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EFFECTS OF APPLYING LINEAR AND NONLINEAR FILTERS ON TESTS FOR UNIT ROOTS WITH ADDITIVE OUTLIERS

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Abstract. Conventional univariate Dickey-Fuller tests tend to produce spurious stationarity when there exist additive outlying observations in the time series. Correct critical values are usually obtained by adding dummy variables to the Dickey-Fuller regression. This is a ni theoretical result but not attractive from the empirical point of view since almost any result ca.. be obtained just by a convenient selection of dummy variables. In this paper we suggest a robust procedure based on running Dickey-Fuller tests on the trend component instead of the original series. We provide both finite-sample and large-sample justifications. Practical implementation is illustrated through an empirical example based on the US/Finland real exchange rate series.

Key words: Additive outliers, Dickey-Fuller test, linear and nonlinear filtering, bootstrap.

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1. Introduction

Univariate and multivariate unit root testing procedures are sensitive to the occurrence of anomalous events such as structural breaks and outliers. From the univariate point of view, the effects of having breaks when applying unit root tests are well documented. Perron (1989) shows that standard unit root tests break down (finding too many unit roots) if there is a structural break in the data generating process, such as a level shift. The intuitive idea is that the unit root hypothesis is closely associated with shocks having a permanent effect. A structural break essentially corresponds to a shock with a lasting effect on the series (see Perron and Vogelsang, 1992). Thus, if this shock is not explicitly taken into account, standard unit root tests will identify the structural break as a unit root. On the contrary, if the break occurs early in the series, unit root tests can lead to over-rejection of the unit root hypothesis (see Leybourne et al., 1998).

A possible solution to this problem requires the use of dummy variables and segmented trends. For instance, Perron (1989) included a set of deterministic regressors to allow for an alternative hypothesis having a trend break at a known date. The main problem with this procedure is that,

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even if we know when the structural break occurs, the critical values obtained depend on the size as well as on the timing of the break. Additionally, the asymptotic distributions of unit root tests also depend on whether the location of the breaks is known from the outset or not (see, e.g. Christiano, 1992).

Another type of anomalous events are the additive transitory outliers. These are events with a (possible) large but temporary effect on the series. Provided these additive outliers (AO's) are sufficiently large or sufficiently frequent, such an effect can dominate the remaining information contained in the series and biases unit root inference towards rejection of the unit root hypothesis, as reported by Martin and Yohai (1986), Franses and Haldrup (1994), Lucas (1995*a*,*b*), Shin et al. (1996), and Vogelsang (1999), among others.

In principle, it is possible to include dummy variables for these transitory shocks as is usually done in the case of structural breaks and calculate the corresponding new critical values. See Franses and Haldrup (1994), Shin et al. (1996), and Vogelsang (1999). A crucial step in this approach, however, turns out to be the detection of the AO's. Tsay (1986), Chang et al. (1988), Chen and Liu (1993), Gómez and Maravall (1996) and Kaiser (1998) provide some of the most common procedures to identify isolate outliers. For practical purposes, however, it requires a considerable skill to identify the outliers and evaluate the many different test statistics. Alternatively, one could avoid the use of dummy variables by considering robust estimation techniques following the approach by Lucas (1995*a*,*b*). Hence, the main idea of outlier detection is to throw them out, whereas the basic idea of robust estimation is to leave them in. See Maddala and Yin (1996) and Yin and Maddala (1997) for further insights.

Within the robust approach, in this article we explore a different route for testing for a unit root when AO's are a possibility in the data. Assume the series of interest, z_t, is split in two unobserved components, $z_t = z_t^g + z_t^c$, where z_t^g denotes the permanent or trend component and z_t^c stands for the transitory or cyclical component. Our proposal is to apply the unit root tests to z_t^g instead of the observed series z_t, the idea being that, since the AO's are transitory, they could be incorporated, to a large extent, into the cyclical component z_t^c and not into the trend. To estimate the growth component z_t^g, we suggest the use of low-pass filters.

The paper is organized as follows. In Section 2 we introduce the model of interest and some analytical results concerning the effects of frequency and size of AO's on the well-known Dickey-Fuller test statistic. Section 3 presents some basic results on filtering and signal extraction. In Section 4 we discuss the effects of filtering on the Dickey-Fuller tests. Section 5 presents some Monte Carlo evidence to illustrate the finite sample implications of our analytical findings. Section

6 provides an empirical example and Section 7 concludes. Proof of Theorem 1 is collected in an Appendix.

2. Unit Roots and Additive Outliers

Let y_t be generated by a random walk with $y_0 = 0$,

$$y_t = y_{t-1} + \varepsilon_t, \qquad t = 1, 2, ..., T, \tag{2.1}$$

where ε_t are independent and identically distributed (i.i.d.) $N\left(0, \sigma_\varepsilon^2\right)$ variates, and suppose systematic AO's of size $\pm\theta$ may occur with probability π, so that the time series we observe is

$$x_t = y_t + \theta\delta_t, \tag{2.2}$$

where δ_t are i.i.d. Bernoulli random variables such that $P\left(\delta_t = 1\right) = \frac{1}{2}\pi$, $P\left(\delta_t = -1\right) = \frac{1}{2}\pi$ and $P\left(\delta_t = 0\right) = 1 - \pi$, $\delta_0 = 0$.

Consider the unit root regression

$$\Delta y_t = \rho y_{t-1} + \varepsilon_t. \tag{2.3}$$

As is well-known, the Dickey-Fuller unit root test takes as null hypothesis $H_0 : \rho = 0$, unit root, against $H_1 : \rho < 0$, stable root. The test is implemented by means of the least squares estimate of ρ,

$$\widehat{\rho} = \left(\sum_{t=1}^{T} y_{t-1}\Delta y_t\right) \Big/ \left(\sum_{t=1}^{T} y_{t-1}^2\right), \tag{2.4}$$

and the corresponding t statistic for a zero coefficient,

$$t_{\widehat{\rho}} = \widehat{\rho} \Big/ \widehat{\sigma_y}\left(\sum_{t=1}^{T} y_{t-1}^2\right)^{-1/2}, \tag{2.5}$$

where $\widehat{\sigma}_y^2 = T^{-1}\sum_{t=1}^{T}\left(\Delta y_t - \widehat{\rho} y_{t-1}\right)^2$ is the residual variance, $\Delta = (1-L)$ and L is the lag operator defined so that $L^n y_t = y_{t-n}$ for positive and negative values of n.

Note from equation (2.2) that, in the presence of AO's, regression (2.3) becomes

$$\Delta x_t = \rho x_{t-1} + u_t, \tag{2.6}$$

where $u_t = \varepsilon_t + \theta\Delta\delta_t$ with corresponding least squares statistics

$$\widehat{\rho}_{AO} = \left(\sum_{t=1}^{T} x_{t-1}\Delta x_t\right) \Big/ \left(\sum_{t=1}^{T} x_{t-1}^2\right), \tag{2.7}$$

and

$$t_{\widehat{\rho}_{AO}} = \widehat{\rho}_{AO} \Big/ \widehat{\sigma}_x \left(\sum_{t=1}^{T} x_{t-1}^2 \right)^{-1/2}, \tag{2.8}$$

where $\widehat{\sigma}_x^2 = T^{-1} \sum_{t=1}^{T} \left(\Delta x_t - \widehat{\rho}_{AO} x_{t-1} \right)^2$. Therefore, under H_0, the observed series turns out to be an $I(1)$ process with $MA(1)$ innovations, since $E\left(u_t u_{t-1}\right) = -\pi\theta^2$, $E\left(u_t u_{t-j}\right) = 0$ for $j > 1$. Consequently, u_t satisfies Phillips (1987) general mixing conditions, and a functional central limit theorem applies to the partial sums of u_t, so that the asymptotic distribution of (2.7) and (2.8) can be derived. In particular, it can be proved that, as $T \to \infty$,

$$\begin{aligned} T\widehat{\rho}_{AO} \Rightarrow & \left(\int_0^1 W\left(r\right) dW\left(r\right) \right) \Big/ \left(\int_0^1 W^2\left(r\right) dr \right) \\ & - \left(\theta/\sigma_\varepsilon\right)^2 \pi \left(\int_0^1 W^2\left(r\right) dr \right)^{-1} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} t_{\widehat{\rho}_{AO}} \Rightarrow & \left(1 + 2\left(\theta/\sigma_\varepsilon\right)^2 \pi \right)^{-1/2} \\ & \times \left(\int_0^1 W\left(r\right) dW\left(r\right) \right) \Big/ \left(\int_0^1 W^2\left(r\right) dr \right)^{1/2} \\ & - \left(\theta/\sigma_\varepsilon\right)^2 \pi \left(\int_0^1 W^2\left(r\right) dr \right)^{-1/2}, \end{aligned} \tag{2.10}$$

where $\Rightarrow$ denotes weak convergence and where $W\left(r\right)$ is a standard Brownian motion defined on the unit interval $r \in [0,1]$.

Equations (2.9) and (2.10) were first derived by Franses and Haldrup (1994). From these equations we learn that if $\pi > 0$, ρ is still estimated superconsistently, but the asymptotic distributions of the Dickey-Fuller statistics shift to the left, leading to tests with exact size greater than asymptotic nominal size and, consequently, to over-reject the unit root hypothesis in favor of the stationary alternative. Note that a positive AO and a negative AO with the same magnitude have the same effect on the limiting distributions. Note also that distributions are the same for any combination of θ, π and σ_ε^2 giving rise to the same value of $\left(\theta/\sigma_\varepsilon\right)^2 \pi$. In other words, large shocks with small chance of occurrence have the same effect as that of small shocks with high probability of occurrence. The Dickey-Fuller tests simply cannot distinguish infrequent large shocks from frequent small shocks. On the other hand, Franses and Haldrup's findings are readily extended to models allowing for the presence of deterministic components as well. See Yin and Maddala (1997).

3. Filters and Signal Extraction

Filter techniques have been used for a long time in estimating the states of stochastic dynamic systems or in extracting information from noisy observations. These techniques are also widely adopted in economics, specially with macroeconomics series. The usual aim of filtering in this latter case is to isolate the business cycle component of the series from slowly evolving secular trends and rapidly varying seasonal or irregular component.

For simplicity, assume that we are interested in splitting an observed time series, z_t, in two unobserved components,

$$z_t = z_t^g + z_t^c, \tag{3.1}$$

where z_t^g is the growth or trend component and z_t^c stands for the cyclical component. Note that (3.1) can be rewritten as

$$z_t = z_t^g + \left(z_t - z_t^g\right), \tag{3.2}$$

or

$$z_t = \left(z_t - z_t^c\right) + z_t^c. \tag{3.3}$$

Equation (3.2) requires a definition of the trend component while (3.3) needs a definition of the business cycle component. Burns and Mitchell (1946) is the seminal contribution to the measurement of business cycles. They proposed a decomposition that is no longer in common use. Instead, modern empirical macroeconomics employ a variety of detrending and smoothing techniques to carry out trend-cycle decompositions. Examples of these techniques are application of two-sided moving averages, first differencing, removal of linear or quadratic time trends and applications of the Hodrick and Prescott (1997) filter.

