On preference, freedom and diversity

Abstract We study the problem of ranking sets of options in terms of freedom of choice. We propose a framework in which both the diversity of the options and the preferences of the agent over the options do play a role. We formulate some axioms that reflect these two aspects of freedom and we study their logical implications. Two different criteria for ranking sets are characterized, which generalize some of the rankings proposed so far in the literature.

1 Motivation

Recent research in the field of normative economics has been characterized by various attempts to depart from the standard welfaristic framework and move towards wider foundations based on non-utilitarian values. One of the most active research programs in this area aims at introducing liberty as an intrinsically relevant principle in the evaluation of alternative states of affairs. The interpretations of freedom advanced in this literature revolve around the idea that to be free an agent has to enjoy access to options to choose from. Hence the problem of measuring a person's freedom is handled by finding a suitable measure of the set of options she faces. There is now a extensive literature concerned with the problem of ranking opportunity sets in terms of freedom of choice. A non-exhaustive list would include Arrow (1995), Bossert et al. (1994), Jones and Sugden (1982), Klemisch-Ahlert (1993), Pattanaik and Xu (1990, 1998a, 2000), Puppe (1996), Sen (1991, 1993), Suppes (1987). Surveys of this literature may be found in Barberà et al. (2004) and Peragine (1999).

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The analytical framework generally used in this literature is the following: X (with generic element x, y, z , etc.) is a finite set of all conceivable objects of concern for any individual, and $P(X)$ (with generic elements A, B, C , etc.) denotes the set of all non-empty subsets of X . An element of X is an opportunity or an option, and an element of $P(X)$ is an opportunity set. The problem is that of ranking opportunity sets on the basis of freedom of choice. Hence, let $\succsim$ be a binary relation over $P(X)$, interpreted as “offers at least as much freedom as”, with $\sim$ and $\succ$ being, respectively, the symmetric and asymmetric components of $\succsim$. For any set A , its cardinality is denoted by $\#A$.

A starting point is the article by Pattanaik and Xu (1990): they characterize axiomatically a criterion for ranking opportunity sets in terms of freedom of choice, introducing three axioms as follows. First (Indifference), every unit set should be freedom-wise indifferent to every other: for all options $x, y \in X$, $\{x\} \sim \{y\}$. Second (Monotonicity), more opportunity should mean more freedom: $\forall x \neq y \in Z$, $\{x, y\} \succ \{x\}$; third (Independence), if a set A is judged to give at least as much freedom as another set B , then that ranking will be unaffected by the addition to or subtraction from each of an alternative x not contained in either: for all $A, B \in P(X)$ and for all $x \in X - \{A \cup B\}$, $A \succsim B$ if and only if $A \cup \{x\} \succsim B \cup \{x\}$. These axioms yield the simple cardinality result (Pattanaik and Xu 1990): *for all $A, B \in P(X)$, $\succsim$ satisfies the axioms Indifference, Monotonicity and Independence if and only if*

$$A \succsim B \Leftrightarrow \#A \geq \#B.$$

This criterion, which applies only to finite opportunity sets, ranks them according to the number of options they contain. It is a quantity-based ranking rule, in which there is no role for information about the nature or the value of different alternatives. This cardinality-based rule is not supported by the authors: rather they judged it to be a trivial rule.

A first explanation of the triviality result is given by Sen (1990, 1991, 1993): the root of the problem is that we find it absurd to dissociate the extent of our freedom from our preferences over the alternatives. According to Sen’s view, the axiomatic structure of Pattanaik and Xu (1990) fails to capture that aspect of freedom linked to the possibility of choosing what is valuable to us. This is what Sen calls the ‘opportunity aspect’ of freedom. From this point of view, an individual is free if she has access to alternatives that she regards as valuable in terms of some criteria. These criteria may be her own actual preferences – as Sen seems to suggest – or, alternatively, a given set of potential preferences.

Jones and Sugden (1982) first suggested the use of potential preferences in assessing a person’s freedom and interpreted the potential preferences as the preferences of a “reasonable” person. For them, the intrinsic value of freedom of choice should be judged not in terms of the preferences that the agent actually has or in terms of his future preference ordering, but in terms of the preference orderings that a reasonable person can possibly have. According to them, for instance, if any reasonable person would be indifferent between two particular alternatives, then offering the choice between these two alternatives to any person would contribute little to her freedom of choice. Pattanaik and Xu (1998) build on Jones and Sugden’s idea of reasonable preferences and construct a ranking consistent with it. In comparing two opportunity sets, A and B , Pattanaik and Xu (1998) concentrate

on $\max(A)$ and $\max(B)$, where $\max(A)$ is the set of all alternatives in A which reasonable persons may choose from the feasible set A , and similarly for B . The model they propose has the virtue of capturing the opportunity aspect of freedom – i.e., the value of the different alternatives – without collapsing into an indirect utility ranking.¹

