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Noether symmetries in the phase space

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The constants of motion of a mechanical system with a finite number of degrees of freedom are related to the variational symmetries of a Lagrangian constructed from the Hamiltonian of the original system. The configuration space for this Lagrangian is the phase space of the original system. The symmetries considered in this manner include transformations of the time and may not be canonical in the standard sense. [http://dx.doi.org/10.1063/1.4896601]

I. INTRODUCTION

In analytical mechanics and in other areas of physics, the usefulness of the continuous symmetry groups is well known. Perhaps the simplest example is encountered in Lagrangian mechanics, where the groups of point transformations that leave a Lagrangian invariant up to a total derivative are associated with constants of motion (Noether–Bessel-Hagen theorem). In its elementary version, the Noether theorem only includes symmetries of the Lagrangian induced by transformations of the extended configuration space onto itself, known as point symmetries (see, e.g., Ref. 1), and the associated constants of motion are of a restricted class (see Eq. (7), below); in many cases, not all of the constants of motion can be obtained in this way, and in some cases, not even one constant of motion is associated with point symmetries of the Lagrangian (see, e.g., Ref. 2). By contrast, in the Hamiltonian formalism, any constant of motion is the generator of a group of canonical transformations that leave the Hamiltonian invariant. (Even though in the standard definition of a canonical transformation, the time is not transformed.)

In this paper we make use of the fact that the Noether theorem does not require the regularity of the Lagrangian, and that the Hamilton equations can be obtained from a Lagrangian whose configuration space is the phase space of the original system.[3] In that way, all the constants of motion are obtainable with the elementary version of the Noether theorem provided, of course, that one is able to find enough symmetries of this Lagrangian. Though this problem is as complicated as that of finding the groups of canonical transformations that leave a Hamiltonian invariant, a simplification is obtained from the fact that there are two subgroups of the point symmetries of the Lagrangian that can be determined in a simple manner. One of these subgroups is formed by the transformations in the phase space induced by the point transformations of the extended configuration space, i.e., in the space (q_i, t), and the other is formed by the transformations induced by transformations in the space (p_i, t).

In Section II we present some elementary concepts and results about the variational symmetries of a Lagrangian and their relationship with constants of motion; in Section III we apply these results to a Lagrangian that reproduces the Hamilton equations and we derive some useful expressions that

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simplify the search for constants of motion. Section IV contains several illustrative examples and in Section V we present some conclusions.

II. CONSTANTS OF MOTION ASSOCIATED WITH VARIATIONAL SYMMETRIES

We begin by summarizing the definition of the variational symmetries of a Lagrangian and the basic equation that relates the generator of a variational symmetry with a constant of motion.

If $q_1, q_2, \ldots, q_n$ is a system of generalized coordinates for some mechanical system, a one-parameter family of point transformations is given by expressions of the form

$$q'_i = q'_i(q_1, \ldots, q_n, t, s), \qquad t' = t'(q_1, \ldots, q_n, t, s), \tag{1}$$

where $i = 1, 2, \ldots, n$ and $s \in \mathbb{R}$ is a parameter that can take values in some neighborhood of 0. The family of point transformations (1) is a *variational symmetry*, or a *Noether symmetry*, of the Lagrangian $L(q_i, \dot{q}_i, t)$ if

$$L\left(q'_i, \frac{\mathrm{d}q'_i}{\mathrm{d}t'}, t'\right)\frac{\mathrm{d}t'}{\mathrm{d}t} = L\left(q_i, \frac{\mathrm{d}q_i}{\mathrm{d}t}, t\right) + \frac{\mathrm{d}}{\mathrm{d}t}F(q_i, t, s), \qquad \text{for all } s, \tag{2}$$

where F is some real-valued function. (Usually, the term Noether symmetry is employed in the case where $F = 0$.) Assuming that $q'_i(q_1, \ldots, q_n, t, 0) = q_i$ and $t'(q_1, \ldots, q_n, t, 0) = t$, from Eq. (2) one finds that

$$\sum_{i=1}^{n}\left[\frac{\partial L}{\partial q_i}\eta_i + \frac{\partial L}{\partial \dot{q}_i}\left(\frac{\mathrm{d}\eta_i}{\mathrm{d}t} - \dot{q}_i\frac{\mathrm{d}\xi}{\mathrm{d}t}\right)\right] + \frac{\partial L}{\partial t}\xi + L\frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{\mathrm{d}G}{\mathrm{d}t}, \tag{3}$$

where

$$\eta_i(q_j, t) \equiv \left.\frac{\partial q'_i(q_j, t, s)}{\partial s}\right|_{s=0}, \qquad \xi(q_i, t) \equiv \left.\frac{\partial t'(q_i, t, s)}{\partial s}\right|_{s=0}, \tag{4}$$

and

$$G(q_i, t) \equiv \left.\frac{\partial F(q_i, t, s)}{\partial s}\right|_{s=0}.$$

The functions η_i and ξ are the components of the *generator* of the transformations (1),

$$\mathbf{X} = \sum_{i=1}^{n}\eta_i\frac{\partial}{\partial q_i} + \xi\frac{\partial}{\partial t}. \tag{5}$$

Equation (3) can be written in the form

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \sum_{i=1}^{n}(\eta_i - \xi\dot{q}_i)\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}\right), \tag{6}$$

where

$$\varphi(q_i, \dot{q}_i, t) \equiv \sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\eta_i + \xi\left(L - \sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) - G. \tag{7}$$

Hence, if the Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0,$$

are satisfied, the function φ is a constant of motion.

