Convergence and asymptotic of multi-level Hermite-Padé polynomials

by

Luis Giraldo González Ricardo

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in

Mathematical Engineering

Universidad Carlos III de Madrid

Advisor:

Guillermo López Lagomasino

June 2022

This work is licensed under Creative Commons **Attribution – Non Commercial – Non Derivatives**

To Adriana

Acknowledgments

First, my sincere thanks to Guillermo, who was my advisor for some time, and now is more like a father. For all his commitment during these harsh pandemic times, for all those fruitful conversations about mathematics and life. Bur, above all things, thanks for making this research more a joy than a work.

I must thank the Department of Mathematics as a whole for the support during the last five years. Particularly, I want to show my appreciation to Fernando Lledó, for those long talks mixing mathematics and life experiences.

In addition, I have to express my gratitude to the scholarships sponsored by the Department of Mathematics: without them, this dissertation would have never existed. Also, I must thank GAMA for taking me in, as another member.

I do not want to forget to mention the kind welcome I received at KU Leuven by professors Walter van Assche and Arno Kuijlaars. Though the Omicron variant made my stay in Leuven a bit odd, it was an amazing experience anyway.

I am indebted to my friends, specially to the ones here in Spain that made me forget loneliness and sadness. Also to my family overseas, my mom, my brother and cousin.

Finally, Adriana, I cannot thank you enough, because you are home, you are my strength and an example to follow.

Published and submitted content

The content of this doctoral thesis is based on the following articles:

1. L.G. González Ricardo, G. López Lagomasino, S. Medina Peralta: On the convergence of multi-level Hermite-Padé approximants for a class of meromorphic functions. *Mediterr. J. Math.* **17**, 149 (2020). https://doi.org/10/g4jr.

   I am one of the authors of this article and it is included in Chapter 2 of this work.

2. L.G. González Ricardo, G. López Lagomasino, S. Medina Peralta: Logarithmic asymptotic of multi-level Hermite-Padé polynomials. *Integral Transform. Spec. Funct* **32**, 493-511. https://doi.org/10/g4js

   I am one of the authors of this article and it is included in Chapter 2 of this work.

3. S. Medina Peralta, L.G. González Ricardo, G. López Lagomasino: On the convergence of multi-level Hermite-Padé approximants. Submitted to *Physica D*.

   I am one of the authors of this article and it is included in Chapter 3 of this work.

4. L.G. González Ricardo, G. López Lagomasino: Strong asymptotic of Cauchy biorthogonal polynomials and orthogonal polynomials with varying measures. To appear in *Const. Approx.*

   I am one of the authors of this article and it is included in Chapter 1, Section 1.4 and Chapter 4 of this work.

Summary and main contributions

The present dissertation will focus on a certain mixed-type Hermite-Padé approximation problem for a Nikishin system of functions, that was introduced recently in [62]. This approximation scheme appeared in the search of discrete solutions for the Degasperis-Procesi equation and is also connected with Cauchy biorthogonal polynomials.

In [62, Def. 1.3] the following problem was posed: Let a Nikishin system of measures $\mathcal{N}(\sigma_1, \ldots, \sigma_m)$ and its associated Nikishin system of functions $(\widehat{s}_{1,1}, \ldots, \widehat{s}_{1,m})$ (for details see Definition 1.15 below) be given. Then, for each $n \in \mathbb{N}$, there exist polynomials $a_{n,0}, a_{n,1}, \ldots, a_{n,m}$, with $\deg a_{n,j} \leq n-1$, $j = 0, 1, \ldots, m-1$, $\deg a_{n,m} \leq n$, not all identically equal to zero, called multi-level (ML) Hermite-Padé polynomials that verify:

$$\mathscr{A}_{n,0} := \left[a_{n,0} + \sum_{k=1}^{m} (-1)^k a_{n,k} \widehat{s}_{1,k}\right] \in \mathcal{O}\left(\frac{1}{z^{n+1}}\right)$$

$$\mathscr{A}_{n,j} := \left[(-1)^j a_{n,j} + \sum_{k=j+1}^{m} (-1)^k a_{n,k} \widehat{s}_{j+1,k}\right] \in \mathcal{O}\left(\frac{1}{z}\right), \; j = 1, \ldots, m-1.$$

Here and in the sequel $\mathcal{O}(\cdot)$ is as $z \to \infty$ along paths non tangential to the support of the measures involved. For completeness write $\mathscr{A}_{n,m} := (-1)^m a_{n,m}$.

The text has been organized as follows:

- In Chapter 1, the basic aspects of Padé and Hermite-Padé approximation are introduced. In addition, it is discussed the historical development of the most important notions and results related to the present dissertation. In this way, the state of the art that precedes the present research is also given.

- In Chapter 2 it is studied a rational perturbation of the Nikishin system of functions. More precisely, given a fixed Nikishin system $\mathcal{N}(\sigma_1, \ldots, \sigma_m)$. and for each $n \in \mathbb{N}$, there exist polynomials $a_{n,0}, a_{n,1}, \ldots, a_{n,m}$, with $\deg a_{n,j} \leq n-1$, $j = 0, 1, \ldots, m-1$, $\deg a_{n,m} \leq n$, not all identically equal to zero that verify:

$$\mathscr{A}_{n,0} := \left[a_{n,0} + \sum_{k=1}^{m}(-1)^k a_{n,k}(\widehat{s}_{1,k} + r_k)\right] \in \mathcal{O}\left(\frac{1}{z^{n+1}}\right)$$

$$\mathscr{A}_{n,j} := \left[(-1)^j a_{n,j} + \sum_{k=j+1}^{m}(-1)^k a_{n,k}\widehat{s}_{j+1,k}\right] \in \mathcal{O}\left(\frac{1}{z}\right), j = 1, \ldots, m-1.$$

Here, r_k, $k = 1, \ldots, m$ are rational fractions with real coefficients, $r_k(\infty) = 0$ and poles outside $\operatorname{supp}\sigma_m$.

This chapter has two main theorems, the first deals with the uniform convergence of the Hermite-Padé approximants (Theorem 2.2), while the second is devoted to the limit of the zero counting measures of the linear forms $\mathscr{A}_{n,j}$, $j = 1, \ldots, m$ (Theorem 2.3).

Theorem 2.2 is situated in the tradition of Stieltjes-type theorems for simultaneous approximants. One important asset to obtain convergence results for Hermite-Padé approximants is to have a good control on the location of the zeros of the linear forms $\mathscr{A}_{n,j}$, $j = 0, 1, \ldots, m$. But the introduction of the rational fractions r_k in the first level $\mathscr{A}_{n,0}$ provokes that certain amount of the zeros of $A_{n,j}$ have a "wild" behavior. Nevertheless, it was sufficient to add some mild restrictions on the set of poles of r_k's in order to prove that only a fixed number of zeros of $\mathscr{A}_{n,j}$, independent of n, can leave $\operatorname{supp}\sigma_j$. Consequently, it could be proved the uniform convergence of the fractions $a_{n,j}/a_{n,m}$, $j = 0, 1, \ldots, m-1$. This result is a natural generalization of [62, Th. 1.6]. An important corollary of the convergence is to know the limit behavior of the zeros of $a_{n,j}$ that leave $\operatorname{supp}\sigma_m$.

Hereafter, it is studied the multiple orthogonality relations arising from this mixed-type Hermite-Padé approximation problem. It is well known that Hermite-Padé approximation of Nikishin systems are strongly related to multiple orthogonal polynomials. For ML Hermite-Padé this fact was made explicit in [32, Lemma 2.4]. Given the perturbation of the fractions r_k some orthogonalities are lost, and the associated multi-orthogonal polynomials satisfy incomplete orthogonality relations. This difficulty was overcome and the logarithmic asymptotic of the multiple orthogonal polynomials is given in Theorem 2.3. A fundamental consequence of Theorem 2.3 is the precise knowledge of the rates of convergence of the fractions $a_{n,j}/a_{n,m}$, $j = 1, \ldots, m-1$. The aforementioned theorem extends [32, Th. 3.4].

- In Chapter 3 it is studied a modification made to ML Hermite-Padé approximation problem by V.G. Lysov in [66]. There, it was considered a problem with more freedom in the interpolation conditions at infinity. That is, given a Nikishin system $\mathcal{N}(\sigma_1, \ldots, \sigma_m)$ and a multi-index $\vec{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$, there exist polynomials $a_{\vec{n},0}, a_{\vec{n},1}, \ldots, a_{\vec{n},m}$, where $\deg a_{\vec{n},j} \leq |\vec{n}| - 1$, $j = 0, 1, \ldots, m-1$, and $\deg a_{\vec{n},m} \leq |\vec{n}|$, not all identically equal to zero, that verify

$$\mathscr{A}_{\vec{n},j}(z) := \left((-1)^j a_{\vec{n},j} + \sum_{k=j+1}^{m}(-1)^k a_{\vec{n},k}\widehat{s}_{j+1,k}\right)(z) = \mathcal{O}\left(\frac{1}{z^{n_{j+1}+1}}\right), \qquad z \to \infty.$$

Firstly, the Markov-type theorem [66, Prop. 1.2] is extended to a wider class of measures. Previously, V.G. Lysov only considered measures σ_j, $j = 1, \ldots, m$ supported on compact intervals Δ_j and $\sigma_j' > 0$ a.e. on Δ_j, $j = 1, \ldots, m$. Then, Theorem 3.2 extends Lysov's convergence result, because the restrictions over the measures are weaker, namely σ_j has constant sign on Δ_j, $j = 1, \ldots, m$.

Moreover, the asymptotic result presented in [66, Cor. 1.1] is complemented in this chapter with the study of the ratio asymptotic of the multi-orthogonal polynomials associated to the ML Hermite-Padé approximation problem. Hence, this theorem constitutes a direct generalization of [32, Th. 1.2].

- Chapter 4 is devoted to the study of the strong asymptotic of Cauchy biorthogonal polynomials (see Theorem 4.1), which were introduced in [14]. This family of polynomials has appeared in various applications, like in the search of discrete solutions of the Degasperis-Procesi equation [14]. In [15, 16], the strong asymptotic for Laguerre-type weights was studied. Here, the goal is to obtain the strong asymptotic only imposing the Szegő condition on measures supported on compact intervals of the real line.

  In order to do so, first was needed the refinement of some previous results regarding the asymptotic of orthogonal polynomials with varying weights. In this sense, Theorems 4.2 and 4.3 are improvements of [23, Th. 4] and [95, Th. 14.3], respectively. Afterwards, it is discussed the connection of Cauchy biorthogonal polynomials with ML Hermite-Padé polynomials for a Nikishin system of two measures. Thanks to this link, the main tool to prove the aforementioned asymptotic is a technique developed by A.I. Aptekarev to find the strong asymptotic behavior of multi-orthogonal polynomials with respect to Angelesco and Nikishin systems (see [4, 5]). Aptekarev's method relies on topological reasoning; in particular, the Banach and Brouwer fixed point theorems.

  Though the proof of Theorem 4.1 rests on Aptekarev's ideas, it also contains important simplifications (see Proposition 4.9 and compare with [5, Section 2.2]) with respect to the one given in [5], mainly thanks to Theorem 4.3. On the other hand, some weaknesses appearing in the demonstrations in [5] are corrected, particularly during the proof of Theorem 4.15.

- Finally, in Chapter 5 some open problems for future research are given.

Contents

Chapter

1 Introduction

{ch:intro}

Approximation theory is a wide branch of Mathematical Analysis. It borrows techniques from other areas such as Potential Theory, Real and Complex Analysis, and Functional Analysis. It finds applications in other areas such as Differential Equations and Number Theory. One of the goals of the present chapter is to introduce the theoretical framework that serves as a basis for the results discussed along the dissertation. In addition, the mathematical tools given in the following pages are complemented with some historical remarks, in order to explain the origin and context of this research.

1.1 Padé approximation and orthogonality

The driving force behind approximation theory is, given a certain function f to approximate it with another function g, which is "simpler" than f, and such that the "difference" between them is rather "small". Of course, the procedure has its pros and cons, on one hand g is easier to manipulate than f but, on the other hand, some information about f in necessarily lost.

We will be concerned with the approximation of functions $f : \Omega \subset \mathbb{C} \to \mathbb{C}$, holomorphic on an open region Ω of the complex plane $\mathbb{C}$. The historical development of complex function theory has three different schools, following ideas of A.L. Cauchy, K. Weierstrass and B. Riemann, respectively. Curiously, the equivalence of Weierstrass' and Cauchy's approach is given through Taylor's theorem: that is, every holomorphic function (in Cauchy's sense) f in a neighborhood of $z_0 \in \mathbb{C}$ has a convergent power series expansion (Weierstrass' approach), i.e. f is analytic and vice versa. So,

$$f(z) = \sum_{k\geq 0} f_k (z - z_0)^k, \quad f_k = \frac{f^{(k)}(z_0)}{k!}. \tag{1.1}$$ {taylor}

Taylor's theorem allows to construct a sequence of polynomials that approximates f near z_0 by truncating the series expansion after n terms:

$$P_n(z) := \sum_{k=0}^{n} f_k (z - z_0)^k.$$

It is easy to see that these polynomials can be characterized by the asymptotic formula

$$f(z) - P_n(z) = \mathcal{O}\left((z - z_0)^{n+1}\right), \ z \to z_0.$$

It can be proved, as a consequence of Cauchy's integral formula, that P_n converges uniformly to f in closed disks $\{z \mid |z - z_0| \leq r\}$, $r < R_{z_0}(f)$ where $R_{z_0}(f)$ stands for the radius of convergence of the series (1.1).

This technique has a drawback. As polynomials are entire functions, they are not good if the function to be approximated has singularities. It is well known that the Taylor series is convergent only in the disk of radius equal to the distance from z_0 to the nearest singular point of f. The simplest functions with singularities are rational functions so, it is natural to ask: Is it possible to construct good rational approximations to a given analytic function f? This question was formulated and –answered– by Henri Padé (1863-1953) in his doctoral thesis from 1892. It is important to remark that this idea did not come "out of nowhere", because many mathematicians before Padé had studied variations of the same problem; for example, C. Hermite (1822-1901), L. Kronecker (1823-1891), or G. Frobenius (1849-1917) to name a few (see [19, Section 5.2.5], for more information).

The problem of the representation of an analytic function by rational fractions was suggested to Padé by his doctoral advisor, Charles Hermite. As he pointed out: "we were led to deal with this question by a word from Mr. Hermite, collected in one of his lessons, in which he gave a glimpse of the riches that this theory undoubtedly still conceals" [80, p. 5]. The problem studied by Padé, in modern terminology, is stated as follows [80, p. 9].

{pr:pade}

Problem 1:

Given a formal power series about $z = 0$

$$f(z) \sim \sum_{n \geq 0} f_n z^n,$$

and a pair of non-negative integers (n, m)*, find polynomials* $P_{n,m}$ *and* $Q_{n,m}$ *such that*:

i) $\deg P_{n,m} \leq n$, $\deg Q_{n,m} \leq m$, $Q_{n,m} \not\equiv 0$,

ii) $(Q_{n,m} f - P_{n,m})(z) = \mathcal{O}(z^{n+m+1})$, $z \to 0$.

Padé proved that this problem has solution, because it is equivalent to solving a homogeneous system of $n+m+1$ linear equations on $n+m+2$ unknowns, and he easily deduced that $P_{n,m}/Q_{n,m}$ is uniquely determined. In his honor, the fraction $\pi_{n,m}(f) := P_{n,m}/Q_{n,m}$ is called the *Padé approximant of type* (m, n) of f. The main contribution made by Padé, was the systematic and structural study of the properties of these approximants. This endeavor was possible in large part thanks to another tool due to the French scholar: the Padé table, which resembles Cantor's proof that $\mathbb{Q}$ is countable.

The amount of problems related to the Padé table and Padé approximants are of cyclopean dimensions. To mention a few, there is the question of the convergence of horizontal sequences of Padé approximants and the location of the singularities of the approximated function. There is a similar problem regarding the convergence of diagonal sequences of the Padé table, particularly for a special kind of analytic functions: the so called Cauchy transforms of measures.

Let μ be a finite positive Borel measure, whose support $\operatorname{supp}\mu$ is contained in $\mathbb{R}$ and has an infinite set of points and set $\Delta = \operatorname{Co}(\operatorname{supp}\mu)$ (the convex hull of the support of μ). Further, assume that the sequence of moments $\{c_n\}_{n\geq 0}$ is such that $c_n = \int x^n \,\mathrm{d}\,\mu(x) < \infty$, for all $n \in \mathbb{Z}_+$. Denote this class of measures as $\mathscr{M}(\Delta)$. The *Markov function* or the *Cauchy transform of the measure* μ is defined as

$$\widehat{\mu}(z) := \int \frac{\mathrm{d}\,\mu(x)}{z-x}. \tag{1.2}$$ {def:cauchy}

It is not difficult to check that $\widehat{\mu} \in \mathbf{H}(\overline{\mathbb{C}} \setminus \Delta)$, where $\mathbf{H}(\Omega)$ stands for the set of holomorphic functions on the open set $\Omega \subset \mathbb{C}$. Furthermore, we can associate to $\widehat{\mu}$ its formal Taylor expansion at infinity

$$\widehat{\mu}(z) \sim \sum_{n\geq 0} \frac{c_n}{z^{n+1}}.$$

When Δ is a half-line (that is, an interval of the form $[a, +\infty)$ or $(-\infty, a]$, $a \in \mathbb{R}$) the function $\widehat{\mu}$ is also called *Stieltjes function*.

This class of analytic functions is quite interesting. Many elementary functions can be written in terms of Cauchy transforms of measures. In addition, if complex weights are considered, a large number of analytic functions with a finite number of algebraic singularities can be represented in that form.

For convenience, in the particular case of Markov and Stieltjes functions, the Taylor expansion is usually taken at ∞. So, the rational function $\pi_n(\widehat{\mu}) = P_{n-1}/Q_n$ is called the n-th diagonal Padé approximant of $\widehat{\mu}$ if the pair of polynomials (P_{n-1}, Q_n), $n \geq 1$, verifies the following conditions:

i') $\deg P_{n-1} \leq n-1$, $\deg Q_n \leq n$ with $Q_n \not\equiv 0$,

ii') $(Q_n\widehat{\mu} - P_{n-1})(z) = \mathscr{O}(1/z^{n+1})$, $z \to \infty$.

The sequence of denominators $\{Q_n\}_{n\geq 1}$ is unique if we normalize Q_n to be monic. It satisfies several interesting properties. First, the polynomial Q_n fulfills the orthogonality relations

$$\int x^{\nu} Q_n(x) \,\mathrm{d}\,\mu(x) = 0, \qquad \nu = 0, 1, \ldots, n-1.$$

That is, the sequence of denominators $\{Q_n\}_{n\geq 1}$ coincides with the sequence $\{Q_n(\cdot;\mu)\}_{n\geq 1}$ of monic orthogonal polynomials with respect to the measure μ. The associated orthonormal polynomials are determined by

$$q_n(x;\mu) := \alpha_n Q_n(x;\mu), \quad \alpha_n := \left(\int Q_n^2(x) \,\mathrm{d}\,\mu(x)\right)^{-1/2}. \tag{1.3}$$ {def:ortnor}

Consequently, $\deg Q_n = n$, Q_n has n simple zeros inside Δ [94, Th. 3.3.1], and the zeros of Q_n and Q_{n+1} interlace [94, Th. 3.3.3]. Moreover, the polynomials P_{n-1} can be expressed in terms of Q_n

$$P_{n-1}(z) = \int \frac{Q_n(z) - Q_n(x)}{z-x} \,\mathrm{d}\,\mu(x). \tag{1.4}$$ {def:2Kind}

Some natural questions arise. Does the sequence $\{\pi_n\}_{n\geq 0}$ converge? If it does, what is the limit? Can we estimate the speed of convergence?