The criterion of reasonableness for defining the set of potential preferences has been criticized, among others, by Bavetta and Peragine (2000) and by Bavetta and Guala (2004). Specifically, Bavetta and Peragine (2000), while agreeing with the idea that freedom should be assessed by looking at options and potential preferences, nonetheless, settle for an alternative point of view on potential preferences, according to which all and only the preference relations available to an agent in a given choice situation should be taken into account in the assessment of her own freedom. Hence they admit cases in which not only the set of options differ from one individual (one situation) to the other, but also the set of available preference relations. They define an opportunity situation as a pair composed of the set of options available to a decision maker and the set of potential preference orderings that she is confronting in a specific choice situation. Then, in comparing two opportunity situations, they say that (Bavetta and Peragine 2000, Theorem 2) an individual i enjoys more freedom than another individual j if and only if the choice set that his preference profiles elicit from his own opportunity set A has at least as many elements as the choice set that j can elicit by means of his own preference profiles from B . This approach, apart from the specific interpretation advanced by Bavetta and Peragine, is very general from an analytical viewpoint, in that it contains, as a special case, the situation where the sets of potential preferences available to agents are all the same – i.e., the case considered by Pattanaik and Xu (1998). Therefore, in dealing with the role of preferences in assessing a person's freedom, this last approach will be followed in the present paper.

A different explanation of the triviality of the pure cardinality rule is suggested by the authors themselves (Pattanaik and Xu 1990): it is not necessarily dependent on the dissociation of the extent of freedom from the preferences over the alternatives. Instead, they put the blame on the last, independence axiom: it does not take into account the extent to which the different alternatives are “close” or similar to each other. Thus, a development along this line of criticism amounts to introducing formally a notion of closeness or similarity of options and to reformulate the independence axiom. This is the direction undertaken by the same authors in a subsequent paper (Pattanaik and Xu 2000) and, more recently, by Bossert et al. (2003).

To sum up, the axiomatic structure of Pattanaik and Xu can be questioned on the basis of two considerations. First, it does not consider the fundamental link between freedom and preferences, therefore failing to capture the “opportunity” aspect of freedom. Second, as the authors point out, it does not take into account the “similarity” or “closeness” aspect of the available alternatives; this is the “variety” aspect of freedom. Each of the two questions have been addressed very much in isolation from the other. The main purpose of this paper is to provide a unified

¹ A different approach is proposed by Romero-Medina (2001): he first considers the set $\max(X)$ from all alternatives in the universal set X that reasonable people will choose if the universal set was feasible. Then he concentrates on $\max(X) \cap A$ and $\max(X) \cap B$ when comparing A and B .

framework for the analysis of the two themes. One might claim that the two aspects are very closely related. They are, in fact, so closely related that two options one preferred to another have to be necessarily dissimilar. However, this is not the case. Imagine that one has to choose between two rings that are exactly identical by any conceivable criterion of similarity. Let us call them ring a and ring b . In this case one might think that any reasonable preference should declare the two objects as indifferent. Nevertheless, one of them, say ring a , might be a wedding ring. In this case, although there are no justifiable differences between the two rings in terms of similarities, there is an important circumstance that makes a preferred to b .

2 The analytical framework

The basic ingredients of our framework are the following: (i) a notion of similarity which can be represented by means of a binary similarity relation on the universal set of options (see Pattanaik and Xu 2000) and (ii) a set of preference relations over the options, interpreted along the lines of Bavetta and Peragine (2000). It should be noticed that our formulation of the notion of similarity allows for only two levels of similarity: two options are either similar or they are not. In that sense our framework is different from the one in Bossert et al. (2003) where different degrees of similarity are allowed.

2.1 Notation

We are denoting by X the universal finite set of opportunities, by $\mathcal{P}(X)$ the set of opportunity sets and by $N = \{1, \dots, n\}$ the set of agents.

We first introduce information about the individual preferences over the opportunities.

A decision maker's preference relation is denoted by R . R is a complete, reflexive and transitive binary relation defined over X . The set of all possible preference profiles is denoted by $\Pi = \{R_1, \dots, R_m\}$. Thus, $\forall x, y \in X, \forall h \in \{1, \dots, m\}$, $x R_h y$ means that “ x is preferred to y according to the preference ordering R_h ”. In our framework each of the n individuals in the society holds a set $\Pi_i, i \in N$, of preference orderings, where $\Pi_i \subseteq \Pi$. $\mathcal{P}(\Pi) = 2^\Pi - \{\emptyset\}$ represents the set of all subsets of preference profiles.