Equation (3) is a partial differential equation for $n + 2$ functions of $n + 1$ variables. The method usually employed to solve Eq. (3) is based on the fact that ξ and η_i are functions of q_i and t only and, in many cases, the left-hand side of Eq. (3) is a polynomial in the $\dot{q}_i$, with coefficients that depend on q_i and t only. Since Eq. (3) must hold for all values of q_i, $\dot{q}_i$, and t, without imposing the equations of motion, by equating the coefficients of the products of the $\dot{q}_i$ on each side of the equation, one obtains a system of equations that only involve the variables q_i and t.

In this manner one obtains some expressions for the partial derivatives, $\partial G/\partial t$ and $\partial G/\partial q_i$, of the unknown function G in terms of L, ξ, η_i and their first partial derivatives. From the equality of the mixed second partial derivatives of G with respect to q_i and t, one finds $n(n+1)/2$ equations, that do not contain G. Once ξ and η_i are determined from the set of equations thus obtained, the functions G and φ can be finally calculated.

It can be mentioned that, since Eq. (6) must be valid for all values of t, q_i, $\dot{q}_i$ and $\ddot{q}_i$, a direct computation shows that this equation is equivalent to

$$\frac{\partial \varphi}{\partial \dot{q}_i} = \sum_{j=1}^{n} (\eta_j - \xi \dot{q}_j) \left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right), \tag{8}$$

$$\frac{\partial \varphi}{\partial t} + \sum_{j=1}^{n} \frac{\partial \varphi}{\partial q_j} \dot{q}_j = \sum_{j=1}^{n} (\eta_j - \xi \dot{q}_j) \left(\frac{\partial^2 L}{\partial t \partial \dot{q}_j} + \sum_{k=1}^{n} \frac{\partial^2 L}{\partial q_k \partial \dot{q}_j} \dot{q}_k - \frac{\partial L}{\partial q_j} \right). \tag{9}$$

Observe that by using Eq. (7) in the above equations, Eq. (8) turns out to be an identity and Eq. (9) reduces to Eq. (3). That is, they are equivalent, as must be, to Eq. (3). From Eq. (8) we see that when the Lagrangian L is not singular; that is,

$$\det \left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right) \neq 0, \tag{10}$$

then we can solve for the functions $(\eta_j - \xi \dot{q}_j)$ and substituting the resulting expressions into Eq. (9) one obtains a partial differential equation for the constant of motion φ, which determines the constants of motion that can be associated with variational symmetries of L. By solving the resulting equation one has the constant of motion and, then, the corresponding one-parameter family of point transformations. In the case of a singular Lagrangian, if m is the rank of the Hessian matrix, $(\partial^2 L/\partial \dot{q}_i \partial \dot{q}_j)$, only m functions $(\eta_j - \xi \dot{q}_j)$ will be determined by Eq. (8). By substituting these m functions into Eq. (9) we obtain a partial differential equation for the constant of motion φ, which contains $n - m$ unknown functions $(\eta_j - \xi \dot{q}_j)$, and Eq. (8) leads to $n - m$ further conditions for φ.

III. VARIATIONAL SYMMETRIES IN THE PHASE SPACE

The Hamilton equations can be derived from the auxiliary Lagrangian

$$\mathfrak{L}(q_i, p_i, \dot{q}_i, \dot{p}_i, t) \equiv p_i \dot{q}_i - H(q_i, p_i, t). \tag{11}$$

In fact, substituting the expression (11) into the Euler–Lagrange equations one obtains

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathfrak{L}}{\partial \dot{q}_i} - \frac{\partial \mathfrak{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t} p_i + \frac{\partial H}{\partial q_i},$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathfrak{L}}{\partial \dot{p}_i} - \frac{\partial \mathfrak{L}}{\partial p_i} = 0 - \dot{q}_i + \frac{\partial H}{\partial p_i}$$

$(i = 1, 2, \ldots, n)$, which are the Hamilton equations corresponding to the Hamiltonian $H(q_i, p_i, t)$. It may be noticed that Eq. (11) is similar to the usual relationship between the Hamiltonian and the Lagrangian of a mechanical system with n degrees of freedom, namely,

$$L(q_i, \dot{q}_i, t) = p_i \dot{q}_i - H(q_i, p_i, t), \tag{12}$$

but the usual Lagrangian is a function of q_i, $\dot{q}_i$, and t only. Equation (12) defines the Hamiltonian in terms of L, provided that the equations

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{13}$$

can be solved for the $\dot{q}_i$, as functions of q_i, p_i, and t, which amounts to the condition (10).

If one substitutes the Lagrangian $L(q_i, \dot{q}_i, t)$ into the Euler–Lagrange equations, and the regularity condition (10) is satisfied, one obtains a system of n second-order ordinary differential equations (ODEs), which are equivalent to the Hamilton equations. By contrast, as shown above, from $\mathfrak{L}$ one obtains a system of $2n$ first-order ODEs, which are precisely the Hamilton equations. The Lagrangian $\mathfrak{L}$ is *singular* in the sense that it does not satisfy the analog of the condition (10); however, as can be seen from the discussion above, this fact does not affect the validity of the analogs of Eqs. (3) and (4) to find the variational symmetries of $\mathfrak{L}$, and relate them with constants of motion by means of the analog of Eq. (7). (A detailed treatment of the Noether theorem for singular Lagrangians can be found in Refs. 4–7. Cf. also Ref. 8.)