1.2 Convergence of diagonal Padé approximants

The first two problems stated above were answered by the Russian mathematician A.A. Markov (1856-1922) in a short article from 1895, [68]. There, Markov studied the convergence of the continued fractions of the function given by

$$\widehat{f}(z) := \int_a^b \frac{f(x)\,\mathrm{d}x}{z-x},$$

where f is real and positive. Due to the uniqueness of the Padé approximants, the convergents of the continued fraction associated with $\widehat{f}$ coincide with the sequence $\{\pi_n\}_{n\geq 0}$. Notice that, in modern mathematical language, Markov's proof requires that the measure $\mathrm{d}\mu(x) := f(x)\,\mathrm{d}x$ be absolutely continuous with respect to the Lebesgue measure but, as C. Berg said in [12], this is a consequence of the historical context and it is not essential in the proof itself. So, Markov's result is valid for a more general class of measures.

{remain:1} Before the discussion of Markov's theorem, some identities are needed.

Proposition 1.1:
Let $\mu \in \mathscr{M}(\Delta)$, then for $z \in \mathbb{C}\setminus\Delta$

$$\widehat{\mu}(z) - \frac{P_{n-1}(z)}{Q_n(z)} = \int \frac{Q_n(x)}{Q_n(z)}\frac{\mathrm{d}\mu(x)}{z-x} = \int \frac{Q_n^2(x)}{Q_n^2(z)}\frac{\mathrm{d}\mu(x)}{z-x}.$$

Proof. From (1.4) it is immediate that

{mar:1}
$$P_{n-1}(z) = Q_n(z)\widehat{\mu}(z) - \int \frac{Q_n(x)}{z-x}\,\mathrm{d}\mu(x), \tag{1.5}$$

which gives the first equality. Moreover, notice that, as a consequence of orthogonality

$$0 = \int \frac{Q_n(z)-Q_n(x)}{z-x} Q_n(x)\,\mathrm{d}\mu(x).$$

Hence,

$$Q_n(z)\int \frac{Q_n(x)}{z-x}\,\mathrm{d}\mu(x) = \int \frac{Q_n^2(x)}{z-x}\,\mathrm{d}\mu(x),$$

Dividing this last expression by $Q_n(z)$, we get

$$\int \frac{Q_n(x)}{z-x}\,\mathrm{d}\mu(x) = \frac{1}{Q_n(z)}\int \frac{Q_n^2(x)}{z-x}\,\mathrm{d}\mu(x).$$

Substituting this relation in (1.5) the second equality is easily deduced.

□

Let $\{x_{n,i}\}_{i=1}^n$ denote the roots of the polynomial Q_n. It is very well known the existence of positive constants $\{\lambda_{n,i}\}_{i=1}^n$, called Christoffel coefficients, such that

{Gauss_Jacobi}
$$\int p(x)\,\mathrm{d}\,\mu(x) = \sum_{i=1}^{n} \lambda_{n,i}\, p(x_{n,i}), \tag{1.6}$$

for every polynomial p with $\deg p \leq 2n-1$. The Christoffel coefficients are given by

{Christoffel}
$$\lambda_{n,i} = \int \frac{Q_n(x)\,\mathrm{d}\,\mu(x)}{Q_n'(x)(x - x_{n,i})}. \tag{1.7}$$

Formula (1.6) is called Gauss-Jacobi quadrature [94, Sec. 3.4] and from it we can infer a simple representation for the fractions $\pi_n(\widehat{\mu})$. In fact, from the simplicity of the zeros of Q_n the partial fraction decomposition of $\pi_n(\widehat{\mu})$ is written as

$$\pi_n(\widehat{\mu})(z) = \sum_{i=1}^{n} \frac{\beta_{n,i}}{z - x_{n,i}}.$$

Using the residue theorem

$$\begin{aligned}
\beta_{n,i} &= \operatorname{Res}\left[\pi_n(\widehat{\mu}), x_{n,i}\right] = \lim_{z\to x_{n,i}} (z - x_{n,i})\pi_n(\widehat{\mu})(z) \\
&= \lim_{z\to x_{n,i}} (z - x_{n,i}) \int \frac{Q_n(z) - Q_n(x)}{Q_n(z)(z - x)}\,\mathrm{d}\,\mu(x) = \int \frac{Q_n(x)\,\mathrm{d}\,\mu(x)}{Q_n'(x_{n,i})(x - x_{n,i})} = \lambda_{n,i}.
\end{aligned}$$

Therefore, we get

$$\pi_n(\widehat{\mu}; z) = \sum_{i=1}^{n} \frac{\lambda_{n,i}}{z - x_{n,i}}. \tag{1.8}$$ {Pade:PFD}

Now we are in conditions to prove Markov's theorem. In the following, we denote by $\|\cdot\|_K$ the sup-norm on the compact set K. {Markov}

Theorem 1.2:
Let $\mu \in \mathscr{M}(\Delta)$. We have

$$\limsup_n \|\widehat{\mu} - \pi_n(\widehat{\mu})\|_K^{1/2n} \leq \|\Psi^{-1}\|_K,$$

where K is a compact subset of $\overline{\mathbb{C}} \setminus \Delta$ and Ψ is the conformal map from $\overline{\mathbb{C}} \setminus \Delta$ onto the exterior of the unit disc such that $\Psi(\infty) = \infty, \Psi'(\infty) > 0$.

Proof. First, let us prove that the family $\{\pi_n\}_{n\geq 0}$ is normal. Fix a compact $K \subset \overline{\mathbb{C}} \setminus \Delta$ then, by (1.8)

$$|\pi_n(\widehat{\mu}; z)| \leq \sum_{i=1}^{n} \frac{\lambda_{n,i}}{|z - x_{n,i}|} \leq \frac{1}{d(\Delta, K)},$$

where $d(\Delta, K)$ stands for the distance between K and Δ.

Take the level curve $\Gamma_\rho = \{z \mid |\Psi(z)| = \rho\}$, $1 < \rho < +\infty$ with ρ close enough to 1 so that K lies outside Γ_ρ. On the curve Γ_ρ we get

$$\left|\Psi^{2n+1}(z)(\widehat{\mu} - \pi_n(\widehat{\mu}))(z)\right|_{z\in\Gamma_\rho} \leq C(\rho)\rho^{2n+1},$$

where the constant $C(\rho)$ is independent from n, because $\{\pi_n(\widehat{\mu})\}_{n\geq 0}$ is normal. But the analyticity of $\Psi^{2n+1}(\widehat{\mu}-\pi_n(\widehat{\mu}))$ on $\overline{\mathbb{C}}\setminus\Delta$ allows us to use the maximum modulus principle and assure that the bound also holds on K. Consequently,

$$|(\widehat{\mu}-\pi_n(\widehat{\mu}))(z)| \leq C(\rho)\left|\frac{\rho}{\Psi(z)}\right|^{2n+1}, \quad z\in K.$$

This is equivalent to

$$\|\widehat{\mu}-\pi_n(\widehat{\mu})\|_K \leq C(\rho)\left(\rho\|\Psi^{-1}\|_K\right)^{2n+1}.$$

Therefore, for all ρ sufficiently near 1,

$$\limsup_n \|\widehat{\mu}-\pi_n(\widehat{\mu})\|_K^{1/2n} \leq \rho\|\Psi^{-1}\|_K,$$

and the statement readily follows letting $\rho \to 1$.

□

Markov's theorem is a classical result in approximation theory and, as one can expect, it can be proved in different ways. For a proof closer to Markov's see [94, Sec. 3.5, Th. 3.5.4]. For an alternative proof where measure-theoretical arguments are used, see [77, Ch. 2, §6].

On the other hand, since 1895 Markov's theorem has been profusely studied and extended in several directions. One of the directions explored has been to obtain analogous results for measures with "larger" support, that is with measures supported on a half-line and on all $\mathbb{R}$. Here appears a connection between orthogonal polynomials, approximation theory, and moment problems, since Markov's theorem is strongly linked to moment problems.

Given a positive Borel measure λ with $\operatorname{supp}\lambda \subset \mathbb{R}$ its sequence of moments is defined as $c_n = \int x^n \,\mathrm{d}\lambda$, $n \in \mathbb{Z}_+$. The moment problem is, given a sequence $\{c_n\}_{n\geq 0}$ of real numbers to find, should it exist, a measure λ whose moments are $\{c_n\}_{n\geq 0}$. In case the moment problem is solvable we say that the moment problem is *determinate* if the solution is unique; otherwise, it is said to be *indeterminate*. The classical study of moment problems has three fundamental cases: when the support of the measure is a finite interval (*Hausdorff problem*); the support is a half-line (*Stieltjes problem*); and when $\operatorname{supp}\lambda = \mathbb{R}$ (*Hamburger problem*). In each one of these cases there are necessary and sufficient conditions for the moment problem to be solvable see, for example [89, 46].

It is well known that every solvable Hausdorff moment problem is determinate (see [89, p. xi]). It is a direct consequence of the Weierstrass theorem on the uniform approximation by means of polynomials of any continuous function on a bounded interval and the Riesz representation theorem for continuous positive linear functionals on the space of continuous functions on a compact set.

However, there are striking differences between the moment problem when $\operatorname{supp}\lambda$ is bounded and when it is not. Stieltjes and Hamburger moment problems may be indeterminate. One of the most remarkable works on the subject was due to T.J. Stieltjes (1856-1894) in the paper [93].

Here, Stieltjes introduced an enormous amount of very fruitful ideas (for more information related to the importance of Stieltjes' work see [96]). Among other things, he proved that if the moment problem is determinate for a measure λ supported on a half-line (let us say $\mathbb{R}_+$) then, the sequence of diagonal Padé approximants converges to $\widehat{\lambda}$. Since for measures of bounded support the moment problem is alway determinate, Stieltjes' theorem implies that of Markov.

Note that the problem studied by Stieltjes has an additional difficulty with respect to Markov's. In this case, the function to be approximated $\widehat{\lambda}$ is not holomorphic at the interpolation point ($z = \infty$). Moreover, the restriction imposed over the measure, i.e. that the moment problem be determinate, naturally leads to the following question (also posed by Stieltjes). Can we give a condition on the sequence $\{c_n\}_{n\geq 0}$ so that the (Stieltjes) moment problem be determinate? A sufficient condition was given by the Swedish mathematician T. Carleman (1892-1949) in 1926 ([21]). The so called Carleman condition

$$\sum_{n=1}^{\infty} c_n^{-1/2n} = \infty \tag{1.9}$$ {carl:cond}

implies determinacy. So, the condition (1.9) complements Stieltjes' theorem on the convergence of Padé approximants for the Cauchy transform of a measure supported on a half-line.

Another important extension to Markov's theorem, following the spirit of Stieltjes, is due to H. Hamburger (1889-1956), who obtained it as part of his doctoral dissertation and appeared in [43]. There, the German mathematician proved that the convergence of the Padé approximants to $\widehat{\lambda}$, when λ is supported on the real line is equivalent to the determinacy of the Hamburger moment problem. For a modern exposition of Hamburger's method, the interested reader can consult [12] and references therein. In addition, for a detailed discussion of Hamburger's solution of the moment problem and orthogonal polynomials see [77, Ch. 2 §7]. For a more classical approach to the Stieltjes and Hamburger moment problems and orthogonal polynomials see [34, Ch. II].

As is common in mathematics, the interest in Padé approximants was more or less dormant during the first half of the XX-th century. But, by the seventies they attracted much attention, particularly from members of the Soviet mathematical school led by A.A. Gonchar (1931-2012), who became one of the most important specialists in approximation theory, orthogonal polynomials, and Padé approximation.

1.3 Multipoint Padé approximants and Markov's theorem

The study of rational interpolation of functions is a mathematical problem with a long history, and can be traced back as far as the first half of the XIX-th century (see, [19, Sec. 5.2.5]). In [7, Sec. 1.1], rational interpolants are called multipoint Padé approximants, but are also called N-point Padé approximants. However, since the 1970's grew a renewed interest in multipoint Padé approximation. One of the reasons to this resurgence was the link that A.A. Gonchar observed between these type of approximants and the rate of convergence of best rational approximants to

Markov functions associated with measures of bounded support, see [38]. This connection led Gonchar to study the following interpolation problem for a Markov function $\widehat{\mu}$ as in (1.2), with Δ
{pr:MPade} a finite interval.

Problem 2:

Let $\{w_{2n}\}_{n\in\mathbb{Z}_+}$, $\deg w_{2n} = 2n$ *be a sequence of monic polynomials with real coefficients, whose zeros* $\{x_{2n,i}\}_{i=1}^{2n}$ *(counting multiplicities) lie in* $\mathbb{C}\setminus\Delta$*. Find a pair of polynomials* (P_{n-1}, Q_n)*,* $n \geq 1$ *such that*

{MPade:1}

{MPade:2}

1. $\deg P_{n-1} \leq n-1$ *and* $\deg Q_n \leq n$*,* $Q_n \not\equiv 0$;
2. $\mathscr{O}\left(\dfrac{1}{z^{n+1}}\right) = \dfrac{(Q_n\widehat{\mu} - P_{n-1})(z)}{w_{2n}(z)} \in \mathbf{H}(\overline{\mathbb{C}}\setminus\Delta)$.

The fraction P_{n-1}/Q_n is called the n-th *multipoint Padé approximant* of $\widehat{\mu}$. From condition 1 we have $2n+1$ unknowns (the coefficients of the polynomials) and from 2 we have $2n$ equations. So, the existence of the pair (P_{n-1}, Q_n) is reduced to solving a homogeneous linear system of equations and by the Rouché-Frobenius Theorem it always has a solution. If we normalize Q_n to be monic the solution is unique.

Given a smooth Jordan curve Γ, surrounding Δ and $\{x_{n,i}\}_{i=1}^{2n}$ lying outside of it, by the Residue and Fubini Theorems for $\nu = 0, 1, \ldots, n-1$ we have [38, §2.1]

$$\begin{aligned} 0 &= \int_\Gamma \frac{(Q_n\widehat{\mu} - P_{n-1})(z)}{w_{2n}(z)} z^\nu \,\mathrm{d}z = \int_\Gamma \frac{(Q_n\widehat{\mu})(z)}{w_{2n}(z)} z^\nu \,\mathrm{d}z = \int_\Gamma\int_\Delta \frac{z^\nu Q_n(z)}{(z-x)w_{2n}(z)}\,\mathrm{d}\mu(x)\,\mathrm{d}z \\ &= \int_\Delta\int_\Gamma \frac{z^\nu Q_n(z)}{(z-x)w_{2n}(z)}\,\mathrm{d}z\,\mathrm{d}\mu(x) = \int_\Delta x^\nu Q_n(x)\frac{\mathrm{d}\mu(x)}{w_{2n}(x)}. \end{aligned}$$

Consequently, the polynomial Q_n coincides with the n-th monic orthogonal polynomial with respect to the measure (with differential notation) $\mathrm{d}\mu_n := \mathrm{d}\mu/|w_{2n}|$. In order to avoid confusion with $Q_n(\cdot;\mu)$, we use the notation $Q_n(\cdot;\mu_n)$ and $\{Q_n(\cdot;\mu_n)\}_{n\geq 0}$ is the sequence of orthogonal polynomials with respect to the *varying measure* $\mathrm{d}\mu_n$, and the respective orthonormal polynomials are $q_n := \tau_n Q_n$ where

{normal}
$$\tau_n := \left(\int x^\nu Q_n(x)\frac{\mathrm{d}\mu_n(x)}{w_{2n}(x)}\right)^{-1/2}. \tag{1.10}$$

This connection revealed a series of useful facts: $Q_n(\cdot;\mu_n)$ has maximal degree; all its zeros are real, simple and lie inside Δ and the polynomials $P_{n-1}(\cdot;\mu_n)$ and $Q_n(\cdot;\mu_n)$ are mutually prime.

At the same time, new questions originated from Gonchar's paper because the introduction of orthogonal polynomials with respect to varying measures posed new difficulties in order to extend results such as the Markov and Stieltjes theorems. Both problems were attacked by one of his students G. López Lagomasino, in [58] and [49, 50], respectively. One can corroborate the difficulties added by varying measures comparing the proofs of a result analogous to Proposition 1.1 for multipoint Padé approximants. The original result can be found in [58], and the present proof is inspired by [59, Lemma 2.1].

{remain:2}

Proposition 1.3:
Let $\mu \in \mathscr{M}(\Delta)$ and w_{2n} is a monic polynomial with real coefficients whose zeros $\{x_{2n,i}\}_{i=1}^{2n}$ lie outside Δ. For the polynomials Q_n, P_{n-1} defined by Problem 2 we get,

$$\widehat{\mu}(z) - \frac{P_{n-1}(z)}{Q_n(z)} = \frac{w_{2n}(z)}{Q_n(z)} \int \frac{Q_n(x)}{z-x} \frac{\mathrm{d}\mu(x)}{w_{2n}(x)} = \frac{w_{2n}(z)}{Q_n^2(z)} \int \frac{Q_n^2(x)}{z-x} \frac{\mathrm{d}\mu(x)}{w_{2n}(x)}, \quad z \in \mathbb{C} \setminus \Delta.$$

Proof. Write $A_n(z) = (Q_n\widehat{\mu} - P_{n-1})(z)$. Then,

$$\begin{aligned} A_n(z) &= -P_{n-1}(z) + \int Q_n(z) \frac{\mathrm{d}\mu(x)}{z-x} - w_{2n}(z) \int \frac{Q_n(x)}{z-x} \frac{\mathrm{d}\mu(x)}{w_{2n}(x)} + w_{2n}(z) \int \frac{Q_n(x)}{z-x} \frac{\mathrm{d}\mu(x)}{w_{2n}(x)} \\ &= -P_{n-1}(z) + \int \frac{w_{2n}(x)Q_n(z) - w_{2n}(z)Q_n(x)}{z-x} \mathrm{d}\mu(x) + w_{2n}(z) \int \frac{Q_n(x)}{z-x} \frac{\mathrm{d}\mu(x)}{w_{2n}(x)}. \end{aligned}$$

Notice that

$$\int \frac{w_{2n}(x)Q_n(z) - w_{2n}(z)Q_n(x)}{z-x} \mathrm{d}\mu(x)$$

is a polynomial in z. Consequently,

$$L_n(z) = -P_{n-1}(z) + \int \frac{w_{2n}(x)Q_n(z) - w_{2n}(z)Q_n(x)}{z-x} \mathrm{d}\mu(x)$$

is also a polynomial and

$$A_n(z) = L_n(z) + w_{2n}(z) \int \frac{Q_n(x)}{z-x} \frac{\mathrm{d}\mu(x)}{w_{2n}(x)} = w_{2n}(z)\mathscr{O}\left(\frac{1}{z^{n+1}}\right).$$

Necessarily, $\deg L_n < \deg w_{2n}$ and at the same time L_n equals zero at the roots of w_{2n}. Therefore, $L_n \equiv 0$. From this, the first equality of the statement readily follows.

The second equality is obtained by orthogonality. Note that

$$\int \frac{Q_n(z) - Q_n(x)}{x-z} Q_n(x) \frac{\mathrm{d}\mu(x)}{w_{2n}(x)} = 0.$$

Hence,

$$Q_n(z) \int \frac{Q_n(x)}{x-z} \frac{\mathrm{d}\mu(x)}{w_{2n}(x)} = \int \frac{Q_n^2(x)}{x-z} \frac{\mathrm{d}\mu(x)}{w_{2n}(x)},$$

and the statement is immediate.

□

The Markov-type theorem proved in [49, Th. 1] is the following. {LagoMar78}

Theorem 1.4:
Let $\{w_{2n}\}_{n\geq 0}$ be a sequence of monic polynomials with real coefficients with zeros $\{x_{2n,i}\}_{i=1}^{2n}$ and let $X = \cup_{n\geq 0}\{x_{2n,i}\}$. If (the accumulation points of X) $X' \subset \mathbb{C} \setminus \Delta$, then for the rational fractions P_{n-1}/Q_n defined by Problem 2 we have

$$\lim_n \frac{P_{n-1}(z)}{Q_n(z)} = \widehat{\mu}(z), \quad z \in \mathbb{C} \setminus \Delta,$$

and the convergence is uniform on compact subsets $K \subset \mathbb{C} \setminus \Delta$ with geometric rate.

