Some discrete multiple orthogonal polynomials☆

J. Arvesú[a], J. Coussement[b,1] , W. Van Assche[b,*]

[a] *Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. de la Universidad 30, E-28911 Leganés, Madrid, Spain*

[b] *Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200B, B-3001 Leuven, Belgium*

Abstract

In this paper, we extend the theory of discrete orthogonal polynomials (on a linear lattice) to polynomials satisfying orthogonality conditions with respect to r positive discrete measures. First we recall the known results of the classical orthogonal polynomials of Charlier, Meixner, Kravchuk and Hahn (T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978; R. Koekoek and R.F. Swarttouw, Reports of the Faculty of Technical Mathematics and Informatics No. 98-17, Delft, 1998; A.F. Nikiforov et al., Classical Orthogonal Polynomials of a Discrete Variable, Springer, Berlin, 1991). These polynomials have a lowering and raising operator, which give rise to a Rodrigues formula, a second order difference equation, and an explicit expression from which the coefficients of the three-term recurrence relation can be obtained. Then we consider r positive discrete measures and define two types of multiple orthogonal polynomials. The continuous case (Jacobi, Laguerre, Hermite, etc.) was studied by Van Assche and Coussement (J. Comput. Appl. Math. 127 (2001) 317–347) and Aptekarev et al. (Multiple orthogonal polynomials for classical weights, manuscript). The families of multiple orthogonal polynomials (of type II) that we will study have a raising operator and hence a Rodrigues formula. This will give us an explicit formula for the polynomials. Finally, there also exists a recurrence relation of order $r+1$ for these multiple orthogonal polynomials of type II. We compute the coefficients of the recurrence relation explicitly when $r=2$.

Keywords: Multiple orthogonal polynomials; Discrete orthogonality; Charlier polynomials; Meixner polynomials; Kravchuk polynomials; Hahn polynomials

---

☆ This research was supported by INTAS project 00-272, Dirección General de Investigación del Ministerio de Ciencia y Tecnología of Spain under grants BFM-2000-0029 and BFM-2000-0206-C04-01, Dirección General de Investigación de la Comunidad Autónoma de Madrid, and by project G.0184.02 of FWO-Vlaanderen.

* Corresponding author.

[1] Research Assistant of FWO-Vlaanderen.

1. Classical discrete orthogonal polynomials

Orthogonal polynomials $\{p_n(x)\colon n=0,1,2,\ldots\}$ corresponding to a positive measure μ on the real line are such that p_n has degree n and satisfies the conditions

$$\int p_n(x)x^j\,\mathrm{d}\mu(x)=0, \quad j=0,1,\ldots,n-1.$$

This defines the polynomial up to a multiplicative factor. In the case of discrete orthogonal polynomials, we have a discrete measure μ (with finite moments)

$$\mu=\sum_{k=0}^{N}\rho_k\delta_{x_k}, \quad \rho_k>0,\ x_k\in\mathbb{R} \text{ and } N\in\mathbb{N}\cup\{+\infty\},$$

which is a linear combination of Dirac measures on the $N+1$ points $x_0,\ldots,x_N$. The orthogonality conditions of a discrete orthogonal polynomial p_n on the set $\{x_k=k\colon k=0,1,\ldots,N\}$ (linear lattice) are more conveniently written as

$$\sum_{k=0}^{N}p_n(k)(-k)_j\,\rho_k=0, \quad j=0,1,\ldots,n-1,$$

with $(a)_j=a(a+1)\ldots(a+j-1)$ if $j>0$ and $(a)_0=1$ (the Pochhammer symbol).

The *classical discrete orthogonal polynomials (on a linear lattice)* are those of Charlier, Meixner, Kravchuk and Hahn [4,6,7]. In this paper, we will always be considering monic polynomials, which is often different from the normalisation in the literature. In the case of the (monic) discrete orthogonal polynomials of Charlier we have

$$x_k=k \quad\text{and}\quad \rho_k=\frac{a^k}{k!}, \qquad k\in\mathbb{N},\ a>0,$$

which is a Poisson distribution on $\mathbb{N}=\{0,1,2,\ldots\}$. We denote these monic polynomials by $C_n(x;a)$. They satisfy the conditions

$$\sum_{k=0}^{+\infty}C_n(k;a)(-k)_j\frac{a^k}{k!}=0, \quad j=0,1,\ldots,n-1.$$

The monic discrete orthogonal polynomials of Meixner $M_n(x;\beta,c)$ (with $\beta>0$ and $0<c<1$) are those with

$$x_k=k \text{ and } \rho_k=\frac{(\beta)_k}{k!}\,c^k, \quad k\in\mathbb{N},$$

which is a negative binomial distribution (Pascal distribution) on $\mathbb{N}$. They have the orthogonality conditions

$$\sum_{k=0}^{+\infty}M_n(k;\beta,c)(-k)_j\frac{(\beta)_k}{k!}\,c^k=0, \quad j=0,1,\ldots,n-1.$$

The monic Kravchuk polynomials $K_n(x;p,N)$ (with $0<p<1$ and $N\in\mathbb{N}$) are the discrete orthogonal polynomials that satisfy the conditions

$$\sum_{k=0}^{N}K_n(k;p,N)(-k)_j\binom{N}{k}p^k(1-p)^{N-k}=0, \quad j=0,1,\ldots,n-1.$$

Here the weights ρ_k form a binomial distribution. Finally, the monic discrete orthogonal polynomials of Hahn $Q_n(x;\alpha,\beta,N)$ (with $\alpha,\beta > -1$ and $N \in \mathbb{N}$) have the orthogonality conditions

$$\sum_{k=0}^{N} Q_n(k;\alpha,\beta,N)(-k)_j \binom{\alpha+k}{k}\binom{\beta+N-k}{N-k} = 0, \quad j = 0,1,\ldots,n-1.$$

In this case we have a hypergeometric distribution.

If we define the weight function

$$w(x) = \begin{cases} \dfrac{a^x}{\Gamma(x+1)} & \text{Charlier,} \\[2ex] \dfrac{\Gamma(\beta+x)}{\Gamma(\beta)} \dfrac{c^x}{\Gamma(x+1)} & \text{Meixner,} \\[2ex] \dfrac{\Gamma(N+1)}{\Gamma(x+1)\Gamma(N-x+1)} p^x (1-p)^{N-x} & \text{Kravchuk,} \\[2ex] \dfrac{\Gamma(\alpha+x+1)}{\Gamma(x+1)\Gamma(\alpha+1)} \dfrac{\Gamma(\beta+N-x+1)}{\Gamma(N-x+1)\Gamma(\beta+1)} & \text{Hahn,} \end{cases}$$

then in each of the four examples this function satisfies a first order difference equation

$$\Delta(\sigma(x)w(x)) = \tau(x)w(x) \tag{1.1}$$

with σ a polynomial of degree $\leqslant 2$ and τ a polynomial of degree 1. This equation is known as *Pearson's equation*. Here we define $\Delta f(x) = f(x+1) - f(x)$ as the forward difference of f in x. The backward difference of f in x is defined as $\nabla f(x) = f(x) - f(x-1)$. The results are as follows:

	$C_n(x;a)$	$M_n(x;\beta,c)$	$K_n(x;p,N)$	$Q_n(x;\alpha,\beta,N)$
σ	x	x	$(1-p)x$	$x(\beta+N-x+1)$
τ	$a-x$	$(c-1)x+\beta c$	$Np-x$	$N(\alpha+1)-(\alpha+\beta+2)x$

When we take the forward difference of the classical discrete orthogonal polynomials, one can show that these polynomials are again orthogonal polynomials of the same family, but not monic and with a different set of parameters. As such, the Δ operator acts as a *lowering operator* on these families of polynomials. The explicit expressions are

$$\Delta C_n(x;a) = nC_{n-1}(x;a),$$

$$\Delta M_n(x;\beta,c) = nM_{n-1}(x;\beta+1,c),$$

$$\Delta K_n(x;p,N) = nK_{n-1}(x;p,N-1),$$

$$\Delta Q_n(x;\alpha,\beta,N) = nQ_{n-1}(x;\alpha+1,\beta+1,N-1).$$

One can easily prove these identities by applying summation by parts

$$\sum_{k=M}^{N} u(k)\Delta v(k) = u(N+1)v(N+1) - u(M)v(M) - \sum_{k=M}^{N} v(k+1)\Delta u(k) \tag{1.2}$$

on the orthogonality relations.

For each of the classical discrete orthogonal polynomials we also have *raising operators* which can also be found using summation by parts. The results are

$$\nabla\left(\frac{a^x}{\Gamma(x+1)}C_n(x;a)\right) = -\frac{a^{x-1}}{\Gamma(x+1)}C_{n+1}(x;a),$$

$$\nabla\left(\frac{(\beta)_x}{x!}c^x M_n(x;\beta,c)\right) = \frac{(\beta)_{x-1}c^{x-1}(c-1)}{x!}M_{n+1}(x;\beta-1,c),$$

$$\nabla\left(\binom{N}{x} p^x(1-p)^{N-x}K_n(x;p,N)\right) = -\frac{\binom{N+1}{x} p^x(1-p)^{N+1-x}}{p(1-p)(N+1)}K_{n+1}(x;p,N+1)$$

and

$$\nabla\left(\binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}Q_n(x;\alpha,\beta,N)\right)$$
$$= -\frac{n+\alpha+\beta}{\alpha\beta}\binom{\alpha+x-1}{x}\binom{\beta+N-x}{N-x+1}Q_{n+1}(x;\alpha-1,\beta-1,N+1).$$

When we use this raising operator several times, we get a *Rodrigues formula* for the polynomials. If we work out this formula, we get an explicit expression for the classical discrete orthogonal polynomials which we can link to a hypergeometric function. These expressions are

$$C_n(x;a) = (-a)^n {}_2F_0\left(\left.\begin{matrix}-n,-x\\ -\end{matrix}\,\right|\, -\frac{1}{a}\right),$$

$$M_n(x;\beta,c) = \frac{c^n}{(c-1)^n}(\beta)_n {}_2F_1\left(\left.\begin{matrix}-n,-x\\ \beta\end{matrix}\,\right|\, 1-\frac{1}{c}\right),$$

$$K_n(x;p,N) = p^n(-N)_n {}_2F_1\left(\left.\begin{matrix}-n,-x\\ -N\end{matrix}\,\right|\, \frac{1}{p}\right),$$

$$Q_n(x;\alpha,\beta,N) = \frac{(\alpha+1)_n(-N)_n}{(n+\alpha+\beta+1)_n} {}_3F_2\left(\left.\begin{matrix}-n,-x,n+\alpha+\beta+1\\ -N,\alpha+1\end{matrix}\,\right|\, 1\right).$$

