of DPRL(α) and (iii), let $t \leq t'$, and denote $a = t' - t$. Then condition (iii) is equivalent to

$$X \leq_{\alpha\text{-rl}} X + a \quad \text{for all } a > 0. \tag{4.1}$$

Now, from (2.4) we have

$$q_{X,\alpha}(t) = \overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t)) - t \quad \text{for all } t < u_X,$$

and, for $a > 0$ we have

$$q_{X+a,\alpha}(t) = \overline{F}_{X+a}^{-1}(\overline{\alpha}\overline{F}_{X+a}(t)) - t = \overline{F}_X^{-1}(\overline{\alpha}\overline{F}_X(t-a)) - t + a = q_{X,\alpha}(t-a)$$
$$\text{for all } t < u_X + a.$$

That is, condition (4.1) is equivalent to the decreasingness of $q_{X,\alpha}$. □

In the literature there are results that are similar to Theorem 4.1, but which involve aging notions other than DPRL(α). For example, Theorems 1.A.30, 1.B.38, 3.B.24, 3.B.25, and 4.A.53 in Shaked and Shanthikumar (2007) give similar characterizations for the IFR aging notion. Theorems 2.A.23, 2.B.17, 3.A.56, 3.C.13, and 4.A.51 in Shaked and Shanthikumar (2007), as well as a result in Belzunce, Gao, Hu, and Pellerey (2004), give similar characterizations for the decreasing mean residual life (DMRL) aging notion.

A classical result (Joe and Proschan, 1984b) states that

$$X \leq_{\alpha\text{-rl}} Y \quad \text{for all } \alpha \in (0,1)$$

if, and only if,

$$X \leq_{\text{hr}} Y, \tag{4.2}$$

where $\leq_{\text{hr}}$ denotes the hazard rate stochastic order (see Shaked and Shanthikumar, 2007). This result was strengthened in Franco-Pereira, Lillo, Romo, and Shaked (2010) who showed that if X is a continuous random variable, and if for some fixed $\varepsilon \in (0,1)$ we have that

$$X \leq_{\alpha\text{-rl}} Y \quad \text{for all } \alpha \in (0,\varepsilon) \tag{4.3}$$

then (4.2) still holds. Now, suppose that some continuous random variable X is DPRL(α) for all $\alpha \in (0,\varepsilon)$. From Theorem 4.1(iii) we see that (4.3) holds if we replace X and Y there by $X+t$ and $X+t'$, for any t and t' such that $t \leq t'$. Thus, from (4.2) we get

$$X + t \leq_{\text{hr}} X + t' \quad \text{whenever } t \leq t',$$

which means, by Shaked and Shanthikumar (2007, Theorem 1.B.38(iii)), that X is IFR. We have thus proven the following positive result that may be contrasted with the negative result shown in Example 3.1.

Proposition 4.2. *Let X be a random variable with a continuous distribution function, and let $\varepsilon \in (0,1)$. If X is* DPRL(α) *for all $\alpha \in (0,\varepsilon)$ then X is* IFR.

5 Further properties

Intuitively speaking, the order $\leq_{\alpha\text{-rl}}$ is an order of magnitude in the sense that a "larger" random variable may be expected to be larger with respect to this order. However, from Theorem 4.1 (iii) it follows that that is not always the case. A natural condition under which indeed $X+t'$ is larger than $X+t$ with respect to this order, when $t \leq t'$, is that X is DPRL(α). The next result highlights the usefulness of the DPRL(α) notion in a similar situation. The following result is an analog of Theorem 1.B.21 in Shaked and Shanthikumar (2007) which involves the IFR aging notion, and of Theorem 2.A.17 in Shaked and Shanthikumar (2007) which involves the DMRL aging notion.

Theorem 5.1. *Let X be a positive, absolutely continuous,* DPRL(α) *random variable with interval support. Then*

$$X \leq_{\alpha\text{-rl}} aX \quad \textit{for all } a > 1. \tag{5.1}$$

Proof. By (2.4),

$$q_{X,\alpha}(t) = \overline{F}_X^{-1}(\overline{\alpha}\,\overline{F}_X(t)) - t \quad \text{for all } t > 0,$$

and

$$q_{aX,\alpha}(t) = \overline{F}_{aX}^{-1}(\overline{\alpha}\,\overline{F}_{aX}(t)) - t = a\overline{F}_X^{-1}\left(\overline{\alpha}\,\overline{F}_X\left(\frac{t}{a}\right)\right) - t = aq_{X,\alpha}\left(\frac{t}{a}\right)$$
$$\text{for all } t > 0 \text{ and all } a > 1.$$

If X is DPRL(α) then

$$q_{X,\alpha}(t) \leq q_{X,\alpha}\left(\frac{t}{a}\right) \leq aq_{X,\alpha}\left(\frac{t}{a}\right) = q_{aX,\alpha}(t) \quad \text{for all } t > 0 \text{ and all } a > 1,$$

which yields (5.1). $\square$

If X is not DPRL(α) then (5.1) may not hold. More explicitly, for any $\alpha \in (0,1)$, if X is not DPRL(α) then (5.1) need not hold for any $a > 1$; this is shown in the following example.

