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GROMOV HYPERBOLICITY THROUGH DECOMPOSITION OF METRIC SPACES

José M. Rodríguez¹ and Eva Tourís²

Abstract. In this paper we study the hyperbolicity in the Gromov sense of metric spaces. We deduce the hyperbolicity of a space from the hyperbolicity of its “building block components”. These results are valuable since they simplify notably the topology of the space and allow to obtain global results from local information. We also study how the punctures and the decomposition of a Riemann surface in Y -pieces and funnels affect on the hyperbolicity of the surface.

1. Introduction

A good way to understand the important connections between graphs and Potential Theory on Riemannian manifolds (see e.g. [4], [11], [16], [20], [21], [22], [23], [31], [32], [36]) is to study the Gromov hyperbolic spaces. This approach allows to establish a general setting to work simultaneously with graphs and manifolds, in the context of metric spaces. Besides, the idea of Gromov hyperbolicity grasps the essence of negatively curved spaces, and has been successfully used in the theory of groups (see e.g. [17], [18], [19] and the references therein).

Although there exist some interesting examples of hyperbolic spaces (see the examples after Definition 2.1), the literature gives no good guide about how to determine whether or not a space is hyperbolic. This limitation can be somehow got round, since the theory allows to obtain powerful results about non-hyperbolic spaces which have hyperbolic universal coverings. As topological “obstacles” may prevent a space from being hyperbolic, the possibility of studying its universal covering instead, which is always free of obstacles, implies a substantial simplification, and sometimes let us extract important information about the space itself (see e.g. [26]).

However, as was stated above, the characterization of hyperbolic spaces remains open. Recently, some interesting results of Balogh and Buckley [6] about the hyperbolicity of Euclidean bounded domains with their quasihyperbolic metric have made significant progress in this direction (see also [9] and the references therein).

Originally, the main aim of the present work was to study when non-exceptional Riemann surfaces equipped with its Poincaré metric were Gromov hyperbolic. However, we have proved several theorems on hyperbolicity for general metric spaces, which are interesting by themselves (see Section 2) and have important consequences for Riemann surfaces (see Section 3). Although one should expect Gromov hyperbolicity in non-exceptional Riemann surfaces due to its constant curvature -1 , this turns out not to be true in general, since topological obstacles can impede it: for instance, the two-dimensional jungle-gym (a $\mathbf{Z}^2$ -covering of a torus with genus two) is not hyperbolic. Let us recall that in the case of modulated plane domains, quasihyperbolic metric and Poincaré metric are equivalent.

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In [34] we prove that there is no inclusion relationship between hyperbolic Riemann surfaces and the usual classes of Riemann surfaces, such as O_G , O_{HP} , O_{HB} , O_{HD} , surfaces with hyperbolic isoperimetric inequality (see Section 3), or the complements of these classes (even in the case of plane domains). This fact makes the study of hyperbolic Riemann surfaces more complicated and interesting. One can find some other results on hyperbolicity of metric spaces and Riemann surfaces in [27], [28], [33] and [34].

Here we present the outline of the main results. We refer to the next sections for the definitions and the precise statements of the theorems.

We can create or delete infinitely many topological obstacles in a metric space, preserving its hyperbolicity (see Theorem 2.2). Furthermore, some results in this paper let us deduce the hyperbolicity of a space from the hyperbolicity of its “building block components” (see theorems 2.3, 2.4 and 2.5). These results are valuable since they simplify notably the topology and allow to obtain global results from local information.

As a corollary, we obtain particular results on Riemann surfaces (see theorems 3.1 and 3.2). Besides, we study how the punctures (see Theorem 3.3) and the decomposition of a Riemann surface in Y -pieces and funnels (see Theorem 3.4 and Corollary 3.1) affect on the hyperbolicity of the surface. In fact, these results allow, in many cases, to forget the punctures in order to study the hyperbolicity of a Riemann surface; this fact can be a significant simplification in the topology of the surface, and therefore makes easier the study of its hyperbolicity. As a consequence of these results, we have obtained many examples of hyperbolic Riemann surfaces (see Lemma 3.4, propositions 3.1, 3.2 and 3.3, and Corollary 3.2).

It is a remarkable fact that almost every constant appearing in the results of this paper depends just on a small number of parameters. This is a common place in the theory of hyperbolic spaces (see e.g. theorems A and B) and is also typical of surfaces with curvature -1 (see the collar lemma in [29] and [35], and lemmas 3.1, 3.2 and 3.3).

NOTATIONS. We denote by X , X_i or X^i geodesic metric spaces. By d_X and B_X we shall denote, respectively, the distance and the balls in the metric of X .

We denote by S or S_i non-exceptional Riemann surfaces. We assume that the metric defined on these surfaces is the Poincaré metric, unless the contrary is specified.

If Ω is a plane domain, we shall denote by λ_Ω the conformal density of the Poincaré metric in Ω , *i.e.* the function such that $ds = \lambda_\Omega(z)|dz|$ is the Poincaré metric in Ω .

We denote by $\text{int } A$ the interior of the set A .

Finally, we denote by c and c_i , positive constants which can assume different values in different theorems.

2. Results in metric spaces

In our study of hyperbolic Gromov spaces we use the notations of [17]. We give now the basic facts about these spaces. We refer to [17] for more background and further results.

DEFINITION 2.1. Let us fix a point w in a metric space (X, d) . We define the *Gromov product* of $x, y \in X$ with respect to the point w as

$$(x|y)_w := \frac{1}{2} \{d(x, w) + d(y, w) - d(x, y)\} \geq 0.$$

We say that the metric space (X, d) is δ -hyperbolic ($\delta \geq 0$) if

$$(x|z)_w \geq \min \{(x|y)_w, (y|z)_w\} - \delta,$$

for every $x, y, z, w \in X$. We say that X is *hyperbolic* (in the Gromov sense) if the value of δ is not important.

It is convenient to remark that this definition of hyperbolicity is not universally accepted, since sometimes the word hyperbolic refers to negative curvature or to the existence of Green's function. However, in this paper we only use the word *hyperbolic* in the sense of Definition 2.1.

EXAMPLES: (1) Every bounded metric space X is $(\text{diam } X)$ -hyperbolic (see e.g., [17, p.29]).

(2) Every complete simply connected Riemannian manifold with sectional curvature which is bounded from above by $-k$, with $k > 0$, is hyperbolic (see e.g., [17, p.52]).

(3) Every tree with edges of arbitrary length is 0-hyperbolic (see e.g., [17, p.29]).

DEFINITION 2.2. If $\gamma : [a, b] \rightarrow X$ is a continuous curve in a metric space (X, d) , we can define the *length* of γ as

$$L(\gamma) := \sup \left\{ \sum_{i=1}^n d(\gamma(t_{i-1}), \gamma(t_i)) : a = t_0 < t_1 < \dots < t_n = b \right\}.$$

We say that γ is a *geodesic* if it is an isometry, i.e. $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $t, s \in [a, b]$. We say that X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y ; we denote by $[x, y]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic space is path-connected. A geodesic metric space is *proper* if every closed ball is compact.

DEFINITION 2.3. If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of three geodesics $[x_1, x_2]$, $[x_2, x_3]$ and $[x_3, x_1]$. A geodesic triangle T is δ -thin (or satisfies the *Rips condition* with constant δ) if for every $x \in [x_i, x_j]$ we have that $d(x, [x_j, x_k] \cup [x_k, x_i]) \leq \delta$ for any permutation $\{x_i, x_j, x_k\}$ of $\{x_1, x_2, x_3\}$. The space X is δ -thin if every geodesic triangle in X is δ -thin.

A basic result is that hyperbolicity is equivalent to Rips condition:

THEOREM A. ([17, p.41]) *Let us consider a geodesic metric space X .*

- (1) *If X is δ -hyperbolic, then it is 4δ -thin.*
- (2) *If X is δ -thin, then it is 4δ -hyperbolic.*

We present now the class of maps which play the main role in the theory.

DEFINITION 2.4. A function between two metric spaces $f : X \rightarrow Y$ is a *quasiisometry* if there are constants $a \geq 1, b \geq 0$ with

$$\frac{1}{a} d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq a d_X(x_1, x_2) + b, \quad \text{for every } x_1, x_2 \in X.$$

A such function is called an (a, b) -quasiisometry. We say that the image of f is ε -full (for some $\varepsilon \geq 0$) if for every $y \in Y$ there exists $x \in X$ with $d_Y(y, f(x)) \leq \varepsilon$. We say that X and Y are *quasiisometrically equivalents* if there exists a quasiisometry between X and Y , with image ε -full, for some $\varepsilon \geq 0$. An (a, b) -quasiisometry in X is an (a, b) -quasiisometry between an interval of $\mathbf{R}$ and X .

Observe that a quasiisometry can be discontinuous.

REMARK. It is well known (see e.g. [21] or [22]) that quasiisometrical equivalence is an equivalence relation. In fact, if $f : X \rightarrow Y$ is an (a, b) -quasiisometry with image ε -full, then there exists a function $g : Y \rightarrow X$ which is an $(a, 2a\varepsilon + ab)$ -quasiisometry. In particular, if f is a surjective (a, b) -quasiisometry, then g is an (a, ab) -quasiisometry (in this case we can choose as $g(y)$ any point in $f^{-1}(y)$).

Quasiisometries are important since they are the maps which preserve hyperbolicity:

THEOREM B. ([17, p.88]) *Let us consider an (a, b) -quasiisometry between two geodesic metric spaces $f : X \rightarrow Y$. If Y is δ -hyperbolic, then X is δ' -hyperbolic, where δ' is a constant which only depends on δ , a and b . Besides, if the image of f is ε -full for some $\varepsilon \geq 0$, then X is hyperbolic if and only if Y is hyperbolic.*

Along the paper we will work with spaces in which we have defined the length of curves (see e.g. theorems 2.1, 2.2, 2.3, 2.4 and 2.5). There is a natural way to define a pseudodistance in these spaces.

DEFINITION 2.5. If X is a space in which we have defined the length of curves, then we associate to it the *canonical distance* (or *pseudodistance*)

$$d_X(x, y) := \inf \{L(\gamma) : \gamma \subset X \text{ is a continuous curve joining } x \text{ and } y\},$$

where $L(\gamma)$ is the length of γ , if x and y belong to the same path-connected component of X . If x and y belong to distinct path-connected components of X , we define $d_X(x, y) := \infty$. If A is a subset of X , we define for x and y in the same path-connected component of A ,

$$d_X|_A(x, y) := \inf \{L(\gamma) : \gamma \subset A \text{ is a continuous curve joining } x \text{ and } y\}.$$

If x and y belong to distinct path-connected components of A , we define $d_X|_A(x, y) := \infty$.

The following theorem is a main tool in order to obtain some results of this paper (see Corollary 2.1 and Theorem 2.2).

THEOREM 2.1. *Let us consider a geodesic metric space X . Let us assume that $\{K_n\}_n$ are compact subsets of X , and that there are positive constants c_1, c_2 , such that $\text{diam}_X K_n \leq c_1$ and $d_X(K_n, K_m) \geq c_2$ if $n \neq m$. We denote by X' the quotient space obtained from X by identifying the points of each K_n in a single point k_n . Then the canonical projection of X in X' is a $((c_1 + c_2)/c_2, c_1 c_2/(c_1 + c_2))$ -quasiisometry. Consequently, if X' is a geodesic metric space, then X is hyperbolic if and only if X' is hyperbolic. In particular, if X is δ -hyperbolic, then X' is δ' -hyperbolic, with δ' a universal constant which only depends on δ, c_1 and c_2 . If furthermore each ball in X intersects only a finite number of K_n 's (this is the case if X is proper), then X' is a geodesic metric space.*

REMARK. The proof of Theorem 2.1 gives directly that in the case of a single compact set K_1 , we obtain a $(1, c_1)$ -quasiisometry.