Thus, a one-parameter family of point transformations in the extended phase space, analogous to (1), is a variational symmetry of $\mathfrak{L}$ if

$$\sum_{i=1}^{n}\left[\frac{\partial \mathfrak{L}}{\partial q_i}\eta_i + \frac{\partial \mathfrak{L}}{\partial p_i}\tilde{\eta}_i + \frac{\partial \mathfrak{L}}{\partial \dot{q}_i}\left(\frac{\mathrm{d}\eta_i}{\mathrm{d}t} - \dot{q}_i\frac{\mathrm{d}\xi}{\mathrm{d}t}\right)\right] + \frac{\partial \mathfrak{L}}{\partial t}\xi + \mathfrak{L}\frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{\mathrm{d}G}{\mathrm{d}t}, \tag{14}$$

where G is some function of q_i, p_i, and t, and

$$\eta_i \equiv \left.\frac{\partial q_i'(q_j, p_j, t, s)}{\partial s}\right|_{s=0}, \qquad \tilde{\eta}_i \equiv \left.\frac{\partial p_i'(q_j, p_j, t, s)}{\partial s}\right|_{s=0}, \qquad \xi \equiv \left.\frac{\partial t'(q_i, p_i, t, s)}{\partial s}\right|_{s=0}.$$

In this case, the functions η_i, $\tilde{\eta}_i$ and ξ are the components of the generator of the variational symmetry

$$\mathbf{X} = \sum_{i=1}^{n}\left(\eta_i\frac{\partial}{\partial q_i} + \tilde{\eta}_i\frac{\partial}{\partial p_i}\right) + \xi\frac{\partial}{\partial t}. \tag{15}$$

According to Eqs. (7) and (11), the constant of motion associated with this symmetry is

$$\begin{aligned}\varphi &= \sum_{i=1}^{n}\frac{\partial \mathfrak{L}}{\partial \dot{q}_i}\eta_i + \xi\left(\mathfrak{L} - \sum_{i=1}^{n}\frac{\partial \mathfrak{L}}{\partial \dot{q}_i}\dot{q}_i\right) - G \\ &= \sum_{i=1}^{n} p_i\eta_i - \xi H - G.\end{aligned} \tag{16}$$

It may be noticed that, by virtue of Eqs. (12) and (13), the constant of motion (16) has the same form as (7), the only difference is that in the case of Eq. (16), the functions η_i and G may depend on p_i. In fact, as we shall see below, *any* constant of motion can be obtained from Eqs. (14) and (16).

Eliminating $\mathfrak{L}$ and G from Eq. (14) with the aid of Eqs. (11) and (16), one finds that

$$\sum_{i=1}^{n}\left[-\frac{\partial H}{\partial q_i}\eta_i + \left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right)\tilde{\eta}_i\right] - \frac{\partial H}{\partial t}\xi = \sum_{i=1}^{n}\eta_i\frac{\mathrm{d}p_i}{\mathrm{d}t} - \xi\frac{\mathrm{d}H}{\mathrm{d}t} - \frac{\mathrm{d}\varphi}{\mathrm{d}t}.$$

Equating the coefficients of $\dot{q}_i$ and $\dot{p}_i$ on both sides of the last equation one obtains

$$\tilde{\eta}_i = -\xi\frac{\partial H}{\partial q_i} - \frac{\partial \varphi}{\partial q_i}, \tag{17}$$

$$\eta_i = \xi\frac{\partial H}{\partial p_i} + \frac{\partial \varphi}{\partial p_i} \tag{18}$$

and, therefore,

$$\sum_{i=1}^{n}\left(-\frac{\partial H}{\partial q_i}\eta_i - \frac{\partial H}{\partial p_i}\tilde{\eta}_i\right) = -\frac{\partial \varphi}{\partial t}. \tag{19}$$

Then, the substitution of Eqs. (17) and (18) into (19) yields the condition

$$\frac{\partial \varphi}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial \varphi}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial \varphi}{\partial p_i}\frac{\partial H}{\partial q_i}\right) = 0. \tag{20}$$

Thus, *if* one assumes that the Hamilton equations are satisfied, then Eq. (20) amounts to $\mathrm{d}\varphi/\mathrm{d}t = 0$ [cf. Eq. (6)].

After using Eqs. (17) and (18), Eq. (15) amounts to

$$\mathbf{X} = \xi \left[\sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right) + \frac{\partial}{\partial t} \right] + \sum_{i=1}^{n} \left(\frac{\partial \varphi}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial \varphi}{\partial q_i} \frac{\partial}{\partial p_i} \right). \tag{21}$$

The vector field between braces in Eq. (21), which only involves the Hamiltonian, is tangent to the curves in the extended phase space that represent the evolution of the system. In other words, this vector field generates transformations that map a solution curve of the Hamilton equations into itself, while that the part depending on φ generates transformations that map a solution curve into a different one.

In the standard treatment of the Hamiltonian formalism, the attention is focussed on *canonical transformations*, $q_i' = q_i'(q_j, p_j, t)$, $p_i' = p_i'(q_j, p_j, t)$, without transforming the time ($t' = t$) (see, e.g., Refs. 3 and 9); the variational symmetries considered here reduce to this restricted class of transformations when $\xi = 0$.