The ideas behind [38, 58] led to the extension of the Stieltjes theorem to the case of multipoint Padé approximants in [50]. Let $\mu_{\mathbb{R}_+}$ be a finite Borel measure on $[0,\infty)$ and assume that the set of zeros of the polynomial w_{2n} are contained in $(-\infty, a]$, $a < 0$. If we write

$$c_{n,k} = \int_0^\infty \frac{x^k \,\mathrm{d}\,\mu_{\mathbb{R}_+}(x)}{w_{n,k+1}(x)}, \quad w_{n,k+1}(x) = \prod_{i=1}^{k+1}(1 - x_{2n,i}^{-1}x), \quad k = 0, 1, \ldots, 2n-1$$

the extension of the Carleman-Stieltjes theorem in [50, Th. 1] says that if

{carl:gen}

$$\lim_n \sum_{k=1}^{2n-1} \frac{1}{\sqrt[2n]{c_{n,k}}} = \infty, \tag{1.11}$$

we get uniform convergence of P_{n-1}/Q_n to $\widehat{\mu}_{\mathbb{R}_+}$ on compact subsets of $\mathbb{C} \setminus [0,\infty)$. It is easy to check that if $\mu_{\mathbb{R}_+}$ satisfies Carleman's condition (1.9) then (1.11) holds. Therefore, for measures satisfying Carleman's condition the multipoint Padé approximants converge to $\widehat{\mu}_{\mathbb{R}_+}$ no matter how you choose the points $x_{2n,i} \in (-\infty, a]$, $a < 0$. This is mentioned as a corollary. It is also mentioned that if the zeros $x_{2n,i}$ verify $\lim_n \sum_{i=1}^n (1/\sqrt{|x_{2n,i}|}) = \infty$ then (1.11) takes place for any $\mu_{\mathbb{R}_+}$. Therefore, an analogue of Markov's theorem is valid for multipoint Padé approximants as well.

Another twist in the theory of Padé approximation was given in [20], where a more general approximation scheme was introduced.

{def:pade:inc}

Definition 1.5:
Let $\mu \in \mathscr{M}(\Delta)$ where Δ is contained in a half line of the real axis. Fix an arbitrary $\kappa \geq -1$. Consider a sequence of polynomials $\{w_n\}_{n\in\Lambda}$, $\Lambda \subset \mathbb{Z}_+$, such that $\deg w_n = \kappa_n \leq 2n + \kappa + 1$, whose zeros lie in $\mathbb{R} \setminus \Delta$. Let $R_n = p_n/q_n$ be a sequence of rational functions with real coefficients satisfying the following conditions for each $n \in \Lambda$:

a) $\deg p_n \leq n + \kappa$, $\deg q_n \leq n$, $q_n \not\equiv 0$,

b) $\dfrac{q_n\widehat{\mu} - p_n}{w_n}(z) = \mathcal{O}\left(\dfrac{1}{z^{n+1-l}}\right) \in \mathbf{H}(\mathbb{C} \setminus \Delta)$, $z \to \infty$, *where $l \in \mathbb{Z}_+$ is fixed.*

We say that $\{R_n\}_{n\in\Lambda}$ is a sequence of incomplete diagonal multi-point Padé approximants of $\widehat{\mu}$.

The existence of incomplete diagonal multipoint Padé approximants is always guaranteed, but they are not necessarily unique, as it happens with "complete" multipoint Padé approximants. For sequences of this kind of rational approximants a Stieltjes-type theorem [20, Lemma 2] was obtained in terms of (logarithmic) capacity. We rewrite it using 1-Hausdorff content. The proof for 1-Hausdorff content is simpler, because this concept is easier to manipulate that the logarithmic capacity. The aforementioned result will be fundamental in the following chapters, and we enunciate it here. First we need to introduce the notion of 1-Hausdorff convergence.

def:hausdorff}

Definition 1.6:
Let A be a subset of $\mathbb{C}$. By $\mathscr{U}(A)$ we denote the class of all coverings of A by at most a numerable set of disks.

i) *Set*

$$h(A) = \inf\left\{\sum_{i=1}^{\infty} |U_i| \mid \{U_i\} \in \mathscr{U}(A)\right\},$$

where $|U_i|$ *stands for the radius of the disk* U_i*. The quantity* $h(A)$ *is called the* 1*-dimensional Hausdorff content of the set* A*.*

ii) *Let* $\{\phi_n\}_{n\in\mathbb{N}}$ *be a sequence of complex-valued functions defined on a region* $D \subset \mathbb{C}$ *and* ϕ *another function defined on* D *(the value* ∞ *is permitted). We say that* $\{\phi_n\}_{n\in\mathbb{N}}$ *converges in Hausdorff content to the function* ϕ *inside* D *if for each compact subset* K *of* D *and for each* $\epsilon > 0$*, we have*

$$\lim_n h\{z \in K \mid |\phi_n(z) - \phi(z)| > \epsilon\} = 0$$

(by convention $\infty \pm \infty = \infty$*). We denote this writing* $h - \lim_n \phi_n = \phi$ *inside* D*.*

Following, we have the Stieltjes-type result for incomplete diagonal multi-point Padé approximants. {lm:pade:incom

Lemma 1.7:
Let $\mu \in \mathscr{M}(\Delta)$ *be given, where* Δ *is contained in a half line. Assume that* $\{R_n\}_{n\in\Lambda}$*,* $\Lambda \subset \mathbb{N}$ *satisfies* a)-b) *in Definition* 1.5 *and either the number of zeros of* w_n *lying on a bounded segment of* $\mathbb{R} \setminus \Delta$ *tends to infinity as* $n \to \infty$*,* $n \in \Lambda$ *or* μ *satisfies Carleman's condition* (1.9)*. Then*

$$h - \lim_{n\in\Lambda} R_n = \widehat{\mu}, \text{ inside } \mathbb{C} \setminus \Delta.$$

1.3.1 Markov-type theorems for meromorphic functions

A central problem in the study of Padé approximants is its convergence to meromorphic functions. Again, these questions can be studied with diagonal or horizontal sequences of the Padé table. A very well known result on the convergence of horizontal sequences is Montessus de Ballore's theorem (see [72]), and it served as a starting point to further research in this topic. The convergence of diagonal sequences of Padé approximants of meromorphic functions presented new challenges. The first theorems on the convergence of diagonal (or near diagonal) sequences in measure or capacity were obtained by J. Nuttal [78] and C. Pommerenke [81], respectively.

This problem attracted the attention of A.A. Gonchar, and in [36] solved it for a wide class of meromorphic functions. He considered functions of the form $f = \widehat{\mu} + r$ where r is a rational function with poles outside $\Delta = \mathrm{Co}(\operatorname{supp}\mu)$ and $r(\infty) = 0$. Then, if Δ is a real finite interval and the sequence $\{q_n\}_{n\geq 0}$ of orthonormal polynomials with respect to μ has ratio asymptotic

$$\lim_{n\to\infty} \frac{q_{n+1}(z)}{q_n(z)} = \Psi(z) \qquad (1.12)$$ {ratio}

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \Delta$, we get

$$\lim_n \pi_n(f)(z) = f(z) = \widehat{\mu}(z) + r(z), \qquad (1.13)$$ {markov:rat}

where the convergence is uniform on compact subsets of the region $\mathbb{C} \setminus (\Delta \cup [r(z) = \infty])$ (with the poles of r deleted). To rightly assess Gonchar's contribution we must comment one of the difficulties he encountered to obtain (1.13). The "perturbation" introduced by the rational fraction r makes it possible that some (or even all) zeros of Q_n, i.e. the poles of $\pi_n(f)$, abandon Δ. Hence, the location of these "wild" zeros becomes a major question. He was able to tackle the problem with sophisticated asymptotic relations and bounds, and he proved that each pole of r attracts as many zeros of Q_n as its multiplicity, and the remaining zeros accumulate on Δ. Gonchar's proof was later simplified in [56] thanks to a lemma proved by Gonchar in [37], which predates [36], but not used there (!). Gonchar's lemma allows to locate poles of Padé approximants of meromorphic functions and, in addition, also serves to deduce uniform convergence from a weaker type of
{lm:gonchar} convergence. This result is part of our theoretical background, so we will state it here.

Lemma 1.8 (A.A. Gonchar)**:**
Assume that $h - \lim\limits_n \phi_n = \phi$ *inside a region* Ω.

i) *If* $\phi_n \in \mathbf{H}(\Omega)$ *for all n, then*

$$\lim_n \phi_n = \phi, \qquad K \subset \Omega,$$

uniformly on compact subset $K \subset \Omega$ *and* $\phi \in \mathbf{H}(\Omega)$ (*more precisely,* ϕ *differs from a certain* $\phi_0 \in \mathbf{H}(\Omega)$ *at most on a set E of null* 1*-Hausdorff content*).

ii) *If for all* $n \in \mathbb{N}$, $\phi_n \in \mathbf{M}_m(\Omega)$ *–the class of all meromorphic functions in* Ω *with at most m poles counting multiplicity– then* $\phi \in \mathbf{M}_m(\Omega)$.

iii) *If for all* $n \in \mathbb{N}$, $\phi_n \in \mathbf{M}_m(\Omega)$ *and* ϕ *has exactly m poles in* Ω, *then there exists* $n_0 \in \mathbb{N}$ *such that for all* $n > n_0$ *each* ϕ_n *has exactly m poles in* Ω. *Moreover, if* ζ *is a pole of* ϕ *of order* ν, *then for each* $\epsilon > 0$ *sufficiently small there exists* $n_0(\zeta)$ *such that for all* $n > n_0(\zeta)$ *the functions* ϕ_n *have exactly* ν *poles in the disk* $\{z : |z - \zeta| < \epsilon\}$. *We express this saying that the poles of* ϕ_n *converge as* $n \to \infty$ *to the poles of* ϕ *in* Ω *according to their order. Finally,*

$$\lim_n \phi_n = \phi, \qquad K \subset \Omega',$$

where Ω' *is the region obtained deleting from* Ω *the poles of* ϕ.

In [36], in order to guarantee that the sequence of orthonormal polynomials of μ verifies (1.12), A.A. Gonchar imposed the Szegő condition

{szego:cond}
$$\int_a^b \frac{\ln \mu'(x)\,\mathrm{d}x}{\sqrt{(b-x)(x-a)}} > -\infty, \qquad \Delta = [a, b], \tag{1.14}$$

which implies the so called strong asymptotic of the sequence of orthonormal polynomials, see [94, Th. 12.1.2]. However, as the author underlines, only (1.12) is used in the proof.

A natural problem was to find a condition weaker than Szegő's which implied (1.12). A student of Gonchar, E.A. Rakhmanov asserted in [83] that it was sufficient to require $\mu' > 0$ a.e. in Δ, where μ' stands for the Radon-Nikodym derivative of μ with respect to the Lebesgue measure.

However, A. Maté and P. Nevai observed in [69] that there was a gap in Rakhmanov's proof due to a typo in formula (xiii. 10) of [35]. E.A. Rakhmanov removed the gap in [84]. Simplified proofs of Rakhmanov's theorem may be found in [85, 74].

In a separate paper, E.A. Rakhmanov showed in [82] that if the fraction r has real coefficients then, (1.13) can be obtained for arbitrary positive Borel measures supported on Δ (for multipoint Padé approximants see [51]). In the same paper, he also showed that if r has complex coefficients the Padé approximants may fail to converge uniformly in $\mathbb{C} \setminus \Delta$ with the poles of r deleted, even for very simple measures supported on two disjoint intervals of $\mathbb{R}$. That is, in the complex case it is mandatory to impose additional restrictions on the measure μ.

The study of meromorphic functions of the form $\widehat{\mu} + r$ for measures supported in $\mathbb{R}_+$ was done by G. López Lagomasino in two different papers. The first one (see [52]), dealt with the case of rational fractions r with real coefficients, poles outside the positive half-line, and $r(\infty) = 0$, while the moments of the measure satisfy Carleman's condition. If this extension of Gonchar's result on diagonal sequences of Padé approximants of Markov-type meromorphic functions appeared shortly after Gonchar's, the case of r with complex coefficients appeared much later. The main reason was that for measures with unbounded support the orthogonal polynomials do not verify ratio asymptotic. It was necessary to translate the problem to one with orthogonal polynomials with respect to varying measures with bounded support and extend for such sequences of orthogonal polynomials Rakhmanov's theorem on ratio asymptotic. The problem was finally settled in [56] under the additional assumption, apart from Carleman's condition, that $\mu' > 0$ a.e. on $\mathbb{R}_+$.

As a final remark to the present section, we emphasize that the results obtained in [36, 82, 52, 56] as extensions of the Markov and Stieltjes theorems show the importance of the study of the asymptotic properties of orthogonal polynomials, with particular interest in orthogonal polynomials with varying measures. They allow not only to estimate the rate of convergence, but also to prove the convergence of Padé and multipoint Padé approximants in certain classes of meromorphic functions.

1.4 Asymptotic of orthogonal polynomials

The study of the asymptotic properties of orthogonal polynomials, is a wide field where several branches of mathematical analysis meet. Since the first decades of the xx-th century, this has been a subject that has drawn a lot of attention, and rivers of ink have flowed around it. It is sufficient to take a look at the extensive literature dedicated to orthogonal polynomials with respect to a "fixed" measures and to "varying" measures see, for example, [98, 95], [77, Ch. 3] and [64, 79].

To fix ideas, let us recall that we are dealing, in general, with measures $\mu \in \mathscr{M}(\Delta)$, where $\Delta = \mathrm{Co}(\operatorname{supp} \mu) \subset \mathbb{R}$ with the additional assumption that Δ is compact. The analysis of the asymptotic of "general" orthogonal polynomials is divided into the exterior asymptotic and the interior asymptotic. The latter studies the sequence of orthogonal polynomials $\{Q_n\}_{n\geq 0}$ for $x \in \Delta$

with respect to certain norm, usually $L_2(\mu, \Delta)$; and the former captures the behavior of $\{Q_n\}_{n\geq 0}$ in $\mathbb{C} \setminus \Delta$. Furthermore, exterior asymptotic can be divided into three types, each one imposing different restrictions on the measure:

(L) *Weak* or *logarithmic asymptotic*. Here the sequence $\{|Q_n(z)|^{1/n}\}_{n\geq 0}$ is studied. It is also known as n-th root asymptotic.

(R) *Ratio asymptotic*. The sequence under consideration is $\{Q_{n+1}/Q_n\}_{n\geq 0}$.

(S) *Strong* or *Szegő asymptotic*. The interest is focused on $\{Q_n/\Phi^n\}_{n\geq 0}$ where Φ is a certain analytic function on $\overline{\mathbb{C}} \setminus \Delta$.

It is well known and easy to verify that

$$\text{(S)} \Rightarrow \text{(R)} \Rightarrow \text{(L) and (L)} \not\Rightarrow \text{(R)} \not\Rightarrow \text{(S)}.$$

We will start by discussing an analytic function which plays a central role in the study of strong asymptotic and some developments of this dissertation. Then, we are going to review the main results related to the exterior asymptotic of orthogonal polynomials.

1.4.1 Szegő function

{Szego:fun}

Let Δ be a compact interval of $\mathbb{R}$ and μ a measure such that $\operatorname{supp}\mu \subset \Delta$. If μ satisfies *Szegő's condition*:

{szeg:cond}

$$\int_\Delta \ln \mu'(x)\, \mathrm{d}\,\eta_\Delta(x) > -\infty, \tag{1.15}$$

we write $\mu \in \mathscr{S}(\Delta)$, where μ' denotes the Radon-Nikodym derivative of μ with respect to the Lebesgue measure and

{df:cheb}

$$\mathrm{d}\,\eta_\Delta(x) := \frac{\mathrm{d}\,x}{\sqrt{(x-a)(b-x)}} \tag{1.16}$$

stands for the Chebyshev measure on the interval $\Delta = [a, b]$.

Let $\mu \in \mathscr{S}([-1, 1])$. On the unit circle $\mathbb{T}$, a symmetric measure σ can be defined such that $\sigma(B) = \mu(B^*)$ whenever B is a Borel set contained either in the upper or lower half of the unit circle and B^* is its orthogonal projection on $[-1, 1]$. It readily follows that

$$\sigma'(e^{it}) = |\sin t|\, \mu'(\cos t), \qquad t \in [0, 2\pi]$$

where σ' and μ' denote the Radon-Nikodym derivatives of σ and μ with respect to the Lebesgue measure on $\mathbb{T}$ and $[-1, 1]$, respectively. If $\zeta = e^{it}$ and $x = \operatorname{Re}(\zeta) = \cos t$, we can also write

$$\sigma'(\zeta) = \sqrt{1 - x^2}\, \mu'(x), \qquad \zeta \in \mathbb{T}, \qquad x = \operatorname{Re}(\zeta).$$

Let

$$\mathsf{S}(\sigma, z) := \exp\left[\frac{1}{4\pi} \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \ln \sigma'(\zeta) |\mathrm{d}\,\zeta|\right],$$

be the (standard) Szegő function associated with the measure σ. Note that if μ satisfies the Szegő condition on $[-1,1]$ then $\int_{\mathbb{T}} \ln\sigma'(\zeta)|\,\mathrm{d}\zeta| > -\infty$; that is, σ verifies Szegő's condition on $\mathbb{T}$ (and vice versa).

In general, when $\operatorname{supp}\mu = \Delta = [a,b]$ (not necessarily $[-1,1]$), we define σ as it was done before out of the measure $\widetilde{\mu}$ supported on $[-1,1]$ such that $\widetilde{\mu}(B) = \mu(\{x \in [a,b] : \frac{2}{b-a}\left(x - \frac{b+a}{2}\right) \in B\})$, for every Borel set $B \subset [-1,1]$. In this case

$$\sigma'(e^{it}) = \sqrt{(b-x)(x-a)}\mu'(x), \qquad x = \frac{b-a}{2}\cos t + \frac{b+a}{2}.$$

We wish to define a Szegő function $\mathsf{G}(\mu,\cdot)$ with respect to the measure μ so that

$$\mathsf{G}(\mu,u) = \mathsf{S}(\sigma,\Psi(u)), \qquad u \in \overline{\mathbb{C}} \setminus \Delta,$$

where Ψ is the conformal map defined previously. Then, from known properties of the Szegő function for measures on the unit circle, we have

$$\lim_{u\to x} |\mathsf{G}(\mu,u)|^2 = \lim_{u\to x} |\mathsf{S}(\sigma,\Psi(u))|^2 = 1/\sigma'(\zeta) = (\sqrt{(b-x)(x-a)}\,\mu'(x))^{-1}, \quad \text{a.e. on } \Delta, \tag{1.17}$$ {Glimit}

where $\zeta = \Psi(x)$ (Ψ can be extended continuously to Δ as usual assuming that the interval has two sides and since σ is symmetric with respect to the real line we can take ζ either on the upper half or the lower half of $\mathbb{T}$). Straightforward calculations show that the explicit expression of $\mathsf{G}(\mu,u)$ (the Szegő function for μ supported on Δ) is

$$\mathsf{G}(\mu,u) := \exp\left[\frac{\sqrt{(u-b)(u-a)}}{2\pi}\int_\Delta \frac{\ln(\sqrt{(b-x)(x-a)}\,\mu'(x))}{x-u}\,\mathrm{d}\eta_\Delta(x)\right]. \tag{1.18}$$ {green}

The square root outside the integral is taken to be positive for $u > b$ and those inside the integral are positive when $x \in (a,b)$.

The property stated in (1.17) also serves to characterize the Szegő function associated with a measure $\mu \in \mathscr{S}(\Delta)$ by a boundary value problem. This is, given $\mu \in \mathscr{S}(\Delta)$ find a holomorphic function g in $\overline{\mathbb{C}} \setminus \Delta$ such that

$$\begin{cases} g(u) \neq 0 \text{ for } u \in \overline{\mathbb{C}} \setminus \Delta; \\ g(\infty) > 0; \\ \lim_{u\to x} |g(u)|^2 = (\sqrt{(b-x)(x-a)}\,\mu'(x))^{-1}, \quad \text{a.e. on } \Delta. \end{cases} \tag{1.19}$$ {szego:BVP}

This problem has as solution $g(u) = \mathsf{G}(\mu,u)$ (for an expanded exposition see [94, Ch.xvi]).

