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Electing a Parliament

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Abstract

We present a model where a society elects a parliament by voting for candidates belonging to two parties. The electoral rule determines the seats distribution between the two parties. We analyze two electoral rules, multidistrict majority and single-district proportional. In this framework, the policy outcome is simply a function of the number of seats parties take in the election. We prove that in both systems there is a unique pure strategy perfect equilibrium outcome. Finally, we compare the outcomes in the two systems.

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1 Introduction

In parliamentary democracies policies reflect a legislative debate where all political parties, in power and at the opposition, contribute to the final outcome. The role parties have in shaping the national policy, arguably, depends on their strength, i.e. on the number of seats they have in parliament. Recently the political economics literature has shown a growing interest in understanding the role of political institutions in shaping national economic policy (see Persson, 2002). In this paper we focus on the rules for electing a legislature. The political science literature (see Cox, 1997) has identified in the electoral formula the main dimension in which legislative elections differ.[1] In this paper we analyze a multidistrict majoritarian election and a single-district proportional election, which are the two most studied systems (see, among many others, Persson and Tabellini 2000, Cox 1997). The common viewpoint has been that parties relative strength is given by their share of votes. We believe it is time to let the number of seats in parliament play a role.

Specifically, we study a society, composed by policy motivated strategic citizens, electing a parliament of k members by voting for representatives of two parties (L and R, say). Seats in parliament are allocated to the two parties according to the electoral result. In the literature on parliamentary election (see, among many others, Persson and Tabellini 2000, Alesina and Rosenthal 1996, De Sinopoli and Iannantuoni 2007) the simplifying assumption is that the vote share taken in the election is equal to the seats share in parliament, and thus policies directly depend on the share of votes. We take the point of view that the policy outcome is defined on the number of seats parties win in the election. We hence explore what we believe to be the intriguing consequences of a seemingly minor departure from this common feature of the literature.

Multidistrict majoritarian elections

We first study a situation where citizens (with single-peaked preferences) are distributed in k districts (a generic district is denoted by d) and vote strategically in each one by majority rule. The electoral result (i.e. a pure strategy combination) determines the number of seats for the two parties in parliament. We capture the idea of a parliamentary compromise between the two parties with different strengths by assuming that the policy outcome is a function decreasing in the number of districts won by the leftist party. To solve such a voting game the issue of the solution concept needs to be addressed. Given the typical weakness of the Nash solution concept in this type of games, we turn to refinements. To see why this is the case, consider an election in one district: the election of any candidate is a Nash equilibrium outcome when there are more than three voters. Differently from standard models with two

[1] Another important dimension is naturally the district size (see Cox, 1997). Papers analyzing how differences in district size affect the policies are among others, Persson and Tabellini (1999), Myerson (1993), Milesi-Ferretti et al (2000).

parties also the concept of undominated equilibria fails to have byte in such a voting game. Let us define by α_k the policy outcome obtained by averaging the outcomes when L takes k and $(k-1)$ seats. Similarly, denote α_1 the average policy when party L wins 1 and 0 seats. Only voters located on the left/right of the α_k/α_1 policy have dominant strategies, i.e. voting for the L/R party. We define an outcome as "pure" if it assigns probability one to a given policy. We then propose a natural solution concept in this context, which we call district-sincerity. A strategy combination is district sincere if each player who strictly prefers (given the strategies of the players in the other districts) that party L/R wins in his district, votes for party L/R. We prove that the voting game has a unique district-sincere outcome in pure strategy, which is also the unique district-sincere "pure" outcome. Such an outcome is characterized by a number of seats for the leftist party equal to the number of districts d whose medians m_d are smaller or equal to the corresponding α_d (that is again the average policy when party L wins d and $d-1$ seats).

We then turn our attention to tremblig-hand-perfection, which is a refinement of the undominated equilibrium concept and, as we will show, of the district-sincerity concept as well. We do so because we want to compare the results obtained in the multidistrict majoritarian case with the proportional one, in which a concept like district sincerity does not make any sense. We prove the existence of a unique pure strategies perfect equilibrium outcome, which is obviously the unique district-sincere outcome in pure strategies, and the unique "pure" outcome induced by perfect equilibria.

Proportional election.

We then turn our attention to a situation in which citizens (with single-peaked preferences) are distributed in one national district electing k representatives. There are various mechanisms to transform votes into seats under proportional rule, we use a very general one by simply assuming a minimum number of votes needed to get a certain number of seats for the leftist party.[2] Again, the policy outcome is simply a decreasing function of the number of seats won by the leftist party. Similarly to the majoritarian election and even if we have a two-party scenario, also in this case the undominated equilibrium concept does not help in solving the game. Except for voters located on the left/right of the α_k/α_1 policy, voters do not have dominant strategies. Also in this context, we prove that there exists a unique pure strategy perfect equilibrium outcome, which is also the unique "pure" outcome induced by perfect equilibria.

The main advantage of having a unique equilibrium outcome is naturally in that we can compare the outcomes in the two systems. We carry out such a comparison upon various distributions of players' bliss policies. We consider two leading cases. The first one identifies a situation where each district of the majoritarian system is a replica, in terms of medians' bliss policies distribution,

[2] Such a formulation allows, for example, any majority premium.

of the national district of the proportional system: the case with full homogeneity across districts. The second case investigates the circumstance under which the districts (of the majoritarian system) are equally sized and ordered according to the political preferences of their voters from left to right:[3] this is the case of maximal dishomogeneity across districts. We find that in the case of homogeneity across districts, the outcome may differ depending on which electoral system is adopted. A single district proportional system favors a more moderate outcome, since it protects minorities dispersed in different districts more than a multidistrict majority system. In the case of extreme heterogeneity across districts, the outcomes are instead the same independently of the electoral system. Hence, differences in electoral outcomes are a joint product of the electoral system and the distribution of voters. In societies where leftist voters are concentrated in some districts and rightist voters in others the choice of the electoral system - proportional vs. multidistrict majority- will tend not to affect the political outcome, while in societies where electoral districts are similar to each other in terms of the political preferences of their voters, the outcome will tend to be more moderate when elections are held with a proportional system than when elections are held with a multidistrict majority system. This is fairly intuitive since with a lower concentration of like-minded voters, in a multidistrict majority system fewer votes are wasted on a candidate who would win anyway.

From the analysis of the above voting games we want to emphasize two points. The first one deals with the solution concept. The second one relates to the use of strategic voting per se.

As already explained in the previous lines, the Nash solution concept is not adequate to solve the voting games we define. Moreover, regardless of the two-party structure, also the undominated principle is not helpful. Nevertheless, if we resort to the trembling-hand-perfect solution concept we obtain a unique outcome in pure strategies. As already pointed out, extending to mixed strategies such an outcome is the only one assigning probability 1 to a given policy.

Let us spend now few words on the use of strategic voting. A common criticism to such an approach is that strategic voting models have multiple outcomes, and such multiplicity "is more severe the larger the size of the electorate ... regardless of the solution concept that is used" (Merlo, 2005, p. 15). This paper points out that this is not necessarily the case.

Related literature.

As pointed out at the beginning of this introduction, this paper belongs to the "non-majoritarian" literature of legislative election (Alesina and Rosenthal,

[3] Specifically equally sized districts have been ordered according to the political preferences of their voters, with the first district being inhabited by the first $\frac{n}{k}$ most leftwing voters, the second by the next $\frac{n}{k}$ most leftwing voters and the following districts being inhabited each by $\frac{n}{k}$ increasingly more rightwing voters.

1996), which focuses not on which party has the majority in parliament, but rather on the composition of it. The main contribution of this paper is to carry out such an analysis by not relying on the assumption that the vote share taken in the election is translated into an equal seats share in parliament. We define instead the policy outcome as function of the number of seats parties win in parliament.

Furthermore, this paper relates to three strands of the political economy literature: the multidistrict majoritarian, the proportional elections and the recent growing literature on the comparison of policy outcomes under different electoral rules.