Every system of monic orthogonal polynomials p_n satisfies a recurrence relation of the form

$$x p_n(x) = p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad n \geqslant 0,$$

with $a_n > 0$ or, when the orthogonality is on $\mathbb{R}^+$,

$$x p_n(x) = p_{n+1}(x) + (A_n + C_n) p_n(x) + A_{n-1} C_n p_{n-1}(x), \quad n \geqslant 0,$$

with $A_n, C_n > 0$ and initial conditions $p_0 = 1$ and $p_{-1} = 0$. From the explicit expression of the polynomials, one can compute these recurrence coefficients by comparing coefficients in the recurrence relation, and for discrete orthogonal polynomials on a linear lattice it is most convenient to use the basis $\{(-x)_j\colon\ j = 0, 1, 2, \ldots\}$. For the classical monic discrete orthogonal polynomials we have the following results:

$C_n(x;a)$	$M_n(x;\beta,c)$	$K_n(x;p,N)$	$Q_n(x;\alpha,\beta,N)$
$a_n = an$	$a_n = \frac{cn(\beta+n-1)}{(1-c)^2}$	$a_n = n(1-p)p(N-n+1)$	$A_n = \frac{(\alpha+n+1)(N-n)(n+\alpha+\beta+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}$
$b_n = a + n$	$b_n = \frac{n+(\beta+n)c}{1-c}$	$b_n = p(N-n) + n(1-p)$	$C_n = \frac{n(\beta+n)(N+\alpha+\beta+n+1)}{(\alpha+\beta+2n)(\alpha+\beta+2n+1)}$

Finally, we can prove (by combining the lowering and raising operator) that the classical discrete orthogonal polynomials satisfy a second order difference equation of the form

$$\sigma(x)\nabla\Delta y(x) + \tau(x)\Delta y(x) + \lambda_n y(x) = 0, \tag{1.3}$$

with σ and τ the same as in Pearson's equation and λ_n a constant depending on n. For the Charlier polynomials we have $\lambda_n = n$, for the Meixner polynomials $\lambda_n = n(1-c)$, for the Kravchuk polynomials $\lambda_n = n$ and for the Hahn polynomials $\lambda_n = n(n+\alpha+\beta+1)$.

Note that the Meixner, Kravchuk and Charlier polynomials are limiting cases of the Hahn polynomials. Indeed we have that

$$M_n(x;\beta,c) = \lim_{N\to+\infty} Q_n\left(x; \beta-1, \left(\frac{1-c}{c}\right)N, N\right),$$

$$K_n(x;p,N) = \lim_{t\to+\infty} Q_n(x; pt, (1-p)t, N),$$

$$C_n(x;a) = \lim_{\beta\to+\infty} M_n\left(x; \beta, \frac{a}{\beta}\right).$$

2. Discrete multiple orthogonal polynomials

2.1. *Definitions*

Suppose we have r positive measures $\mu_1, \ldots, \mu_r$ on $\mathbb{R}$. For each of them we define the support as

$$\operatorname{supp}(\mu_j) = \{x \in \mathbb{R} \mid \forall \varepsilon > 0,\ \mu_j((x-\varepsilon, x+\varepsilon)) > 0\}.$$

These are closed subsets of $\mathbb{R}$. We also define

$$\Delta_j = \text{the smallest interval that contains } \operatorname{supp}(\mu_j).$$

Finally, we introduce a multi-index $\vec{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$ and its length $|\vec{n}| = n_1 + n_2 + \cdots + n_r$. Multiple orthogonal polynomials can now be defined as follows [8, Chapter 4.3], [2,10]:

Definition 2.1 (Type I). An r-vector of type I multiple orthogonal polynomials $(A_{\vec{n},1}, \ldots, A_{\vec{n},r})$, corresponding to the multi-index $\vec{n} \in \mathbb{N}^r$, is such that each $A_{\vec{n},j}$ is a polynomial of degree $\leqslant n_j - 1$ and the following orthogonality conditions hold:

$$\int x^k \sum_{j=1}^{r} A_{\vec{n},j}(x)\, \mathrm{d}\mu_j(x) = 0, \qquad k = 0, 1, \ldots, |\vec{n}| - 2. \tag{2.1}$$

Each $A_{\vec{n},j}$ has n_j coefficients, so we have a linear system of $|\vec{n}| - 1$ homogeneous relations and $|\vec{n}|$ unknowns. This system has a unique solution up to a multiplicative factor if and only if the matrix of the system has rank $|\vec{n}| - 1$ (which means that the measures must satisfy some conditions). In that case we can determine the type I vector uniquely and we call the multi-index $\vec{n}$ a *normal index* for type I.

Definition 2.2 (Type II). A type II multiple orthogonal polynomial $P_{\vec{n}}$, corresponding to the multi-index $\vec{n} \in \mathbb{N}^r$, is a polynomial of degree $\leqslant |\vec{n}|$ which satisfies the orthogonality conditions

$$\begin{aligned} \int_{\Delta_1} P_{\vec{n}}(x) x^k \, \mathrm{d}\mu_1(x) &= 0, \qquad k = 0, 1, \ldots, n_1 - 1, \\ &\vdots \\ \int_{\Delta_r} P_{\vec{n}}(x) x^k \, \mathrm{d}\mu_r(x) &= 0, \qquad k = 0, 1, \ldots, n_r - 1. \end{aligned} \tag{2.2}$$

Here we have a linear system of $|\vec{n}|$ homogeneous relations for the $|\vec{n}| + 1$ unknown coefficients of $P_{\vec{n}}$. Again we would like to have that the solution $P_{\vec{n}}$ is unique up to a multiplicative factor and also that this polynomial has exactly degree $|\vec{n}|$ (then the monic multiple orthogonal polynomial exists and will be unique). When we have this, we call $\vec{n}$ a *normal index* for type II. Let

$$M = \begin{pmatrix} M_1(n_1) \\ \vdots \\ M_r(n_r) \end{pmatrix} \tag{2.3}$$

be the matrix of system (2.2), with

$$M_j(n_j) = \begin{pmatrix} m_0^{(j)} & m_1^{(j)} & \ldots & m_{|\vec{n}|}^{(j)} \\ \vdots & \vdots & & \vdots \\ m_{n_j-1}^{(j)} & m_{n_j}^{(j)} & \ldots & m_{|\vec{n}|+n_j-1}^{(j)} \end{pmatrix}$$

an $n_j \times (|\vec{n}|+1)$ matrix of moments of the measure μ_j. Then the index $\vec{n}=(n_1,\dots,n_r)$ is normal for type II if and only if the matrix B, which you get by omitting the last column in M, has rank $|\vec{n}|$. We can also observe that, when the index $\vec{n}=(n_1,\dots,n_r)$ is normal, the corresponding unique monic multiple orthogonal polynomial of type II is of the form

$$P_{\vec{n}}(x)=\frac{1}{\det B}\det\begin{pmatrix} m_0^{(1)} & m_1^{(1)} & \dots & m_{|\vec{n}|}^{(1)} \\ \vdots & \vdots & & \vdots \\ m_{n_1-1}^{(1)} & m_{n_1}^{(1)} & \dots & m_{|\vec{n}|+n_1-1}^{(1)} \\ \vdots & \vdots & & \vdots \\ m_0^{(r)} & m_1^{(r)} & \dots & m_{|\vec{n}|}^{(r)} \\ \vdots & \vdots & & \vdots \\ m_{n_r-1}^{(r)} & m_{n_r}^{(r)} & \dots & m_{|\vec{n}|+n_r-1}^{(r)} \\ 1 & x & \dots & x^{|\vec{n}|} \end{pmatrix}.$$

In this paper we will only study the type II multiple orthogonal polynomials. We also suppose that we have r positive discrete measures on $\mathbb{R}$:

$$\mu_j=\sum_{k=0}^{N_j}\rho_{j,k}\delta_{x_{j,k}},\qquad \rho_{j,k}>0,\ x_{j,k}\in\mathbb{R},\ N_j\in\mathbb{N}\cup\{+\infty\},\ j=1,\dots,r,$$

with all the $x_{j,k}$, $k=0,\dots,N_j$, different and this for each j. In this case we have that $\operatorname{supp}(\mu_j)$ is the closure of $\{x_{j,k}\}_{k=0}^{N_j}$ and that Δ_j is the smallest closed interval on $\mathbb{R}$ which contains $\{x_{j,k}\}_{k=0}^{N_j}$. The corresponding polynomials are then discrete multiple orthogonal polynomials. The discrete measures in this paper will be supported on $\mathbb{N}$ or a subset, which is achieved by taking $x_{j,k}=k$ for $j=1,\dots,r$. The orthogonality conditions are then more conveniently expressed in terms of the polynomials $(-x)_i$.

Definition 2.3 (Discrete Type II). A discrete multiple orthogonal polynomial of type II on the linear lattice, corresponding to the multi-index $\vec{n}\in\mathbb{N}^r$, is a polynomial $P_{\vec{n}}$ of degree $\leqslant|\vec{n}|$ that satisfies the orthogonality conditions

$$\begin{aligned} &\sum_{k=0}^{N_1}P_{\vec{n}}(k)(-k)_i\,\rho_{1,k}=0, \qquad i=0,1,\dots,n_1-1,\\ &\qquad\vdots\\ &\sum_{k=0}^{N_r}P_{\vec{n}}(k)(-k)_i\,\rho_{r,k}=0, \qquad i=0,1,\dots,n_r-1. \end{aligned} \tag{2.4}$$

We now introduce a systems of measures for which every multi-index is normal, namely an *AT system* [8, p. 140]. Note that we have to tone down this a little bit. When N_j is finite and $n_j - N_j - 1 = \ell$ is greater than zero, then every polynomial of the form

$$P_{\vec{n}}(x) = (x - x_{j,0}) \dots (x - x_{j,N_j})(x - a_1) \dots (x - a_\ell) R_{|\vec{n}|-n_j}(x)$$

with $R_{|\vec{n}|-n_j}$ a polynomial of degree $\leqslant |\vec{n}| - n_j$, satisfies the conditions in (2.4) corresponding to the measure μ_j. Now it is easy to see that for every $a_1, \dots, a_\ell$ we can find a $R_{|\vec{n}|-n_j}$ so that $P_{\vec{n}}$ also satisfies the other conditions in (2.4). So only in the cases that $n_j \leqslant N_j + 1$, $j = 1, \dots, r$, we can have a unique solution.

