



Working Paper 09-68  
Business Economics Series 08  
October, 2009

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## OPTIMAL RISK IN MARKETING RESOURCE ALLOCATION\*

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### Abstract

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Marketing resource allocation is increasingly based on the optimization of expected returns on investment. If the investment is implemented in a large number of repetitive and relatively independent simple decisions, it is an acceptable method, but risk must be considered otherwise. The Markowitz classical mean-deviation approach to value marketing activities is of limited use when the probability distributions of the returns are asymmetric (a common case in marketing). In this paper we consider a unifying treatment for optimal marketing resource allocation and valuation of marketing investments in risky markets where returns can be asymmetric, using coherent risk measures recently developed in finance. We propose a set of first order conditions for the solution, and present a numerical algorithm for the computation of the optimal plan. We use this approach to design optimal advertisement investments in sales response management.

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**Keywords:** Resource allocation, coherent risk measures, optimization, sales response models.

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\* This research has been partly supported by the Ministerio de Educación y Ciencia of Spain, through the projects SEJ2007-65897/ECON and SEJ2006-15401-C04, and Comunidad Autónoma de Madrid, through the project S-0505/TIC/000230.

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# 1 Introduction

Firms spend billions of dollars on management strategic and tactic investments, ranging from new product development, to logistics and supply chain decisions, sales force scheduling, market research, communication, sales promotion, and other marketing expenditures. In uncertain markets firm stakeholders increasingly want proof of their marketing investment payback. Traditionally, many companies allocate resources using widespread rules-of-thumb. For example, funding advertising or sales-forces with a “percentage-of-sales”, or a “bottom-up” rule investing until some output-measurement level is achieved, or just imitating the decisions taken by strong competitors or industry-benchmarks. The caveat with these practices is that they are not based on controlled studies. Just because the company is doing well, does not mean that the investment is cause for it, and even so, it does not mean that the same strategy will work in the future or for a different company.

The economic value of any investment is the return on investment (ROI); i.e., the net present value or sum of discounted future returns minus the discounted value of the expenditure (including all variable costs and the amortization of the proportional part of any fixed costs). If this value is computed at customer level, it is known as the Customer Lifetime Value (CLV) for the firm. The return on investment, at firm/specific customers level, should be maximized to determine the resource allocation, and the optimal value must be positive or the investment should not be undertaken. Under perfect certainty, a marketing action will have only one possible return and marketing decisions can be easily planned. However the assumption of perfect foresight is unrealistic and can give misleading insights – the actual ROI and CLV fluctuates randomly.

In an uncertain context, the optimal marketing plan involves choosing between alternative projects assessing their random values. The marketing literature essentially focuses on expected returns maximization. Following the seminal works by Little (1970, 1975), the developments in statistics & econometric theory, OR, and computational capacity for storing and processing data, have spurred the development of management science decision methods for resource allocation (see e.g., Little and Lodish 1969, Lodish 1971, Little 1975, Blattberg and Hoch 1990, Wierenga et al. 1999, Wierenga 2008, Eisenstein and Lodish 2002, Divakar et al. 2005, Natter et al. 2007, Tirenni et al. 2007). The management science procedures essentially consist of two stages. In stage one, the available information (e.g., market data, experiments, and managerial expertise) is used to estimate the firm expected return conditionally on the investment decision and other exogenous variables, and in the second one the optimal decision is made by maximizing the expected economic outcome. If the model includes exogenous variables, then what-if optimal decisions can be considered for different scenarios. Nowadays, these tasks are increasingly implemented by firms using computerized platforms known as Decision Support Systems (DSS) integrating statistical data and managerial judgment to estimate demands and then to apply optimization tools, see e.g. Divakar et al. (2005). For a literature review see Little, (2004a, 2004b), Gupta and Steenburgh (2008),

Shankar (2008) and Wierenga (2008).

Expected value is the most basic form of risk analysis. If the investment is implemented in a large number of repetitive and relatively independent simple decisions, it is an acceptable method. But for distinctive decisions it is not so convenient, since maximizing expected value of investments managers do not take into account the returns' fluctuations and can, therefore, expose companies to considerable risks. Good investment is not just a matter of "pocketing money" on a short-term horizon, and we observe that after some time a proportion of companies fail with little control about the dimension of failure. The recent world-wide financial crisis shows how badly an investor may be positioned without notice due to the emphasis on average returns. In an uncertain market, risk adverse executives must optimize the expected payback on their investments whilst penalize the intrinsic risks. Additionally, if the model includes exogenous variables, then the optimal resource allocation should anticipate the impact on the plan of exogenous scenario changes.

The idea is to invest in projects that have the minimum level of risk with the highest possible return. However, most of the customer relationships and marketing valuation literature use the expected ROI and the balance between return and risk is not considered. As Hogan et al. (2002) point out, an unresolved challenge for marketers is how to adjust for the differential of risk of different customers. Even if risk is taken into account, the decision is based on the variance (see, e.g., Holthausen and Assmus 1982, Zhou and Pham, 2004, and Prakhya, Rajiv and Srinivasan, 2006), giving misleading results when the returns have asymmetric probability distributions.

In the financial context, Markowitz (1952) lays the basis for valuing a portfolio of investments in terms of expected returns and standard-deviations. However, financial and marketing investments usually have asymmetric probability distributions. In this case, the variance is not an accurate measure of the investor risk preferences if a downside risk is more weighted than an upside risk. Markowitz (1991) acknowledges this shortcoming. Since returns are usually asymmetric, the use of mean-variance methods often provide misleading results. This fact is nowadays widespread in finance theory, where a set of alternative valuation methods such as Value at Risk, and Conditional Value at Risk are usually considered. This leads to the notions of *coherent risk and deviation measures* developed by Artzner et al. (1997, 1999) and Rockafellar et al. (2006).

In this paper we present a procedure for marketing resource allocation, accounting for associated risk. In particular we consider coherent risk measures, and present a numerical procedure for the computation of this problem in resource allocation.

The rest of the paper is structured as follows. In Section 2 we review the literature about optimal planning under uncertainty and coherent risk measures. Section 3 presents the general method for resource allocation. In Section 4, we present an application to illustrate the behavior of the method for sales response management. In the concluding remarks section we summarize the findings.

## 2 Optimal resource investment strategies

In this section we consider the problem of optimal investment in a management context, using coherent risk measures. Consider a probabilistic space  $(\Omega, \mathcal{A}, P)$ , where  $\omega$  belongs to a set  $\Omega$  representing states of nature with probability  $P$ . The space of feasible investment decisions is  $\mathcal{X} \subset \mathbb{R}^n$ . Often  $\mathcal{X}$  is a compact set, for example a budget set. We assume that the uncertain return outcome associated to each decision  $x \in \mathcal{X}$  is a random variable  $Y_x = Y(x, \omega)$  with finite variance, and denote by  $\mathcal{Y} = \{Y_x : x \in \mathcal{X}\}$ . Each random variable  $Y_x \in \mathcal{Y}$  induces a Borel probability measure  $\pi_x = P \circ Y_x^{-1}$  on  $\mathbb{R}$ , describing the uncertainty associated to the decision  $x$ . We will denote the cumulative distribution by  $F_{Y_x}(y) = \pi_x \{(-\infty, y]\}$ . Usually, some preliminary statistical analysis allows the decision maker to study the distribution  $\pi_x$ . This setup can be applied to the majority of the management decision problems, where company returns to decisions  $x \in \mathcal{X}$  is given by the random variables  $Y_x$ .