As explained in the previous sections, we are interested in ranking *opportunity situations*. An opportunity situation is a pair $(A, \Pi_i) \in \mathcal{P}(X) \times \mathcal{P}(\Pi)$. Hence our ranking is represented by a binary relation $\succsim$ over $\mathcal{P}(X) \times \mathcal{P}(\Pi)$. The expression $(A, \Pi_i) \succsim (B, \Pi_j)$ should be read as “the opportunity situation (A, Π_i) offers at least as much freedom as the opportunity situation (B, Π_j) ”.

We then introduce the following definition: $\forall (A, \Pi_i) \in \mathcal{P}(X) \times \mathcal{P}(\Pi)$, we define the choice set as:

$$\max_i(A) = \{x \in A : \exists R \in \Pi_i \text{ such that } x R y \forall y \in A\}.$$

We now introduce the notion of similarity among alternatives.

Let S be a reflexive and symmetric binary relation over X : $x S y$ means that x is similar to y ; $x \not S y$ means that x is not similar to y . For all $A \in \mathcal{P}(X)$, A

is *homogeneous* if and only if $\forall a, a' \in A, aSa'$. For all $A \in \mathcal{P}(X)$, a similarity based partition of A is defined as a class $\{A_1, \dots, A_m\}$ such that: (1) $A_k \subseteq A, \forall k = 1, \dots, m$; (2) $\cup_{k=1}^m A_k = A$; (3) $A_1, \dots, A_m$ are pairwise disjoint; (4) A_k is homogeneous $\forall k = 1, \dots, m$. The similarity-based partition is denoted by $\phi(A), \phi'(A), \phi''(A)$, etc. Let $\Phi(A)$ be the set of all similarity partitions $\phi(A)$ of A such that for every similarity partition $\phi'(A), \#\phi'(A) \geq \#\phi(A)$. So, $\Phi(A)$ is the set of all smallest similarity-based partitions of A . For all $x \in X$ and all $A \in \mathcal{P}(X)$ we say that xSa if and only if $xSa \forall a \in A$. For all $A, B \in \mathcal{P}(X)$, with A homogeneous, we say that A does not mimic B if and only if, for all $\phi(B) \in \Phi(B)$, there exists $a \in A$ such that, for all $B_i \in \phi(B)$, it is not true that aSB_i .

Next we impose some axioms on $\succsim$, which capture our intuition on the extent of freedom enjoyed by the decision maker under alternative situations. We assume first that the relation $\succsim$ is transitive.

3 The axioms

In this section we consider several axioms that define properties of the binary relation $\succsim$ over X . These properties embody both the opportunity aspect of freedom and the variety aspect.

Axiom 1 *Indifference between no-freedom situations (INF).* $\forall x, y \in X, \forall \Pi_i, \Pi_j \in \mathcal{P}(\Pi), (x, \Pi_i) \sim (y, \Pi_j)$.

Axiom INF is inspired by the *principle of no choice situation*, introduced by Jones and Sugden (1982) and used by Pattanaik and Xu (1990). This principle states that if two opportunity sets are singleton, then the degree of freedom offered by them is identical, being nil. It is intended to capture the intrinsic value of freedom. Axiom INF extends this basic indifference principle to our framework: if two opportunity situations are characterized by as many choice sets that are singletons, then the degree of freedom that they offer is the same.

Axiom 2 *Addition of irrelevant alternatives (AIA).* $\forall A, \in \mathcal{P}(X), \forall \Pi_i \in \mathcal{P}(\Pi), \forall x \in X$ such that $\max_i(A \cup \{x\}) = (\max_i(A)), (A \cup \{x\}, \Pi_i) \sim (A, \Pi_i)$.

The axiom AIA deals with the description of the alternatives that are irrelevant in terms of freedom. According to AIA, the addition of alternatives that are not eligible in a given set A is declared to be not freedom enhancing.