When h is a function on Δ such that $\ln h$ is integrable with respect to $\mathrm{d}\eta_\Delta(x)$ we also write

$$\mathsf{G}(h,u) = \exp\left[\frac{\sqrt{(u-b)(u-a)}}{2\pi}\int_\Delta \frac{\ln h(x)}{x-u}\,\mathrm{d}\eta_\Delta(x)\right], \quad u \in \overline{\mathbb{C}} \setminus \Delta.$$

These functions are related with outer functions (see [87, Def. 17.14]) whose analytical representation is

$$g(h,u) = c\exp\left[\frac{\sqrt{(u-b)(u-a)}}{\pi}\int_\Delta \frac{\ln h(x)}{x-u}\,\mathrm{d}\eta_\Delta(x)\right], \quad u \in \overline{\mathbb{C}} \setminus \Delta,$$

where $|c| = 1$.

Notice that in the definition of the Szegő function (which squared is an outer function) with respect to a measure μ we do not take the Radon-Nikodym derivative μ' of μ with respect to the Lebesgue measure $\mathrm{d}\,x$ but instead with respect to $\mathrm{d}\,\eta_\Delta(x)$ which is precisely $\sqrt{(b-x)(x-a)}\mu'(x)$.

1.4.2 Asymptotic

The study of the strong asymptotic of general orthogonal polynomials was initiated by G. Szegő (1895-1985) and S. Bernstein (1880-1968), although results in this direction were obtained earlier for classical orthogonal polynomials. Anyway, the work of Szegő has a remarkable generality and an intrinsic beauty due to the elegance of his arguments. The core of Szegő's method is to study the orthogonal polynomials on the unit circle instead of Δ, and using the tools from Hardy space theory he was able to obtain the strong asymptotic of these kind of polynomials. Afterwards, with a very clever ploy he established a direct connection between orthogonal polynomials on a real interval and orthogonal polynomials on the unit circle. Nowadays [94, Ch. xi, Ch. xii] is still a basic reference on the subject, that can be complemented and expanded with [90].

With the previous discussion we are in condition to state Szegő's theorem on the strong
{szeg:th} asymptotic of orthogonal polynomials (see [94, Th. 12.1.2]).

Theorem 1.9 (Szegő)**:**
Let $\mu \in \mathscr{S}(\Delta)$ and $\{q_n(\cdot,\mu)\}_{n\geq 0}$ the sequence of orthonormal polynomials with respect to μ (see (1.3)). Moreover, let G be defined as in (1.18) and Ψ be the conformal map from $\overline{\mathbb{C}} \setminus \Delta$ onto $\{|z| > 1\}$ with $\Psi(\infty) = \infty$ and $\Psi'(\infty) > 0$. Then,

$$\lim_n \frac{q_n(z;\mu)}{\Psi^n(z)} = \frac{1}{\sqrt{2\pi}}\mathsf{G}(z,\mu), \quad z \in \overline{\mathbb{C}} \setminus \Delta$$

where the limit is uniform on compact subsets of $\overline{\mathbb{C}} \setminus \Delta$. In addition,

$$\lim_n \alpha_n \operatorname{cap}^n(\Delta) = \frac{1}{\sqrt{2\pi}}\mathsf{G}(\infty,\mu),$$

where $\operatorname{cap}(\Delta) = 1/\Psi'(\infty)$ denotes the logarithmic capacity of Δ and α_n is the leading coefficient of q_n.

Later, L.Ya. Geronimus (1898-1984) established an equivalence between the existence of asymptotic formulas as the ones in Theorem 1.9 and Szegő's condition (see [35, Th. 9.2]). This is, if a subsequence of $\{\alpha_n \operatorname{cap}^n(\Delta)\}_{n\geq 0}$ is bounded above or, for some $z \in \overline{\mathbb{C}} \setminus \Delta$ a subsequence of $\{q_n(z;\mu)/\Psi^n(z)\}$ is bounded, then (1.15) must hold. Although the statement of Theorem 1.9 is given for general measures in the class $\mathscr{S}(\Delta)$, Szegő originally proved it for absolutely continuous measures, A.N. Kolmogorov (1903-1987) and M.G. Krein (1907-1989) obtained it for more general measures. Furthermore, a general treatment can be found in G. Freud's book ([34, §v.2]), though Szegő's theorem is relatively hidden.

A research as complete as Szegő's was published by H. Widom (1932-2021) in a lengthy article from 1969, [99]. There, he studied sequences of orthogonal polynomials on a system of Jordan curves and arcs, among other related subjects as extremal polynomials. One of the problems Widom overcame was the multi-valuedness of the functions appearing in the asymptotic formulas, using powerful techniques from complex analysis and potential theory.

According to [64], it seems that P. Erdős (1913-1996) and P. Turán (1910-1976) were the first interested in the study of the n-th root asymptotic and the zero distribution of general orthogonal polynomials, while they were researching the convergence of Lagrange interpolation ([27]). The Hungarian mathematicians proved that given a measure μ verifying the so called *Erdős-Turán condition*, i.e., $\mu' > 0$ a.e. on Δ, then

$$\lim_n |q_n(z;\mu)|^{1/n} = |\Psi(z)|, \quad z \in \mathbb{C} \setminus \Delta, \tag{1.20}$$ {eq:root}

where the convergence is uniform on compact subsets of $\mathbb{C} \setminus \Delta$, and

$$\lim_n \alpha_n^{1/n} = \frac{1}{\operatorname{cap}(\Delta)}.$$

An interesting consequence of Erdős-Turán's result is the following. Let δ_x denote Dirac's delta at x, and $\{x_{n,j}\}_{j=1}^n$ the zeros of the polynomial $Q_n(z,\mu)$. Define the n-th zero counting measure:

$$\mu_n := \frac{1}{n}\sum_{j=1}^{n} \delta_{x_{n,j}}. \tag{1.21}$$ {def:zero_coun

Then,

$$* - \lim_n \mu_n = \frac{1}{\pi}\frac{\mathrm{d}x}{\sqrt{(x-a)(b-x)}}$$

where the convergence is considered in the weak-* topology of measures. This is, the asymptotic zero distribution converges to the unitary Chebyshev measure, independently of the measure μ.

Much research has been done related to logarithmic asymptotic, we suggest the general survey [64] and references therein, as well as the monograph [92]. In particular, the latter introduced an important notion in the study of the logarithmic asymptotic of orthogonal polynomials: *the regular (n-th root) asymptotic behavior*. Taking into account [92, Th. 3.1.1] and [92, Def. 3.1.2], we give the following definition. {Reg}

Definition 1.10:
Let μ be a finite positive Borel measure with compact support. Let α_n be as in (1.3)*. We say that μ is regular, and write $\mu \in$* **Reg***, if*

$$\lim_n \alpha_n^{1/n} = \frac{1}{\operatorname{cap}(\operatorname{supp}\mu)}.$$

In [92] it is proved that when $\mu \in$ **Reg** then the sequence $\{q_n\}_{n\geq 0}$ of orthonormal polynomials with respect to μ have what is called regular n-th root asymptotic behavior (this is a formula similar to (1.20) with an appropriate right hand side).

It is curious that strong and n-th root asymptotic attracted so much attention in the first half of the xx-th century while no one showed interest for ratio asymptotic, at least as far as the known literature is concerned. According to [64], P. Nevai (1948-) was the first to look at this type of asymptotic, though he was primarily interested in the behavior of the coefficients of the recurrence relation associated with the sequence of orthogonal polynomials. Recall that if $\{q_n(\cdot,\mu)\}_{n\geq 0}$ is the sequence of orthonormal polynomials associated with μ, then

$$xq_n(x) = A_{n+1}q_{n+1}(x) + B_n q_n(x) + A_n q_{n-1}(x), \quad n = 1, 2, \ldots,$$

where $A_n > 0$ and $B_n \in \mathbb{R}$. In the book [73] P. Nevai introduced what is known today as the *Nevai class*. It is said that μ is in the Nevai class $\mathscr{M}(A, B)$ if

$$\lim_n A_n = A > 0, \qquad \lim_n B_n = B.$$

Nevai was able to prove that if $\mu \in \mathscr{M}(A, B)$ then $\operatorname{supp}\mu$ equals an interval $\Delta = [B - 2A, B + 2A]$ plus a denumerable set of isolated points in $\overline{\mathbb{R}} \setminus \Delta$. Additionally,

{nevai}

$$\lim_n \frac{q_{n+1}(z,\mu)}{q_n(z,\mu)} = \Psi(z), \quad z \in \mathbb{C} \setminus \operatorname{supp}(\mu) \tag{1.22}$$

uniformly on compact subsets of $\mathbb{C} \setminus \operatorname{supp}(\mu)$ and Ψ is the conformal map from $\overline{\mathbb{C}} \setminus \Delta$ onto the complement of the unit disk defined above. Moreover, if (1.22) is verified pointwise for $\{z_m\}_{m\geq 0}$ with $z_m \underset{m}{\longrightarrow} \infty$, then the measure μ is in the Nevai class.

As was mentioned earlier in the analysis of Gonchar's extension of Markov's Theorem, E.A. Rakhmanov proved that the Erdős-Turán condition implies ratio asymptotic and, therefore, such measures belong to the Nevai class.

1.5 Asymptotic for varying measures

subsection1.5}

The interesting properties and many applications of orthogonal polynomials with respect to varying measures revealed by the research of A.A. Gonchar and his circle of collaborators, worked as a catalytic to the study of the asymptotic properties of these polynomial sequences.

It seems that the path was opened in [40] (originally published in Russian in 1984), a seminal research that definitely linked the weak asymptotic as well as the asymptotic zero distribution of orthogonal polynomials with potential theory. Before continuing we need to introduce some potential theoretic notions. The main references for potential theory and its connection with approximation theory are [77, 86, 88].

Let $\mathscr{M}_1(K)$ be the class of all unitary positive Borel measures such that $\operatorname{supp}\mu \subset K$, where K is a compact subset of $\mathbb{C}$. The (logarithmic) potential of the measure μ is defined by

$$V^{\mu}(z) := \int \ln \frac{1}{|t - z|} \, \mathrm{d}\mu(t), \quad z \in \mathbb{C}.$$

It is well known that V^μ is superharmonic on $\mathbb{C}$ and subharmonic on $\overline{\mathbb{C}} \setminus \operatorname{supp} \mu$. The energy of the measure is given by

$$I(\mu) := \int V^\mu(z) \, \mathrm{d}\mu(z) = \iint \ln \frac{1}{|t-z|} \, \mathrm{d}\mu(t) \, \mathrm{d}\mu(z).$$

Let K be a compact subset of the complex plane and consider the class $\mathscr{M}_1(K)$. Define the *Robin's constant* of K as

$$I(K) := \inf\{I(\mu) \mid \mu \in \mathscr{M}_1(K)\},$$

and the *logarithmic capacity* of the compact K by

$$\operatorname{cap}(K) := \exp(-I(K)).$$

If a compact $K \subset \mathbb{C}$ verifies $\operatorname{cap}(K) > 0$ then, there exist a unique measure $\overline{\mu} \in \mathscr{M}_1(K)$ such that $I(\overline{\mu}) = I(K)$, and this measure is called the *equilibrium measure* of K (see [86, Th. 3.7.6]).

A very important result in potential theory is O. Frostman's (1907-1977) theorem, also known as the fundamental theorem of potential theory, which describes the behavior of the equilibrium measure's potential. {frostman}

Theorem 1.11 (Frostman)**:**
Let $K \subset \mathbb{C}$ be compact with $\operatorname{cap}(K) > 0$. Then, there exists a unique measure $\lambda \in \mathscr{M}_1(K)$ and a constant γ such that

$$V^\lambda(z) \begin{cases} \leq \gamma & z \in \mathbb{C} \\ = \gamma & z \in K \setminus A, \end{cases} \tag{1.23}$$

where A is a Borel set with $\operatorname{cap}(A) = 0$. Moreover, $\lambda = \overline{\mu}$ and $\gamma = I(\overline{\mu})$.

Potential theoretic considerations found fruitful applications in the study of orthogonal polynomials, see for example [92]. In particular, it was proved that if $\mu \in \mathbf{Reg}$ and μ is supported on the real line, then

$$\lim_n |Q_n(z;\mu)|^{1/n} = e^{-V^\lambda(z)},$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta$, where λ is the equilibrium measure of $\operatorname{supp} \mu$ and Δ denotes the smallest interval which contains $\operatorname{supp} \mu$. This result is the potential theoretic reformulation of (1.20).

The appearance of varying measures was a call to look for new techniques in order to obtain asymptotic results for its associated orthogonal polynomials. Gonchar and Rakhmanov's research revived the study of potentials with external fields. Fix a compact $E \subset \mathbb{R}$, by an external field acting on E we mean a continuous function $\varphi : E \to \mathbb{R}$. In [40], without a detailed proof, the following result was stated. {lm:extfield}

Lemma 1.12:
Let $E \subset \mathbb{R}$ be a regular compact set (with respect to the Dirichlet problem) and φ a continuous

function on E. Then, there exists a unique measure $\lambda \in \mathscr{M}_1(E)$ and a constant γ such that

$$(V^\lambda + \varphi)(z) \begin{cases} \leq \gamma, & z \in \operatorname{supp} \lambda \\ \geq \gamma, & z \in E \end{cases} .$$

A detailed proof (with a slightly more general formulation) can be found in [88, Th. I.1.3]. Notice that if $\varphi \equiv 0$, Lemma 1.12 reduces to Theorem 1.11. From Lemma 1.12, Gonchar and Rakhmanov were able to obtain the n-th root asymptotic for orthogonal polynomials with varying measures, in a more general setting than the one associated to multipoint Padé approximation, assuming that $\mu' > 0$ a.e. on the interval of orthogonality. A more relaxed version of the result was published in [92, Th. 3.3.3], but it does not cover the class of external fields we need in the
{th:vw:lasym} present dissertation. The extension we need appeared in [33, Lemma 4.2].

Lemma 1.13:
*Assume that $\mu \in$ **Reg** and $\operatorname{supp} \mu \subset \mathbb{R}$ is regular. Let $\{\varphi_n\}_{n\in\Lambda\subset\mathbb{Z}_+}$ be a sequence of positive continuous functions on $\operatorname{supp} \mu$ such that*

th:vw:lasym_1}
$$\lim_{n\in\Lambda} \frac{1}{2n} \ln \frac{1}{\varphi_n(x)} = \varphi(x) > -\infty, \tag{1.24}$$

uniformly on $\operatorname{supp} \mu$. Let $\{q_n\}_{n\in\Lambda}$ be a sequence of monic polynomials such that $\deg q_n = n$ and

$$\int x^\nu q_n(x)\varphi_n(x) \, \mathrm{d}\, \mu(x) = 0, \quad \nu = 0, 1, \ldots, n-1.$$

Then

th:vw:lasym_2}
$$* - \lim_{n\in\Lambda} \mu_{q_n} = \lambda, \tag{1.25}$$

and

th:vw:lasym_3}
$$\lim_{n\in\Lambda} \left(\int |q_n(x)|^2 \varphi_n(x) \, \mathrm{d}\, \mu(x) \right)^{1/2n} = e^{-\gamma}, \tag{1.26}$$

where λ and γ are the equilibrium measure and equilibrium constant in the presence of the external field φ on $\operatorname{supp} \mu$, and μ_{q_n} is as in (1.21). We also have

th:vw:lasym_4}
$$\lim_{n\in\Lambda} \left(\frac{|q_n(z)|}{\|q_n \varphi_n^{1/2}\|_E} \right)^{1/n} = \exp\left(\gamma - V^\lambda(z)\right), \quad K \subset \mathbb{C} \setminus \Delta, \tag{1.27}$$

where $\|\cdot\|_E$ denotes the sup-norm on $E = \operatorname{supp} \mu$ and $\Delta = \operatorname{Co}(\operatorname{supp} \mu)$.

A year after [40] appeared, G. López Lagomasino in [53] proved a result on the ratio asymptotic of orthogonal polynomials with respect to varying measures. Assuming that $Q_n(\cdot, \mu_n)$ is the n-th monic orthogonal polynomial with respect to $\mathrm{d}\, \mu(x)/w_{2n}$ where $\mu' > 0$ a.e. on $\operatorname{supp} \mu = [-1, 1]$ and w_{2n} is a polynomial of degree $\leq 2n$ whose zeros lie on an interval J disjoint from $\operatorname{supp} \mu$ (for all $n \in \mathbb{Z}_+$), then

$$\lim_n \frac{Q_{n+1}(z, \mu_{n+1})}{Q_n(z, \mu_n)} = \frac{1}{2}(z + \sqrt{z^2 - 1}), \quad z \in \mathbb{C} \setminus [-1, 1].$$

Later, the restrictions over the varying part of the measure were weakened in [55].

By the end of the 1980s the first results on the strong asymptotic of orthogonal polynomials with varying measures were obtained in [54], being later improved in [24]. The authors imposed certain restrictions on the varying part of the measures, which we need to complement to achieve more accurate results. We will consider sequences of measures $\{(\mu_n, w_{2n})\}_{n\in\mathbb{Z}_+}$, where $\mu_n \in \mathscr{M}(\Delta)$ and, w_{2n} are polynomials with real coefficients and $\deg w_{2n} \leq 2n$ verifying:

{S1}

(S1). There exists a finite positive Borel measure μ supported on Δ such that $\lim_n \mu_n = \mu$ in the weak star topology of measures, whose absolutely continuous part satisfies $\mu' > 0$ a.e. on Δ, and

$$\lim_n \int |\mu_n' - \mu'| \,\mathrm{d}x = 0.$$

{S2}

(S2). The measure μ satisfies Szegő's condition on Δ; that is,

$$\int_\Delta \ln \mu'(x) \,\mathrm{d}\eta_\Delta(x) > -\infty,$$

and

$$\liminf_n \int \ln \mu_n'(x) \,\mathrm{d}\eta_\Delta(x) \geq \int \ln \mu'(x) \,\mathrm{d}\eta_\Delta(x).$$

{S3}

(S3). Let Ψ be the conformal map from $\Omega = \overline{\mathbb{C}} \setminus \Delta$ onto the exterior of the unit circle such that $\Psi(\infty) = \infty$ and $\Psi'(\infty) > 0$. The zeros of the polynomials w_{2n} verify

$$\lim_{n\to\infty} \sum_{i=1}^{2n} \left(1 - \frac{1}{|\Psi(x_{2n,i})|}\right) = \infty.$$

By convention, $x_{2n,i} = \infty$, $1 \leq i \leq 2n - i_n$, when $i_n < 2n$.

{S4}

(S4). There exist non negative continuous functions φ and ψ on (a, b), $\Delta = [a, b]$, such that

$$\lim_n \varphi^n(x)|w_{2n}(x)| = 1/\psi(x) \tag{1.28}$$ {def:psi}

uniformly on compact subsets of (a, b) and

$$\lim_{n\to\infty} \int_a^b \ln(\varphi^n(x)|w_{2n}(x)|) \,\mathrm{d}\eta_\Delta(x) = -\int_a^b \ln \psi(x) \,\mathrm{d}\eta_\Delta(x) < +\infty. \tag{1.29}$$ {rest_var_w}

The main result in [24] compares the asymptotic behavior of the orthonormal polynomials q_n with respect to $\mathrm{d}\mu_n/|w_{2n}|$ with the sequence w_{2n}. The limit relation depends on Blaschke products. Let $x_{2n,i}$, $2n - i_n + 1 \leq i \leq 2n$, denote the zeros of w_{2n}. If $i_n < 2n$ we define $x_{2n,i} = \infty$, $1 \leq i \leq 2n - i_n$. Set

$$B_{2n}(z) := \prod_{i=1}^{2n} \frac{\Psi(z) - \Psi(x_{2n,i})}{1 - \overline{\Psi(x_{2n,i})}\Psi(z)}.$$

When $x_{2n,i} = \infty$ the corresponding factor in the Blaschke product is replaced by $1/\Psi(z)$. {th:st:blasch}

Theorem 1.14:

Let $\{(\mu_n, w_{2n})\}_{n\in\mathbb{Z}_+}$ be a sequence of measures verifying $(S1)$-$(S3)$ on Δ and q_n is the n-th orthonormal polynomial such that

$$\int_\Delta x^\nu q_n(x) \frac{\mathrm{d}\mu_n(x)}{|w_{2n}(x)|} = 0, \quad \nu = 0, 1, \ldots, n-1,$$

and

$$\int_{\Delta} q_n^2(x) \frac{\mathrm{d}\,\mu_n(x)}{|w_{2n}(x)|} = 1.$$

Then,

$$\lim_{n} \frac{q_n^2(z)}{w_{2n}(z)} B_{2n}(z) = \frac{1}{2\pi} \mathsf{G}^2(\mu, z)$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \Delta$.