The central investment decision making problem consists of choosing  $x \in \mathcal{X}$  so as to minimize risks in  $Y_x$ . Risk is an ambiguous word. It has been associated with statistical variances and volatility, but in the investment literature risk is generally considered as an overall assessment of potential losses. Henceforth, we will use the expression “deviation measure” when considering generalizations of standard deviation designed to assess variability around the mean, whereas the expression “risk measures” will be used in the assessment of losing scenarios ( $Y_x < 0$ ). The manager must select the investment  $x^*$  solving

$$\min_{x \in \mathcal{X}} \rho(Y_x) \quad (1)$$

where  $\rho$  is a risk measure penalizing losses. Classical risk functions are the minus expected return  $\rho_M(Y_x) = E[-Y_x]$  which is insensitive to risks, and even for investments with  $E[-Y_x] = 0$  investors are exposed to large losses for tail-outcomes  $-Y_x > 0$ . Expectations are suitable for some long-range strategies where stochastic ups and downs tend to safely average out, but not in a short-range operation. Markowitz’s (1952) criteria

$$\rho_{MD}(Y_x) = E[-Y_x] + \gamma \sqrt{\text{Var}[Y_x]} \quad (2)$$

where  $\gamma > 0$  is a “safety” scaling, is a widespread method to introduce safety margins using standard deviations. It enhances solutions  $x^*$  with expected loss  $E[-Y_{x^*}]$  not just zero but reassuringly negative. This is due to the fact that losses  $-Y_x > 0$  will occur only in states associated to the high end of the distribution of  $-Y_x$  lying more than  $\gamma \sqrt{\text{Var}[Y_x]}$  units above the mean loss.

Unfortunately, the mean-deviation risk penalizes investments with high returns fluctuations around the mean. As the distribution  $\pi_x$  is generally asymmetric (a distributional asymmetry is found in the returns of almost any managerial investment), the use of mean-variance methods often provides misleading results. One of the difficulties is to construct a proper measure of risk  $\rho(Y_x)$  allowing a higher weight for downside risks than for upside ones.

Over the past decades, the important limitations of the mean-variance and expected utility approaches have led banks and financial regulators to develop a variety of alternative risk measures for the purpose of better quantifying the financial risks that they face. Most of these methods can be extended to the management and marketing context. A popular risk measure is the Value at Risk (VaR) with confidence level  $\alpha \in (0, 1)$ , defined as a distributional percentile of losses,

$$VaR_\alpha(Y_x) = -F_{Y_x}^{-1}(\alpha) = -\inf\{z : P(Y_x \leq z) > \alpha\},$$

which use was proposed by the *Global Derivatives Study Group* (GDSG, 1993). This criterion has been already used to measure risk in marketing investments, by Dickson and Giglierano (1986) who call it *sinking-the-boat-risk*. If  $Y_x$  is normally distributed, the minimization of VaR is almost equivalent to the Markowitz mean-deviation model, as  $VaR_\alpha(Y_x) = E[-Y_x] + \Phi^{-1}(\alpha) \sqrt{Var[Y_x]}$ , where  $\Phi^{-1}(\alpha)$  is the  $\alpha$ -quantile of a standard normal distribution. But for asymmetric distributions it has a better performance.

A major problem of the VaR is that it does not inform about the likely size of the loss that we can have when that value is achieved, as a consequence the Conditional Value-at-Risk (CVaR) risk measure with confidence level  $\alpha$  has been proposed in the finance literature, given by

$$CVaR_\alpha(Y) = -\frac{1}{\alpha} \int_0^\alpha F_Y^{-1}(\epsilon) d\epsilon = \frac{1}{\alpha} \int_0^\alpha VaR_\epsilon(Y) d\epsilon.$$

If  $Y$  has an absolutely continuous probability distribution, then

$$CVaR_\alpha(Y) = -E[Y|Y \leq F_Y^{-1}(\alpha)] = -E[Y|Y \leq -VaR_\alpha(Y)].$$

and we can interpret the *CVaR* as the expected shortfall or expected loss for an investment whose return does not exceed a predetermined quantile threshold. This risk measure has been recognized and encouraged by the Basel Committee on Banking Supervision of the *Bank for International Settlement* (BIS, 2006). Once again, if  $Y$  is normally distributed, the minimization of CVaR is almost equivalent to the Markowitz mean-deviation model<sup>1</sup>.

There exists an alternative approach for measuring risk, based on von Neumann and Morgenstern's (1944) expected utility theory, which stems from Bernoulli's work (1738). Regular preferences on losses (satisfying the completeness, transitiveness, continuity and independence axioms) can be expressed by

$$\rho(Y) = -\int u(y) dF_Y(y) = -\int_0^1 u(F_Y^{-1}(y)) dy,$$

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<sup>1</sup>For normally distributed returns,

$$CVaR_\alpha(Y) = -E[Y_x] + \frac{\phi(\Phi^{-1}(\alpha))}{\alpha} \sqrt{Var[Y_x]}$$

where  $\phi$  is the density of a standard normal distribution.

where  $u$  is a non-negative function of the returns. A concave function  $u$  was considered to introduce risk-aversion. The expected utility theory is refuted in experiments showing that this approach is not compatible with the decision-makers behavior (e.g., Tversky and Kahneman, 1979). Wang (2000) proposed an alternative procedure, applying weights  $\nu(\cdot)$  on  $F_Y$  as in  $\int u(y) d(\nu \circ F_Y)$ , and considering also  $u(y) = y$ , leading to a more realistic risk-measure

$$\rho(Y) = - \int y d(\nu \circ F_Y)(y) = - \int_0^1 F_Y^{-1}(y) d\nu(y).$$

This family can be found in the statistical literature as L-statistics. In risk-analysis context, the weight measure  $\nu$  is usually asymmetric around some anchor point. The VaR is a particular case with

$$\nu(y) = \begin{cases} 0 & y \leq \alpha \\ 1 & y > \alpha, \end{cases}$$

and the CVaR with

$$\nu(y) = \begin{cases} y/\alpha^{-1} & y \leq \alpha \\ 1 & y > \alpha. \end{cases}$$

A variety of measures  $\nu$  can be considered alike.

Fueled by these developments, all the classical financial problems are being revisited by the researchers, and the new risk measures are being infused into insurance and finance daily practice. But not all these risk measures are equally popular, and some seem less useful for investment decision planning. To clarify these issues, finance scholars have developed a conceptual theory of risk measurement.