PROOF. Since $d_X(K_n, K_m) \geq c_2$, we have that $d_{X'}$ (defined by Definition 2.5) is a distance. If $p: X \rightarrow X'$ is the canonical projection, it is clear that $d_{X'}$ verifies

$$(2.1) \quad d_{X'}(p(x), p(y)) = \min \left\{ d_X(x, y), \inf \left\{ d_X(x, K_{n_0}) + \sum_{j=1}^r d_X(K_{n_{j-1}}, K_{n_j}) + d_X(K_{n_r}, y) \right\} \right\},$$

for every $x, y \in X$, where the infimum is taken for every (possibly non-ordered) finite subset of natural numbers $\{n_j\}_{j=0}^r$.

Let us see that

$$(2.2) \quad \frac{c_2}{c_1 + c_2} d_X(x, y) - \frac{c_1 c_2}{c_1 + c_2} \leq d_{X'}(p(x), p(y)), \quad \text{for every } x, y \in X.$$

In order to see this inequality, we observe that for every (possibly non-ordered) finite subset of natural numbers $\{n_j\}_{j=0}^r$, we have

$$d_X(x, y) \leq d_X(x, K_{n_0}) + \sum_{j=1}^r d_X(K_{n_{j-1}}, K_{n_j}) + d_X(K_{n_r}, y) + \sum_{j=0}^r \text{diam}_X(K_{n_j}).$$

Observe that

$$\sum_{j=0}^r \text{diam}_X(K_{n_j}) \leq (r+1)c_1 = \frac{c_1}{c_2} r c_2 + c_1 \leq \frac{c_1}{c_2} \sum_{j=1}^r d_X(K_{n_{j-1}}, K_{n_j}) + c_1.$$

Then

$$\begin{aligned} d_X(x, y) &\leq d_X(x, K_{n_0}) + \left(1 + \frac{c_1}{c_2}\right) \sum_{j=1}^r d_X(K_{n_{j-1}}, K_{n_j}) + d_X(K_{n_r}, y) + c_1 \\ &\leq \frac{c_1 + c_2}{c_2} \left[d_X(x, K_{n_0}) + \sum_{j=1}^r d_X(K_{n_{j-1}}, K_{n_j}) + d_X(K_{n_r}, y) \right] + c_1, \end{aligned}$$

for every (possibly non-ordered) finite subset of natural numbers $\{n_j\}_{j=0}^r$, and we conclude

$$d_X(x, y) \leq \frac{c_1 + c_2}{c_2} d_{X'}(p(x), p(y)) + c_1,$$

for every $x, y \in X$, which gives (2.2).

The inequality $d_{X'}(p(x), p(y)) \leq d_X(x, y)$ follows from (2.1), and then p is a quasiisometry.

We prove now that X' is a geodesic metric space if each ball in X intersects only a finite number of K_n 's. We have that X' is a geodesic metric space if and only if the infimum in (2.1) is always a minimum. In order to take this infimum it is enough to consider (possibly non-ordered) finite subset of natural numbers $\{n_j\}_{j=0}^r$ with

$$d_X(x, K_{n_0}) + \sum_{j=1}^r d_X(K_{n_{j-1}}, K_{n_j}) + d_X(K_{n_r}, y) \leq d_X(x, y).$$

Since the left hand side of this inequality is greater or equal than $r c_2$, we have that $r \leq c_2^{-1} d_X(x, y)$.

Then we obtain

$$\begin{aligned} d_X(x, K_{n_i}) &\leq d_X(x, K_{n_0}) + \sum_{j=1}^i d_X(K_{n_{j-1}}, K_{n_j}) + \sum_{j=0}^{i-1} \text{diam}_X(K_{n_j}) \\ &\leq d_X(x, y) + \sum_{j=0}^{i-1} \text{diam}_X(K_{n_j}) \leq d_X(x, y) + r c_1 \\ &\leq \left(1 + \frac{c_1}{c_2}\right) d_X(x, y). \end{aligned}$$

This implies that, in order to take this infimum, it is enough to consider sets K_{n_j} verifying $d_X(x, K_{n_j}) \leq (1 + c_1 c_2^{-1})d_X(x, y)$. Since only a finite number of K_n intersect $\overline{B_X(x, (1 + c_1 c_2^{-1})d_X(x, y))}$, the infimum in (2.1) is in fact a minimum.

It is easy to see that if X is a proper space, then each ball in X intersects only a finite number of K_n 's: If $H > 0$ and $Y \subseteq X$, we define $V_H(Y) := \{x \in X : d(x, Y) \leq H\}$. We consider a closed ball $\overline{B_X(p, r)}$ and its covering by the open sets $\{\{\text{int } V_{c_2/2}(K_n)\}_n, B_X(p, 2r) \setminus \cup_n V_{c_2/4}(K_n)\}$. If we choose a finite subcovering with N open sets, we deduce that $\overline{B_X(p, r)}$ intersects at most N compact sets in $\{K_n\}_n$.

The conclusions about hyperbolicity are a consequence of these facts and Theorem B. $\square$

REMARK. If we denote by p^{-1} any function $p^{-1} : X' \rightarrow X$ satisfying $p(p^{-1}(x)) = x$ for every $x \in X'$, we have by (2.2) that p^{-1} is a $((c_1 + c_2)/c_2, c_1)$ -quasiisometry whose image is c_1 -full.

We have the following particular case. It allows to create (or delete) infinitely many topological obstacles in a metric space, preserving its hyperbolicity (we paste by identifying points). We need a definition.

DEFINITION 2.6. Let us consider a geodesic metric space X and $\{\eta_n^1, \eta_n^2\}_n$ pairwise disjoint compact subsets of X . If c_1, c_2 are positive constants, we say that $\{\eta_n^1, \eta_n^2\}_n$ are (c_1, c_2) -identified if $(\eta_n^1, d_X|_{\eta_n^1})$ and $(\eta_n^2, d_X|_{\eta_n^2})$ are isometric for each n , $\text{diam}_X(\eta_n^1 \cup \eta_n^2) \leq c_1$ for every n and $d_X(\eta_n^1 \cup \eta_n^2, \eta_m^1 \cup \eta_m^2) \geq c_2$ for every $n \neq m$. We denote by X_0 a space obtained by identifying in X the compact sets η_n^1 and η_n^2 by an isometry, for each n .

REMARK. Since $d_X(\eta_n^1 \cup \eta_n^2, \eta_m^1 \cup \eta_m^2) \geq c_2$, we have that d_{X_0} (defined by Definition 2.5) is a distance.

COROLLARY 2.1. *Let us consider a geodesic metric space X and $\{\eta_n^1, \eta_n^2\}_n$ (c_1, c_2) -identified. If each η_n^i is a single point, then the canonical projection of X in X_0 is a $((c_1 + c_2)/c_2, c_1 c_2/(c_1 + c_2))$ -quasiisometry. Consequently, if X_0 is a geodesic metric space, then X is hyperbolic if and only if X_0 is hyperbolic. In particular, if X is δ -hyperbolic, then X_0 is δ' -hyperbolic, with δ' a universal constant which only depends on δ, c_1 and c_2 .*

Identifying points is a very particular way to paste spaces. The following theorem allows to create infinitely many topological obstacles in a metric space ("genus", if the space is a surface), preserving its hyperbolicity, in a more general way.

There is a more useful point of view to appreciate the next theorem: we can delete infinitely many topological obstacles in a metric space, preserving its hyperbolicity. This fact allows a big simplification in the topology of the space (recall that the topological obstacles make difficult the hyperbolicity of a space).

THEOREM 2.2. *Let us consider a geodesic metric space X and $\{\eta_n^1, \eta_n^2\}_n$ (c_1, c_2) -identified. Then the canonical projection of X in X_0 is a $((c_1 + c_2)/c_2, c_1 c_2/(c_1 + c_2))$ -quasiisometry. Consequently, if X_0 is a geodesic metric space, then X is hyperbolic if and only if X_0 is hyperbolic. In particular, if X is δ -hyperbolic, then X_0 is δ' -hyperbolic, with δ' a universal constant which only depends on δ, c_1 and c_2 .*

REMARKS.

1. A similar argument to the one in the proof of Theorem 2.1 gives that X_0 is a geodesic metric space if each ball in X intersects only a finite number of η_n^i 's (this is the case if X is proper).

2. If η_n^i are simple closed curves, the condition that $(\eta_n^1, d_X|_{\eta_n^1})$ and $(\eta_n^2, d_X|_{\eta_n^2})$ are isometric is equivalent to $L_X(\eta_n^1) = L_X(\eta_n^2)$.

PROOF. It is clear that $d_X|_{\eta_n^1}$, $d_X|_{\eta_n^2}$ and d_{X_0} (defined by Definition 2.5) are distances.

Let us consider the canonical projection $p : X \rightarrow X_0$. It is clear that for every curve γ in X we have $L_X(\gamma) = L_{X_0}(p(\gamma))$. Then for every $x, y \in X$ we have $d_{X_0}(p(x), p(y)) \leq d_X(x, y)$, since there are more curves joining $p(x)$ and $p(y)$ in X_0 than curves joining x and y in X .

In order to prove the other inequality, let us fix $x, y \in X$ and let us consider a geodesic γ in X_0 joining $p(x)$ and $p(y)$, if there is such geodesic (in other case, we can take γ_n with $L_{X_0}(\gamma_n) \leq d_{X_0}(p(x), p(y)) + 1/n$). Let us define $\eta_n := p(\eta_n^1) = p(\eta_n^2)$. Then $d_{X_0}(\eta_n, \eta_m) \geq c_2$ if $n \neq m$.

If $L_{X_0}(\gamma) = d_X(x, y)$, then $d_{X_0}(p(x), p(y)) = d_X(x, y)$. If $L_{X_0}(\gamma) < d_X(x, y)$, then γ meets some η_n . In this case let us choose a curve $\gamma_0 \subseteq \gamma$ as follows: Since $d_{X_0}(\eta_n, \eta_m) \geq c_2$, γ intersects only a finite number of η_n 's, which we denote by $\eta_{n_1}, \dots, \eta_{n_r}$. We consider γ as a oriented curve from $p(x)$ to $p(y)$; then we can assume that γ meets $\eta_{n_1}, \dots, \eta_{n_r}$ in this order (we only bear in mind the first intersection).

If $\gamma : [0, l] \rightarrow X_0$, let us define

$$t_1^1 := \min\{0 \leq t \leq l : \gamma(t) \in \eta_{n_1}\}, \quad t_1^2 := \max\{0 \leq t \leq l : \gamma(t) \in \eta_{n_1}\}.$$

In a similar way we define recursively

$$t_i^1 := \min\{t_{i-1}^2 \leq t \leq l : \gamma(t) \in \eta_{n_i}\}, \quad t_i^2 := \max\{t_{i-1}^2 \leq t \leq l : \gamma(t) \in \eta_{n_i}\},$$

if $\gamma([t_{i-1}^2, l]) \cap \eta_{n_i} \neq \emptyset$; in other case, we choose instead of i the least natural number j with $i < j \leq r$ and $\gamma([t_{i-1}^2, l]) \cap \eta_{n_j} \neq \emptyset$.

In order to simplify the notation we can assume that we can define t_i^1 and t_i^2 for every $i = 1, 2, \dots, r$ (in other case, we extract a subsequence).

We define γ_0 as the restriction of γ to the closed set $[0, t_1^1] \cup [t_1^2, t_2^1] \cup \dots \cup [t_{r-1}^2, t_r^1] \cup [t_r^2, l]$.

