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Departamento de Economía
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (341) 624-98-75

ACCURACY OF NUMERICAL SOLUTIONS USING THE EULER EQUATION RESIDUALS

Manuel S. Santos *

Abstract
In this paper we derive some asymptotic properties on the accuracy of numerical solutions. We show that the approximation error of the policy function is of the same order of magnitude as the size of the Euler equation residuals. Moreover, for bounding this approximation error the most relevant parameters are the discount factor and the curvature of the return function. These findings provide theoretical foundations for the construction of tests that can assess the performance of alternative computational methods.

Keywords: Accuracy, Euler equation residuals, value and policy functions.

* Santos, Department of Economics, University of Minnesota and Department of Economics, Universidad Carlos III de Madrid. E-mail: msantos@atlas.socsci.umn.edu

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1. Introduction

Accuracy seems a minimal requirement for a rigorous application of computational techniques, and an analysis of the approximation error is generally very useful for the optimal design and evaluation of alternative numerical procedures. The current literature, however, is not particularly helpful for making these critical comparisons. There are some standard, regular optimization problems in which discretized versions of the dynamic programming algorithm may yield arbitrarily accurate approximations. In such situations, and in the absence of closed-form solutions, a typical evaluation procedure is to compare the outcome of a computational method against the solution of a discretized version of the dynamic programming algorithm. This is nevertheless a roundabout accuracy test, which is generally very costly and in some cases becomes infeasible. We certainly need more operative methods for checking the accuracy properties of numerical solutions and for evaluating the performance of competing algorithms.

In this paper we derive some accuracy properties of numerical solutions based upon the size of the Euler equation residuals. If $\varepsilon > 0$ is the maximum size of the Euler equation residuals, then we show that the approximation error involved in a numerical solution is of the same order of magnitude, and correspondingly the approximation error of the value function generated by such a numerical solution is of order ε^2. Since these residuals can be easily computed [cf., Christiano and Fisher (1997), Judd (1992)], these asymptotic results should be useful to derive accuracy tests which are valid for any proposed algorithm. Furthermore, the constants involved in our orders of convergence can be bounded from primitive data of the model, independently of the numerical solution. These constants depend on the discount factor and on the curvature of the return function. As is typical in theoretical results of this nature, these bounds are usually very conservative. Hence, at a later stage of this research we suggest some further practical ways to sharpen these error estimates.

Our analysis is restricted to a stochastic model of economic growth in which every decentralized solution may be derived as the optimal program of a social planning problem. This study could in principle be extended to alternative decentralized settings in which the equivalence between competitive solutions and optimal allocations may break down, although one should bear in mind that strong concavity and interiority of the individual decision problem are key assumptions in all our results.

An early procedure for checking the accuracy of numerical solutions was proposed by den Haan and Marcet (1994). Invoking standard statistical techniques these authors concoct a test for the orthogonality of the Euler equation residuals over current and past information, and consider that such a statistic provides an accuracy measure for a given numerical solution. The problem with this approach, however, is that orthogonal Euler equation residuals may be compatible with large deviations from the optimal policy; moreover, for a given numerical solution these residuals can be computed numerically at a relatively low cost without resorting to formal statistical techniques.

Being aware of these criticisms, Judd (1992) suggested an alternative test that entails numerical computation of the Euler equation residuals over the whole state space. But from these computations, neither Judd nor den Haan and Marcet have attempted to infer the size of the approximation errors of the computed value and policy functions, without specific knowledge of the true functions.

To understand the nature of this contribution, it may be helpful to provide the following illustration. For a given one-dimensional model, let ε be the maximum absolute value of the Euler equation residuals, and let E be the corresponding value for the approximation error of the numerical solution. Suppose that both values are related by the simple functional form, $E = a\varepsilon^{1/n}$, where a and n are positive numbers, and for concreteness we let $a = 1$. Given our present computational abilities, machine precision imposes a sixteen-digit accuracy; consequently, let us assume that $\varepsilon > 10^{-16}$. Now, if $n = 1$ then E would be of the same order of magnitude as ε, but if $n = 16$, then $E > 10^{-1}$. Note that $E > 10^{-1}$ would mean in this case that there are deviations from the true policy which may be greater than ten percent, and this is not generally acceptable in economic applications. Hence, in this simple situation the Euler equation residuals could be the basis for good accuracy estimates only if n is a small number. Statements of the sort that $E \to 0$ as $\varepsilon \to 0$ would not be useful to appraise the approximation error involved in a given numerical solution.

The remaining sections proceed as follows. Section 2 introduces the model and some basic methods for characterizing optimal solutions. Section 3 is devoted to our main results. Section 4 presents some numerical experiments that illustrate the nature of our theoretical findings along with some other relevant implementational issues. Some concluding remarks follow in the final section.

2. The model and preliminary considerations

We begin with a reduced form version of a standard stochastic model of economic growth [cf., Brock and Mirman (1972), and Stokey and Lucas with Prescott (1989)]. Let $(K, \mathcal{K})$ and $(Z, \mathcal{Z})$ be measurable spaces, and let $(K \times Z, \mathcal{K} \times \mathcal{Z})$ be the product space. The set K contains all possible values for the endogenous state variable, and Z is the set of possible values for the exogenous shock. The technological constraints are summarized by a given feasible set $\Omega \subset K \times K \times Z$, which is the graph of a correspondence $\Gamma : K \times Z \rightarrow K$. The intertemporal objective is characterized by a return function v on Ω and a given discount factor $0 < \beta < 1$.

The exogenous random variable z follows a Markov process defined by a transition function Q on $(Z, \mathcal{Z})$, which is assumed to be weakly continuous. It follows that for each given $z_0 \in Z$ one can define a probability measure $\mu^t(z_0, \cdot)$ on every t-fold product space $(Z^t, \mathcal{Z}^t) = (Z \times Z \ldots \times Z, \mathcal{Z} \times \mathcal{Z} \ldots \times \mathcal{Z})$ comprising all partial histories of the form $z^t = (z_1, z_2, \ldots, z_t)$.

The optimization problem is to find a sequence of measurable functions $\{\pi_t\}_{t=0}^{\infty}$, $\pi_t : Z^{t-1} \rightarrow K$, as a solution to

$$W(k_0, z_0) = \sup_{\{\pi_t\}_{t \geq 0}} \sum_{t=0}^{\infty} \beta^t \int_{Z^t} v(\pi_t, \pi_{t+1}, z_t) \mu^t(z_0, dz^t) \tag{2.1}$$

$$s.\ t. \quad (\pi_t, \pi_{t+1}, z_t) \in \Omega$$

$$(k_0, z_0) \text{ fixed}, \ \pi_0 = k_0, \ 0 < \beta < 1, \ t = 0, 1, 2, \ldots .$$

ASSUMPTION 1: *The set $K \times Z \subset R^l \times R^m$ is compact. The set Ω has nonempty interior, and for each fixed z the projection $\Omega_z = \{(k, k') \mid (k, k', z) \in \Omega\}$ is convex.*

ASSUMPTION 2: *The mapping $v : \Omega \rightarrow R$ is bounded, continuous, and differentiable of class C^2 with bounded derivatives. Also, for all z there exists some constant $\eta > 0$ such that $v(k, k', z) + \frac{\eta}{2} \|k'\|^2$ is concave as a function on (k, k').*

ASSUMPTION 3: *For each interior point (k_0, z_0) in $K \times Z$ there exists an optimal solution to (2.1) such that every realization $\{k_t, z_t\}_{t \geq 0}$ has the property that $(k_t, k_{t+1}, z_t) \in int(\Omega)$ for each $t \geq 1$.*

These hypotheses are fairly standard. In Assumption 1, the condition that the set $K \times Z$ is compact has been adopted for analytical convenience, and such condition becomes pertinent for computing optimal solutions. Using more involved arguments, our results could be extended to non-compact domains. For present purposes, a key postulate is a mild form of strong concavity of the return function v on the endogenous state variables (Assumption 2).[1] Also, excepting some simple unidimensional cases it seems difficult to relax the interiority of optimal solutions in Assumption 3.

Under the foregoing regularity assumptions, one readily shows that the value function $W(k_0, z_0)$, given in (2.1), is well defined and jointly continuous on $K \times Z$. Moreover, for each fixed z_0 the mapping $W(\cdot, z_0)$ is concave, and satisfies the Bellman equation

$$W(k_0, z_0) = \max_{k_1} \; v(k_0, k_1, z_0) + \beta \int_Z W(k_1, z_1) Q(z_0, dz_1) \tag{2.2}$$

The optimal value $W(k_0, z_0)$ is attained at a unique point k_1 given by the policy function $k_1 = g(k_0, z_0)$. Furthermore, $\{k_t, z_t\}_{t \geq 0}$ is an optimal solution to problem (2.1) if and only if it satisfies at all times Bellman's equation (2.2).

An alternative way to approach problem (2.1) is via classical variational methods. Thus, every optimal contingency plan $\{k_t, z_t\}_{t \geq 0}$ must fulfil at all times the system of Euler equations

$$D_2 v(k_{t-1}, k_t, z_{t-1}) + \beta \int_Z D_1 v(k_t, k_{t+1}, z_t) Q(z_{t-1}, dz_t) = 0 \quad t = 1, 2, \ldots. \tag{2.3}$$

where $D_j v(k, k', z)$ denotes the partial derivative of the function v with respect to the jth component variable, for $j = 1, 2, 3$. Likewise, under the above basic assumptions one readily shows that the set of optimal solutions is completely characterized by the system of Euler equations [cf. Benveniste and Scheinkman (1982)].