The applications of the asymptotic properties of orthogonal polynomials with varying measures were unleashed with the study of a hard problem in approximation theory; namely, the simultaneous approximation of analytic functions.

1.6 Hermite-Padé approximation

Hermite's proof of the irrationality of π is a landmark in the history of mathematics; not only because of the importance of the problem itself, but also for the fruitfulness of the techniques developed for the proof. There, Hermite introduced simultaneous rational approximation of systems of exponentials to crack a centuries old problem. The book [44] marked a definitive inflection in the application of complex analysis to solve hard number theory problems.

The study of simultaneous approximation of systems of analytic functions stayed more or less dormant until the 1930's when in a series of lectures delivered at Rijksuniversiteit Groningen, Kurt Mahler (1903-1988) gave a systematic approach. Mahler's lectures were published decades later in [67]. Further contributions to the subject were made by two mathematicians closely acquainted with Mahler: J. Coates (1945-) and H. Jager (1933-) in [67] and [45], respectively. Mahler's approach, seen from a more modern point of view, is to study the following two approximation problems, known as Type I and Type II Hermite-Padé approximants, though according to Mahler's terminology they were called Latin and German polynomials, respectively.

Let $\vec{f} = (f_1, \ldots, f_m)$ be a family of analytic functions on a certain domain $D \subset \overline{\mathbb{C}}$, such that
{pr:type:i} $\infty \in D$. Fix a non-zero multi-index $\vec{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$, $|\vec{n}| = n_1 + \cdots + n_m$.

Problem 3 (Type I)**:**

There exist polynomials $a_{\vec{n},1}, \ldots, a_{\vec{n},m}$, not all identically equal to zero, such that

{I.i}

{I.ii} *I.i.* $\deg a_{\vec{n},j} \leq n_j - 1$, $j = 2, \ldots, m$ ($\deg a_{\vec{n},j} \leq -1$ *means* $a_{\vec{n},j} \equiv 0$);

I.ii. $a_{\vec{n},0}(z) + \sum_{j=1}^{m} a_{\vec{n},j}(z) f_j(z) = \mathcal{O}(1/z^{|\vec{n}|})$, $z \to \infty$

{pr:type:ii} *for some polynomial $a_{\vec{n},0}$.*

Problem 4 (Type II)**:**

There exist a polynomial $Q_{\vec{n}} \not\equiv 0$ such that

{II.i}

II.i. $\deg Q_{\vec{n}} \leq |\vec{n}|$;

{II.ii}

II.ii. $Q_{\vec{n}}(z)f_j(z) - P_{\vec{n},j}(z) = \mathcal{O}(1/z^{n_j+1})$, $z \to \infty$, $j = 1, \dots, m$.

for some polynomials $P_{\vec{n},j}$, $j = 1, \dots, m$.

Notice that if $m = 1$ Problems 3 and 4 coincide and both are equivalent to Problem 1. Apart from that, the polynomials $a_{\vec{n},0}$ and $P_{\vec{n},j}$, $j = 1, \dots, m$ are uniquely determined once their counterparts are found. From an algebraic point of view both constructions are closely related as it was pointed out in [67], [45] and [22].

After Hermite's proof of the irrationality of π, Type I, Type II and a combination of both (called mixed type) have been used to prove the irrationality of other numbers. We may cite F. Beukers' paper [17], where he showed that R. Apéry's proof of $\zeta(3) \notin \mathbb{Q}$ can be interpreted in terms of a mixed type Hermite-Padé approximation. For further information in this line of applications we recommend [97].

A very important question related with Hermite-Padé approximation is to know if the associated polynomials attain the maximal degree possible. So, we say that a multi-index $\vec{n}$ is *normal* for the system $\vec{f}$ for Type I approximation (respectively, for Type II) if $\deg a_{\vec{n},j} = n_j - 1$, $j = 1, \dots, m$ (respectively, $\deg Q_{\vec{n}} = |\vec{n}|$). If every multi-index $\vec{n}$ is normal, the system of functions $\vec{f}$, is said to be *perfect*.

An easy consequence of perfectness is that $(a_{\vec{n},1}, \dots, a_{\vec{n},m})$ and $Q_{\vec{n}}$ are uniquely determined up to a constant factor. Hence, if the system $\vec{f}$ is perfect, the orders of interpolation at infinity in Problems 3 and 4 are exact for every $\vec{n}$. There are a few systems known to be perfect, for example $(e^{w_1 z}, \dots, e^{w_m z})$ with $w_i \neq w_j$ for $i \neq j$ and $((1-z)^{w_1}, \dots, (1-z)^{w_m})$ with $w_i - w_j \notin \mathbb{Z}$, where in both cases the interpolation conditions are taken at the origin. Other examples of perfect systems may be constructed in terms of Markov functions.

1.6.1 Angelesco and Nikishin systems

Between 1918 and 1919 M.A. Angelesco introduced an interesting type of system of functions (see [1, 2]), which in his honor are called *Angelesco system*. They are constructed as follows. Consider the family of pairwise disjoint bounded intervals $\Delta_j \subset \mathbb{R}$, $j = 1, \dots, m$ and a system of measures σ_j, $j = 1, \dots, m$ such that $\mathrm{Co}(\operatorname{supp} \sigma_j) = \Delta_j$. Then, the system of m Markov functions

$$\vec{f}_A := (\widehat{\sigma}_1, \dots, \widehat{\sigma}_m) = \left(\int \frac{\mathrm{d}\sigma_1(x)}{z-x}, \dots, \int \frac{\mathrm{d}\sigma_m(x)}{z-x} \right)$$

is the Angelesco system generated by $(\sigma_1, \dots, \sigma_m)$. Angelesco's papers remained unnoticed for over 60 years until they were rediscovered by the Russian mathematician E.M. Nikishin (1945-1986). In [75], Nikishin deduced some of the formal properties of such systems.

Fix a multi-index $\vec{n} \in \mathbb{Z}_+^m$ and consider the Type II Hermite-Padé approximants for $(\widehat{\sigma}_1, \dots, \widehat{\sigma}_m)$.

It is not difficult to obtain that

$$\int x^\nu Q_{\vec{n}}(x)\,\mathrm{d}\,\sigma_j(x) = 0, \quad \nu = 0,1,\dots,n_j-1,\ \ j = 1,\dots,m.$$

An easy consequence of the above orthogonality conditions verified by $Q_{\vec{n}}$ is that $Q_{\vec{n}}$ has n_j simple zeros in each interval Δ_j. Hence, $\deg Q_{\vec{n}} = |\vec{n}|$ and the Angelesco systems are perfect for Type II approximation.

Angelesco systems do not have a nice behavior in terms of convergence (about the asymptotic we will return later). In this respect, another system of functions, built in terms of Markov functions too, are much more interesting and they are at the core of this dissertation.

Almost right after the rediscovery of Angelesco systems, Nikishin introduced in [76] (the original Russian version is from 1980) what he called MT-systems, but renamed after him as *Nikishin systems*. Let Δ_α, Δ_β be two intervals contained in the real line such that $\Delta_\alpha \cap \Delta_\beta = \emptyset$. Consider the measures $\sigma_\alpha \in \mathscr{M}(\Delta_\alpha)$, $\sigma_\beta \in \mathscr{M}(\Delta_\beta)$, $\widehat{\sigma}_\beta \in L_1(\sigma_\alpha)$. With these two measures we construct a third one as follows (using differential notation)

$$\mathrm{d}\langle \sigma_\alpha, \sigma_\beta\rangle(x) := \widehat{\sigma}_\beta(x)\,\mathrm{d}\,\sigma_\alpha(x).$$

When we consider consecutive products of measures, a.e. $\langle \sigma_\alpha, \sigma_\beta, \sigma_\gamma\rangle := \langle \sigma_\alpha, \langle \sigma_\beta, \sigma_\gamma\rangle\rangle$ we implicitly assume not only that $\widehat{\sigma}_\gamma \in L_1(\sigma_\beta)$, but also $\widehat{\langle \sigma_\beta, \sigma_\gamma\rangle} \in L_1(\sigma_\alpha)$, where $\widehat{\langle \sigma_\beta, \sigma_\gamma\rangle}$ denotes the Cauchy transform of $\langle \sigma_\beta, \sigma_\gamma\rangle$. It is important to remark that this product is neither commutative nor associative.

{def:nikishin}

Definition 1.15:
Take a collection Δ_j, $j = 1,\dots,m$ of intervals such that

$$\Delta_j \cap \Delta_{j+1} = \emptyset, \ \ j = 1,\dots,m-1.$$

Let $(\sigma_1,\dots,\sigma_m)$ be a system of measures such that $\mathrm{Co}(\operatorname{supp}\sigma_j) = \Delta_j$, $\sigma_j \in \mathscr{M}(\Delta_j)$, $j = 1,\dots,m$. *We say $(s_{1,1},\dots,s_{1,m}) = \mathscr{N}(\sigma_1,\dots,\sigma_m)$, where*

$$s_{1,1} = \sigma_1, \ \ s_{1,2} = \langle \sigma_1, \sigma_2\rangle, \ \ \dots, \ \ s_{1,m} = \langle \sigma_1, \langle \sigma_2, \dots, \sigma_m\rangle\rangle$$

is the Nikishin system of measures generated by $(\sigma_1,\dots,\sigma_m)$. The vector $\vec{s} = (\widehat{s}_{1,1},\dots,\widehat{s}_{1,m})$ is called the Nikishin system of functions.

Notice that any sub-system of $(\sigma_1,\dots,\sigma_m)$ of consecutive measures is also a generator of some Nikishin system. In the sequel for $1 \le j \le k \le m$ we will write

$$s_{j,k} := \langle \sigma_j, \sigma_{j+1}, \dots, \sigma_k\rangle, \qquad s_{k,j} := \langle \sigma_k, \sigma_{k-1}, \dots, \sigma_j\rangle.$$

The perfectness of Nikishin systems was not a simple problem. Fix $\vec{n} \in \mathbb{Z}_+^m$ and consider Type II approximation for the system $(\widehat{s}_{1,1},\dots,\widehat{s}_{1,m})$. It is not difficult to check that

$$\int x^\nu Q_{\vec{n}}(x)\,\mathrm{d}\,s_{1,j}(x) = 0, \quad \nu = 0,1,\dots,n_j-1, \quad j = 1,\dots,m.$$

The main difference with Angelesco systems is that all the measures $s_{1,j},\ j = 1,\ldots,m$ have the same support, and it is not obvious that $\deg Q_{\vec{n}} = |\vec{n}|$.

In [76], Nikishin stated without proof, that the multi-indices $\vec{n} \in \mathbb{Z}_+^m$ verifying $n_1 \geq n_2 \geq \cdots \geq n_m, n_1 - n_m \leq 1$ are normal for type II approximation. Since then, the perfectness of Nikishin systems remained as an important open problem. Nikishin's assertion was proved by K. Driver and H. Stahl (1942-2013) in [26, Th. 4.1] and later slightly improved in [18]. Finally, the problem was solved in [30]. For the case of generating measures with unbounded and/or touching supports see also [31].

Among approximation problems of mixed-type, very recently was introduced the following one in [62] for Nikishin system of functions: {pr:ML_HP}

Problem 5 (ML Hermite-Padé)**:**
Given a Nikishin system $\mathcal{N}(\sigma_1,\ldots,\sigma_m)$, for each $n \in \mathbb{N}$, there exist polynomials $a_{n,0}, a_{n,1},\ldots,a_{n,m}$, with $\deg a_{n,j} \leq n-1,\ j = 0,1,\ldots,m-1, \deg a_{n,m} \leq n$, not all identically equal to zero, called multi-level (ML) Hermite-Padé polynomials that verify:

$$\mathcal{A}_{n,0} := \left[a_{n,0} + \sum_{k=1}^{m}(-1)^k a_{n,k}\widehat{s}_{1,k}\right] \in \mathcal{O}\left(\frac{1}{z^{n+1}}\right)$$

$$\mathcal{A}_{n,j} := \left[(-1)^j a_{n,j} + \sum_{k=j+1}^{m}(-1)^k a_{n,k}\widehat{s}_{j+1,k}\right] \in \mathcal{O}\left(\frac{1}{z}\right), j = 1,\ldots,m-1.$$

Here and in the sequel $\mathcal{O}(\cdot)$ is as $z \to \infty$ along paths non tangential to the support of the measures involved. For completeness write $\mathcal{A}_{n,m} := (-1)^m a_{n,m}$.

Notice that in this scheme of approximation the interpolation conditions involve all Nikishin systems of the "inner levels", i.e. $\mathcal{N}(\sigma_1,\ldots,\sigma_m)$, $\mathcal{N}(\sigma_2,\ldots,\sigma_m)$, $\ldots$, $\mathcal{N}(\sigma_m) = (s_{m,m})$. Finding the polynomials $a_{n,0}, a_{n,1},\ldots,a_{n,m}$ is equivalent to solving a homogeneous linear system of $n(m+1)$ equations, given by the interpolation conditions, on $n(m+1)+1$ unknowns, corresponding with the coefficients of the polynomials. Consequently, the system of equations has a non trivial solution.

1.7 Markov-type theorems and asymptotic for Nikishin systems

When E.M. Nikishin introduced the homonymous system of functions, he was able to prove the convergence of Type II approximants for $m = 2$ (see [76, Th. 4]). Since then, the question of convergence remained open until the early 90's when G. López Lagomasino and J. Bustamante succeeded to extend Nikishin's result for systems with $m > 2$ measures ([20]) and sequences of multi-indices $\Lambda \subset \mathbb{Z}_+$ satisfying $n_i \geq |\vec{n}|/m - c, i = 1,\ldots,m$, where c is a constant independent of $\vec{n} \in \Lambda$ and $i = 1,\ldots,m$. For a larger class of multi-indices the convergence was obtained in capacity. An important ingredient in the proof was to show that to a great extent the convergence can be reduced to that of multipoint Padé approximation of Stieltjes type functions. This occurs due to the appearance of extra interpolation points which are not implicit in the definition. This phenomenon was discovered by A.A. Gonchar in the case of Nikishin systems of two measures. This idea leads to the introduction of so called incomplete multipoint Padé approximants in Definition 1.5 and Lemma 1.7 which constitute important tools in the proof of convergence theorems. Improvements of the results in [20] may be found in [29].

In [41] a general system of functions was introduced constructed as combinations of Nikishin and Angelesco systems, and were called *generalized Nikishin systems*. The authors proved the normality for certain sets of multi-indices and the convergence of Type II approximants assuming that the generating measures verify the Erdős-Turán condition. Convergence was derived after proving the logarithmic asymptotic of the associated Hermite-Padé polynomials.

Regarding Type I approximation of Nikishin systems the first results were obtained in [59]. This article was followed by an extensions in the spirit of Gonchar's result on the convergence of Padé approximants to meromorphic functions of the form $\widehat{\mu}+r$ (see [60, 61]), where r is a rational function with real coefficients to vector function of the form $\vec{s}+\vec{r}$, where $\vec{s}$ is a Nikishin system of functions and $\vec{r}$ is a vector of rational fractions, where the component have disjoint sets of poles.

The path to the study of weak asymptotic of multi-orthogonal polynomials associated to Nikishin systems was started by a A.A. Gonchar and E.A. Rakhmanov, where they linked this type of asymptotic with equilibrium problems with vector potentials. They used this technique first in [39] to study the convergence of Type II approximants of Angelesco systems, and later with strong restrictions on the generating measures they used it to describe the weak asymptotic and the convergence of generalized Nikishin systems in [41]. A clear exposition of the logarithmic asymptotic of multiple orthogonal polynomials associated to Type II approximation can be found in [57, Th. 5].

The ratio asymptotic was studied in [6], and some other important results were obtained also, for example, the interlacing property of the zeros of the multiple orthogonal polynomials associated to Type II approximation of Nikishin systems. Later, the aforementioned research was supplemented in [47].

On the other hand, the strong asymptotic of multi orthogonal polynomials with respect to Angelesco and Nikishin systems were described in two articles by A.I. Aptekarev, separated each other by a period of ten years: [4] and [5]. Aptekarev's approach to tackle this problem relies heavily on fixed point theorems and topological reasoning. A recommended survey on this topic and weak asymptotic, previous to Aptekarev's article of 1999, is [3].

CHAPTER

2 Rational perturbation of multi-level Hermite-Padé polynomials

{ch:pert}

In the present chapter, as we stated previously, we deal with the proof of Markov [68] and Stieltjes [93] type theorems for the convergence of simultaneous Padé approximants of a certain class of meromorphic functions. We have seen that this study was started in [36], in the context of Padé approximation. We also commented that in [62] a new approximation scheme was introduced (see Problem 5) motivated by the study of peakon solutions of the Degasperis-Procesi equation (see, for example, [65]).

The aforementioned works inspired us to study the convergence of such interpolation processes for the case of general Nikishin systems. We wish to see the consequences of perturbing a Nikishin system with rational functions with real coefficients. A similar question was raised and solved in [60] for Type I Hermite-Padé approximation (see also [61]).

2.1 Statement of the problem and auxiliary results

In the sequel, we will restrict to Borel measures $\mu \in \mathscr{M}(\Delta)$. Consider a collection of intervals Δ_j, $j = 1, \ldots, m$ such that $\Delta_j \cap \Delta_{j+1} = \varnothing$ for $j = 1, \ldots, m-1$. Fix a system of measures $(\sigma_1, \ldots, \sigma_m)$ with $\mathrm{Co}(\operatorname{supp} \sigma_j) \subset \Delta_j$, $\sigma_j \in \mathscr{M}(\Delta_j)$, $j = 1, \ldots, m$. With these elements at hand we construct the Nikishin system of measures $\mathscr{N}(\sigma_1, \ldots, \sigma_m)$ (see Definition 1.15) and its associated Nikishin system of functions $(\widehat{s}_{1,1}, \widehat{s}_{1,2}, \ldots, \widehat{s}_{1,m})$.

2.1.1 Convergence of the approximants

Now we are in position to describe the approximation objects. {def:ML:per}

Definition 2.1:
Consider the Nikishin system $\mathscr{N}(\sigma_1, \ldots, \sigma_m)$. Let $r_j = \dfrac{v_j}{t_j}$, $k = 1, \ldots, m$, be rational fractions with real coefficients, $\deg v_k < \deg t_k = d_k$, $(v_k, t_k) = 1$ (coprime) for all $k = 1, \ldots, m$. For each $n \in \mathbb{N}$, there exist polynomials $a_{n,0}, a_{n,1}, \ldots, a_{n,m}$, with $\deg a_{n,j} \leq n-1$, $j = 0, 1, \ldots, m-1$, $\deg a_{n,m} \leq n$, not all identically equal to zero, called multi-level (ML) Hermite-Padé polynomials

that verify

$$\mathscr{A}_{n,0} := \left[a_{n,0} + \sum_{k=1}^{m}(-1)^k a_{n,k}(\widehat{s}_{1,k} + r_k)\right] \in \mathscr{O}\left(\frac{1}{z^{n+1}}\right), \tag{2.1}$$ {ML:per:1}

$$\mathscr{A}_{n,j} := \left[(-1)^j a_{n,j} + \sum_{k=j+1}^{m}(-1)^k a_{n,k}\widehat{s}_{j+1,k}\right] \in \mathscr{O}\left(\frac{1}{z}\right), \quad j = 1, \ldots, m-1. \tag{2.2}$$ {ML:per:2}

Here and in the sequel $\mathscr{O}(\cdot)$ is as $z \to \infty$ along paths non tangential to the support of the measures involved. For completeness we denote $\mathscr{A}_{n,m} := (-1)^m a_{n,m}$.

When $r_k \equiv 0, k = 1, \ldots, m$, this construction coincides with the one in Problem 5. Recall that in this scheme of approximation the interpolation conditions involve all Nikishin systems of the "inner levels", i.e. $\mathscr{N}(\sigma_1, \ldots, \sigma_m)$, $\mathscr{N}(\sigma_2, \ldots, \sigma_m)$, $\ldots$, $\mathscr{N}(\sigma_m) = (s_{m,m})$. To find the vector polynomial $(a_{n,0}, a_{n,1}, \ldots, a_{n,m})$ is equivalent to solving a homogeneous linear system of $n(m+1)$ equations on $n(m+1)+1$ unknowns. Therefore, the system of equations has a non trivial solution. However, the solution does not need to be unique.