We define $K_1 := \{p^{-1}(\gamma(t_1^1)), p^{-1}(\gamma(t_1^2))\}, \dots, K_r := \{p^{-1}(\gamma(t_r^1)), p^{-1}(\gamma(t_r^2))\} \subset X$; since p is not injective, we take $p^{-1}(\gamma(t_i^1)) := \lim_{t \rightarrow (t_i^1)^-} p^{-1}(\gamma(t))$ and $p^{-1}(\gamma(t_i^2)) := \lim_{t \rightarrow (t_i^2)^+} p^{-1}(\gamma(t))$; if $t_1^1 = 0$ (and/or $t_r^2 = l$) we take $p^{-1}(\gamma(t_1^1)) = x$ (and/or $p^{-1}(\gamma(t_r^2)) = y$). With this choice of $K_1, \dots, K_r$, we define X' according to Theorem 2.1. If $p' : X \rightarrow X'$ is the canonical projection, then (2.2) gives

$$d_{X'}(p'(x), p'(y)) \geq \frac{c_2}{c_1 + c_2} d_X(x, y) - \frac{c_1 c_2}{c_1 + c_2},$$

for every $x, y \in X$. If $\gamma_1 := p'(p^{-1}\gamma_0) \subset X'$, then we have the following

$$d_{X_0}(p(x), p(y)) := L_{X_0}(\gamma) \geq L_{X_0}(\gamma_0) = L_{X'}(\gamma_1) = d_{X'}(p'(x), p'(y)) \geq \frac{c_2}{c_1 + c_2} d_X(x, y) - \frac{c_1 c_2}{c_1 + c_2},$$

for every $x, y \in X$. The equality $L_{X'}(\gamma_1) = d_{X'}(p'(x), p'(y))$ is a consequence of the following facts: γ_1 is a continuous curve in X' and the restriction of γ_0 to any interval $[0, t_1^1], [t_1^2, t_2^1], \dots, [t_{r-1}^2, t_r^1], [t_r^2, l]$ is a geodesic in X_0 ; therefore γ_1 is also a geodesic in X' .

Then, we have that p is a $((c_1 + c_2)/c_2, c_1 c_2/(c_1 + c_2))$ -quasiisometry.

The conclusions about hyperbolicity are a consequence of this fact and Theorem B. $\square$

Since hyperbolicity is invariant under quasiisometries, a compact set in X does not play a significant role in the hyperbolicity of X . Furthermore, we can delete even infinitely many compacts, under some weak conditions:

PROPOSITION 2.1. *Let us consider a geodesic metric space X and compact subsets $\{K_n\}_n$ of X such that $X_0 := X \setminus \bigcup_n \text{int } K_n$ is path-connected. We assume that there are positive constants c_1, c_2 , such that $\text{diam}_X K_n \leq c_1$ and $d_X(K_n, K_m) \geq c_2$ if $n \neq m$. Then, there exists a quasiisometry f of X_0 in X with constants which only depend on c_1 and c_2 ; furthermore, the image of f is c_1 -full. Consequently, if X_0 is a geodesic metric space, then X is hyperbolic if and only if X_0 is hyperbolic. In particular, if X is δ -hyperbolic, then X_0 is δ' -hyperbolic, with δ' a universal constant which only depends on δ, c_1 and c_2 .*

PROOF. Let us denote by X' the metric space obtained from X by identifying the points of each K_n in a single point q_n . By Theorem 2.1 and the remark before Definition 2.6, there is a $((c_1 + c_2)/c_2, c_1)$ -quasiisometry $j : X' \rightarrow X$, whose image is c_1 -full. We denote by $(X_0)'$ the metric space obtained from X_0 by identifying the points of each ∂K_n in a single point q_n . Since $X' = (X_0)'$, we can consider the composition f of the canonical projection $p : X_0 \rightarrow (X_0)'$ and j . Theorem 2.1 gives that p is a $((c_1 + c_2)/c_2, c_1 c_2/(c_1 + c_2))$ -quasiisometry. Then f is a quasiisometry of X_0 in X , with constants which only depend on c_1 and c_2 . Since p is surjective, the image of f is c_1 -full.

The conclusions about hyperbolicity are a consequence of these facts and Theorem B. $\square$

It is also possible to change infinitely many compact sets by other compact sets.

PROPOSITION 2.2. *Let us consider two geodesic metric spaces X^1, X^2 , and compact subsets $\{K_n^1\}_n \subset X^1, \{K_n^2\}_n \subset X^2$, such that $X_0 := X^1 \setminus \bigcup_n \text{int } K_n^1 = X^2 \setminus \bigcup_n \text{int } K_n^2$ is path-connected, with $L_{X^1}(\gamma) = L_{X^2}(\gamma)$, for every curve $\gamma \subset X_0$. We also assume that there are positive constants c_1, c_2 , such that $\text{diam}_{X^i} K_n^i \leq c_1$ and $d_{X^i}(K_n^i, K_m^i) \geq c_2$ if $n \neq m$. Then there exists a quasiisometry $f : X^1 \rightarrow X^2$ with constants which only depend on c_1 and c_2 . Consequently, X^1 is hyperbolic if and only if X^2 is hyperbolic. In particular, if X^1 is δ -hyperbolic, then X^2 is δ' -hyperbolic, with δ' a universal constant which only depends on δ, c_1 and c_2 .*

PROOF. By Proposition 2.1, there exists a quasiisometry f_i of X_0 in X^i with constants which only depend on c_1 and c_2 . Since the image of f_1 is c_1 -full, the remark after Definition 2.4 gives that there exists a quasiisometry f_3 of X^1 in X_0 with constants which only depend on c_1 and c_2 . Therefore $f = f_2 \circ f_3$ is the required quasiisometry.

Since the argument is symmetric in X^1 and X^2 , there exists a similar quasiisometry g of X^2 in X^1 .

The conclusions about hyperbolicity are a consequence of these facts and Theorem B. $\square$

DEFINITION 2.7. By a *graph* $R = (V, E)$ we mean a set of points $V = V(R)$ (called vertices), with a set of edges $E = E(R)$ connecting pairs of vertices; the edges are non-oriented, i.e. $[v_1, v_2] = [v_2, v_1]$; it is allowed any finite number of edges between two vertices (in particular, between a vertex and itself); it is allowed for any vertex to be an end of infinitely many edges. We assume also that R is connected.

A *cycle* in a graph is an edge connecting a vertex with itself or a sequence of distinct edges $[v_1, v_2], [v_2, v_3], \dots, [v_{n-1}, v_n], [v_n, v_1]$.

By a *tree* we mean a graph without cycles; therefore, between two distinct vertices there is at most one edge, and there are no edges connecting a vertex with itself.

A useful tool in the theory of Riemann surfaces is “cut and paste” (see e.g. [12, chapter X.3], [16], [31], [32]). The following result allows to paste infinitely many hyperbolic spaces. We need a definition.

DEFINITION 2.8. Let us consider a metric space X , a family of geodesic metric spaces $\{X_n\}_n \subseteq X$ such that $\eta_{nm} := \eta_{mn} := X_n \cap X_m$ are compact sets, and positive constants c_1, c_2 . We consider the graph $R = (V, E)$ with vertices $V = \{v_n\}_n$ and edges E , such that $[v_n, v_m] \in E$ if and only if $\eta_{nm} \neq \emptyset$. We say that $\{X_n\}_n$ is a (c_1, c_2) -decomposition of X if R is a tree, $\text{diam}_{X_n}(\eta_{nm}) \leq c_1$ for every n, m , and $d_{X_n}(\eta_{nm}, \eta_{nk}) \geq c_2$ for every n and $m \neq k$.

REMARK. Since R is a tree and η_{nm} are compact sets, it is clear that X is a geodesic metric space.

THEOREM 2.3. *Let us consider a metric space X and a family of geodesic metric spaces $\{X_n\}_n$ which is a $(0, 0)$ -decomposition of X . If δ_n is the sharpest constant for X_n to be thin, then X is δ -thin with the sharpest constant $\delta = \sup_n \delta_n$. Then X is hyperbolic if and only if there exists a constant c_1 such that X_n is c_1 -hyperbolic for every n .*

REMARK. If we want to create topological obstacles by pasting $\{X_n\}_n$ (i.e. to take a graph instead of a tree), we can apply Theorem 2.3 with a tree and, after that, we can use Corollary 2.1.

PROOF. Let us observe that each η_{nm} is a single point; then the restriction of d_X to any X_n coincides with d_{X_n} , since R is a tree.

We consider a geodesic triangle T in X . If the three vertices of T are in the same X_n for some n , it is clear that $T \subseteq X_n$, since in other case some geodesic $[x_i, x_{i+1}] \subset T$ exits of X_n and enters again at the same point (since R is a tree), and this is impossible because then $[x_i, x_{i+1}] \cap X_n$ is shorter than $[x_i, x_{i+1}]$; then T is δ_n -thin. This shows that $\delta \geq \sup_n \delta_n$.

If T intersects several X_n we define $T_n := T \cap X_n$. If there are two vertices of T in the same X_n , then T_n is a geodesic triangle if we consider as the third vertex of T_n the unique point from which T exits of X_n ; then T_n is δ_n -thin (observe that distinct sides of T_n are included in distinct sides of T). The third vertex of T is in some X_m ; then T_m is a geodesic “biangle” if we consider as the second vertex of T_m the unique point from which T exits of X_m ; let us observe again that distinct sides of T_m are included in distinct sides of T . We denote by γ_1 and γ_2 the two sides of T_m . Let us fix a point $x \in T_m$. Without loss of generality, $x \in \gamma_1$. We choose now some point of γ_2 ; this point splits γ_2 in two geodesics γ_2^1 and γ_2^2 ; now we can see T_m as a geodesic triangle of sides γ_1 , γ_2^1 and γ_2^2 . Then we have

$$d_X(x, \gamma_2) = d_{X_m}(x, \gamma_2^1 \cup \gamma_2^2) \leq \delta_m.$$

If T is not contained in $T_n \cup T_m$, then there is some T_k without vertices of T . In this case T_k is a geodesic “biangle”, choosing as vertices of T_k the two points from which T exits of X_k , and we can proceed as in the last case. Then, T is $(\sup_n \delta_n)$ -thin.

If T has the three vertices in distinct X_n then, we can proceed as in the last cases. The unique difference is that if T_k does not have vertices of T , T_k can be now a geodesic “biangle” or a geodesic triangle. Then, T is $(\sup_n \delta_n)$ -thin, and this shows that $\delta \leq \sup_n \delta_n$.

Now, the last statement in the theorem is a consequence of Theorem A and the equality $\delta = \sup_n \delta_n$. $\square$

The following result allows to reduce the study of the hyperbolicity of a geodesic metric space to the study of the hyperbolicity of their “pieces” or “components”; it is useful since it allows to approach the hyperbolicity as a local problem, although it is a global one.

From another point of view, the next result allows to paste infinitely many spaces, preserving their hyperbolicity, in a more general way than Theorem 2.3; observe that the sets η_{nm} below do not need to be connected and therefore we can create a finite number of topological obstacles each time we paste two spaces.

THEOREM 2.4. *Let us consider a metric space X and a family of geodesic metric spaces $\{X_n\}_n \subseteq X$ which is a (c_1, c_2) -decomposition of X . Then X is δ -hyperbolic if and only if there exists a constant c_3 such that X_n is c_3 -hyperbolic for every n . Furthermore, δ is a universal constant which only depends on c_1, c_2 and c_3 .*

It is natural to consider a graph G instead of the tree R . A canonical way to deal with this problem is to take a two-steps strategy: we can consider a tree R with $V(R) = V(G)$ and $E(R) \subset E(G)$, and apply Theorem 2.4 to R ; so we obtain a connected space and consequently we can apply Theorem 2.2 to finish the gluing process following the combinatorial design of the edges in $E(G) \setminus E(R)$. In this way theorems 2.2 and 2.4 can be combined to obtain a more useful result than separately.