It follows then that under quite general conditions a complete characterization of the set of optimal solutions can be achieved by either the methods of dynamic programming or by variational techniques. For this fundamental reason, these two analytical approaches have been the basic starting point for the design of

[1] In Assumption 2, $\|\cdot\|$ denotes the Euclidean norm. For further discussion of this specific concavity assumption, see for instance Montrucchio (1987).

computational algorithms in dynamic economic models. Thus, there exists a family of methods that focus on the computation of the value and policy functions [e.g., Christiano (1990), Rust (1997), Santos and Vigo (1998), Tauchen (1990)], and a rival family of methods that seek to approximate the Euler equations [cf., Christiano and Fisher (1997), Judd (1992), Marcet (1994), McGrattan (1996)].

The most reliable algorithms are those based upon the methodology of dynamic programming, since convergence to the fixed point W is guaranteed by the familiar method of successive approximations, and arbitrary levels of accuracy can be achieved by sufficiently fine approximations of the functional space [cf., Santos and Vigo (1998)]. In contrast, algorithms approximating the Euler equations are less amenable to the derivation of error bounds or accuracy properties, and there is no operational iterative procedure that can insure global convergence. Although these latter algorithms may lack convergence to a desired solution, they are sometimes more practical. Hence, researchers are often faced with the basic problem of sorting out the most appropriate numerical procedure.

3. Main results

In order to evaluate the performance of competing algorithms, it becomes then imperative to have available accuracy tests that can be applied universally. As a step in this direction, we shall derive in this section some theoretical results on the accuracy of solutions based upon the size of the Euler equation residuals. To evaluate these findings, one should bear in mind that sharper error estimates are usually obtained at the expense of more stringent assumptions. Indeed, approximation errors may easily propagate in models displaying complex dynamic behavior.

Taking into account these considerations, we first derive some accuracy results under a condition that limits the degree of divergence between the computed and optimal orbits. (As argued later, this condition seems appropriate for models featuring global convergence to a steady state or invariant distribution.) Then, we consider the more general case with no restrictions on the degree of divergence of the computed and optimal orbits; here, the obtained error estimates are more conservative, and so these bounds would not be operational for models with simple dynamics. There is, therefore, a clear trade-off pervading these accuracy results: More generality may be attained at the cost of rather loose error estimates.

Let $\widehat{g}$ be a measurable selection of the technological correspondence Γ. Define

$W_{\widehat{g}}$ as

$$W_{\widehat{g}}(k_0, z_0) = \sum_{t=0}^{\infty} \beta^t \int_{Z^t} v(\widehat{g}^t(k_0, z_0), \widehat{g}^{t+1}(k_0, z_0), z_t) \mu^t(z_0, dz^t)$$

where $\widehat{g}^t(k_0, z_0)$ denotes the composite mapping $\widehat{g}(\widehat{g}(\ldots, \widehat{g}(k_0, z_0), \ldots), z_{t-2}), z_{t-1})$ for each possible realization $(z_1, z_2, \ldots, z_{t-1})$. Let $\varepsilon > 0$. Assume that

$$\left\| \begin{array}{c} D_2 v(k_0, \widehat{g}(k_0, z_0), z_0) \\ +\beta \int_Z D_1 v(\widehat{g}(k_0, z_0), \widehat{g}^2(k_0, z_0), z_1) Q(z_0, dz_1) \end{array} \right\| \leq \varepsilon \tag{3.1}$$

for all (k_0, z_0).[2]

The interpretation is that $\widehat{g}$ is the computed policy function, and $W_{\widehat{g}}$ is the resulting value function under the plan generated by $\widehat{g}$. Likewise, constant $\varepsilon > 0$ is an upper bound for the Euler equation residuals generated by policy $\widehat{g}$.

In our search for tight error estimates for models with simple dynamic behavior, we now impose the following condition that limits the degree of divergence between computed and optimal orbits.

CONDITION NDIV: *For all $\delta > 0$ and (k_0, z_0), there is $H > 0$ such that if $\|g - \widehat{g}\| \leq \delta$, then $\|g^t - \widehat{g}^t\| \leq H\delta$ for all $t > 1$.*

That is, the condition asserts that if both functions g and $\widehat{g}$ are close to each other, then their orbits cannot be far apart. (In particular, if $H = 1$, then $\|g^t(k_{0,}, z_{0,}) - \widehat{g}^t(k_{0,} z_0)\| \leq \delta$ for all $t \geq 1$.) For simple, deterministic models, the asymptotic behavior of these orbits is driven by the speed of convergence around the unique, stable steady state. The speed of convergence will provide us with an operational estimate for H.

In the following two lemmas, we shall present some simple results relating the approximation errors of the value and policy functions. Then, using an iterative argument, Theorem 3.3 provides upper bounds for these approximations under conditions (3.1) and NDIV.

LEMMA 3.1: *Assume that conditions (3.1) and NDIV are satisfied. Let $\|g - \widehat{g}\| \leq \delta$. Then, under Assumptions (1)-(3), we have $\|W - W_{\widehat{g}}\| \leq \frac{H\delta}{(1-\beta)}\varepsilon$*

PROOF: For a given z_0, let $\{\widehat{k}_t\}_{t \geq 0}$ be the solution generated by function $\widehat{g}$ and let $\{k_t^*\}_{t \geq 0}$ be the optimal solution for $k_0^* = \widehat{k}_0 = k_0$. Then,

[2] In the sequel, for a continuous function $g : K \times Z \to K$, we let $\|g\| = \max_{(k,z) \in K \times Z} \|g(k, z)\|$.

$$\begin{aligned}
W(k_0,z_0) - W_{\widehat{g}}(k_0,z_0) &= \lim_{T\to\infty}\sum_{t=0}^{T}\beta^t \int_{Z_t}\left[v(k_t^*,k_{t+1}^*,z_t) - v(\widehat{k}_t,\widehat{k}_{t+1},z_t)\right]\mu^t(z_0,dz^t) \\
&\leq \lim_{T\to\infty}\sum_{t=0}^{T}\beta^t \int_{Z_t}[D_1 v(\widehat{k}_t,\widehat{k}_{t+1},z_t)\cdot(k_t^* - \widehat{k}_t) + \\
&\quad + D_2 v(\widehat{k}_t,\widehat{k}_{t+1},z_t)\cdot(k_{t+1}^* - \widehat{k}_{t+1})]\mu^t(z_0,dz^t) \\
&\leq \sum_{t=0}^{\infty}\beta^t(H\delta\varepsilon) = \frac{H\delta}{1-\beta}\varepsilon
\end{aligned} \tag{3.2}$$

Here, the first inequality follows from the concavity of v. Then, after rearranging terms and using the fact that $k_0^* = \widehat{k}_0$, the second inequality follows from conditions (3.1) and NDIV.

LEMMA 3.2: *Let $E > 0$. Assume that $\left\|W - \widehat{W}_{\widehat{g}}\right\| \leq E$. Then, under Assumptions (1)-(3), we have $\|g - \widehat{g}\| \leq \left(\frac{2}{\eta}E\right)^{1/2}$.*

PROOF: Write $f(k,k',z) = v(k,k',z) + \beta\int_Z W(k',z')Q(z,dz')$. Let $k^* = g(k,z)$ and $\widehat{k} = \widehat{g}(k,z)$. Then, since $W(k,z) \geq W_{\widehat{g}}(k,z)$, we get

$$f(k,k^*,z) - f(k,\widehat{k},z) \leq E \tag{3.3}$$

Moreover, by virtue of the asserted concavity of v (Assumption 2),

$$f(k,k^*,z) - f(k,\widehat{k},z) \geq \frac{\eta}{2}\left\|k^* - \widehat{k}\right\|^2 \tag{3.4}$$

as $D_2 f(k,k^*,z) = 0$. Combining inequalities (3.3) and (3.4), we finally obtain $\left\|k^* - \widehat{k}\right\| \leq \left(\frac{2}{\eta}E\right)^{1/2}$. Since k and z are arbitrarily chosen vectors, we conclude that $\|g - \widehat{g}\| \leq \left(\frac{2}{\eta}E\right)^{1/2}$.

THEOREM 3.3: *Assume that conditions (3.1) and NDIV are satisfied. Then, under Assumptions (1)-(3), we have $\|W - W_{\widehat{g}}\| \leq \frac{2H^2}{\eta(1-\beta)^2}\varepsilon^2$ and $\|g - \widehat{g}\| \leq \frac{2H}{\eta(1-\beta)}\varepsilon$.*

This theorem is established in the Appendix. The corresponding error bounds are obtained as the fixed points from successive iterations of Lemmas 3.1 and 3.2. Observe that these bounds only depend on observable data of the model, and conditions (3.1) and NDIV.

The applicability of the previous results is certainly limited by condition NDIV. This condition would not be satisfied around unstable steady states, since small deviations from the optimal policy may change drastically the asymptotic dynamics of the computed and optimal solutions (cf. Figure 1). Indeed, in the present illustration for any initial condition k_0 in the interval $(k^*, \widehat{k}^*)$ orbit $\{g^t(k_0)\}_{t\geq 1}$converges to $\overline{k}$ and orbit $\{\widehat{g}^t(k_0)\}_{t\geq 1}$ converges to $\underline{k}$, and such limit points do not converge to each other as the computed function $\widehat{g}$ gets closer to the optimal policy g. But even at an unstable steady state, if the computed and optimal policies are arbitrarily close to each other, then for any initial condition k_0 the corresponding orbits will initially remain close. Thus, for small Euler equation residuals, a more natural way to bound the distance $\|k_t^* - \widehat{k}_t\|$ for each $t \geq 1$ is by an iterated application of the mean-value theorem, invoking some asymptotic properties of the derivative of the policy function.