## 2.1 Coherent risk measures

To successfully incorporate the new risk measures into decision models, they must be compatible with the axiomatic required for decision making problems. Artzner et al. (1997, 1999) first present and justify a set of risk-measure axioms – the axioms of coherency – and this axiomatic has become one of the most important recent achievements in the financial risk area, and they were reformulated later by Rockafellar et al. (2006). Consider the class  $L^2$  of random variables  $Y$  with  $E[|Y|^2] < \infty$ . Having  $Y \in L^2$  ensures that both the mean and the standard deviation of  $Y$  are well defined and finite. According to Rockafellar et al. (2006),

**Definition 1** *A function  $\rho$  defined over  $L^2$  is said to be a coherent measure (in the basic sense) if it satisfies:*

- (R1)  $\rho(c) = -c$  for all  $c \in R$ .
- (R2) *Convexity:*  $\rho((1 - \lambda)Y + \lambda Y') \leq (1 - \lambda)\rho(Y) + \lambda\rho(Y')$  for all  $Y, Y' \in L^2$  and  $\lambda \in (0, 1)$
- (R3) *Monotonicity:*  $\rho(Y) \leq \rho(Y')$  for all  $Y, Y' \in L^2$  s.t.  $\Pr(Y \geq Y') = 1$

(R4) *Closedness*: for all  $Y_n, Y \in L^2$ , if  $E \left[ |Y_n - Y|^2 \right] \rightarrow 0$  and  $\rho(Y_n) \leq c$ , then  $\rho(Y) \leq c$

(R5) *Positive homogeneity*:  $\rho(\lambda Y) = \lambda \rho(Y)$  for all  $Y \in L^2$ ,  $\lambda > 0$

$\rho$  is a coherent measure in the extended sense if it satisfies (R1) – (R4) but not necessarily (R5).

Note that R1 with R2 imply that  $\rho(Y + c) = \rho(Y) - c$  for all  $Y \in L^2$ ,  $c \in \mathbb{R}$ , and R1 with R5 imply that  $\rho(Y + Y') \leq \rho(Y) + \rho(Y')$  for all  $Y, Y' \in L^2$ , furthermore this subadditive property with R5 implies R2. The original definition of coherency in Artzner et al. (1999) requires (R5), but some authors have questioned it and that is why the “extended sense” concept is sometimes used.

For example, for any weight  $\gamma \geq 0$  the following risk measures

|                               |  |
|-------------------------------|--|
| LEAST ABSOLUTE DEVIATION      | $\rho(Y) = E[-Y] + \gamma \cdot \sqrt{E[ Y - E[Y] ]}$ ,              |
| SEMIDEVIATION                 | $\rho(Y) = E[-Y] + \gamma \cdot \sqrt{E[(\max\{Y - E[Y], 0\})^2]}$ , |
| CVAR WITH $\alpha \in (0, 1)$ | $\rho(Y) = E[-Y] + \gamma \cdot CVaR_\alpha(Y - E[Y])$ ,             |

are coherent risk measures in the basic sense. The mean-deviation measure  $\rho(Y) = E[-Y] + \gamma \cdot \sqrt{Var[Y]}$  is not coherent since the monotonicity condition is not satisfied. The value at risk  $\rho(Y) = VaR_\alpha Y$  is not coherent either (the subadditivity is not satisfied, and convexity fails), and for that reason is less commonly used than the CVaR. We will emphasize the CVaR, which satisfies

$$\rho(Y) = E[-Y] + \gamma \cdot CVaR_\alpha(Y - E[Y]) = (\gamma - 1) E[Y] + \gamma \cdot CVaR_\alpha(Y).$$

In particular, for  $\gamma = 1$  we obtain  $\rho(Y) = CVaR_\alpha(Y)$ . The worst-case risk measure  $\rho(Y) = \sup_{\omega \in \Omega} Y(\omega)$  is also coherent, but very conservative. Note that when  $\alpha \nearrow 1$  the  $CVaR_\alpha(Y) \searrow E[Y]$  (risk insensitive), and when  $\alpha \searrow 0$  the  $CVaR_\alpha(Y) \nearrow \sup Y$  (conservative assessment), and typically a significance level of 0.05 is used.

A quite general way to define coherent risk measures is to correct the mean-deviation measure by replacing the element  $\gamma \sqrt{Var[Y_x]}$  in (2) by a more general measure of deviation with respect to the mean,  $D(Y)$ . Rockafellar et al. (2006) introduce an axiomatic approach to deviation measures of financial outcomes around its mean, inspired by the work of Artzner et al. (1999). A coherent risk measure  $\rho$  in the basic sense is “risk averse” if it satisfies the lower bound (R6) :  $\rho(Y) > E[-Y]$  for all nonconstant  $Y$  (aversity). Rockafellar et al. (2006) prove a one-to-one match between deviation and risk averse measures, considering

$$\begin{aligned} \rho(Y) &= E[-Y] + D(Y), \\ D(Y) &= \rho(Y - E[Y]). \end{aligned}$$

There is a useful characterization for coherent risk measures in the basic sense. Any coherent risk measure  $\rho(Y)$  defined in  $L^2$  can be written as

$$\rho(Y) = \max_{q \in Q} E[-Yq]$$

where  $Q$  is a uniquely defined<sup>2</sup> convex and weakly compact set of functions in  $L^2$ . The set  $Q$  can be characterized as  $Q = \{q \in L^2 : E[-Yq] \leq \rho(Y), \forall Y \in L^2\}$ . There are also extensions of this result for coherent measures in the extended case, see Rockafellar (2007, Theorem 4). The set  $Q$  has been computed for the many coherent risk measures (see, e.g., Rockafellar et al. 2006). For example, if  $\rho(Y) = E[-Y]$  then  $Q = \{1\}$ . If  $\rho(Y) = \sup Y$ , then  $Q = \{q \in L^2 : q \geq 0, E[q] = 1\}$ . For the  $CVaR_\alpha$  with  $\alpha \in (0, 1)$ , the set  $Q$  is given by

$$Q = \{q \in L^2 : 0 \leq q \leq \alpha^{-1}, E[q] = 1\}. \quad (3)$$

Unfortunately, in most cases  $\rho(Y)$  is not differentiable, and specific algorithms are required to minimize this measure. In the next section we present a numerical method for risk minimization based on this characterization.

### 3 A scenario planning approach for optimal-risk managerial investment

In most practical stochastic decision problems, the probability distribution is approximated by discrete distributions with a countable number of outcomes called scenarios. Optimal investment planning based on coherent risk measures can be easily handled with scenarios. In this section we present an algorithm for coherent risk optimization based on “scenario planning.” In particular, we consider that  $x$  is an investment in monetary units,  $\mathcal{X}$  is the budget set, and  $\mathcal{Y} = \{Y_x : x \in \mathcal{X}\}$  are the possible financial return outcomes.