Example 5.2. For a fixed $\alpha \in (0,1)$ and a fixed $a > 1$, let X be a random variable with the following distribution function:

$$F_X(t) = \begin{cases} 0, & t < 0, \\ (1+\alpha)at, & 0 \leq t < \frac{1}{2a}, \\ 1 - \frac{a(1-\alpha)(1-t)}{(2a-1)}, & \frac{1}{2a} \leq t < 1. \end{cases}$$

A lengthy straightforward computation yields

$$q_{X,\alpha}(t) = \begin{cases} \frac{\alpha}{a(1+\alpha)} - t, & t < 0, \\ \alpha\left(\frac{1}{a(1+\alpha)} - t\right), & 0 \leq t < \frac{1}{2a(1+\alpha)}, \\ \left[(1+\alpha)(2a-1) - 1\right]t - \frac{a-1}{a}, & \frac{1}{2a(1+\alpha)} \leq t < \frac{1}{2a}, \\ \alpha(1-t), & \frac{1}{2a} \leq t < 1, \\ 0, & t \geq 1. \end{cases}$$

The graph of $q_{X,\alpha}(t)$ is shown in Figure 2. Obviously X is not DPRL(α) $\big[q_{X,\alpha}(t)$ is increasing in the interval $\big(\frac{1}{2a(1+\alpha)}, \frac{1}{2a}\big)\big]$. We want to show (see the proof of Theorem 5.1) that

$$aq_{X,\alpha}\left(\frac{\tilde{t}}{a}\right) < q_{X,\alpha}(\tilde{t}) \tag{5.2}$$

for some $\tilde{t} \in (0,1)$. In order to do that take $\tilde{t} = \frac{1}{2(1+\alpha)}$; again, see Figure 2. Then

$$aq_{X,\alpha}\left(\frac{\tilde{t}}{a}\right) = aq_{X,\alpha}\Big(\frac{1}{2a(1+\alpha)}\Big) = \frac{\alpha}{2(1+\alpha)},$$

and

$$q_{X,\alpha}(\tilde{t}) = \begin{cases} \alpha\big(1 - \frac{1}{2(1+\alpha)}\big) = \frac{\alpha(1+2\alpha)}{2(1+\alpha)}; & \text{if } a \geq 1+\alpha, \\ \frac{2(a^2+1)(1+\alpha)-3a\alpha-4a}{2a(1+\alpha)}; & \text{if } a < 1+\alpha. \end{cases}$$

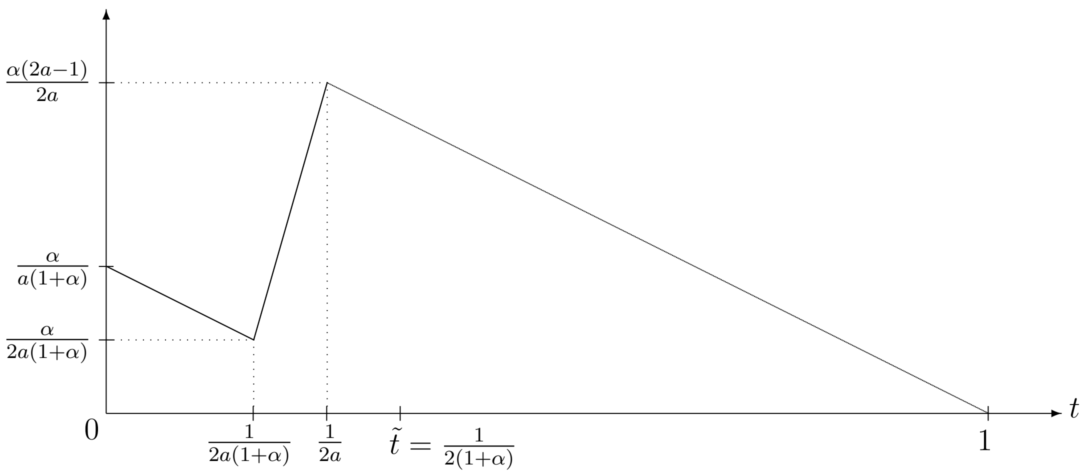

Figure 2: The graph of $q_{X,\alpha}(t)$ in Example 5.2

If $a \geq 1+\alpha$ then

$$aq_{X,\alpha}\left(\frac{\tilde{t}}{a}\right) = \frac{\alpha}{2(1+\alpha)} < \frac{\alpha(1+2\alpha)}{2(1+\alpha)} = q_{X,\alpha}(\tilde{t}),$$

where the inequality follows from $1+2\alpha > 1$. So inequality (5.2) holds in this case.