Axiom 3 *Addition of relevant alternatives (ARA).* $\forall A \in \mathcal{P}(X), A$ homogeneous, $\forall \Pi_i \in \mathcal{P}(\Pi), \forall x \in X - A$ such that $x \in \max_i(A \cup \{x\})$,

$$[xS \max_i(A)] \Rightarrow (A \cup \{x\}, \Pi_i) \sim (A, \Pi_i),$$

and

$$[xS \max_i(A)] \Rightarrow (A \cup \{x\}, \Pi_i) \succ (A, \Pi_i).$$

Addition of relevant alternatives is a modified version of an axiom that was introduced in Pattanaik and Xu (2000). ARA explicitly takes into account the information about similarity of alternatives in the case of addition of an alternative which is eligible according to the relevant set of preferences. ARA requires that, given a homogeneous set A and $x \in \max_i(A \cup \{x\})$, if x is similar to the elements in $\max_i(A)$, then the addition of x to A does not change the degree of freedom already offered by the opportunity set A , and if x is dissimilar to the elements in A , then, adding x to A will actually increase the degree of freedom offered by A .

The axioms AIA and ARA together determine when the addition of new alternatives to a given opportunity situation is relevant (ARA) or irrelevant (AIA) in terms of freedom.

Axiom 4 *Similarity composition (SC).* $\forall A, B, C, D \in \mathcal{P}(X), \forall \Pi_i, \Pi_j \in \mathcal{P}(\Pi)$, such that C and D are homogeneous and $\max_i(C \cup A)$ does not mimic $\max_i(A)$, and $\max_i(A) \cap \max_i(C) = \max_j(B) \cap \max_j(D) = \emptyset$,

$$[(A, \Pi_i) \succsim (B, \Pi_j) \text{ and } (C, \Pi_i) \succsim (D, \Pi_j)] \Rightarrow [(A \cup C, \Pi_i) \succsim (B \cup D, \Pi_j)],$$

$$[(A, \Pi_i) \succ (B, \Pi_j) \text{ and } (C, \Pi_i) \succsim (D, \Pi_j)] \Rightarrow [(A \cup C, \Pi_i) \succ (B \cup D, \Pi_j)].$$

Similarity composition is a weaker version of an axiom proposed originally by Sen (1991). Sen's axiom requires that, given $A \cap C = B \cap D = \emptyset$, if $[A \succsim B$ and $C \succsim D]$, then $[A \cup C \succsim B \cup D]$, and, if $[A \succ B$ and $C \succsim D]$, then $[A \cup C \succ B \cup D]$. However, in our context, a modification of Sen's axiom seems to be warranted. Suppose $A = \{a\}$, $B = \{b\}$, $C = \{c\}$ and $D = \{d\}$ and suppose the elements of these four sets are all distinct. Let aSc and bSd i.e., a and c are similar but b and d are dissimilar. In view of this, we can justifiably feel that adding c to the set $\{a\}$ does not significantly increase the degree of freedom while adding d to the set $\{b\}$ does. In that case, we may be unwilling to accept that $[A \cup C \succsim B \cup D]$ even if $[A \succsim B$ and $C \succsim D]$. SC deals with this problem of Sen's axiom by restricting the applicability of the axiom to the case where both C and D are homogeneous, C does not mimic A , and the intersection of their maximal sets is empty.

4 The first ranking

We start our exercise by characterizing an ordering which incorporates both information about preferences and the similarity of alternatives using a cardinal approach. First we select the alternatives that, in a given set A , are relevant according to the available preferences Π_i , i.e., $\max_i(A)$; then we count the number of elements in the similarity-based partition of $\max_i(A)$, i.e., the number of elements contained in $\phi(\max_i(A))$.

Definition 1 $\succsim = \succsim_1$ iff and only if for all $(A, \Pi_i), (B, \Pi_j) \in \mathcal{P}(X) \times \mathcal{P}(\Pi)$, given $\phi(A) \in \Phi(A)$ and $\phi(B) \in \Phi(B)$,

$$(A, \Pi_i) \succsim (B, \Pi_j) \Leftrightarrow \# \phi(\max_i(A)) \geq \# \phi(\max_j(B)).$$

In other words, an opportunity situation (A, Π_i) offers more freedom of choice than another opportunity situation (B, Π_j) if and only if the number of alternatives contained in $\phi(\max_i(A))$ is bigger than the number of alternatives contained in $\phi(\max_j(B))$.

We now characterize the ordering just introduced.

Theorem 1 $\succsim \succsim_1$ if and only if $\succsim$ satisfies the axioms INF, AIA, ARA and SC.

Proof The necessity part of the proposition is straightforward; we prove only the sufficiency part². Hence, let $\succsim$ satisfy INF, ARA, AIA and SC. We first prove that:

(1) $\forall A \in \mathcal{P}(X), \forall \Pi_i \in \mathcal{P}(\Pi), (A, \Pi_i) \sim (\max_i(A), \Pi_i)$.