We want to remark that the conclusion of Theorem 2.4 is not true if we delete the hypothesis $\text{diam}_{X_n}(\eta_{nm}) \leq c_1$ or $d_{X_n}(\eta_{nm}, \eta_{nk}) \geq c_2$.

We wish to emphasize that condition $\text{diam}_{X_n}(\eta_{nm}) \leq c_1$ is not very restrictive: if the space is “wide” at every point (in the sense of big injectivity radius, as is the case of simply connected spaces) or “narrow” at every point (as is the case of trees), it is easier to study its hyperbolicity; if we can found narrow parts (as η_{nm}) and wide parts, the problem is more difficult and interesting.

REMARK. Observe that the conditions

$$(1) \text{diam}_{X_n}(\eta_{nm}) \leq c_1 \text{ and } d_{X_n}(\eta_{nm}, \eta_{nk}) \geq c_2,$$

imply

$$(2) \text{diam}_{X_n}(\eta_{nm}) \leq c_1 \text{ and } \text{diam}_{X_n}(\eta_{nm}) \leq c_4 d_{X_n}(\eta_{nm}, \eta_{nk}), \text{ with constant } c_4 = c_1/c_2, \text{ and each curve with finite length in } X \text{ intersects only a finite number of } \eta_{nm} \text{'s.}$$

In fact, in the proof of Theorem 2.4 we only use (2); therefore, the conclusion of Theorem 2.4 is true if we change hypothesis (1) by (2).

PROOF. The idea of the proof is to construct a space X' verifying the hypothesis of Theorem 2.3, which is quasiisometric to X . In order to do this, we choose $x_{nm} = x_{mn} \in \eta_{nm} = \eta_{mn}$. Let us define a geodesic metric space X' (as in Theorem 2.3) as the union of $\{X_n\}_n$ by identifying x_{nm} and x_{mn} for each $[v_n, v_m] \in E$. Let us consider the canonical projection $p : X' \rightarrow X$, which is a continuous and surjective function. It is clear that for every curve γ in X' we have $L_{X'}(\gamma) = L_X(p(\gamma))$. Then for every $x, y \in X'$ we have $d_X(p(x), p(y)) \leq d_{X'}(x, y)$, since there are more curves joining $p(x)$ and $p(y)$ in X than curves joining x and y in X' .

In order to prove the other inequality, let us fix $x, y \in X'$ and let us consider a geodesic γ in X joining $p(x)$ and $p(y)$. Let us define $\gamma_{nm} := p(\eta_{nm}) = p(\eta_{mn})$. Then,

$$d_X(\gamma_{nm}, \gamma_{nk}) = d_{X_n}(\eta_{nm}, \eta_{nk}) \geq c_2 \geq \frac{c_2}{c_1} \text{diam}_{X_n}(\eta_{nm}),$$

if $m \neq k$, and we conclude

$$(2.3) \quad \text{diam}_{X_n}(\eta_{nm}) \leq c_4 d_X(\gamma_{nm}, \gamma_{nk}),$$

if $m \neq k$, with $c_4 = c_1/c_2$.

If γ does not meet $\cup_{nm} \gamma_{nm}$, then $\gamma \subset X_n$ for some n , and consequently, $d_X(p(x), p(y)) = d_{X'}(x, y)$. In other case, since $d_X(\gamma_{nm}, \gamma_{nk}) = d_{X_n}(\eta_{nm}, \eta_{nk}) \geq c_2$, γ intersects only a finite number of γ_{nm} 's, which we denote by $\gamma_{n_1 n_2}, \gamma_{n_3 n_4}, \dots, \gamma_{n_{2r-1} n_{2r}}$.

We consider γ as a oriented curve from $p(x)$ to $p(y)$; then we can assume that γ meets $\gamma_{n_1 n_2}, \gamma_{n_3 n_4}, \dots, \gamma_{n_{2r-1} n_{2r}}$ in this order (we only bear in mind the first intersection).

If $\gamma : [0, l] \rightarrow X$, let us define

$$t_1^1 := \min\{0 \leq t \leq l : \gamma(t) \in \gamma_{n_1 n_2}\}, \quad t_1^2 := \max\{0 \leq t \leq l : \gamma(t) \in \gamma_{n_1 n_2}\}.$$

In a similar way, we define recursively

$$t_i^1 := \min\{t_{i-1}^2 \leq t \leq l : \gamma(t) \in \gamma_{n_{2i-1} n_{2i}}\}, \quad t_i^2 := \max\{t_{i-1}^2 \leq t \leq l : \gamma(t) \in \gamma_{n_{2i-1} n_{2i}}\},$$

if $\gamma([t_{i-1}^2, l]) \cap \gamma_{n_{2i-1} n_{2i}} \neq \emptyset$; in other case, we choose instead of i the least natural number j with $i < j \leq r$ and $\gamma([t_{i-1}^2, l]) \cap \gamma_{n_{2j-1} n_{2j}} \neq \emptyset$.

In order to simplify the notation we can assume that we can define t_i^1 and t_i^2 for every $i = 1, 2, \dots, r$ (in other case, we extract a subsequence). Then, we have for $1 \leq i < r$

$$n_{2i+1} = \begin{cases} n_{2i}, & \text{if } \gamma([t_i^2, l]) \cap X_{n_{2i-1}} = \emptyset, \\ n_{2i-1}, & \text{if } \gamma([t_i^2, l]) \cap X_{n_{2i-1}} \neq \emptyset. \end{cases}$$

We remark that $n_{2i+1} = n_{2i}$ if and only if any curve joining x and y in X intersects $X_{n_{2i}}$.

We define γ_1 as the restriction of γ to the closed set $[0, t_1^1] \cup [t_1^2, t_2^1] \cup \dots \cup [t_{r-1}^2, t_r^1] \cup [t_r^2, l]$.

We define η_1 as the preimage of γ_1 by p , such that the preimage of each compact subset of γ_1 is another compact subset in X' . For any $i = 1, 2, \dots, r$, it is clear that we can choose an arc $g_i \subset X'$ connecting $p^{-1}(\gamma(t_i^1))$ and $p^{-1}(\gamma(t_i^2))$. We define $\eta := \eta_1 \cup g_1 \cup \dots \cup g_r$; it is a continuous curve joining x and y in X' .

We show now that we can choose g_i such that

$$(2.4) \quad \begin{aligned} L_{X'}(g_1) &\leq c_1 + c_4 L_X(\gamma[t_1^2, t_2^1]), & L_{X'}(g_r) &\leq c_1 + c_4 L_X(\gamma[t_{r-1}^2, t_r^1]), \\ L_{X'}(g_i) &\leq c_4 L_X(\gamma[t_{i-1}^2, t_i^1]) + c_4 L_X(\gamma[t_i^2, t_{i+1}^1]), & \text{if } 1 < i < r. \end{aligned}$$

If $n_3 = n_2$, by (2.3) we can choose g_1 verifying

$$L_{X'}(g_1) \leq \text{diam}_{X_{n_1}}(\eta_{n_1 n_2}) + \text{diam}_{X_{n_2}}(\eta_{n_2 n_1}) \leq c_1 + c_4 d_X(\gamma_{n_2 n_1}, \gamma_{n_3 n_4}) \leq c_1 + c_4 L_X(\gamma[t_1^2, t_2^1]).$$

If $n_3 = n_1$, by (2.3) we can choose g_1 verifying

$$L_{X'}(g_1) \leq \text{diam}_{X_{n_1}}(\eta_{n_1 n_2}) \leq c_4 d_X(\gamma_{n_1 n_2}, \gamma_{n_3 n_4}) \leq c_4 L_X(\gamma[t_1^2, t_2^1]),$$

and then, we have the first inequality in (2.4). The second inequality in (2.4) is similar.

Assume now that $1 < i < r$. If $n_{2i+1} = n_{2i}$, by (2.3) we can choose g_i verifying

$$\begin{aligned} L_{X'}(g_i) &\leq \text{diam}_{X_{n_{2i-1}}}(\eta_{n_{2i-1}n_{2i}}) + \text{diam}_{X_{n_{2i}}}(\eta_{n_{2i}n_{2i-1}}) \\ &\leq c_4 d_X(\gamma_{n_{2i-1}n_{2i}}, \gamma_{n_{2i-3}n_{2i-2}}) + c_4 d_X(\gamma_{n_{2i-1}n_{2i}}, \gamma_{n_{2i+1}n_{2i+2}}) \\ &\leq c_4 L_X(\gamma[t_{i-1}^2, t_i^1]) + c_4 L_X(\gamma[t_i^2, t_{i+1}^1]). \end{aligned}$$

If $n_{2i+1} = n_{2i-1}$, by (2.3) we can choose g_i verifying

$$L_{X'}(g_i) \leq \text{diam}_{X_{n_{2i-1}}}(\eta_{n_{2i-1}n_{2i}}) \leq c_4 d_X(\gamma_{n_{2i-1}n_{2i}}, \gamma_{n_{2i-3}n_{2i-2}}) \leq c_4 L_X(\gamma[t_{i-1}^2, t_i^1]),$$

and then, we have the last inequality in (2.4).

Therefore, by (2.4) we have that

$$\begin{aligned} d_{X'}(x, y) &\leq L_{X'}(\eta) = L_{X'}(\eta_1) + \sum_{i=1}^r L_{X'}(g_i) \\ &\leq L_X(\gamma[0, t_1^1]) + \sum_{i=1}^{r-1} L_X(\gamma[t_i^2, t_{i+1}^1]) + L_X(\gamma[t_r^2, l]) \\ &\quad + c_1 + c_4 L_X(\gamma[t_1^2, t_2^1]) + c_4 \sum_{i=2}^{r-1} (L_X(\gamma[t_{i-1}^2, t_i^1]) + L_X(\gamma[t_i^2, t_{i+1}^1])) + c_1 + c_4 L_X(\gamma[t_{r-1}^2, t_r^1]) \\ &\leq (1 + 2c_4) \left(L_X(\gamma[0, t_1^1]) + \sum_{i=1}^{r-1} L_X(\gamma[t_i^2, t_{i+1}^1]) + L_X(\gamma[t_r^2, l]) \right) + 2c_1 \\ &\leq (1 + 2c_4) L_X(\gamma) + 2c_1 = (1 + 2c_4) d_X(p(x), p(y)) + 2c_1. \end{aligned}$$

Therefore, the canonical projection $p : X' \rightarrow X$ is a surjective $(1 + 2c_4, 2c_1 / (1 + 2c_4))$ -quasiisometry, and then, there is an “inverse” $j : X \rightarrow X'$, which is a $(1 + 2c_4, 2c_1)$ -quasiisometry.

Now, theorems 2.3 and B give the result. $\square$

As we have said just below Theorem 2.4, if we want to create more general topological obstacles by pasting $\{X_n\}_n$, (i.e. to take a graph instead of a tree), firstly we can apply Theorem 2.4 with a tree and secondly, Theorem 2.2. A simple application of this idea is given by the following result.