3.1. **The Hodrick-Prescott filter.**

The Hodrick-Prescott (HP) filter is particularly widely used in real business cycle models to detrend macroeconomic time series and concentrate on the stylized facts of an economy along the business cycle. See, e.g., Harvey and Jaeger (1993), King and Rebelo (1993), Cogley and Nason (1995), Guay and St-Amant (1997), Canova (1998), Ehglen (1998), Baxter and King (1999), and Kaiser and Maravall (1999).

The basis of the filter is as follows: Starting from equation (3.2), they define the permanent component z_t^g of the series as the solution to the optimization problem

$$\min_{z_t^g} \sum_{t=1}^{T} \left[\left(z_t - z_t^g\right)^2 + \lambda \left(\Delta^2 z_{t+1}^g\right)^2 \right]. \tag{3.4}$$

The first term of (3.4) might be regarded as a measure of the goodness of fit of the trend component to the observed series, while the second one imposes a penalty in order to obtain a smooth trend component. King and Rebelo (1993) show that the infinite-sample version of the HP filter defines the cyclic component of z_t as $z_t^c = H(L)z_t$, where

$$H(L) = \frac{\lambda(1-L)^2\left(1-L^{-1}\right)^2}{1+\lambda(1-L)^2\left(1-L^{-1}\right)^2}, \tag{3.5}$$

that is an optimal linear filter in the sense of minimum mean squared error.

On the other hand, letting $F(L) = 1 - H(L)$, from (3.5) it follows that the growth HP filter has expression

$$\begin{aligned} F(L) = &\left[1+\lambda\left(1-L^{-1}\right)^2\left(1-L^2\right)\right]^{-1} \\ &\times\left[\lambda L^2 - 4\lambda L + (1+6\lambda) - 4\lambda L^{-1} + \lambda L^{-2}\right]^{-1}, \end{aligned} \tag{3.6}$$

which turns out to be a symmetric infinite order low-pass filter that must be approximated for practical purposes. In this respect, one possibility would be to truncate its weights at some fixed lag n. However, in actual practice, an alternative procedure is as follows. First, stack the data into a column vector X. Second, define a matrix Γ that links the corresponding vector of trend components, X^g, to the data: $X = \Gamma X^g$. The matrix Γ is implied by equation (3.6), i.e.,

$$x_t = \lambda x_{t+2}^g - 4\lambda x_{t+1}^g + (1+6\lambda)x_t^g - 4\lambda x_{t-1}^g + \lambda x_{t-2}^g, \tag{3.7}$$

with some modifications near the endpoints:

$$x_1 = (1+\lambda)x_1^g - 2\lambda x_2^g + (1+\lambda)x_3^g,$$

and

$$x_2 = -2\lambda x_1^g + (1+5\lambda)x_2^g - 4\lambda x_3^g + \lambda x_4^g.$$

Comparable modifications must be made near the end of the sample. Finally, the growth HP filter is given by $X^g = \Gamma^{-1}X$. This alternative procedure has the attractive property that there is no loss of data from filtering.

The values of the smoothing parameter λ suggested by Kydland and Prescott (1990) are $\lambda = 1600$ for quarterly data and $\lambda = 400$ for annual data, since these are the values for the ratios of the volatility of the cycle component relative to the volatility of the growth component. However, for reasons that will become clear later on, we also analyze other values of λ.

3.2. **The Baxter and King filter.** Application of the HP filter is frequently ad hoc in the sense that the researcher only looks for a stationary business cycle component without explicitly specify the statistical characteristics of business cycles. By contrast, Baxter and King (1999) develop methods for measuring business cycles that require the researcher begin by specifying such as characteristics. Technically, they develop approximated *band-pass filters,* i.e., filters passing only frequencies between $\underline{\omega}$ and $\bar{\omega}$ of the corresponding spectrum. See, e.g., Christiano and Fitzgerald (1999). These band-pass filters turn out to be moving averages of infinite order. In fact, most of the filters used in macroeconomic time series are two-sided moving averages, say $h(L) = \sum_{j=-\infty}^{\infty} h_j L^j$, that, for practical purposes, have to be approximated by a finite two-sided moving average of order p, $h_p(L) = \sum_{j=-n}^{n} h_j L^j$. We will further specialize the analysis to symmetric moving averages where $h_j = h_{-j}$ for all j.

The implications of those two-sided filters are clearly seen in the frequency domain by looking at the corresponding *frequency response functions.* The frequency domain function of the two-sided $MA(\infty)$ equation is defined as

$$\beta(\omega) = \sum_{j=-\infty}^{\infty} b_j e^{-i\omega j}, \tag{3.8}$$

while for the finite two-sided $MA(n)$, the frequency domain function is

$$\alpha(\omega) = \sum_{j=-n}^{n} a_j e^{-i\omega j}. \tag{3.9}$$

King and Rebelo (1993) show that the optimal (in the sense of minimizing the mean squared error of the discrepancy $\delta(\omega) = \beta(\omega) - \alpha(\omega)$) approximating filter for given maximum lag length n, is constructed by simply setting $a_j = b_j$ for $j = 0, 1, ..., n$, and $a_j = 0$ for $j \geq n+1$. Moreover, they also prove that symmetric moving average with weights summing up to zero, i.e., $\sum_{j=-n}^{n} a_j = 0$, has trend elimination properties. In fact, these moving averages make stationary series containing quadratic deterministic trends or even I(1) or I(2) processes. Note that $\sum_{j=-n}^{n} a_j = 0$ if and only if $\alpha(0) = 0$.

These trend reducing filters are called *high-pass filters* because they pass components of the data with frequency larger than a predetermined value $\bar{\omega}$ close to zero, i.e, $\beta(\omega) = 0$ for $|\omega| < \bar{\omega}$ and $\beta(\omega) = 1$ for $|\omega| \geq \bar{\omega}$. By contrast, the so-called *low-pass filters* retain only slow-moving components of the data. An ideal symmetric low-pass filter passes only frequencies $-\underline{\omega} \leq \omega \leq \underline{\omega}$, having frequency-response function given by $\beta(\omega) = 1$ for $|\omega| \leq \underline{\omega}$ and $\beta(\omega) = 0$ for $|\omega| > \underline{\omega}$. Therefore, low frequencies (long term movements) remain unchanged while others are canceled

out. In terms of the finite symmetric $MA(n)$ filter, this means that low-pass filters must satisfy $\sum_{j=-n}^{n} a_j = 1$.

Letting $b(L) = \sum_{j=-\infty}^{\infty} b_j L^j$ denote the time-domain representation of the ideal infinite-order low-pass filter, King and Rebelo (1993) show that the filter weights b_j can be found by the inverse Fourier transform of the frequency response function, i.e.,

$$b_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} \beta(\omega) e^{i\omega j} d\omega,$$

yielding $b_0 = \underline{\omega}/\pi$ and $b_j = \sin(j\underline{\omega})/j\pi$, $j = 1, 2, ...$ The complementary high-pass filter has coefficients $(1 - b_0)$ at $j = 0$ and $-b_j$ for $j = 1, 2, ...$ While the weights tend to zero as j becomes large, notice that an infinite-order moving average is necessary to construct the ideal filter. This leads to consider approximation of the ideal filter with a finite moving average $a(L) = \sum_{j=-n}^{n} a_j L^j$.

Usually, the definition of business cycle is associated with the NBER business cycle duration as defined by Burns and Mitchell (1946) where $\underline{\omega}$ corresponds to 32 quarters (eight years) and $\bar{\omega}$ to six quarters (eighteen months). So, the trend component is obtained with a low-pass filter with cycles of duration of eight years or less, $\underline{\omega} = \pi/4$. This low-pass filter is what we call BK filter in the Monte Carlo simulations.

3.3. **Nonlinear filters.** The HP and BK filters are examples of linear filters. For completeness, herein we will also consider a class of nonlinear filters, called the *median filters* (Wen and Zeng, 1999), that has been proven very useful in recent years in signal processing in the field of electrical engineering. Median filters have two interesting properties: edge (sharp change) preservation and efficient noise attenuation with robustness against impulsive-type noise. Neither can be achieved by traditional linear filtering techniques. To compute the output of a median filter, an odd number of sample values are sorted, and the middle or median value is used as the filter output. If the filter length is $2n + 1$, the filtering procedure is denoted as

$$med\{x_{t-n}, x_{t-n+1}, ..., x_t, ..., x_{t+n}\}. \tag{3.10}$$

To be able to filter also the outmost input samples, where parts of the filter window fall outside the input signal, the time series is appended to the required size.

Frequency analysis and impulse response have no meaning in median filtering since the impulse response of a median filter is zero. Nonetheless, one very important property of the median filter is the so-called *root-convergence* property, namely, any finite sample time series contains a root signal set that is invariant to the median filtering. From an economic point of view, this invariant property

is of interest because it makes possible that possible structural shifts of economic fundamentals be not disturbed by the filtering operation. See Wen and Zeng (1999) for further details.

4. Filtering Unit Root Processes

From Section 2 we learn that testing for a unit root in the presence of systematic AO's is similar to testing for a unit root in the presence of MA errors. This might suggest the use of more general tests like the ADF and PP tests or the use of unit root statistics robust to MA errors, such as the modifications to the PP statistics proposed by Perron and Ng (1996). Indeed, Vogelsang (1999) reports some experimental evidence showing the good size and power properties of these modified PP statistics when there are outliers. In this sense, his paper and the present article can be viewed as complements rather than substitutes.

Herein we study two different possibilities to test for a unit root in the presence of AO's. First, we are interested in trying to approximate the temporary AO's by adding lags of the contaminated series x_t in equation (2.2), and therefore estimate the model

$$\Delta x_t = c + \rho x_{t-1} + \sum_{i=1}^{p} \phi_i \Delta x_{t-i} + \eta_t, \tag{4.1}$$

where the number of lags, p, chosen for instance by means of an order selection criteria, is such that η_t is a white noise process. Of course, using the t statistic of ρ in equation (4.1) to test for the null hypothesis $\rho = 0$ is nothing else that the customary ADF test.

Second, we want to analyze the impact of applying unit root tests to the trend component, x_t^g, of the observed series x_t. The basic goes as follows. Let x_t be split into the unobserved components x_t^g and x_t^c as in equation (3.1). By substituting in (2.6) we have

$$\Delta x_t^g = \rho x_{t-1}^g + \xi_t, \tag{4.2}$$

where $\xi_t = u_t - \Delta x_t^c = \varepsilon_t - \Delta\left(x_t^c - \theta\delta_t\right)$ under the null $\rho = 0$. Now, since the AO's generated by δ_t are transitory, they should be incorporated, to a large extent, into the cyclical component and not into the trend. Consequently, $(x_t^c - \theta\delta_t)$ would be an I(0) series free of AO's that can be well approximated by adding autoregressive terms of Δx_t^g in equation (4.2).