If $\max_i(A) = A$, then the result clearly follows. If not, $\max_i(A) \sim \max_i(A)$. Let $x \in (A \setminus \max_i(A))$. By AIA $(x \cup \max_i(A), \Pi_i) \sim (\max_i(A), \Pi_i)$. Therefore by AIA and transitivity

$$(A, \Pi_i) \sim (\max_i(A), \Pi_i).$$

By (1) and transitivity, we obtain: (2) $(A, \Pi_i) \succsim (B, \Pi_j) \Leftrightarrow (\max_i(A), \Pi_i) \succsim (\max_j(B), \Pi_j)$.

Now consider that INF corresponds to axiom INS in Pattanaik and Xu (2000); moreover, if we restrict ourselves to the maximal sets, axioms ARA and SC imply, respectively, the S-monotonicity and the S-composition axioms of Pattanaik and Xu (2000). Therefore, on the basis of Theorem 4.6 of Pattanaik and Xu (2000), we know that, for all $\phi(\max_i(A)) \in \Phi(\max_i(A))$ and for all $\phi(\max_j(B)) \in \Phi(\max_j(B))$, $(\max_i(A), \Pi_i) \succsim (\max_j(B), \Pi_j)$ if and only if $\#\phi(\max_i(A)) \geq \#\phi(\max_j(B))$. Hence we have:

(3) $(\max_i(A), \Pi_i) \succsim (\max_j(B), \Pi_j) \Leftrightarrow \#\phi(\max_i(A)) \geq \#\phi(\max_j(B))$.

Finally, transitivity (2) and (3) imply:

$$(A, \Pi_i) \succsim (B, \Pi_j) \Leftrightarrow \#\phi(\max_i(A)) \geq \#\phi(\max_j(B)).$$

□

5 The second ranking

In this section we characterize a second ordering. This ranking also incorporates both information about preferences and similarity, using a cardinal approach. For a given set A , we first study a similarity-based partition $\phi(A) = \{A_1, \dots, A_m\}$ belonging to the set of the smallest similarity-based partitions of A , that is $\Phi(A)$; then, for each $A_k \in \phi(A)$, we concentrate on the relevant alternatives, as elicited by the available set of preferences: i.e., on $\max_i(A_k)$; finally, we aggregate the sets $\max_i(A_k)$ for all the $k = 1, \dots, m$, and for all the partitions in $\Phi(A)$. We denote the resulting set by $\max_i(\Phi(A))$. Formally,

Definition 2 Given $A \in \mathcal{P}(X)$, $\Pi_i \in \mathcal{P}(\Pi)$, and $\Phi(A)$,

$$\max_i(\Phi(A)) := \{x \in \max_i(A_k) \mid A_k \in \phi(A) \text{ and } \phi(A) \in \Phi(A)\}$$

We now define our second ordering:

Definition 3 $\succsim = \succsim_2$ if and only if for all $(A, \Pi_i), (B, \Pi_j) \in \mathcal{P}(X) \times \mathcal{P}(\Pi)$,

$$(A, \Pi_i) \succsim (B, \Pi_j) \Leftrightarrow \#\max_i(\Phi(A)) \geq \#\max_j(\Phi(B)).$$

² This proof makes extensive use of the results in Pattanaik and Xu (2000).

In other words, an opportunity situation (A, Π_i) offers more freedom of choice than other opportunity situations (B, Π_j) if and only if the number of alternatives contained in $\max_i(\Phi(A))$ is bigger than the number of alternatives contained in $\max_j(\Phi(B))$.

We introduce some axioms that will be used to characterize the ranking $\succsim_2$.

Axiom 5 *Subtraction of irrelevant alternatives (SIA).* $\forall A \in \mathcal{P}(X), \forall \Pi_i \in \mathcal{P}(\Pi)$, given S and $x \in A$, if $\nexists A_k \in \phi(A), \phi(A) \in \Phi(A)$, such that $x \in \max_i(A_k)$, then $(A - \{x\}, \Pi_i) \sim (A, \Pi_i)$.

The axiom SIA is a version of AIA adapted to an environment where the relevant alternatives are those that are maximal in at least one element belonging to one of the smallest similarity-based partitions of a set A .

Axiom 6 *Subtraction of relevant alternatives (SRA).* $\forall A \in \mathcal{P}(X), \forall \Pi_i \in \mathcal{P}(\Pi)$, given S , if $\exists A_k \in \phi(A), \phi(A) \in \Phi(A)$, such that $x \in \max_i(A_k)$, then $(A - \{x\}, \Pi_i) < (A, \Pi_i)$.