Under the foregoing assumptions it is easy to show from the analysis of Santos and Vigo (1998) that the policy function is differentiable on K, and the asymptotic exponential growth of the derivative is bounded by $\frac{1}{\sqrt{\beta}}$. More precisely, let $D_1 g^t(k_0, z_0)$ represent the derivative of the function $g(g(\ldots(g(k_0, z_0)\ldots) z_{t-2}) z_{t-1})$ with respect to k_0 for every possible realization $(z_1, z_2, \ldots, z_{t-1})$. Let

$$L = \max_{(k_0,k_1,z_0)\in\Omega} \|D_{11}v(k_0, k_1, z_0)\| \tag{3.1}$$

where $D_{11}v(k_0, k_1, z_0)$ is the second-order own partial derivative of v with respect to k_0, evaluated at the point (k_0, k_1, z_0)

PROPOSITION 3.4 [CF., SANTOS AND VIGO (1998)]: *Under Assumptions (1)-(3), the derivative function $D_1 g^t(k_0, z_0)$ is well defined and continuous on $K \times Z$. Moreover,*

$$\sum_{t=0}^{\infty} \beta^t \int_{Z^t} \left\|D_1 g^{t+1}(k_0, z_0)\right\|^2 \mu(z_0, dz^t) \leq \frac{L}{\eta} \tag{3.5}$$

In light of these considerations, we now establish the following result.

THEOREM 3.5: *Assume that condition (3.1) is satisfied. Then, under assumptions (1)-(3), we have* $\|W - W_{\widehat{g}}\| \leq \frac{2}{\eta\left(\frac{1}{\sqrt{\beta}}-1\right)^2(1-\sqrt{\beta})^2}\left(\frac{L}{\eta}\right)\varepsilon^2$ *and* $\|g - \widehat{g}\| \leq \frac{2}{\eta\left(\frac{1}{\sqrt{\beta}}-1\right)(1-\sqrt{\beta})}\left(\frac{L}{\eta}\right)^{1/2}\varepsilon$ *for $\varepsilon > 0$ small enough.*

The theorem is proven in the Appendix. Observe that again the approximation error of the policy function is of the same order of magnitude as the size of the Euler equation residuals ε, and the approximation error of the value function is of order ε^2. Also, the discount factor and the curvature of the return function are the only pertinent parameters involved in these orders of convergence.

As in Lemma 3.1, the approximation error of the value function is in this case the discounted sum of current and future deviations from the true policy function times ε [cf., (3.2)]. However, in Theorem 3.5 we consider an additional channel of influence of the discount factor β, since the term $||k_t^* - \widehat{k}_t||$ is now bounded by ε times an estimate that depends on the derivatives of the policy function, and as asserted in Proposition 3.4, such derivatives have an exponential growth factor no greater than $\frac{1}{\sqrt{\beta}}$. Likewise, there are two possible channels in which parameter η may affect the approximation errors of the computed value and policy functions. First, as in Lemma 3.2, for a given value ε for the residuals, the approximation error of the policy function decreases with η. Second, as stated in Proposition 3.4 the ratio $\frac{L}{\eta}$ limits the asymptotic growth of the derivative of the policy function.

It should be observed that the bounds established in Theorem 3.5 hold for small ε, whereas the bounds derived in Theorem 3.3 hold uniformly. However, this is a relatively minor difference, since for results of this nature the main problem is to bound the asymptotic rate of convergence of the approximation error.

REMARK 3.6 (THE INVERSE DIRECTION): Note that if $||g - \widehat{g}|| \leq \varepsilon$, then under the postulated differentiability conditions on v it follows that there are constants M and N such that $||W - W_{\widehat{g}}|| \leq M\varepsilon^2$ and

$$\left\| D_2 v(k_0, \widehat{g}(k_0, z_0), z_0) + \beta \int_Z D_1 v(\widehat{g}(k_0, z_0), \widehat{g}^2(k_0, z_0), z_1) Q(z_0, dz_1) \right\| \leq N\varepsilon$$

for all (k_0, z_0). The purpose of the present theorems is to establish the less trivial, inverse direction: For small Euler equation residuals one can derive uniform bounds for the approximation error of the computed value and policy functions, without specific knowledge of their true values. In infinite horizon models an added analytical difficulty for results of this kind is that each Euler equation contains three different capital vectors and there is no terminal condition.

REMARK 3.7 (THE FINITE DIMENSIONAL CASE): It may be instructive to redo this analysis for the finite dimensional case. For instance, let us assume that $F : R^l \to R$ is a strongly concave, C^2 mapping. Let

$$x^* = \arg\max_{x \in R^l} F(x)$$

Then, the derivative

$$DF(x^*) = 0$$

Moreover, by virtue of the strong concavity of this function,

$$\|DF(x)\| \leq \varepsilon \Rightarrow \|x - x^*\| \leq M\varepsilon \tag{3.6}$$

where constant M is inversely related to the curvature of F, and so this estimate can be chosen independently of ε, for ε small enough.[3] In addition, concavity implies that

$$\begin{aligned} F(x^*) - F(x) &\leq DF(x) \cdot (x^* - x) \\ &\leq \|DF(x)\| \, \|x^* - x\| \\ &\leq M\varepsilon^2 \end{aligned}$$

where the last inequality follows from (3.6). Therefore, in the finite dimensional case, if $\|DF(x)\| \leq \varepsilon$, then there exists a constant M such that

$$F(x^*) - F(x) \leq M\varepsilon^2 \qquad \text{and} \qquad \|x - x^*\| \leq M\varepsilon \tag{3.7}$$

Matters are not so simple in the infinite horizon model, since asymptotically discounting brings down the curvature to zero. On the other hand, the bounds in (3.7) are valid for the whole domain of optimal decisions, whereas the bounds in Theorems 3.3 and 3.5 build upon the recursive nature of the dynamic optimization problem, and only apply for the approximation errors of the value and policy functions. That is, these latter bounds hold for the computed solution $\widehat{k}_1 = \widehat{g}(k_0, z_0)$, but not for the entire orbit $\{\widehat{k}_t\}_{t\geq 1}$.

[3]To prove (3.6), consider the function $(x - x^*) \cdot DF(y)$, where the vector $(x - x^*)$ is held fixed. Then, for $y = x$ an application of the mean value theorem yields

$$(x - x^*) \cdot DF(x) = (x - x^*) \cdot DF(x^*) + (x - x^*) \cdot D^2F(s) \cdot (x - x^*)$$

for some vector s. Therefore

$$\|x - x^*\| \, \|DF(x)\| \geq |(x - x^*) \cdot DF(x)| \geq \eta \, \|x - x^*\|^2$$

where η is a lower bound on the curvature of F. It follows that

$$\|DF(x)\| \geq \eta \, \|x - x^*\|$$

Consequently, $\|DF(x)\| \leq \varepsilon \Rightarrow \|x - x^*\| \leq \frac{1}{\eta}\varepsilon$.

As one can observe from these arguments, interiority of optimal solutions and strong concavity of the objective are the most fundamental assumptions for establishing these results. The interiority assumption together with concavity allows us to characterize optimal solutions by the system of Euler equations. Likewise, the lower bound η on the curvature of the return function v guarantees that the constants associated with these orders of convergence can be chosen independently of the optimization problem. Without such a uniform bound on the curvature of the return function, small Euler equation residuals are compatible with arbitrarily large deviations from *both* the true value and policy functions. This can be illustrated by the following simple example.

EXAMPLE 3.8: Consider the family of parameterized unidimensional functions $f_\eta : R \to R$, defined by $f_\eta(x) = -\frac{\eta}{2}x^2$, $\eta > 0$. Then, the maximum value is attained at $x^* = 0$, and for given $\varepsilon > 0$ the restriction on an approximate solution $\widehat{x}$ is that $|-\eta\widehat{x}| \leq \varepsilon$. Without loss of generality, let $\widehat{x} = \frac{\varepsilon}{\eta}$. Then, for fixed ε we get that $\widehat{x} \to \infty$ as $\eta \to 0$. Moreover, $f_\eta(\widehat{x}) = -\frac{\eta}{2}\widehat{x}^2 = -\frac{\varepsilon^2}{2\eta}$. Hence, $f_\eta(\widehat{x}) \to -\infty$ as $\eta \to 0$. Consequently, even in the finite dimensional case it is necessary to control the curvature of the function in order to bound uniformly the approximation error for *both* the value and policy functions.

Finally, we would like to close this section with some comments on our choice of metric for the approximation errors. In all our analysis, we have always considered the maximum of these values, and this is just the so called sup norm in the corresponding space of functions. This norm is the standard one in numerical analysis, and it seems natural in our case where there is not a specified prior distribution on the set of initial values (k_0, z_0). Furthermore, our most basic presumption is that a desired level of accuracy for the value and policy functions should be determined by the researcher guided by economic and analytical considerations. (These considerations include a variety of issues such as sensitivity of the solution to variations in parameter values or initial conditions, computational and approximation errors, feasible range for calibrating the parameters, units of measurement for the chosen variables, measurement errors, sample variability of the analyzed variables, and importance of the economic issue under study.)[4] Once

[4]It has been widely recognized [e.g., see the discussion in Cooley and Prescott (1995, p. 35)] that testing the output of a computer simulation (or for that matter, selecting a desired accuracy level) is an issue with many dimensions to it. Hence, the desired accuracy level is bound to depend on the particularities of the issue under concern, and there is no hard-and-fast rule that will work well in all cases.

the appropriate accuracy level for the computed policy and value functions has been selected, in our case one can easily compute the constants involved in the orders of convergence so as to set an appropriate bound for the Euler equation residuals. Of course, the numerical algorithm must be formulated so as to conform with the tolerance level imposed on these residuals.