Minimizing  $\rho(Y_x)$  in  $x \in \mathcal{X}$  is not trivial, since the functional  $\rho(Y_x)$  is not differentiable. In this paper we present a scenario-based algorithm to minimize  $\rho(Y_x)$  in  $x \in \mathcal{X}$  which can be used for any coherent risk measure. We assume henceforth that

$$\rho(Y_x) = E[-Y_x] + \gamma \cdot CVaR_\alpha(Y_x - E[Y_x]), \quad (4)$$

being  $0 < \alpha < 1$  the level of confidence, and  $\gamma \geq 0$ . Furthermore, we will assume that all the random variables in  $Y_x \in \mathcal{Y}$  take discrete values on the scenario revenues set  $\{y_0, y_1, y_2, y_3, \dots\}$ , with probability

$$\pi_x(n) = \Pr\{Y_x = y_n\}$$

for  $n = 0, 1, 2, 3, \dots$ . In many cases the scenario set is infinite, but sometimes a finite discretization  $\{y_0, y_1, y_2, \dots, y_K\}$  suffices to obtain a good insight. Note that firm managers tend to do a softer version of “scenario planning,” considering one or at most two alternatives to a “most likely” base case. The consequence of this simplification an extrapolative image of the future in which business risks

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<sup>2</sup>Alternative notations can be found. Define  $\Pi = \{\pi = qP : q \in Q\}$ . Since  $\int qdP = 1$ , then  $\Pi$  is a family of probability distributions and  $\rho(Y) = \max\{E_\pi[-Y] : \pi \in \Pi\}$  can be interpreted as a type of preferences under ambiguity of probabilities (see Ellsberg 1961, Gilboa and Schmeidler 1989).



and opportunities are more or less similar to the present average. To avoid these limitations, we will discuss a general algorithm that works for large scenario sets, and can be applied even in some infinite scenario cases, as we illustrate in the next subsections.

For all variables in  $Y_x \in L^2$  with probability  $\pi_x(n)$ , the risk of the return  $Y_x$ , is given by:

$$\begin{aligned} \rho(Y_x) &= (\gamma - 1) E[Y_x] + \gamma \cdot \max_{q \in Q} E[-Y_x q] \\ &= (\gamma - 1) E[Y_x] - \gamma \cdot \min_{\{q_n\} \in Q} \sum_{n=0}^{\infty} y_n q_n \pi_x(n), \quad (5) \\ Q &= \left\{ \{q_n\} \in L^2 : \sum_{n=0}^{\infty} q_n \pi_x(n) = 1, 0 \leq q_n \leq \alpha^{-1} \right\} \end{aligned}$$

using that  $E[q] = 1$ . Since we consider sequences  $\{q_n\}$  which are bounded, we are actually optimizing in the space of bounded sequences  $L^\infty$ . The risk measure can be addressed solving the linear minimization problem in  $q \in Q$ , as the solution to the Lagrangian saddle point in  $L^2 \cap L^\infty$ , with

$$\begin{aligned} L\left(\{q_n, \lambda_n^L, \lambda_n^U\}, \lambda^o, x\right) &= \sum_{n=0}^{\infty} y_n q_n \pi_x(n) + \lambda^o \left( \sum_{n=0}^{\infty} q_n \pi_x(n) - 1 \right) \\ &\quad - \sum_{n=0}^{\infty} \lambda_n^U (\alpha^{-1} - q_n) - \sum_{n=0}^{\infty} \lambda_n^L q_n. \end{aligned}$$

The Karush-Kuhn-Tucker (KKT) first order conditions (necessary and sufficient) for a solution are:

$$\begin{aligned} y_n \pi_x(n) &= \lambda^o \pi_x(n) + \lambda_n^U - \lambda_n^L, & n = 0, 1, 2, \dots \\ \sum_{n=0}^{\infty} q_n \pi_x(n) - 1 &= 0, \\ \lambda_n^L q_n &= 0, & n = 0, 1, 2, \dots \\ \lambda_n^U (\alpha^{-1} - q_n) &= 0, & n = 0, 1, 2, \dots \\ \lambda_n^L, \lambda_n^U &\leq 0, & n = 0, 1, 2, \dots \end{aligned} \quad (6)$$

The optimal solution  $\{q_n^*(x)\} \in Q$  depends on  $x \in \mathcal{X}$ .

**Theorem 2** *Whenever  $k \leq \lambda^o < k + 1$ , with  $k \in \mathbb{N}$ , the solution of problem (5) is defined as:*

$$\begin{aligned} q_n^*(x) &= \frac{1}{\alpha}, \text{ for } n = 0, 1, 2, \dots, k - 1, \\ q_k^*(x) &= \left( 1 - \frac{1}{\alpha} \sum_{n=0}^{k-1} \pi_x(n) \right) \frac{1}{\pi_x(k)}, \\ q_n^*(x) &= 0, \text{ for } n = k + 1, k + 2, \dots \end{aligned}$$

**Proof.**

First, note that  $\lambda_0^L = 0$ . Assume that  $\lambda_0^L \neq 0$ , then  $q_0 = 0$ , and  $\sum_{n=1}^{\infty} q_n \pi_x(n) = 1$ , implying that there are infinite solutions for problem (5).

Assume that  $\lambda^o = 0$ , then  $\lambda_0^U = 0$ , and  $q_0^* \leq 1/\alpha$ ,  $q_0^* \neq 0$ . Furthermore, for all  $n = 1, 2, \dots$ ,  $\lambda_n^U = y_n \pi_x(n)$  or  $\lambda_n^L = -y_n \pi_x(n)$ , since at least one of them is not equal to zero. As,  $\lambda_n^U \leq 0$ , necessarily  $\lambda_n^U = 0$ ,  $\lambda_n^L = -y_n \pi_x(n)$  and  $q_n^* = 0$  for all  $n = 1, 2, \dots$ . Then,  $q_0^* = \frac{1}{\pi_x(0)}$ .

Assume that  $k \leq \lambda^o < k + 1$ . If  $\lambda_n^L = 0$ , for  $n = k + 1, k + 2, \dots$ , then  $\lambda_n^U = (y_n - \lambda^o) \pi_x(n) \leq 0$ , implying  $y_n \leq \lambda^o$  which is a contradiction, and therefore,  $\lambda_n^L \neq 0$  for  $n = k + 1, k + 2, \dots$  and  $q_n^* = 0$  for  $n = k + 1, k + 2, \dots$ ; as a consequence  $\lambda_n^U = 0$  for  $n = k + 1, k + 2, \dots$ . Therefore, problem (5) can be rewritten as:

$$\min \left\{ \sum_{n=1}^k q_n y_n \pi_x(n) : \sum_{n=0}^k q_n \pi_x(n) = 1, 0 \leq q_n \leq \alpha^{-1} \right\}.$$