As in the Lagrangian case, we can follow two different approaches. One approach consists in solving Eq. (20) for the constant of motion and with that information to compute the generator of the corresponding symmetry by means of Eq. (21), with ξ completely arbitrary. Notice that the general solution to Eq. (20) is an arbitrary function of $2n$ functionally independent constants of motion, and the knowledge of this set is equivalent to having the solution of the Hamilton equations, which, in general is not an easy task. The second approach consists in making an ansatz about the components of the generator of the symmetry and using Eqs. (17), (18), and (20) to look for the constant of motion. In the examples given below we assume that η_i and ξ are functions of q_i and t only (which corresponds to the transformations induced by the point transformations in the extended configuration space), or that $\tilde{\eta}_i$ and ξ are functions of p_i and t only.

IV. EXAMPLES

We now present some illustrative examples of the results derived in Section III.

A. First example

As a first example we consider the Hamiltonian for a particle in a uniform gravitational field

$$H = \frac{p^2}{2m} + mgx. \tag{22}$$

Substituting Eq. (22) into Eqs. (17), (18), and (20) we obtain

$$\tilde{\eta} = -mg\xi - \frac{\partial \varphi}{\partial x}, \tag{23}$$

$$\eta = \frac{p}{m}\xi + \frac{\partial \varphi}{\partial p} \tag{24}$$

and

$$\frac{\partial \varphi}{\partial t} = mg\frac{\partial \varphi}{\partial p} - \frac{p}{m}\frac{\partial \varphi}{\partial x}. \tag{25}$$

As stated above, we consider symmetries induced by transformations of the extended configuration space or of the extended "momentum space." (Note that in this simple example, the general solution of Eq. (25) can be readily obtained.)

1. The Lagrangian solution

First we look for particular solutions to the conditions (23)–(25) such that

$$\eta = \eta(x, t), \qquad \xi = \xi(x, t).$$

Under these conditions, Eq. (24) implies that

$$\varphi = -\frac{p^2}{2m}\xi + p\eta + a_1(x,t), \tag{26}$$

where a_1 is a function of x and t only. Substituting this result into Eq. (25) we obtain

$$\frac{p^3}{2m^2}\frac{\partial\xi}{\partial x} + \frac{p^2}{m}\left(\frac{1}{2}\frac{\partial\xi}{\partial t} - \frac{\partial\eta}{\partial x}\right) - p\left(\frac{\partial\eta}{\partial t} + \frac{1}{m}\frac{\partial a_1}{\partial x} + g\xi\right) + mg\eta - \frac{\partial a_1}{\partial t} = 0. \tag{27}$$

This equation implies that

$$\xi = \xi(t), \tag{28}$$

$$\eta = \frac{x}{2}\frac{\mathrm{d}\xi}{\mathrm{d}t} + a_2(t), \tag{29}$$

$$a_1 = -\frac{mx^2}{4}\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} - mx\frac{\mathrm{d}a_2}{\mathrm{d}t} - mgx\xi + a_3(t), \tag{30}$$

$$0 = \frac{mx^2}{4}\frac{\mathrm{d}^3\xi}{\mathrm{d}t^3} + x\left(\frac{3}{2}mg\frac{\mathrm{d}\xi}{\mathrm{d}t} + m\frac{\mathrm{d}^2 a_2}{\mathrm{d}t^2}\right) + mga_2 - \frac{\mathrm{d}a_3}{\mathrm{d}t}, \tag{31}$$

where a_2, a_3 are functions of t only. Equation (31) implies that

$$\xi = c_1 t^2 + c_2 t + c_3, \tag{32}$$

$$\eta = c_1\left(xt - \frac{1}{2}gt^3\right) + c_2\left(\frac{1}{2}x - \frac{3}{4}gt^2\right) + c_4 t + c_5, \tag{33}$$

$$\begin{aligned} a_1 = {} & c_1\left(\frac{1}{2}mgxt^2 - \frac{1}{2}mx^2 - \frac{1}{8}mg^2t^4\right) + c_2\left(\frac{1}{2}mgxt - \frac{1}{4}mg^2t^3\right) - c_3(mgx) \\ & + c_4\left(\frac{1}{2}mgt^2 - mx\right) + c_5(mgt), \end{aligned} \tag{34}$$

where $c_1, \ldots, c_5$ are arbitrary constants.

The corresponding constant of motion is given by

$$\begin{aligned} \varphi = {} & c_1\left[-\frac{(2pt+mgt^2-2mx)^2}{8m}\right] + c_2\left[-\frac{(p+mgt)(2pt+mgt^2-2mx)}{4m}\right] + c_3\left(-\frac{p^2}{2m} - mgx\right) \\ & + c_4\left(\frac{2pt+mgt^2-2mx}{2}\right) + c_5(p+mgt). \end{aligned} \tag{35}$$

On the other hand,

$$\tilde{\eta} = c_1\left(mx - pt - \frac{3}{2}mgt^2\right) + c_2\left(-\frac{1}{2}p - \frac{3}{2}mgt\right) + c_4 m \tag{36}$$

and the generator of the symmetry takes the form

$$\begin{aligned} \mathbf{X} = {} & c_1\left[t^2\frac{\partial}{\partial t} + \left(tx - \frac{1}{2}gt^3\right)\frac{\partial}{\partial x} + \left(mx - pt - \frac{3}{2}mgt^2\right)\frac{\partial}{\partial p}\right] \\ & + c_2\left[t\frac{\partial}{\partial t} + \left(\frac{1}{2}x - \frac{3}{4}gt^2\right)\frac{\partial}{\partial x} + \left(-\frac{1}{2}p - \frac{3}{2}mgt\right)\frac{\partial}{\partial p}\right] + c_3\frac{\partial}{\partial t} \\ & + c_4\left(t\frac{\partial}{\partial x} + m\frac{\partial}{\partial p}\right) + c_5\frac{\partial}{\partial x}, \end{aligned} \tag{37}$$

whose projection to the extended configuration space reduces to that obtained in the standard Lagrangian formulation.[10]