Many scholars analyzed elections under multidistrict majority rule. Palfrey (1989) seminal paper proves that in an uncertain framework strategic voting under plurality rule leads to duvergerian equilibria.[4] Cox (1994, 1987, 1997) in many contributions analyzes a k-candidates m-seats model, finding duvergerian equilibria, based on a rational expectations condition. Austen-Smith (1986) develops a model, with two candidates, in which voters, located in various districts, vote taking into account the policy outcome, which is a probabilistic function of the set of winning districts representatives. Callander (2005) studies parties competing over a continuum of districts under plurality rule, but when voters act sincerely. He finds a duvergerian result, and policy does not converge to the median position.

Concerning the proportional elections, the most related paper is Alesina and Rosenthal (1996) in which they analyze the strategic voting behavior of a continuum of voters facing an institutional context where there are two branches of the government: the executive, elected by plurality rule, and the legislature, elected by proportional rule. The policy outcome is the result of the compromise between these two branches: hence the composition of the legislature is crucial. The main implication of this model is that "divided government" (that is the situation in which the majority in Congress is in favor of the party who has lost the presidential election) can be explained through the behavior of voters with intermediate (that is, situated in between parties' announced positions) preferences, who take advantage of the institutional structure above, balancing the plurality of the winning party in the executive by voting in favor of the opposite party in the legislative election. The solution of such a voting game relies on a coalition-proof behavior (to be precise they use a refinement of coalition-proof, the abstract stable set), the main reason being the necessity to circumvent the difficulties arising from the assumption of a continuum of voters.[5]

Finally, there is a growing interest in the issue of how different institutions may affect national policies. Persson-Tabellini (1999, 2000) model two-party

[4]By duvergerian equilibria is meant that under plurality rule only two-party equilibria emerge (Duverger, 1954).

[5]A different strand of the literature takles the question of voting under proportional representation by defining also a post-electoral bargaining stage among elected parties (see, among others, Baron and Diermeier, 2001).

competition in two scenarios defined by the electoral rule: proportional versus majoritarian elections. Proportional rule is characterized by a single national district, and, hence, winning more than 50% of the votes of the population means winning the election. Majoritarian elections are defined under plurality rule (first past the post) in three single-candidate districts, each one coinciding with a specific group of the population. Winning means winning two out of the three districts. The model is solved by assuming two forms of uncertainty (for whole population and group-specific). The main result is that in majoritarian countries (as opposed to proportional ones) elections increase competition in key districts, leading to an increase in targeted redistribution at the expense of programs benefiting a large majority of voters. Finally, Morelli (2004) analyzes a more complex game (with party formation) in which there are three equal sized districts and a continuum of voters. He only considers sincere voting and a simple form of strategic voting. The main result is that for asymmetric distribution of policy preferences the policy outcome under proportional election is more moderate than the one with plurality, while when preferences are similar across districts both plurality and proportional rules lead to the median voter's preferred policy outcome.

The paper is organized as follows. Section 2 introduces the general setup of the model, and then we solve the multidistrict majoritarian election in Section 3, and the proportional one in Section 4. We devote Section 5 to the comparison of the policy outcomes under the two systems, and, finally, Section 6 concludes the paper.

2 The model

Consider a society electing a parliament of k members.

The policy space. The unidimensional policy space $\mathbb{X}$ is a closed interval of the real line, and without loss of generality we assume $\mathbb{X} = [0, 1]$.

Parties. There are two parties, indexed by $p \in P = \{L, R\}$. Each party p is characterized by a policy position $\theta_p \in \mathbb{X}$, such that $\theta_L < \theta_R$.

Voters. There is a finite set of voters $N = \{1, 2, ..., n\}$. Each voter $i \in N$ has a most preferred policy (his bliss point, sometimes referred to as his location) $\theta_i \in \mathbb{X}$. Voters' preferences are single peaked and symmetric. Let us denote as $u_i(X)$ player i's utility function over the policy space. Given the set of parties P, each voter i casts his vote for one of them. Hence, the pure strategy set of voter i is given by $S_i = \{L, R\}$, and let denote $S = S_1 \times S_2 \times ... \times S_n$. A mixed strategy of player i is a vector $\sigma_i = (\sigma_i^L, \sigma_i^R)$ where each σ_i^p represents the probability that player i votes for party $p \in P$. As usual, the mixed strategy which assigns probability one to a pure strategy will be denoted by such a pure strategy.

The electoral rule. Voters vote to elect a parliament composed by k representatives. Given a pure strategy combination $s \in S$, the electoral rule determines the composition of the parliament, that is to say the seats allocated to each party. We consider two different electoral rules: majority rule and proportional rule (see Persson and Tabellini, 2000). Let $\varphi : S \rightarrow \{0, 1, ..., k\}$ be the function that maps votes into the number of seats allocated to party L, being the seats of R simply $k - \varphi(s)$.

The policy outcome. The final policy outcome is the result of a bargaining process among parties. We do not explicitly model this bargaining process but we assume that it depends only on the number of seats each party has in the parliament. In other words we assume the existence of a function $X(.)$ that maps the number of seats obtained by party L into the policy space, i.e., $X : \{0, 1, ..., k\} \rightarrow \mathbb{X}$. We assume that $X(.)$ is a decreasing function, that is to say the more seats L obtains, the more leftist the policy is.

Given the electoral rule φ and the policy outcome function X, the utility that voter $i \in N$ gets under the pure strategy combination s is:

$$U_i(s) = u_i(X(\varphi(s))).$$

Given a mixed strategy combination $\sigma = (\sigma_1, ..., \sigma_n)$, because players make their choice independently of each other, the probability that $s = (s_1, s_2, ..., s_n)$ occurs is:

$$\sigma(s) = \prod_{i \in N} \sigma_i^{s_i}.$$

The expected utility that player i gets under the mixed strategy combination σ is:

$$U_i(\sigma) = \sum \sigma(s) U_i(s).$$

In the following, as usual, we shall write $\sigma = (\sigma_{-i}, \sigma_i)$, where $\sigma_{-i} = (\sigma_1, ...\sigma_{i-1}, \sigma_{i+1}, ...\sigma_n)$ denotes the $(n-1)-$tuple of strategies of the players other than i. Furthermore s_i will denote the mixed strategy σ_i that gives probability one to the pure strategy s_i.

For $j \in \{1, 2, ..., k\}$, define $\alpha_j = \frac{X(j)+X(j-1)}{2}$. If a voter $i \in N$ has his bliss point equal to α_j such a voter is indifferent between a parliament with j members of L and one with just $(j-1)$ members of L. In order to simplify the reading, and the writing, of the paper we assume that no such a voter exists.[6]

An outcome is a probability distribution over policies, we'll call "pure" an outcome that assigns probability one to a given policy, and we'll denote it by that policy.[7]

[6] Case by case we will discuss what happens if such an assumption is not satisfied, but the general discuss and the main propositions will be developed under such an assumption.

[7] We should have used the term *degenerate* outcome, but we have preferred the above terminology.