Since

$$\sum_{n=1}^k q_n y_n \pi_x(n) = \lambda^o + \sum_{n=1}^k q_n (\lambda_n^U - \lambda_n^L),$$

and  $\lambda_n^L, \lambda_n^U \leq 0$ , the optimal solution of problem (5) is  $\lambda_n^U = 0$ ,  $\lambda_n^L \neq 0$ , for all  $n = 0, 1, \dots, k - 1$ , therefore,  $q_n^* = 1/\alpha$ , for all  $n = 0, 1, \dots, k - 1$ , and the constraint  $\sum_{n=0}^{\infty} q_n \pi_x(n) - 1 = 0$ , define the optimal solution of  $q_k$  as

$$q_k^* = \frac{1}{\pi_x(k)} \left( 1 - \frac{1}{\alpha} \sum_{n=0}^{k-1} \pi_x(n) \right) \leq \frac{1}{\alpha}.$$

□

**Corollary 3** *Under the conditions of Theorem 2, the risk function is given by*

$$\rho(Y_x) = (\gamma - 1) E[Y_x] + \frac{\gamma}{\alpha} \sum_{n=0}^{k-1} (y_k - y_n) \pi_x(n) - \gamma y_k.$$

**Proof.**

It follows directly from Theorem 2, using that

$$\begin{aligned}
\rho(Y_x) &= (\gamma - 1) E[Y_x] - \gamma \cdot \min_{\{q_n\} \in Q} \sum_{n=0}^{\infty} y_n q_n \pi_x(n) = \\
&= (\gamma - 1) E[Y_x] - \frac{\gamma}{\alpha} \sum_{n=0}^{k-1} y_n \pi_x(n) - \frac{\gamma}{\pi_x(k)} \left( 1 - \frac{1}{\alpha} \sum_{n=0}^{k-1} \pi_x(n) \right) y_k \pi_x(k) = \\
&= (\gamma - 1) E[Y_x] + \frac{\gamma}{\alpha} \sum_{n=0}^{k-1} (y_k - y_n) \pi_x(n) - \gamma y_k.
\end{aligned}$$

□

**Corollary 4**  $k$  is the solution of problem

$$\min \left\{ k : \sum_{n=0}^k \pi_x(n) \geq \alpha \right\}.$$

**Proof.**

It follows directly from Theorem 2, using that:

$$\frac{1}{\pi_x(k)} \left( 1 - \frac{1}{\alpha} \sum_{n=0}^{k-1} \pi_x(n) \right) \leq \frac{1}{\alpha}, \text{ for all } x;$$

i.e.,

$$\sum_{n=0}^k \pi_x(n) \geq \alpha, \text{ for all } x.$$

Then, we should consider the minimum  $k$  such that  $\sum_{n=0}^k \pi_x(n) \geq \alpha$ . □

The previous results can be used to compute  $\rho(Y_x)$ . However, it is not immediate how to solve Problem (1) since changes in  $x$  often imply changes in  $k$ . Since a grid evaluation procedure is particularly inefficient in high dimensions, we propose a two-step algorithm to solve  $\min_{x \in \mathcal{X}} \rho(Y_x)$  using Theorem 2 and Corollary 4. A summary of the proposed algorithm is:

**Algorithm 5**

**Step 1.** Select the final stop tolerance  $\epsilon_{TOL}$ . Initialize variables  $x \in X$ .

**Step 2.** Find  $k = k(x)$  as the solution of problem

$$\min \left\{ k : \sum_{n=0}^k \pi_x(n) \geq \alpha \right\}. \quad (7)$$

**Step 2.1.** Compute  $q = q(k, x)$  as follows:

$$\begin{aligned} q_n &= \frac{1}{\alpha}, \text{ for } n = 0, 1, 2, \dots, k-1, \\ q_k &= \frac{1}{\pi_x(k)} \left( 1 - \frac{1}{\alpha} \sum_{n=0}^{k-1} \pi_x(n) \right), \\ q_n &= 0, \text{ for } n = k+1, k+2, \dots \end{aligned}$$

**Step 2.2.** Solve the problem

$$\min_{x \in \mathcal{X}} \left\{ (\gamma - 1) E[Y_x] + \frac{\gamma}{\alpha} \sum_{n=0}^{k-1} (y_k - y_n) \pi_x(n) - \gamma y_k \right\}. \quad (8)$$

Denote by  $\bar{x}$  the solution of this problem.

**Step 2.3.** Update new point  $x \leftarrow \bar{x}$ . Repeat until

$$|x - \bar{x}| \leq \epsilon_{TOL}.$$

Next let us show that the convergence of sequences  $(x_j)_{j=1}^{\infty} \subset \mathcal{X}$  and  $(q_j)_{j=1}^{\infty} \subset Q$ , as constructed in Algorithm (5), implies that the limit of  $(x_j)_{j=1}^{\infty}$  solves Problem (1). Actually we can prove a more general result.

**Lemma 6 Lemma.** Consider two topological spaces  $A$  and  $B$  and a continuous function  $V : A \times B \rightarrow \mathbb{R} \cup \{+\infty\}$ . Consider also two sequences  $(a_j)_{j=1}^{\infty} \subset A$  and  $(b_j)_{j=1}^{\infty} \subset B$  such that

$$V(a_j, b_j) = \text{Max} \{V(a_j, b); b \in B\} \quad (9)$$

and

$$V(a_{j+1}, b_j) = \text{Min} \{V(a, b_j); a \in A\} \quad (10)$$

for every  $j \in \mathbb{N}$ . Suppose that there exists

$$(a_0, b_0) = \lim_{j \rightarrow \infty} (a_j, b_j).$$

Then,

i)  $(a_0, b_0)$  is a saddle point of  $V$ , i.e.,

$$V(a_0, b) \leq V(a_0, b_0) \leq V(a, b_0) \quad (11)$$

holds for every  $a \in A$  and every  $b \in B$ .

ii)  $a_0$  solves the optimization problem

$$\left\{ \begin{array}{l} \text{Min} \\ a \in A \end{array} \left( \text{Sup} \{V(a, b); b \in B\} \right) \right.$$

**Proof.**

*i)* If  $a \in A$  we have that (10) implies the inequality

$$V(a, b_0) = \lim_{j \rightarrow \infty} V(a, b_j) \geq \lim_{j \rightarrow \infty} V(a_{j+1}, b_j) = V(a_0, b_0).$$

Besides, if  $b \in B$  (9) leads to

$$V(a_0, b) = \lim_{j \rightarrow \infty} V(a_j, b) \leq \lim_{j \rightarrow \infty} V(a_j, b_j) = V(a_0, b_0).$$

*ii)* We must prove the expression

$$\text{Sup} \{V(a_0, b); b \in B\} \leq \text{Sup} \{V(a, b); b \in B\}$$

for every  $a \in A$ . The first inequality of (11) leads to

$$\text{Sup} \{V(a_0, b); b \in B\} = V(a_0, b_0).$$

On the other hand, it is obvious that

$$\text{Sup} \{V(a, b); b \in B\} \geq V(a, b_0)$$

for every  $a \in A$ , and bearing in mind the second inequality in (11) we have that

$$\text{Sup} \{V(a, b); b \in B\} \geq V(a_0, b_0)$$

for every  $a \in A$ . □

## 4 Risk minimization in sales response management

Successful marketing budget allocation requires an optimal use of sales response models. To get firm money's worth from any marketing campaign, managers must control the risk of the expected loss. We will consider a firm planning the marketing expenditure  $x$ , where the range of feasible decisions is the budget set  $\mathcal{X} = \{x \in \mathbb{R}^n : 0 \leq x \leq M\}$  and  $\{Y_x : x \in \mathcal{X}\}$  are the associated outcomes. Note that the returns are given by  $Y_x = mS_x - x$  where  $m > 0$  is the unit margin and  $S_x$  is the sales response associated with the decision  $x$ . In particular  $S_x$  is a random variable taking discrete values  $\{0, 1, 2, \dots\}$ , and following a probability distribution  $\pi_x$ . We consider that the firm considers the risk of loss as given by (4).