2. Another solution

Now we look for new particular solutions to the conditions (23)–(25). We start with

$$\tilde{\eta} = \tilde{\eta}(p, t), \qquad \xi = \xi(p, t).$$

Under these conditions Eq. (23) implies that

$$\varphi = -x(\tilde{\eta} + mg\xi) + b_1(p, t), \tag{38}$$

where b_1 is a function of p and t only. Substituting this result into Eq. (25) we have that

$$-\frac{p}{m}(\tilde{\eta} + mg\xi) + \left(\frac{\partial}{\partial t} - mg\frac{\partial}{\partial p}\right) b_1 - \left[\left(\frac{\partial}{\partial t} - mg\frac{\partial}{\partial p}\right)(\tilde{\eta} + mg\xi)\right] x = 0. \tag{39}$$

Then

$$\begin{aligned} \tilde{\eta} + mg\xi &= f(v), \\ b_1 &= -\frac{p^2}{2m^2 g} f(v) + h(v), \end{aligned} \tag{40}$$

where f and h are arbitrary functions of v and

$$v \equiv p + mgt. \tag{41}$$

Therefore, from Eqs. (38) and (40) we find that

$$\varphi = -\frac{1}{mg}\left(\frac{p^2}{2m} + mgx\right) f(v) + h(v) = -\frac{1}{mg} H f(v) + h(v). \tag{42}$$

Note that in this case the constant of motion is given in terms of arbitrary functions of v. In fact, v and H are functionally independent constants of motion and any function of them is also a constant of motion.

As in the case of Eq. (42), the constant of motion given in Eq. (35) can be expressed in terms of H and v

$$\varphi = c_1\left[-\frac{1}{8}\left(\frac{v^2}{mg} - \frac{2H}{g}\right)^2\right] + c_2\left[-\frac{v}{4m}\left(\frac{v^2}{mg} - \frac{2H}{g}\right)\right] + c_3(-H) + c_4\left(\frac{v^2}{2mg} - \frac{H}{g}\right) + c_5(v).$$

Even though in this particular example one can find the general solution of Eq. (25), the present approach allow us to distinguish subgroups of the variational symmetries admitted in this example.

In order to get the generator of the symmetry given by this second solution we compute

$$\eta = \xi\frac{p}{m} - f\frac{p}{m^2 g} - \frac{H}{mg}\frac{\partial f}{\partial v} + \frac{\partial h}{\partial v}, \tag{43}$$

and the generator of the symmetry acquires the form

$$\mathbf{X} = \xi\left(\frac{p}{m}\frac{\partial}{\partial x} - mg\frac{\partial}{\partial p} + \frac{\partial}{\partial t}\right) + \left(\frac{\partial h}{\partial v} - \frac{H}{mg}\frac{\partial f}{\partial v} - \frac{pf}{m^2 g}\right)\frac{\partial}{\partial x} + f\frac{\partial}{\partial p}. \tag{44}$$

B. Second example

Consider now the Hamiltonian associated with the time-dependent oscillator given by

$$H = \frac{1}{2m}p^2 + \frac{m\omega^2(t)}{2}x^2, \tag{45}$$

where ω is a function of t. Using Eq. (45) in Eqs. (17), (18), and (20) we obtain

$$\tilde{\eta} = -m\omega^2 x\xi - \frac{\partial \varphi}{\partial x}, \tag{46}$$

$$\eta = \frac{p}{m}\xi + \frac{\partial \varphi}{\partial p}, \tag{47}$$

and

$$\frac{\partial \varphi}{\partial t} = m\omega^2 x \frac{\partial \varphi}{\partial p} - \frac{p}{m}\frac{\partial \varphi}{\partial x}. \tag{48}$$

1. The Lagrangian solution

We construct particular solutions to the conditions Eqs. (46)–(48). For this purpose, we assume that $\xi = \xi(x, t)$, $\eta = \eta(x, t)$. Then, Eq. (47) implies that

$$\varphi = -\frac{p^2}{2m}\xi + p\eta + a_1(x, t). \tag{49}$$

where $a_1(x, t)$ is a function of x and t only. Using this result into Eq. (48), we obtain

$$\frac{p^3}{2m^2}\frac{\partial \xi}{\partial x} + \frac{p^2}{m}\left(\frac{1}{2}\frac{\partial \xi}{\partial t} - \frac{\partial \eta}{\partial x}\right) - p\left(\frac{\partial \eta}{\partial t} + \frac{1}{m}\frac{\partial a_1}{\partial x} + \omega^2 \xi x\right) + m\omega^2 x\eta - \frac{\partial a_1}{\partial t} = 0. \tag{50}$$

This equation implies

$$\xi = \xi(t), \tag{51}$$

$$\eta = \frac{x}{2}\frac{d\xi}{dt} + a_2(t), \tag{52}$$

$$\frac{\partial \eta}{\partial t} + \frac{1}{m}\frac{\partial a_1}{\partial x} + \omega^2 x\xi = 0, \tag{53}$$

$$m\omega^2 x\eta - \frac{\partial a_1}{\partial t} = 0, \tag{54}$$

where a_2 is a function of t only. From Eq. (53) making use of Eq. (52) we get

$$\frac{a_1}{m} = -\frac{x^2\omega^2\xi}{2} - \frac{d^2\xi}{dt^2}\frac{x^2}{4} - \frac{da_2}{dt}x + a_3(t), \tag{55}$$

where a_3 is another function of t only. Substituting Eqs. (52) and (55) into Eq. (54) we obtain

$$\frac{x^2}{4}\left(\frac{d^3\xi}{dt^3} + 4\omega^2\frac{d\xi}{dt} + 4\omega\frac{d\omega}{dt}\xi\right) + x\left(\frac{d^2a_2}{dt^2} + \omega^2 a_2\right) + \frac{da_3}{dt} = 0. \tag{56}$$