2.2. *AT systems*

Definition 2.4. An AT system of r positive discrete measures is a system where the measures are of the form

$$\mu_j = \sum_{k=0}^{N} \rho_{j,k} \delta_{x_k}, \quad \rho_{j,k} > 0,\ x_k \in \mathbb{R},\ N \in \mathbb{N} \cup \{+\infty\},\ j = 1, \dots, r,$$

so that $\operatorname{supp}(\mu_j)$ is the closure of $\{x_k\}_{k=0}^{N}$ and $\Delta_j = \Delta$ for each $j = 1, \dots, r$. We also assume that there exist r continuous functions $w_1, \dots, w_r$ on Δ with $w_j(x_k) = \rho_{j,k}$, $k = 1, \dots, N$, $j = 1, \dots, r$, such that the $|\vec{n}|$ functions

$$\begin{gathered} w_1(x), xw_1(x), \dots, x^{n_1-1}w_1(x), \\ w_2(x), xw_2(x), \dots, x^{n_2-1}w_2(x), \\ \vdots \\ w_r(x), xw_r(x), \dots, x^{n_r-1}w_r(x) \end{gathered}$$

form a Chebyshev system on Δ for each multi-index $\vec{n}$ with $|\vec{n}| < N + 1$. This means that every linear combination (except the one with each coefficient equal to 0)

$$\sum_{j=1}^{r} Q_{n_j-1}(x) w_j(x)$$

with Q_{n_j-1} a polynomial of degree at most $n_j - 1$, has at most $|\vec{n}| - 1$ zeros on Δ.

For such a system of measures the following theorem holds.

Theorem 2.1. *Suppose we have an AT system of r positive discrete measures. Then every discrete multiple orthogonal polynomial $P_{\vec{n}}$ of type* II, *corresponding to the multi-index $\vec{n}$ with $|\vec{n}| < N+1$, has exactly $|\vec{n}|$ different zeros on Δ.*

Proof. Suppose $P_{\vec{n}}$ has $m < |\vec{n}|$ sign changes on Δ at the points $y_1, \ldots, y_m$. Consider a multi-index $\vec{m} = (m_1, \ldots, m_r)$ so that $|\vec{m}| = m$, $m_i \leqslant n_i$ for each $i = 1, \ldots, r$ and $m_j < n_j$ for some j. Then construct the function

$$Q(x) = \sum_{i=1}^{r} Q_i(x) w_i(x),$$

with Q_i a polynomial of degree $m_i - 1$ whenever $i \neq j$ and Q_j a polynomial of degree m_j, satisfying the interpolation conditions

$$Q(y_k) = 0, \quad k = 1, \ldots, m, \qquad Q(y_0) = 1,$$

with $y_0 \notin \{y_1, \ldots, y_m\}$ a point in Δ. Here Q is a linear combination of a Chebyshev system of order $m+1$ on Δ, and this interpolation problem has a unique solution. We also know that Q has at most m zeros on Δ since $Q \not\equiv 0$ (because $Q(y_0) = 1$). Hence Q has exactly m different zeros on Δ, namely $y_1, \ldots, y_m$, and Q changes sign at these points. Now we have that $P_{\vec{n}} Q$ does not change sign on Δ and we also know that $P_{\vec{n}} Q$ does not vanish at all the points $\{x_k\}_{k=0}^N$ because of $|\vec{n}| < N + 1$. So we have that

$$\sum_{k=0}^{N} P_{\vec{n}}(x_k) Q(x_k) = \sum_{i=1}^{r} \sum_{k=0}^{N} P_{\vec{n}}(x_k) Q_i(x_k) \rho_{i,k} \neq 0,$$

which is in contradiction with the orthogonality relations of $P_{\vec{n}}$ (Q_i is for each i a polynomial of degree less than n_j). Hence $P_{\vec{n}}$ has at least $|\vec{n}|$ sign changes and so at least $|\vec{n}|$ different zeros on Δ. We know that $P_{\vec{n}}$ is a polynomial of degree at most $|\vec{n}|$, so $P_{\vec{n}}$ has exactly $|\vec{n}|$ different zeros on Δ. $\square$

Now we have in an AT system that every discrete multiple orthogonal polynomial $P_{\vec{n}}$ of type II, corresponding to the multi-index $\vec{n}$, with $|\vec{n}| < N + 1$, has exactly degree $|\vec{n}|$. Let $P_{\vec{n},1}$ and $P_{\vec{n},2}$ be two linearly independent discrete multiple orthogonal polynomials, corresponding to the multi-index $\vec{n}$. Then we can find a linear combination of these polynomials which is of degree $< |\vec{n}|$, different from zero. This polynomial also satisfies the linear system (2.4) and so this is in contradiction with the preceding theorem. Hence in an AT system every multi-index $\vec{n}$, with $|\vec{n}| < N + 1$, is normal. Note that only the multi-indices with $|\vec{n}| \leqslant N + 1$ can be normal. If $|\vec{n}| - N - 1 = \ell > 0$, then the polynomial

$$(x - x_0) \ldots (x - x_N)(x - a_1) \ldots (x - a_\ell)$$

satisfies the conditions of a multiple orthogonal polynomial, corresponding to the multi-index $\vec{n}$, and this for all $a_1, \ldots, a_\ell$. So we do not have a unique solution (and the multi-index is not normal). If $|\vec{n}| = N + 1$ then $P_{\vec{n}} = (x - x_0) \ldots (x - x_N)$ is certainly a solution. Then, if the multi-index $\vec{n}$ is normal, we are dealing with a function that is zero at all the points $\{x_k\}_{k=0}^N$, so we will drop this case.

Finally, we give two examples of Chebyshev systems which appear in our examples. The first example can be found in [8, p. 138, Example 3].

Example 2.1. The functions

$$
\begin{gathered}
v(x)c_1^x, v(x)xc_1^x, \dots, v(x)x^{n_1-1}c_1^x, \\
\vdots \\
v(x)c_r^x, v(x)xc_r^x, \dots, v(x)x^{n_r-1}c_r^x,
\end{gathered}
$$

with all the $c_i > 0$, $i = 1, \dots, r$, different and v a continuous function which has no zeros on $\mathbb{R}^+$, form a Chebyshev system on $\mathbb{R}^+$ for every $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$.

Example 2.2. The functions

$$
\begin{gathered}
v(x)\Gamma(x+\beta_1), v(x)x\Gamma(x+\beta_1), \dots, v(x)x^{n_1-1}\Gamma(x+\beta_1), \\
\vdots \\
v(x)\Gamma(x+\beta_r), v(x)x\Gamma(x+\beta_r), \dots, v(x)x^{n_r-1}\Gamma(x+\beta_r),
\end{gathered}
\tag{2.5}
$$

with $\beta_i > 0$ and $\beta_i - \beta_j \notin \mathbb{Z}$ whenever $i \neq j$ and v a continuous function with no zeros on $\mathbb{R}^+$, form a Chebyshev system on $\mathbb{R}^+$ for every $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$. If $\beta_i - \beta_j \notin \{0, 1, \dots, N-1\}$ whenever $i \neq j$, then this still gives a Chebyshev system for every $\vec{n} = (n_1, \dots, n_r)$ for which $n_i < N+1$, $i = 1, 2, \dots, r$.

We refer to the appendix for a proof. Note that the indices $\vec{n}$ with $|\vec{n}| \leqslant N$ are such that $n_i < N+1$ for $i = 1, \dots, r$.

3. Recurrence relation

Suppose that all the multi-indices are normal for the r measures $\mu_1, \dots, \mu_r$. Then there exists an interesting recurrence relation of order $r+1$ for the monic multiple orthogonal polynomials of type II with nearly diagonal multi-indices [10]. Here the nearly diagonal multi-index, corresponding to n, is given by

$$
\vec{s}(n) = \left(\underbrace{k+1, k+1, \dots, k+1}_{s \text{ times}}, \underbrace{k, k, \dots, k}_{r-s \text{ times}} \right),
$$

with $n = kr + s$, $0 \leqslant s < r$. If we write $P_n(x) = P_{\vec{s}(n)}(x)$, then the following recurrence relation holds:

$$
xP_n(x) = P_{n+1}(x) + \sum_{j=0}^{r} a_{n,j} P_{n-j}(x),
\tag{3.1}
$$

with initial conditions $P_0 = 1$ and $P_j = 0$, $j = -1, -2, \ldots, -r$. Each zero of the polynomial P_{n+1} is an eigenvalue of the matrix

$$A_{n+1} = \begin{pmatrix} a_{0,0} & 1 & 0 & \ldots & & & & & \ldots & 0 \\ a_{1,1} & a_{1,0} & 1 & 0 & \ldots & & & & \ldots & 0 \\ a_{2,2} & a_{2,1} & a_{2,0} & 1 & 0 & \ldots & & & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & & & & \vdots \\ a_{r,r} & a_{r,r-1} & \ldots & & a_{r,0} & 1 & 0 & \ldots & \ldots & 0 \\ 0 & a_{r+1,r} & \ddots & & & a_{r+1,0} & 1 & 0 & \ldots & 0 \\ \vdots & 0 & \ddots & \ddots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & & & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & & \ddots & 1 \\ 0 & 0 & \ldots & \ldots & 0 & a_{n,r} & a_{n,r-1} & \ldots & a_{n,1} & a_{n,0} \end{pmatrix}.$$

By expanding the determinant $\det(A_{n+1} - xI_{n+1})$ along the last column (do this r times), one can show that $(-1)^n \det(A_n - xI_n)$ satisfies the same recurrence relation as the P_n. So also the converse holds: each eigenvalue of A_{n+1} is a zero of P_{n+1}. Observe that when all the multi-indices $\vec{n}$ with $|\vec{n}| \leqslant N$ are normal, recursion (3.1) still holds for $n < N$.

A more general form of this recurrence relation is given by

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_1}(x) + b_{\vec{n},0}P_{\vec{n}}(x) + \sum_{j=1}^{r} b_{\vec{n},j}P_{\vec{n}-\vec{v}_j}(x), \tag{3.2}$$

where $\vec{e}_i$ is the ith standard unit vector in $\mathbb{R}^r$ and $\vec{v}_j = \sum_{k=0}^{j-1} \vec{e}_{r-k}$. In the case $r = 2$ the recurrence relation for the polynomials with nearly diagonal multi-indices (3.1) gives the relations $(n+1, n) \to (n, n) \to (n, n-1) \to (n-1, n-1)$ and $(n+1, n+1) \to (n+1, n) \to (n, n) \to (n, n-1)$. The first relation follows from the general case $(n_1 + 1, n_2) \to (n_1, n_2) \to (n_1, n_2 - 1) \to (n_1 - 1, n_2 - 1)$ by setting $n_1 = n_2 = n$. To obtain the second one we set $n_1 = n$ and $n_2 = n + 1$ and interchange the measures μ_1 and μ_2.