On the other hand, if $a < 1+\alpha$ then a straightforward computation shows that inequality (5.2) is equivalent to $(a-1)^2 > 0$, which is always true. Therefore $X \nleq_{\alpha\text{-rl}} aX$. ◀

Another situation in which the DPRL(α) aging notion arises as a natural condition will be described next. The result below (Theorem 5.4), again, indicates a useful property of the order $\leq_{\alpha\text{-rl}}$ when one of the compared random variables is "larger in magnitude" than the other one. The following result from Franco-Pereira, Lillo, Romo, and Shaked (2010) will be used in the proofs of Theorems 5.4 and 5.5.

Proposition 5.3. *Let* $\{X_\theta,\ \theta \in \Theta\}$ *and* $\{Y_\theta,\ \theta \in \Theta\}$ *be two families of random variables with continuous distribution functions. Let* V *and* W *be random variables with distribution functions given by*

$$F_V(t) = \int_\Theta F_{X_\theta}(t) dH(\theta) \quad \textit{and} \quad F_W(t) = \int_\Theta F_{Y_\theta}(t) dH(\theta), \quad t \in \mathbb{R},$$

where H *is some distribution function on* Θ*. If*

$$X_\theta \leq_{\alpha\text{-rl}} Y_{\theta'} \quad \textit{for all } \theta, \theta' \in \Theta, \tag{5.3}$$

then $V \leq_{\alpha\text{-rl}} W$*.*

The following result is a generalization of the sufficiency part of Theorem 4.1(iii).

Theorem 5.4. *Let* X *be a continuous* DPRL(α) *random variable. Let* Z *be a nonnegative continuous random variable that is independent of* X*. Then*

$$X \leq_{\alpha\text{-rl}} X + Z. \tag{5.4}$$

Proof. We write

$$F_X(x) = \int_{z=0}^{\infty} F_X(x)\, dF_Z(\theta)$$

and

$$F_{X+Z}(x) = \int_{z=0}^{\infty} F_{X+\theta}(x)\, dF_Z(\theta).$$

Denote $X_\theta = X$ and $Y_\theta = X + \theta$. Now, in Proposition 5.3, take $\Theta = [0, \infty)$ and $H = F_Z$. Then $V = X$ and $W = X + Z$. By Theorem 4.1(iii) we see that (5.3) holds. Therefore the stated result follows from Proposition 5.3. □

It is worthwhile to point out that if X in Theorem 5.4 is not DPRL(α) then the conclusion of that theorem need not hold. In order to see this, note that Theorem 4.1(iii) actually says that X is DPRL(α) if, and only if, $X \leq_{\alpha\text{-rl}} X + a$ for every $a \geq 0$. Thus, if X in Theorem 5.4 is not DPRL(α) then there exists a degenerate Z such that (5.4) does not hold.

The DPRL(α) aging notion is also useful as a condition under which the order $\leq_{\alpha\text{-rl}}$ is preserved under certain random additions. This is shown next.

Theorem 5.5. *Let* X *and* Y *be two* DPRL(α) *random variables. Let* Z *be a random variable, independent of* X *and* Y*, with support in* $[l, u]$*, where* $-\infty < l < u < \infty$*. If* $X + u \leq_{\alpha\text{-rl}} Y + l$*, then*

$$X + Z \leq_{\alpha\text{-rl}} Y + Z.$$

Proof. Write

$$F_{X+Z}(x) = \int_{\theta=0}^{\infty} F_{X+\theta}(x)\, dF_Z(\theta)$$

and

$$F_{Y+Z}(x) = \int_{\theta=0}^{\infty} F_{Y+\theta}(x)\, dF_Z(\theta).$$

Denote $X_\theta = X + \theta$ and $Y_\theta = Y + \theta$. Take any $\theta, \theta' \in [l, u]$. Then

$$\begin{aligned}
X_\theta = X + \theta &\leq_{\alpha\text{-rl}} X + u && \text{(by Theorem 4.1(iii) and } \theta \leq u) \\
&\leq_{\alpha\text{-rl}} Y + l && \text{(by assumption)} \\
&\leq_{\alpha\text{-rl}} Y + \theta' = Y_{\theta'} && \text{(by Theorem 4.1(iii) and } l \leq \theta');
\end{aligned}$$

that is, (5.3) holds for $\Theta = [l, u]$. So, taking $H = F_Z$ in Proposition 5.3, we obtain the stated result. □

6 Estimation of a decreasing PRL function

In many applications it is reasonable to assume that the system life is monotonically degenerating or improving with age. Kochar, Mukerjee, and Samaniego (2000) have studied the estimation of the mean residual life function under decreasing or increasing restrictions. To the best of our knowledge, estimation of a percentile residual life function under monotone restrictions does not appear to have been considered in the literature. In this section we initiate a study of such estimation procedures following an approach that is similar to the approach of Kochar, Mukerjee, and Samaniego (2000). We propose an estimator of the percentile residual life function under the condition that it decreases, we look at its computation, and then we prove its consistency.