This equation implies

$$\frac{d^3\xi}{dt^3} + 4\omega^2\frac{d\xi}{dt} + 4\omega\frac{d\omega}{dt}\xi = 0, \tag{57}$$

$$\frac{d^2a_2}{dt^2} + \omega^2 a_2 = 0, \tag{58}$$

and that a_3 is an irrelevant constant. Equation (57) integrates to

$$\frac{d^2 f}{dt^2} + \omega^2 f = \frac{c_3}{f^3} \tag{59}$$

where $f^2 \equiv \xi$ and c_3 is an integration constant.[11] Using these results into Eq. (52) we have

$$\eta(x, t) = f\frac{df}{dt}x + a_2. \tag{60}$$

Thus, from Eqs. (55), (60), with $\xi = f^2$, we obtain that the constant of motion is given by

$$\varphi = \frac{1}{2m}\left(fp - m\frac{df}{dt}x\right)^2 + \frac{c_3 m}{2}\left(\frac{x}{f}\right)^2 - m\frac{da_2}{dt}x + a_2 p. \tag{61}$$

A direct computation shows that

$$\tilde{\eta} = \frac{df}{dt}\left(fp - \frac{df}{dt}x\right) - \frac{c_3 m x}{f^2} - f^2\omega^2 x - m\frac{da_2}{dt}. \tag{62}$$

Therefore, the generator of the symmetry is given by

$$\begin{aligned}\mathbf{X}_H = {} & xf\frac{df}{dt}\frac{\partial}{\partial x} + \left[\frac{df}{dt}\left(fp - \frac{df}{dt}x\right) - \frac{c_3 m x}{f^2} - f^2\omega^2 x\right]\frac{\partial}{\partial p} + f^2\frac{\partial}{\partial t} \\ & + a_2\frac{\partial}{\partial x} + m\frac{da_2}{dt}\frac{\partial}{\partial p}\end{aligned} \tag{63}$$

whose projection to the extended configuration space reduces to that obtained in the Lagrangian formulation[11] (in this reference, a_2 is set to zero, cf. also Ref. 12). Notice that $\frac{1}{2m}\left(fp - m\frac{df}{dt}x\right)^2 + \frac{c_3 m}{2}\left(\frac{x}{f}\right)^2$ corresponds to the Lewis invariant, which is associated with conservation of angular momentum for two-dimensional systems;[13] we must remark that the constant of motion $-m\frac{da_2}{dt}x + a_2 p$ could be used along the ideas of Ref. 11.

Note also that there exist five constants of motion associated to the time-dependent oscillator regardless of the form of ω, this is because ξ satisfies a third-order differential equation, whose general solution must contain three arbitrary constants and the solution for a_2 must contain two additional arbitrary constants (see the following example).

2. Another solution

We find a new particular solution to Eqs. (46)–(48). For this purpose, we take $\tilde{\eta} = \tilde{\eta}(p, t)$ and $\xi = \xi(p, t)$. Then, under these conditions, Eq. (46) implies that

$$\varphi = -\frac{m\omega^2 x^2}{2}\xi - x\tilde{\eta} + b_1(p, t). \tag{64}$$

where b_1 is a function of p and t only.

Now from Eq. (48) and using (64), we have a third-order polynomial equation in the variable x given by

$$\begin{aligned} & x^3\frac{m^2\omega^4}{2}\frac{\partial\xi}{\partial p} - x^2 m\left(\frac{\omega^2}{2}\frac{\partial\xi}{\partial t} + \xi\omega\frac{d\omega}{dt} - \omega^2\frac{\partial\tilde{\eta}}{\partial p}\right) \\ & - x\left(\omega^2\xi p + m\omega^2\frac{\partial b_1}{\partial p} + \frac{\partial\tilde{\eta}}{\partial t}\right) + \frac{\partial b_1}{\partial t} - \frac{p}{m}\tilde{\eta} = 0,\end{aligned} \tag{65}$$

where b_2 is a function of t only. Therefore,

$$\xi = \xi(t), \tag{66}$$

$$\tilde{\eta} = p\left(\frac{1}{2}\frac{d\xi}{dt} + \frac{\xi}{\omega}\frac{d\omega}{dt}\right) + b_2(t), \tag{67}$$

$$\omega^2\xi p + m\omega^2\frac{\partial b_1}{\partial p} + \frac{\partial\tilde{\eta}}{\partial t} = 0, \tag{68}$$

$$\frac{\partial b_1}{\partial t} - \frac{p}{m}\tilde{\eta} = 0. \tag{69}$$

Now from Eqs. (67) and (68), a direct calculation shows that

$$\begin{aligned} b_1 = {} & -\frac{p^2}{2m}\left[\frac{1}{2\omega^2}\frac{d^2\xi}{dt^2} + \frac{1}{\omega^3}\frac{d\omega}{dt}\frac{d\xi}{dt} + \frac{\xi}{\omega^3}\frac{d^2\omega}{dt^2} - \frac{\xi}{\omega^4}\left(\frac{d\omega}{dt}\right)^2\right] \\ & - \frac{p}{m\omega^2}\frac{db_2}{dt} + b_3(t),\end{aligned} \tag{70}$$