4. Some examples of discrete multiple orthogonal polynomials

4.1. Multiple Charlier polynomials

The discrete *multiple orthogonal polynomials of Charlier* are associated with the r discrete measures

$$\mu_i = \sum_{k=0}^{+\infty} \frac{a_i^k}{k!}\,\delta_k, \quad a_i > 0,\ i = 1, \ldots, r,$$

where all the a_i are different. For each measure the weights form a Poisson distribution on $\mathbb{N}$. So we have that $\operatorname{supp}(\mu_1) = \cdots = \operatorname{supp}(\mu_r) = \mathbb{N}$ and that $\Delta_1 = \cdots = \Delta_r = \mathbb{R}^+$. These r measures form an AT system. This follows from the fact that we can define r functions

$$w_i(x) = \frac{a_i^x}{\Gamma(x+1)}, \quad x \in \mathbb{R}^+, \; i = 1, \ldots, r,$$

so that $w_i(k) = a_i^k/k!$, $k \in \mathbb{N}$, $i = 1, \ldots, r$, and that

$$\begin{gathered} w_i(x), xw_i(x), \ldots, x^{n_1-1}w_i(x), \\ \vdots \\ w_r(x), xw_r(x), \ldots, x^{n_2-1}w_r(x) \end{gathered}$$

is a Chebyshev system on $\mathbb{R}^+$ for every multi-index $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$. This easily follows from Example 2.1. So every multi-index is normal.

The monic discrete multiple orthogonal polynomial of Charlier $C_{\vec{n}}^{\vec{a}}$, corresponding to the multi-index $\vec{n}=(n_1, \ldots, n_r)$ and the set of parameters $\vec{a}=(a_1, \ldots, a_r)$, is the monic polynomial of degree $|\vec{n}|$ which satisfies the orthogonality conditions

$$\begin{gathered} \sum_{k=0}^{+\infty} C_{\vec{n}}^{\vec{a}}(k)(-k)_j \frac{a_1^k}{k!} = 0, \quad j = 0, \ldots, n_1 - 1, \\ \vdots \\ \sum_{k=0}^{+\infty} C_{\vec{n}}^{\vec{a}}(k)(-k)_j \frac{a_r^k}{k!} = 0, \quad j = 0, \ldots, n_r - 1. \end{gathered}$$

The Gamma function, defined for $x > 0$ by

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} \mathrm{e}^{-t} \, \mathrm{d}t$$

can be continued analytically to $\mathbb{C} \setminus \{0, -1, -2, -3, \ldots\}$ by the property $\Gamma(z+1) = z\Gamma(z)$. The points $0, -1, -2, \ldots$ are simple poles and the Gamma function is never zero [1, Chapter 6]. Using this, we can extend each function w_i to a C^∞-function by

$$w_i(x) = \begin{cases} \dfrac{a_i^x}{\Gamma(x+1)} & x \in \mathbb{R} \setminus \{-1, -2, -3, \ldots\}, \\ 0 & x \in \{-1, -2, -3, \ldots\}. \end{cases}$$

Summation by parts easily shows that

$$\frac{a_i}{w_i(x)} \nabla(w_i(x) C_{\vec{n}}^{\vec{a}}(x)) = -C_{\vec{n}+\vec{e}_i}^{\vec{a}}(x), \quad i = 1, \ldots, r. \tag{4.1}$$

For the proof we check that the left-hand side of the equation satisfies the orthogonality conditions of the right-hand side (here we use the fact that $w_i(-1) = 0$). Observe that on the left-hand side the zeros of the denominator are cancelled by the zeros of the numerator. We call $w_i^{-1}(x)\nabla w_i(x)$ a *raising operator* because the ith component of the multi-index is increased by 1. Repeatedly using the raising operators gives us the *Rodrigues formula* for the polynomials $C_{\vec{n}}^{\vec{a}}$, namely

$$C_{\vec{n}}^{\vec{a}}(x) = \left[\prod_{j=1}^{r}(-a_j)^{n_j}\right] \Gamma(x+1) \left[\prod_{i=1}^{r}\left(\frac{1}{a_i^x}\nabla^{n_i} a_i^x\right)\right] \left(\frac{1}{\Gamma(x+1)}\right). \tag{4.2}$$

Here the product of the difference operators $a_i^{-x}\nabla^{n_i}a_i^x$ can be taken in any order because these operators commute. For the backward difference operator ∇ we have the property

$$\nabla^n f(x) = \sum_{k=0}^{n}\binom{n}{k}(-1)^k f(x-k). \tag{4.3}$$

Combining the Rodrigues formula and (4.3) we can obtain an explicit expression for the polynomials $C_{\vec{n},\vec{a}}$. For the case $r=2$ we get

$$\begin{aligned}
C_{n_1,n_2}^{a_1,a_2}(x) &= (-a_1)^{n_1}(-a_2)^{n_2}\Gamma(x+1)\frac{1}{a_1^x}\,\nabla^{n_1}a_1^x\left(\frac{1}{a_2^x}\,\nabla^{n_2}\frac{a_2^x}{\Gamma(x+1)}\right)\\
&= (-a_1)^{n_1}(-a_2)^{n_2}\Gamma(x+1)\frac{1}{a_1^x}\,\nabla^{n_1}a_1^x\left(\frac{1}{a_2^x}\sum_{l=0}^{n_2}\binom{n_2}{l}(-1)^l\frac{a_2^{x-l}}{\Gamma(x-l+1)}\right)\\
&= (-a_1)^{n_1}(-a_2)^{n_2}\Gamma(x+1)\sum_{l=0}^{n_2}\binom{n_2}{l}(-1)^l a_2^{-l}\frac{1}{a_1^x}\,\nabla^{n_1}\frac{a_1^x}{\Gamma(x-l+1)}\\
&= (-a_1)^{n_1}(-a_2)^{n_2}\Gamma(x+1)\sum_{l=0}^{n_2}\binom{n_2}{l}(-1)^l a_2^{-l}\sum_{k=0}^{n_1}\binom{n_1}{k}(-1)^k\frac{a_1^{-k}}{\Gamma(x-k-l+1)}.
\end{aligned}$$

When we work this out, we find

$$\begin{aligned}
C_{n_1,n_2}^{a_1,a_2}(x) &= (-a_1)^{n_1}(-a_2)^{n_2}\sum_{k=0}^{n_1}\sum_{l=0}^{n_2}\frac{(-n_1)_k a_1^{-k}}{k!}\,\frac{(-n_2)_l a_2^{-l}}{l!}\,\frac{\Gamma(x+1)}{\Gamma(x-k-l+1)}\\
&= (-a_1)^{n_1}(-a_2)^{n_2}\sum_{k=0}^{n_1}\sum_{l=0}^{n_2}(-n_1)_k(-n_2)_l(-x)_{k+l}\frac{(-1/a_1)^k}{k!}\frac{(-1/a_2)^l}{l!}\\
&= (-a_1)^{n_1}(-a_2)^{n_2}\lim_{\gamma\to+\infty}F_2\left(-x,-n_1,-n_2;\gamma,\gamma;-\frac{\gamma}{a_1},-\frac{\gamma}{a_2}\right),
\end{aligned}$$

where

$$F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m!n!}\,x^m y^n$$

is the second of Appell's hypergeometric functions of two variables [5]. Finally, we use the substitution $l \to j - k$ to find the expression

$$
\begin{aligned}
C_{n_1,n_2}^{a_1,a_2}(x) &= (-a_1)^{n_1}(-a_2)^{n_2}\sum_{k=0}^{n_1}\sum_{j=k}^{k+n_2}(-n_1)_k(-n_2)_{j-k}(-x)_j\frac{(-1/a_1)^k}{k!}\frac{(-1/a_2)^{j-k}}{(j-k)!}\\
&= (-a_1)^{n_1}(-a_2)^{n_2}\sum_{j=0}^{n_1+n_2}\sum_{k=\max(0,j-n_2)}^{\min(j,n_1)}(-n_1)_k(-n_2)_{j-k}\frac{(-1/a_1)^k}{k!}\frac{(-1/a_2)^{j-k}}{(j-k)!}(-x)_j\\
&= (-a_1)^{n_1}(-a_2)^{n_2}\sum_{j=0}^{n_1+n_2}\sum_{k=0}^{j}(-n_1)_k(-n_2)_{j-k}\frac{(-1/a_1)^k}{k!}\frac{(-1/a_2)^{j-k}}{(j-k)!}(-x)_j.
\end{aligned}
$$

If we use the basis $\{(-x)_j\}_{j=0}^{+\infty}$ and write

$$
C_{n_1,n_2}^{a_1,a_2}(x) = \sum_{j=0}^{n_1+n_2} c_{n_1,n_2}^{(j)}(-x)_j,
$$

then the coefficients $c_{n_1,n_2}^{(j)}$ can be used to compute the coefficients in the recurrence relation

$$
xP_{n_1,n_2}(x) = P_{n_1+1,n_2}(x) + b_{n_1,n_2}P_{n_1,n_2}(x) + c_{n_1,n_2}P_{n_1,n_2-1}(x) + d_{n_1,n_2}P_{n_1-1,n_2-1}(x),
$$

where $P_{n_1,n_2}(x) = C_{n_1,n_2}^{a_1,a_2}(x)$. Indeed, by comparing coefficients we have

$$
\begin{aligned}
b_{n_1,n_2} &= n_1 + n_2 - (-1)^{n_1+n_2}(c_{n_1,n_2}^{(n_1+n_2-1)} + c_{n_1+1,n_2}^{(n_1+n_2)}),\\
c_{n_1,n_2} &= (-1)^{n_1+n_2}(c_{n_1,n_2}^{(n_1+n_2-2)} + c_{n_1+1,n_2}^{(n_1+n_2-1)} + (b_{n_1,n_2} - n_1 - n_2 + 1)c_{n_1,n_2}^{(n_1+n_2-1)}),\\
d_{n_1,n_2} &= (-1)^{n_1+n_2-1}(c_{n_1,n_2}^{(n_1+n_2-3)} + c_{n_1+1,n_2}^{(n_1+n_2-2)}\\
&\quad + (b_{n_1,n_2} - n_1 - n_2 + 2)c_{n_1,n_2}^{(n_1+n_2-2)} + c_{n_1,n_2}c_{n_1,n_2-1}^{(n_1+n_2-2)}).
\end{aligned}
\tag{4.4}
$$

From the explicit expression of $C_{n_1,n_2}^{a_1,a_2}$ we then get, after some calculations, that

$$
\begin{aligned}
b_{n_1,n_2} &= a_1 + n_1 + n_2,\\
c_{n_1,n_2} &= n_1 a_1 + n_2 a_2,\\
d_{n_1,n_2} &= a_1 n_1 (a_1 - a_2).
\end{aligned}
\tag{4.5}
$$