To illustrate the proposed approach, we will assume that sales response  $S_x$  follows: (case 1) Poisson distribution with parameter  $\mu_x > 0$ , and (case 2) Negative Binomial distribution with parameter  $p_x = r / (r + \mu_x)$ , where  $r, \mu_x > 0$ . Furthermore, we assume that  $\mu_x$  follows an ADBUG function

$$\mu(x) = \beta_0 + (\beta_1 - \beta_0) \frac{x^\gamma}{(\beta_2 + x)^\gamma} \in (\beta_0, \beta_1)$$

If  $x = 0$ , then the mean of sales is equal to the floor  $\mu(x) = \beta_0$ . When  $x$  grows it tends to the maximum level  $\mu(x) = \beta_1 > \beta_0$ . If  $\gamma > 1$ , the curve is S-shaped, whereas for  $0 < \gamma \leq 1$ , we obtain a concave function, (see e.g., Little, 2004a). We have implemented the algorithm 5 using MATLAB 7.6 on a Mobile Workstation Intel<sup>®</sup> Centrino<sup>®</sup> Pro<sup>™</sup> with machine precision  $10^{-16}$ .

#### 4.1 Case 1. The Poisson distribution

Assume that the sales response  $S_x$  follows a Poisson distribution,

$$\pi_n(x) = \pi(S_x = n) = \frac{\mu_x^n}{n!} \exp(-\mu_x), \quad n = 0, 1, 2, \dots$$

where the expected sales response  $\mu_x$  follows the ADBUG model with  $\beta_0 = 0.1$ ,  $\beta_1 = 50$ ,  $\beta_2 = 2$  and  $\gamma = 3$ . We consider a unit margin  $m = 100$ . The budget constraint is  $\mathcal{X} = \{x \in \mathbb{R} : 0 \leq x \leq M\}$  with  $M = 1000$ .

Using Theorem 2, whenever  $k \leq \lambda^o \leq k + 1$ , with  $k \in \mathbb{N}$ , the solution of problem is defined by:

$$\begin{aligned} q_n^*(x) &= \frac{1}{\alpha}, \text{ for } n = 0, 1, 2, \dots, k-1, \\ q_k^*(x) &= \left( e^{\mu_x} - \frac{1}{\alpha} \sum_{n=0}^{k-1} \frac{\mu_x^n}{n!} \right) \frac{k!}{\mu_x^k}, \\ q_n^*(x) &= 0, \text{ for } n = k+1, k+2, \dots \end{aligned}$$

and using Corollary 4,  $k$  is the solution of problem

$$\min \left\{ k : \alpha \leq e^{-\mu_x} \left( \sum_{n=1}^k \frac{\mu_x^n}{n!} \right) \right\}.$$

Table 1 summarizes the optimal decision  $x^*$  for the (4) risk measure with different weights  $\gamma \geq 0$ , and significance level  $\alpha = 0.05$ . We compare the results with the ones obtained from the mean-deviation criteria,

$$\begin{aligned} \rho_{MD}(Y_x) &= E[-Y_x] + \gamma \sqrt{\text{Var}[Y_x]} = E[x - m S_x] + \gamma m \sqrt{\text{Var}[S_x]} \\ &= x - m \mu(x) + \gamma m \sqrt{\mu(x)}. \end{aligned}$$

For  $\gamma = 0$  we obtain  $\rho_{CVaR} = \rho_{MD} = E[-Y_x]$  the expected loss criteria.

TABLE 1: Optimal decisions for (4), and different mean-deviation measures

|                               | $\gamma = 0$              |               | $\gamma = 0.1$ |               | $\gamma = 0.4$ |               | $\gamma = 1$ |  |
|-------------------------------|---------------------------|---------------|----------------|---------------|----------------|---------------|--------------|--|
| Risk measure                  | $\rho_{CVaR} = \rho_{MD}$ | $\rho_{CVaR}$ | $\rho_{MD}$    | $\rho_{CVaR}$ | $\rho_{MD}$    | $\rho_{CVaR}$ | $\rho_{MD}$  |  |
| $x^*$                         | 169.00                    | 169.21        | 168.38         | 169.61        | 166.49         | 170.02        | 162.66       |  |
| $E[-Y_{x^*}]$                 | -4657.95                  | -4657.95      | -4657.94       | -4657.94      | -4657.91       | -4657.94      | -4657.71     |  |
| $\sqrt{\text{Var}[-Y_{x^*}]}$ | 694.76                    | 694.77        | 694.71         | 694.80        | 694.57         | 694.83        | 694.28       |  |

When  $\gamma$  increases the CVaR tends to invest more than the mean-deviation criteria. The mean and variance of the optimal investments shows little difference, with gaps increasing with the weight  $\gamma$ . Note that the loss  $-Y_{x^*}$  associated to the optimal decision is generally an asymmetric random variable. The third order moment is an indicator of tail-asymmetry of losses distribution from the origin,

$$E \left[ (-Y_x)^3 \right] = E \left[ (x - mS_x)^3 \right] = x^3 - 3mx^2E[S_x] - m^3E[S_x^3] + 3m^2xE[S_x^2],$$

which will be negative when higher profits (negative losses tail) are more likely than losses (positive losses tail). The more negative this value is, the better the financial outcomes of the investment tend to be.

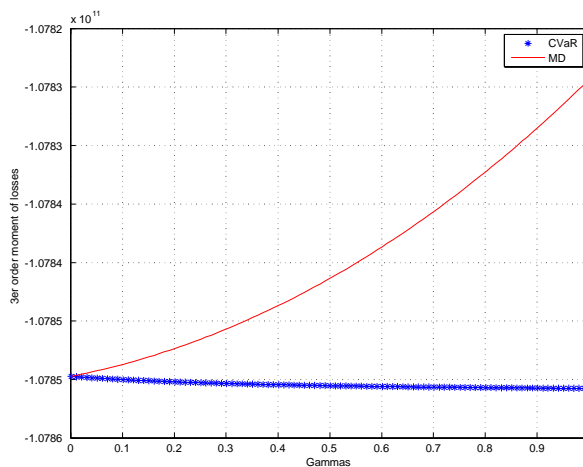


Figure 1. Third order moment for optimal losses using (4) and mean-deviation, using different weights  $\gamma \geq 0$ .