3 The multidistrict majoritarian election

We first consider a situation in which there are k districts, indexed by $d \in D = \{1, 2, ..., k\}$. Voters are hence distributed in the k districts and let N_d be the set of voters in district d, i.e. $N_1, N_2, ..., N_k$ is the partition of N in the k districts.[8] We assume that in each district d there is an odd number of voters n_d. Let $m_d \in M = \{m_1, ..., m_k\}$ be the median voter in district d, and, without loss of generality, assume that $m_1 \leq m_2 \leq ... \leq m_k$. Let us define the distribution $F^m(\theta) = \{\# m_d \in M \text{ s.t. } m_d \leq \theta\}$.[9]

In each district voters elect a representative belonging either to party L or to party R by majority rule. Given a pure strategy combination $s = (s_1, s_2, ..., s_n)$, let $s_d = (s_i)_{i \in N_d}$ be the pure strategy combination of the voters in district d. District d is won by the party which gets more votes and let $D^L(s)$ be the districts where L wins, hence the electoral rule φ^M is simply:

$$\varphi^M(s) = \# D^L(s).$$

3.1 The solution

The above game is a typical example of a situation in which the use of the Nash solution concept is completely inadequate. As a matter of fact, in every district, the election of any candidate is a Nash equilibrium outcome, if there are at least three voters. Differently from standard models with two parties, in this case not even the concept of undominated equilibria seems appropriate. As a matter of fact, if a voter's bliss point is located (strictly) in between α_k and α_1, it follows that such a voter does not have any dominated strategy. As a consequence, if all the bliss points are in between α_k and α_1, not even sophisticated voting can help us shape the set of solutions, that is to say for every possible composition of the parliament there exists a sophisticated equilibrium leading to that composition of the parliament. We then need to use a solution concept stronger than undominated equilibrium. Limiting the analysis to pure strategies we will show the existence of a unique perfect equilibrium outcome. Allowing for mixed strategies uniqueness cannot be hoped for, nevertheless the above outcome is the only "pure" one, i.e. the only one assigning probability 1 to a given policy.

Instead of working directly with perfect equilibria we prefer to introduce the weaker (as we will show later) concept of district sincerity. In words, a strategy combination is district sincere if, given the strategies of the players in

[8] Notice that with uppercase we denote sets, and with lowercase their cardinality. Hence N_1 is the set of voters in district 1, and by n_1 we denote the number of them.

[9] The assumption about the oddness of the number of voters in each district assures that the electoral result does not end in a tie. This implies two things. First, a pure strategy combination leads to what we have defined as a "pure" outcome. Second, the median is uniquely defined. We could have skipped this assumption by dealing with a deterministic tie-breaking rule and by defining accordingly the median. A preliminary cost-benefit analysis suggested us to make use of this assumption.

the other districts, every voter who strictly prefers party L/R winning in his district votes for party L/R. Formally, given σ, we shall write $\sigma = \left(\sigma^{-d}, \sigma^{d}\right)$, where $\sigma^{-d} = (\sigma_i)_{i \in N \diagup N_d}$ denotes the $(n - n_d)$ −tuple of strategies of the players outside the district d while $\sigma^{d} = (\sigma_i)_{i \in N_d}$ denotes the n_d−tuple of strategies of the players in the district d. Moreover, let L^d $\left(R^d\right)$ denote the n_d−tuple of pure strategies of the players in the district d where everybody votes for L[10] (R).[11]

Definition 1 *District-sincerity. A strategy combination σ is district-sincere if for every district d and for every player i in district d the following holds:*

$$\begin{aligned} U_i\left(\sigma^{-d}, L^d\right) - U_i\left(\sigma^{-d}, R^d\right) &> 0 \quad \textit{then } \sigma_i = L \\ U_i\left(\sigma^{-d}, L^d\right) - U_i\left(\sigma^{-d}, R^d\right) &< 0 \quad \textit{then } \sigma_i = R \end{aligned}$$

Notice that every district-sincere strategy combination is an equilibrium, because a player affects the outcome only if he is pivotal in his district and district sincerity implies that the outcome is affected in the "right" direction.

Now, we will prove that there is only a pure strategy district sincere outcome. To this end let us define:[12]

$$\bar{d}^M = \begin{cases} 0 & if\ m_1 > \alpha_1 \\ \max d \text{ s.t } m_d \leq \alpha_d & if\ m_1 \leq \alpha_1. \end{cases} \tag{1}$$

In words, given all districts d such that the median voter location m_d is on the left of α_d (i.e. the average of the outcomes when L wins d and $(d-1)$ districts), we take the rightmost of them. In the following we prove that the unique pure strategy district sincere outcome is the outcome where party L wins exactly $\bar{d}^M$ districts.

Proposition 1 $X\left(\bar{d}^M\right)$ *is the unique pure strategy district-sincere equilibrium outcome.*[13]

[10] Hence, L wins district d.

[11] For simplicity, we write the definition of district sincerity with the n_d-tuple of strategies of the players in district d given by everybody voting for L/R. Obviously, we could, at a cost of an heavier notation, have written any n_d-tuple of strategies leading to the winning of L/R in district d.

[12] We remind that we have assumed that no bliss point equal to α_1 exists and so $m_1 \neq \alpha_1$, and analogously $m_d \neq \alpha_d$. However since we are going to discuss in some cases also what happens if these condidions do not hold, we prefer to define $\bar{d}$ independently from the above conditions.

[13] In case $m_{\bar{d}} = \alpha_{\bar{d}}$, we would have two different possible outcome $X(\bar{d})$ and $X(\bar{d}-1)$.

Proof. We first prove that it exists a pure strategy district sincere equilibrium (PDSE) with outcome $X\left(\bar{d}^M\right)$.

Consider the following strategy combination $\bar{s} = (\bar{s}_1, ..., \bar{s}_n)$ with:

$$\bar{s}_i = L \ \ if \ i \in d \leq d^M \ and \ \theta_i < \alpha_{\bar{d}^M} \ or \ i \in d > d^M \ and \ \theta_i < \alpha_{\bar{d}^M+1}$$

$$\bar{s}_i = R \ \ if \ i \in d \leq d^M \ and \ \theta_i > \alpha_{\bar{d}^M} \ or \ i \in d > d^M \ and \ \theta_i > \alpha_{\bar{d}^M+1}$$

(i.e., in every district $d \leq \bar{d}^M$, every voter i with $\theta_i < \alpha_{\bar{d}^M}$ votes for party L, and every voter i with $\theta_i > \alpha_{\bar{d}^M}$ votes for party R; in every district $d > \bar{d}^M$: every voter i with $\theta_i < \alpha_{\bar{d}^M+1}$ votes for L, and every voter i with $\theta_i > \alpha_{\bar{d}^M+1}$ votes for party R).

Notice that under $\bar{s}$ party L wins every district $d \leq \bar{d}^M$, because in such a case $m_d < \alpha_d$, while R wins all the district $d > \bar{d}^M$, because in such a case $m_d > \alpha_d \geq \alpha_{\bar{d}^M+1}$, hence the outcome of $\bar{s}$ is $X(\bar{d}^M)$. Furthermore $\bar{s}$ is district sincere, because in every district where L wins voters who prefer $X(\bar{d}^M)$ to $X(\bar{d}^M - 1)$ vote for L and the others for R, while in the district where R wins voters vote accordingly to their preferences over $X(\bar{d}^M)$ and $X(\bar{d}^M + 1)$.

We now prove that no other PDSE outcome exists. Suppose we have an equilibrium with $\hat{d} \neq \bar{d}^M$ districts won by L. District-sincerity implies that in districts won by L, every voter i with $\theta_i < \alpha_{\hat{d}}$ votes for L, and every voter i with $\theta_i > \alpha_{\hat{d}}$ votes in favor of party R. Moreover, in districts in which R is getting the majority, voter i with $\theta_i < \alpha_{\hat{d}+1}$ votes for L, and voter i with $\theta_i > \alpha_{\hat{d}+1}$ votes for R. Suppose first that $\hat{d} < \bar{d}^M$, then it must be $\alpha_{\bar{d}^M} \leq \alpha_{\hat{d}+1} < \alpha_{\hat{d}}$ and hence district-sincerity implies that party L gets at least $\bar{d}^M$ districts, which contradicts $X\left(\hat{d}\right)$ being a district sincere equilibrium outcome. Mutatis mutandis, $\hat{d} > \bar{d}^M$ implies $\alpha_{\hat{d}+1} < \alpha_{\hat{d}} \leq \alpha_{\bar{d}^M+1}$ and this with district sincerity and the fact that $\alpha_{\bar{d}^M+1} < m_{\bar{d}^M+1}$implies party R wins at least $\left(k - \bar{d}^M\right)$ districts, and, hence, party L wins at most $\bar{d}^M$ districts contradicting $X\left(\hat{d}\right)$ being a district sincere equilibrium outcome. ■