4.2. Multiple Meixner polynomials (first kind)

Here we consider r measures $\mu_1, \ldots, \mu_r$ which in each case form a negative binomial distribution. For the *multiple orthogonal polynomials of Meixner I* we take the same parameter $\beta > 0$, but different values for the parameter $0 < c < 1$. So we have that

$$\mu_i = \sum_{k=0}^{+\infty} \frac{(\beta)_k c_i^k}{k!}\, \delta_k, \quad 0 < c_i < 1,\ i = 1, \ldots, r,$$

with all the c_i different. Just as in the previous example, the support of these measures is $\mathbb{N}$ and we have $\Delta_1 = \cdots = \Delta_r = \mathbb{R}^+$. If $\beta \notin \mathbb{N}$, we define the functions w_i^β, $i = 1, \ldots, r$, as

$$w_i^\beta(x) = \begin{cases} \dfrac{\Gamma(\beta + x)}{\Gamma(\beta)} \dfrac{c_i^x}{\Gamma(x+1)} & \text{if } x \in \mathbb{R} \setminus (\{-1, -2, -3, \ldots\} \cup \{-\beta, -\beta - 1, -\beta - 2, \ldots\}), \\ 0 & \text{if } x \in \{-1, -2, -3, \ldots\}. \end{cases}$$

Then these are functions in $C^\infty(\mathbb{R} \setminus \{-\beta, -\beta - 1, -\beta - 2, \ldots\})$ with simple poles in $-\beta, -\beta - 1, -\beta - 2, \ldots$. If $\beta \in \mathbb{N}$ we define

$$w_i^\beta(x) = \frac{(x+1)_{\beta-1}}{(\beta-1)!}\, c_i^x$$

which are functions in C^∞. With this definition these functions satisfy $w_i^\beta(k) = (\beta)_k c_i^k / k!$, $k \in \mathbb{N}$, $i = 1, \ldots, r$. Then by Example 2.1 we know that the measures $\mu_1, \ldots, \mu_r$ form an AT system which gives us that every multi-index $\vec{n} = (n_1, \ldots, n_r)$ is normal for these measures.

The monic discrete multiple orthogonal polynomials of Meixner I, corresponding to the multi-index $\vec{n} = (n_1, \ldots, n_r)$ and the parameters β, $\vec{c} = (c_1, \ldots, c_r)$, is the unique monic polynomial $M_{\vec{n}}^{\beta;\vec{c}}$ of degree $|\vec{n}|$ which satisfies the orthogonality conditions

$$\sum_{k=0}^{+\infty} M_{\vec{n}}^{\beta;\vec{c}}(k)(-k)_j w_1^\beta(k) = 0, \quad j = 0, \ldots, n_1 - 1,$$

$$\vdots$$

$$\sum_{k=0}^{+\infty} M_{\vec{n}}^{\beta;\vec{c}}(k)(-k)_j w_r^\beta(k) = 0, \quad j = 0, \ldots, n_r - 1.$$

Using summation by parts we can again show that for these polynomials, we have the following raising operations:

$$\frac{1}{w_i^{\beta-1}(x)} \nabla\left(w_i^\beta(x) M_{\vec{n}}^{\beta;\vec{c}}(x)\right) = \frac{c_i - 1}{c_i(\beta - 1)} M_{\vec{n}+\vec{e}_i}^{\beta-1;\vec{c}}(x), \quad i = 1, \ldots, r. \tag{4.6}$$

Observe that on the left-hand side of the equation the zeros of the denominator are cancelled by the zeros of the numerator and the poles of the numerator by the poles of the denominator. A repeated application of these operators gives the *Rodrigues formula*, namely

$$M_{\vec{n}}^{\beta;\vec{c}}(x) = (\beta)_{|\vec{n}|} \left[\prod_{k=1}^{r} \left(\frac{c_k}{c_k - 1}\right)^{n_k}\right] \frac{\Gamma(\beta)\Gamma(x+1)}{\Gamma(\beta + x)} \times \left[\prod_{i=1}^{r} \left(\frac{1}{c_i^x} \nabla^{n_i} c_i^x\right)\right] \left(\frac{\Gamma(\beta + |\vec{n}| + x)}{\Gamma(\beta + |\vec{n}|)\Gamma(x+1)}\right). \tag{4.7}$$

To find an explicit expression of these polynomials in the case $r = 2$ we again use property (4.3) and after some calculations we have

$$M_{n_1,n_2}^{\beta;c_1,c_2}(x) = \left(\frac{c_1}{c_1 - 1}\right)^{n_1} \left(\frac{c_2}{c_2 - 1}\right)^{n_2} (\beta)_{n_1+n_2}\ F_1\left(-x, -n_1, -n_2; \beta; 1 - \frac{1}{c_1}, 1 - \frac{1}{c_2}\right)$$

$$= \frac{c_1^{n_1} c_2^{n_2} (\beta)_{n_1+n_2}}{(c_1 - 1)^{n_1} (c_2 - 1)^{n_2}} \sum_{j=0}^{n_1+n_2} \sum_{k=0}^{j} \frac{(-n_1)_k (-n_2)_{j-k}}{(\beta)_j} \frac{((c_1-1)/c_1)^k}{k!} \frac{((c_2-1)/c_2)^{j-k}}{(j - k)!} (-x)_j.$$

Here

$$F_1(\alpha, \beta, \beta'; \gamma; x, y) = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_{m+n} m! n!} x^m y^n$$

is the first of Appell's hypergeometric functions of two variables [5]. From this explicit expression we can find the coefficients of the recurrence relation

$$x P_{n_1,n_2}(x) = P_{n_1+1,n_2}(x) + b_{n_1,n_2} P_{n_1,n_2}(x) + c_{n_1,n_2} P_{n_1,n_2-1}(x) + d_{n_1,n_2} P_{n_1-1,n_2-1}(x),$$

with $P_{n_1,n_2}(x) = M_{n_1,n_2}^{\beta;c_1,c_2}(x)$. We denote

$$M_{n_1,n_2}^{\beta;c_1,c_2}(x) = \sum_{j=0}^{n_1+n_2} c_{n_1,n_2}^{(j)} (-x)_j$$

and formulas (4.4) then give us

$$b_{n_1,n_2} = n_1(2a_1 + 1) + n_2(a_1 + a_2 + 1) + a_1\beta,$$

$$c_{n_1,n_2} = (n_1(a_1^2 + a_1) + n_2(a_2^2 + a_2))(n_1 + n_2 + \beta - 1),$$

$$d_{n_1,n_2} = (\beta + n_1 + n_2 - 1)(\beta + n_1 + n_2 - 2)(a_1 + 1)(a_1 - a_2) a_1 n_1 \tag{4.8}$$

with $a_1 = c_1/(1 - c_1)$ and $a_2 = c_2/(1 - c_2)$.

4.3. Multiple Meixner polynomials (second kind)

In the case of *multiple orthogonal polynomials of Meixner II* we also have r measures $\mu_1, \ldots, \mu_2$ which form a negative binomial distribution, but here we change only the value of the parameter $\beta > 0$. So we have that

$$\mu_i = \sum_{k=0}^{+\infty} \frac{(\beta_i)_k c^k}{k!} \delta_k, \quad \beta_i > 0,\ i = 1, \ldots, r,$$

with $0 < c < 1$ and all the β_i different. The support of these measures is again $\mathbb{N}$. It follows from Example 2.2 that every multi-index $\vec{n} = (n_1, \ldots, n_r)$ is normal for the measures $\mu_1, \ldots, \mu_r$, whenever $\beta_i - \beta_j \notin \mathbb{Z}$ for all $i \neq j$.

Define

$$w^{\beta_i}(x) = \begin{cases} \dfrac{\Gamma(\beta_i + x)}{\Gamma(\beta)} \dfrac{c^x}{\Gamma(x + 1)} & \text{if } x \in \mathbb{R} \setminus (\{-1, -2, -3, \ldots\} \cup \{-\beta, -\beta - 1, -\beta - 2, \ldots\}), \\ 0 & \text{if } x \in \{-1, -2, -3, \ldots\}, \end{cases}$$

then the monic discrete multiple orthogonal polynomials of Meixner II, corresponding to the multi-index $\vec{n}$ and the parameters $\vec{\beta}=(\beta_1,\ldots,\beta_r)$, $\beta_i>0$ ($\beta_i-\beta_j\notin\mathbb{Z}$ for all $i\neq j$) and $0<c<1$, is the unique monic polynomial $M_{\vec{n}}^{\vec{\beta};c}$ of degree $|\vec{n}|$ that satisfies the orthogonality conditions

$$\sum_{k=0}^{+\infty} M_{\vec{n}}^{\vec{\beta};c}(k)(-k)_j w^{\beta_1}(k)=0, \quad j=0,\ldots,n_1-1,$$

$$\vdots$$

$$\sum_{k=0}^{+\infty} M_{\vec{n}}^{\vec{\beta};c}(k)(-k)_j w^{\beta_r}(k)=0, \quad j=0,\ldots,n_r-1.$$

In the same way as in Section 4.2 we can show that the following raising operations exist for these polynomials:

$$\frac{1}{w^{\beta_i-1}(x)}\nabla\left(w^{\beta_i}(x)M_{\vec{n}}^{\vec{\beta};c}(x)\right)=\frac{c-1}{c(\beta_i-1)}M_{\vec{n}+\vec{e}_i}^{\vec{\beta}-\vec{e}_i;c}(x), \quad i=1,\ldots,r. \tag{4.9}$$

After a repeated application of these raising operators we get the *Rodrigues formula*, namely

$$M_{\vec{n}}^{\vec{\beta};c}(x)=\left(\frac{c}{c-1}\right)^{|\vec{n}|}\left[\prod_{k=1}^{r}(\beta_i)_{n_i}\right]\frac{\Gamma(x+1)}{c^x} \times\left[\prod_{i=1}^{r}\left(\frac{\Gamma(\beta_i)}{\Gamma(\beta_i+x)}\nabla^{n_i}\frac{\Gamma(\beta_i+n_i+x)}{\Gamma(\beta_i+n_i)}\right)\right]\left(\frac{c^x}{\Gamma(x+1)}\right). \tag{4.10}$$

Now we can again use property (4.3) to find an explicit expression for these polynomials. After some calculations in the case $r=2$ we finally get the expression

$$\begin{aligned} M_{n_1,n_2}^{\beta_1,\beta_2;c}(x)&=\left(\frac{c}{c-1}\right)^{n_1+n_2}(\beta_2)_{n_2}(\beta_1)_{n_1}F_{1:0;1}^{1:1;2}\left(\begin{matrix}(-x):(-n_1);(-n_2,\beta_1+n_1); \\ \\ (\beta_1):-;(\beta_2);\end{matrix}\ \frac{c-1}{c},\frac{c-1}{c}\right)\\ &=\left(\frac{c}{c-1}\right)^{n_1+n_2}(\beta_2)_{n_2}(\beta_1)_{n_1}\sum_{j=0}^{n_1+n_2}\sum_{k=0}^{j}\frac{(-n_1)_k(-n_2)_{j-k}(\beta_1+n_1)_{j-k}}{k!(j-k)!(\beta_2)_{j-k}}\frac{((c-1)/c)^j}{(\beta_1)_j}(-x)_j.\end{aligned}$$