Let $X_1, X_2, \ldots, X_n$ be independent random variables with a common distribution function F_X, and let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the corresponding order statistics. The resulting empirical distribution function is

$$F_{X,n}(t) = \frac{\#\{k : X_k \leq t, 1 \leq k \leq n\}}{n}, \quad t \in \mathbb{R},$$

and the corresponding left continuous inverse (that is, the quantile function) is

$$F^{-1}_{X,n}(p) = X_{k:n} \quad \text{if} \quad \frac{k-1}{n} < p \leq \frac{k}{n}, \quad k = 1, 2, \ldots, n.$$

Following (2.5), a natural empirical counterpart of $q_{X,\alpha}$ is the sample α-percentile residual life function, which is given by

$$\hat{q}_{X,n,\alpha}(t) = F^{-1}_{X,n}(\alpha + (1-\alpha)F_{X,n}(t)) - t, \quad t < X_{n:n}.$$

Note that $\hat{q}_{X,n,\alpha}$ is a piecewise linear function with jump discontinuities. It consists of line segments with slope equal to -1 with jump discontinuities (which gives rise to a rather ragged estimator). The estimator $\hat{q}_{X,n,\alpha}$ was introduced and studied in Csörgö and Csörgö (1987). Further properties of it were obtained in Barabás, Csörgö, Horváth, and Yandell (1986), Csörgö and Mason (1989), Aly (1992), and Csörgö and Viharos (1992).

The estimator that we suggest is based on the fact that

$$q_{X,\alpha} \text{ is DPRL}(\alpha) \Longleftrightarrow q_{X,\alpha}(t) = \inf_{y \leq t} q_{X,\alpha}(y). \tag{6.1}$$

Thus, the estimator $\hat{q}^*_{X,n,\alpha}$ that we propose is given by

$$\hat{q}^*_{X,n,\alpha}(t) = I_{(t,\infty)}(X_{n:n}) \inf_{y \le t} \hat{q}_{X,n,\alpha}(y), \quad t \in \mathbb{R}, \tag{6.2}$$

where $I_{(t,\infty)}$ denotes the indicator function of the indicated interval. Note that $\hat{q}^*_{X,n,\alpha}$ is the largest decreasing function that lies below the empirical $\hat{q}_{X,n,\alpha}$.

The computation of the estimator is quite simple. Let $X_1, X_2, \dots, X_n$ be the random variables that make up the sample, and let $X_{1:n} \le X_{2:n} \le \dots \le X_{n:n}$ be the corresponding order statistics. First we find the number of different observational values in the sample, k, say. Next, let $Y_1 < Y_2 < \dots < Y_k$ be the resulting ordered values with no ties; that is,

$$X_{1:n} = Y_1 < Y_2 < \dots < Y_k = X_{n:n}.$$

Then we explicitly have

$$\hat{q}^*_{X,n,\alpha}(t) = \begin{cases} \hat{q}_{X,n,\alpha}(Y_1-) + Y_1 - t, & \text{if } t < Y_1, \\ \min\{\hat{q}_{X,n,\alpha}(Y_1-), \hat{q}_{X,n,\alpha}(Y_2-), \dots, & \\ \quad \hat{q}_{X,n,\alpha}(Y_j-), \hat{q}_{X,n,\alpha}(Y_{j+1}-) + Y_{j+1} - t\}, & \text{if } Y_j \le t < Y_{j+1},\ j = 1, 2, \dots, k-1, \\ 0 & \text{if } t \ge Y_k. \end{cases}$$

To illustrate how the estimator looks like, consider a sample of size $n = 11$ with the ordered observed values $X_{1:11} = -5$, $X_{2:11} = X_{3:11} = -2$, $X_{4:11} = 1$, $X_{5:11} = X_{6:11} = 7$, $X_{7:11} = 11$, $X_{8:11} = 15$, $X_{9:11} = 16$, $X_{10:11} = 18$, and $X_{11:11} = 21$. Then there are $k = 9$ resulting ordered values with no ties:

$$Y_1 = -5,\ Y_2 = -2,\ Y_3 = 1,\ Y_4 = 7,\ Y_5 = 11,\ Y_6 = 15,\ Y_7 = 16,\ Y_8 = 18,\ \text{and } Y_9 = 21.$$

In Figures 3, 4, and 5 the estimators $\hat{q}_{X,n,1/3}$ and $\hat{q}^*_{X,n,1/3}$, $\hat{q}_{X,n,0.5}$ and $\hat{q}^*_{X,n,0.5}$, and $\hat{q}_{X,n,0.8}$ and $\hat{q}^*_{X,n,0.8}$, respectively, are shown. For the purpose of comparisons, the three DPRL estimators $\hat{q}^*_{X,n,1/3}$, $\hat{q}^*_{X,n,0.5}$, and $\hat{q}^*_{X,n,0.8}$ are put together in Figure 6; note that, obviously, $\hat{q}^*_{X,n,0.8}$ is the highest among these three estimators, and $\hat{q}^*_{X,n,1/3}$ is the lowest.

In Theorem 6.2 below we show that $\hat{q}^*_{X,n,\alpha}$ is a strongly uniform consistent estimator of $q_{X,\alpha}$. In order to do that, we need to recall some definitions and a technical result. First we recall the definition of several types of consistency.