Given the assumption that no voter is located in α_j $(j = 1, ..., k)$, if σ is district sincere and assigns probability one to a given policy, then σ is a pure strategy combination. Hence, we have:

Corollary 2 $X\left(\bar{d}^M\right)$ *is the unique "pure" outcome induced by district-sincere equilibria.*

3.1.1 Perfect equilibrium

The concept of perfect equilibrium was introduced by Selten (1975):

Definition 2 *A completely mixed strategy* σ^ε *is an* ε*-perfect equilibrium if*

$$
\begin{array}{rcl}
\forall i & \in & N, \quad \forall s_i, s_i^{'} \in S_i \\
if \quad U_i\left(s_i, \sigma_{-i}^{\varepsilon}\right) & > & U_i\left(s_i^{'}, \sigma_{-1}^{\varepsilon}\right) \quad then \\
\sigma_i^{\varepsilon}\left(s_i^{'}\right) & \leq & \varepsilon.
\end{array}
$$

A strategy combination σ is a perfect equilibrium if there exists a sequence $\{\sigma^{\varepsilon}\}$ of ε-perfect equilibria converging (for $\varepsilon \to 0$) to σ.

Because a dominated strategy is never a best reply to a completely mixed strategy of the opponent and, hence, in every ε-perfect equilibrium it is played with probability less than ε, the perfect equilibrium concept is a refinement of the undominated equilibrium concept. The next proposition shows that, in this model, it is a refinement also of district sincerity.

Proposition 3 *Every perfect equilibrium σ is district sincere.*

Proof. Let $f_i(\sigma)$ denote the probability player i is pivotal under the strategy combination σ in his district d. Clearly, we can write:

$$
U_i(L, \sigma_{-i}) - U_i(R, \sigma_{-i}) = f_i(\sigma)\left[U_i\left(\sigma^{-d}, L^d\right) - U_i\left(\sigma^{-d}, R^d\right)\right] \tag{2}
$$

and, if $\sigma >> 0$, then $f_i(\sigma)$ is strictly positive. Suppose now σ is not district sincere. This implies there exists a district d and a player $i \in N_d$ such that either $U_i\left(\sigma_{-d}, L^d\right) - U_i\left(\sigma_{-d}, R^d\right) > 0$ and $\sigma_i(R) > 0$ or $U_i\left(\sigma_{-d}, L^d\right) - U_i\left(\sigma_{-d}, R^d\right) < 0$ and $\sigma_i(L) > 0$. Let us consider the first case. Take a sequence of completely mixed strategy combinations σ^{ε} converging to σ. Sufficiently close to σ, $f_i(\sigma^{\varepsilon})$ is strictly positive as well as $\left[U_i\left(\sigma^{\varepsilon^{-d}}, L^d\right) - U_i\left(\sigma^{\varepsilon^{-d}}, R^d\right)\right]$ and hence R is not a best reply for player i. It follows that if σ^{ε} is a sequence of $\varepsilon-$perfect equilibria, $\sigma_i^{\varepsilon}(R) \leq \varepsilon$, and hence $\sigma_i(R) = 0$. *Mutatis mutandis* the second case.

■

Propositions 1 and 3 directly imply that the only possible pure strategy perfect equilibrium outcome of the model can be $X\left(\bar{d}^M\right)$. Because not every district-sincere equilibrium is perfect, we still have to prove that there exists a pure strategies perfect equilibrium whose outcome is $X\left(\bar{d}^M\right)$. This is accomplished considering $\bar{s}$ as defined in the proof of Proposition 1. From (2), it is immediate that $\bar{s}$ is a best reply to every strategy combination sufficiently close to it, hence perfect.[14] Then, we have:

Proposition 4 $X\left(\bar{d}^M\right)$ *is the unique pure strategy perfect equilibrium outcome.*

[14] This shows also that $\bar{s}$ is a strictly perfect equilibrium (Okada, 1981) and a stable set as defined in Kholberg and Mertens (1986). Notice that $\bar{s}$ is an absorbing retract (Kalai and Samet, 1984) and, hence, also a stable set accordingly to the definition of Mertens (1989).

Moreover, from Corollary 2, Propositions 3 and 4 immediately follows that:

Corollary 5 $X\left(\bar{d}^M\right)$ *is the unique "pure" outcome induced by perfect equilibria.*

We now introduce an example that will be useful in discussing all the main features of this type of voting games. Despite the fact that for every possible outcome there is an undominated equilibrium of the example with that outcome, the game has a unique pure strategy district-sincere equilibrium outcome, which is also the only pure strategy perfect equilibrium outcome. Nevertheless, such a unique outcome may result from two different equilibria. Hence, a uniqueness result (in terms of equilibrium strategies) cannot be hoped for. Furthermore, also a mixed strategy equilibrium exists, supporting the district-sincere equilibrium outcome with some probability (positive, but different from one). Hence, the uniqueness of the outcome must rely either on the use of pure strategies, or, when mixed strategies are allowed, on limiting the analysis to outcomes assigning probability one to a given policy.

3.2 Example 1

The parties' positions are $\theta_L = 0.1$ and $\theta_R = 0.9$. There are two districts 1 and 2 with three voters each. Both districts have one voter with bliss point in 0.31 and one in 0.69. The medians are located in $m_1 = 0.4$ and $m_2 = 0.6$. Policies are: $X(0) = 0.9 > X(1) = 0.5 > X(2) = 0.1$. Every voter i's utility is simply minus the distance between his bliss point and the policy X. Because $\alpha_2 = 0.3$ and $\alpha_1 = 0.7$ the game has no dominated strategies and, hence, everybody voting for L is an undominated equilibrium with outcome $X(2) = 0.1$. Analogously, we have an undominated equilibrium where everybody votes for R with outcome $X(0) = 0.9$.

According to Proposition 1, $X(1) = 0.5$ is the unique pure strategy district-sincere equilibrium outcome and according to Proposition 3 is the only pure strategy perfect equilibrium outcome.

Nevertheless, there are two different pure strategy district sincere and perfect equilibria.[15] In one every voter in district 1 votes for L and every voter in district 2 votes for R, in the other every voter in district 1 votes for R and every voter in district 2 votes for L.

The game has also a mixed equilibrium $(\bar{\sigma})$ in which voters in 0.31 vote for L, voters in 0.69 vote for R, while the median voter in district 1 plays the mixed strategy $\frac{1}{3}L + \frac{2}{3}R$ and the median voter in district 2 plays $\frac{2}{3}L + \frac{1}{3}R$. Under $\bar{\sigma}$, $X(0)$ occurs with probability $\frac{2}{9}$, $X(2)$ with probability $\frac{2}{9}$ and $X(1)$ with probability $\frac{5}{9}$.

[15] Both of them are also strictly perfect and stable.

It is easy to verify that this equilibrium (i.e. $\bar{\sigma}$) is district sincere. Consider voters in district 1: the strategy combination of the voters in district 2 implies party L wins with probability equal to $\frac{2}{3}$ in district 2. In such a case the median voter of district 1 is indifferent between a leftist or a rightist winning in his district, while the voter located in 0.31 strictly prefers that district 1 is won by L,[16] while the voter located in 0.69 will prefer that district 1 is won by R.[17] Similarly for voters in district 2.

Now we want to prove that $\bar{\sigma}$ is perfect and that even applying stronger solution concept than perfection as strategic stability (Mertens, 1989) we cannot eliminate it. Notice that $\bar{\sigma}$ is also quasi-strict (this easily follows from $\bar{\sigma}$ being district-sincere and from the fact that, given that in each district the median voter randomizes and the other two voters vote one for L and one for R, voters are pivotal with positive probability). From that it easily follows that it is isolated because the equilibria near $\bar{\sigma}$ can be studied simply analyzing the following 2×2 game (Γ) among the two median voters (the row player being the one in district 1).