THEOREM 2.5. *Let us consider a family of geodesic metric spaces $\{X_n\}_n$. Let us consider a graph $R := (V, E)$ with vertices $V = \{v_n\}_n$ and edges E , such that there exists a finite subset of edges $E_1 \subset E$, with $R' := (V, E \setminus E_1)$ being a tree. We denote by $\{[v_n, v_m]^i\}_{i=1}^{r(n,m)}$ the edges connecting v_n and v_m . We construct a metric space X by pasting $\{X_n\}_n$ following the combinatorial design of R in the following way: if $[v_n, v_m]^i \in E$ we choose compact subsets $\eta_{nm}^i \subseteq X_n$, $\eta_{mn}^i \subseteq X_m$, with $(\eta_{nm}^i, d_{X_n}|_{\eta_{nm}^i})$ and $(\eta_{mn}^i, d_{X_m}|_{\eta_{mn}^i})$ isometric; we define X as the union of $\{X_n\}_n$ by identifying η_{nm}^i and η_{mn}^i by an isometry, for each $[v_n, v_m]^i \in E$. Assume that X is a geodesic metric space and that there are positive constants c_1, c_2 such that $\text{diam}_{X_n}(\eta_{nm}^i) \leq c_1$ and $d_{X_n}(\eta_{nm}^i, \eta_{nk}^j) \geq c_2$ if $m \neq k$ or $i \neq j$. Then X is hyperbolic if and only if there exists a constant c_3 such that X_n is c_3 -hyperbolic for every n .*

The following result is very simple, but it will be useful in the proof of Proposition 3.1.

LEMMA 2.1. *Let us consider a geodesic metric space X . If every geodesic triangle in X which is a simple closed curve, is δ -thin, then X is δ -thin.*

PROOF. First, we observe that a geodesic can not intersect itself. Let us consider a geodesic triangle T in X ; we denote its sides by $\gamma_1, \gamma_2, \gamma_3$, and we assume that $\gamma_1 \cup \gamma_2 \cup \gamma_3$ is not a simple closed curve. In the points in which two sides intersects, the distance of one point to the union of the other sides is zero. We consider now a point x which is in only one side of T (without loss of generality, $x \in \gamma_1$); there is a simple closed curve T_x verifying $x \in T_x \subset T$ and which is the union of two or three segments $\eta_i \subseteq \gamma_i$. In order to see this, consider the maximal open segment $g \subseteq \gamma_1$ with $x \in g$ and $g \cap (\gamma_2 \cup \gamma_3) = \emptyset$. We choose as η_1 the closure of g and we consider $\partial\eta_1 = \{y, z\}$. Let us assume that $y \in \gamma_2$ and $z \in \gamma_3$; since $\gamma_2 \cap \gamma_3$ has at least one point, we can choose segments $\eta_2 \subseteq \gamma_2$, $\eta_3 \subseteq \gamma_3$, with $\partial\eta_2 = \{y, w\}$, $\partial\eta_3 = \{z, w\}$ and $\eta_2 \cap \eta_3 = \{w\}$; then, $T_x = \eta_1 \cup \eta_2 \cup \eta_3$. We assume now $y, z \in \gamma_2$ (the case $y, z \in \gamma_3$ is similar); since γ_2 can not intersect itself, we can choose a segment $\eta_2 \subseteq \gamma_2$ with $\partial\eta_2 = \{y, z\}$, and then $T_x = \eta_1 \cup \eta_2$.

If T_x is the union of three segments $\eta_i \subseteq \gamma_i$, then T_x is a geodesic triangle in X which is a simple closed curve. The hypothesis gives that the distance of x to the union of the other sides is less or equal than δ .

If T_x is the union of two segments $\eta_i \subseteq \gamma_i$, then by a change in the notation we can assume that $T_x = \eta_1 \cup \eta_2$ and $x \in \eta_1$. We choose some point of η_2 ; this point splits η_2 in two geodesics η_2^1 and η_2^2 ; now we can see T_x as a geodesic triangle of sides η_1 , η_2^1 and η_2^2 , which is a simple closed curve. Then we have by hypothesis

$$d_X(x, \eta_2) = d_X(x, \eta_2^1 \cup \eta_2^2) \leq \delta. \quad \square$$

3. Results in Riemann surfaces

Here we present the results which allow to “paste” Riemann surfaces, preserving hyperbolicity. In this section we always work with the Poincaré metric; consequently, curvature is always -1 . In fact, many concepts appearing here (as punctures or funnels) only make sense with the Poincaré metric.

The intuition would say that negative curvature must imply hyperbolicity; in fact this is what happens when there are no topological “obstacles” (as in the case of the Poincaré disk $\mathbf{D}$) or there is a finite number of them (see Proposition 3.2). However, if there are infinitely many topological “obstacles”, the hyperbolicity can fail, as is the case of the two-dimensional jungle gym (a $\mathbf{Z}^2$ -covering of a torus with genus two), which is quasiisometric to the Euclidean plane.

The results in this section are useful since they not only provide many examples of hyperbolic Riemann surfaces, but also allow to establish criteria in order to decide whether a Riemann surface is hyperbolic or not.

Below we collect some definitions concerning to Riemann surfaces which will be referred to afterwards.

An *open non-exceptional* Riemann surface (or a non-exceptional Riemann surface without boundary) S is a Riemann surface whose universal covering space is the unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$, endowed with its Poincaré metric, i.e. the metric obtained by projecting the Poincaré metric of the unit disk

$$ds = \lambda_{\mathbf{D}}(z)|dz| = \frac{2|dz|}{1-|z|^2},$$

or, equivalently, the upper half plane $\mathbf{U} = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$, with the metric $ds = \lambda_{\mathbf{U}}(z)|dz| = |dz|/\operatorname{Im} z$. Observe that, with this definition, every non-bordered compact non-exceptional Riemann surface is open. With this metric, S is a complete Riemannian manifold with constant curvature -1 , and therefore S is a geodesic metric space. The only Riemann surfaces which are left out are the sphere, the plane, the punctured plane and the tori. It is easy to study the hyperbolicity of these particular cases; if we give to these surfaces metrics with constant curvature ($K = 1$ for the sphere and $K = 0$ in the other cases), the sphere and the tori are hyperbolic since they are compact, the punctured plane is hyperbolic since it is isometric to a cylinder, and the plane is not hyperbolic ($\mathbf{R}^n$ is hyperbolic if and only if $n = 1$).

It is well-known (see e.g. [5, p.100], [7, p.131], [25, p.18]) that

$$(3.1) \quad d_{\mathbf{D}}(0, z) = \log \frac{1 + |z|}{1 - |z|} = 2 \operatorname{Arctanh} |z|.$$

Let S be an open non-exceptional Riemann surface with a puncture q (if $S \subset \mathbf{C}$, every isolated point in ∂S is a puncture). A *collar* in S about q is a doubly connected domain in S “bounded” both by q and a Jordan curve (called the boundary curve of the collar) orthogonal to the pencil of geodesics emanating from q . It is well known that the length of the boundary curve is equal to the area of the collar (see e.g. [8]).

A collar in S about q of area α will be called an α -*collar* and it will be denoted by $C_S(q, \alpha)$. A theorem of Shimizu [35] gives that for every puncture in any open non-exceptional Riemann surface, there exists an α -collar for every $0 < \alpha \leq 1$ (see also [24, p.60-61]).

We say that a curve is *homotopic to a puncture* q if it is freely homotopic to $\partial C_S(q, \alpha)$ for some (and then for every) $0 < \alpha < 1$.

We have used the word *geodesic* in the sense of Definition 2.2, that is to say, as a global geodesic or a minimizing geodesic; however, we need now to deal with a special type of local geodesics: simple closed geodesics, which obviously can not be minimizing geodesics. We will continue using the word *geodesic* with the meaning of Definition 2.2, unless we are dealing with closed geodesics.

A *collar* in S about a simple closed geodesic γ is a doubly connected domain in S “bounded” by two Jordan curves (called the boundary curves of the collar) orthogonal to the pencil of geodesics emanating from γ ; such collar is equal to $\{p \in S : d_S(p, \gamma) < d\}$, for some positive constant d . The constant d is called the *width* of the collar. Collar Lemma [29] says that there exists a collar of γ of width d , for every $0 < d \leq d_0$, where $\cosh d_0 = \coth(L_S(\gamma)/2)$.

We say that S is a *bordered non-exceptional Riemann surface* (or a non-exceptional Riemann surface with boundary) if it can be obtained deleting an open set V of an open non-exceptional Riemann surface R , such that:

- (1) $d_S := d_R|_S$ (recall Definition 2.5),
- (2) any ball in R intersects at most a finite number of connected components of V ,
- (3) the boundary of S is locally Lipschitz.

Any such surface S is a bordered orientable Riemannian manifold of dimension 2 and its Riemannian metric has constant negative curvature -1 . It is not difficult to see that S is a geodesic metric space.

A *funnel* is a bordered non-exceptional Riemann surface which is topologically a cylinder and whose boundary is a simple closed geodesic. Given a positive number a , there is a unique (up to conformal

mapping) funnel such that its boundary curve has length a . Every funnel is conformally equivalent, for some $\beta > 1$, to the subset $\{z \in \mathbf{C} : 1 \leq |z| < \beta\}$ of the annulus $\{z \in \mathbf{C} : 1/\beta < |z| < \beta\}$.

Every doubly connected final of an open non-exceptional Riemann surface is a puncture (if there are homotopically non-trivial curves with arbitrary small length) or a funnel (in other case).

A *Y-piece* is a bordered non-exceptional Riemann surface which is conformally equivalent to a sphere without three open disks and whose boundary curves are simple closed geodesics. Given three positive numbers a, b, c , there is a unique (up to conformal mapping) Y-piece such that their boundary curves have lengths a, b, c (see e.g. [30, p.410]). They are a standard tool for constructing Riemann surfaces. A clear description of these Y-pieces and their use is given in [12, chapter X.3] and [10, chapter 1].

A *generalized Y-piece* is a non-exceptional Riemann surface (with or without boundary) which is conformally equivalent to a sphere without n open disks and m points, with integers $n, m \geq 0$ and $n + m = 3$, so that the n boundary curves are simple closed geodesics and the m deleted points are punctures. Observe that a generalized Y-piece is topologically the union of a Y-piece and m cylinders, with $0 \leq m \leq 3$.

DEFINITION 3.1. A set I in an open non-exceptional Riemann surface S is called *r -uniformly separated* if the balls $\{B_S(p, r)\}_{p \in I}$ are pairwise disjoint. We say that I is *uniformly separated* if the value of r is not important.

A set I in an open non-exceptional Riemann surface S is called *r -strongly uniformly separated* if the balls $\{B_S(p, r)\}_{p \in I}$ are simply connected and pairwise disjoint. We say that I is *strongly uniformly separated* if the value of r is not important.

The sets strongly uniformly separated play a central role in the study of hyperbolic isoperimetric inequalities (HII) in open Riemann surfaces. In fact, we have the following result:

THEOREM C. [2, Theorem 1] *Let S be an open non-exceptional Riemann surface, let I be a closed and countable subset of S and $S^* := S \setminus I$. Then, S^* has HII if and only if S has HII and I is strongly uniformly separated in S .*

We recall that an open non-exceptional Riemann surface S satisfies a hyperbolic isoperimetric inequality if there is a positive constant h such that

$$A_S(D) \leq h L_S(\partial D)$$

holds for every relatively compact domain $D \subset S$ with smooth boundary, where $A_S(D)$ denotes the area of D in the Poincaré metric of S .

There are interesting relations of the hyperbolic isoperimetric inequality with other conformal invariants of a Riemann surface (see e.g. [2], [12, p.95], [13], [15], [25, p.145], [37, p.333]).

Theorems 2.2 and 2.4 have a simpler statement in the context of non-exceptional Riemann surfaces (with or without boundary) if we “paste” these surfaces by identifying simple closed geodesics:

THEOREM 3.1. *Let us consider a non-exceptional Riemann surface S with boundary, and $\{\eta_n^1, \eta_n^2\}_n$ $(c_1, 0)$ -identified, such that $\{\eta_n^1, \eta_n^2\}_n \subseteq \partial S$ are pairwise disjoint simple closed geodesics and $L_S(\eta_n^1) \leq c_1$ for every n . Then S is hyperbolic if and only if S_0 is hyperbolic. In particular, if S is δ -hyperbolic, then S_0 is δ' -hyperbolic, with δ' a universal constant which only depends on δ and c_1 .*

REMARK. Since η_n^i are simple closed curves, the condition that $(\eta_n^1, d_S|_{\eta_n^1})$ and $(\eta_n^2, d_S|_{\eta_n^2})$ are isometric is equivalent to $L_S(\eta_n^1) = L_S(\eta_n^2)$.