Figure 1 shows the  $E \left[ (-Y_x)^3 \right]$  for the optimal decisions based on (4) with  $\alpha = 0.05$ , and different mean-deviation measures when  $\gamma$  increases. The results show a clear advantage for the CVaR. In all the cases, as the returns have asymmetric probability distributions, the optimal decision based on the CVaR risk measure has higher asymmetry coefficient, which suggests lower losses.

## 4.2 Case 2. The Negative Binomial

For a Poisson distribution, the mean is equal to the variance. Empirical researchers often find this assumption unrealistic, as conditional variance of data exceeds the conditional mean, which is usually referred to as “overdispersion” (relative to the Poisson model). The standard model for count data model with overdispersion is the negative binomial. Assume that the sales  $S_x$  have a

negative Binomial distribution, so that

$$\pi_n(x) = \pi(S_x = n) = \frac{\Gamma(n+r)}{\Gamma(n+1)\Gamma(r)} \left(1 + \frac{\mu_x}{r}\right)^{-r} \left(\frac{\mu_x}{r + \mu_x}\right)^n, \quad n = 0, 1, 2, \dots$$

with  $r, \mu_x > 0$ , and  $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$  the Gamma function, with  $\Gamma(n+1) = n!$  When  $r \rightarrow \infty$  the negative Binomial tends to the Poisson distribution with parameter  $\mu_x > 0$ . We assume that  $\mu_x$  has an ADBUG model with  $\beta_0 = 1$ ,  $\beta_1 = 1000$ ,  $\beta_2 = 2$  and  $\gamma = 3$ . We consider a unit margin  $m = 20$ , and a budget constraint  $\mathcal{X} = \{x \in \mathbb{R} : 0 \leq x \leq M\}$  with  $M = 1000$ .

Using Theorem 2, whenever  $k \leq \lambda^o \leq k+1$ , with  $k \in \mathbb{N}$ :

$$\begin{aligned} q_n^*(x) &= \frac{1}{\alpha}, \text{ for } n = 0, 1, 2, \dots, k-1, \\ q_k^*(x) &= \left(1 - \frac{1}{\alpha} \frac{\Gamma(k+1)}{\Gamma(k+r)} \frac{1}{(1-p_x)^k} \sum_{n=0}^{k-1} \frac{\Gamma(n+r)}{\Gamma(n+1)} (1-p_x)^n\right), \\ q_n^*(x) &= 0, \text{ for } n = k+1, k+2, \dots \end{aligned}$$

Using Corollary 4,  $k$  is the solution of problem

$$\min \left\{ k : \frac{p_x^r}{\Gamma(r)} \sum_{n=0}^k \frac{\Gamma(n+r)}{\Gamma(n+1)} (1-p_x)^n \geq \alpha \right\}.$$

We consider two cases for the Binomial negative:  $r = 1$  (we obtain relatively symmetric distributions), and  $r = 0.1$  (where we obtain asymmetric distributions).

TABLE 2: Optimal decisions for (4), and mean-deviation measures for  $r = 1$  (relatively symmetric distributions)

|                        | $\gamma = 0$              | $\gamma = 0.1$ |             | $\gamma = 0.4$ |             | $\gamma = 1$  |             |
|------------------------|---------------------------|----------------|-------------|----------------|-------------|---------------|-------------|
| Risk measure           | $\rho_{CVaR} = \rho_{MD}$ | $\rho_{CVaR}$  | $\rho_{MD}$ | $\rho_{CVaR}$  | $\rho_{MD}$ | $\rho_{CVaR}$ | $\rho_{MD}$ |
| $x^*$                  | 342.22                    | 342.16         | 324.45      | 342.03         | 264.17      | 341.90        | 0           |
| $E[-Y_{x^*}]$          | -19311.53                 | -19311.53      | -19310.57   | -19311.53      | -19288.82   | -19311.53     | -19.99      |
| $\sqrt{Var}[-Y_{x^*}]$ | 19663.75                  | 19663.69       | 19645.02    | 19663.57       | 19562.99    | 19663.43      | 28.28       |

As we can observe, even with  $r = 1$  where the Binomial negative is a relatively symmetric distribution, when  $\gamma$  increases to one the Mean-Deviation criteria stops the investment, whereas the CVaR varies more smoothly when the penalty  $\gamma$  is increased.

Figure 2 shows a clear advantage of the CVaR in terms of third order moment  $E[(-Y_{x^*})^3]$  from the origin, as the negative loss tail is heavier for optimal



investments based on the CVaR than for the optimal decisions using Mean-Deviation risk measures.

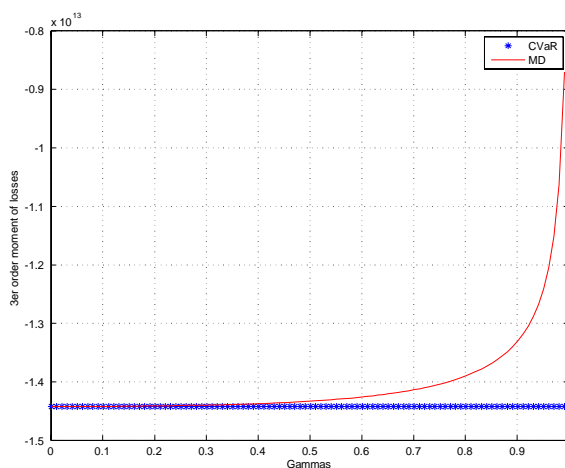


Figure 2. Third order moment for optimal losses using (4) and mean-deviation, using different weights  $\gamma \geq 0$ , for a Negative Binomial with  $r = 1$ .

We have computed the experiment for relatively more asymmetric distributions. Table 3 summarizes the findings.

TABLE 3: Optimal decisions for (4), and mean-deviation measures for  $r = 0.1$  (asymmetric distributions)

|                        | $\gamma = 0$              | $\gamma = 0.1$ |             | $\gamma = 0.4$ |             | $\gamma = 1$  |             |
|------------------------|---------------------------|----------------|-------------|----------------|-------------|---------------|-------------|
| Risk measure           | $\rho_{CVaR} = \rho_{MD}$ | $\rho_{CVaR}$  | $\rho_{MD}$ | $\rho_{CVaR}$  | $\rho_{MD}$ | $\rho_{CVaR}$ | $\rho_{MD}$ |
| $x^*$                  | 342.22                    | 326.37         | 282.29      | 289.55         | 0           | 242.79        | 0           |
| $E[-Y_{x^*}]$          | -19311.53                 | -19310.77      | -19298.98   | -19302.08      | -19.99      | -19271.47     | -19.99      |
| $\sqrt{Var}[-Y_{x^*}]$ | 62153.80                  | 62101.28       | 61924.60    | 61957.37       | 66.33       | 61712.7       | 66.33       |

For  $r = 0.1$  the results are even more advantageous for the CVaR than in the relatively more symmetric case considered in Table 2. Figure 3 shows a clear advantage of the CVaR in terms of third order moment  $E[(-Y_{x^*})^3]$  from the origin.