Although we maintain that the desired level of accuracy is bound to depend on the situation under consideration, it may be worth pointing out that our accuracy results are invariant to linear transformations (or measurement units) of the variables. Thus, for illustrative purposes, assume that

$$\left\| D_2 v\left(k, k^{'}, z\right) + \beta \int_Z D_1 v\left(k^{'}, k^{''}, z^{'}\right) Q\left(z, dz^{'}\right) \right\| \leq \varepsilon.$$

Now, let us consider a linear transformation $y = Ak$, for $A > 0$. Then, differentiating with respect to y, it follows that

$$\left\| D_2 v\left(y, y^{'}, z\right) + \beta \int_Z D_1 v\left(y^{'}, y^{''}, z^{'}\right) Q\left(z, dz^{'}\right) \right\| \leq \frac{\varepsilon}{A}$$

if and only if

$$\left\| D_2 v\left(k, k^{'}, z\right) + \beta \int_Z D_1 v\left(k^{'}, k^{''}, z^{'}\right) Q\left(z, dz^{'}\right) \right\| \leq \varepsilon.$$

Also, as it is to be expected, under the new measurement units, and using the corresponding bound $\frac{\varepsilon}{A}$ for the residuals, the estimates for the approximation error $\|W - W_{\widehat{g}}\|$ in Theorems 3.3 and 3.5 remain unchanged, since the corresponding parameters concerning the curvature of the return function are in this case, $\frac{\eta}{A^2}$, and $\frac{L}{A^2}$. Further, the estimate of the approximation error $\|g - \widehat{g}\|$ for the policy function is now A times the previous one, which is consistent with the change of units.

In conclusion, these simple calculations illustrate that after a change of measurement units the same accuracy results are obtained by modifying accordingly the tolerance allowed for the Euler equation residuals. Similar considerations can be drawn for linear transformations of the objective function.

4. Numerical experiments

In this section we consider some numerical exercises in order to illustrate the behavior of the approximation error of the policy function in a simple growth model. Although for our benchmark parameterizations the model does not feature an exact analytical solution, we shall use a standard numerical procedure to estimate the evolution of the approximation error, and correspondingly the constants involved in the orders of convergence. In all our numerical experiments these constants display a relatively stable behavior for our best convergence orders.

4.1. The growth model

For simplicity, our numerical experiments are confined to the following standard, one-sector growth model

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{(c_t)^{1-\sigma}}{1-\sigma}$$

$$\begin{aligned} s.\,t. \quad k_{t+1} &= Az_t k_t^{\alpha} + (1-\pi)k_t - c_t \\ \log z_{t+1} &= \rho \log z_t + \varepsilon_t \end{aligned} \tag{4.1}$$

$$k_0 \text{ and } z_0 \text{ fixed, } t = 0, 1, \ldots.$$

$$0 < \beta < 1, \quad A > 0, \quad 0 < \alpha < 1, \quad 0 \leq \pi \leq 1, \quad 0 < \rho < 1, \quad k_t, c_t \geq 0$$

where E_0 is the expectations operator at time 0.

In order to solve this model, we focus on the system of Euler equations

$$c_t^{-\sigma} - \beta E_t c_{t+1}^{-\sigma}(\alpha A z_{t+1} k_{t+1}^{\alpha-1} + (1-\pi)) = 0 \qquad \text{for all } t \geq 0 \tag{4.2}$$

We shall first consider a deterministic version of this simple growth model in which we fix the following values

$$\alpha = 0.34 \qquad \pi = 0.05 \qquad \text{and} \qquad z_t = 1 \quad \text{for all } t \geq 0$$

and we let parameters σ and β vary across experiments. As explained below, in order to illustrate the sensitivity of our error estimates to variations in the curvature of the utility function, we select two alternative steady-state values for consumption, $c^* = 0.3$ and $c^* = 3.0$, over three different values for σ [cf., Tables

1-2]. These steady-state values are obtained after adjusting suitably parameter A. Parameter β will be equal to 0.95 in all cases considered, except for Table 3 where $\beta = 0.99$. And, all our computations will be restricted to the domain of capital values, $k \in [\frac{1}{2}k^*, \frac{3}{2}k^*]$, where k^* is the corresponding steady-state value for the deterministic model.

4.2. The numerical algorithm

Computation of the model solutions will follow a standard numerical procedure for approximating the Euler equation [cf., Christiano and Fisher (1997)]. As discussed in Marcet (1994), the idea here is to approximate the right-hand-side term on (4.2) by a polynomial function $\widehat{\Psi}(k_t, z_t)$. Once function $\widehat{\Psi}$ has been computed, the optimal consumption policy $\widehat{c}$ is obtained as

$$\widehat{c}(k, z) = [\widehat{\Psi}(k, z)]^{-1/\sigma} \tag{4.3}$$

Then, the law of motion of capital $\widehat{g}(k, z)$ can be derived from (4.1). Moreover, the Euler equation residuals are subsequently calculated as

$$\varepsilon(k_0, z_0) = \left|[\widehat{c}(k_0, z_0)]^{-\sigma} - \beta E_0[\widehat{c}(k_1, z_1)]^{-\sigma}(\alpha A z_1 k_1^{\alpha-1} + (1-\pi))\right|$$

for each (k_0, z_0) for $k_1 = \widehat{g}(k_0, z_0)$. As already stressed, computation of the residuals for a given finite sample of points is a relatively easy task, since it just entails functional evaluations. In our algorithm, function $\widehat{\Psi}$ belongs to the space of Chebyshev polynomials [cf. Judd (1992)]. Alternative numerical experiments were carried out with $\widehat{\Psi}$ in the space of piecewise-linear functions [cf., McGrattan (1996)] without substantial changes in our error estimates. (For our one-sector growth model, however, finer grids were needed under piecewise-linear interpolations so as to achieve the same accuracy levels.)

4.3. Error bounds for the computed policy function

We shall consider three alternative approaches for bounding the approximation error of the policy function. First, we calculate the worst-case errors derived in Theorem 3.5. These bounds are expected to be fairly pessimistic, since they do not take into account stability properties of our optimization problem. Then, from the above analysis we derive tighter error bounds using condition NDIV. This condition seems fairly appropriate for the present, globally convergent model. Likewise, to have a further sense of these approximations we also present a numerical estimation of these error bounds.

(a) *Worst-case error bounds*: From Theorem 3.5 the approximation error of the policy function is bounded by $\frac{2}{\eta\left(\frac{1}{\sqrt{\beta}}-1\right)\left(1-\sqrt{\beta}\right)}\left(\frac{L}{\eta}\right)^{\frac{1}{2}}\varepsilon$. For the benchmark parameterization reported in Table 1(a), we have computed the values $\frac{L}{\eta}$ and η for a thousand of equally spaced points in K, using for $g(k)$ the value of the approximate solution $\widehat{g}(k)$. We have considered from Prop. 3.4 that a good proxy for $\frac{L}{\eta}$ is the maximum value of the ratio of the largest over the smallest eigenvalue of the matrix second-order derivatives D^2v evaluated at points $(k, \widehat{g}(k))$. Also, from the proof of Lemma 3.2 a good proxy for η is the minimum absolute value of the second order derivative of the utility function over the computed solution with smallest Euler equation residuals. Under these computations, we obtain $\frac{2}{\eta}\left(\frac{L}{\eta}\right)^{\frac{1}{2}} = 2.4 \times 10$, and $\frac{1}{\left(\frac{1}{\sqrt{\beta}}\right)\left(1-\sqrt{\beta}\right)} = 1.52 \times 10^3$ for $\beta = 0.95$. Hence, $\frac{2}{\eta\left(\frac{1}{\sqrt{\beta}}-1\right)\left(1-\sqrt{\beta}\right)}\left(\frac{L}{\eta}\right)^{\frac{1}{2}} = 3.648 \times 10^4$. Likewise, for the calibrated values considered in Table 3(a) below with $\beta = 0.99$, we obtain $\frac{2}{\eta\left(\frac{1}{\sqrt{\beta}}-1\right)\left(1-\sqrt{\beta}\right)}\left(\frac{L}{\eta}\right)^{\frac{1}{2}} = 1.6 \times 10^6$. Hence, for Euler equation residuals $\varepsilon \backsimeq 10^{-10}$, the approximation error of the policy function $\|g - \widehat{g}\| \backsimeq 3 \times 10^{-6}$ for $\beta = 0.95$, and $\|g - \widehat{g}\| \backsimeq 10^{-4}$ for $\beta = 0.99$. Even though these theoretical estimates will turn out to be fairly conservative, all these values can still be useful in some practical situations. Of course, these bounds depend on the calibrated parameter values. Thus, for the parameterization reported in Table 2(a), the preceding estimates would have to be factored by 10^2 , since the level of consumption is roughly 10 times higher.

(b) *Error bounds under the assumption of non-divergent orbits*: The previous bounds are valid on a worst-case basis, and hence they are not expected to be tight in all applications. Sharper estimates can be derived from further properties of the solution. For instance, in the present model there is global convergence to the unique steady state, k^*.[5] Invoking this property of global convergence in our method of proof [cf., Prop. 3.4], we get that $\|g - \widehat{g}\| \leq \frac{M}{\eta(1-\beta)}\varepsilon$, where constant M

[5]Using information of the curvature of the utility function, Montrucchio (1995) has derived a lower bound $\underline{\beta}$, such that for discount factors above this lower bound any multisectoral model would exhibit a global turnpike.

would depend on the speed of convergence to the steady-state solution. Rather than going through this estimate, we propose here an alternative derivation using condition NDIV. At the steady-state solution, constant H is determined by the stable root of the Euler equation. Thus, if $1 > \lambda_1 > 0$ is such a stable root, the difference between the steady states of the computed and optimal solutions is locally bounded by $H = \frac{1}{1-\lambda_1}$. Then, let

$$N^{ndiv} = \frac{2H}{\eta(1-\beta)} \tag{4.4}$$

be the upper bound derived in Theorem 3.3. To compute this error bound,[6] we need to provide upper estimates for H and η. As already explained, we let $H = \frac{1}{1-\lambda_1}$, where λ_1 is the smallest root of the Euler equation evaluated at the steady-state solution. Also, as in our previous computations in part (a) our estimate for η is the minimum absolute value of the second order derivative of the utility function over a thousand of equally spaced points k evaluated at the numerical solution with smallest Euler residuals. Using these estimates, we obtain that $N^{ndiv} = 5.79 \times 10^1$ for the parameterization considered in Table 1(a) with $\beta = 0.95$, and $N^{ndiv} = 4.28 \times 10^2$ for the parameterization considered in Table 3(a) with $\beta = 0.99$. These are therefore much sharper bounds than those derived in part (a).