In order to obtain x_t^g, one can use (approximate) low-pass linear filters, i.e.,

$$x_t^g = a\left(L\right) x_t = \sum_{j=-n}^{n} a_j x_{t-j}, \tag{4.3}$$

with $\sum_{j=-n}^{n} a_j = 1$. There is no "best" choice of n. Increasing n leads to a better approximation to the ideal infinite-dimensional filter, but results in more lost observations at the beginning and end of the sample. Thus, the choice of n in a particular instance will depend on the length of the sample and the importance attached to obtaining an accurate approximation to the ideal filter. Nowadays, after some experimentation, Baxter and King (1999) suggest to use moving averages based on three years of past data and three years of future data, as well as the current observation, when working with both quarterly and annual time series. Moreover, following Baxter and King's suggestions, we require that the low-pass filter be *symmetric* ($a_j = a_{-j}$, $j = 1, 2, ..., n$) and *time invariant*, with coefficients not depending on the point in the sample. These requirements prevent from filters introducing *phase shift* (i.e., altering the timing relationships between series at some frequency) and dependencies on the length of the sample period.

Now, denote by $\widehat{\rho}^{g}_{AO}$ and $t_{\widehat{\rho}^{g}_{AO}}$ the Dickey-Fuller statistics when applied to equation (4.2), i.e.,

$$\widehat{\rho}^{g}_{AO} = \left(\sum_{t=1}^{T} x^{g}_{t-1} \Delta x^{g}_{t} \right) \Big/ \left(\sum_{t=1}^{T} \left(x^{g}_{t-1} \right)^2 \right), \tag{4.4}$$

and

$$t_{\widehat{\rho}^{g}_{AO}} = \widehat{\rho}^{g}_{AO} \Big/ \widehat{\sigma}^{g}_{x} \left(\sum_{t=1}^{T} \left(x^{g}_{t-1} \right)^2 \right)^{-1/2}, \tag{4.5}$$

with $\widehat{\sigma}^{g}_{x} = \left(T^{-1} \sum_{t=1}^{T} \left(\Delta x^{g}_{t} - \widehat{\rho}^{g}_{AO} x^{g}_{t-1} \right)^2 \right)^{1/2}$. Further, assume, without loss of generality, the initial condition $x^{g}_{0} = 0$. The limiting behavior of $\widehat{\rho}^{g}_{AO}$ and its t ratio can now be stated as follows.

Theorem 1. *Let $T > 2n + 1$ for fixed n. Then, as the sample size grows large, we have that*

$$T\widehat{\rho}^{g}_{AO} \Rightarrow \left(\int_0^1 W(r)\, dW(r) + \Phi \right) \left(\int_0^1 W^2(r)\, dr \right)^{-1} \tag{4.6}$$

$$t_{\widehat{\rho}^{g}_{AO}} \Rightarrow \Psi \left(\int_0^1 W(r)\, dW(r) + \Phi \right) \left(\int_0^1 W^2(r)\, dr \right)^{-1/2} \tag{4.7}$$

where

$$\Phi = \frac{1}{2} \sum_{j=-n}^{n} a_j (1 - a_j) - (\theta/\sigma_\varepsilon)^2 \pi \sum_{j=-n}^{n} a_j (a_j - a_{j-1})$$

$$\Psi = \left[\sum_{j=-n}^{n} a_j^2 + 2(\theta/\sigma_\varepsilon)^2 \pi \sum_{j=-n}^{n} a_j (a_j - a_{j-1}) \right]^{-1/2}.$$

Therefore, as in the unfiltered case, ρ is estimated (super-) consistently with the limiting distributions having nuisance parameters depending now on $\theta, \pi, \sigma^2_\varepsilon, n$ and the weights $\{a_j\}_{j=-n}^{n}$. When

$n = 0, a_0 = 1$, we recover expressions (2.9) and (2.10). Note that, if we choose a sequence of weights such that $\sum_{j=-n}^{n} a_j^2 \approx \sum_{j=-n}^{n} a_j a_{j-1} \approx 1$, then $\Phi \approx 0$ and $\Psi \approx 1$, independently of the AO's and for all n. In the particular case where $a_j = 1/(2n+1)$ for $j = -n, ..., n$, Φ and Ψ become

$$\Phi = \frac{n}{2n+1}, \tag{4.8}$$

$$\Psi = \left[\frac{1}{2n+1}\right]^{-1/2}. \tag{4.9}$$

Actually, this *uniform* or *moving average* low-pass filter,

$$x_t^g = \frac{1}{2n+1} \sum_{j=-n}^{n} x_{t-j}\,, \tag{4.10}$$

where the growth or trend component is defined as a two-sided or centered moving average, is a widely used method of detrending economic time series.

In Figure 1 we displayed the empirical density functions for the Dickey-Fuller (no constant, no lags) t test using equation (2.5) and equation (4.5) with the HP ($\lambda = 10$), the BK and the median filters, for a sample size of $T = 1000$ and for $\pi = 0.1, \theta = 16$, i.e, in the presence of large and frequent AO's. For comparison purposes, we have also included the customary Dickey-Fuller distribution without AO's. It is clear the huge shift to the left of the Dickey-Fuller distribution in the presence of AO's, as well as the apparent gains from using the different filtered versions of the Dickey-Fuller distribution.

[Figure 1 about here.]

5. Experimental Evidence

Herein we provide some Monte Carlo evidence on the numerical implications of our analytical findings. The data is generated according to the following data generating process (DGP),

$$\begin{aligned} \Delta y_t =& \rho y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim N(0,1)\,, \\ x_t =& y_t + \theta \delta_t, \end{aligned}$$

with $\pi = 0, 0.01, 0.05, 0.10$, $\theta = 1, 6, 16$ and $T = 100, 200, 500, 1000$. We considered that at most 10% of the series y_t is contaminated, which is standard in this kind of studies, see, e.g., Franses and Haldrup (1994). However, they consider breaks of sizes 3, 4 and 5, while we consider more abrupt ones. The full factorial design means that we have 48 cases for the Monte Carlo experiment. In order to obtain the critical values, we simulate the DGP under $H_0 : \rho = 0$, and consider the lower 5% tail of the ordered Dickey-Fuller t-statistics $t_{\widehat{\rho}_{AO}}$ and $t_{\widehat{\rho}_{AO}^g}$ as the critical values of interest. The

power of the tests has been obtained by simulating the DGP under $H_1 : \rho = -0.2$, and computing the percentage of rejections using the critical values previously obtained. For all experiments, 10,000 replications of the DGP were used.

In our experiments, HP stands for the HP filter, BK stands for the low-pass Baxter and King filter, U stands for the uniform or moving average filter, and MD stands for the median filter. With respect to the HP filter, we choose different values of the smoothing parameter λ, namely, $\lambda = 10, 100, 400$ and 1600. The window parameter of the BK, U and MD filters were set at $n = 3$, implying a window size of $2n + 1 = 7$. The number of lags (p) of the ADF test were chosen by means of the SIC criterion. The results are reported in Tables 1 to 5.

[Table 1 about here.]

[Table 2 about here.]

Consider first the effects of AO's on the standard Dickey-Fuller tests. The estimated regressions are given by

$$\Delta x_t = c + \rho x_{t-1} + \eta_t, \tag{5.1}$$

and

$$\Delta x_t = c + \rho x_{t-1} + \sum_{i=1}^{p} \phi_i \Delta x_{t-i} + \eta_t. \tag{5.2}$$

From Table 1 we learn that the critical values of $t_{\widehat{\rho}_{AO}}$ are very unstable in the case of model (5.1), ranging from -2.85 ($\pi = 0$) to -17.319 ($\pi = 0.1$) for $T = 1000$ and $\theta = 16$. With respect to model (5.2), we can see in Table 1 that in the case of no AO's ($\pi = 0$), critical values are virtually the same as in the case of no lags, which indicates that the lag order selection criterion works well. Nonetheless, the critical values are still very unstable, ranging from -2.9 to -11, for $T < 500$, but more stable for $T > 500$, ranging from -2.85 to -6. In turn, the size adjusted power of $t_{\widehat{\rho}_{AO}}$ against $H_1 : \rho = -0.2$ is good for $T > 100$ in both cases as we can see from Table 2.

Consider now the behavior of $t_{\widehat{\rho}^g_{AO}}$. The estimated model is

$$\Delta x^g_t = c + \rho x^g_{t-1} + \sum_{i=1}^{p} \phi_i \Delta x^g_{t-i} + \xi_t. \tag{5.3}$$

[Table 3 about here.]

[Table 4 about here.]

As we can see in Table 3, the critical values are very stable for sample sizes as small as 100 in the case of the HP filter and 200 when using the BK filter. For instance, the critical values range from -2.8 to -3.22 in this latter case. As regards the HP filter, the critical values obtained from

the different values of λ are not very different, specially for medium to large sample sizes. Critical values are also fairly stable with respect to π, θ and T when using the U and MD filters. Overall, from our simulations it shows up that the most stable critical values are obtained when using the MD filter. It is interesting to note that when there are no AO's, the detrending procedure or the value of λ does not affect, to a large extent, the critical value. In fact, in that case the critical values obtained with the trend component are similar to those obtained with the observed series. On the other hand, (size adjusted) power is high in all cases for medium to large sample sizes. See Table 4. It is not as high as the unfiltered case for small sample sizes, which is an expected result (cf. Ghysels and Perron, 1993).

Finally, it is also of interest to analyze the robustness of the critical values obtained so far in Tables 1 and 3. In particular, in Table 5 we present some results concerning the empirical sizes obtained from the different testing procedures when using the critical values for the $\pi = 0$ case. It is particularly noticeable the size distortions of the ADF test in model (5.2) specially for $\theta = 16$, and the good performance of the MD filter for any value of π, θ and T included in the Monte Carlo design.

[Table 5 about here.]

Figures 2 and 3 provide the graphical counterpart to the previous comments on the stability of the critical values. Note in particular that the distribution of the MD filtered version of the ADF test is virtually the same with and without AO's (Figures 2.e and 3.e), in contrast with the unfiltered version of the ADF test (Figures 2.a and 3.a). On the other hand, the size problems of the BK filtered version of the ADF test come up after looking at the different left tails for $T = 100$ (Figure 2.d) and $T = 200$ (Figure 3.d). All these findings are in full agreement with the results obtained in Table 5.

[Figure 2 about here.]

[Figure 3 about here.]

6. Empirical Application

The above procedures were applied to the US/Finland exchange rate series (in logarithms) based on the consumer price index (CPI). See Figure 4. The data is annual and spans the years 1900-1988. This series was analyzed by Perron and Vogelsang (1992) in the context of testing for a unit root in the presence of a changing mean at an unknown date. They did not consider the possibility of AO's in the data. By conducting an ADF test as in model (5.2) with $p = 1$, they found a t ratio of -5.74 (5% critical value =-2.89). Thus, the unit root hypothesis was rejected for this series.

[Figure 4 about here.]

Franses and Haldrup (1994) (FH94, henceforth) pointed out that the series may contain additive outlying observations so that the results by Perron and Vogelsang (1992) may be biased towards false rejection of the unit root hypothesis. Using the methods to detect outliers of Chen and Liu (1993) as implemented in the TRAMO Program (Gómez and Maravall, 1996), FH94 found evidence of AO's in the series.

To remove the influence of outliers, FH94 suggested to conduct the model

$$\Delta x_t = c + \rho x_{t-1} + \sum_{i=1}^{p} \phi_i \Delta x_{t-i} + \sum_{i=0}^{p+1} \omega_i D^i_{t-i} + \eta_t, \tag{6.1}$$

where we test whether $\rho = 0$ by using the ADF t ratio. D^i indicates impulse dummy variable taking the value 1 at the time the outlier occurrence takes place and 0 otherwise.