Here

$$F_{l:m;n}^{p:q;k}\left(\begin{matrix}\vec{a}:\vec{b};\vec{c}; \\ \\ \vec{\alpha}:\vec{\beta};\vec{\gamma};\end{matrix}\ x,y\right)=\sum_{r,s=0}^{+\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}$$

with $\vec{a}=(a_1,\ldots,a_p)$, $\vec{b}=(b_1,\ldots,b_q)$, $\vec{c}=(c_1,\ldots,c_k)$, $\vec{\alpha}=(\alpha_1,\ldots,\alpha_l)$, $\vec{\beta}=(\beta_1,\ldots,\beta_m)$, $\vec{\gamma}=(\gamma_1,\ldots,\gamma_n)$. These are (the generalisations of) the Kampé de Fériet's series [9] which are a generalisation of the four Appell series in two variables. To find the coefficients of the recurrence relation

$$xP_{n_1,n_2}(x)=P_{n_1+1,n_2}(x)+b_{n_1,n_2}P_{n_1,n_2}(x)+c_{n_1,n_2}P_{n_1,n_2-1}(x)+d_{n_1,n_2}P_{n_1-1,n_2-1}(x)$$

with $P_{n_1,n_2}(x)=M^{\beta_1,\beta_2;c}_{n_1,n_2}(x)$, we make use of this explicit expression. If we denote

$$M^{\beta_1,\beta_2;c}_{n_1,n_2}(x)=\sum_{j=0}^{n_1+n_2} c^{(j)}_{n_1,n_2}(-x)_j,$$

one can first compute $c^{(n_1+n_2-1)}_{n_1,n_2}$, $c^{(n_1+n2-2)}_{n_1,n_2}$ and $c^{(n_1+n_2-3)}_{n_1,n_2}$ and then use property (4.4) to find

$$\begin{aligned}
b_{n_1,n_2}&=n_1(2a+1)+n_2(a+1)+a\beta_1,\\
c_{n_1,n_2}&=a(a+1)(n_1n_2+n_1(n_1+\beta_1-1)+n_2(n_2+\beta_2-1)),\\
d_{n_1,n_2}&=a^2(a+1)n_1(n_1+\beta_1-1)(n_1+\beta_1-\beta_2),
\end{aligned}\tag{4.11}$$

where $a=c/(1-c)$.

4.4. *Multiple Kravchuk polynomials*

Consider r different measures $\mu_1,\dots,\mu_r$ which in each case form a binomial distribution on the integers $0,1,\dots,N$. So they are of the form

$$\mu_i=\sum_{k=0}^{N}\binom{N}{k}p_i^k(1-p_i)^{N-k}\delta_k,\quad 0<p_i<1,\ i=1,\dots,r,$$

where all the $0<p_i<1$ are different. The support of these measures is $\{0,1,\dots,N\}$ and we have that $\Delta_1=\cdots=\Delta_r=[0,N]$. Define the functions

$$v_i^N(x)=\begin{cases}\dfrac{N!\,p_i^x(1-p_i)^{N-x}}{\Gamma(x+1)\Gamma(N-x+1)} & \text{if } x\in\mathbb{R}\setminus(\{-1,-2,\dots\}\cup\{N+1,N+2,\dots\}),\\ 0 & \text{if } x\in\{-1,-2,\dots\}\cup\{N+1,N+2,\dots\},\end{cases}$$

where $i=1,\dots,r$, which are functions in C^∞. Then the following holds: $v_i^N(k)=\binom{N}{k}p_i^k(1-p_i)^{N-k}$, $k\in\mathbb{N}$, $i=1,\dots,r$. Now from Example 2.1 it follows that $\mu_1,\dots,\mu_r$ is an AT system, so that every multi-index $\vec{n}=(n_1,\dots,n_r)$ with $|\vec{n}|\leqslant N$ is normal for these measures.

The monic discrete *multiple orthogonal polynomials of Kravchuk*, corresponding to the multi-index $\vec{n}=(n_1,\dots,n_r)$ with $|\vec{n}|\leqslant N$ and the parameters N and $\vec{p}=(p_1,\dots,p_r)$, is the unique monic polynomial $K^{\vec{p};N}_{\vec{n}}$ of degree $|\vec{n}|$ which satisfies the conditions

$$\sum_{k=0}^{N}K^{\vec{p};N}_{\vec{n}}(k)(-k)_jv_1^N(k)=0,\quad j=0,\dots,n_1-1,$$

$$\vdots$$

$$\sum_{k=0}^{N}K^{\vec{p};N}_{\vec{n}}(k)(-k)_jv_r^N(k)=0,\quad j=0,\dots,n_r-1.$$

Using summation by parts we can show that for these polynomials there exist some raising operators, namely

$$\frac{p_i(1-p_i)(N+1)}{v_i^{N+1}(x)} \nabla(v_i^N(x) K_{\vec{n}}^{\vec{p};N}(x)) = -K_{\vec{n}+\vec{e}_i}^{\vec{p};N+1}(x), \quad i=1,\ldots,r. \tag{4.12}$$

One should use the fact that $v_i^N(-1) = v_i^N(N+1) = 0$ to show that the left-hand side of the equations satisfies the orthogonality conditions of $K_{\vec{n}+\vec{e}_i}^{\vec{p};N+1}$. (Also note that on the left-hand side the simple zeros of the denominator are also zeros of the numerator.) After a repeated application of these raising operators we get the *Rodrigues formula* which is of the form

$$K_{\vec{n}}^{\vec{p};N}(x) = (-N)_{|\vec{n}|} \left[\prod_{k=1}^{r} p_k^{n_k}\right] \frac{\Gamma(x+1)\Gamma(N-x+1)}{N!}$$
$$\times \left[\prod_{i=1}^{r} \left(\left(\frac{1-p_i}{p_i}\right)^x \nabla^{n_i} \left(\frac{p_i}{1-p_i}\right)^x\right)\right] \left(\frac{(N-|\vec{n}|)!}{\Gamma(x+1)\Gamma(N-|\vec{n}|-x+1)}\right). \tag{4.13}$$

In the case of $r=2$ we can again use property (4.3) to find an explicit expression in the form

$$K_{n_1,n_2}^{p_1,p_2;N}(x) = \sum_{j=0}^{n_1+n_2} c_{n_1,n_2}^{(j)} (-x)_j$$

for these polynomials. After some calculations we get

$$K_{n_1,n_2}^{p_1,p_2;N}(x) = p_1^{n_1} p_2^{n_2} (-N)_{n_1+n_2}\ F_1\left(-x, -n_1, -n_2; -N; \frac{1}{p_1}, \frac{1}{p_2}\right)$$
$$= p_1^{n_1} p_2^{n_2} (-N)_{n_1+n_2} \sum_{j=0}^{n_1+n_2} \sum_{k=0}^{j} \frac{(-n_1)_k}{k!} \left(\frac{1}{p_1}\right)^k \frac{(-n_2)_{j-k}}{(j-k)!} \left(\frac{1}{p_2}\right)^{j-k} \frac{(-x)_j}{(-N)_j}.$$

These are the same expressions as in the example of Meixner I with $\beta = -N$, $c_1 = p_1/(p_1-1)$ and $c_2 = p_2/(p_2-1)$, which gives $a_1 = -p_1$ and $a_2 = -p_2$ (one can see this already in the Rodrigues formula). The coefficients of the recurrence relation

$$xP_{n_1,n_2}(x) = P_{n_1+1,n_2}(x) + b_{n_1,n_2} P_{n_1,n_2}(x) + c_{n_1,n_2} P_{n_1,n_2-1}(x) + d_{n_1,n_2} P_{n_1-1,n_2-1}(x),$$

with $P_{n_1,n_2}(x) = K_{n_1,n_2}^{p_1,p_2;N}(x)$, are then

$$b_{n_1,n_2} = -n_1(2p_1-1) - n_2(p_1+p_2-1) + p_1 N,$$
$$c_{n_1,n_2} = (n_1(p_1^2-p_1) + n_2(p_2^2-p_2))(n_1+n_2-N-1),$$
$$d_{n_1,n_2} = (n_1+n_2-N-1)(n_1+n_2-N-2)(1-p_1)(p_1-p_2)p_1 n_1. \tag{4.14}$$

4.5. Multiple Hahn polynomials

Take r measures $\mu_1,\ldots,\mu_r$ which in each case form a hypergeometric distribution on the integers $0,\ldots,N$. We change the value of the parameter $\alpha > -1$ and keep $\beta > -1$ fixed. We write

$$\mu_i = \sum_{k=0}^{N} \frac{(\alpha_i+1)_k}{k!}\,\frac{(\beta+1)_{N-k}}{(N-k)!}\,\delta_k, \quad \alpha_i > -1,\ i=1,\ldots,r$$

with all the α_i different. The case where we change the parameter β and keep $\alpha > -1$ fixed is obtained when we substitute $x \to N-x$. The support of these measures is $\{0,\ldots,N\}$ and we again have that $\Delta_1 = \cdots = \Delta_r = [0,N]$.

The monic discrete *multiple orthogonal polynomial of Hahn*, corresponding to the multi-index $\vec{n}=(n_1,\ldots,n_r)$ and the parameters $\vec{\alpha}=(\alpha_1,\ldots,\alpha_r)$, β, and N, is the unique monic polynomial $Q_{\vec{n}}^{\vec{\alpha};\beta;N}$ of degree $|\vec{n}|$ which satisfies the conditions

$$\sum_{k=0}^{N} Q_{\vec{n}}^{\vec{\alpha};\beta;N}(k)(-k)_j v^{\alpha_1,\beta;N}(k) = 0, \quad j=0,\ldots,n_1-1,$$

$$\vdots$$

$$\sum_{k=0}^{N} Q_{\vec{n}}^{\vec{\alpha};\beta;N}(k)(-k)_j v^{\alpha_r,\beta;N}(k) = 0, \quad j=0,\ldots,n_r-1$$

with

$$v^{\alpha,\beta;N}(x) = \begin{cases} \dfrac{\Gamma(\alpha+x+1)}{\Gamma(\alpha+1)\Gamma(x+1)}\,\dfrac{\Gamma(\beta+N-x+1)}{\Gamma(\beta+1)\Gamma(N-x+1)} & \text{if } x\in\mathbb{R}\setminus(X_1\cup X_2), \\ 0 & \text{if } x\in X_1, \end{cases}$$

where $X_1 = \{-1,-2,\ldots\}\cup\{N+1,N+2,\ldots\}$ and $X_2 = \{-\alpha-1,-\alpha-2,\ldots\}\cup\{\beta+N+1,\beta+N+2,\beta+N+3,\ldots\}$. Here $v^{\alpha,\beta;N}$ is a $C^\infty(\mathbb{R}\setminus X_2)$-function with simple poles at the points of X_2. We see from Example 2.2 that every multi-index $\vec{n}=(n_1,\ldots,n_r)$ with $|\vec{n}|\leqslant N$ is normal whenever $\alpha_i-\alpha_j\notin\{0,1,\ldots,N-1\}$ for $i\neq j$.