An estimator $\hat{q}_n(\cdot)$ of a function $q(\cdot)$ is said to be *consistent* if

$$|\hat{q}_n(t) - q(t)| \xrightarrow{\text{p}} 0 \quad \text{for all } t,$$

where $\xrightarrow{\text{p}}$ denotes convergence in probability. The estimator $\hat{q}_n(\cdot)$ of the function $q(\cdot)$ is said to be *strongly consistent* if

$$|\hat{q}_n(t) - q(t)| \xrightarrow{\text{a.s.}} 0 \quad \text{for all } t,$$

where $\xrightarrow{\text{a.s.}}$ denotes almost sure convergence. Finally, the estimator $\hat{q}_n(\cdot)$ of the function $q(\cdot)$ is said to be *strongly uniform consistent* if

$$\sup_t |\hat{q}_n(t) - q(t)| \xrightarrow{\text{a.s.}} 0.$$

We will also need the following variation of Lemma 1 of Rojo and Samaniego (1993).

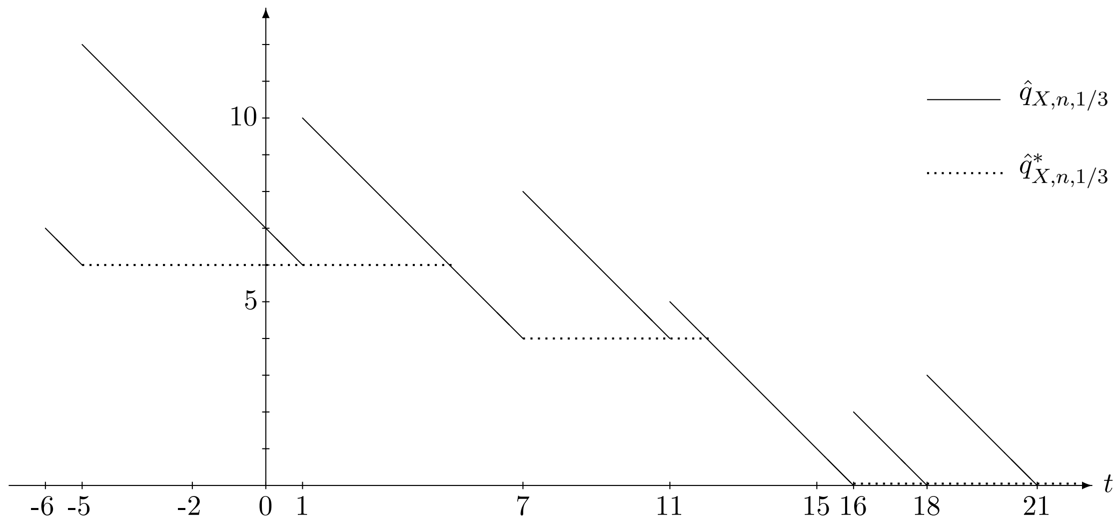

Figure 3: Illustration of the estimators $\hat{q}_{X,n,1/3}$ and $\hat{q}^*_{X,n,1/3}$.

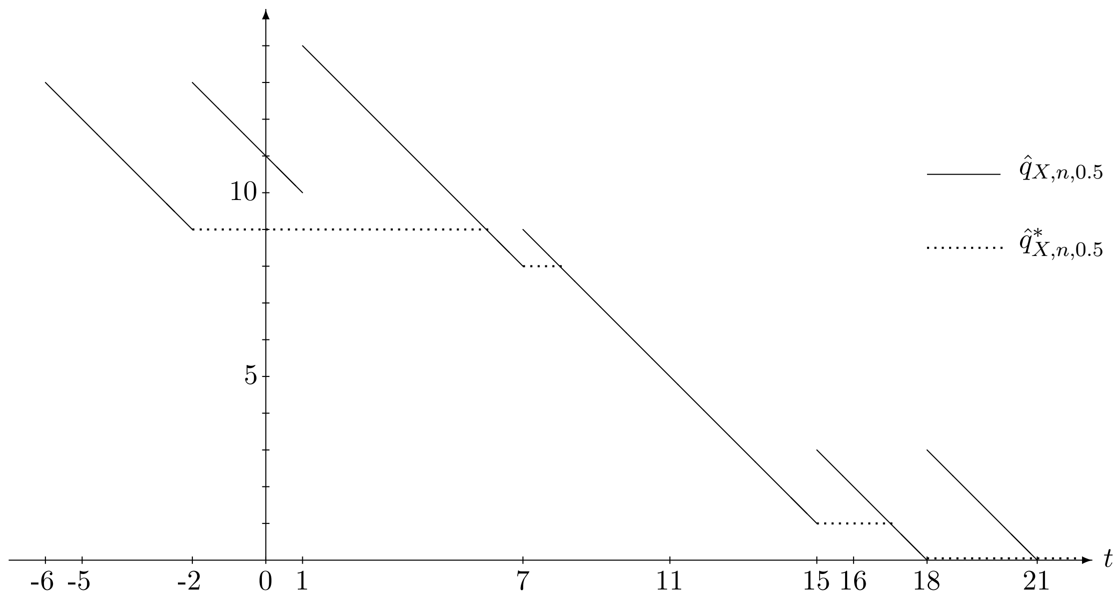

Figure 4: Illustration of the estimators $\hat{q}_{X,n,0.5}$ and $\hat{q}^*_{X,n,0.5}$.