{Th:conv:unif} Let $T = \mathrm{lcm}(t_1, \ldots, t_m)$, $\deg T = D$, where lcm stands for least common multiple.

Theorem 2.2:
For each $n \in \mathbb{N}$ let $a_{n,0}, a_{n,1}, \ldots, a_{n,m}$ be Hermite-Padé polynomials associated with the Nikishin system $\mathscr{N}(\sigma_1, \ldots, \sigma_m)$ and $(r_1, \ldots, r_m)$ such that (2.1) *and* (2.2) *holds. Assume that the zeros of the polynomial T lie in the complement of $\Delta_1 \cup \Delta_m$ and f has exactly D poles in $\mathbb{C} \setminus \Delta_m$, where*

$$f := \widehat{s}_{m,1} - \sum_{k=1}^{m-1}(-1)^k \widehat{s}_{m,k+1} r_k - (-1)^m r_m.$$

Suppose that either the sequence of moments of σ_m satisfies Carleman's condition (1.9) *or Δ_{m-1} is a bounded interval. Then,*

{convergencia}
$$\lim_n \frac{a_{n,j}}{a_{n,m}} = \widehat{s}_{m,j+1}, \qquad j = 1, \ldots, m-1, \tag{2.3}$$

and

{polos}
$$\lim_n \frac{a_{n,0}}{a_{n,m}} = f, \tag{2.4}$$

uniformly on each compact subset of $\mathbb{C} \setminus (\Delta_m \cup \{z : T(z) = 0\})$. For all sufficiently large n, $\deg a_{n,m} = n$, $a_{n,m}$ has exactly $n - D$ simple zeros in the interior of Δ_m and D zeros in $\mathbb{C} \setminus \Delta_m$ which converge to the poles of f in this region according to their order. For $j = 1, \ldots, m-1$ and all sufficiently large n the polynomial $a_{n,j}$ has at least $n - D - m + j$ sign changes in Δ_m and at least D zeros in $\mathbb{C} \setminus \Delta_m$ of which D converge to the zeros of T according to their multiplicity and the remaining ones accumulate on $\Delta_m \cup \{\infty\}$.

The fact that $\deg a_{n,m} = n$ for n large enough implies that for such indices the vector polynomial $(a_{n,0}, \ldots, a_{n,m})$ is unique up to a constant factor. Indeed, from two non-collinear solutions of (2.1)-(2.2) one can construct a non-trivial solution whose last polynomial has degree smaller that n.

Notice that $f(z) \equiv \widehat{s}_{m,1}$ when $r_k \equiv 0$, $k = 1,\ldots,m$, and Theorem 2.2 gives the main statement in [62]; namely, relation (1.23) of Theorem 1.6. The expressions of the limit relations in Theorem 2.2 are similar to those in [60, Theorem 1.2] where Type I Hermite-Padé approximants of meromorphic functions were studied.

Obviously, the poles of f in $\mathbb{C}\setminus\Delta_m$ are the zeros of T. Therefore, the total number of poles of f in that region equals D if and only if for each zero ζ of T, say of multiplicity τ, we have

$$\lim_{z\to\zeta}(z-\zeta)^\tau f(z) = -\sum_{k=1}^{m-1}(-1)^k\widehat{s}_{m,k+1}(\zeta)\lim_{z\to\zeta}(z-\zeta)^\tau r_k(z) - (-1)^m\lim_{z\to\zeta}(z-\zeta)^\tau r_m(z) \neq 0.$$

Therefore, sufficient conditions for f to have D poles in $\mathbb{C}\setminus\Delta_m$ is that $(t_j,t_k) = 1, 1\le j,k\le m$ or, more generally, that for each $\zeta, T(\zeta) = 0$, there is only one polynomial t_k which has ζ as zero of degree τ. Indeed, in this case all the terms in the previous sum cancel except one which is trivially different from zero. (The functions $\widehat{s}_{m,j}$, $j = 1,\ldots,m$, are never zero in $\mathbb{C}\setminus\Delta_m$.)

2.1.2 Logarithmic asymptotic

To obtain the general asymptotic of the ML Hermite-Padé polynomials we restrain ourselves to the case when the intervals Δ_j (in particular Δ_m) are bounded. In addition, we assume that $\operatorname{supp}\sigma_k$ is a regular compact set for $k = 1,\ldots,m$; that is, Green's function of the region $\mathbb{C}\setminus\operatorname{supp}\sigma_k$ with singularity at ∞ can be extended continuously to $\operatorname{supp}\sigma_k$. Let $\mathscr{M}_1(\operatorname{supp}\sigma_k)$ be the subclass of probability measures in $\mathscr{M}(\operatorname{supp}\sigma_k)$. Define

$$\mathscr{M}_1 = \mathscr{M}_1(\operatorname{supp}\sigma_1)\times\cdots\times\mathscr{M}_1(\operatorname{supp}\sigma_m).$$

It is well known (see, for example, [11, Section 4]), that there exists a unique vector measure $\vec{\lambda} = (\lambda_1,\ldots,\lambda_m)\in\mathscr{M}_1$ and a unique vector constant $\omega^{\vec{\lambda}} = (\omega_1^{\vec{\lambda}},\ldots,\omega_m^{\vec{\lambda}})$ such that

$$-\frac{1}{2}V_{j-1}^{\vec{\lambda}}(x) + V_j^{\vec{\lambda}}(x) - \frac{1}{2}V_{j+1}^{\vec{\lambda}}(x) = \omega_j^{\vec{\lambda}}, \qquad x\in\operatorname{supp}\lambda_j, \qquad j = 1,\ldots,m. \tag{2.5}$$ {vecequil}

(By convention $V_0^{\vec{\lambda}} \equiv V_{m+1}^{\vec{\lambda}} \equiv 0$.) The vector measure $\vec{\lambda}$ is called equilibrium measure for the system of compact sets $\operatorname{supp}\sigma_k$, $k = 1,\ldots,m$ with interaction matrix $\mathscr{C}_{\mathscr{N}} = (c_{j,k})$, $1\le, j,k\le m$, where $c_{j,j} = 1$, $j = 1,\ldots,m$, $c_{j-1,j} = -1/2$, $j = 2,\ldots,m$, $c_{j,j+1} = -1/2$, $j = 1,\ldots,m-1$, and the rest of the entries equal zero. Notice that the left hand of (2.5) is the product of the j-th row of $\mathscr{C}_{\mathscr{N}}$ times the vector potential $(V_1^{\vec{\lambda}},\ldots,V_m^{\vec{\lambda}})$.

The vector equilibrium measure allows to describe the normalized distribution of the zeros of the polynomials $a_{n,m}$ and the roots of the forms $\mathscr{A}_{n,j}$, $j = 1,\ldots,m-1$. In this section, we also require that the zeros of T lie in the complement of $\Delta_1\cup\Delta_m$. From Theorem 2.2 we know that under these conditions for all sufficiently large $n > N$:

- $\deg a_{n,m} = n$ with exactly $n - D$ simple zeros on Δ_m and the remaining D zeros of $a_{n,m}$ converge to the poles of f in $\mathbb{C}\setminus\Delta_m$ according to their multiplicity.

- $\mathscr{A}_{n,j}$, $j = 1, \ldots, m-1$, has exactly $n - D$ zeros in $\mathbb{C} \setminus \Delta_{j+1}$ they are simple and lie in Δ_j.

Let $Q_{n,j}$, $j = 1, \ldots, m$ be the monic polynomial of degree $n - D$ whose zeros are the roots of $\mathscr{A}_{n,j}$ on Δ_j. (Recall that $\mathscr{A}_{n,m} = (-1)^m a_{n,m}$.)

$$\mathscr{H}_{n,j} := \frac{Q_{n,j+1} T \mathscr{A}_{n,j}}{Q_{n,j}}, \quad j = 0, 1, \qquad \mathscr{H}_{n,j} := \frac{Q_{n,j+1} \mathscr{A}_{n,j}}{Q_{n,j}}, \quad j = 2, \ldots, m.$$

By convention $Q_{n,0} \equiv Q_{n,m+1} \equiv 1$ and $\Delta_{m+1} = \emptyset$. Notice that $\mathscr{A}_{n,j}/Q_{n,j} \in \mathbf{H}(\mathbb{C} \setminus \Delta_{j+1})$,
{teo:1} $j = 0, \ldots, m$. Recall that μ_Q is the zero counting measure of Q (see (1.21)).

Theorem 2.3:

*Assume that all the zeros of T lie in the complement of $\Delta_1 \cup \Delta_m$, and f has exactly D poles in $\mathbb{C} \setminus \Delta_m$. Suppose that $\sigma_j \in$ **Reg** and $\operatorname{supp} \sigma_j$, $j = 1, \ldots, m$. Then,*

{weak_Qnj}
$$* \lim_{n\to\infty} \mu_{Q_{n,j}} = \lambda_j, \quad j = 1, \ldots, m \tag{2.6}$$

where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_m) \in \mathscr{M}_1$ is the vector equilibrium measure determined by the matrix $\mathscr{C}_{\mathscr{N}}$ on the system of compact sets $\operatorname{supp} \sigma_j$, $j = 1, \ldots, m$. Moreover,

{limit_nesimo}
$$\lim_{n\to\infty} \left| \int Q_{n,j}^2(x) \frac{\mathscr{H}_{n,j}(x)\, \mathrm{d}\,\sigma_j(x)}{Q_{n,j-1}(x) Q_{n,j+1}(x)} \right|^{1/2n} = \exp\left(-\sum_{k=j}^{m} \omega_k^{\vec{\lambda}} \right), \tag{2.7}$$

where $\omega^{\vec{\lambda}} = (\omega_1^{\vec{\lambda}}, \ldots, \omega_m^{\vec{\lambda}})$ is the vector equilibrium constant.

symptotic_Anj} From this result the logarithmic asymptotic behavior of the forms $\mathscr{A}_{n,j}$ can be derived.

Theorem 2.4:

Suppose that the assumptions of Theorem 2.3 are satisfied. Then,

it_nrooth_Anj}
$$\lim_{n\to\infty} |\mathscr{A}_{n,j}(z)|^{1/n} = A_j(z), \qquad \mathscr{K} \subset \mathbb{C} \setminus (\Delta_j \cup \Delta_{j+1}), \qquad j = 1, \ldots, m-1 \tag{2.8}$$

where

$$A_j(z) = \exp\left(V^{\lambda_{j+1}}(z) - V^{\lambda_j}(z) - 2 \sum_{k=j+1}^{m} \omega_k^{\vec{\lambda}} \right), \qquad j = 1, \ldots, m-1.$$

Moreover,

$$\lim_{n\to\infty} |\mathscr{A}_{n,m}(z)|^{1/n} = \exp\left(-V^{\lambda_m}(z) \right), \qquad \mathscr{K} \subset \mathbb{C} \setminus (\Delta_m \cup Z)$$

where $Z = \{z : T(z) = 0\}$, and

$$\lim_{n\to\infty} |\mathscr{A}_{n,0}(z)|^{1/n} = \exp\left(V^{\lambda_1}(z) - 2 \sum_{k=1}^{m} \omega_k^{\vec{\lambda}} \right), \qquad \mathscr{K} \subset \mathbb{C} \setminus (\Delta_1 \cup Z).$$

Here, $\vec{\lambda} = (\lambda_1, \ldots, \lambda_m)$ is the vector equilibrium measure and $(\omega_1^{\vec{\lambda}}, \ldots, \omega_m^{\vec{\lambda}})$ is the vector equilibrium constant for the vector potential problem determined by the interaction matrix $\mathscr{C}_{\mathscr{N}}$ acting on the system of compact sets $\operatorname{supp} \sigma_j$, $j = 1, \ldots, m$.

2.1.3 Auxiliary results

In this subsection we introduce some definitions and results needed in our developments. We start with a useful Lemma whose proof in the case of measures with bounded support is a straightforward consequence of Cauchy's integral formula and Fubini's theorem but in the unbounded case is more elaborated and can be found in [59, Lemma 2.1]. {lm:rem:t-i}

Lemma 2.5:
Let $(s_{1,1},\ldots,s_{1,m}) = \mathcal{N}(\sigma_1,\ldots,\sigma_m)$ be given. Assume that there exist polynomials with real coefficients $a_0,\ldots,a_m$ and a polynomial w with real coefficients whose zeros lie in $\mathbb{C}\setminus\Delta_1$ such that

$$\frac{\mathscr{A}(z)}{w(z)} \in \mathbf{H}(\mathbb{C}\setminus\Delta_1) \ \ \textit{and} \ \ \frac{\mathscr{A}(z)}{w(z)} = \mathcal{O}\left(\frac{1}{z^N}\right), \qquad z\to\infty,$$

where $\mathscr{A} := a_0 + \sum_{k=1}^{m} a_k \widehat{s}_{1,k}$ and $N \geq 1$. Let $\mathscr{A}_1 := a_1 + \sum_{k=2}^{m} a_k \widehat{s}_{2,k}$. Then,

$$\frac{\mathscr{A}(z)}{w(z)} = \int \frac{\mathscr{A}_1(x)}{z-x}\frac{\mathrm{d}\,\sigma_1(x)}{w(x)}. \tag{2.9}$$ {relint}

If $N \geq 2$, we also have

$$\int x^{\nu}\mathscr{A}_1(x)\frac{\mathrm{d}\,\sigma_1(x)}{w(x)} = 0, \qquad \nu = 0,1,\ldots,N-2. \tag{2.10}$$ {lm:rem:orth}

In particular, $\mathscr{A}_1$ has at least $N-1$ sign changes in $\mathring{\Delta}_1$ (the interior of Δ_1 in $\mathbb{R}$ with the usual topology).

In the following, we need some relations involving reciprocals and ratios of Cauchy transforms of measures. It is well known that for each measure $\sigma\in\mathscr{M}(\Delta)$, where Δ is contained in a half line (that is, an interval of the form $[c,+\infty)$ or $(-\infty,c]$, $c\in\mathbb{R}$), there exist a measure $\tau\in\mathscr{M}(\Delta)$ and a polynomial $\ell(z) = az+b$, $a = 1/|\sigma|$, $b\in\mathbb{R}$, such that

$$\frac{1}{\widehat{\sigma}(z)} = \ell(z) + \widehat{\tau}(z),$$

where $|\sigma|$ is the total variation of the measure σ. For more information in the case of measures with compact support see [46, Appendix] and [89, Theorem 6.3.5], when the measure is supported in a half line see [31, Lemma 2.3]. If σ satisfies Carleman's condition $\sum_{n=0}^{\infty}|c_n|^{-1/2n} = \infty$, then τ satisfies the same condition, [59, Theorem 1.5]. We call τ the inverse measure of σ.

Such measures appear frequently in our arguments, so we will fix a notation to differentiate them. In relation with the measures denoted by s they will carry over to them the corresponding sub-indices. The same goes for the polynomials ℓ. For example,

$$\frac{1}{\widehat{s}_{j,k}(z)} = \ell_{j,k}(z) + \widehat{\tau}_{j,k}(z). \tag{2.11}$$ {inv:cauchy:tr

We also use

$$\frac{1}{\widehat{\sigma}_{\alpha}(z)} = \ell_{\alpha}(z) + \widehat{\tau}_{\alpha}(z).$$

On some occasions, we write $\widehat{\langle \sigma_\alpha, \sigma_\beta \rangle}$ in place of $\widehat{s}_{\alpha,\beta}$. In the paper [30, Lemma 2.10] (see also [31]) several formulas involving Cauchy transforms of measures were proved. For our reasoning, the most important ones establish that

$$\frac{\widehat{s}_{1,k}}{\widehat{s}_{1,1}} = \frac{|s_{1,k}|}{|s_{1,1}|} - \langle \tau_{1,1}, \widehat{\langle s_{2,k}, \sigma_1 \rangle\rangle}, \tag{2.12}$$ {rat:cauchy:tr

where $|s|$ denotes the total variation of the measure s.

2.2 General properties of the zeros

Now will focus on the location of the zeros of the polynomials $a_{n,j}$ and the forms $\mathscr{A}_{n,j}$. As above

lm:zeros:a_nm} $T = \operatorname{lcm}(t_1, \ldots, t_m)$ and $D = \deg T$.

Lemma 2.6:
For each $n \geq 2D$, the form $\mathscr{A}_{n,j}$, $j = 1, \ldots, m$, has at least $n - 2D$ sign changes in $\mathring{\Delta}_j$ and at most n zeros in $\mathbb{C} \setminus \Delta_{j+1}$ ($\Delta_{m+1} = \emptyset$). If the zeros of T lie outside of Δ_1 then $\mathscr{A}_{n,j}$, $j = 1, \ldots, m$, has at least $n - D$ sign changes in $\mathring{\Delta}_j$. The form $\mathscr{A}_{n,0}$ has at most $2D$ zeros in $\mathbb{C} \setminus \Delta_1$ and this number reduces to D should the zeros of T lie in the complement of Δ_1. If the zeros of T lie outside Δ_1 and for some n we know that $a_{n,m}$ has exactly $n - D$ sign changes on Δ_m then, $\mathscr{A}_{n,0}$ cannot have zeros in $\mathbb{C} \setminus \Delta_1$ and $\mathscr{A}_{n,j}$, $j = 1, \ldots, m-1$, has exactly $n - D$ zeros in $\mathbb{C} \setminus \Delta_{j+1}$ they are all simple and lie on Δ_j.

Proof. Fix $n \geq 2D$. Consider the linear form

$$\begin{aligned}
\mathscr{L}_{n,0}(z) :=& T(z)\mathscr{A}_{n,0}(z) = \left[a_{n,0}T + \sum_{k=1}^{m} (-1)^k a_{n,k} T r_k + \sum_{k=1}^{m} (-1)^k a_{n,k} T \widehat{s}_{1,k} \right](z) \\
=& \left[p_{n,0} + \sum_{k=1}^{m} (-1)^k p_{n,k} \widehat{s}_{1,k} \right](z) = \mathcal{O}\left(\frac{1}{z^{n-D+1}} \right),
\end{aligned}$$

where

{ps} $$p_{n,0} = a_{n,0}T + \sum_{k=1}^{m} (-1)^k a_{n,k} T r_k, \qquad p_{n,k} = a_{n,k}T, \quad k = 1, \ldots, m. \tag{2.13}$$

Using Lemma 2.5, in particular (2.10), we obtain the following orthogonality relations

$$\int x^\nu \mathscr{L}_{n,1}(x) \, \mathrm{d}\sigma_1(x) = 0, \qquad \nu = 0, 1, \ldots, n - D - 1,$$

and $\mathscr{L}_{n,1} := -p_{n,1} + \sum_{k=2}^{m} (-1)^k p_{n,k} \widehat{s}_{2,k}$ has at least $n - D$ sign changes on $\mathring{\Delta}_1$.

Notice that

$$\mathscr{L}_{n,1} = -p_{n,1} + \sum_{k=2}^{m} (-1)^k p_{n,k} \widehat{s}_{2,k} = -T a_{n,1} + \sum_{k=2}^{m} (-1)^k T a_{n,k} \widehat{s}_{2,k} = \mathscr{A}_{n,1} T.$$

Therefore, $\mathscr{A}_{n,1}$ has at least $n - 2D$ sign changes in the interior of Δ_1 (D sign changes may be on account of T). However, if the zeros of T are in the complement of Δ_1 then we can affirm that $\mathscr{A}_{n,1}$

has at least $n-D$ sign changes in the interior of Δ_1. These two situations are accountable for the different statements on the number of sign changes of $\mathscr{A}_{n,j}$ on Δ_j.