(c) *Numerical error bounds*: The above theoretical estimates are usually too conservative, since they are derived on a worst-case error basis. In order to have a better sense of these approximations, these bounds will be estimated numerically. Thus, one can view that a main implication of Theorems 3.3 and 3.5 is that the approximation error of the policy function is of the same order of magnitude as the size of the Euler equation residuals, and with this theory now available we can try to estimate the effective constant associated with the convergence order.

Since for our parameterizations the model does not feature a closed-form solution, we shall follow a standard numerical procedure in order to estimate the approximation error and the constant involved in the convergence order.[7] To this

[6]There are two main differences between the estimate in (4.4) derived in Theorem 3.3 and that in Theorem 3.5. First, the bound in (4.4) applies uniformly (i.e., for all $\varepsilon > 0$), whereas the bound in Theorem 3.5 is only valid for small $\varepsilon > 0$. Second, parameter η in (4.4) comes from the proof of Lemma 3.2, and such curvature parameter is not exactly the same as that required in the proof of Prop. 3.4. In Lemma 3.2, parameter η corresponds to the minimum curvature of the utility function over the optimal solution.

[7]To the best of our knowledge this is the first application of this procedure for accuracy tests

end, we first consider as the true policy the solution $\widehat{g}_f$ obtained from a relatively fine approximation with arbitrarily small Euler equation residuals. Then, for any numerical solution, we derive $\widehat{N}_{\widehat{g}}$ as

$$\widehat{N_{\widehat{g}}} = \sup_{(k_0,z_0)\in K\times Z} \frac{\|\widehat{g}(k_0,z_0) - \widehat{g}_f(k_0,z_0)\|}{\varepsilon_{\widehat{g}}(k_0,z_0)} \tag{4.5}$$

where $\varepsilon_{\widehat{g}}\ (k_0,z_0) \neq 0$ is the value of the Euler equation residuals under $\widehat{g}$ at the point (k_0,z_0), and $\|\widehat{g}(k_0,z_0) - \widehat{g}_f(k_0,z_0)\|$ is the deviation of $\widehat{g}$ from our estimate $\widehat{g}_f$ of the true policy at the given point. Of course, in order to obtain good estimates for $\widehat{N_{\widehat{g}}}$, it is crucial to have variability of the tolerance allowed for the residuals $\varepsilon_{\widehat{g}}$, and if for different discretizations constant $N_{\widehat{g}}$ settles down around a certain value $\widehat{N}$, one would be more confident to believe that this is the relevant constant for the computational problem at hand. For such a given $\widehat{N}$ our estimate of the approximation error for $\widehat{g}_f$ would then be

$$\|g - \widehat{g}_f\| \leq \widehat{N}\varepsilon_{\widehat{g}_f}$$

where g is the true policy function and $\varepsilon_{\widehat{g}_f}$ is the maximum size of the residuals under $\widehat{g}_f$.

Tables 1(a)–(c) report the approximation errors and estimated constant $\widehat{N}_{\widehat{g}}$ for several degrees of interpolation, for three different values for σ, for parameterizations of the deterministic version of the model with steady-state consumption, $c^* = 0.3$. Tables 2(a)–(c) replicate the same numerical experiments for parameterizations of the model with steady-state consumption, $c^* = 3.0$. In all these computations we also indicate the estimate N^{ndiv} from part (b), along with its component values $\frac{2}{\eta}$ and $\frac{2}{\eta(1-\beta)}$. (In these tables, the notation $me + n$ means $m \times 10^n$ and $me - n$ means $m \times 10^{-n}$.)

There are several useful points to be drawn from these exercises. First, the residuals are arbitrarily small (in some cases, with values close to 10^{-10}), and so are the estimated errors for the policy function.[8] Second, assuming linear

using the Euler equation residuals, but similar approaches have been adopted in other numerical experiments; for instance, see Bona and Santos (1997, p. 266) and Kahaner *et al.* (1989, Ch. 5).

[8]In each of these exercises constant $\widehat{N}_{\widehat{g}}$ was derived by considering a solution $g_{\widehat{f}}$ whose Euler equation residuals are at least 10 times smaller than any of the values for the residuals reported in each table. In some cases, we also compared the estimates $\widehat{N}_{\widehat{g}}$ with those obtained from a solution $\widehat{g}_f$ of a sufficiently accurate dynamic programming algorithm, and the results remained basically unchanged.

convergence for the policy function [cf. (4.5)] as in Theorems 3.3 and 3.5, constant $\widehat{N}_{\widehat{g}}$ settles down as we refine the interpolations. Third, as it is to be expected, constant $\widehat{N}_{\widehat{g}}$ becomes smaller as we increase the curvature of the utility function [cf., Tables 1 and 2]. Indeed, in all these experiments parameter value η seems to be the main driving force for the estimate $\widehat{N}_{\widehat{g}}$, and remarkably the rule of thumb, $\widehat{N}_{\widehat{g}} \backsimeq \frac{2}{\eta} \times 10$, appears to be of some practical use; furthermore, $\frac{2}{\eta(1-\beta)}$ is a tight upper bound. Observe that for $c < 1$ parameter η increases with σ, whereas for $c > 1$ parameter η may decrease with σ. All these changes are reflected accordingly in our numerical experiments in Tables 1 and 2, with steady-state consumption $c^* = 0.3$ and $c^* = 3.0$, respectively.[9]

These calculations seem to suggest that our estimate $H = \frac{1}{1-\lambda_1}$ from part (b) is not tight enough in the deterministic version of the model. In order to trace out the separate effects of β and H in N^{ndiv}, we have considered a further increase in the discount factor and the introduction of uncertainty. Thus, in Tables 3(a)–(b), we report the same numerical experiment for Chebyshev interpolation, with $\sigma = 1$ and $\beta = 0.99$ (a higher discount factor) for parameterizations of the model with $c^* = 0.3$ and $c^* = 3.0$. The increase in the discount factor changes slightly the speed of convergence, λ_1, and has a direct effect on the term $\frac{1}{1-\beta}$ in (4.4). Indeed, following (4.4) we should expect constant $\widehat{N}_{\widehat{g}}$ to experience a five-fold increase, since $\frac{\frac{1}{1-0.99}}{\frac{1}{1-0.95}} = 5$. This increase, however, is not reflected in the actual estimations of $\widehat{N}_{\widehat{g}}$. Part of the problem is that an increase in β leads to a further adjustment of parameter η (for high capital values an increase in β fosters savings and consequently raises η) and to further changes in the domain due to a shift in the steady state value k^*.

Finally, in Tables 4 and 5 we replicate the computational experiment for the original stochastic growth model with uncertainty. In our first experiment, ε_t is an i.i.d. process, with mean equal to 0 and standard deviation equal to 0.008. This random process ε_t is coming from a normal distribution, where the domain has been restricted to $\varepsilon \in [-0.032, 0.032]$, and the density function has been appropriately rescaled in order to get a cumulative mass equal to unity. (Observe

[9] Table 2(c) seems to be the only case where actual estimate $\widehat{N}_{\widehat{g}}$ is greater than N^{ndiv}. This calibration of the model is where $\frac{2}{\eta}$ takes on the highest value. Hence, in this case the Euler equation residuals are less accurate, and correspondingly the estimate $\widehat{N}_{\widehat{g}}$ may be subject to a non-negligible approximation error.

that the end-points of the domain are four standard deviations away from the mean.) Also, we let $\rho = 0.9$, and restrict the domain of feasible values to $k \in [\frac{1}{2}k^*, \frac{3}{2}k^*]$ and $z \in [\exp(-.32), \exp(.32)]$, where k^* is the deterministic steady state.

The introduction of stochastic variable z does not affect the curvature of the optimization problem, but it may influence the degree of convergence of the orbits $\{g^t(k_0, z_0)\}_{t\geq 0}$ and $\{\widehat{g}^t(k_0, z_0)\}_{t\geq 0}$. In this case, however, the stochastic innovation has a small variance, and consequently it is expected that the final effect on the above estimates will be rather small.[10] As shown in Tables 4(a)–(b), constant $\widehat{N}_{\widehat{g}}$ is of the same order of magnitude as that in Tables 1(a)–2(a). Of course, the degree of convergence of the orbits $\{g^t(k_0, z_0)\}_{t\geq 0}$ and $\{\widehat{g}^t(k_0, z_0)\}_{t\geq 0}$ may change substantially with further increases in the variance of the innovation process. Tables 5(a)–(b) replicate the above numerical experiments for the case with $\rho = 0.95$ and $\sigma_\varepsilon = 0.08$. For such a relatively large value of the variance of the stationary distribution of z, we just observe a two-fold increase in constant $\widehat{N}_{\widehat{g}}$. This is in fact the only case where the constant $\widehat{N}_{\widehat{g}}$ is greater than the estimate $\frac{2}{\eta(1-\beta)}$, and correspondingly it becomes pertinent to take into account the convergence properties of the orbits. Here, constant N^{ndiv} becomes a much tighter upper bound.