Lemma 6.1. *Let g and h be two bounded functions on $\mathbb{R}$. Then*

$$\left|\inf_y g(y) - \inf_y h(y)\right| \leq \sup_y |g(y) - h(y)|.$$

We are now ready to state and prove the strongly uniform consistency of $\hat{q}^*_{X,n,\alpha}$.

Theorem 6.2. *Let X be a* DPRL(α) *random variable. If F_X has a continuous positive density function f_X such that $\inf_{0\leq p\leq 1} f_X(F_X^{-1}(p)) > 0$, then $\hat{q}^*_{X,n,\alpha}$ is a strongly uniform consistent estimator of $q_{X,\alpha}$.*

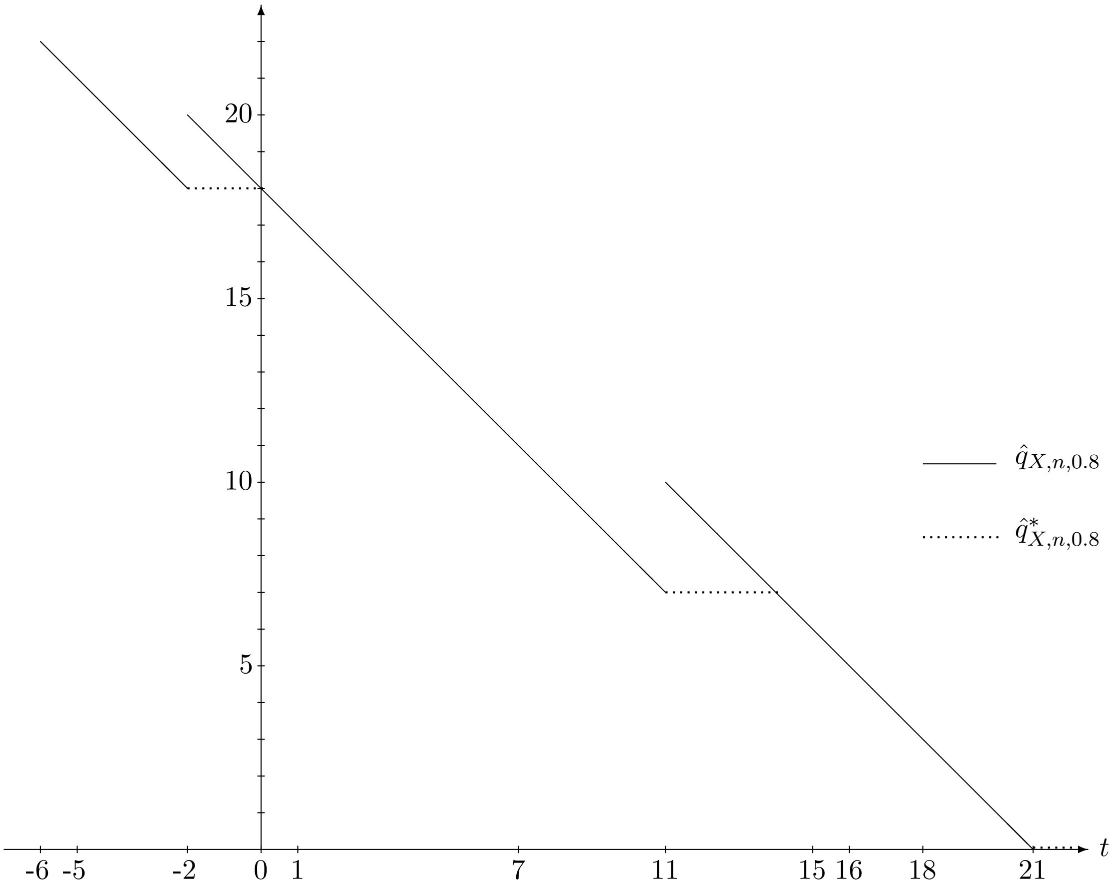

Figure 5: Illustration of the estimators $\hat{q}_{X,n,0.8}$ and $\hat{q}^*_{X,n,0.8}$.