Using summation by parts (and the fact that $v^{\alpha_i,\beta;N}(-1)=v^{\alpha_i,\beta;N}(N+1)=0$) we can find the following raising operations:

$$\frac{1}{v^{\alpha_i-1,\beta-1;N+1}(x)}\,\nabla\left(v^{\alpha_i,\beta;N}(x)Q_{\vec{n}}^{\vec{\alpha};\beta;N}(x)\right)$$
$$= -\frac{|\vec{n}|+\alpha_i+\beta}{\alpha_i\beta}\,Q_{\vec{n}+\vec{e}_i}^{\vec{\alpha}-\vec{e}_i;\beta-1;N+1}(x), \quad i=1,\ldots,r. \tag{4.15}$$

Again we apply these operators a few times to the polynomial $Q_{\vec{n}}^{\vec{\alpha};\beta;N}$ to find a *Rodrigues formula* for this polynomial of the form

$$Q_{\vec{n}}^{\vec{\alpha};\beta;N}(x)=\frac{(-1)^{|\vec{n}|}(\beta+1)_{n_1+n_2}}{\prod_{k=1}^{r}(n_1+n_2+\alpha_k+\beta+1)_{n_k}}\frac{\Gamma(x+1)\Gamma(N-x+1)}{\Gamma(\beta+N-x+1)}$$
$$\times\left[\prod_{i=1}^{r}\left(\frac{1}{\Gamma(\alpha_i+x+1)}\nabla^{n_i}\Gamma(\alpha_i+n_i+x+1)\right)\right]\left(\frac{\Gamma(\beta+N-x+1)}{\Gamma(x+1)\Gamma(N-x+1)}\right).\quad(4.16)$$

Now property (4.3) enables us to find an explicit expression for the polynomials starting from the Rodrigues formula. After some calculations in the case $r=2$ we get

$$Q_{n_1,n_2}^{\alpha_1,\alpha_2;\beta;N}(x)$$
$$=\frac{(\alpha_1+1)_{n_1}(\alpha_2+1)_{n_2}(-N)_{n_1+n_2}}{(n_1+n_2+\alpha_1+\beta+1)_{n_1}(n_1+n_2+\alpha_2+\beta+1)_{n_2}}$$
$$\times F_{2:0;2}^{2:1;3}\left(\begin{matrix}(-x,\beta+n_1+\alpha_1+1):(-n_1);(-n_2,\beta+\alpha_2+n_1+n_2+1,\alpha_1+n_1+1); & \\ & 1,1\\ (-N,\alpha_1+1):-;(\alpha_2+1,\beta+n_1+\alpha_1+1); & \end{matrix}\right)$$

or

$$Q_{n_1,n_2}^{\alpha_1,\alpha_2;\beta;N}(x)=\sum_{j=0}^{n_1+n_2}c_{n_1,n_2}^{(j)}(-x)_j$$

with

$$c_{n_1,n_2}^{(j)}=\frac{(\alpha_1+1)_{n_1}(\alpha_2+1)_{n_2}(-N)_{n_1+n_2}}{(n_1+n_2+\alpha_1+\beta+1)_{n_1}(n_1+n_2+\alpha_2+\beta+1)_{n_2}}$$
$$\times\sum_{k=0}^{j}\frac{(-n_1)_k(-n_2)_{j-k}}{k!\,(j-k)!}\frac{(\beta+\alpha_2+n_1+n_2+1)_{j-k}(\alpha_1+n_1+1)_{j-k}}{(\alpha_2+1)_{j-k}(\beta+n_1+\alpha_1+1)_{j-k}}\frac{(\beta+n_1+\alpha_1+1)_j}{(-N)_j(\alpha_1+1)_j}.$$

Property (4.4) now gives an expression for the coefficients of the recurrence relation

$$xP_{n_1,n_2}(x)=P_{n_1+1,n_2}(x)+b_{n_1,n_2}P_{n_1,n_2}(x)+c_{n_1,n_2}P_{n_1,n_2-1}(x)+d_{n_1,n_2}P_{n_1-1,n_2-1}(x),$$

$0\leqslant n_1+n_2\leqslant N$, where $P_{n_1,n_2}(x)=Q_{n_1,n_2}^{\alpha_1,\alpha_2;\beta;N}(x)$. After some calculations we find that

$$b_{n_1,n_2}=A(n_1,n_2,\alpha_1,\alpha_2,N)+A(n_2,n_1,\alpha_2,\alpha_1+1,N)+D(n_1,n_2,\alpha_1,\alpha_2)$$
$$+C(n_1+1,n_2+1,\alpha_1,\alpha_2,N),$$

$$c_{n_1,n_2} = (A(n_1, n_2, \alpha_1, \alpha_2, N) + A(n_2, n_1, \alpha_2, \alpha_1 + 1, N) + D(n_1, n_2, \alpha_1, \alpha_2))$$
$$\times C(n_2, n_1 + 1, \alpha_2, \alpha_1, N) + A(n_1, n_2, \alpha_1, \alpha_2, N)B(n_1, n_2, \alpha_1, \alpha_2, N),$$

$$d_{n_1,n_2} = A(n_1, n_2, \alpha_1, \alpha_2, N)B(n_1, n_2, \alpha_1, \alpha_2, N)C(n_1, n_2, \alpha_1, \alpha_2, N) \tag{4.17}$$

with

$$A(n_1, n_2, \alpha_1, \alpha_2, N) = \frac{n_1(n_1 + n_2 + \beta + \alpha_2)(n_1 + n_2 + \beta)(\beta + n_1 + \alpha_1 + 1 + N)}{(n_1 + 2n_2 + \beta + \alpha_2)(2n_1 + n_2 + \beta + \alpha_1)(2n_1 + n_2 + \beta + \alpha_1 + 1)},$$

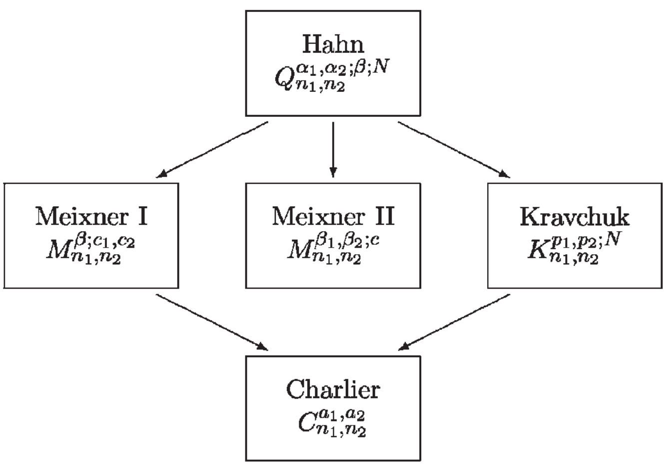
$$B(n_1, n_2, \alpha_1, \alpha_2, N)$$
$$= \frac{(n_1 + \alpha_1 - \alpha_2)(n_1 + n_2 + \beta + \alpha_1)(n_1 + n_2 + \beta - 1)(N - n_1 - n_2 + 1)}{(n_1 + 2n_2 + \beta + \alpha_2 - 1)(2n_1 + n_2 + \beta + \alpha_1)(2n_1 + n_2 + \beta + \alpha_1 - 1)},$$

$$C(n_1, n_2, \alpha_1, \alpha_2, N)$$
$$= \frac{(n_1 + \alpha_1)(n_1 + n_2 + \beta + \alpha_1 - 1)(n_1 + n_2 + \beta + \alpha_2 - 1)(N - n_1 - n_2 + 2)}{(n_1 + 2n_2 + \beta + \alpha_2 - 2)(2n_1 + n_2 + \beta + \alpha_1 - 2)(2n_1 + n_2 + \beta + \alpha_1 - 1)},$$

$$D(n_1, n_2, \alpha_1, \alpha_2) = \frac{(\beta + n_1 + n_2)n_1 n_2}{(2n_1 + n_2 + \alpha_1 + \beta + 1)(n_1 + 2n_2 + \alpha_2 + \beta)}.$$

5. Conclusions

We have studied five examples of discrete multiple orthogonal polynomials. Each system forms an AT system. As in the case of the classical orthogonal polynomials we can consider the multiple



orthogonal polynomials of Charlier, Meixner I, Meixner II and Kravchuk as limiting cases of the multiple Hahn polynomials.

The limit relations are

$$M_{n_1,n_2}^{\beta;c_1,c_2}(x)=(-1)^{n_1+n_2}\lim_{N\to+\infty}Q_{n_1,n_2}^{((1-c_1)/c_1)N,((1-c_2)/c_2)N;\beta-1;N}(N-x),$$

$$M_{n_1,n_2}^{\beta_1,\beta_2;c}(x)=\lim_{N\to+\infty}Q_{n_1,n_2}^{\beta_1-1,\beta_2-1;((1-c)/c)N;N}(x),$$

$$K_{n_1,n_2}^{p_1,p_2;N}(x)=(-1)^{n_1+n_2}\lim_{t\to+\infty}Q_{n_1,n_2}^{-t/p_1,-t/p_2;-N-1;t}(t-x),$$

$$C_{n_1,n_2}^{a_1,a_2}(x)=\lim_{\beta\to+\infty}M_{n_1,n_2}^{\beta;a_1/\beta,a_2/\beta}(x),$$

$$C_{n_1,n_2}^{a_1,a_2}(x)=\lim_{N\to\infty}K_{n_1,n_2}^{a_1/N,a_2/N;N}(x).$$

In the continuous case, studied in [10,3], there are also examples of Angelesco systems, where the measures are supported on disjoint (or touching) intervals. In the discrete case it is not obvious to find such systems which still have some raising operators, a Rodrigues formula, etc.