In summary, these computations illustrate that the curvature of the return function and, to a lesser extent, the discount factor are the main determinants for the accuracy of the Euler equation residuals. Moreover, condition NDIV seems fairly appropriate for the present model, and from comparisons in Tables 1 to 5 it appears that the estimate $H = \frac{1}{1-\lambda_1}$ (where λ_1 is the stable root of the Euler equation at the stationary solution) is not a tight upper bound, even for relatively high values of the variance of the stochastic innovation. Of course, these error estimates may fail to apply for further extensions of the model, such as the existence of taxes, externalities, or other frictions to the economy.

4.4. A comparison with other accuracy tests

It may be worth discussing the present analysis as compared to alternative accuracy checks proposed in the literature. One can certainly argue that the above theoretical bounds (cf., Theorem 3.5) are not sufficiently tight. However, this state of affairs is going to prevail for any other derivation of error bounds on a

[10] In the stochastic model, there is an additional error stemming from the evaluation of the integral. In order to keep this error negligible, we have used very fine partitions for carrying out the integrations.

worst-case basis [cf., Santos and Vigo (1998)]. Theoretical error bounds inform us of the order of convergence—and may be applied in a particularly important computation or to settle some controversial key issue— but these bounds become inoperative when one is interested in sampling the whole parameter space. In situations when one needs to compute the model over a wide range of parameter values, it becomes essential to use more operational procedures. The present paper illustrates that one can sharpen these error estimates by making use of certain appropriate assumptions (e.g., condition NDIV for globally convergent models), or by estimating these errors numerically. These latter procedures are easy to implement, and seem more reliable than some other simple alternative methods to be discussed presently.

One widely used strategy is to test the algorithm against a particular case where the model displays an analytical solution, or against the outcome of a more reliable algorithm. The problem with this approach is that for alternative parameterizations the approximation error of the computed value and policy functions may change substantially, and correspondingly this initial test may become ineffective. As shown in the preceding examples, changes in the curvature of the utility function and in the discount factor may affect considerably the accuracy of the algorithm. These changes in parameter values are indeed accounted for in the accuracy tests proposed in parts (b) and (c) above, which consider specific information of the solutions.

Another commonly employed stopping rule is to fix a tolerance level $\varepsilon > 0$ for the error obtained after successive approximate solutions. For instance, the algorithm may be instructed to stop when $\|W_n - W_{n+1}\| < \varepsilon$, where W_n and $W_{n+1}$are the functions obtained at iterations n and $n+1$, respectively. If the algorithm is generated by a contractive operator with modulus β and W is the true solution, then it is well known that the approximation error $\|W - W_n\| \leq \frac{\varepsilon}{1-\beta}$. But if the algorithm does not satisfy the contraction property, then we cannot infer the magnitude of the error $\|W - W_n\|$ from these differences from successive approximations. In other words, this simple stopping rule seems useful only if it is possible to pin down the speed of convergence of the algorithm.

Finally, we should also refer to the aforementioned work of den Haan and Marcet (1994) and Judd (1992). These authors were the first to advocate for the use of the Euler equation residuals as diagnostic tests for the accuracy of numerical solutions, but they fail to relate these measures to the approximation errors of the value and policy functions. Judd (1992) further argues that the residuals should be error free. Error-free measures are particularly convenient and informative in some situations, but one should realize that such simple rules cannot yield a complete account of the accuracy of the residuals, since accuracy depends on the curvature of the utility function, the discount factor, and on further possible features of the economy such as the existence of taxes and externalities.

5. Concluding remarks

In this paper we have shown that the approximation error of the policy function is of the same order of magnitude as the Euler equation residuals, and correspondingly the approximation error of the value function exhibits a quadratic order of convergence. These asymptotic results are the best possible, since these are the orders of convergence observed in the finite-dimensional case. In our dynamic model the method of proof is more convoluted, and involves the search for a fixed point from successive iterations of the approximation errors of the value and policy functions.

The constants obtained in these error bounds only depend on the curvature of the return function and on the discount factor. (Uncertainty plays an indirect role in the actual estimates, since the existence of random shocks does affect the law of motion and convergence properties of both optimal and computed paths.) These findings provide theoretical foundations for the construction of tests that can assess the accuracy of numerical solutions. For the one-sector neoclassical growth model, we have presented an analysis of the evolution of the error using both worst-case error and numerical estimates. In both cases, our accuracy tests can be cheaply implemented, and appear to be much more effective than other indirect, alternative procedures currently used in the literature.

As in many other approximation results, the main tools of our analysis come from differentiable calculus. Thus, the max norm provides an operational notion of distance for the theoretical study. For choosing an appropriate accuracy level in particular applications, it may be more informative to express the error in relative values or in some other unit-free measure or elasticity, and these values can be readily derived from our previous estimates. In our view, however, derivation of unit-free measures is not essential for appraising the accuracy of the Euler equation residuals. Indeed, as this paper illustrates, a more valuable aspect is to understand how the asserted measurement units will affect the estimates involved in the accuracy tests. Besides, unit-free measures for the residuals fail to capture the effects of key determinants of accuracy of solutions such as the curvature of the return function or the discount factor, or other features of the economy not considered in this paper such as the existence of taxes or externalities.

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Appendix

This appendix is devoted to the proofs of Theorems 3.3 and 3.5. The proof of Theorem 3.3 follows from a repeated application of Lemmas 3.1 and 3.2. The same argument is then applied to prove Theorem 3.5, once the corresponding analogue of Lemma 3.1 is established for the more general situation contemplated in this theorem.

PROOF OF THEOREM 3.3: Let $\gamma = diam(K)$. Then, by a simple modification of the proof of Lemma 3.1 we can get $\|W - W_{\widehat{g}}\| \leq \frac{\gamma}{(1-\beta)}\varepsilon$; moreover, by Lemma 3.2 we obtain $\|g - \widehat{g}\| \leq T_0 \varepsilon^{1/2}$, where $T_0 = \left(\frac{2\gamma}{\eta(1-\beta)}\right)^{1/2}$. Then, again by Lemma 3.1 it follows that $\|W - W_{\widehat{g}}\| \leq \frac{HT_0}{(1-\beta)}\varepsilon^{3/2}$. Also, a further application of Lemma 3.2 yields that $\|g - \widehat{g}\| \leq T_1 \varepsilon^{3/4}$ for $T_1 = \left(\frac{2HT_0}{\eta(1-\beta)}\right)^{1/2}$.

Now, with the new estimate for T_1 we can derive

$$T_2 = \left(\frac{2HT_1}{\eta(1-\beta)}\right)^{1/2}$$

And after a repeated application of this same argument we deduce that

$$\|g - \widehat{g}\| \leq T_n \varepsilon^{\frac{2^{n+1}-1}{2^{n+1}}}, \ n = 1, 2, ... \tag{6.1}$$

where

$$T_n = \left(\frac{2HT_{n-1}}{\eta(1-\beta)}\right)^{1/2} \tag{6.2}$$

for T_0 given above.

Observe that the sequence T_n converges to the fixed point

$$T = \left(\frac{2HT}{\eta(1-\beta)}\right)^{1/2} \tag{6.3}$$

Hence,

$$T = \frac{2H}{\eta(1-\beta)}$$

From (6.1)–(6.3) we may then conclude that

$$\|g - \widehat{g}\| \leq \frac{2H}{\eta(1-\beta)}\varepsilon$$

Moreover, by Lemma 3.1

$$\left\|W - \widehat{W_{\widehat{g}}}\right\| \leq \frac{2H^2}{\eta(1-\beta)^2}\varepsilon^2$$

It remains now to prove Theorem 3.5. In order to apply the previous argument, we first need to extend Lemma 3.1 to the more general case contemplated in the theorem.

LEMMA A.1: *Assume that condition (3.1) is satisfied. Let* $\|g - \widehat{g}\| \leq \delta$ *for* $\delta > 0$. *Then, under Assumptions (1)–(3), we have* $\|W - W_{\widehat{g}}\| \leq \frac{1}{\left(\frac{1}{\sqrt{\beta}}-1\right)(1-\sqrt{\beta})}\left(\frac{L}{\eta}\right)^{1/2}\delta\varepsilon$ *for sufficiently small* δ.

PROOF: As in the proof of Lemma 3.1, observe that

$$\begin{aligned}
\|W(k_0, z_0) - W_{\widehat{g}}(k_0, z_0)\| &\leq \lim_{T\to\infty}\sum_{t=0}^{T}\beta^t\int_{Z^t}\left[D_1 v\left(\widehat{k}_t, \widehat{k}_{t+1}, z_t\right)\cdot\left(k_t^* - \widehat{k}_t\right)\right.\\
&\qquad \left. + D_2 v\left(\widehat{k}_t, \widehat{k}_{t+1}, z_t\right)\cdot\left(k_{t+1}^* - \widehat{k}_{t+1}\right)\right]\mu^t\left(z_0, dz^t\right)\\
&\leq \lim_{T\to\infty}\sum_{t=0}^{T}\beta^t\int_{Z^t}\varepsilon\left\|k_{t+1}^* - \widehat{k}_{t+1}\right\|\mu^t\left(z_0, dz^t\right) \qquad (6.4)
\end{aligned}$$

We now bound the terms $\|k_t^* - \widehat{k}_t\|$ in (6.4) for $\delta > 0$ sufficiently small. Thus,