FH94 reported the following (most significant) AO's: 1917-1919, 1932, 1933, 1945, 1949 and 1950. This means an empirical percentage of AO's of about $\widehat{\pi} = 0.09$. The ADF t ratio in model (6.1) with $p = 2$ was -2.65 which fails to reject the unit root hypothesis at the 5% level.

On the other hand, Vogelsang (1999) (V99, henceforth) proposes a second procedure for detecting AO's which is related to the approach suggested by FH94. The procedure is based on the following regression estimated by OLS:

$$x_t = c + \alpha D^i_t + \eta_t. \tag{6.2}$$

Let t^i_α denote the t statistic for testing $\alpha = 0$ in (6.2). Then, V99 suggests to test for the presence of AO's using $\tau = \sup_{i=1,2,...,T} \left| t^i_\alpha \right|$. The asymptotic distribution of τ is nonstandard but asymptotic critical values has been tabulated by V99.

V99 applied the τ statistic to the CPI-based data in an iterative manner detecting the following AO's: 1917, 1918, 1919, 1921 and 1932. To remove the influence of the outliers on the unit root test, regression (6.1) was used with $p = 0$. The ADF t ratio was -3.51, rejecting the unit root hypothesis at the 5% level in contrast with FH94's results.

These differences clearly illustrate the sensitivity of unit root tests to the choice of dummy variables and choices of lag length and led V99 to suggest the use of the modifications to the Phillips-Perron statistics proposed by Perron and Ng (1996) to the data without searching for or removing outliers. For the CPI-based series the modified PP statistics, MZ_α and MZ_t, defined in (6.3)–(6.5), reject the unit root hypothesis at any reasonable significance level.

Consider now the application of $t_{\widehat{\rho}^g_{AO}}$ in model (5.3) to the CPI-based series. In view of the simulations in Section 5, we only report the results obtained with the BK, the HP10 and the MD filters.

[Table 6 about here.]

In Table 6, the column CV_f stands for the 5% critical values of $t_{\widehat{\rho}^g_{AO}}$ in model (5.3) assuming $\pi = 0$ and a sample size $T = 89$, the number of observations of the CPI-based real exchange rate. Under this assumption, only the BK filter suffers from size distortions, see Table 5. $CV_f^\star$ is the critical value of the tests assuming that the series includes outliers at the same positions and with the sizes reported by V99. From Table 6, it turns out that we fail to reject the unit root hypothesis at the 5% significance level with any of the filter procedures, in contrast with V99's findings and in accordance with FH94's claims. However, with the BK and MD filters, the p–values are around 0.07 and 0.08 when using the corresponding CV_f, and 0.07 and 0.09 when using $CV_f^\star$, respectively.

In light of Table 4, a possible explanation for such discrepancies could be the small power of the filtered unit root tests in small samples. Accordingly, Figures 5 and 6 plot the (size adjusted) power of the ADF tests for the original and the filtered data with ($\pi = 0.1, \theta = 16$) and without AO's, and for $T = 100$ (Figure 5) and $T = 200$ (Figure 6). For completeness, we have also included the MZ_t statistic proposed by Perron and Ng (1996), defined as

$$MZ_\alpha = \frac{T^{-1}\tilde{x}_T^2 - s^2}{2T^{-2}\sum_{t=1}^T \tilde{x}_t^2} \tag{6.3}$$

$$MSB = \sqrt{T^{-2}\frac{\sum_{t=1}^T \tilde{x}_{t-1}^2}{s^2}} \tag{6.4}$$

$$MZ_t = MSB \times MZ_\alpha \tag{6.5}$$

where $\tilde{x}_t$ is the GLS–demeaned series, and we take s^2_{AR} as the estimator of s^2 (see Ng and Perron, 1998, Vogelsang, 1999).

$$s^2_{AR} = \frac{s^2_{ek}}{(1-\hat{\phi}(1))^2}, \tag{6.6}$$

and $s^2_{ek} = T^{-1}\sum_{t=k+1}^T \hat{e}^2_{tk}$, $\hat{\phi}(1) = \sum_{j=1}^k \hat{\phi}_j$, where $\hat{\phi}_j$ are $\{\hat{e}_{jk}\}$ obtained from the autoregression

$$\Delta\tilde{x}_t = b\tilde{x}_{t-1} + \sum_{j=1}^k \phi_j \Delta\tilde{x}_{t-j} + e_{jk}.$$

The order of this autoregregression, k, is chose by the MIC criterion as proposed by Ng and Perron (1998).

[Figure 5 about here.]

[Figure 6 about here.]

A first remarkable conclusion of Figures 5 and 6 is the small power of the MZ_t statistic for non-local alternatives when no outliers are present in the data, in contrast with the filtered unit root tests (Figures 5.a and 6.a). In the presence of AO's and for $T = 100$ (Figure 5.b) the power of the MD and HP10 filtered unit root tests are uniformly smaller than the power of the rest of the ADF tests. The highest power corresponds to the MZ_t statistic for local alternatives and to the ADF and BK filtered tests for non-local alternatives. For $T = 200$ (Figure 6.b) the power of the MZ_t statistic is not the highest one even for local alternatives. In fact, it has the smallest power for non-local alternatives.

From all the preceding analysis, thus, it turns out that in order to test for a unit root in the presence of AO's one can propose to use the MZ_t statistic along with size adjusted versions of the original ADF and BK filtered ADF tests. However, we know that the latter tests have size problems in small to medium sample sizes.

To correct these size problems, Arranz (2001) has recently considered the use of bootstrap resampling techniques. In particular, he shows how the small sample size distortions of the BK filtered ADF test can be corrected to a large extent by means of the following bootstrap procedure:

(1) Estimate the *restricted* model

$$\Delta x_t^g = c + \sum_{i=1}^{p} \phi_i \Delta x_{t-i}^g + \nu_t.$$

(2) Resample residuals ν_t^* and build $x_t^{g^*}$ using the model

$$\Delta x_t^{g^*} = c + \sum_{i=1}^{p} \phi_i \Delta x_{t-i}^{g^*} + \nu_t^*.$$

The size of the reconstructed series is $m = T^{3/4}$ in order to solve the problem of the distribution discontinuity (see Datta, 1996).

(3) Estimate the model

$$\Delta x_t^{g^*} = c + \rho x_{t-1}^{g^*} + \sum_{i=1}^{p} \phi_i \Delta x_{t-i}^{g^*} + \xi_t,$$

obtaining $\widehat{\rho}^*$ and the corresponding customary t–statistic, t^*. Repeat steps $(2) - (3)$ NB times, where NB indicates the number of bootstrap resamples. The bootstrap critical value is obtained by looking at the 5% lower tail of the empirical distribution of t^*.

The bootstrap critical values ($NB = 10,000$) for the CPI-based US/Finland exchange rate series and for the different filtering procedures are collected in Table 6 , column CV_b. Note, as expected,

the similarity of the Monte Carlo and the bootstrap critical values for the MD filtered unit root tests. More interestingly, note that now the BK filtered ADF test rejects the unit root null hypothesis according with the bootstrap 5% critical value (the p–value is 0.045), in agreement with V99. In the case of the MD filtered case, the p–value of the boostrap test is 0.0762.

Finally, the trend component of the CPI-based US/Finland exchange rate series with respect to the HP10, the BK and the median filters are graphed in Figure 7. Note that the linear filters are not so efficient as the median filter in retaining the signal component of the CPI-based series in terms of effectively capturing sharp changes in the trend component.

[Figure 7 about here.]

7. Conclusions

Time series observations are often influenced by interruptive events such as strikes, outbreaks of wars, sudden political or economic crises, or even unnoticed errors of typing and recording. The consequences of these interruptive events create spurious observations, which are inconsistent with the rest of the series. Such observations are usually referred to as outliers.

In this paper we propose two different ways of addressing the question of testing for unit roots in the presence of additive outliers. The first approach considers the possibility of adding extra dynamic terms to the Dickey-Fuller equation. By means of Monte Carlo simulations we show that the critical values are more stable than the ones obtained without any dynamic term as suggested by the data generating process. Nonetheless, the stability (robustness) of the critical values is further increased by our second approach which suggest to run the Dickey-Fuller tests on the trend or permanent component of the series, obtained by either linear or nonlinear filtering techniques. Simulations show that this procedure is robust to additive outliers in terms of size and power. Specially remarkable is the robustness of the nonlinear median filter.

In order to test for a unit root in the presence of additive outliers, our analysis suggests the use of the MZ_t statistic for local alternatives, the bootstrap version of the BK filtered ADF test in small samples and for global alternatives, and the normal (or bootstrap) version of the BK filtered ADF test in medium to large samples and global alternatives.

On the other hand, it appears that the median filter can be more useful than the linear filters in obtaining the trend-cycle decompositions of the underlying series because of its greater capability of tracing sharp changes in removing the stochastic trend component of a time series.

Appendix A. Proof of Theorem 1

Denote by $a(L)$ a general symmetric low-pass filter of order n, i.e., a linear filter $a(L) = \sum_{j=-n}^{n} a_j L^j$ for which $a(L) = a(L^{-1})$ and $a(1) = 1$. Let $x_t^g = a(L)x_t$ denote the filtered version of x_t in equation (2.2), so that $x_t^g = x_{t-1}^g + \xi_t, \xi_t = a(L)u_t$ under the unit root hypothesis, with $u_t = \varepsilon_t + \theta\Delta\delta_t$. Let $\sigma_u^2 = \lim_{T\to\infty} T^{-1}E\left(S_{T,u}^2\right)$ and $\sigma_\xi^2 = \lim_{T\to\infty} T^{-1}E\left(S_{T,\xi}^2\right)$, where $S_{t,u} = \sum_{j=1}^{t} u_j$ and $S_{t,\xi} = \sum_{j=1}^{t} \xi_j$, $t = 1, 2, ..., T$. Note that both u_t and ξ_t satisfy Phillips' (1987) mixing conditions for the application of a functional central limit theorem to their partial sums, yielding

$$T\widehat{\rho}_{AO}^g \Rightarrow \frac{\int_0^1 W(r)\, dW(r) + \lambda_\xi/\sigma_\xi^2}{\int_0^1 W^2(r)\, dr} \tag{A.1}$$

and

$$t_{\widehat{\rho}_{AO}^g} \Rightarrow \left(\frac{\sigma_\xi}{s_\xi}\right) \frac{\int_0^1 W(r)\, dW(r) + \lambda_\xi/\sigma_\xi^2}{\left(\int_0^1 W^2(r)\, dr\right)^{1/2}}, \tag{A.2}$$

where $\lambda_\xi = 1/2\left(\sigma_\xi^2 - s_\xi^2\right)$ and $s_\xi^2 = var(\xi_t)$. From Franses and Haldrup (1994) we have that $\sigma_u^2 = \sigma_\varepsilon^2$,which, in turn, implies that $\sigma_\xi^2 = \sigma_u^2$ using the fact that $a(1) = 1$. The variances of the processes u_t and ξ_t are however, different. Franses and Haldrup prove that $s_u^2 = var(u_t) = \sigma_\varepsilon^2\left(1 + 2(\theta/\sigma_\varepsilon)^2\pi\right)$. By contrast,

$$s_\xi^2 = var(a(L)u_t) = \sum_{j=-n}^{n}\sum_{i=-n}^{n} a_j a_i cov_u(j-i). \tag{A.3}$$

As noted in the main text, u_t is an $MA(1)$ process with autocovariance function

$$cov_u(k) = \begin{cases} \sigma_\varepsilon^2\left(1 + 2\left(\theta/\sigma_\varepsilon\right)^2\pi\right) & k = 0 \\ -\theta^2\pi & k = 1 \\ 0 & k > 1 \end{cases} \tag{A.4}$$

Therefore, using (A.4) and the properties of $a(L)$, it can be proved after some algebra that

$$s_\xi^2 = s_u^2 \sum_{j=-n}^{n} a_j^2 - 2\theta^2\pi \sum_{j=-n}^{n} a_j a_{j-1}, \tag{A.5}$$

which, in turn, implies

$$\lambda_\xi/\sigma_\xi^2 = \frac{1}{2}\left[1 - \sum_{j=-n}^{n} a_j^2 - 2\left(\theta/\sigma_\varepsilon\right)^2 \pi \sum_{j=-n}^{n} a_j^2 + 2\left(\theta/\sigma_\varepsilon\right)^2 \pi \sum_{j=-n}^{n} a_j a_{j-1}\right] = \Phi, \tag{A.6}$$

and

$$\left(\frac{\sigma_\xi}{s_\xi}\right)^2 = \left[\sum_{j=-n}^{n} a_j^2 + 2\left(\theta/\sigma_\varepsilon\right)^2 \pi \sum_{j=-n}^{n} a_j\left(a_j - a_{j-1}\right)\right]^{-1} = \Psi^2. \tag{A.7}$$

The theorem is finally proved by substituting (A.6) and (A.7) in expressions (A.1) and (A.2).