According to SRA, the subtraction of alternatives that are maximal in a similarity partition reduces the freedom that the set offers. SIA and SRA play a role similar to the axioms AIA and ARA used in the previous characterization: they analyze the effect of the addition (or subtraction) of new alternatives to (from) a set A , but now by putting more emphasis on the similarity aspect of freedom. The last axiom we introduce is a composition property.

Axiom 7 *Composition (COM).* $\forall A, B, C, D \in \mathcal{P}(X), \forall \Pi_i, \Pi_j \in \mathcal{P}(\Pi)$, such that $\max_i(\Phi(A)) \cap \max_i(\Phi(C)) = \max_j(\Phi(B)) \cap \max_j(\Phi(D)) = \emptyset$,

$$[(A, \Pi_i) \succsim (B, \Pi_j) \text{ and } (C, \Pi_i) \succsim (D, \Pi_j)] \rightarrow [(A \cup C, \Pi_i) \succsim (B \cup D, \Pi_j)],$$

and,

$$[(A, \Pi_i) \succsim (B, \Pi_j) \text{ and } (C, \Pi_i) > (D, \Pi_j)] \rightarrow [(A \cup C, \Pi_i) > (B \cup D, \Pi_j)].$$

6 Characterization of the second ranking

Theorem 2 $\succsim = \succsim_2$ if and only if $\succsim$ satisfies the axioms INF, SIA, SRA, and COM.

Proof The necessity part of the proposition is straightforward; we prove only the sufficiency part. Let $\succsim$ satisfy INF, SIA, SRA, and COM. The proof has two stages. First, we show that:

$$\begin{aligned} \text{for all } A, B \in \mathcal{Z}, \quad & \text{if } \# \max_i(\Phi(A)) = \# \max_j(\Phi(B)), \\ & \text{then } (A, \Pi_i) \sim (B, \Pi_j). \end{aligned} \quad (1)$$

Suppose $A, B \in \mathcal{P}(X)$ and $\# \max_i(\Phi(A)) = \# \max_j(\Phi(B)) = g$. Let $\max_i(\Phi(A)) = \{a_1, \dots, a_g\}$ and $\max_j(\Phi(B)) = \{b_1, \dots, b_g\}$. By INF,

$$(\{a_1\}, \Pi_i) \sim (\{b_1\}, \Pi_j) \quad (2)$$

and

$$(\{a_2\}, \Pi_i) \sim (\{b_2\}, \Pi_j). \quad (3)$$

$\{a_1\} \cap \{b_1\} = \{a_2\} \cap \{b_2\} = \emptyset$ and further $\max_i (\Phi \{a_1\}) = \{a_1\}$, $\max_i (\Phi \{a_2\}) = \{a_2\}$ and $\max_j (\Phi \{b_1\}) = \{b_1\}$, $\max_j (\Phi \{b_2\}) = \{b_2\}$, since $a_1, a_2 \in \max_i (\Phi(A))$ and $b_1, b_2 \in \max_j (\Phi(B))$. Hence by Eqs. (2) and (3) and COM, we have

$$(\{a_1, a_2\}, \Pi_i) \sim (\{b_1, b_2\}, \Pi_j). \quad (4)$$

By INF, again,

$$(\{a_3\}, \Pi_i) \sim (\{b_3\}, \Pi_j). \quad (5)$$

By Eqs. (4) and (5) and COM,

$$(\{a_1, a_2, a_3\}, \Pi_i) \sim (\{b_1, b_2, b_3\}, \Pi_j). \quad (6)$$

Proceeding in this way, we finally have $(\{a_1, \dots, a_g\}, \Pi_i) \sim (\{b_1, \dots, b_g\}, \Pi_j)$, i.e., $(\max_i (\Phi(A)), \Pi_i) \sim (\max_j (\Phi(B)), \Pi_j)$. If $A = \max_i (\Phi(A))$, then $(A, \Pi_i) \sim (\max_j (\Phi(B)), \Pi_j)$. Suppose $\{A \setminus \max_i (\Phi(A))\} \neq \emptyset$. Let $\{A \setminus \max_i (\Phi(A))\} = \{\bar{a}_1, \dots, \bar{a}_m\} \neq \emptyset$. Let $T_1 = \max_i (\Phi(A)) \cup \{\bar{a}_1\}$, then by SIA $(T_1, \Pi_i) \sim (\max_j (\Phi(A)), \Pi_i)$. We proceed iteratively adding the components in $\{\bar{a}_1, \dots, \bar{a}_m\}$ using SIA and transitivity, until $(A, \Pi_i) \sim (\max_i (\Phi(A)), \Pi_i)$. Hence by transitivity we have

$$(A, \Pi_i) \sim (\max_j (\Phi(B)), \Pi_j). \quad (7)$$

Similarly, by transitivity and SIA, from Eq. (7), we have $(A, \Pi_i) \sim (B, \Pi_j)$ which proves Eq. (1).