(a) For $t = 1$, we have

$$\left\|k_1^* - \widehat{k}_1\right\| \leq \delta \qquad (6.5)$$

(b) For $t = 2$, we get

$$\begin{aligned}
\beta\int_Z\left\|k_2^* - \widehat{k}_2\right\|\mu(z_0, dz_1) &= \beta\int_Z\left\|g(k_1^*, z_1) - \widehat{g}\left(\widehat{k}_1, z_1\right)\right\|\mu(z_0, dz_1)\\
&\leq \beta\int_Z\left(\left\|g(k_1^*, z_1) - g\left(\widehat{k}_1, z_1\right)\right\|\right.\\
&\qquad \left. + \left\|g\left(\widehat{k}_1, z_1\right) - \widehat{g}\left(\widehat{k}_1, z_1\right)\right\|\right)\mu(z_0, dz_1)
\end{aligned}$$

$$
\begin{aligned}
&\leq \beta\left(\int_Z \left\|Dg\left(k_1^*, z_1\right)\cdot\left(k_1^* - \widehat{k}_1\right)\right\| \mu\left(z_0, dz_1\right) + \delta\right) \\
&\leq \beta\left(\int_Z \left\|Dg\left(k_1^*, z_1\right)\right\| \delta\mu\left(z_0, dz_1\right) + \delta\right) \\
&\leq \beta\left[1 + \left(\frac{L}{\eta}\right)^{1/2}\right]\delta \qquad (6.6)
\end{aligned}
$$

Here, the first and third inequalities follow from well known properties of the integral and the norm. The second inequality follows from the mean value theorem and part (a). (In this inequality we have evaluated the derivative at the optimal point (k_1^*, z_1), and for present purposes, this later estimate entails no loss of generality since δ is considered to be sufficiently small.) And, the last inequality follows from Prop. 3.4. Now, using the same arguments

(c) For $t = 3$ we obtain

$$
\begin{aligned}
\beta^2\int_{Z^2}\left\|k_3^* - \widehat{k}_3\right\| \; \mu\left(z_0, dz^2\right) &\leq \beta^2\int_{Z^2}\left\|g(k_2^*, z_2) - \widehat{g}(\widehat{k}_2, z_2)\right\| \mu(z_0, dz^2) \\
&\leq \beta^2\int_{Z^2}\left(\left\|g(k_2^*, z_2) - g(\widehat{k}_2, z_2)\right\| + \left\|g(\widehat{k}_2, z_2) - \widehat{g}(\widehat{k}_2, z_2)\right\|\right)\mu(z_0, dz^2) \\
&\leq \beta^2\left[\int_{Z^2}\left\|Dg(k_2^*, z_2)\cdot(k_2^* - \widehat{k}_2)\right\| \mu(z_0, dz^2) + \delta\right] \qquad (6.7)
\end{aligned}
$$

Then, as in part (b) we can write the term $k_2^* - \widehat{k}_2$ as

$$
\begin{aligned}
k_2^* - \widehat{k}_2 &= g(k_1^*, z_1) - g(\widehat{k}_1, z_1) + g(\widehat{k}_1, z_1) - \widehat{g}(\widehat{k}_1, z_1) \\
&\simeq Dg(k_1^*, z_1)\cdot(k_1^* - \widehat{k}_1) + g(\widehat{k}_1, z_1) - \widehat{g}(k_1, z_1) \qquad (6.8)
\end{aligned}
$$

Now, plugging in (6.8) into (6.7), and making use of Proposition 3.4, it follows that

$$
\begin{aligned}
\beta^2\int_{Z^2}\left\|k_3^* - \widehat{k}_3\right\|\mu(z_0, dz^2) &\leq \beta^2\left[1 + \left(\frac{L}{\eta}\right)^{1/2}\left(1 + \frac{1}{\sqrt{\beta}}\right)\right]\delta \\
&\leq \beta^2\left[1 + \left(\frac{L}{\eta}\right)^{1/2}\left(\frac{\frac{1}{\sqrt{\beta^2}} - 1}{\frac{1}{\sqrt{\beta}} - 1}\right)\right]\delta \qquad (6.9)
\end{aligned}
$$

Finally,

(d) For an arbitrary $t \geq 4$, an extension of these arguments yields

$$\begin{aligned}\beta^{t-1} \int_{Z^t} \left\| k_t^* - \widehat{k}_t \right\| \mu\left(z_0, dz^{t-1}\right) &\leq \beta^{t-1} \left[1 + \left(\frac{L}{\eta}\right)^{1/2} \left(1 + \frac{1}{\sqrt{\beta}} + \ldots + \frac{1}{\sqrt{\beta^{t-2}}}\right)\right] \delta \\ &\leq \beta^{t-1} \left[1 + \left(\frac{L}{\eta}\right)^{1/2} \left(\frac{\frac{1}{\sqrt{\beta^{t-1}}} - 1}{\frac{1}{\sqrt{\beta}} - 1}\right)\right] \delta \qquad (6.10)\end{aligned}$$

We remark that the previous bounds apply for $\widehat{k}_t$ sufficiently close to k_t^*, i.e. for δ small enough.

Without loss of generality, we assume that $\beta > 1/4$ and $\left(\frac{L}{\eta}\right)^{1/2} > 1$. Then, the infinite sum of terms (6.4)–(6.6) and (6.9)–(6.10), for all t, is bounded by

$$\frac{1}{\left(\frac{1}{\sqrt{\beta}} - 1\right)\left(1 - \sqrt{\beta}\right)} \left(\frac{L}{\eta}\right)^{1/2} \delta\varepsilon \qquad (6.11)$$

This is the bound asserted in the lemma. However, in order to complete this proof, we need to establish that (6.11) holds for sufficiently small $\delta > 0$. For the sake of brevity, our arguments for proving this rather technical point will be restricted to the case of no uncertainty (i.e., to the case where functions W and g only depend on state variable k). This method of proof can be extended to our stochastic framework, and such a proof entails repeated use of Chebyshev's inequality.[11]

Since the derivative $Dg(k, z)$ is continuous over the compact set $K \times Z$, this function must be uniformly continuous. Then for every $E > 0$, there is $\alpha > 0$ such that if $\left\|(k, z) - (\widehat{k}, \widehat{z})\right\| < \alpha$, it must hold that $\left\|Dg(k, z) - Dg(\widehat{k}, \widehat{z})\right\| < E$.

For fixed $\alpha > 0$, choose T and δ_T such that

$$\delta_T = \frac{\alpha}{\left[1 + \left(\frac{L}{\eta}\right)^{1/2} \left(\frac{\frac{1}{\sqrt{\beta^{T-1}}} - 1}{\frac{1}{\sqrt{\beta}} - 1}\right)\right]} \qquad (6.12)$$

[11] For any non-negative measurable function $f : Z \rightarrow R$, and any constant $a > 0$, Chebyshev's inequality asserts that $P[\varepsilon \in Z : f(\varepsilon) \geq a] \leq \frac{1}{a} \int f(\varepsilon)\mu(d\varepsilon)$.

Hence, by (6.10) we have that $\left\|k_t^* - \widehat{k}_t\right\| \leq \alpha$ for all $0 \leq t \leq T$. Since $\|k_t^* - k_t\| \leq \gamma$, for γ =diam (K), it follows from (6.4) and (6.11) that for sufficiently small α

$$\|W(k_0, z_0) - W_{\widehat{g}}(k_0, z_0)\| \leq \frac{1}{\left(\frac{1}{\sqrt{\beta}} - 1\right)\left(1 - \sqrt{\beta}\right)} \left(\frac{L}{\eta}\right)^{1/2} \delta_T \, \varepsilon + \frac{\beta^T \gamma}{1-\beta} \qquad (6.13)$$

Thus, in order to establish the lemma, it remains to show that the remainder term $\frac{\beta^T\gamma}{1-\beta}$in (6.13) vanishes as δ_T goes to 0 (and T goes to ∞). That is, $\frac{\frac{\beta^T\gamma}{1-\beta}}{\delta_T}$converges to zero as $\delta_T \rightarrow 0$. However, this follows immediately from (6.12).

Finally, the method of proof of Theorem 3.3 can now be applied *mutatis mutandis* to substantiate Theorem 3.5, once the upper bound in Lemma 3.1 is replaced by that in Lemma A1.

Vertex points	$\varepsilon_{\widehat{g}}$	$\widehat{N}_{\widehat{g}}$	$\frac{2}{\eta}$	$\frac{2}{\eta(1-\beta)}$	N^{ndiv}
3	5.5311e-05	1.3735	0.2824	5.6485	5.7957e+01
4	2.2548e-06	2.4819	0.2824	5.6485	5.7957e+01
5	3.5750e-07	3.8452	0.2824	5.6485	5.7957e+01
6	2.8974e-08	1.1331	0.2824	5.6485	5.7957e+01

Table 1(a).- Steady-state consumption $c^* = 0.3$; Parameter values: $\beta = 0.95$ and $\sigma = 1$.

Vertex points	$\varepsilon_{\widehat{g}}$	$\widehat{N}_{\widehat{g}}$	$\frac{2}{\eta}$	$\frac{2}{\eta(1-\beta)}$	N^{ndiv}
3	1.0049e-03	2.3148e-02	2.4664e-03	4.9329e-02	1.1944
4	4.9691e-05	1.0533e-02	2.4664e-03	4.9329e-02	1.1944
5	6.4451e-06	5.8136e-02	2.4664e-03	4.9329e-02	1.1944
6	1.2084e-06	2.0356e-02	2.4664e-03	4.9329e-02	1.1944

Table 1(b).- Steady-state consumption $c^* = 0.3$; Parameter values: $\beta = 0.95$ and $\sigma = 4$.

Vertex points	$\varepsilon_{\widehat{g}}$	$\widehat{N}_{\widehat{g}}$	$\frac{2}{\eta}$	$\frac{2}{\eta(1-\beta)}$	N^{ndiv}
3	4.9980e+01	1.5394e-05	1.1967e-06	2.3934e-05	1.1184e-03
4	4.2533	7.6553e-06	1.1967e-06	2.3934e-05	1.1184e-03
5	0.3928	2.5241e-05	1.1967e-06	2.3934e-05	1.1184e-03
6	8.2850e-03	4.3287e-08	1.1967e-06	2.3934e-05	1.1184e-03

Table 1(c).- Steady-state consumption $c^* = 0.3$; Parameter values: $\beta = 0.95$ and $\sigma = 10$.