where b_3 is function of t only. Substituting into (69), we obtain

$$\frac{db_3}{dt} = -\frac{p^2}{4}\left\{\frac{d^3\xi}{dt^3} + \frac{d\xi}{dt}\left[\frac{4}{\omega}\frac{d^2\omega}{dt^2} - \frac{8}{\omega^2}\left(\frac{d\omega}{dt}\right)^2 + 4\omega^2\right] + \xi\left[\frac{2}{\omega}\frac{d^2\omega}{dt^2} - \frac{10}{\omega^2}\frac{d^2\omega}{dt^2}\frac{d\omega}{dt} + \frac{8}{\omega^3}\left(\frac{d\omega}{dt}\right)^3\right]\right\} - \frac{p}{\omega^2}\left(\frac{d^2b_2}{dt^2} - \frac{2}{\omega}\frac{d\omega}{dt}\frac{db_2}{dt} + \omega^2 b_2\right). \tag{71}$$

This equation implies that

$$\frac{d^3\xi}{dt^3} + \frac{d\xi}{dt}\left[\frac{4}{\omega}\frac{d^2\omega}{dt^2} - \frac{8}{\omega^2}\left(\frac{d\omega}{dt}\right)^2 + 4\omega^2\right] + \xi\left[\frac{2}{\omega}\frac{d^2\omega}{dt^2} - \frac{10}{\omega^2}\frac{d^2\omega}{dt^2}\frac{d\omega}{dt} + \frac{8}{\omega^3}\left(\frac{d\omega}{dt}\right)^3\right] = 0, \tag{72}$$

$$\frac{d^2b_2}{dt^2} - \frac{2}{\omega}\frac{d\omega}{dt}\frac{db_2}{dt} + \omega^2 b_2 = 0 \tag{73}$$

and that b_3 is an irrelevant constant. Notice that these equations are more complicated than (57), (58). Finally, the constant of motion is

$$\varphi = -\frac{m\omega^2x^2}{2}\xi - xp\left(\frac{1}{2}\frac{d\xi}{dt} + \frac{\xi}{\omega}\frac{d\omega}{dt}\right) - \frac{p^2}{2m}\left[\frac{1}{2\omega^2}\frac{d^2\xi}{dt^2} + \frac{1}{\omega^3}\frac{d\omega}{dt}\frac{d\xi}{dt} + \frac{\xi}{\omega^3}\frac{d^2\omega}{dt^2} - \frac{\xi}{\omega^4}\left(\frac{d\omega}{dt}\right)^2\right] - xb_2(t) - \frac{p}{m\omega^2}\frac{db_2}{dt}. \tag{74}$$

We do not write down η and the vector field $\mathbf{X}$, because they do not help to the discussion. These constants of motion have been obtained using other approaches.[14,15]

C. Third example

Consider now the previous example with constant frequency $\omega = \omega_0$.

1. The Lagrangian solution

In this case, we can see that the Eqs. (57) and (58) reduce to

$$\frac{d^3\xi}{dt^3} + 4\omega_0^2\frac{d\xi}{dt} = 0,$$

$$\frac{d^2a_2}{dt^2} + \omega_0^2 a_2 = 0.$$

These equations imply that

$$\xi(t) = \frac{1}{\omega_0}\big[-c_1\cos(2\omega_0 t) + c_2\sin(2\omega_0 t)\big] + c_3, \tag{75}$$

$$a_2(t) = c_4\cos(\omega_0 t) + c_5\sin(\omega_0 t), \tag{76}$$

where $c_1, \ldots, c_5$ are arbitrary constants. Substituting Eqs. (75) and (76) into Eqs. (52) and (55) we get

$$\eta(x,t) = x\big[c_1\sin(2\omega_0 t) + c_2\cos(2\omega_0 t)\big] + c_4\cos(\omega_0 t) + c_5\sin(\omega_0 t), \tag{77}$$

$$a_1(x,t) = \frac{m\omega_0 x^2}{2}\big[-c_1\cos(2\omega_0 t) + c_2\sin(2\omega_0 t)\big] - \frac{kx^2}{2}c_3 + m\omega_0 x\big[c_4\sin(\omega_0 t) - c_5\cos(\omega_0 t)\big]. \tag{78}$$

Thus, substituting Eqs. (75), (77) and (78) into Eq. (49) we obtain that the constant of motion is given by

$$\begin{aligned}\varphi = {}& c_1\left[\left(\frac{p^2}{2m\omega_0} - \frac{m\omega_0 x^2}{2}\right)\cos(2\omega_0 t) + px\sin(2\omega_0 t)\right]\\ &+ c_2\left[\left(-\frac{p^2}{2m\omega_0} + \frac{m\omega_0 x^2}{2}\right)\sin(2\omega_0 t) + px\cos(2\omega_0 t)\right]\\ &- c_3 H + c_4\big[p\cos(\omega_0 t) + mx\omega_0\sin(\omega_0 t)\big] + c_5\big[p\sin(\omega_0 t) - mx\omega_0\cos(\omega_0 t)\big].\end{aligned} \tag{79}$$