References

Arranz, M. (2001), Unit Root and Cointegration with Additive Outliers and Structural Breaks: Small Sample Solutions, PhD thesis, Universidad Complutense de Madrid.

Arranz, M. and A. Escribano (1998), Unit root tests on the trend components are robust to additive outliers. Unpublished manuscript.

Baxter, M. and R. King (1999), 'Measuring business cycles: Approximate band–pass filters for economic time series', *Review of Economics and Statistics* **81**, 575–593.

Burns, A. and W. Mitchell (1946), *Measuring Business Cycles*, NBER.

Canova, F. (1998), 'Detrending and business cycle facts', *Journal of Monetary Economics* **41**, 475–512.

Chang, I., G. Tsiao and C. Chen (1988), 'Estimation of time series parameters in the presence of outliers', *Technometrics* **30**, 193–204.

Chen, C. and L. Liu (1993), 'Joint estimation of model parameters and outlier effects in time series', *Journal of the American Statistical Association* **88**, 284–297.

Christiano, L. (1992), 'Searching for a break in GNP', *Journal of Business and Economic Statistics* **10**, 237–250.

Christiano, L. and T. Fitzgerald (1999), The band pass filter, Working Paper 7257, NBER.

Cogley, F. and J. Nason (1995), 'Effects of the Hodrick-Prescott filter on trend and difference stationary time series: Implications for business cycle research', *Journal of Economic Dynamics and Control* **19**, 253–278.

Datta, S. (1996), 'On asymptotic properties of bootstrap for AR(1) processes', *Journal of Statistical Planning and Inference* **53**, 361–374.

Ehglen, J . (1998), 'Distortionary effects of the optimal Hodrick–Prescott filter', *Economics Letters* **61**, 345–349.

Franses, P. and N. Haldrup (1994), 'The effects of additive outliers on unit roots and cointegration', *Journal of Business and Economic Statistics* **12**, 471–478.

Ghysels, E. and P. Perron (1993), 'The effect of seasonal adjustment filters on tests for a unit root', *Journal of Econometrics* **55**, 57–98.

Gómez, V. and A. Maravall (1996), Programs TRAMO and SEATS, Working Paper 9628, Bank of Spain.

Guay, A. and P. St-Amant (1997), Do the Hodrick-Prescott and Baxter-King filters provide a good approximation of business cycles?, Cahiers de Recherche 53, CREFE.

Harvey, A. C. and A. Jaeger (1993), 'Detrending, business cycles facts and the business cycle', *Journal of Applied Econometrics* **8**, 231–47.

Hodrick, R. and E. Prescott (1997), 'Post–war U.S. business cycle: An empirical investigation', *Journal of Money, Credit and Banking* **29**, 1–16.

Kaiser, R. (1998), Detection and estimation of structural changes and outliers, Working Paper 98–23, Universidad Carlos III de Madrid.

Kaiser, R. and A. Maravall (1999), 'Estimation of the business cycle: A modified Hodrick–Prescott filter', *Spanish Economic Review* **1**, 175–206.

King, R. and S. Rebelo (1993), 'Low frequency filtering and real business cycles', *Journal of Economic Dynamics and Control* **17**, 207–31.

Kydland, F. and E. C. Prescott (1990), 'Business cycles: Real facts and a monetary myth', *Federal Reserve Bank of Minneapolis Quarterly Review* **14**, 3–18.

Leybourne, S., T. Mills and P. Newbold (1998), 'Spurious rejections by Dickey–Fuller tests in the presence of a break under the null', *Journal of Econometrics* **87**, 191–203.

Lucas, A. (1995*a*), 'An outlier robust unit root tests with an application to the extended Nelson–Plosser data', *Journal of Econometrics* **66**, 153–173.

Lucas, A. (1995*b*), 'Unit roots tests based on M estimators', *Econometric Theory* **11**, 331–346.

Maddala, G. and Y. Yin (1996), Outliers, unit roots and robust estimation of non–stationary time series, *in* G.Maddala and C.Rao, eds, 'Handbook of Statistics', Vol. 15, North–Holland.

Martin, R. and V. Yohai (1986), 'Influence functionals for time series', *The Annals of Statistics* **14**, 781–818.

Ng, S. and P. Perron (1998), Lag length selection and the construction of unit root tests with good size and power, Working Paper 319, Dpt. of Economics, Boston College.

Perron, P. (1989), 'The great crash, the oil price shock and the unit root hypothesis', *Econometrica* **57**, 1361–1401.

Perron, P. and S. Ng (1996), 'Useful modifications to some unit root tests with dependent errors and their local asymptotic properties', *Review of Economic Studies* **63**, 435–465.

Perron, P. and T. Vogelsang (1992), 'Nonstationarity and level shifts with an application to purchasing power parity', *Journal of Business and Economic Statistics* **10**, 301–320.

Phillips, P. (1987), 'Time series regression with a unit root', *Econometrica* **58**, 277–301.

Shin, D., S. Sarkar and J. Lee (1996), 'Unit root tests for time series with outliers', *Stochastic and Probability Letters* **30**, 189–197.

Tsay, R. (1986), 'Time series model specification in the presence of outliers', *Journal of the American Statistical Association* **81**, 132–141.

Vogelsang, T. (1999), 'Two simple procedures for testing for a unit root when there are additive outliers', *Journal of Time Series Analysis* **20**, 173–192.

Wen, Y. and B. Zeng (1999), 'A simple nonlinear filter for economic time series analysis', *Economics Letters* **64**, 151–160.

Yin, Y. and G. Maddala (1997), The effects of different outliers on unit root tests, *in* T.Fornby and R.Hill, eds, 'Advances in Econometrics', Vol. 13, JAI Press.

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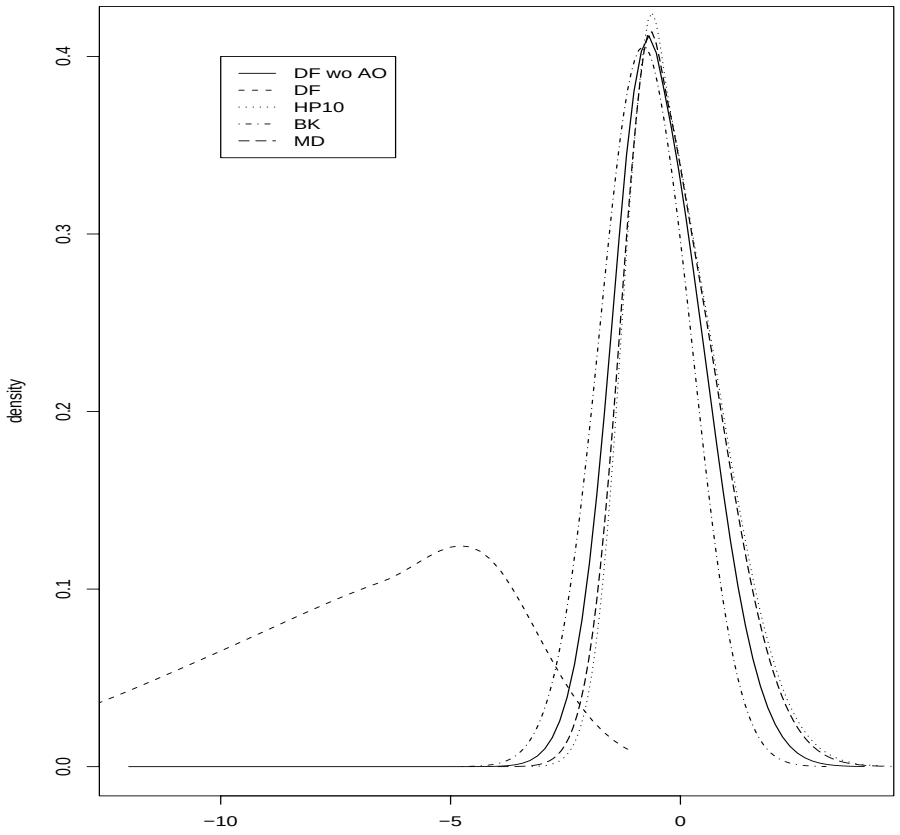

FIGURE 1. Density Estimation of the t DF test (no constant, no lags) under the unit root null hypothesis, based on 50000 replications. $T = 1000$. $\pi = 0.1$ and $\theta = 16$. DF wo AO and DF refer to the distributions of the t–stat in expressions (2.5) and (2.8), respectively; HP10, BK and MD refer to the distribution of the t–stat in expression (4.5), using the Hodrick–Prescott (with $\lambda = 10$), the Baxter–King and the median filters