Vertex points	$\varepsilon_{\widehat{g}}$	$\widehat{N}_{\widehat{g}}$	$\frac{2}{\eta}$	$\frac{2}{\eta(1-\beta)}$	N^{ndiv}
3	4.1292e-06	1.1518e+02	2.4966e+01	4.9932e+02	5.1234e+03
4	1.7309e-07	2.0058e+02	2.4966e+01	4.9932e+02	5.1234e+03
5	2.4044e-08	1.0950e+03	2.4966e+01	4.9932e+02	5.1234e+03
6	1.6817e-09	1.5300e+02	2.4966e+01	4.9932e+02	5.1234e+03

Table 2(a).- Steady-state consumption $c^* = 3.0$; Parameter values: $\beta = 0.95$ and $\sigma = 1$.

Vertex points	$\varepsilon_{\widehat{g}}$	$\widehat{N}_{\widehat{g}}$	$\frac{2}{\eta}$	$\frac{2}{\eta(1-\beta)}$	N^{ndiv}
3	3.0586e-04	6.7501e+03	2.5309e+02	5.0619e+03	1.2256e+05
4	4.9240e-04	5.7024e+03	2.5309e+02	5.0619e+03	1.2256e+05
5	2.1589e-03	8.0919e+03	2.5309e+02	5.0619e+03	1.2256e+05
6	2.0085e-04	1.5159e+03	2.5309e+02	5.0619e+03	1.2256e+05

Table 2(b).- Steady-state consumption $c^* = 3.0$; Parameter values: $\beta = 0.95$ and $\sigma = 4$.

Vertex points	$\varepsilon_{\widehat{g}}$	$\widehat{N}_{\widehat{g}}$	$\frac{2}{\eta}$	$\frac{2}{\eta(1-\beta)}$	N^{ndiv}
3	1.6661e-08	9.1585e+05	8.5087e+04	1.7017e+06	7.9521e+07
4	2.8803e-10	5.6024e+05	8.5087e+04	1.7017e+06	7.9521e+07
5	2.5645e-11	5.2188e+06	8.5087e+04	1.7017e+06	7.9521e+07
6	1.4276e-13	5.9703e+08	8.5087e+04	1.7017e+06	7.9521e+07

Table 2(c).- Steady-state consumption $c^* = 3.0$; Parameter values: $\beta = 0.95$ and $\sigma = 10$.

Vertex points	$\varepsilon_{\widehat{g}}$	$\widehat{N}_{\widehat{g}}$	$\frac{2}{\eta}$	$\frac{2}{\eta(1-\beta)}$	N^{ndiv}
3	1.3676e-04	3.8760	0.2718	2.7181e+01	4.2804e+02
4	4.2590e-06	1.7074	0.2718	2.7181e+01	4.2804e+02
5	5.0893e-07	2.8132	0.2718	2.7181e+01	4.2804e+02
6	1.1294e-07	3.4914	0.2718	2.7181e+01	4.2804e+02

Table 3(a).- Steady-state consumption $c^* = 0.3$; Parameter values: $\beta = 0.99$ and $\sigma = 1$.

Vertex points	$\varepsilon_{\widehat{g}}$	$\widehat{N}_{\widehat{g}}$	$\frac{2}{\eta}$	$\frac{2}{\eta(1-\beta)}$	N^{ndiv}
3	6.9633e-06	3.4096e+02	2.7589e+01	2.7589e+03	4.3447e+04
4	2.3065e-07	1.9599e+02	2.7589e+01	2.7589e+03	4.3447e+04
5	1.1238e-08	1.7596e+02	2.7589e+01	2.7589e+03	4.3447e+04
6	6.1241e-09	2.9434e+02	2.7589e+01	2.7589e+03	4.3447e+04

Table 3(b).- Steady-state consumption $c^* = 3.0$; Parameter values: $\beta = 0.99$ and $\sigma = 1$.

Vertex points	$\varepsilon_{\widehat{g}}$	$\widehat{N_{\widehat{g}}}$	$\frac{2}{\eta}$	$\frac{2}{\eta(1-\beta)}$	N^{ndiv}
2×2	1.6479e-02	6.1995	3.9458e-01	7.8917	8.0941e+01
3×2	1.3916e-02	1.4615	3.9458e-01	7.8917	8.0941e+01
3×3	1.0534e-04	1.5700	3.9458e-01	7.8917	8.0941e+01

Table 4(a).- Steady-state consumption $c^* = 0.3$; Parameter values: $\beta = 0.95$, $\rho = 0.9$, $\sigma_\varepsilon = 0.008$ and $\sigma = 1$.

Vertex points	$\varepsilon_{\widehat{g}}$	$\widehat{N_{\widehat{g}}}$	$\frac{2}{\eta}$	$\frac{2}{\eta(1-\beta)}$	N^{ndiv}
2×2	1.6900e-03	2.2786e+03	3.9128e+01	7.8256e+02	8.0263e+03
3×2	1.4258e-03	1.4584e+02	3.9128e+01	7.8256e+02	8.0263e+03
3×3	5.3793e-05	1.3812e+02	3.9128e+01	7.8256e+02	8.0263e+03

Table 4(b).- Steady-state consumption $c^* = 3.0$; Parameter values: $\beta = 0.95$, $\rho = 0.9$, $\sigma_\varepsilon = 0.008$ and $\sigma = 1$.

Vertex points	$\varepsilon_{\widehat{g}}$	$\widehat{N_{\widehat{g}}}$	$\frac{2}{\eta}$	$\frac{2}{\eta(1-\beta)}$	N^{ndiv}
2×2	0.3752	1.7154e+02	5.7151	1.1430e+02	1.1723e+03
3×3	7.9374e-03	1.3625e+02	5.7151	1.1430e+02	1.1723e+03
5×3	1.0500e-03	2.4192e+01	5.7151	1.1430e+02	1.1723e+03

Table 5(a).- Steady-state consumption $c^* = 0.3$; Parameter values: $\beta = 0.95$, $\rho = 0.95$, $\sigma_\varepsilon = 0.08$ and $\sigma = 1$.

Vertex points	$\varepsilon_{\widehat{g}}$	$\widehat{N_{\widehat{g}}}$	$\frac{2}{\eta}$	$\frac{2}{\eta(1-\beta)}$	N^{ndiv}
2×2	3.1858e-02	2.2786e+03	2.4117e+02	4.8234e+03	1.1549e+05
3×2	3.1858e-02	1.4584e+02	2.4117e+02	4.8234e+03	1.1549e+05
3×3	1.1737e-04	7.2293e+03	2.4117e+02	4.8234e+03	1.1549e+05

Table 5(b).- Steady-state consumption $c^* = 3.0$; Parameter values: $\beta = 0.95$, $\rho = 0.95$, $\sigma_\varepsilon = 0.08$ and $\sigma = 1$.

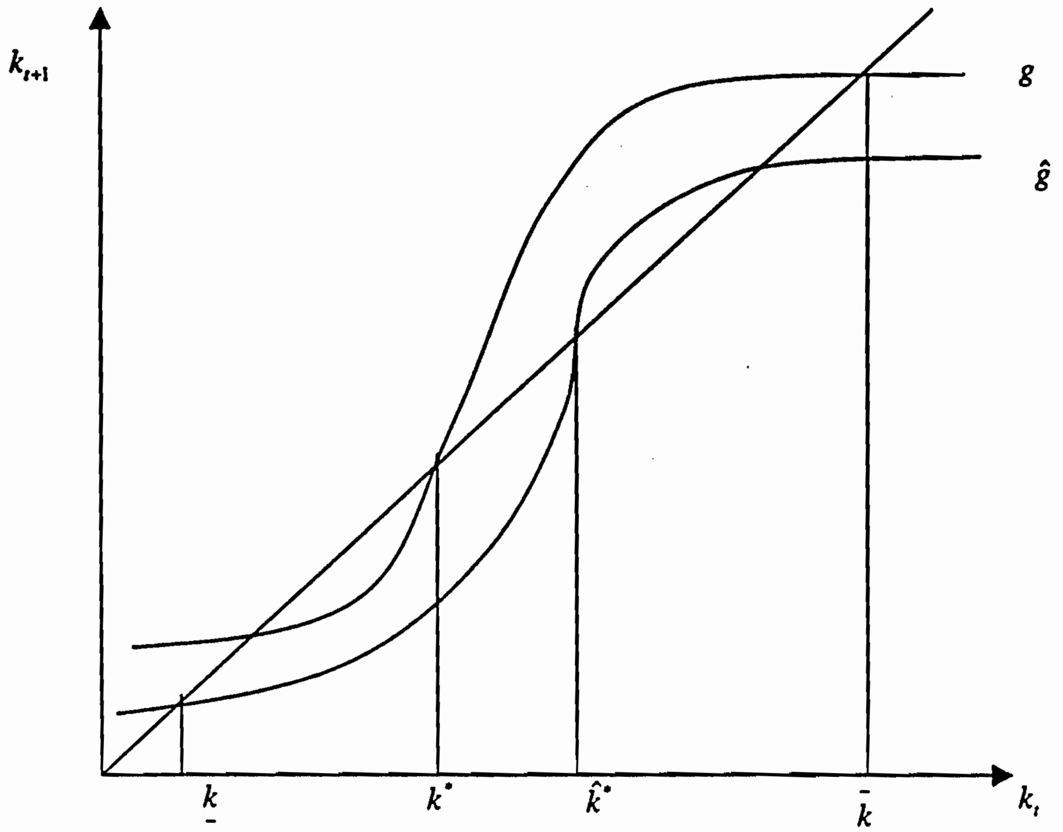

Figure 1 - Divergence of computed and optimal orbits. Condition NDIV fails to hold at the unstable steady state k^*.