THEOREM 3.2. *Let us consider a non-exceptional Riemann surface S (with or without boundary) and a family of non-exceptional Riemann surfaces $\{S_n\}_n$ with boundary which is a $(c_1, 0)$ -decomposition of S , with $\{\eta_{nm}\}_m \subseteq \partial S_n$ pairwise disjoint simple closed geodesics for every n and $L_S(\eta_{nm}) \leq c_1$ for every n, m . Then S is δ -hyperbolic if and only if there exists a constant c_2 such that S_n is c_2 -hyperbolic for every n . Furthermore, δ is a universal constant which only depends on c_1 and c_2 .*

REMARK. The conclusion of Theorem 3.2 is still true if we substitute the simple closed geodesics $\eta_{nm} \subseteq \partial S_n$, by families of simple closed geodesics $\{\eta_{nm}^j\}_{j=1}^{r(n,m)} \subseteq \partial S_n$, with $r(n, m) = r(m, n)$, $L_S(\eta_{nm}^j) = L_S(\eta_{mn}^j) \leq c_1$, and $d_{S_n}(\eta_{nm}^j, \eta_{nm}^l) \leq c_1$.

PROOF OF THEOREMS 3.1 AND 3.2. These results are direct consequence, respectively, of theorems 2.2 and 2.4, and the following facts:

Every non-exceptional Riemann surface (with or without boundary) is a geodesic metric space.

If γ_1, γ_2 , are disjoint simple closed geodesics contained in an open non-exceptional Riemann surface, with length less or equal than a , collar Lemma [29] says that there exist disjoint collars of γ_i of width d_0 , where $\cosh d_0 = \coth(a/2)$. Therefore, $d(\gamma_1, \gamma_2) \geq 2 \operatorname{Argcosh}(\coth(a/2))$; it is clear that this inequality is also true if S has boundary, since then S is contained in an open non-exceptional Riemann surface. $\square$

We collect here the technical results that we will need in the proof of Theorem 3.3.

LEMMA 3.1. *If η is a curve joining $z, w \in \partial B_{\mathbf{D}}(p, r)$, with $0 < r \leq R$ and $\eta \subset \overline{B_{\mathbf{D}}(p, r)}$, then there exists a curve $\eta_0 \subset \partial B_{\mathbf{D}}(p, r)$ joining z and w , with $L_{\mathbf{D}}(\eta_0) \leq c_1 L_{\mathbf{D}}(\eta)$, where c_1 is a universal constant which only depends on R .*

PROOF. Without loss of generality we can assume that $p = 0$. Then $B_{\mathbf{D}}(0, r) = B(0, \tanh(r/2))$, by (3.1). Since the density of the Poincaré metric is $\lambda_{\mathbf{D}}(z) = 2/(1 - |z|^2)$, we have $2L(g) \leq L_{\mathbf{D}}(g) \leq 2L(g)/(1 - \tanh^2(R/2)) = 2L(g) \cosh^2(R/2)$, for any curve $g \subset \overline{B_{\mathbf{D}}(0, r)}$, if L denotes the Euclidean length. Therefore, it is enough to prove the lemma for the Euclidean length instead of the Poincaré length. This result is clear for the Euclidean length with constant $\pi/2$, since $t|\theta - \phi| \leq \pi/2 |te^{i\theta} - te^{i\phi}|$ for $t > 0$ and $|\theta - \phi| \leq \pi$. $\square$

LEMMA 3.2. *Let S be an open non-exceptional Riemann surface and I be a r -strongly uniformly separated set in S . If $x, y \in S_1 := S \setminus \cup_{p \in I} B_S(p, r)$ and γ is a curve joining x and y in S , then there exists a curve $\gamma_0 \subset S_1$ joining x and y with $L_S(\gamma_0) \leq c_1 L_S(\gamma)$, where c_1 is a universal constant which only depends on r . In particular, $d_S|_{S_1}(x, y) \leq c_1 d_S(x, y)$.*

PROOF. Let us assume that $x, y \in \partial B_S(p, r)$ and $\gamma \subset \overline{B_S(p, r)}$, for some $p \in I$; since $B_S(p, r)$ is simply connected, it is isometric to $B_{\mathbf{D}}(0, r)$ and then it is enough to apply Lemma 3.1.

In the general case, the curve γ can intersect many balls $B_S(p_1, r), B_S(p_2, r), \dots, B_S(p_k, r)$. Then we can substitute each connected component η^{ij} of $\eta^i := \gamma \cap \overline{B_S(p_i, r)}$ by a curve $\eta_0^{ij} \subset \partial B_S(p_i, r)$ as in the last case. Since the constant c_1 is the same for any i , the curve γ_0 is the union of η_0^{ij} and $\gamma \setminus \cup_{i=1}^k \eta^i$. $\square$

LEMMA 3.3. *Let S be an open non-exceptional Riemann surface with a puncture p , and $0 < \alpha \leq 1$. Then, we have that*

$$C_S\left(p, \frac{4\alpha}{\sqrt{\alpha^2 + 16}}\right) \cap \gamma = \emptyset,$$

for any geodesic γ joining points $w_1, w_2 \in S \setminus C_S(p, \alpha)$.

PROOF. It is well known that S can be expressed as a quotient $\mathbf{U}/\Gamma$, where Γ is a discrete group of Möbius transformations preserving the upper half plane $\mathbf{U}$, and the set $D_\alpha = \{z \in \mathbf{U} : 0 \leq \operatorname{Re} z < 1, \operatorname{Im} z > 1/\alpha\}$ is projected injectively onto $C_S(p, \alpha)$, for $0 < \alpha \leq 1$.

Let γ be a geodesic joining points $w_1, w_2 \in S \setminus C_S(p, \alpha)$. If γ does not intersect $C_S(p, \alpha)$, then there is nothing to prove, since $4\alpha/\sqrt{\alpha^2 + 16} < \alpha$. We can assume that γ intersects $C_S(p, \alpha)$. Then, there exists a geodesic η in $\mathbf{U}$ (a semicircle centered at a point in the real axis) whose projection is γ and such that the projection of $\eta \cap D_\alpha$ is $\gamma \cap C_S(p, \alpha)$. Without loss of generality we can assume that η meets the horizontal line $\{\operatorname{Im} z = 1/\alpha\}$ in the points $i/\alpha, t + i/\alpha$, with $0 < t \leq 1/2$ (if $t > 1/2$, the geodesic joining $i/\alpha, t - 1 + i/\alpha$ would be shorter).

It is clear that the worse case corresponds to $t = 1/2$. So, we take $t = 1/2$, and we will compute the Euclidean radius r of the semicircle η ; then, $\gamma \cap C_S(p, \beta) = \emptyset$ for $\beta = 1/r$. If $z = x + iy$, the equation of η is $(x - 1/4)^2 + y^2 = r^2$. Since $i/\alpha \in \eta$, we have $1/16 + 1/\alpha^2 = r^2$, which gives $r = \sqrt{\alpha^2 + 16}/(4\alpha)$ and $\beta = 4\alpha/\sqrt{\alpha^2 + 16}$. $\square$

The following lemma gives the first family of examples of hyperbolic non-exceptional Riemann surfaces.

LEMMA 3.4. *Any α -collar of a puncture in an open non-exceptional Riemann surface is c_0 -thin, for $0 < \alpha \leq 1$, with $c_0 = \frac{1}{2} + \frac{1}{2} \log \frac{17}{16}$. Also, the closure of any α -collar of a puncture is c_0 -thin, for $0 < \alpha < 1$.*

PROOF. Let us observe that, given $0 < \alpha \leq 1$, any two α -collars of punctures are isometric. Since $C_S(p, \alpha) \subseteq C_S(p, 1)$ for any $0 < \alpha \leq 1$ and the geodesics (in the metric of $C_S(p, 1)$) joining $w_1, w_2 \in C_S(p, \alpha)$ are contained in $C_S(p, \alpha)$, we have that if $C_S(p, 1)$ is c_0 -thin, then $C_S(p, \alpha)$ is c_0 -thin, for any $0 < \alpha \leq 1$. The same argument works for $\overline{C_S(p, \alpha)}$, with $0 < \alpha < 1$. Therefore, it is enough to show that $C_S(p, 1)$ is c_0 -thin.

Let us observe that $C_S(p, 1) = \cup_{0 < \alpha < 1} \partial C_S(p, \alpha)$. Given a geodesic triangle $T = \{x_1, x_2, x_3\}$ in $C_S(p, 1)$, we denote by α_i the positive number with $x_i \in \partial C_S(p, \alpha_i)$. By a change of notation, we can assume that $0 < \alpha_1 \leq \alpha_2 \leq \alpha_3 < 1$. By Lemma 3.3, $T \cap C_S(p, 4\alpha_1/\sqrt{\alpha_1^2 + 16}) = \emptyset$; then, $T \subset \cup_{4\alpha_1/\sqrt{\alpha_1^2 + 16} \leq \alpha \leq \alpha_3} \partial C_S(p, \alpha)$.

It is clear that every $\partial C_S(p, \alpha)$, with $\alpha_1 \leq \alpha \leq \alpha_3$ intersects at least two sides of T ; since $L_S(\partial C_S(p, \alpha)) = \alpha$, if $x \in T \cap \partial C_S(p, \alpha)$, with $\alpha_1 \leq \alpha \leq \alpha_3$, then the distance of x to the union of the other sides of T is less or equal than $\alpha/2 \leq 1/2$. If $x \in T \cap \partial C_S(p, \alpha)$, with $4\alpha_1/\sqrt{\alpha_1^2 + 16} \leq \alpha < \alpha_1$, then

$$d_S(x, \partial C_S(p, \alpha_1)) \leq d_S\left(\partial C_S(p, \alpha_1), \partial C_S\left(p, \frac{4\alpha_1}{\sqrt{\alpha_1^2 + 16}}\right)\right) = \frac{1}{2} \log\left(1 + \frac{\alpha_1^2}{16}\right) \leq \frac{1}{2} \log \frac{17}{16},$$

and, since there is a point of the union of the other sides of T in $\partial C_S(p, \alpha_1)$, then the distance of x to the union of the other sides of T is less or equal than $\frac{\alpha_1}{2} + \frac{1}{2} \log \frac{17}{16} \leq \frac{1}{2} + \frac{1}{2} \log \frac{17}{16}$. This finishes the proof. $\square$

The following theorem allows, in many cases, to forget the punctures in order to study the hyperbolicity of a Riemann surface; this fact can be a significant simplification in the topology of the surface, and therefore makes easier the study of its hyperbolicity.

THEOREM 3.3. *Let S be an open non-exceptional Riemann surface and I be a r -strongly uniformly separated set in S . Then, S is hyperbolic if and only if $S^* := S \setminus I$ is hyperbolic. In particular, if S is δ -hyperbolic then S^* is δ' -hyperbolic, with δ' a universal constant which only depends on δ and r .*

REMARK. Recall that $d_{S^*} \neq d_S|_{S^*}$, since (S^*, d_{S^*}) is a complete Riemannian manifold (the points of I are at infinite d_{S^*} -distance of the points of S^*). This fact also implies that (S^*, d_{S^*}) is a geodesic metric space.

PROOF. Let us assume that S is δ -hyperbolic. The idea of the proof is to split S into two sets: in a neighbourhood of the set I (which will be dealt with the previous lemmas), and the complement of this set (in which d_S and d_{S^*} are comparable).