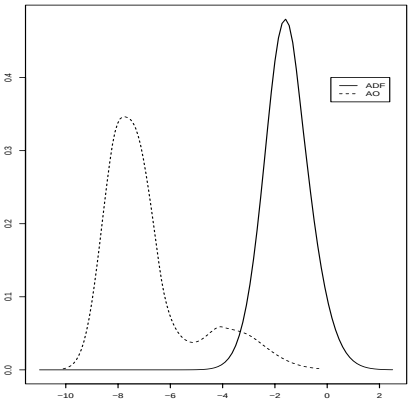

Figure 2.a: ADF

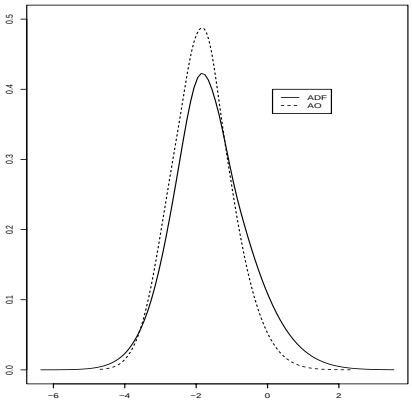

Figure 2.b: HP10

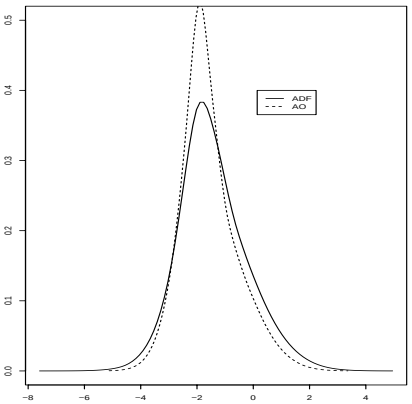

Figure 2.c: HP100

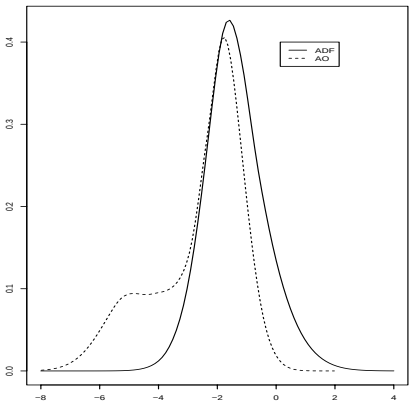

Figure 2.d: BK

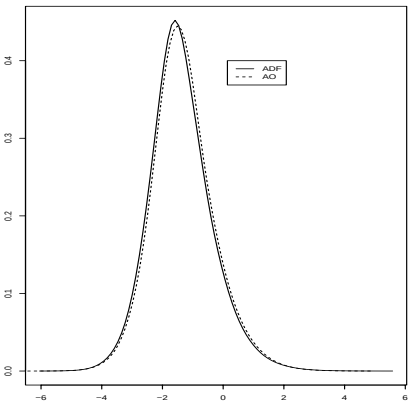

Figure 2.e: MD

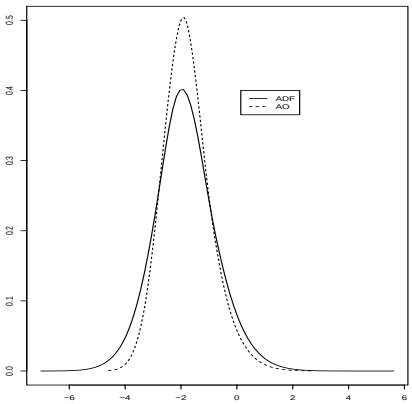

Figure 2.f: U

FIGURE 2. Density estimation of the ADF test and the filtered unit root tests with and without AO's. $T = 100$, $\pi = 0.1$, $\theta = 16$.

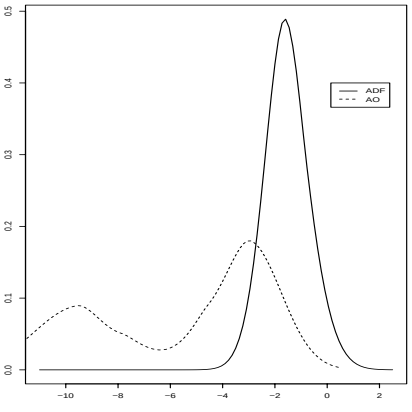

Figure 3.a: ADF

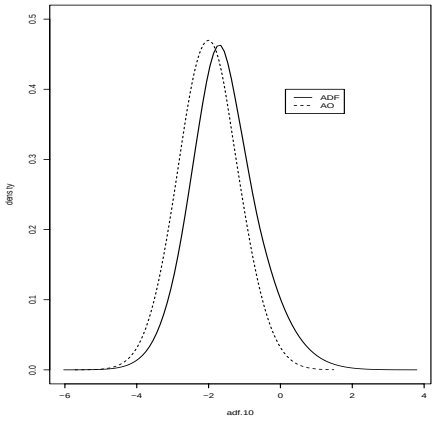

Figure 3.b: HP10

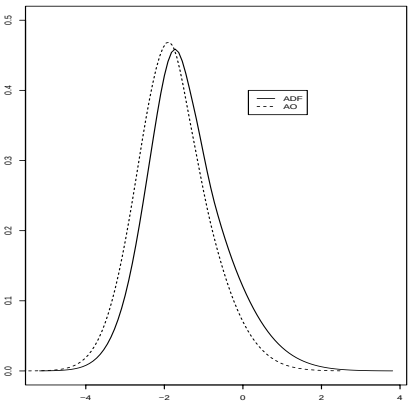

Figure 3.c: HP100

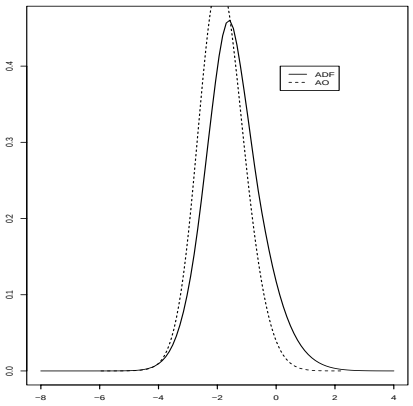

Figure 3.d: BK

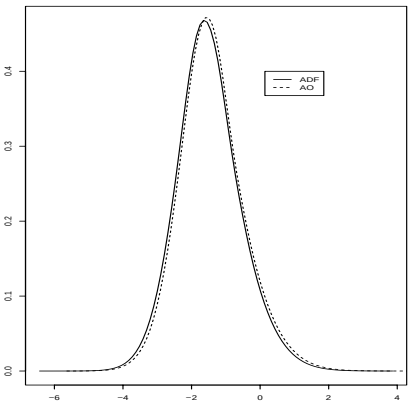

Figure 3.e: MD

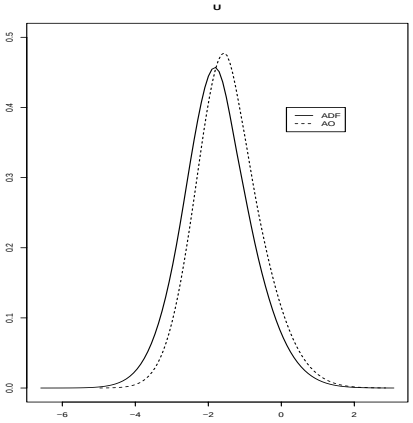

Figure 3.f: U

FIGURE 3. Density estimation of the ADF test and the filtered unit root tests with and without AO's. $T = 200$, $\pi = 0.1$, $\theta = 16$.

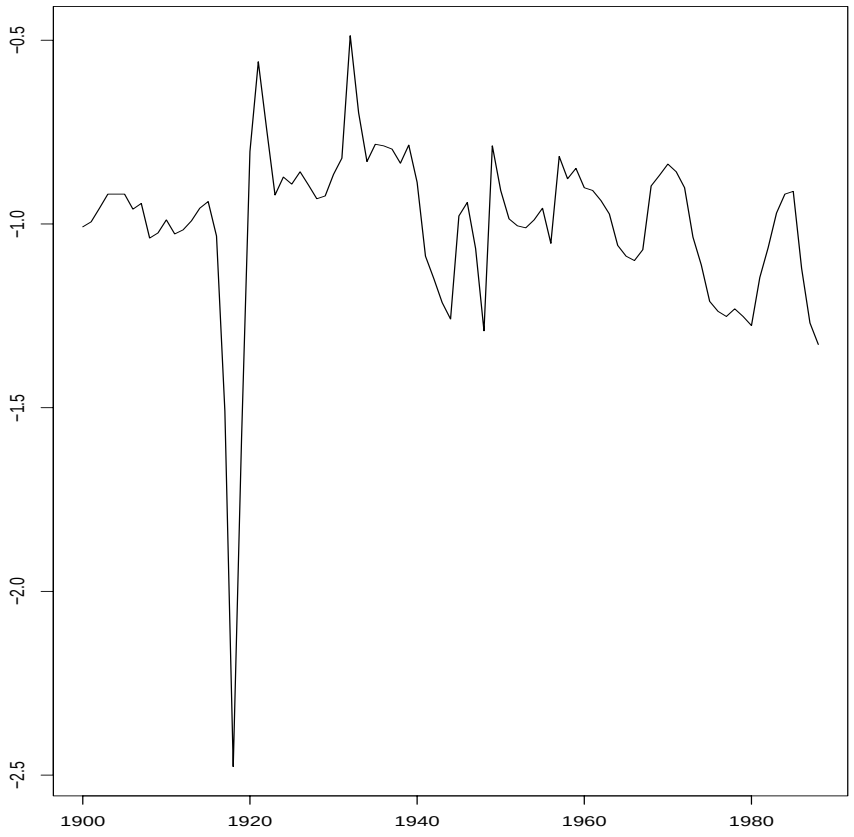

FIGURE 4. Log of CPI–based US/Finland real exchange rate, 1980–88

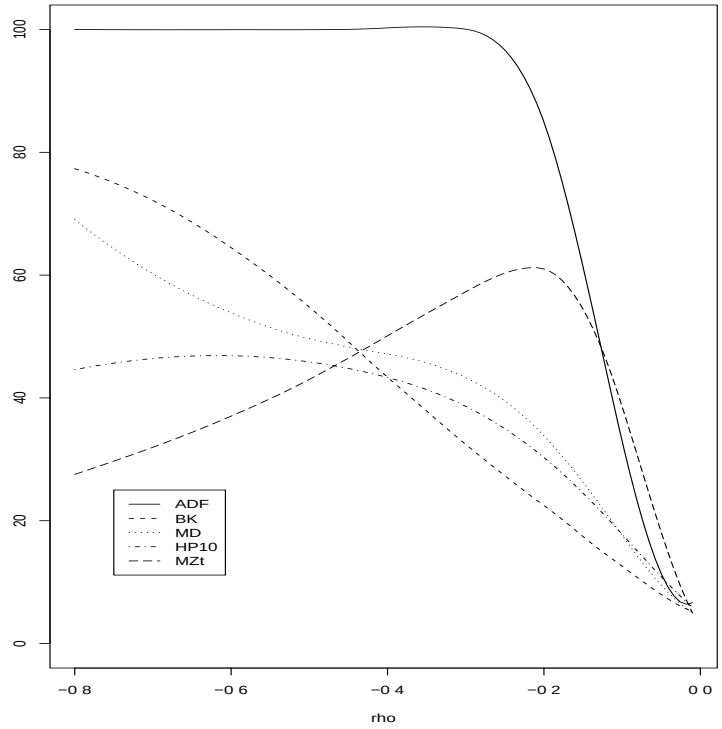

Figure 5.a: No outliers

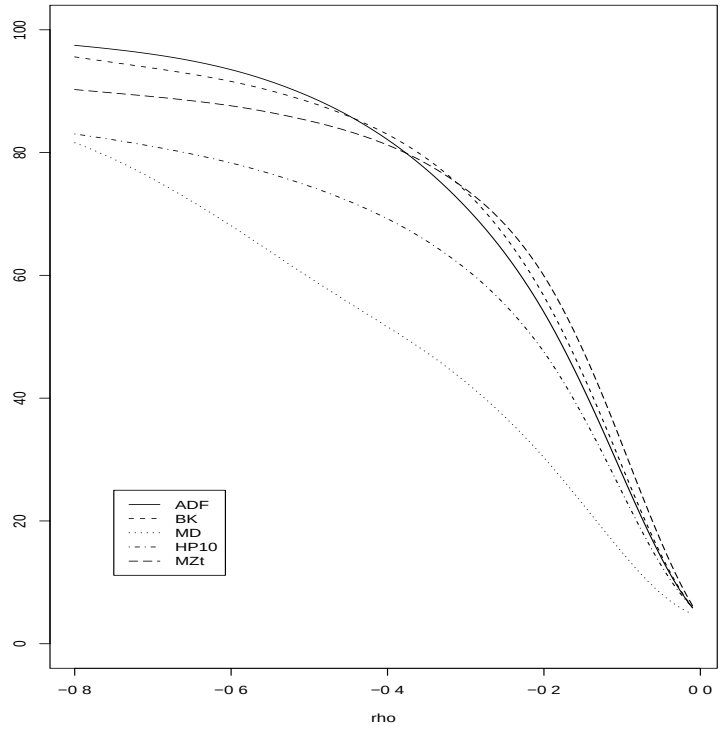

Figure 5.b: AO's, $\pi = 0.1$, $\theta = 16$

FIGURE 5. Size–adjusted power of the ADF, MZ_t and filtered unit root tests with and without additive outliers, $T = 100$, $\pi = 0.1$ and $\theta = 16$