Proof. First we note, from Corollary 1.4.1 of Csörgö (1983) and the paragraph that follows it (in page 6 of Csörgö (1983)), that if F_X has a continuous positive density function f_X such that $\inf_{0\le p\le 1} f_X(F_X^{-1}(p)) > 0$, then

$$\sup_{0\le p\le 1}\left|F_{X,n}^{-1}(p)-F_X^{-1}(p)\right| \xrightarrow{\text{a.s.}} 0,$$

that is, $F_{X,n}^{-1}$ is a strongly uniform consistent estimator of F_X^{-1}. Furthermore, by the Glivenko-Cantelli theorem, $F_{X,n}$ is a strongly uniform consistent estimator of F_X. Combining these two results it follows that

$$\sup_t |F_{X,n}^{-1}(\alpha+(1-\alpha)F_{X,n}(t)) - F_X^{-1}(\alpha+(1-\alpha)F_X(t))| \xrightarrow{\text{a.s.}} 0$$

or, equivalently, that

$$\sup_t |\hat{q}_{X,n,\alpha}(t) - q_{X,\alpha}(t)| \xrightarrow{\text{a.s.}} 0. \tag{6.3}$$

In other words, we see that $\hat{q}_{X,n,\alpha}$ is a strongly uniform consistent estimator of $q_{X,\alpha}$.

Next, note from Lemma 6.1 that

$$\left|\inf_{y\le t}\hat{q}_{X,n,\alpha}(y) - \inf_{y\le t} q_{X,\alpha}(y)\right| \le \sup_{y\le t}|\hat{q}_{X,n,\alpha}(y) - q_{X,\alpha}(y)| \quad \text{for all } t < X_{n:n}. \tag{6.4}$$

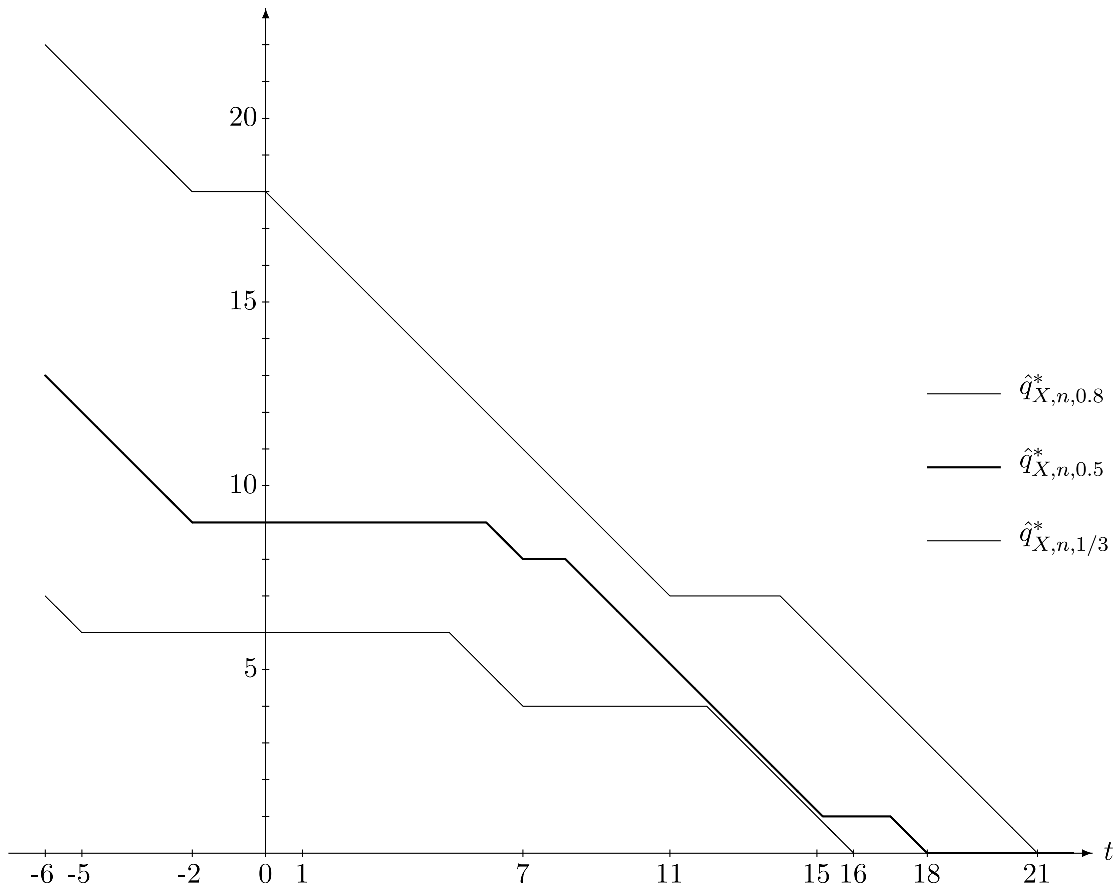

Figure 6: Illustration of the estimators $\hat{q}^*_{X,n,1/3}$, $\hat{q}^*_{X,n,0.5}$, and $\hat{q}^*_{X,n,0.8}$.

Combining this with (6.3) it is seen that

$$\Big|\inf_{y\le t}\hat{q}_{X,n,\alpha}(y)-\inf_{y\le t}q_{X,\alpha}(y)\Big|\xrightarrow{\text{a.s.}}0\quad\text{for all } t<u_X.$$

So, from (6.1) and (6.2) we see that $\hat{q}^*_{X,n,\alpha}$ is a strongly consistent estimator of $q_{X,\alpha}$.