Let $w_{n,1}$ be a polynomial with simple zeros at the points of sign change of $\mathscr{A}_{n,1}$ on $\mathring{\Delta}_1$. In general $\deg w_{n,1} \geq n-2D$, but $\deg w_{n,1} \geq n-D$ if the zeros of T lie outside Δ_1. Therefore,

$$\mathbf{H}(\mathbb{C}\setminus\Delta_2) \ni \frac{\mathscr{A}_{n,1}}{w_{n,1}} = \mathscr{O}\left(\frac{1}{z^{\deg(w_{n,1})+1}}\right).$$

Notice that $\mathscr{A}_{n,1}$ and $w_{n,1}$ satisfy the hypothesis of Lemma 2.5, so

$$\frac{\mathscr{A}_{n,1}(z)}{w_{n,1}(z)} = \int \frac{\mathscr{A}_{n,2}(x)}{z-x}\frac{\mathrm{d}\,\sigma_2(x)}{w_{n,1}(x)},$$

and

$$\int x^\nu \mathscr{A}_{n,2}(x)\frac{\mathrm{d}\,\sigma_2(x)}{w_{n,1}(x)} = 0, \qquad \nu = 0,1,\ldots,\deg(w_{n,1})-1.$$

This yields $\mathscr{A}_{n,2}$ has at least $\deg(w_{n,1})$ sign changes in the interior of Δ_2.

Again, let $w_{n,2}$ be a polynomial with simple zeros at the points of sign change of $\mathscr{A}_{n,2}$ in Δ_2. Hence, $\deg(w_{n,2}) \geq \deg(w_{n,1})$ and

$$\mathbf{H}(\mathbb{C}\setminus\Delta_3) \ni \frac{\mathscr{A}_{n,2}}{w_{n,2}} = \mathscr{O}\left(\frac{1}{z^{\deg(w_{n,1})+1}}\right).$$

Then, we have deduced the same conclusions for $\mathscr{A}_{n,2}$ that we had for $\mathscr{A}_{n,1}$, and we can repeat the same reasoning inductively obtaining that for each $j=1,\ldots,m-1$, there exists a polynomial $w_{n,j}, \deg(w_{n,j}) \geq \deg(w_{n,1})$, with simple zeros at the points of sign change of $\mathscr{A}_{n,j}$ on Δ_j such that

$$\mathbf{H}(\mathbb{C}\setminus\Delta_{j+1}) \ni \frac{\mathscr{A}_{n,j}}{w_{n,j}} = \mathscr{O}\left(\frac{1}{z^{\deg(w_{n,1})+1}}\right). \tag{2.14}$$ {Amj}

For $j=m-1$, we have

$$\mathbf{H}(\mathbb{C}\setminus\Delta_m) \ni \frac{a_{n,m}\widehat{s}_{m,m} - a_{n,m-1}}{w_{n,m-1}}(z) = \mathscr{O}\left(\frac{1}{z^{\deg(w_{n,1})+1}}\right), \tag{2.15}$$ {HP_s_mm}

and using again Lemma 2.5, we obtain

$$\int x^\nu a_{n,m}(x)\frac{\mathrm{d}\,s_{m,m}(x)}{w_{n,m-1}(x)} = 0, \qquad \nu = 0,1,\ldots\deg(w_{n,1})-1.$$

Whence, $a_{n,m}$ has at least $\deg(w_{n,1})$ sign changes on Δ_m. Recall that in general $\deg(w_{n,1}) \geq n-2D$ and its degree is $\geq n-D$ if the zeros of T lie outside Δ_1. This settles the question on the number of sign changes of the forms on the different intervals.

Now let us consider the question of an upper bound on the total number of zeros that $\mathscr{A}_{n,j}$, $j=0,\ldots,m-1$, may have in $\mathbb{C}\setminus\Delta_{j+1}$. The arguments are pretty much the same. We will play on the fact that $\deg(a_{n,m}) \leq n$ and $a_{n,m} \not\equiv 0$.

Assume that $a_{n,m} \equiv 0$. From (2.9) with $w \equiv 1$ it follows that for each $j=1,\ldots,m-1$,

$$\mathscr{A}_{n,j}(z) = \int \frac{\mathscr{A}_{n,j+1}(x)}{z-x}\mathrm{d}\,\sigma_{j+1}(x).$$

Since $\mathscr{A}_{n,m} = (-1)^m a_{n,m}$, this formula with $j = m-1$ readily implies that $a_{n,m-1} \equiv 0$ and $\mathscr{A}_{n,m-1} \equiv 0$ if $a_{n,m} \equiv 0$. Going down on the indices j we conclude that $a_{n,j} \equiv 0$ and $\mathscr{A}_{n,j} \equiv 0$ for all $j = 1, \dots, m$. Formula (2.9) also implies that

$$\mathscr{L}_{n,0}(z) = \int \frac{\mathscr{L}_{n,1}(x)}{z-x} \mathrm{d}\sigma_1(x).$$

If $\mathscr{A}_{n,1} \equiv 0$ so too $\mathscr{L}_{n,1} \equiv 0$; consequently, $\mathscr{L}_{n,0} \equiv 0$ and $a_{n,0} \equiv 0$. In particular, should $a_{n,0} \equiv 0$ then necessarily $a_{n,j} \equiv 0, j = 0, \dots, m$. However, we explicitly excluded the trivial solution in Definition 2. So $a_{n,m} \not\equiv 0$.

Suppose that $\mathscr{A}_{n,0}$ has at least $2D+1$ zeros in $\mathbb{C} \setminus \Delta_1$. Then, there exists a polynomial with real coefficients $w_{n,0}$ of degree $\geq 2D+1$ whose zeros lie in $\mathbb{C} \setminus \Delta_1$ such that

$$\frac{\mathscr{L}_{n,0}(z)}{w_{n,0}(z)} = \frac{T(z)\mathscr{A}_{n,0}(z)}{w_{n,0}(z)} = \mathscr{O}\left(\frac{1}{z^{n+D+2}}\right) \in \mathbf{H}(\mathbb{C} \setminus \Delta_1).$$

Using 2.10, we obtain

$$\int x^\nu \mathscr{L}_{n,1}(x) \frac{\mathrm{d}\sigma_1(x)}{w_{n,0}(x)} = 0, \qquad \nu = 0, 1, \dots, n+D.$$

This means that $\mathscr{L}_{n,1}$ has at least $n+D+1$ sign changes on Δ_1 and $\mathscr{A}_{n,1}$ at least $n+1$ sign changes on Δ_1. Continuing as in the proof of the first part of the lemma we arrive at the conclusion that $a_{n,m}$ has at least $n+1$ sign changes on Δ_m which is not possible since it is a polynomial of degree $\leq n$ not identically equal to zero. Therefore, $\mathscr{A}_{n,0}$ has at most $2D$ zeros in $\mathbb{C} \setminus \Delta_1$. Notice that when the zeros of T are in the complement of Δ_1 in order to conclude that $\mathscr{A}_{n,1}$ has $n+1$ sign changes on Δ_1 it is sufficient to assume that $\deg(w_{n,0}) \geq D+1$, so in this case one can prove that $\mathscr{A}_{n,0}$ has at most D zeros in $\mathbb{C} \setminus \Delta_1$.

Suppose that $\mathscr{A}_{n,k}$ has at least $n+1$ zeros in $\mathbb{C} \setminus \Delta_{k+1}$ for some specific $k \in \{1, \dots, m-1\}$ and n. Then there exists a polynomial $w_{n,k}$ with real coefficients of degree $\geq n+1$ such that

$$\frac{\mathscr{A}_{n,k}(z)}{w_{n,k}(z)} = \mathscr{O}\left(\frac{1}{z^{n+2}}\right) \in \mathbf{H}(\mathbb{C} \setminus \Delta_{k+1}),$$

which, reasoning as above, implies that $\mathscr{A}_{n,k+1}$ has at least $n+1$ sign changes on Δ_{k+1}. Continuing the process one proves that for $j = k+1, \dots, m$, the forms $\mathscr{A}_{n,j}$ also have at least $n+1$ sign changes on Δ_j which contradicts the fact that $a_{n,m}$ cannot have more than n zeros.

Finally, suppose that for some n we know that $a_{n,m}$ has exactly $n-D$ sign changes on Δ_m and $\mathscr{A}_{n,k}$ has at least $n-D+1$ zeros in $\mathbb{C} \setminus \Delta_{k+1}$ for some $k \in \{1, \dots, m-1\}$. Then there exists a polynomial $w_{n,k}$ with real coefficients with zeros in $\mathbb{C} \setminus \Delta_{k+1}$ and degree $\geq n-D+1$ such that

$$\frac{\mathscr{A}_{n,k}(z)}{w_{n,k}(z)} = \mathscr{O}\left(\frac{1}{z^{n-D+2}}\right) \in \mathbf{H}(\mathbb{C} \setminus \Delta_{k+1}).$$

Repeating the arguments used above it follows that for $j = k+1, \dots, m$, the forms $\mathscr{A}_{n,j}$ have at least $n-D+1$ sign changes on Δ_j. In particular, $a_{n,m}$ would have $n-D+1$ sign changes on Δ_m against our assumption. Thus, $\mathscr{A}_{n,j}, j = 1, \dots, m-1$ has at most $n-D$ zeros on $\mathbb{C} \setminus \Delta_{j+1}$. Since it has $n-D$ sign changes on Δ_j the statement readily follows. That $\mathscr{A}_{n,0}$ has no zeros in $\mathbb{C} \setminus \Delta_1$ is proved analogously.

□

2.3 Convergence results

We underline that in the next result no assumption is made on the rational functions r_k except that they have real coefficients. {th:conv:H}

Theorem 2.7:
For each $n \geq 2D$, let $a_{n,0}, a_{n,1}, \ldots, a_{n,m}$ be the Hermite-Padé polynomials associated with the Nikishin system $\mathcal{N}(\sigma_1, \ldots, \sigma_m)$ and $(r_1, \ldots, r_m)$ such that (2.1) *and* (2.2) *holds. Suppose that either σ_m satisfies Carleman's condition* (1.9) *or Δ_{m-1} is a bounded interval. Then,*

$$h-\lim_{n\to\infty} \frac{a_{n,j}}{a_{n,m}} = \widehat{s}_{m,j+1}, \qquad h-\lim_{n\to\infty} \frac{a_{n,m}}{a_{n,j}} = \widehat{s}_{m,j+1}^{-1}, \qquad j = 1, \ldots, m-1,$$

and

$$h-\lim_{n\to\infty} \frac{a_{n,0}}{a_{n,m}} = f = \widehat{s}_{m,1} - \sum_{k=1}^{m-1} (-1)^k \widehat{s}_{m,k+1} r_k - (-1)^m r_m, \tag{2.16}$$ {limit_an0_anm

on each compact subset $\mathcal{K} \subset \mathbb{C} \setminus \Delta_m$. Moreover, the polynomial $a_{n,j}$, $j = 1, \ldots, m-1$, has at least $n - 2D - m + j$ sign changes on Δ_m. If the zeros of the polynomial T lie in the complement of Δ_1 then the polynomial $a_{n,j}$, $j = 1, \ldots, m-1$, has at least $n - D - m + j$ sign changes in Δ_m.

Proof. Let us point out that if σ_m satisfies Carleman's condition so do the measures $s_{m,j}$ and $\tau_{m,j}$, $j = 1, \ldots, m$, see [59, Theorem 1.5]. We reduce the proof of the limit relations to Lemma 1.7.

Assume that $n \geq 2D$. Notice that (2.15) means that the polynomials $a_{n,m-1}$, $a_{n,m}$ and $w_{n,m-1}$ satisfy the conditions of Definition 1.5. Therefore, the rational fractions $a_{n,m-1}/a_{n,m}$ form a sequence of incomplete diagonal multi-point Padé approximants of $\widehat{s}_{m,m}$.

Using Lemma 1.7 we have convergence in Hausdorff content on each compact subset of $\mathbb{C} \setminus \Delta_m$. That is,

$$h-\lim_n \frac{a_{n,m-1}}{a_{n,m}} = \widehat{s}_{m,m}.$$

Dividing $\frac{\mathcal{A}_{n,m-1}}{w_{n,m-1}}$ by $\widehat{s}_{m,m} = \widehat{\sigma}_m$ and using (2.11), we obtain

$$\frac{(a_{n,m-1}\ell_m - a_{n,m}) + a_{n,m-1}\widehat{\tau}_m}{w_{n,m-1}}(z) = \mathcal{O}\left(\frac{1}{z^{n-D}}\right).$$

So, we have again a sequence of incomplete multi-point approximants of $\widehat{\tau}_m$ and, consequently,

$$h-\lim_n \left(\ell_m - \frac{a_{n,m}}{a_{n,m-1}}\right) = \widehat{\tau}_m,$$

which is equivalent to

$$h-\lim_n \frac{a_{n,m}}{a_{n,m-1}} = \widehat{\sigma}_m^{-1}$$

on compact subsets of $\mathbb{C} \setminus \Delta_m$.

Now, using (2.11) and (2.12), for $j = 1, \ldots, m-2$, we have

$$\frac{\mathscr{A}_{n,j}}{\widehat{\sigma}_{j+1}} = \left((-1)^j \ell_{j+1} a_{n,j} + (-1)^{j+1} a_{n,j+1} + \sum_{k=j+2}^{m} (-1)^k \frac{|s_{j+1,k}|}{|\sigma_{j+1}|} a_{n,k} \right) + (-1)^j a_{n,j} \widehat{\tau}_{j+1} - \sum_{k=j+2}^{m} (-1)^k a_{n,k} \widehat{\langle \tau_{j+1}, \langle s_{j+2,k}, \sigma_{j+1} \rangle \rangle}.$$

The quotient $\frac{\mathscr{A}_{n,j}}{\widehat{\sigma}_{j+1}}$ has the same structure as $\mathscr{A}$ in Lemma 2.5. Moreover, using (2.14), we obtain

$$\frac{\mathscr{A}_{n,j}(z)}{(\widehat{\sigma}_{j+1} w_{n,j})(z)} = \mathscr{O}\left(\frac{1}{z^{n-2D}}\right) \in \mathbf{H}(\mathbb{C} \setminus \Delta_{j+1}),$$

and, as consequence of (2.10), for $\nu = 0, \ldots, n-2D-2$, it follows that

$$0 = \int_{\Delta_{j+1}} x^{\nu} \left((-1)^j a_{n,j} - \sum_{k=j+2}^{m} (-1)^k a_{n,k} \widehat{\langle s_{j+2,k}, \sigma_{j+1} \rangle} \right)(x) \frac{\mathrm{d}\,\tau_{j+1}(x)}{w_{n,j}(x)}.$$

The expression in parenthesis under the integral sign has at least $n-2D-1$ sign changes in $\mathring{\Delta}_{j+1}$. Thus, there exists a polynomial $w^*_{n,j}$ of degree $n-2D-1$ whose zeros are simple and lie in $\mathring{\Delta}_{j+1}$ such that

$$\frac{1}{w^*_{n,j}} \left((-1)^j a_{n,j} - \sum_{k=j+2}^{m} (-1)^k a_{n,k} \widehat{\langle s_{j+2,k}, \sigma_{j+1} \rangle} \right) \in \mathbf{H}(\mathbb{C} \setminus \Delta_{j+2}).$$

Direct computation or [62, Lemma 2.1] allows to deduce

$$\mathscr{A}_{n,j} - \widehat{s}_{j+1,j+1} \mathscr{A}_{n,j+1} = (-1)^j a_{n,j} - \sum_{k=j+2}^{m} (-1)^k a_{n,k} \widehat{\langle s_{j+2,k}, \sigma_{j+1} \rangle}.$$

From the statement of our problem we know that $\mathscr{A}_{n,j} - \widehat{s}_{j+1,j+1} \mathscr{A}_{n,j+1}$ is $\mathscr{O}(1/z)$. Hence,

$$\frac{1}{w^*_{n,j}(z)} \left((-1)^j a_{n,j} - \sum_{k=j+2}^{m} (-1)^k a_{n,k} \widehat{\langle s_{j+2,k}, \sigma_{j+1} \rangle} \right)(z) = \mathscr{O}\left(\frac{1}{z^{n-2D}}\right), \qquad z \to \infty.$$

Notice that if $j = m-2$ we have

$$\frac{a_{n,m-2} - a_{n,m} \widehat{s}_{m,m-1}}{w^*_{n,j}}(z) = \mathscr{O}\left(\frac{1}{z^{n-2D}}\right).$$

Thus, $a_{n,m-2}/a_{n,m}$ is an incomplete diagonal multi-point Padé approximant of $\widehat{s}_{m,m-1}$ and we obtain convergence in Hausdorff convergence on compact subsets of $\mathbb{C} \setminus \Delta_m$

$$h - \lim_{n\to\infty} \frac{a_{n,m-2}}{a_{n,m}} = \widehat{s}_{m,m-1}.$$

Dividing by $\widehat{s}_{m,m-1}$ and arguing as we did above it also follows that

$$h - \lim_{n\to\infty} \frac{a_{n,m}}{a_{n,m-2}} = \widehat{s^{-1}_{m,m-1}}.$$

Using the identity $\langle s_{j+2,k}, s_{j+1,j+1}\rangle = \langle s_{j+2,j+1}, s_{j+3,k}\rangle$ for $k = j+3, \ldots, m$, we deduce

{Prep_Elim}

$$(-1)^j a_{n,j} - \sum_{k=j+2}^{m} (-1)^k a_{n,k} \widehat{\langle s_{j+2,k}, \sigma_{j+1}\rangle}$$
$$= (-1)^j a_{n,j} - (-1)^{j+2} a_{n,j+2} \widehat{s}_{j+2,j+1} - \sum_{k=j+3}^{m} (-1)^k a_{n,k} \widehat{\langle s_{j+2,j+1}, s_{j+3,k}\rangle}. \quad (2.17)$$

Since we wish to eliminate $\widehat{s}_{j+2,j+1}$ in the right hand side of (2.17), we divide both sides by it and use again (2.11) and (2.12). Then,

$$\left((-1)^j a_{n,j} \ell_{j+2,j+1} - (-1)^{j+2} a_{n,j+2} - \sum_{k=j+3}^{m} (-1)^k \frac{|\langle s_{j+2,j+1}, s_{j+3,k}\rangle|}{|s_{j+2,j+1}|} \right) +$$
$$(-1)^j a_{n,j} \widehat{\tau}_{j+2,j+1} + \sum_{k=j+3}^{m} (-1)^k a_{n,k} \widehat{\langle \tau_{j+2,j+1}, \langle s_{j+3,k}, s_{j+2,j+1}\rangle\rangle}$$

which is a linear form as those in Lemma 2.5. Thus

$$\mathbf{H}(\mathbb{C} \setminus \Delta_{j+2}) \ni \frac{1}{(w^*_{n,j} \widehat{s}_{j+2,j+1})} \left((-1)^j a_{n,j} + \sum_{k=j+3}^{m} (-1)^k a_{n,k} \widehat{\langle s_{j+3,k}, s_{j+2,j+1}\rangle} \right) \in \mathcal{O}\left(\frac{1}{z^{n-2D-1}}\right).$$

Moreover, for $\nu = 0, 1, \ldots, n-2D-3$,

$$\int x^\nu \left((-1)^j a_{n,j} + \sum_{k=j+3}^{m} (-1)^k a_{n,k} \widehat{\langle s_{j+3,k}, s_{j+2,j+1}\rangle} \right)(x) \frac{\mathrm{d}\,\tau_{j+2,j+1}(x)}{w^*_{n,j}(x)} = 0.$$

So, the expression in parenthesis has at least $n-2D-2$ sign changes in the interior of Δ_{j+2}, and we can assure the existence of a polynomial $w^*_{n,j+1}$, $\deg w^*_{n,j+1} = n-2D-2$, with simple zeros located at the points of sign change inside Δ_{j+2} so that

$$\frac{1}{w^*_{n,j+1}} \left((-1)^j a_{n,j} + \sum_{k=j+3}^{m} (-1)^k a_{n,k} \widehat{\langle s_{j+3,k}, s_{j+2,j+1}\rangle} \right) \in \mathbf{H}(\mathbb{C} \setminus \Delta_{j+3}).$$

Using [62, Lemma 2.1] with $r = j+2$ (or direct calculation), we have

$$(-1)^j a_{n,j} + \sum_{k=j+3}^{m} (-1)^k a_{n,k} \widehat{\langle s_{j+3,k}, s_{j+2,j+1}\rangle} = \mathscr{A}_{n,j} - \widehat{s}_{j+1,j+1} \mathscr{A}_{n,j+1} + \widehat{s}_{j+1,j+1} \mathscr{A}_{n,j+2},$$

and taking into account the definition of the forms $\mathscr{A}_{n,j}$ the right hand side is $\mathcal{O}(1/z)$; thus,

$$\frac{1}{w^*_{n,j+1}(z)} \left((-1)^j a_{n,j} + \sum_{k=j+3}^{m} (-1)^k a_{n,k} \widehat{\langle s_{j+3,k}, s_{j+2,j+1}\rangle} \right)(z) = \mathcal{O}\left(\frac{1}{z^{n-2D-1}}\right).$$

In particular, if $j = m-3$, it is not difficult to see that the fraction $a_{n,m-3}/a_{n,m}$ is an incomplete diagonal multi-point Padé approximant of $\widehat{s}_{m,m-2}$ from where we can deduce the Hausdorff convergence on compact subsets of $\mathbb{C} \setminus \Delta_m$

$$h - \lim_n \frac{a_{n,m-3}}{a_{n,m}} = \widehat{s}_{m,m-2},$$

and similarly

$$h - \lim_n \frac{a_{n,m}}{a_{n,m-3}} = \widehat{s}_{m,m-2}^{-1}.$$

This process can be continued inductively. After $m - j - 1$ reductions we obtain the existence of a polynomial $\widetilde{w}_{n,j}$ with degree $\geq n - 2D - m + j$ with simple zeros inside Δ_{m-1} such that

$$\frac{a_{n,j} - a_{n,m}\widehat{s}_{m,j+1}}{\widetilde{w}_{n,j}}(z) = \mathscr{O}\left(\frac{1}{z^{n-2D-m+j+2}}\right) \in \mathbf{H}(\mathbb{C} \setminus \Delta_m), \qquad z \to \infty, \tag{2.18}$$ {sign_changes}

which allows us to deduce that

$$h - \lim_n \frac{a_{n,j}}{a_{n,m}} = \widehat{s}_{m,j+1},$$

on compact subsets of $\mathbb{C} \setminus \Delta_m$.