By Proposition 1 in [2], there exist $0 < r_1 < r_2 < r$ and $0 < \alpha < 1$, which only depend on r , such that

$$B_S(p, r_1) \subseteq C_{S^*}(p, \alpha) \cup \{p\} \subseteq B_S(p, r_2).$$

We define $S_0 := S^* \setminus \bigcup_{p \in I} C_{S^*}(p, \alpha)$. Then we have that

$$S^* = S_0 \cup \left(\bigcup_{p \in I} \overline{C_{S^*}(p, \alpha)} \right).$$

By Lemma 3.1 in [2], we have that

$$\tanh \frac{r_1}{2} < \frac{L_S(\gamma)}{L_{S^*}(\gamma)} < 1,$$

for every curve $\gamma \subset S_0$ with finite length in S . Then we obtain for $x, y \in S_0$

$$\tanh \frac{r_1}{2} \leq \frac{d_S|_{S_0}(x, y)}{d_{S^*}|_{S_0}(x, y)} \leq 1.$$

Then the identity $i : (S_0, d_{S^*}|_{S_0}) \rightarrow (S_0, d_S|_{S_0})$ is a quasiisometry with constants which only depend on r . Observe that $(S_0, d_{S^*}|_{S_0})$ and $(S_0, d_S|_{S_0})$ are geodesic metric spaces (they are bordered non-exceptional Riemann surfaces).

We prove now that the inclusion $j : (S_0, d_S|_{S_0}) \rightarrow (S, d_S)$ is a quasiisometry with constants which only depend on r .

It is clear that for every $x, y \in S_0$ we have $d_S(x, y) \leq d_{S^*}|_{S_0}(x, y)$, since there are more curves joining x and y in S than in S_0 .

We prove now the other inequality. If $x, y \in S \setminus \bigcup_{p \in I} B_S(p, r_2)$, then Lemma 3.2 gives

$$(3.2) \quad d_S|_{S_0}(x, y) \leq d_S|_{S \setminus \bigcup_{p \in I} B_S(p, r_2)}(x, y) \leq c_1 d_S(x, y),$$

since $S \setminus \bigcup_{p \in I} B_S(p, r_2) \subseteq S_0$, where c_1 is a constant which only depends on r .

If $x, y \in S_0$, we can choose $x', y' \in S \setminus \bigcup_{p \in I} B_S(p, r_2)$, with $d_S(x, x') \leq d_{S^*}|_{S_0}(x, x') \leq r_2 - r_1$ and $d_S(y, y') \leq d_{S^*}|_{S_0}(y, y') \leq r_2 - r_1$. Then (3.2) gives that

$$d_S|_{S_0}(x, y) \leq d_S|_{S_0}(x', y') + 2(r_2 - r_1) \leq c_1 d_S(x', y') + 2(r_2 - r_1) \leq c_1 d_S(x, y) + 2(r_2 - r_1)(1 + c_1).$$

Consequently, $j : (S_0, d_S|_{S_0}) \longrightarrow (S, d_S)$ is a quasiisometry with constants which only depend on r .

Since S is δ -hyperbolic and $j \circ i$ is a quasiisometry, then Theorem B gives that $(S_0, d_{S^*}|_{S_0})$ is δ_1 -hyperbolic, with δ_1 a universal constant which only depends on δ and r .

We recall that, by Lemma 3.4, $\overline{C_{S^*}(p, \alpha)}$ is c_0 -hyperbolic for every $p \in I$.

If $p, q \in I$ and $p \neq q$, we have

$$\begin{aligned} \text{diam}_{S^*}(\partial C_{S^*}(p, \alpha)) &\leq L_{S^*}(\partial C_{S^*}(p, \alpha)) = \alpha < 1, \\ d_{S^*}(\partial C_{S^*}(p, \alpha), \partial C_{S^*}(q, \alpha)) &\geq d_{S^*}(\partial C_{S^*}(p, \alpha), \partial C_{S^*}(p, 1)) + d_{S^*}(\partial C_{S^*}(q, 1), \partial C_{S^*}(q, \alpha)) = 2 \log \frac{1}{\alpha}. \end{aligned}$$

Then, Theorem 2.4 gives that S^* is δ' -hyperbolic, with δ' a universal constant which only depends on δ and r .

We assume now that S^* is hyperbolic. Then, Theorem 2.4 also gives that $(S_0, d_{S^*}|_{S_0})$ is hyperbolic. Since i is a surjective quasiisometry, then Theorem B gives that $(S_0, d_S|_{S_0})$ is hyperbolic.

If $p, q \in I$ and $p \neq q$, we have

$$\begin{aligned} \text{diam}_S(\partial C_{S^*}(p, \alpha)) &\leq \text{diam}_S(B_S(p, r_2)) = 2r_2, \\ d_S(\partial C_{S^*}(p, \alpha), \partial C_{S^*}(q, \alpha)) &\geq d_S(B_S(p, r_2), B_S(q, r_2)) \geq 2(r - r_2). \end{aligned}$$

Since $\text{diam}_S(C_{S^*}(p, \alpha) \cup \{p\}) \leq 2r_2$, then $C_{S^*}(p, \alpha) \cup \{p\}$ (with its metric in S) is $2r_2$ -hyperbolic, for every $p \in I$. Therefore, Theorem 2.4 gives that S is hyperbolic. $\square$

The following proposition gives the second family of examples of hyperbolic non-exceptional Riemann surfaces, with a qualitative behaviour of the hyperbolicity constants.

PROPOSITION 3.1. *Let A be an annulus with its simple closed geodesic γ verifying $L_A(\gamma) \leq a$, and let F be a funnel with boundary γ . Then F and A are c_1 -hyperbolic, where c_1 is a constant which only depends on a . Also, every doubly connected non-exceptional Riemann surface with compact boundary or without boundary, is hyperbolic.*

PROOF. We consider the punctured disc $\mathbf{D}^* := \{z \in \mathbf{C} : 0 < |z| < 1\}$ and the annulus $A_\varepsilon := \{z \in \mathbf{C} : \varepsilon < |z| < 1\}$, for $0 < \varepsilon < 1$. It is well known that every annulus is conformally equivalent to A_ε for some $0 < \varepsilon < 1$. As $\mathbf{D}^*$ is hyperbolic (by Theorem 3.3) we will show that the funnels of A_ε (for $0 < \varepsilon < \varepsilon_0$) are hyperbolic using a quasiisometry; an iterative process will give the result for $0 < \varepsilon < 1$.

The Poincaré length of any curve γ in a plane domain Ω can be computed by the formula $L_\Omega(\gamma) = \int_\gamma \lambda_\Omega(z) |dz|$. In these particular cases we have

$$\lambda_{\mathbf{D}^*}(z) = \frac{1}{|z| \log \frac{1}{|z|}}, \quad \lambda_{A_\varepsilon}(z) = \frac{c_\varepsilon}{|z| \operatorname{sen}\left(c_\varepsilon \log \frac{1}{|z|}\right)},$$

with $c_\varepsilon := \pi / \log(1/\varepsilon)$ (see e.g. [1, p.17]). The simple closed geodesic in A_ε is $\{|z| = \sqrt{\varepsilon}\}$, and has length $2\pi c_\varepsilon$. We also have $C_{\mathbf{D}^*}(0, 2\pi / \log(1/r)) = \{0 < |z| < r\}$ and $C_{\mathbf{D}^*}(0, \alpha) = \{0 < |z| < \exp(-2\pi/\alpha)\}$.

If $F := \{\sqrt{\varepsilon} \leq |z| < 1\}$, we show now that the injection $i : (F, d_{A_\varepsilon}) \longrightarrow (\mathbf{D}^*, d_{\mathbf{D}^*})$ is a quasiisometry, if ε is small enough. Let us observe that F is a convex set for the distance d_{A_ε} , since $\{\sqrt{\varepsilon} \leq |z| < 1\}$ and $\{\varepsilon < |z| \leq \sqrt{\varepsilon}\}$ are isometric for d_{A_ε} .

However, F is not a convex set for the distance $d_{\mathbf{D}^*}$. We assume that a $d_{\mathbf{D}^*}$ -geodesic γ joining two points in F , intersects $\{0 < |z| < \sqrt{\varepsilon}\} = C_{\mathbf{D}^*}(0, 4\pi/\log(1/\varepsilon))$. By Lemma 3.3, if $r_\varepsilon := \exp(-\sqrt{\pi^2 + \log^2(1/\varepsilon)}/2)$, we have that

$$(3.3) \quad \{0 < |z| < r_\varepsilon\} \cap \gamma = C_{\mathbf{D}^*}\left(0, \frac{4\pi}{\sqrt{\pi^2 + \log^2(1/\varepsilon)}}\right) \cap \gamma = \emptyset.$$

Observe that $t/\operatorname{sen} t$ is an increasing function in $(0, \pi)$. This gives, in $B_\varepsilon := \{r_\varepsilon \leq |z| < 1\}$,

$$1 < \frac{\lambda_{A_\varepsilon}(z)}{\lambda_{\mathbf{D}^*}(z)} = \frac{c_\varepsilon \log \frac{1}{z}}{\operatorname{sen}(c_\varepsilon \log \frac{1}{z})} \leq \frac{c_\varepsilon \log \frac{1}{r_\varepsilon}}{\operatorname{sen}(c_\varepsilon \log \frac{1}{r_\varepsilon})} = \frac{\frac{\pi}{2} \sqrt{1 + \frac{\pi^2}{\log^2(1/\varepsilon)}}}{\operatorname{sen}\left(\frac{\pi}{2} \sqrt{1 + \frac{\pi^2}{\log^2(1/\varepsilon)}}\right)} =: M_\varepsilon,$$

if $\frac{\pi}{2} \sqrt{1 + \frac{\pi^2}{\log^2(1/\varepsilon)}} < \pi$, i.e. $0 < \varepsilon < \exp(-\pi/\sqrt{3})$ (this condition is equivalent to $\varepsilon < r_\varepsilon$). Then, we conclude $d_{\mathbf{D}^*}|_{B_\varepsilon}(z_1, z_2) \leq d_{A_\varepsilon}|_{B_\varepsilon}(z_1, z_2) \leq M_\varepsilon d_{\mathbf{D}^*}|_{B_\varepsilon}(z_1, z_2)$, for every $z_1, z_2 \in F$, if $0 < \varepsilon < \exp(-\pi/\sqrt{3})$.

By (3.3), we have that $d_{\mathbf{D}^*}(z_1, z_2) \leq d_{A_\varepsilon}(z_1, z_2) \leq M_\varepsilon d_{\mathbf{D}^*}(z_1, z_2)$, for every $z_1, z_2 \in F$, if $0 < \varepsilon < \exp(-\pi/\sqrt{3})$. Then, $i: (F, d_{A_\varepsilon}) \rightarrow (\mathbf{D}^*, d_{\mathbf{D}^*})$ is a $(M_\varepsilon, 0)$ -quasiisometry, if $0 < \varepsilon < \exp(-\pi/\sqrt{3})$.