	L	R
L	$-0.3, -0.5$	$-0.1, -0.1$
R	$-0.1, -0.1$	$-0.5, -0.3$

This game has two pure strategy equilibria (L, R), (R, L) and a mixed one $\left(\frac{1}{3}L + \frac{2}{3}R, \frac{2}{3}L + \frac{1}{3}R\right)$ which correspond to $\bar{\sigma}$. Since $\left(\frac{1}{3}L + \frac{2}{3}R, \frac{2}{3}L + \frac{1}{3}R\right)$ is isolated and quasi-strict then it is a strongly stable equilibrium of Γ (cf. van Damme, 1991:55, th 3.4.4). Moreover, because the other players are using their strict best reply in $\bar{\sigma}$, it follows that $\bar{\sigma}$ is a strongly stable equilibrium (Kojima *et. al.*, 1985) of the voting game, and, hence, a Mertens' stable set.

4 Proportional representation

We study now the electoral rule corresponding to proportional representation. We analyze the case (see Persson and Tabellini, 2000) where there is only one voting district electing k representatives. We assume, without loss of generality, that voters' bliss policies are ordered such that $\theta_1 \leq \theta_2 \leq ... \leq \theta_n$, and are distributed in such a national district accordingly to the distribution $F(\theta) = \{\#i \in N \text{ s.t. } \theta_i \leq \theta\}$.

Voters elect representatives belonging to party L and R by proportional rule. There are various rules used in proportional system to transform votes into seats, we use a very general one, which allows, for example, any majority

[16] Because $\frac{1}{3}(-0.19) + \frac{2}{3}(-0.21) > \frac{1}{3}(-0.59) + \frac{2}{3}(-0.19.)$
[17] Because $\frac{1}{3}(-0.21) + \frac{2}{3}(-0.19) > \frac{1}{3}(-0.19) + \frac{2}{3}(-0.59)$

premium. To get d representatives, $d = 0, 1, ..., k$, party L needs at least n_d number of votes (i.e. to elect exactly d representatives party L needs a number of votes in between $[n_d, n_{d+1})$).[18]

Given a pure strategy combination $s = (s_1, s_2, ..., s_n)$ let $N_d^L(s)$ be the set of citizens voting for party L under s, and let us define by $n_d^L(s)$ its cardinality. Hence, there exists a unique d^* such that $n_{d^*}^L \in [n_{d^*}, n_{d^*+1})$, and the electoral rule φ^P is simply:

$$\varphi^P(s) = d^*.$$

4.1 The solution

Similarly to the majoritarian case previously studied, because voters located in between α_1 and α_k do not have any dominant strategy, also in this case we need a stronger solution concept than undominated equilibrium. Limiting the analysis to pure strategy equilibria we prove that there exists a unique perfect equilibrium outcome. Moreover, this is the unique "pure" outcome.

To this end let us give the following definition:

$$\bar{d}^P = \begin{cases} 0 & if\ F(\alpha_1) < n_1 \\ \max d \text{ s.t } F(\alpha_d) \geq n_d & if\ F(\alpha_1) \geq n_1 \end{cases} \tag{3}$$

In words, $\bar{d}^P$ is the maximum number of seats for the left party such that the number of voters whose bliss points are on the left of α_d (that is the outcome averaging a parliament with d and $d-1$ seats for L) is greater or equal to the minimum number of votes needed to elect d representatives for party L.

Proposition 6 $X\left(\bar{d}^P\right)$ *is the unique pure strategy perfect equilibrium outcome and the unique "pure" outcome induced by perfect equilibria.*

Proof. We first prove that there exists a perfect equilibrium with the unique "pure" outcome $X\left(\bar{d}^P\right)$.
We have to analyze three cases:[19]
i) $\bar{d}^P \neq k$ and $\theta_{n_{\bar{d}^P}} > \alpha_{\bar{d}^P+1}$
Consider the following strategy combination $\bar{s} = (\bar{s}_1, ..., \bar{s}_n)$ with:

$$\bar{s}_i = L\ \ if\ i \in [1, 2, ..., n_{\bar{d}^P}]$$

$$\bar{s}_i = R\ \ if\ i \in [n_{\bar{d}^P} + 1, ..., n]$$

[18] Analogously to the multidistrict majoritarian case (see footnote 9) we rely on the use of a deterministic rule to determine the seats' allocation.
[19] In order to avoid duplication of proof, if $\bar{d}^P = 0$, let $\theta_0 = 0$ and hence refer to (ii).

Notice that under $\bar{s}$ exactly $\bar{d}^P$ seats are won by L. Now we show that $\bar{s}$ is perfect

Notice that L is a strict best reply for every $i \in [1, 2, ..., n_{\bar{d}^P}]$, because if one of them vote for R instead L the outcome moves from $X\left(\bar{d}^P\right)$ to $X\left(\bar{d}^P - 1\right)$ which is worst for them because they are located to the left of $\alpha_{\bar{d}^P}$. Consider the completely mixed strategy combination σ^ε :

$$\sigma_i^\varepsilon = (1 - \varepsilon^n) L + \varepsilon^n R \;\; if \; i \in [1, 2, ..., n_{\bar{d}^P}]$$

$$\sigma_i^\varepsilon = (1 - \varepsilon) R + \varepsilon L \;\; if \; i \in [n_{\bar{d}^P} + 1, ..., n]$$

We claim that, for ε sufficiently close to zero, σ^ε is an $\varepsilon - perfect$ equilibrium. Because L is a strict best reply to $\bar{s}$ for $i \in [1, 2, ..., n_{\bar{d}^P}]$ it is also for close-by strategies. Notice that the probability a player $i \in [n_{\bar{d}^P} + 1, ..., n]$ is "pivotal" between the election of $\bar{d}^P$ and $\bar{d}^P + 1$ of L candidates is infinitely greater than every other probability in which his vote matters. Because all these players are located to the right of $\alpha_{\bar{d}^P+1}$ R is preferred for them to L and, hence, σ^ε is an $\varepsilon - perfect$ equilibrium. Therefore $\bar{s}$ is perfect.

ii)$\bar{d}^P \neq k$ and $\theta_{n_{\bar{d}^P}} < \alpha_{\bar{d}^P+1}$

Let $\tilde{n}$ the larger i such that $\theta_i < \alpha_{\bar{d}^P+1}$. By the definition of $\bar{d}^P$ and because $\theta_{n_{\bar{d}^P}} < \alpha_{\bar{d}^P+1}$, we have $\tilde{n} \in \left[n_{\bar{d}^P}, n_{\bar{d}^P+1}\right)$. Consider the following strategy combination $\tilde{s}$:

$$\tilde{s}_i = L \;\; if \; i \in [1, 2, ..., \tilde{n}]$$

$$\tilde{s}_i = R \;\; if \; i \in [\tilde{n} + 1, ..., n]$$

Notice that under $\tilde{s}$ exactly $\bar{d}^P$ seats are won by L. Now we show that $\tilde{s}$ is perfect. To this end consider the completely mixed strategy combination σ^ε :

$$\sigma_i^\varepsilon = (1 - \varepsilon^n) L + \varepsilon^n R \;\; if \; i \in [1, 2, ..., \tilde{n}]$$

$$\sigma_i^\varepsilon = (1 - \varepsilon) R + \varepsilon L \;\; if \; i \in [\tilde{n} + 1, ..., n]$$

We claim that, for ε sufficiently close to zero, σ^ε is an $\varepsilon - perfect$ equilibrium. Notice that the probability a player is "pivotal" between the election of $\bar{d}^P$ and $\bar{d}^P + 1$ of L candidates is infinitely greater than every other probability in which his vote matters. Because for all the players located to the left (right) of $\alpha_{\bar{d}^P+1}$, $L(R)$ is preferred to $R(L)$, σ^{ε} is an $\varepsilon - perfect$ equilibrium. Therefore $\tilde{s}$ is perfect

iii)$\bar{d}^P = k$

Let $\breve{n}$ the larger i such that $\theta_i < \alpha_k$. By the definition of $\bar{d}^P$ we have that $\breve{n} \geq n_k$. Consider the following strategy combination $\breve{s}$:

$$\breve{s}_i = L \ \ if \ i \in [1, 2, ..., \breve{n}]$$

$$\breve{s}_i = R \ \ if \ i \in [\breve{n}+1, ..., n]$$

Notice that under $\breve{s}$ all the k seats are won by L. Moreover for every completely mixed strategy combination close to $\breve{s}$, the probability a player is "pivotal" between the election of k and $k-1$ of L candidates is infinitely greater than every other probability in which his vote matters. Hence, $\breve{s}$ is perfect.