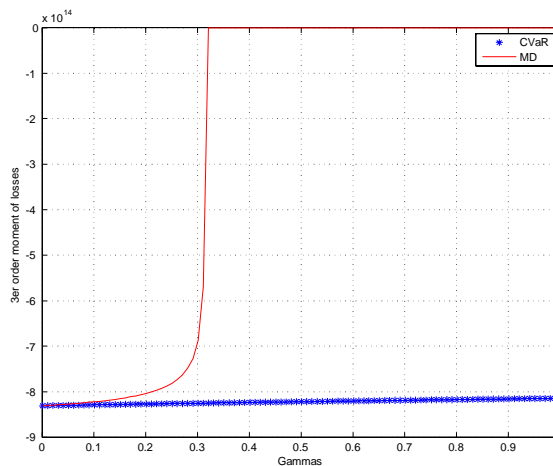


Figure 3. Third order moment for optimal losses using (4) and mean-deviation, using different weights  $\gamma \geq 0$ , for a Negative Binomial with  $r = 0.1$ .

All the results suggest that rational investors with strong risk aversion (high  $\gamma$ ), who are keen on decisions with a high probability of upside returns should use the CVaR instead of Markowitz's Mean-Deviation risk measure for marketing budget allocation planning, even in cases where the loss distributions have moderate level of symmetry.

## 5 Concluding remarks

Under perfect foresight, marketing budget allocation decisions should be based on the net present value of the returns. But due to markets uncertainty, returns are random variables that can be analyzed by analytical models, and marketing decisions are usually based on the expected value of this return. For some years, business schools have formed managers keen on market orientation and customers relationship management driven by expected returns, with little emphasis on risk assessment. Financial safeguards are central to avoid debacles as the one currently shocking the world economy.

Managers and marketers should include the risk analysis in the decision making process. The analysis of risk is not simple. First we discuss why Markowitz's mean-deviation approach is inappropriate when the returns have asymmetric probability distributions, which is a common case in marketing investment management. The class of coherent risk measures, currently used in finance theory and practice, provide an alternative approach suitable for marketing decision making. The procedure is particularly appealing to firms operating in turbulent markets preventing the consequences of averse outcomes.

In this paper we consider a unifying treatment for optimal marketing resource allocation and risk assessment in marketing investments, with particular emphasis on sales response management. Though we propose the use of coher-

ent risk measures for marketing planning and risk valuation under uncertainty, computing the optimal decision is not simple. Finance theory does not have developed computational methods for solving these problems in general. In this paper we present a set of first order conditions for the solution, and provide an algorithm for the numerical computation of the optimal decision. We show how this method can be applied for planning the marketing expenditure with the ADBUG sales response model. The results are useful for both marketing and finance managers, and can be used in a variety of marketing strategic decisions.

We have discussed the application of these techniques for budget allocation, emphasizing sales response to marketing expenditures. But this approach has a variety of applications, for example firm valuation.

Valuations based on expected returns overestimate the actual brand value compared to the coherent risk measure. For example, brand value can be computed as

$$-\rho\left(\sum_{i=1}^n Y_i\right) = E\left[\sum_{i=1}^n Y_i\right] - D\left(\sum_{i=1}^n Y_i\right)$$

where  $\{Y_i\}$  are net returns from different customers, and  $n$  the number of customers. Firms with high risk levels could have significantly lower brand values than suggested by estimations derived from the expected ROI-CLV, as the representativeness of the mean can be overstated. Coherence is essential as it guaranties that  $\rho\left(\sum_{i=1}^n Y_i\right) \leq \sum_{i=1}^n \rho(Y_i)$  introducing the classical risk measure requirement that risk is reduced by increasing our assets, i.e. our customer base.

Finally note that the algorithm covers dynamic problems by using a scenario tree if a "here-and-now" decision is addressed. These ideas can be extended to the case of adaptive decision problems where sequential investments are decided with increasing information. In particular, we can consider that  $x = (x_1, \dots, x_T) \in \mathcal{X}$  is a sequence of investments in monetary units and introduce the notation  $x_s^t = (x_s, \dots, x_t)$  for  $s \leq t$ , and  $\{Y_{x_1}, Y_{x_1^2}, \dots, Y_{x_1^T}\}$  is the cash-flow of financial returns associated with  $x$  (in present values at time zero). If we assume that at time  $t$  we have observed the previous outcomes  $Y_{x_1}, Y_{x_1^2}, \dots, Y_{x_1^{t-1}}$ , then we can update the probability function of the investment value  $Y_x = \sum_{t=1}^T Y_{x_1^t}$ . In particular, the multistage investment considers at each period  $t$  the previous decisions  $x_1^{t-1}$  given, and solves

$$\min_{x_t^T \in \mathcal{X}_t(x_1^{t-1})} \rho_t\left(\sum_{\tau=t}^T Y_{x_1^{\tau-1}}\right)$$

where  $\mathcal{X}_t(x_1^{t-1}) = \{x_t^T : (x_1^{t-1}, x_t^T) \in \mathcal{X}\}$ , and the conditional CVaR risk measure  $\rho_t$  is defined as

$$\rho_t(Y) = (\gamma - 1) E_t[Y] + \gamma \cdot \max_{q \in Q_t} E_t[-Y | q]$$

where  $Q_t = \{q \in L^2 : 0 \leq q \leq \alpha^{-1}, E_t[q] = 1\}$ , and  $E_t[\cdot]$  denotes the conditional expectation. A discrete *scenario tree* can be used to adapt the proposed

algorithm to this context. Under Markovian assumptions, the conditional expectation can be specified using transition probabilities. Note also that we can express the present value of the project as:

$$V_0 = \min_{x_1^T \in \mathcal{X}} \rho_1 \left( Y_{x_1} + \min_{x_2^T \in \mathcal{X}_t(x_1)} \rho_2 \left( Y_{x_1^2} + \dots + \min_{x_{T-2}^T \in \mathcal{X}_t(x_1^{T-2})} \rho_T \left( Y_{x_1^{T-1}} + \min_{x_T \in \mathcal{X}_t(x_1^{T-1})} \rho_T (Y_{x_1^T}) \right) \right) \right).$$

The project value can be also reconsidered as a variation of the Bellman Dynamic Programming equation, see Ruszyński and Shapiro (2006). We leave the details to future research.

Summarizing, budget allocation decisions must balance between the expected profitability and the degree of risk to undertake. Measuring risk is the first dilemma of a company manager. The CVaR and some other coherent risk measures have the ability to differentiate between different sides of risk asymmetries. However, they are not always easy to handle due to the lack of differentiability. By choosing the appropriate risk framework, the presented algorithm lead to profitable marketing decision plans with mitigated downside risk.

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