We choose $0 < \alpha_1 < \alpha_2 < 1$, with $\alpha_2 < \sqrt{\alpha_1}$. Observe that $\operatorname{sen} t/\operatorname{sen}(at)$ is an increasing function in $(0, \pi/2)$, if $1 < a < 2$. Then we have, in $\{\sqrt{\alpha_1} \leq |z| < 1\}$,

$$1 < \frac{\lambda_{A_{\alpha_2}}(z)}{\lambda_{A_{\alpha_1}}(z)} = \frac{c_{\alpha_2} \operatorname{sen}(c_{\alpha_1} \log \frac{1}{z})}{c_{\alpha_1} \operatorname{sen}(c_{\alpha_2} \log \frac{1}{z})} \leq \frac{c_{\alpha_2}}{c_{\alpha_1} \operatorname{sen}(c_{\alpha_2} \log \frac{1}{\sqrt{\alpha_1}})} = \frac{\log \frac{1}{\alpha_1}}{\log \frac{1}{\alpha_2}} \frac{1}{\operatorname{sen}\left(\frac{\pi}{2} \frac{\log \frac{1}{\alpha_1}}{\log \frac{1}{\alpha_2}}\right)} =: M(\alpha_1, \alpha_2).$$

These inequalities give $d_{A_{\alpha_1}}(z_1, z_2) \leq d_{A_{\alpha_2}}(z_1, z_2) \leq M(\alpha_1, \alpha_2) d_{A_{\alpha_1}}(z_1, z_2)$, for every $z_1, z_2 \in \{\sqrt{\alpha_2} \leq |z| < 1\}$, since $\{\sqrt{\alpha_2} \leq |z| < 1\}$ is $d_{A_{\alpha_2}}$ -convex and $\{\sqrt{\alpha_1} \leq |z| < 1\}$ is $d_{A_{\alpha_1}}$ -convex (recall that $\alpha_2 < \sqrt{\alpha_1}$). Then, $i: (\{\sqrt{\alpha_2} \leq |z| < 1\}, d_{A_{\alpha_2}}) \rightarrow (A_{\alpha_1}, d_{A_{\alpha_1}})$ is a $(M(\alpha_1, \alpha_2), 0)$ -quasiisometry, if $\alpha_2 < \sqrt{\alpha_1}$.

We consider the sequence, for $n \geq 0$,

$$\varepsilon_n := \exp\left(\frac{-2\pi}{\sqrt{3}} \left(\frac{3}{4}\right)^n\right).$$

Let us observe that $\varepsilon_0 = \exp(-2\pi/\sqrt{3}) < \exp(-\pi/\sqrt{3})$ and $\varepsilon_{n+1} < \sqrt{\varepsilon_n}$.

We prove now, by induction, that F_ε (a funnel corresponding to A_ε) is c_n -hyperbolic for $0 < \varepsilon \leq \varepsilon_n$, where c_n is a constant which only depends on n . Since $0 < \varepsilon \leq \varepsilon_n$ is equivalent to $L_{A_\varepsilon}(\gamma_\varepsilon) = 2\pi^2/\log(1/\varepsilon) \leq 2\pi^2/\log(1/\varepsilon_n)$ and $\{\varepsilon_n\}$ is an increasing sequence with limit 1, the induction hypothesis is equivalent to Proposition 3.1 for funnels. The conclusion for annuli is a consequence of this fact and Theorem 2.4, since an annulus can be obtained by gluing two isometric funnels.

The inclusion $i: (F, d_{A_\varepsilon}) \rightarrow (\mathbf{D}^*, d_{\mathbf{D}^*})$ is a $(M_{\varepsilon_0}, 0)$ -quasiisometry, for $0 < \varepsilon \leq \varepsilon_0$, since M_ε is an increasing function in ε , and $\varepsilon_0 = \exp(-2\pi/\sqrt{3}) < \exp(-\pi/\sqrt{3})$. $(\mathbf{D}^*, d_{\mathbf{D}^*})$ is hyperbolic by Theorem 3.3. Therefore, Theorem B gives the induction hypothesis for $n = 0$.

Let us assume the induction hypothesis for n . If we consider $\alpha_1 = \varepsilon_n$ and $\alpha_2 = \varepsilon$, the inclusion $i: (\{\sqrt{\varepsilon} \leq |z| < 1\}, d_{A_\varepsilon}) \rightarrow (A_{\varepsilon_n}, d_{A_{\varepsilon_n}})$ is a $(M(\varepsilon_n, \varepsilon_{n+1}), 0)$ -quasiisometry, for $\varepsilon_n \leq \varepsilon \leq \varepsilon_{n+1}$, since $M(\varepsilon_n, \varepsilon)$ is an increasing function in ε , and $\varepsilon \leq \varepsilon_{n+1} < \sqrt{\varepsilon_n}$. $(A_{\varepsilon_n}, d_{A_{\varepsilon_n}})$ is c_n -hyperbolic by the induction hypothesis for n . Therefore, Theorem B gives the induction hypothesis for $n + 1$.

Now, we consider a doubly connected non-exceptional Riemann surface S with compact boundary or without boundary. If S does not have boundary, it is isometric to $\mathbf{D}^*$ or some annulus, and then it is hyperbolic by Theorem 3.3 or the first part of the proposition. If S has compact boundary, it is isometric to a bordered surface S_1 contained in R , where R is $\mathbf{D}$, $\mathbf{D}^*$ or some annulus; R is the union of S_1 and at most two other bordered surfaces. Then, Theorem 2.4 gives that S_1 is hyperbolic, since R is hyperbolic. $\square$

This result allows to find two important classes of hyperbolic non-exceptional Riemann surfaces, which appear in propositions 3.2 and 3.3.

DEFINITION 3.2. We say that a non-exceptional Riemann surface S (with or without boundary) is *of finite type* if their fundamental group is finitely generated. If S has not boundary, then it is simply or doubly connected or it can be obtained from a compact non-exceptional Riemann surface by pasting n collars of punctures and m funnels.

PROPOSITION 3.2. *Every non-exceptional Riemann surface with compact boundary or without boundary of finite type is hyperbolic.*

PROOF. We consider a non-exceptional Riemann surface with compact boundary or without boundary of finite type S . If it is simply connected, it is isometric to $\mathbf{D}$ (then it is hyperbolic) or to a subset of $\mathbf{D}$ bordered by a Jordan curve (then it is compact, and consequently it is hyperbolic). If it is doubly connected, we can apply Proposition 3.1.

We assume now that S is not simply nor doubly connected. If S has not boundary, then it can be obtained from a compact non-exceptional Riemann surface S_0 by pasting n collars of punctures and m funnels. We know by Lemma 3.4 and Proposition 3.1 that collars of punctures and funnels are hyperbolic. Since S_0 has finite diameter, it is also hyperbolic. Now, Theorem 2.4 gives the result. If S has compact boundary, then there is a non-exceptional Riemann surface without boundary of finite type R and compact non-exceptional Riemann surfaces with boundary $S_1, \dots, S_n$ such that $R = S \cup S_1 \cup \dots \cup S_n$; then Theorem 2.5 gives the result. $\square$

PROPOSITION 3.3. *Let Y be a generalized Y -piece bounded by $\gamma_1, \gamma_2, \gamma_3$ (γ_i are simple closed geodesics or punctures). If $L(\gamma_i) \leq a$ for $i = 1, 2, 3$, then Y is c_2 -hyperbolic, where c_2 is a constant which only depends on a .*

REMARK. We recall that we consider a puncture as a geodesic of zero-length.

PROOF. Let us consider a geodesic triangle $T \subset Y$, which is a simple closed curve. Then, T is homotopically trivial or is freely homotopic to γ_i for some $i = 1, 2, 3$.

In the first case, T is the boundary of a simply connected domain Ω . Then, there is a simply connected domain $\Omega_0 \subset \mathbf{D}$, with $\overline{\Omega_0}$ isometric to $\overline{\Omega}$. Since $\partial\Omega_0$ is a geodesic triangle in $\mathbf{D}$, it is $\log(1 + \sqrt{2})$ -thin [5, p.130]. Consequently, T is $\log(1 + \sqrt{2})$ -thin.

If T is freely homotopic to γ_i , then $T \cup \gamma_i$ is the boundary of a doubly connected domain G . Then, there is a doubly connected domain G_0 contained in the annulus A with simple closed geodesic of length $L(\gamma_i)$, with $\overline{G_0}$ isometric to $\overline{G}$. Since one of the connected components of ∂G_0 is a geodesic triangle in A , it is $4c_1$ -thin by Proposition 3.1, where c_1 is a constant which only depends on a . Then, T is $4c_1$ -thin.

Since every geodesic triangle in $\mathbf{D}$ is isometric to a geodesic triangle in A , we have that $\log(1 + \sqrt{2}) \leq 4c_1$. Consequently, every geodesic triangle $T \subset Y$, which is a simple closed curve, is $4c_1$ -thin.

Therefore, Lemma 2.1 and Theorem A give the result. $\square$

Many Riemann surfaces can be decomposed in a union of funnels and generalized Y -pieces (see [14, Theorem 4.1] and [3]). The following result uses this decomposition in order to obtain hyperbolicity.

THEOREM 3.4. *Let us consider a non-exceptional Riemann surface S (with or without boundary) without genus (S can be viewed as a plane domain) and $a > 0$. If there is a decomposition of S in a union of funnels and generalized Y -pieces $\{Y_n\}_n$ with $L_S(\partial Y_n) \leq a$, then S is c_1 -hyperbolic, where c_1 is a constant which only depends on a .*

PROOF. Since S has no genus, S can be obtained by pasting the funnels and the Y -pieces following the combinatorial design of a tree. Proposition 3.3 gives that Y_n is c_2 -hyperbolic, where c_2 is a constant which only depends on a . Since every funnel is connected with some Y -piece, its simple closed geodesic has length least or equal than a ; then, Proposition 3.1 gives that the funnels in S are c_3 -hyperbolic, where c_3 is a constant which only depends on a . Consequently, Theorem 3.2 gives the result. $\square$

Theorems 3.4 and 3.1 give directly the following result.

COROLLARY 3.1. *Let us consider a non-exceptional Riemann surface S (with or without boundary) of finite genus and $a > 0$. If there is a decomposition of S in a union of funnels and generalized Y -pieces $\{Y_n\}_n$ with $L_S(\partial Y_n) \leq a$, then S is hyperbolic.*

Theorem 3.4 also gives many non-trivial examples of hyperbolic non-exceptional Riemann surfaces.

DEFINITION 3.3. Let us define a family of Cantor sets inductively, as usual. Let us fix sequences $\{r_n\}$ and $\{R_n\}$ of positive numbers, and $\{N_n\}$ of natural numbers, with $r_n \geq c_1 R_n N_n$ and $r_{n-1} \geq c_2 r_n R_{n-1}$, for some positive constants c_1, c_2 . We consider first the interval $J_0 := [0, 1]$ in the 0-generation K_0 . If we have an interval J in the $(n-1)$ -generation, we choose N_n closed subintervals $J_1, \dots, J_{N_n}$ of J ; we denote by $I_1, \dots, I_{N_n-1}$ the open subintervals of J which are between the closed subintervals $J_1, \dots, J_{N_n}$ (i.e. $J_1 \cup \dots \cup J_{N_n} \cup I_1 \cup \dots \cup I_{N_n-1}$ is an interval). We assume that $|J_j|/|J| \leq R_n$ and $|I_j|/|J| \geq r_n$. The n -generation K_n is the defined by induction as the union of every subinterval J_j of any interval $J \in K_{n-1}$. We say that $K := \cap_n K_n$ is a $(\{r_n\}, \{R_n\}, \{N_n\})$ -Cantor set.

COROLLARY 3.2. *The complement in the Riemann sphere or in the complex plane of any $(\{r_n\}, \{R_n\}, \{N_n\})$ -Cantor set is δ -hyperbolic, where δ is a constant which only depends on c_1 and c_2 .* $\blacksquare$

PROOF. We can obtain a decomposition of the surface in Y -pieces, by dividing successively into two groups the intervals of each generation (we put in each group the same number of intervals ± 1). Conditions $r_n \geq c_1 R_n N_n$ and $r_{n-1} \geq c_2 r_n R_{n-1}$, guarantees that there are enough big “holes” (depending only on c_1 and c_2) between the intervals in K_n . Then, a standard argument gives that the simple closed geodesics have length bounded by a constant which only depends on c_1 and c_2 . The result is now a direct consequence of Theorem 3.4. $\square$

The same argument gives a similar result for families of two-dimensional Cantor sets.

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