Next, we show:

$$\begin{aligned} \text{for all } A, B \in \mathcal{Z}, \quad & \text{if } \# \max_i (\Phi(A)) > \# \max_j (\Phi(B)), \\ & \text{then } (A, \Pi_i) \succ (B, \Pi_j). \end{aligned} \quad (8)$$

Suppose $A, B \in \mathcal{P}(X)$ and $\# \max_i (\Phi(A)) > \# \max_j (\Phi(B))$. Let $\# \max_j (\Phi(B)) = g$ and $\# \max_i (\Phi(A)) = g + t$ (where $t > 0$). Further, let $\max_j (\Phi(B)) = \{b_1, \dots, b_g\}$ and $\max_i (\Phi(A)) = \{a_1, \dots, a_g, \dots, a_{g+t}\}$. Note that $\max_i (\Phi \{a_1, \dots, a_g\}) = \{a_1, \dots, a_g\}$. Hence, by Eq. (1),

$$(\{a_1, \dots, a_g\}, \Pi_i) \sim (B, \Pi_j) \quad (9)$$

since $\max_i (\Phi(A)) = \{a_1, \dots, a_g, \dots, a_{g+t}\}$ let $T_{g+1} = \max_i (\Phi \{a_1, \dots, a_g\}) \cup \{a_{g+1}\}$. Then by SRA and Eq. (9), it follows that

$$(T_{g+1}, \Pi_i) \succ (T_{g+1} - \{a_{g+1}\}, \Pi_i) \sim (\{a_1, \dots, a_g\}, \Pi_i)$$

and by Eq. (9) and transitivity

$$(T_{g+1}, \Pi_i) \succ (\max_j (\Phi(B)), \Pi_j). \quad (10)$$

Taking Eq. (10) and adding $a_{g+2}, \dots, a_{g+t}$ on the left hand side and using SRA repeatedly, we have

$$(\{a_1, \dots, a_{g+t}\}, \Pi_i) \succ (B, \Pi_j). \quad (11)$$

Taking Eq. (11) and using an argument similar to the one used to establish Eq. (7), by SRA, we have $(A, \Pi_i) \succ (B, \Pi_j)$, which proves Eq. (8). Equations (1) and (8) complete the proof of the proposition. $\square$

7 Discussion

The rankings we have presented capture the two different aspects of freedom highlighted in the motivation of this paper: the diversity and the opportunity aspects. However, while in the first ranking a clear priority is given to the opportunity aspect, quite the contrary happens with the second one. In fact, in ranking two opportunity situations according to $\succsim$, we first elicit the choice set from a given set A on the basis of the prevailing preference relations: i.e., we focus on the opportunity aspect; then we apply the diversity criterion only to this subset, by counting the number of equivalence classes in $\max_i(A)$, defined in terms of the similarity relation S . With the second ranking, on the contrary, priority is given to the variety aspect. This point can be clarified by means of a simple example.

Example 1 Let $A = \{\text{red car, blue car, red train, blue train, red bike, blue bike}\}$ be a set of means of transportations available to an agent, who has the following set of preference orderings: $\Pi_i = \{R_1, R_2\}$, where

$R_1 : [\text{red car} \succ \text{blue car} \succ \text{red train} \succ \text{blue train} \succ \text{red bike} \succ \text{blue bike}]$

$R_2 : [\text{red train} \succ \text{blue train} \succ \text{red car} \succ \text{blue car} \succ \text{red bike} \succ \text{blue bike}].$

Hence the opportunity situation we are considering is (A, Π_i) . Suppose, moreover, that the following similarity relation holds: red car S blue car, red train S blue train, red bike S blue bike. In this case the set of smallest similarity-based partitions of A , $\Phi(A)$, is a singleton, the only element being $\phi(A) = \{\{\text{red car, blue car}\}, \{\text{red train, blue train}\}, \{\text{red bike, blue bike}\}\}$. Clearly, $\max_i(A) = \{\text{red car, red train}\}$,

$$\phi(\max_i(A)) = \{\{\text{red car}\}, \{\text{red train}\}\},$$

and thus $\#\phi(\max_i(A)) = 2$.