Finally we still need to prove the strongly *uniform* consistency of $\hat{q}^*_{X,n,\alpha}$. For this, note that (6.4) yields

$$\sup_t\Big|\inf_{y\le t}\hat{q}_{X,n,\alpha}(y)-\inf_{y\le t}q_{X,\alpha}(y)\Big|\le\sup_{y\le t}|\hat{q}_{X,n,\alpha}(y)-q_{X,\alpha}(y)|\quad\text{for all } t<X_{n:n}.$$

Letting $n\to\infty$ above, and making use of the strong uniform consistency of $\hat{q}_{X,n,\alpha}$, we get

$$\sup_t\Big|\inf_{y\le t}\hat{q}_{X,n,\alpha}(y)-\inf_{y\le t}q_{X,\alpha}(y)\Big|\xrightarrow{\text{a.s.}}0\quad\text{for all } t<u_X,$$

which proves the strongly uniform consistency of $\hat{q}^*_{X,n,\alpha}$. $\square$

We note that in order to estimate the percentile residual life function under the condition that it increases, an estimator that is a straightforward modification of the estimator in (6.2) can be introduced. Its computation is similar to the computation of $\hat{q}^*_{X,n,\alpha}$, and it is also strongly uniform consistent. We do not give here the straightforward details.

References

[1] Aly, E.-E. A. A. (1992). On some confidence bands for percentile residual life functions. *Journal of Nonparametric Statistics* **2**, 59–70.

[2] Arnold, B. C. and Brockett, P. L. (1983). When does the βth percentile residual life function determine the distribution? *Operations Research* **31**, 391–396.

[3] Barabás, B., Csörgö, M., Horváth, L., and Yandell, B. S. (1986). Bootstrapped confidence bands for percentile lifetime. *Annals of the Institute of Statistical Mathematics* **38**, 429–438.

[4] Belzunce, F., Gao, X., Hu, T., and Pellerey, F. (2004). Characterizations of the hazard rate order and IFR aging notion. *Statistics and Probability Letters* **70**, 235–242.

[5] Chung, C.-J. F. (1989). Confidence bands for percentile residual lifetime under random censorship model. *Journal of Multivariate Analysis* **29**, 94–126.

[6] Csörgö, M. (1983). *Quantile Processes with Statistical Applications*, SIAM, Philadelphia.

[7] Csörgö, M. and Csörgö, S. (1987). Estimation of percentile residual life. *Operations Research* **35**, 598–606.

[8] Csörgö, S. and Mason, D. M. (1989). Bootstrapping empirical functions. *Annals of Statistics* **17**, 1447–1471.

[9] Csörgö, S. and Viharos, L. (1992). Confidence bands for percentile residual lifetimes. *Journal of Statistical Planning and Inference* **30**, 327–337.

[10] Feng, Z. and Kulasekera, K. B. (1991). Nonparametric estimation of the percentile residual life function. *Communications in Statistics - Theory and Methods 20*, 87–105.

[11] Franco-Pereira, A. M., Lillo, R. E., Romo, J., and Shaked, M. (2010). Percentile residual life orders. *Applied Stochastic Models in Business and Industry*, to appear.

[12] Gupta, R. C. and Langford, E. S. (1984). On the determination of a distribution by its median residual life function: A functional equation. *Journal of Applied Probability* **21**, 120–128.

[13] Haines, A. L. and Singpurwalla, N. D. (1974). Some contributions to the stochastic characterization of wear. In *Reliability and Biometry, Statistical Analysis of Lifelength* (edited by F. Proschan and R. J. Serfling), SIAM, Philadelphia, 47–80.

[14] Joe, H. (1985). Characterizations of life distributions from percentile residual lifetimes. *Annals of the Institute of Statistical Mathematics* **37**, 165–172.

[15] Joe, H. and Proschan, F. (1984a). Percentile residual life functions. *Operational Research* **32**, 668–678.

[16] Joe, H. and Proschan, F. (1984b). Comparison of two life distributions on the basis of their percentile residual life functions. *Canadian Journal of Statistics* **12**, 91–97.

[17] Kochar, S. C., Mukerjee, H., and Samaniego, F. J. (2000). Estimation of a monotone mean residual life. *Annals of Statistics* **28**, 905–921.

[18] Launer, R. L. (1993). Graphical techniques for analyzing failure data with the percentile residual-life function. *IEEE Transactions on Reliability* **42**, 71–75, 80.

[19] Lillo, R. E. (2005). On the median residual lifetime and its aging properties: A characterization theorem and applications. *Naval Research Logistics* **52**, 370–380.

[20] Lin, G. D. (2009). On the characterization of life distributions via percentile residual lifetimes. *Sankhyā* **71-A**, 64–72.

[21] Rojo, J. and Samaniego, F. J. (1993). On estimating a survival curve subject to a uniform stochastic ordering constraint. *Journal of the American Statistical Association* **88**, 566–572.

[22] Schmittlein, D. C. and Morrison, D. G. (1981). The median residual lifetime: A characterization theorem and an application. *Operations Research* **29**, 392–399.

[23] Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic Orders*, Springer, New York.