A direct computation shows that

$$\begin{aligned}\tilde{\eta} = {}& c_1\big[2m\omega_0 x\cos(2\omega_0 t) - p\sin(2\omega_0 t)\big] - c_2\big[2m\omega_0 x\sin(2\omega_0 t) + p\cos(2\omega_0 t)\big]\\ &- c_4 m\omega_0\sin(\omega_0 t) + c_5 m\omega_0\cos(\omega_0 t).\end{aligned} \tag{80}$$

Therefore, the generator of the symmetry is given by

$$\begin{aligned}\mathbf{X} = {}& c_1\left\{x\sin(2\omega_0 t)\frac{\partial}{\partial x} + \big[2m\omega_0 x\cos(2\omega_0 t) - p\sin(2\omega_0 t)\big]\frac{\partial}{\partial p} - \frac{1}{\omega_0}\cos(2\omega_0 t)\frac{\partial}{\partial t}\right\}\\ &+ c_2\left\{x\cos(2\omega_0 t)\frac{\partial}{\partial x} - \big[2m\omega_0 x\sin(2\omega_0 t) + p\cos(2\omega_0 t)\big]\frac{\partial}{\partial p} + \frac{1}{\omega_0}\sin(2\omega_0 t)\frac{\partial}{\partial t}\right\} + c_3\frac{\partial}{\partial t}\\ &+ c_4\left[\cos(\omega_0 t)\frac{\partial}{\partial x} - m\omega_0\sin(\omega_0 t)\frac{\partial}{\partial p}\right] + c_5\left[\sin(\omega_0 t)\frac{\partial}{\partial x} + m\omega_0\cos(\omega_0 t)\frac{\partial}{\partial p}\right].\end{aligned} \tag{81}$$

Due to the symmetry of the Hamilton equations under $x \mapsto p$, $p \mapsto -m^2\omega_0^2 x$, the case where ξ and $\tilde{\eta}$ are functions of p and t only, can be obtained from the results above, applying this substitution, namely

$$\begin{aligned}\varphi(x,p,t) = {}& -c_1 m^2\omega_0^2\left[xp\sin(2\omega_0 t) - \left(\frac{m\omega_0 x^2}{2} - \frac{p^2}{2m\omega_0}\right)\cos(2\omega_0 t)\right],\\ &- c_2 m^2\omega_0^2\left[xp\cos(2\omega_0 t) + \left(\frac{m\omega_0 x^2}{2} - \frac{p^2}{2m\omega_0}\right)\sin(2\omega_0 t)\right]\\ &- c_3 H - c_4 m^2\omega_0^2\left[x\cos(\omega_0 t) - \frac{p}{m\omega_0}\sin(\omega_0 t)\right]\\ &- c_5 m^2\omega_0^2\left[x\sin(\omega_0 t) + \frac{p}{m\omega_0}\cos(\omega_0 t)\right],\end{aligned} \tag{82}$$

associated to the generator

$$\begin{aligned}\mathbf{X} = {}& -c_1 m^2\omega_0^2\left\{\left[\frac{2p}{m\omega_0}\cos(2\omega_0 t) + x\sin(2\omega_0 t)\right]\frac{\partial}{\partial x} - p\sin(2\omega_0 t)\frac{\partial}{\partial p} + \frac{1}{\omega_0}\cos(2\omega_0 t)\frac{\partial}{\partial t}\right\}\\ &+ c_2 m^2\omega_0^2\left\{\left[\frac{2p}{m\omega_0}\sin(2\omega_0 t) - x\cos(2\omega_0 t)\right]\frac{\partial}{\partial x} + p\cos(2\omega_0 t)\frac{\partial}{\partial p} + \frac{1}{\omega_0}\sin(2\omega_0 t)\frac{\partial}{\partial t}\right\}\\ &+ c_3\frac{\partial}{\partial t} + c_4 m^2\omega_0^2\left[\frac{1}{m\omega_0}\sin(\omega_0 t)\frac{\partial}{\partial x} + \cos(\omega_0 t)\frac{\partial}{\partial p}\right]\\ &- c_5 m^2\omega_0^2\left[\frac{1}{m\omega_0}\cos(\omega_0 t)\frac{\partial}{\partial x} - \sin(\omega_0 t)\frac{\partial}{\partial p}\right].\end{aligned} \tag{83}$$

The constant of motion (82) essentially coincides with (79), in spite of the fact that the vector fields (81) and (83) have different forms.

V. CONCLUSIONS

We have shown that by using the Lagrangian (11), which leads to the Hamilton equations, one can find a relationship between symmetries and constants of motion associated with point transformations in the phase space. In particular, by requiring that $\eta = \eta(q_i, t)$, $\xi = \xi(q_i, t)$ one obtains the transformations in the phase space induced by the point transformations of the extended configuration space. The other important and natural group of transformations is obtained imposing $\tilde{\eta} = \tilde{\eta}(p_i, t)$, $\xi = \xi(p_i, t)$.

After finishing this work we found Ref. 16, where the authors apply the basic idea of this paper, extending the Noether Theorem to its Hamiltonian form; there are some important differences however: In the first place, they eliminate the function G by using the equality of the mixed second partial derivatives of G. Since their approach is restricted to one degree of freedom, they obtain three partial differential equations but, in general case, one would have to deal with too many Eqs. ($n(n + 1)/2$) in contrast with Eqs. (17)–(20). Second, they have to find the function G in order to finally arrive to the constant of motion, which is calculated here in a direct manner.

Another important difference is that they restrict themselves to time-independent canonical transformations, in contrast with our more general discussion.

Also, in their discussion about the time-dependent oscillator they only obtain the Lewis invariant, as a consequence of the assumptions made guided by the problem under consideration, without a more systematic approach.

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