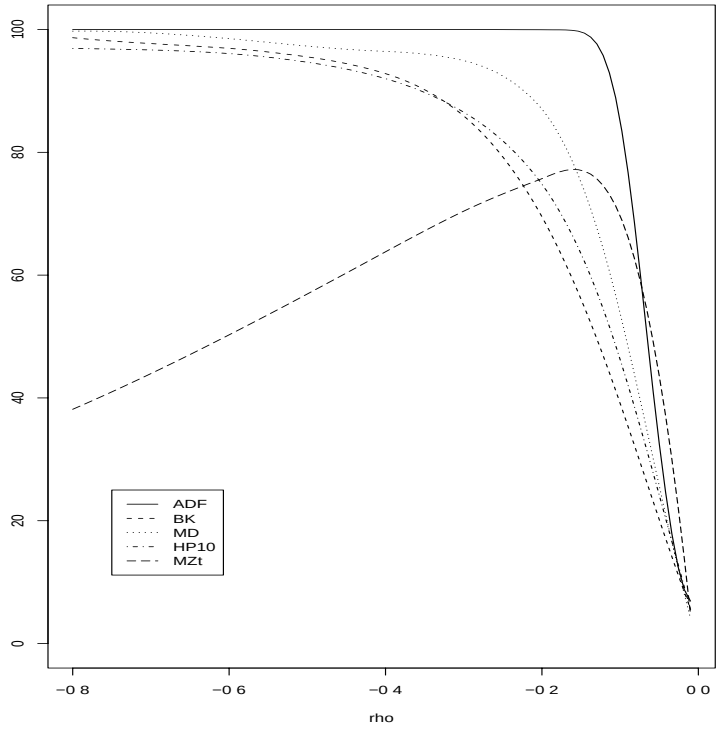

Figure 6.a: No outliers

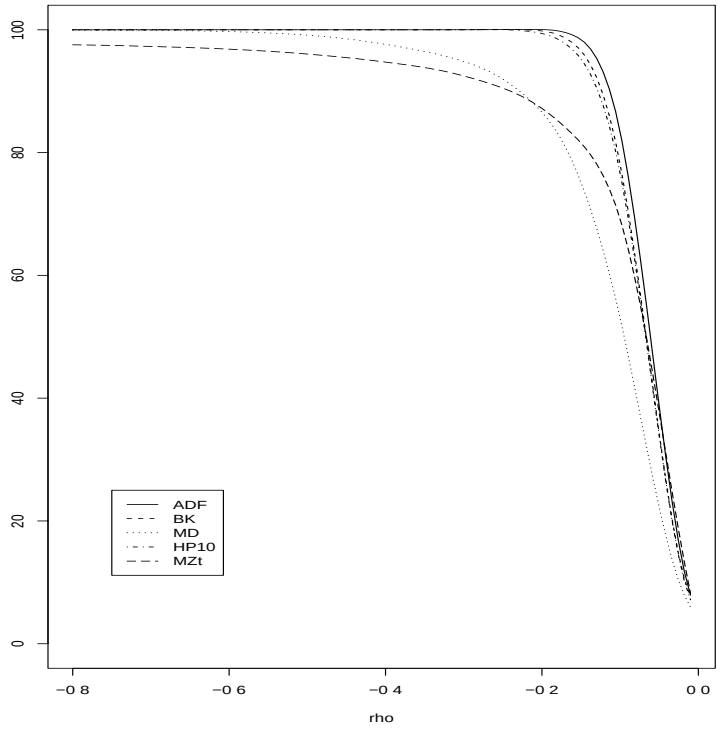

Figure 6.b: AO, $\pi = 0.1$, $s = 16$

FIGURE 6. Size–adjusted power of the ADF, MZ_t and filtered unit root tests with and without additive outliers, $T = 200$, $\pi = 0.1$ and $\theta = 16$

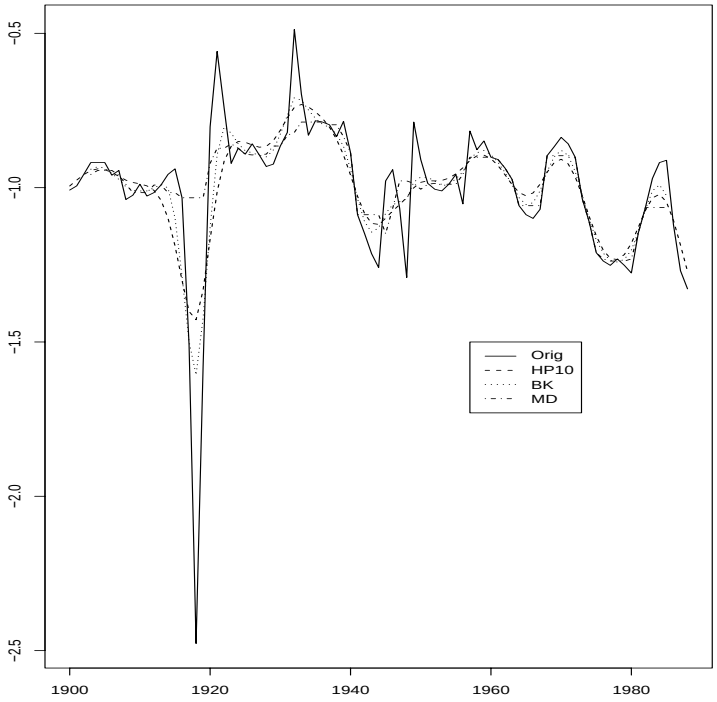

Figure 7.a: Trend

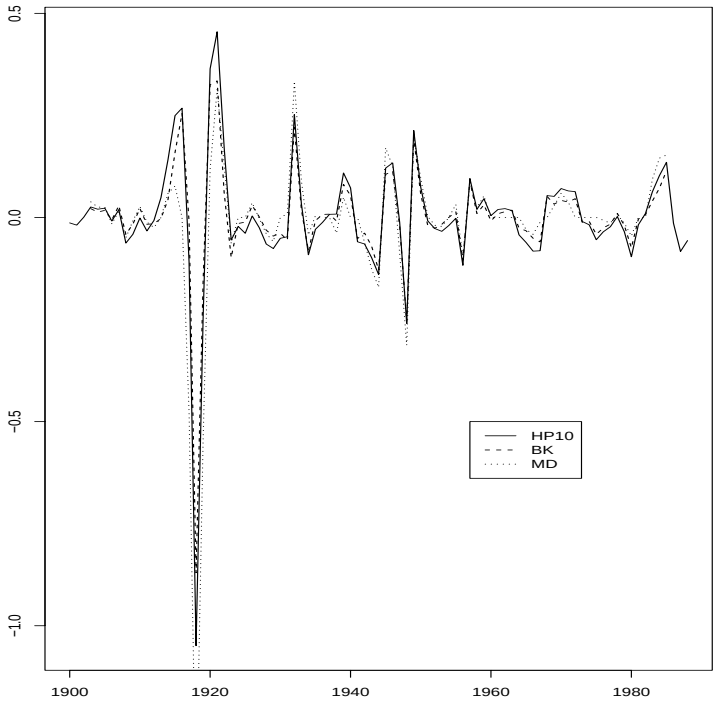

Figure 7.b: Cycle

FIGURE 7. Log of CPI–based US/Finland real exchange rates. Trend and cycle components

		DF				ADF			
π	θ	T=100	T=200	T =500	T =1000	T=100	T=200	T =500	T =1000
0		-2.899	-2.879	-2.866	-2.850	-2.920	-2.889	-2.865	-2.850
	1	-2.927	-2.887	-2.870	-2.873	-2.942	-2.891	-2.870	-2.873
0.01	6	-3.546	-3.227	-3.136	-3.642	-3.429	-3.111	-3.002	-2.975
	16	-5.814	-4.870	-4.586	-6.902	-5.318	-3.388	-3.123	-3.168
	1	-3.057	-2.979	-2.941	-2.952	-3.053	-2.965	-2.933	-2.933
0.05	6	-5.949	-5.539	-5.201	-5.781	-5.501	-3.892	-3.390	-3.081
	16	-8.909	-9.815	-10.886	-13.024	-8.909	-9.571	-3.584	-3.451
	1	-3.139	-3.074	-3.066	-3.069	-3.118	-3.052	-2.994	-2.910
0.10	6	-6.603	-6.845	-7.479	-7.852	-6.570	-4.704	-3.404	-3.334
	16	-8.819	-10.988	-14.975	-17.319	-8.819	-10.988	-5.915	-3.751

TABLE 1. 5% critical values of DF and ADF tests with AO's. Estimated models (5.1)–(5.2).

		DF				ADF			
π	θ	T=100	T=200	T =500	T =1000	T=100	T=200	T =500	T =1000
0		87.55	99.99	100.00	100.00	86.35	99.95	100.0	100.0
	1	95.58	95.63	95.57	95.96	74.14	99.85	100.0	100.0
0.01	6	98.74	97.83	97.45	99.36	76.85	99.61	100.0	100.0
	16	100.00	99.98	99.85	100.00	76.48	99.70	100.0	100.0
	1	96.58	96.49	96.33	96.59	76.57	99.83	100.0	100.0
0.05	6	100.00	100.00	100.00	100.00	76.56	99.84	100.0	100.0
	16	100.00	100.00	100.00	100.00	60.92	99.92	100.0	100.0
	1	97.16	97.10	97.05	97.24	78.04	99.75	100.0	100.0
0.10	6	100.00	100.00	100.00	100.00	80.89	99.80	100.0	100.0
	16	100.00	100.00	100.00	100.00	52.38	99.83	100.0	100.0

TABLE 2. Size–adjusted power of DF and ADF tests with AO's. $\rho = -0.2$. Estimated models (5.1)–(5.2).