It is of interest to look at the limit of the coefficients in the recurrence relation for the multiple orthogonal polynomials (in the case $r=2$) when we set $n_1=\lfloor sn\rfloor$, $n_2=\lfloor tn\rfloor$ and n tends to infinity. These results are useful for an asymptotic study on these polynomials. For our examples of discrete multiple orthogonal polynomials we have

	$C_{n_1,n_2}^{a_1,a_2}$	$M_{n_1,n_2}^{\beta;c_1,c_2}$	$M_{n_1,n_2}^{\beta_1,\beta_2;c}$
$\lim_{n\to+\infty}\frac{b_{\lfloor sn\rfloor,\lfloor tn\rfloor}}{n}$	$s+t$	$s(2a_1+1)+t(a_1+a_2+1)$	$s(2a+1)+t(a+1)$
$\lim_{n\to+\infty}\frac{c_{\lfloor sn\rfloor,\lfloor tn\rfloor}}{n^2}$	0	$(s+t)(s(a_1^2+a_1)+t(a_2^2+a_2))$	$a(a+1)(s^2+st+t^2)$
$\lim_{n\to+\infty}\frac{d_{\lfloor sn\rfloor,\lfloor tn\rfloor}}{n^3}$	0	$a_1(a_1-a_2)(a_1+1)s(s+t)^2$	$a^2(a+1)s^3$

Here $a_1=c_1/(1-c_1)$ and $a_2=c_2/(1-c_2)$ in the third column and $a=c/(1-c)$ in the last column. For the multiple orthogonal polynomials of Krawtchouk and Hahn we can set $n_1=\lfloor sN\rfloor$,

$n_2 = \lfloor tN \rfloor$, $s, t > 0$, $s + t < 1$ and take the limit when N tends to infinity. This gives

	$K_{n_1,n_2}^{p_1,p_2;N}$	$Q_{n_1,n_2}^{\alpha_1,\alpha_2;\beta;N}$
$\lim_{N\to+\infty} \frac{b_{\lfloor sN \rfloor, \lfloor tN \rfloor}}{N}$	$p_1 - s(2p_1 - 1) - t(p_1 + p_2 - 1)$	$A(s,t) + A(t,s) + D(s,t) + B(s,t)$
$\lim_{N\to+\infty} \frac{c_{\lfloor sN \rfloor, \lfloor tN \rfloor}}{N^2}$	$(s + t - 1)(s(p_1^2 - p_1) + t(p_2^2 - p_2))$	$(A(s,t) + A(t,s) + D(s,t))B(t,s)$ $+A(s,t)B(s,t)$
$\lim_{N\to+\infty} \frac{d_{\lfloor sN \rfloor, \lfloor tN \rfloor}}{N^3}$	$p_1(p_1 - p_2)(1 - p_1)s(s + t - 1)^2$	$A(s,t)B(s,t)^2$

where

$$A(s,t) = \frac{s(1+s)(s+t)^2}{(s+2t)(t+2s)^2},$$

$$B(s,t) = \frac{(1-s-t)s(s+t)^2}{(s+2t)(t+2s)^2},$$

$$D(s,t) = \frac{(s+t)st}{(t+2s)(s+2t)}.$$

Appendix A

Here we prove Example 2.2, namely that functions (2.5), with $\beta_i > 0$ and $\beta_i - \beta_j \notin \mathbb{Z}$ if $i \neq j$, and v a continuous function which has no zeros on $\mathbb{R}^+$ form a Chebyshev system on $\mathbb{R}^+$ for every $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$. This means that every linear combination of these functions (except the one with each coefficient equal to 0) has at most $|\vec{n}| - 1$ zeros. Such a linear combination is also a linear combination of the functions

$$v(x)\Gamma(\beta_1 + x), v(x)(x + \beta_1)\Gamma(\beta_1 + x), \ldots, v(x)(x + \beta_1)_{n_1 - 1}\Gamma(\beta_1 + x),$$

$$\vdots$$

$$v(x)\Gamma(\beta_r + x), v(x)(x + \beta_r)\Gamma(\beta_r + x), \ldots, v(x)(x + \beta_r)_{n_r - 1}\Gamma(\beta_r + x)$$

and because of $z\Gamma(z) = \Gamma(z+1)$, these functions are also

$$\begin{gathered} v(x)\Gamma(\beta_1 + x), v(x)\Gamma(\beta_1 + x + 1), \ldots, v(x)\Gamma(\beta_1 + x + n_1 - 1), \\ \vdots \\ v(x)\Gamma(\beta_r + x), v(x)\Gamma(\beta_r + x + 1), \ldots, v(x)\Gamma(\beta_r + x + n_r - 1). \end{gathered} \tag{A.1}$$

So it is sufficient to show that the functions (A.1) form a Chebyshev system on $\mathbb{R}^+$ for every $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$. If we take $m = |\vec{n}|$ and define the matrix $Z(\vec{n}, x_1, \ldots, x_m)$ as

$$\begin{pmatrix} \Gamma(\beta_1 + x_1) & \Gamma(\beta_1 + x_2) & \ldots & \Gamma(\beta_1 + x_m) \\ \vdots & \vdots & & \vdots \\ \Gamma(\beta_1 + x_1 + n_1 - 1) & \Gamma(\beta_1 + x_2 + n_1 - 1) & \ldots & \Gamma(\beta_1 + x_m + n_1 - 1) \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \Gamma(\beta_r + x_1) & \Gamma(\beta_r + x_2) & \ldots & \Gamma(\beta_r + x_m) \\ \vdots & \vdots & & \vdots \\ \Gamma(\beta_r + x_1 + n_r - 1) & \Gamma(\beta_r + x_2 + n_r - 1) & \ldots & \Gamma(\beta_r + x_m + n_r - 1) \end{pmatrix},$$

then we have to show that $v(x_1) \ldots v(x_m) \det Z(\vec{n}, x_1, \ldots, x_m) \neq 0$, which is equivalent to $\det Z(\vec{n}, x_1, \ldots, x_m) \neq 0$ for every m different points $x_1, \ldots, x_m$ in $\mathbb{R}^+$ (the function v has no zeros on $\mathbb{R}^+$). We can now use the definition of the Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt, \quad x > 0$$

to find that

$$\det Z(\vec{n}, x_1, \ldots, x_m) = \underbrace{\int_0^\infty \ldots \int_0^\infty}_{m \text{ times}} e^{-t_1} t_1^{x_1 - 1} \ldots e^{-t_m} t_m^{x_m - 1} \det C(\vec{n}, t_1, \ldots, t_m) \, dt_1 \ldots dt_m \tag{A.2}$$

with

$$C(\vec{n},t_1,\ldots,t_m)=\begin{pmatrix} t_1^{\beta_1} & t_2^{\beta_1} & \ldots & t_m^{\beta_1}\\ t_1^{\beta_1+1} & t_2^{\beta_1+1} & \ldots & t_m^{\beta_1+1}\\ \vdots & \vdots & & \vdots\\ t_1^{\beta_1+n_1-1} & t_2^{\beta_1+n_1-1} & \ldots & t_m^{\beta_1+n_1-1}\\ \vdots & \vdots & & \vdots\\ \vdots & \vdots & & \vdots\\ t_1^{\beta_r} & t_2^{\beta_r} & \ldots & t_m^{\beta_r}\\ t_1^{\beta_r+1} & t_2^{\beta_r+1} & \ldots & t_m^{\beta_r+1}\\ \vdots & \vdots & & \vdots\\ t_1^{\beta_r+n_r-1} & t_2^{\beta_r+n_r-1} & \ldots & t_m^{\beta_r+n_r-1} \end{pmatrix}.$$

From [8, p. 138, Example 4] we know that the functions $t^{\beta_1}, t^{\beta_1+1}, \ldots, t^{\beta_1+n_1-1}, \ldots, t^{\beta_r}, t^{\beta_r+1}, \ldots, t^{\beta_r+n_r-1}$ form a Chebyshev system on $\mathbb{R}^+$ if all the exponents are different, which holds because $\beta_i - \beta_j \notin \mathbb{Z}$, $i \neq j$. If all $n_i < N+1$ then the exponents are different if $\beta_i - \beta_j \notin \{0,1,\ldots,N-1\}$ whenever $i \neq j$. So $\det C(\vec{n},t_1,\ldots,t_m)$ is different from zero if and only all the $t_1,\ldots,t_m$ are different. We can then write

$$\begin{aligned}&\det Z(\vec{n},x_1,\ldots,x_m)\\ &\quad = \int_{0<t_1<\cdots<t_m} \mathrm{e}^{-\sum_{j=1}^m t_j} \det C(\vec{n},t_1,\ldots,t_m) \sum_{\pi\in S_m} (-1)^{\mathrm{sign}(\pi)} t_{\pi(1)}^{x_1-1}\ldots t_{\pi(m)}^{x_m-1}\,\mathrm{d}t_1\ldots\mathrm{d}t_m, \qquad \text{(A.3)}\end{aligned}$$

with S_m the permutation group. From the definition of the determinant of a matrix we see that

$$\sum_{\pi\in S_m} (-1)^{\mathrm{sign}(\pi)} t_{\pi(1)}^{x_1-1}\ldots t_{\pi(m)}^{x_m-1} = \begin{vmatrix} t_1^{x_1-1} & t_1^{x_2-1} & \ldots & t_1^{x_m-1}\\ t_2^{x_1-1} & t_2^{x_2-1} & \ldots & t_2^{x_m-1}\\ \vdots & \vdots & & \vdots\\ t_m^{x_1-1} & t_m^{x_2-1} & \ldots & t_m^{x_m-1} \end{vmatrix}.$$

The $t_1,\ldots,t_m$ are strictly positive and different, and Example 2.1 with multi-index $(1,\ldots,1)$ then implies that this is different from zero if all the $x_1,\ldots,x_m$ are different. So, if the $x_1,\ldots,x_m$ are different, then the integrand of (A.3) has constant sign in the integration area and hence $\det Z(\vec{n},x_1,\ldots,x_m)$ is different from zero.

References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards, Washington, 1964, Dover, New York, 1972.

[2] A.I. Aptekarev, Multiple orthogonal polynomials, J. Comput. Appl. Math. 99 (1998) 423–447.

[3] A.I. Aptekarev, A. Branquinho, W. Van Assche, Multiple orthogonal polynomials for classical weights, manuscript.

[4] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.

[5] A. Erdélyi, Higher Transcendental Functions, Vol. I, McGraw-Hill Book Company, New York, 1953.

[6] R. Koekoek, R.F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Reports of the faculty of Technical Mathematics and Informatics No. 98-17, Delft, 1998 (math.CA/9602214 at arXiv.org).

[7] A.F. Nikiforov, S.K. Suslov, V.B. Uvarov, Classical Orthogonal Polynomials of a Discrete Variable, Springer, Berlin, 1991.

[8] E.M. Nikishin, V.N. Sorokin, Rational Approximations and Orthogonality, in: Translations of Mathematical Monographs, Vol. 92, American Mathematical Society, Providence, RI, 1991.

[9] H.M. Srivastava, Per W. Karlsson, Multiple Gaussian Hypergeometric Series, Ellis Horwood Series in Mathematics and its Applications, Horwood Chichester, New York, 1984.

[10] W. Van Assche, E. Coussement, Some classical multiple orthogonal polynomials, J. Comput. Appl. Math. 127 (2001) 317–347.