Now we prove that no other "pure" outcome is induced by a perfect equilibrium. Suppose we have a perfect equilibrium σ^{δ} which induces $X(\delta)$ as policy outcome. Because for every sequence of completely mixed strategy combination converging to σ^{δ}, for every player, the probability of the event "being pivotal between $X(\delta+1)$ and $X(\delta)$" is infinitely greater than the probability of the event "being pivotal between $X(\delta+j)$ and $X(\delta+1+j)$" $(j = 1, .., k-\delta-1)$ and the probability of the event "being pivotal between $X(\delta)$ and $X(\delta-1)$" is infinitely greater than the the probability of the event "being pivotal between $X(\delta-j)$ and $X(\delta-1-j)$" $(j = 1, .., k-\delta-1)$ we must have:

(α) $\forall i \ s.t. \ \theta_i < \alpha_{\delta+1}$ $\quad \sigma_i^{\delta} = L$

(β) $\forall i \ s.t. \ \theta_i > \alpha_{\delta}$ $\quad \sigma_i^{\delta} = R$

Suppose $\delta < \bar{d}^P$. This would imply that $\alpha_{\delta+1} \geq \alpha_{\bar{d}^P}$, and, by (α), it follows that in σ^{δ} party L would receive at least $n_{\bar{d}^P}$ contradicting the fact that just δ of its candidates are elected.

Suppose $\delta > \bar{d}^P$. Notice that $\delta > \bar{d}^P$ implies that $\alpha_{\bar{d}^P+1} \geq \alpha_{\delta}$ and the above condition (β) implies that in σ^{δ} party R takes at least all the votes of the voters located to the right of $\alpha_{\bar{d}^P+1}$. By the definition of $\bar{d}^P$, it follows that, even if all the others voters vote for L, the leftist party cannot win $\bar{d}^P+1$ seats, which contradicts $\delta > \bar{d}^P$. ■

5 Comparing electoral systems

It is interesting to compare the equilibrium outcome in the single district proportional and the multidistrict majority system. Such a comparison is made straightforward by our uniqueness results. For the sake of the comparison, in this section we specify a particular electoral rule dictating the minimum number of votes required to elect a member of parliament when the single district proportional system is adopted. The minimum number of votes needed to elect d members of parliament with the single district proportional system is $\frac{n}{k}(d-1)+\frac{1}{2}\frac{n}{k}$.[20] In case the multidistrict majority system is used, the electoral rule requires the leftist party to obtain at least half of a district votes in order to carry the district. We remind the reader that we defined with $X\left(\overline{d}^{P}\right)$ the unique perfect equilibrium outcome in the single district proportional and with $X\left(\overline{d}^{M}\right)$ as the unique district sincere equilibrium outcome in the multidistrict majority systems.

When a multidistrict majority system is adopted, the electoral outcome may depend on how voters are distributed across districts. Since the electoral outcome is instead independent of voters' distribution across districts when a single district proportional system is adopted, the comparison of electoral outcomes between the two systems is bound to be affected by the distribution of voters across districts. In order to get to grips with such an issue, we consider two extreme distributions of voters across districts. We first look at a situation of homogeneity across districts. This case represents a society where districts of the multidistrict majority system are similar to each other and similar to the single district of the proportional system, in terms of the political preferences of their voters. More specifically districts are homogeneous in the sense that their median voters have the same preferences which, hence, coincide with the preferences of the median voter of the single district in the proportional system. We then examine a case of heterogeneity across districts. In this alternative society, districts of the multidistrict majority system are characterized by diverse political orientations, with some districts being a stronghold of the leftist party some others a stronghold of the rightist party and some other districts inhabited by voters with more mixed political orientations. Specifically we consider a situation of extreme heterogeneity across districts where - equally sized- districts have been ordered according to the political preferences of their voters, with the first district being inhabited by the first $\frac{n}{k}$ most leftist voters, the second by the next $\frac{n}{k}$ most leftist voters and the following districts being inhabited each by $\frac{n}{k}$ increasingly more rightist voters.

We find that in the case of homogeneity across districts, the outcome may differ depending on which electoral system is adopted. A single district proportional system favours a more moderate outcome, since it protects minorities dispersed in different districts more than a multidistrict majority system. In the case of extreme heterogeneity across districts, the outcomes are instead the same independently of the electoral system. Differences in electoral outcomes are a joint product of the electoral system and the distribution of voters. In societies where leftist voters are concentrated in some districts and rightist voters in others the choice of the electoral system - proportional vs. multidistrict majority- will tend not to affect the political outcome, while in societies where electoral districts are similar to each other in terms of the political preferences

[20] Given our assumptions, we never incur a tie in the remainder.

of their voters, the outcome will tend to be more moderate when elections are held with a proportional system than when elections are held with a multidistrict majority system. This is fairly intuitive since with a lower concentration of like-minded voters, in a multidistrict majority system fewer votes are wasted on a candidate who would win anyway.

5.1 Homogeneity across districts

We first consider a situation in which each district of the multidistrict majority system has the same median voter as the single district of the proportional system, i.e. $m_d = m$ for all d. The example that follows, points out that the two systems may give rise to different outcomes in this case.

Example 2. Consider a society with six voters electing a parliament of two members, i.e. $n = 6$, $k = 2$, two parties with preferred policies $\theta_L = 0$ and $\theta_R = 1$ respectively and the following symmetric outcome function $X(2) = 0 < X(1) = \frac{1}{2} < X(0) = 1$. The averages of consecutive outcomes are thus $\alpha_2 = \frac{X(2)+X(1)}{2} = \frac{1}{4}$ and $\alpha_1 = \frac{X(1)+X(0)}{2} = \frac{3}{4}$. Four of the six voters are leftist, having zero as their preferred policy, i.e. their bliss points are $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$, and the remaining two are rightist, having one as their preferred policy, i.e. $\theta_5 = \theta_6 = 1$. If the multidistrict majority system is adopted, two districts - inhabited by three voters each- elect a member of parliament each. A party carries a district if it obtains at least two votes in the district. District 1 is inhabited by two voters with bliss point in 0 and one voter with bliss point in 1, i.e. the three voters in district one are $\theta_1 = \theta_2 = 0$ and $\theta_5 = 1$, and district 2 is inhabited by two voters with bliss point in 0 and one voter with bliss point in 1, i.e. the three voters in district two are $\theta_3 = \theta_4 = 0$ and $\theta_6 = 1$. Observe that the median voter in each of the two district is a voter with 0 as his preferred policy, i.e. $m_1 = m_2 = 0$. If the single district proportional system is adopted, the six voters all belong to the single district and the electoral rule prescribes that at least $\frac{6}{2}\left(d - \frac{1}{2}\right)$ votes are needed to elect d representatives. Observe that the median voter in the single district is a voter with 0 as his preferred policy, i.e. $m = 0$. The unique district sincere equilibrium outcome of the multidistrict majority system is $X\left(\overline{d}^M\right) = X(2) = 0$, i.e. the leftist party obtains two members of parliament and implements its preferred policy. Indeed, observe that $\alpha_2 = \frac{1}{4} > m_2 = 0$. On the other hand the unique perfect equilibrium outcome of the proportional system is $X\left(\overline{d}^P\right) = X(1) = \frac{1}{2}$, i.e. the leftist party obtains one member of parliament and implements a moderate policy. Indeed, observe that $F(\alpha_1) = 4 > 3\left(\frac{1}{2}\right) = 1.5$ and $F(\alpha_2) = 4 < 3\left(\frac{3}{2}\right) = 4.5$.