BK Filter

	$\pi = 0$	$\pi = 0.01$			$\pi = 0.05$			$\pi = 0.1$		
T		$\theta = 1$	$\theta = 6$	$\theta = 16$	$\theta = 1$	$\theta = 6$	$\theta = 16$	$\theta = 1$	$\theta = 6$	$\theta = 16$
100	-3.023	-2.958	-2.938	-3.344	-2.975	-3.158	-3.295	-2.951	-3.385	-5.630
200	-2.928	-2.945	-2.890	-2.950	-2.917	-2.920	-3.247	-2.923	-2.899	-3.072
500	-2.855	-2.852	-2.818	-2.808	-2.844	-2.789	-2.862	-2.833	-2.823	-3.293
1000	-2.855	-2.870	-2.817	-2.844	-2.861	-2.840	-2.933	-2.847	-2.842	-3.072
HP10 Filter										
100	-3.195	-3.211	-3.081	-2.897	-3.201	-2.963	-2.958	-3.173	-2.967	-3.220
200	-3.038	-3.040	-3.092	-2.932	-3.079	-3.019	-3.106	-3.079	-3.022	-3.416
500	-2.859	-2.877	-2.993	-2.970	-2.912	-2.971	-2.956	-2.995	-2.831	-3.143
1000	-2.858	-2.872	-3.013	-2.931	-2.891	-2.926	-3.041	-2.916	-2.885	-3.350
HP100 Filter										
100	-3.171	-3.188	-3.535	-3.068	-3.329	-3.082	-2.894	-3.367	-3.082	-3.022
200	-2.907	-2.918	-3.030	-3.201	-2.933	-3.181	-3.108	-2.938	-3.108	-3.213
500	-2.861	-2.871	-2.940	-3.137	-2.887	-3.136	-3.160	-2.931	-3.070	-3.036
1000	-2.847	-2.870	-2.965	-3.039	-2.887	-3.102	-3.054	-2.914	-3.013	-3.111
HP400 Filter										
100	-3.232	-3.215	-3.752	-3.385	-3.311	-3.314	-2.818	-3.344	-3.255	-2.864
200	-2.900	-2.943	-3.033	-3.455	-2.963	-3.304	-3.167	-2.977	-3.185	-3.121
500	-2.876	-2.883	-2.963	-3.112	-2.904	-3.021	-3.132	-2.950	-3.193	-3.078
1000	-2.853	-2.867	-2.970	-3.103	-2.896	-2.987	-3.013	-2.918	-3.106	-3.053
HP1600 Filter										
100	-3.389	-3.419	-3.865	-3.990	-3.471	-3.878	-3.024	-3.502	-3.721	-3.145
200	-2.982	-3.008	-3.130	-3.503	-3.043	-3.363	-3.074	-3.071	-3.364	-3.088
500	-2.914	-2.905	-3.000	-3.147	-2.934	-3.052	-3.090	-2.984	-3.223	-3.053
1000	-2.869	-2.878	-2.951	-3.171	-2.913	-2.982	-2.972	-2.932	-3.060	-2.919
U Filter										
100	-3.562	-3.55	-3.22	-2.91	-3.48	-3.03	-2.62	-3.42	-3.01	-3.11
200	-3.234	-3.22	-3.12	-2.90	-3.21	-2.87	-2.69	-3.17	-2.99	-2.88
500	-3.229	-3.22	-3.13	-2.96	-3.21	-2.94	-2.80	-3.15	-3.07	-2.96
1000	-3.157	-3.15	-3.03	-2.98	-3.14	-3.00	-2.90	-3.13	-2.98	-3.05
MD Filter										
100	-2.885	-2.902	-2.899	-2.899	-2.900	-2.860	-2.859	-2.914	-2.831	-2.831
200	-2.935	-2.939	-2.935	-2.935	-2.925	-2.885	-2.884	-2.916	-2.836	-2.836
500	-2.973	-2.973	-2.972	-2.972	-2.968	-2.940	-2.940	-2.969	-2.868	-2.668
1000	-2.994	-3.009	-2.997	-2.997	-3.002	-2.963	-2.964	-3.008	-2.917	-2.847

TABLE 3. 5% critical values of filtered unit root tests with AO's. Estimated model (5.3)

	$\pi = 0$	$\pi = 0.01$			$\pi = 0.05$			$\pi = 0.1$		
T		$\theta = 1$	$\theta = 6$	$\theta = 16$	$\theta = 1$	$\theta = 6$	$\theta = 16$	$\theta = 1$	$\theta = 6$	$\theta = 16$
BK Filter										
100	21.85	23.93	32.43	47.08	25.42	39.28	44.67	26.54	50.23	55.81
200	69.04	68.41	76.42	79.69	71.42	81.57	90.36	73.84	86.21	99.56
500	99.99	99.99	100.00	100.00	99.99	100.00	100.00	99.99	100.00	100.00
1000	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
HP10 Filter										
100	30.42	12.32	14.41	21.70	12.53	24.91	40.53	13.14	25.99	40.77
200	74.81	59.43	59.25	72.10	58.38	76.11	92.52	58.96	83.19	96.04
500	100.00	99.95	99.93	99.99	99.93	100.00	100.00	99.91	100.00	100.00
1000	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
HP100 Filter										
100	15.43	11.89	5.97	11.77	9.33	13.87	20.37	8.59	13.08	16.78
200	44.36	35.16	30.35	23.07	35.17	26.96	38.58	35.01	35.65	50.11
500	98.67	96.45	95.71	93.51	96.37	95.60	99.50	95.87	97.51	99.94
1000	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
HP400 Filter										
100	7.69	20.80	12.50	15.11	18.99	19.08	27.25	18.37	17.19	18.06
200	28.99	30.47	26.35	12.02	29.45	17.44	29.32	28.86	22.35	31.56
500	91.80	84.59	81.85	77.22	84.16	83.16	93.78	82.25	78.21	93.69
1000	99.97	100.00	100.00	99.99	99.99	99.98	100.00	99.99	99.99	100.00
HP1600 Filter										
100	4.78	28.34	20.35	17.75	27.37	21.23	38.01	26.56	22.88	29.77
200	15.75	32.04	29.04	19.78	31.67	26.02	44.72	30.84	23.78	39.01
500	72.66	66.09	62.45	57.80	64.55	59.63	70.84	62.15	47.82	62.28
1000	99.01	99.15	99.11	98.45	99.21	99.13	99.29	99.15	98.99	99.95
U Filter										
100	28.78	29.07	29.36	23.45	28.93	33.92	21.31	28.33	27.25	38.88
200	60.44	61.33	60.30	54.85	59.92	61.16	64.72	60.60	58.70	54.09
500	99.95	99.96	99.96	99.96	99.94	99.94	99.98	99.95	99.99	100.00
1000	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
MD Filter										
100	33.81	32.64	32.29	32.31	32.33	30.24	30.26	31.15	29.68	29.71
200	86.70	86.59	86.53	86.53	86.50	86.35	86.37	87.02	86.57	86.59
500	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
1000	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

TABLE 4. Size–adjusted power of filtered unit root tests with AO's. $\rho = -0.2$. Estimated model (5.3).

	$\pi = 0.01$			$\pi = 0.05$			$\pi = 0.1$		
T	$\theta = 1$	$\theta = 6$	$\theta = 16$	$\theta = 1$	$\theta = 6$	$\theta = 16$	$\theta = 1$	$\theta = 6$	$\theta = 16$
				ADF					
100	5.26	11.78	23.67	6.64	27.91	88.53	7.66	46.36	97.00
200	5.01	7.82	12.30	5.89	20.81	41.52	7.08	23.57	67.50
500	5.05	6.94	8.72	5.81	12.12	12.11	6.79	13.21	34.97
1000	5.35	6.76	9.83	6.09	8.36	15.37	5.86	12.97	20.30
				BK Filter					
100	4.98	4.71	8.67	5.00	6.99	8.76	4.91	9.81	31.67
200	4.96	4.39	5.18	4.62	4.80	8.91	4.70	4.66	7.43
500	4.97	4.39	4.40	4.83	4.06	5.11	4.62	4.47	12.30
1000	4.82	4.24	4.41	4.70	4.48	5.72	4.47	4.60	8.22
				HP10 Filter					
100	5.00	3.69	2.31	4.92	2.76	2.39	4.64	2.86	5.44
200	5.12	5.58	3.85	5.58	4.88	6.05	5.64	4.83	11.22
500	5.03	6.60	6.22	5.44	6.36	6.35	6.71	4.64	9.51
1000	5.10	7.07	5.93	5.31	5.80	7.43	5.73	5.17	13.53
				HP100 Filter					
100	5.24	8.95	3.71	6.21	4.07	2.30	6.63	4.07	3.73
200	5.00	6.38	9.04	5.15	8.56	8.21	5.20	7.94	10.25
500	4.97	5.95	9.02	5.24	9.14	9.53	5.80	8.10	7.74
1000	5.06	6.30	7.40	5.30	8.53	7.87	5.64	7.30	9.19
				HP400 Filter					
100	5.06	9.57	6.31	5.72	5.80	1.51	6.11	5.35	1.82
200	5.01	6.15	12.64	5.30	10.43	8.95	5.53	8.66	8.26
500	5.02	6.27	8.41	5.35	6.88	8.67	5.98	9.71	7.93
1000	5.14	6.62	8.67	5.48	6.83	7.12	5.88	8.79	7.82
				HP1600 Filter					
100	5.13	8.81	11.49	5.68	10.29	1.84	5.96	8.69	2.42
200	5.12	6.45	11.80	5.49	9.76	6.14	5.63	9.33	6.32
500	5.07	6.37	8.55	5.41	6.90	7.40	6.05	9.34	6.84
1000	5.15	6.28	9.75	5.62	6.73	6.46	5.85	7.87	5.63
				U Filter					
100	4.83	2.71	1.08	4.36	1.42	0.16	3.72	1.50	1.03
200	4.83	3.85	2.25	4.77	1.89	0.89	4.45	2.74	1.81
500	4.93	3.97	2.64	4.72	2.60	1.51	4.11	3.46	2.52
1000	4.94	3.68	3.26	4.87	3.17	2.49	4.65	3.17	3.76
				MD Filter					
100	5.13	5.06	5.05	5.21	4.54	4.53	5.28	4.42	4.42
200	4.89	4.84	4.84	4.82	4.21	4.20	4.73	3.84	3.83
500	4.95	4.97	4.97	4.92	4.48	4.46	4.84	3.79	2.06
1000	4.96	4.88	4.88	4.97	4.42	4.43	5.01	3.87	3.23

TABLE 5. Size of the ADF and filtered unit root tests with AO's. Critical values obtained from Tables 1 and 3 with $\pi = 0$

	p	$t_{\widehat{\rho}^g_{AO}}$	CV_f	$CV_f^\star$	CV_b
HP10	4	-1.4576	-3.2314	-3.1592	-2.8572
MD	1	-2.6381	-2.9376	-2.8983	-2.8924
BK	3	-2.8261	-3.0074	-3.0475	-2.7510

TABLE 6. Filtered ADF tests on CPI–based US/Finland real exchange rates.CV_f stands for the 5% critical values of $t_{\widehat{\rho}^g_{AO}}$ in model (5.3) assuming $\pi = 0$ and a sample size $T = 89 CV_f^\star$ is the 5% critical value of the tests assuming that the series includes outliers at the same positions and with the sizes reported by V99. CV_b is the 5% critical value obtained by bootstrap.