On the other hand, $\max_i(\Phi(A)) = \{\max_i\{\text{red car, blue car}\} \cup \max_i\{\text{red train, blue train}\} \cup \{\text{red bike, blue bike}\}\}$, hence

$$\max_i(\Phi(A)) = \{\text{red car, red train, red bike}\}$$

and $\#\max_i(\Phi(A)) = 3$. We can see that, while in the first ranking – which is based on the cardinality of $\phi(\max_i(A))$ – not all the classes of similar options count for the final evaluation, but only those that are contained in the maximal set $\max_i(A)$. In the case of the second ranking all the similarity-based equivalence classes necessarily count in the evaluation, even if they contain options which are among the least preferred by the agent – as it is the case for the options $\{\text{red bike, blue bike}\}$ in the example.

Before concluding we wish to draw the attention of the reader toward the relationship that exists between our rankings and some of the main results achieved so far in the literature. As the following remarks illustrate, the rules proposed in this paper have the advantage of generalizing some important results so far axiomatized in the literature.

The basic ingredients of our analysis are the similarity relation on the set of alternatives and the set of preferences available to the agents. The following remarks show what happens when these aspects of freedom are deemed to be irrelevant.

We first study the case of irrelevance of the similarity relation:

Remark 1 If $\nexists a, b \in X$ such that aSb , then the ranking established in Theorem 2 coincides with the cardinality rule of Pattanaik and Xu (1990), and the ranking established in Theorem 1 coincides with the rule characterized by Bavetta and Peragine (2000) (Proposition 5.2): i.e., given two opportunity situations $(A, \Pi_i), (B, \Pi_j), (A, \Pi_i) \geq (B, \Pi_j)$ if and only if $\# \max_i(A) \geq \# \max_j(B)$

We now turn to the case of irrelevance of the preference relations.

Remark 2 Suppose that the set of preference orderings Π satisfies a “richness” assumption, such that $\forall A \in \Pi(X), \max(A) := \{x \in A : \forall y \in A xRy \text{ for some } R \in \Pi\} = A$. Then, if $\forall i \in N, \Pi_i = \Pi$, all possible preference orderings can be held by the individuals, and the ranking established in Theorem 1 coincides with the simple similarity-based ordering of Pattanaik and Xu (2000): given two opportunity situations $(A, \Pi_i), (B, \Pi_j), (A, \Pi_i) \geq (B, \Pi_j)$ if and only if $\# \phi(A) \geq \# \phi(B)$. Moreover, in this case, the ranking established in Theorem 2 coincides with the simple cardinality-based ordering of Pattanaik and Xu (1990).

We now consider the case of irrelevance of the similarity relation and of the preference relations.

Remark 3 Suppose, again, that Π satisfies the “richness” assumption, such that $\forall A \in \Pi(X), \max(A) := \{x \in A : \forall y \in A xRy \text{ for some } R \in \Pi\} = A$. If, in addition to assuming that $\forall i \in N, \Pi_i = \Pi$, we assume that $\nexists a, b \in X$ such that aSb , then the ranking established in Theorem 1 coincides with the ranking established in Theorem 2 and with the simple cardinality-based ordering of Pattanaik and Xu (1990).

The final remark studies the consequences of adopting the “reasonable” preferences view, in the case of irrelevance of the similarity relation.

Remark 4 If $\forall i \in N, \Pi_i = \Pi^*$, where Π^* stands for the set of reasonable preference profiles *à la* Jones and Sugden (1982) and Pattanaik and Xu (1998), and, moreover, $\nexists a, b \in X$ such that aSb , then the ranking established in Theorem 1 coincides with the rule characterized by Pattanaik and Xu (1998) (Proposition 5.1): given two opportunity situations $(A, \Pi_i), (B, \Pi_j), (A, \Pi_i) \geq (B, \Pi_j)$ if and only if $\# \max_*(A) \geq \# \max_*(B)$, where $\max_*(A)$ represents the choice set of A , as elicited according to the set of reasonable preferences Π^* .

8 Concluding remarks

In this paper we have tried to provide a unified framework for the analysis the opportunity aspect and of the variety aspect of freedom. The basic ingredients of our framework are the following: (i) a notion of similarity, represented by means of a binary similarity relation on the universal set of options (see Pattanaik and Xu, 2000), and (ii) a set of preference relations over the options, interpreted along the lines of Bavetta and Peragine (2000).

We have proposed some axioms that capture both the variety and the opportunity aspect of freedom and studied their logical implications. Two different criteria for ranking opportunity sets have been characterized, where both preferences and

diversity of options do play a role. These criteria generalize some of the rankings proposed so far in the literature.

Our formulation allows for only two levels of similarity. One possible extension of this work would consist of considering a more articulated notion of similarity, for instance along the lines proposed by Bossert et al. (2003). This will be the subject of future research.

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