In the multidistrict majority system two votes are enough to carry a district and thus four votes are enough to elect two members of parliament. The electoral rule of the proportional system, however, requires more than four votes to elect two members of parliament. The election result is markedly different in the two cases, with a two-nil victory for the left in the multidistrict majority system and a one-one draw in the proportional system. The policies implemented, which depend on the parliamentary strength of a party, differ as well in the two cases, with a more moderate policy in the second case. The example suggests that in a multidistrict majority system - with fairly homogeneous districts- a party may obtain a landslide victory in terms of seats in parliament without a corresponding landslide victory in terms of the number of votes, while in a proportional system there would be a closer relationship between number of seats in parliament and number of votes. The proportional system tends to moderate the electoral outcome. This happens because a minority of voters dispersed in different districts will be able to elect fewer members of parliament in a multidistrict majority system than in a single district proportional system. Since the final policy decision that is implemented is closer to a party preferred policy the stronger its parliamentary force is, the single district proportional system is conducive to a more moderate policy outcome. The following proposition proves that this intuition carries over to less special situations. In order to be able to compare leftist and rightist policies to moderate ones in a sensible way, we assume that the outcome function is symmetric around the mid point of the policy interval. We prove that the equilibrium policy outcome - if the single district proportional system is adopted as an electoral system- is not farther away from the mid point of the policy interval than the equilibrium policy outcome in case the electoral system adopted is the multidistrict majority one.

Proposition 7 *Assume that $m_d = m$,$\forall d$, and that $X(d)$ is symmetric around $\frac{1}{2}$,*[21] *then:*

a. if $X\left(\overline{d}^M\right) \leq \frac{1}{2}$, $X\left(\overline{d}^M\right) \leq X\left(\overline{d}^P\right) \leq \frac{1}{2}$;

b. if $X\left(\overline{d}^M\right) > \frac{1}{2}$, $\frac{1}{2} \leq X\left(\overline{d}^P\right) \leq X\left(\overline{d}^M\right)$.

Proof. *Part a.* We first prove that $X\left(\overline{d}^M\right) \leq X\left(\overline{d}^P\right)$. Given that $X\left(\overline{d}^M\right) \leq \frac{1}{2}$, suppose, contrary to the thesis, that $X\left(\overline{d}^P\right) < X\left(\overline{d}^M\right)$, i.e. $\overline{d}^P > \overline{d}^M$ and $\alpha_{\overline{d}^P} < \alpha_{\overline{d}^M}$. Since $X\left(\overline{d}^M\right)$ is the unique district sincere equilibrium outcome, it has to be that $\alpha_{\overline{d}^P} < m_{\overline{d}^P}$, otherwise $X\left(\overline{d}^P\right)$ would be the district sincere equilibrium outcome instead. Since by assumption $m_d = m$, $\forall d$, then $\alpha_{\overline{d}^P} < m$. Since m is the median of every district, $F\left(\alpha_{\overline{d}^P}\right) < \frac{n}{2}$. Moreover, $X\left(\overline{d}^P\right)$ is the equilibrium outcome in the proportional election hence: $F\left(\alpha_{\overline{d}^P}\right) \geq \frac{n}{k}\left(\overline{d}^P - \frac{1}{2}\right)$. These observations together imply:

$$\frac{n}{2} > F\left(\alpha_{\overline{d}^P}\right) \geq \frac{n}{k}\left(\overline{d}^P - \frac{1}{2}\right).$$

[21] That is to say $X(k-j) = 1 - X(j)$, $j = 0, 1, 2, ..., k$.

For $\frac{n}{2} > \frac{n}{k}\left(\overline{d}^P - \frac{1}{2}\right)$ to hold, it has to be that $\overline{d}^P < \frac{k+1}{2}$, which directly implies $\overline{d}^P \leq \frac{k}{2}$. Observe that symmetry of $X(d)$ implies that the number of members of parliament the leftist party obtains is at least half of the total when the outcome is to the left of $\frac{1}{2}$, i.e. $\overline{d}^M \geq \frac{k}{2}$. Since we argued above that $\overline{d}^P$ can be at most equal to $\frac{k}{2}$ and we assumed it is higher than $\overline{d}^M$ which is at least as high as $\frac{k}{2}$, we obtain $\frac{k}{2} \geq \overline{d}^P > \overline{d}^M \geq \frac{k}{2}$. This is impossible because $\overline{d}^P$ and $\overline{d}^M$ are integer numbers. We conclude that $\overline{d}^P \leq \overline{d}^M$ and thus $X\left(\overline{d}^P\right) \geq X\left(\overline{d}^M\right)$.

We are left to show that $X\left(\overline{d}^P\right) \leq \frac{1}{2}$. Assume, contrary to the thesis, that $X\left(\overline{d}^P\right) > \frac{1}{2}$, hence $\overline{d}^M > \overline{d}^P$ and by symmetry of $X(d)$ around $\frac{1}{2}$ we have $\alpha_{\overline{d}^P+1} \geq \frac{1}{2}$. Notice that if k even $\left(\overline{d}^P+1\right) \leq \frac{k}{2}$ as well as if k is odd $\left(\overline{d}^P+1\right) \leq \frac{k+1}{2}$. In both cases $\left(\overline{d}^P+1\right) \leq \frac{k+1}{2}$. Furthermore, notice that $F\left(\alpha_{\overline{d}^P+1}\right) > \frac{n}{2}$, since we know that $\alpha_{\overline{d}^P+1} \geq \alpha_{\overline{d}^M} \geq m$. Then we have that:

$$F\left(\alpha_{\overline{d}^P+1}\right) > \frac{n}{2} = \frac{n}{k}\left(\frac{k+1}{2} - \frac{1}{2}\right) \geq \frac{n}{k}\left(\left(\overline{d}^P+1\right) - \frac{1}{2}\right)$$

which contradicts 3. We conclude that $X\left(\overline{d}^M\right) \leq X\left(\overline{d}^P\right) \leq \frac{1}{2}$.

Part b. We first prove that $X\left(\overline{d}^P\right) \leq X\left(\overline{d}^M\right)$. Given that $X\left(\overline{d}^M\right) > \frac{1}{2}$, suppose, contrary to the thesis, that $X\left(\overline{d}^P\right) > X\left(\overline{d}^M\right)$, i.e. $\overline{d}^P < \overline{d}^M$ and $\alpha_{\overline{d}^P} > \alpha_{\overline{d}^M}$. Since $X\left(\overline{d}^M\right)$ is the district sincere equilibrium outcome it has to be that $m_{\overline{d}^M} \leq \alpha_{\overline{d}^M}$. Since $m_d = m$,$\forall d$, $m \leq \alpha_{\overline{d}^M}$. Since m is the median, $\frac{n}{2} \leq F(m)$. These observations together imply:

$$\frac{n}{2} \leq F(m) \leq F\left(\alpha_{\overline{d}^M}\right).$$

Observe that $\frac{n}{k}\left(\overline{d}^M - \frac{1}{2}\right) < \frac{n}{2}$, since by symmetry of $X(d)$, $\overline{d}^M < \frac{k}{2}$. Then $\overline{d}^M$ is greater than $\overline{d}^P$ and such that $\frac{n}{k}\left(\overline{d}^M - \frac{1}{2}\right) \leq F\left(\alpha_{\overline{d}^M}\right)$, contradicting that $X\left(\overline{d}^P\right)$ is the equilibrium outcome in the proportional election. We conclude that $\overline{d}^P \geq \overline{d}^M$ and thus $X\left(\overline{d}^P\right) \leq X\left(\overline{d}^M\right)$.

We are left to show that $\frac{1}{2} \leq X\left(\overline{d}^P\right)$. Suppose $\frac{1}{2} > X\left(\overline{d}^P\right)$, i.e. $\overline{d}^P \geq \frac{k+1}{2}$. District sincerity implies $\alpha_{\overline{d}^P} < m$ (because $\alpha_{\overline{d}^M} \geq \alpha_{\overline{d}^P}$ and $\alpha_{\overline{d}^M} < m$). This, in turn, implies that $F\left(\alpha_{\overline{d}^P}\right) < \frac{n}{2}$, since m is the median voter. Observe that $\frac{n}{2} \leq \frac{n}{k}\left(\overline{d}^P - \frac{1}{2}\right)$ when $\overline{d}^P \geq \frac{k+1}{2}$. Thus:

$$F\left(\alpha_{\overline{d}^P}\right) < \frac{n}{k}\left(\overline{d}^P - \frac{1}{2}\right)$$

which contradicts 3. We conclude that $X\left(\overline{d}^M\right) \geq X\left(\overline{d}^P\right) \geq \frac{1}{2}$.

5.2 Heterogeneity across districts

We now consider a situation of extreme heterogeneity across districts. We have in mind a society where some districts are the stronghold of the leftist party and some others of the rightist party. Specifically, the k districts of the multidistrict majority system are inhabited by the same odd number of voters, $n_d = \frac{n}{k}$, for all d. Moreover, districts have been ordered according to the political preferences of their voters, with the first district being inhabited by the first $\frac{n}{k}$ most leftist voters, the second by the next $\frac{n}{k}$ most leftist voters and the following districts being inhabited each by $\frac{n}{k}$ increasingly more rightist voters. Thus median voters in each district are ordered, with $m_1 \leq m_2 \leq ... \leq m_d \leq ... \leq m_k$.

Example 2 (Continued). Consider a society identical to the one presented in Example 2 except for the distribution of voters in the two districts of the multidistrict majority system. In this alternative society, district 1 is inhabited by three leftist voters, with bliss points $\theta_1 = \theta_2 = \theta_3 = 0$ and median voter $m_1 = 0$, while district 2 is inhabited by one leftist voter and two rightist voters, i.e. by voters with bliss points $\theta_4 = 0$, $\theta_5 = \theta_6 = 1$ and median voter $m_2 = 1$. The unique district sincere equilibrium outcome of the multidistrict majority system is $X\left(\overline{d}^M\right) = X(1) = \frac{1}{2}$. i.e. the leftist party obtains one member of parliament and implements a moderate policy. Indeed, observe that $\alpha_1 = \frac{3}{4} > m_1 = 0$ and $\alpha_2 = \frac{1}{4} < m_2 = 1$. The unique perfect equilibrium outcome of the proportional system is $X\left(\overline{d}^P\right) = X(1) = \frac{1}{2}$. Indeed, observe that $F(\alpha_1) = 4 > 3\left(\frac{1}{2}\right) = 1.5$ and $F(\alpha_2) = 4 < 3\left(\frac{3}{2}\right) = 4.5$.

The example presents a society were leftist voters are more concentrated in one district of the multidistrict majority system. One of their votes is - so to speak- wasted, in the sense that the leftist candidate in district 1 would be elected even with only two votes in his favour, while an extra vote would be useful to elect the leftist candidate in district 2. The following proposition proves that such an intuition carries over to more general situations and the two electoral systems - i.e. the single district proportional and multidistrict majority system- give rise to the same equilibrium outcome when districts are equally sized and ordered from left to right.

Proposition 8 *If districts are equally sized and ordered from left to right, then* $X\left(\overline{d}^P\right) = X\left(\overline{d}^M\right)$.

Proof. Contrary to the thesis, suppose first $X\left(\overline{d}^P\right) > X\left(\overline{d}^M\right)$, i.e. $\overline{d}^P < \overline{d}^M$ and $\alpha_{\overline{d}^P} > \alpha_{\overline{d}^M}$. Recall that $\overline{d}^M$ is the maximum d satisfying $\alpha_d \geq m_d$. Furthermore, $F\left(\alpha_{\overline{d}^M}\right) \geq F\left(m_{\overline{d}^M}\right)$. Since districts are ordered and of equal size, the total number of voters up to and including the median voter of a generic district d is at least equal to the number of voters in all previous districts - $\frac{n}{k}(d-1)$- plus half of the voters in that district - $\frac{n}{k}\frac{1}{2}$-, i.e. $F(m_d) > \frac{n}{k}(d-1) + \frac{n}{k}\frac{1}{2} = \frac{n}{k}\left(d - \frac{1}{2}\right)$ [22] for all d. Hence, it follows:

$$F\left(\alpha_{\overline{d}^M}\right) > \frac{n}{k}\left(\overline{d}^M - \frac{1}{2}\right).$$

This contradicts $X\left(\overline{d}^P\right)$ being the equilibrium outcome in the proportional election, since we found a higher d satisfying $F(\alpha_d) \geq \frac{n}{k}\left(d - \frac{1}{2}\right)$. We conclude that $X\left(\overline{d}^P\right) \leq X\left(\overline{d}^M\right)$.

Contrary to the thesis, suppose now that $X\left(\overline{d}^P\right) < X\left(\overline{d}^M\right)$, i.e. $\overline{d}^P > \overline{d}^M$ and $\alpha_{\overline{d}^P} < \alpha_{\overline{d}^M}$. Since $\overline{d}^M$ is by definition the maximum d satisfying $\alpha_d \geq m_d$ and we are assuming $\overline{d}^P > \overline{d}^M$, it follows that $\alpha_{\overline{d}^P} < m_{\overline{d}^P}$. Given that districts are equally sized and ordered, the total number of voters strictly to the left of the median voter of a generic district d is strictly smaller than the number of voters in all previous districts - $\frac{n}{k}(d-1)$- plus half of the voters in that district - $\frac{n}{k}\frac{1}{2}$-, i.e. for $\alpha_d < m_d$, $F(\alpha_d) < \frac{n}{k}(d-1) + \frac{n}{k}\frac{1}{2} = \frac{n}{k}\left(d - \frac{1}{2}\right)$ for all d. Since $\alpha_{\overline{d}^P} < m_{\overline{d}^P}$, it follows that:

$$F\left(\alpha_{\overline{d}^P}\right) < \frac{n}{k}\left(\overline{d}^P - \frac{1}{2}\right)$$

which contradicts 3.
We conclude that $X\left(\overline{d}^P\right) = X\left(\overline{d}^M\right)$.

6 Conclusions

We have studied a model of rational voters electing a parliament by voting for candidates belonging to two parties. Such a model contributes to the "non-majoritarian" literature of legislative election, in that it focuses not on which party has the majority in parliament, but rather on the composition of it, where by composition we mean indeed the number of seats parties win in the legislature. Hence, we do not rely on the usual simplifying assumption that translates votes share into equal seats share.

Legislative elections may differ in many dimensions, we focus on what we believe is the most important one: the electoral rules. Specifically, we analyze the two most popular electoral rules in modern democracies: multidistrict majority and purely proportional representation. In both systems we prove the existence of a unique pure strategies perfect equilibrium outcome, which is the unique "pure" outcome induced by perfect equilibria.

[22] The strict inequality sign follows from the fact that $\frac{n}{k}$ is odd.

The uniqueness of the outcome allows us to carry out a comparison of the policies under the two systems. We analyze it upon various distributions of players bliss policies showing that the outcomes do not coincide - except in a peculiar case- and that the proportional system tends to lead to more moderate outcomes.

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