It readily follows that

$$\frac{a_{n,j} - a_{n,m}\widehat{s}_{m,j+1}}{\widehat{s}_{m,j+1}\widetilde{w}_{n,j}}(z) = \mathscr{O}\left(\frac{1}{z^{n-2D-m+j+1}}\right) \in \mathbf{H}(\mathbb{C} \setminus \Delta_m), \qquad z \to \infty,$$

but

$$\frac{a_{n,j} - a_{n,m}\widehat{s}_{m,j+1}}{\widehat{s}_{m,j+1}} = a_{n,j}\widehat{\tau}_{m,j+1} - (a_{n,m} - \ell_{m,j+1}a_{n,j}).$$

Hence,

$$\int x^\nu a_{n,j}(x) \frac{\mathrm{d}\,\tau_{m,j+1}(x)}{\widetilde{w}_{n,j}(x)} = 0, \qquad \nu = 0, 1, \ldots, n - 2D - m + j - 1.$$

Therefore, the polynomial $a_{n,j}$ has at least $n - 2D - m + j$ sign changes in $\mathring{\Delta}_m$. Also, we obtain

$$h - \lim_n \frac{a_{n,m}}{a_{n,j}} = \widehat{s}_{m,j+1}^{-1}$$

on compact subsets of $\mathbb{C} \setminus \Delta_m$.

To find the limit of the sequence $a_{n,0}/a_{n,m}$, $n \geq 0$, we change a little our previous arguments. It is easy to check that the reasoning above do not change substantially if we consider the linear forms $\mathscr{L}_{n,j} := T(z)\mathscr{A}_{n,j}(z)$ instead of $\mathscr{A}_{n,j}$. The main differences are in the asymptotic orders and in the bounds for the number of sign changes in Δ_m, but not in the conclusions.

In consequence, the following holds (see (2.13))

$$\frac{p_{n,0} - p_{n,m}\widehat{s}_{m,1}}{\widetilde{w}_{n,0}}(z) = \mathscr{O}\left(\frac{1}{z^{n-2D-m+j-1}}\right) \in \mathbf{H}(\mathbb{C} \setminus \Delta_m),$$

and we conclude that

$$h - \lim_n \frac{p_{n,0}}{p_{n,m}} = \widehat{s}_{m,1}.$$

However,

$$\frac{p_{n,0}}{p_{n,m}} = \frac{a_{n,0}T + \sum_{k=1}^m (-1)^k a_{n,k}Tr_k}{a_{n,m}T} = \frac{a_{n,0}}{a_{n,m}} + \sum_{k=1}^m (-1)^k \frac{a_{n,k}}{a_{n,m}} r_k.$$

Therefore, (2.16) readily follows.

Throughout the proof, if the zeros of T lie outside Δ_1 then in the right hand side of (2.14) we can write $\mathscr{O}\left(1/z^{n-D+1}\right)$ and we can replace $2D$ with D obtaining $n - D - m + j$ sign changes on Δ_m for $a_{n,j}$ as indicated in the final statement.

□

Once we have proved Theorem 2.7, the statement of Theorem 2.2 follows rather easily.

Proof of Theorem 2.2. In the hypothesis of this theorem the zeros of the polynomial T lie outside Δ_1; consequently, according to the last statement of Lemma 2.6 the rational functions $\frac{a_{n,0}}{a_{n,m}}$ have at most D poles in $\mathbb{C} \setminus \Delta_m$. On the other hand, we are assuming that f has exactly D poles in $\mathbb{C} \setminus \Delta_m$. From (2.16) and Lemma 1.8, we obtain that for all sufficiently large $n \in \mathbb{N}$ the fractions $\frac{a_{n,0}}{a_{n,m}}$ have exactly D poles outside Δ_m. Moreover, Gonchar's lemma asserts that each pole of f in $\mathbb{C} \setminus \Delta_m$ attracts as many zeros of $a_{n,m}$ as its order; that is, if $\zeta \in \mathbb{C} \setminus \Delta_m$ is a pole of f of order τ then for each $\varepsilon > 0$, there exists $n_0(\zeta) \in \mathbb{N}$ such that for all $n \geq n_0(\zeta)$ the polynomial $a_{n,m}$ has exactly τ zeros in the disk $\{z : |z - \zeta| < \varepsilon\}$. Thus the statements about the zeros of $a_{n,m}$ take place.

Fix $\varepsilon > 0$ and let D_ε be $\mathbb{C} \setminus \Delta_m$ minus an ε neighborhood of each pole of f in this region. Then, there exists n_0 such that for all $n \geq n_0$ and $j = 0, \ldots, m-1$, the rational functions $a_{n,j}/a_{n,m}$ are analytic in D_ε. From [37, Lemma 1] it follows that the limits in Lemma 2.7 hold uniformly on each compact subset of D_ε. Since $\varepsilon > 0$ is arbitrary, we obtain the limits in Theorem 2.2.

Fix $j = 1, \ldots, m-1$. Let ζ be a zero of T of multiplicity τ. Choose $\varepsilon > 0$ small enough and N sufficiently large such that $a_{n,m}$ has no zero on $\{|z - \zeta| = \varepsilon\}$ and exactly τ zeros inside the circle $\{|z - \zeta| = \varepsilon\}$ for $n \geq N$. As the function $\widehat{s}_{m,j+1}$ is holomorphic and has no zeros in $\mathbb{C} \setminus \Delta_m$, by the uniform convergence we get

$$\lim_{n\to\infty} \int_{|z-\zeta|=\varepsilon} \frac{(a_{n,j}/a_{n,m})'}{a_{n,j}/a_{n,m}}(z)\,\mathrm{d}z = \int_{|z-\zeta|=\varepsilon} \frac{(\widehat{s}_{m,j+1})'}{\widehat{s}_{m,j+1}}(z)\,\mathrm{d}z = 0.$$

Since $a_{n,m}$ has exactly τ zeros inside $\{|z - \zeta| = \varepsilon\}$ for all sufficiently large n, from the argument principle we obtain that $a_{n,j}, j =, \ldots, m-1$ also has exactly τ zeros inside that disk for all sufficiently large n.

Thus, in the circle $\{|z - \zeta| < \varepsilon\}$ the number of zeros of $a_{n,j}$ and $a_{n,m}$ coincide, i.e. ζ attracts as many zeros of $a_{n,j}$ as its order. We can extend this idea to a smooth Jordan curve Γ that surrounds all zeros of T and lies in $\mathbb{C} \setminus \Delta_m$. Then D zeros of $a_{n,j}$ accumulate at the zeros of T counting multiplicities and the remaining ones accumulate on $\Delta_m \cup \{\infty\}$.

□

Theorem 2.2 has some consequences on the convergence of the forms $\mathscr{A}_{n,j}$. {conforms}

Corollary 2.8:
Under the assumptions of Theorem 2.2, we have

$$\lim_{n\to\infty} \frac{\mathscr{A}_{n,j}}{a_{n,m}} = 0, \qquad j = 0, \ldots, m-1,$$

uniformly on each compact subset of $\mathbb{C} \setminus (\Delta_{j+1} \cup \Delta_m \cup \{z : T(z) = 0\})$.

Proof. From Theorem 2.2 and the expression of the forms $\mathscr{A}_{n,j}$ it follows that for $j = 1, \ldots, m-1$,

$$\lim_{n\to\infty} \frac{\mathscr{A}_{n,j}}{a_{n,m}} = (-1)^j \widehat{s}_{m,j+1} + \sum_{k=j+1}^{m-1} (-1)^k \widehat{s}_{m,k+1}\widehat{s}_{j+1,k} + (-1)^m \widehat{s}_{j+1,m} \equiv 0,$$

uniformly on compact subsets of $\mathbb{C} \setminus (\Delta_{j+1} \cup \Delta_m \cup \{z : T(z) = 0\})$. The equivalence to zero of the last expression is a consequence of a well known formula appearing in [31, Lemma 2.9]. Similarly,

$$\lim_{n\to\infty} \frac{\mathscr{A}_{n,0}}{a_{n,m}} = f + \sum_{k=1}^{m-1} (-1)^k \widehat{s}_{m.k+1}(\widehat{s}_{1,k} + r_k) + (-1)^m(\widehat{s}_{1,m} + r_m) \equiv 0,$$

uniformly on compact subsets of $\mathbb{C} \setminus (\Delta_1 \cup \Delta_m \cup \{z : T(z) = 0\})$. In proving the equality to zero aside from the identity in [31, Lemma 2.9] one uses the expression of f.

□

2.3.1 Rate of convergence

Throughout this subsection we assume that the conditions of Theorem 2.2 are in place. We will begin showing that when Δ_m is a finite interval convergence takes place with geometric rate. We will derive this result using Theorem 2.2 and the maximum principle. Similar arguments were employed in [29] in the case of Type ɪɪ Hermite-Padé approximation. First, we introduce some concepts to be used in the sequel.

Let $\varphi_t, t \in \overline{\mathbb{C}} \setminus \Delta_m$, be the conformal representation of $\overline{\mathbb{C}} \setminus \Delta_m$ onto $\{w : |w| < 1\}$ such that $\varphi_t(t) = 0$, $\varphi_t'(t) > 0$. It is easy to verify that $|\varphi_t(z)|$ can be extended continuously to $\overline{\mathbb{C}}^2$ in the two variables z, t and equals zero only when $z = t$. In fact

$$|\varphi_t(z)| = \left| \frac{\varphi_\infty(z) - \varphi_\infty(t)}{1 - \overline{\varphi_\infty(t)}\varphi_\infty(z)} \right|.$$

Let $0 < \rho < 1$ and

$$\gamma_\rho := \{z : |\varphi_\infty(z)| = \rho\}.$$

Fix a compact set $\mathscr{K} \subset \overline{\mathbb{C}} \setminus (\Delta_m \cup \{z : T(z) = 0\})$. Take ρ sufficiently close to 1 so that $\mathscr{K}$ lies in the unbounded connected component of the complement of γ_ρ. Set

{delta} $$\kappa_\rho := \inf\{|\varphi_t(z)| : t \in \Delta_{m-1}, z \in \gamma_\rho\}, \qquad \delta(\mathscr{K}) = \max\{|\varphi_t(z)| : t \in \Delta_{m-1}, z \in \mathscr{K}\}. \tag{2.19}$$

From the continuity of $|\varphi_t(z)|$ in the two variables it readily follows that

$$\lim_{\rho\to 1} \kappa_\rho = 1, \qquad \delta(\mathscr{K}) < 1.$$

{cor:4} As usual, $\|\cdot\|_{\mathscr{K}}$ denotes the uniform norm on $\mathscr{K}$.

Corollary 2.9:
Under the hypothesis of Theorem 2.2, if we assume additionally that Δ_m is bounded then

{rate3} $$\limsup_n \left\| \frac{a_{n,j}}{a_{n,m}} - \widehat{s}_{m,j+1} \right\|_{\mathscr{K}}^{1/n} \le \delta(\mathscr{K}) \|\varphi_\infty\|_{\mathscr{K}} < 1, \qquad j = 1, \dots, m-1, \tag{2.20}$$

and

{polos*} $$\limsup_n \left\| \frac{a_{n,0}}{a_{n,m}} - f \right\|_{\mathscr{K}}^{1/n} \le \delta(\mathscr{K}) \|\varphi_\infty\|_{\mathscr{K}} < 1 \tag{2.21}$$

for every compact set $\mathscr{K} \subset \mathbb{C} \setminus (\Delta_m \cup \{z : T(z) = 0\})$ and $\delta(\mathscr{K})$ is defined in (2.19).

Proof. Fix $\mathcal{K} \subset \overline{\mathbb{C}} \setminus (\Delta_m \cup \{z : T(z) = 0\})$. According to Theorem 2.2, for all sufficiently large $n > N$ the polynomials $a_{n,m}$ have exactly D zeros in $\mathbb{C} \setminus \Delta_m$ and they lie at a positive distance from $\mathcal{K}$ (independent of $n > D$). In the sequel we only consider such n's.

Let $q_{n,m} = \prod_1^D (z - x_{n,k})$ be the monic polynomial of degree D whose zeros are the roots of $a_{n,m}$ outside Δ_m. From Theorem 2.2 we know that $\lim_{n\to\infty} q_{n,m} = T$. Fix $j = 1, \dots, m$. Assume that $\widetilde{w}_{n,j}(z) = \prod_{k=1}^{\deg(\widetilde{w}_{n,j})} (z - \zeta_{n,j,k})$, where $\widetilde{w}_{n,j}$ is the polynomial introduced in the proof of Theorem 2.7 (see (2.18)). Set

$$\varphi_{n,j}(z) := \prod_{k=1}^{\deg(\widetilde{w}_{n,j})} \varphi_{\zeta_{n,j,k}}(z), \qquad \psi_n(z) := \prod_{k=1}^{D} \varphi_{x_{n,k}}(z).$$

From (2.18) it follows that

$$\psi_n \frac{(a_{n,j}/a_{n,m}) - \widehat{s}_{m,j+1}}{\varphi_\infty^n \varphi_{n,j}} \in \mathbf{H}(\overline{\mathbb{C}} \setminus \Delta_m).$$

Take ρ sufficiently close to 1 so that $\mathcal{K}$ lies in the unbounded connected component of the complement of γ_ρ. On γ_ρ, for all sufficiently large $n > N_1 \geq N$, we have

$$\left\| \psi_n \frac{(a_{n,j}/a_{n,m}) - \widehat{s}_{m,j+1}}{\varphi_\infty^n \varphi_{n,j}} \right\|_{\gamma_\rho} \leq \rho^{-n} \kappa_\rho^{-\deg \widetilde{w}_{n,j}}. \tag{2.22}$$ {rate}

Indeed, $|\psi_n(z)| \leq 1$ for all $z \in \overline{\mathbb{C}} \setminus \Delta_m$, $\varphi_{\zeta_{n,j,k}}(z) \geq \kappa_\rho$ for all $\zeta_{n,j,k} \in \Delta_{m-1}$, and for all sufficiently large $n \geq N_2 \geq N_1$, $\|(a_{n,j}/a_{n,m}) - \widehat{s}_{m,j+1}\|_{\gamma_\rho} \leq 1$ since by Theorem 2.2 the function under the norm sign converges to zero on γ_ρ.

Using the maximum principle, from (2.22) it follows that for all $z \in \mathcal{K}$

$$\left| \frac{a_{n,j}(z)}{a_{n,m}(z)} - \widehat{s}_{m,j+1}(z) \right| \leq \frac{|\varphi_{n,j}(z)}{|\psi_n(z)|} \frac{\varphi_\infty^n(z)}{\rho^n \kappa_\rho^{\deg(\widetilde{w}_{n,j})}} \leq \frac{\|\varphi_\infty\|_{\mathcal{K}}^n}{|\psi_n(z)|\rho^n} \left(\frac{\delta(\mathcal{K})}{\kappa_\rho} \right)^{\deg(\widetilde{w}_{n,j})}. \tag{2.23}$$ {rate2}

Since the points $x_{n,1}, \dots, x_{n,D}$ remain bounded away from $\mathcal{K}$ independently of n, we obtain that

$$\inf_{n > N_3} \{|\psi_n(z)| : z \in \mathcal{K}\} \geq C > 0,$$

where $N_3 \geq N_2$ is sufficiently large. On the other hand, recall that $n - 2D - m + j \leq \deg(\widetilde{w}_{n,j}) \leq n$; consequently, using (2.23), we obtain

$$\limsup_n \left\| \frac{a_{n,j}}{a_{n,m}} - \widehat{s}_{m,j+1} \right\|_{\mathcal{K}}^{1/n} \leq \frac{\delta(\mathcal{K}) \|\varphi_\infty\|_{\mathcal{K}}}{\rho^n \kappa_\rho}.$$

From here we get (2.20) since $\lim_{\rho\to 1} \kappa_\rho = 1$.

The proof of (2.21) is basically the same.

□

We wish to point out that if Δ_m is unbounded but Δ_{m-1} is bounded then it is also possible to prove convergence with geometric rate modifying slightly the arguments. Of course, the estimate of the rate of convergence will differ from the one above.

{cor5}

Corollary 2.10:
Under the hypothesis of Theorem 2.2 *if we assume additionally that* Δ_m *is bounded, then*

$$\limsup_{n\to\infty} \left\| \frac{\mathscr{A}_{n,j}}{a_{n,m}} \right\|_{\mathscr{K}}^{1/n} \le \delta(\mathscr{K}) \|\varphi_\infty\|_{\mathscr{K}}, \qquad j = 0, \ldots, m-1,$$

for every compact $\mathscr{K} \subset \overline{\mathbb{C}} \setminus (\Delta_{j+1} \cup \Delta_m \cup \{z : T(z) = 0\})$.

Proof. Indeed, for $j = 1, \ldots, m-1$,

$$\frac{\mathscr{A}_{n,j}}{a_{n,m}} = (-1)^j \frac{a_{n,j}}{a_{n,m}} + \sum_{k=j+1}^{m} (-1)^k \frac{a_{n,k}}{a_{n,m}} \widehat{s}_{j+1,k} =$$

$$(-1)^j \left(\frac{a_{n,j}}{a_{n,m}} - \widehat{s}_{m,j+1} \right) + \sum_{k=j+1}^{m-1} (-1)^k \left(\frac{a_{n,k}}{a_{n,m}} - \widehat{s}_{m,k+1} \right) \widehat{s}_{j+1,k}$$

because, according to [31, Lemma 2.9]

$$(-1)^j \widehat{s}_{m,j+1} + \sum_{k=j+1}^{m-1} (-1)^k \widehat{s}_{m.k+1} \widehat{s}_{j+1,k} + (-1)^m \widehat{s}_{j+1,m} \equiv 0$$

for all $z \in \mathbb{C} \setminus (\Delta_{j+1} \cup \Delta_m)$. Now it remains to use (2.20) and trivial estimates. The proof for $j = 0$ is similar.

□

When the measures generating the Nikishin system are regular (see Definition 1.10), then more precise estimates of the rate of convergence may be given.

2.4 Multi-orthogonality relations

{zeros_A_nj} We begin by obtaining some integral representations which will be needed.

Lemma 2.11:
Assume that all the zeros of T *lie in the complement of* $\Delta_1 \cup \Delta_m$*,* f *has exactly* D *poles in* $\mathbb{C} \setminus \Delta_m$*, and* $n > N \ge D$*. Then, for each* $j = 1, \ldots, m-1$*,*

{A_njQ_nj_1}
$$\frac{\mathscr{A}_{n,j}}{Q_{n,j}}(z) = \int_{\Delta_{j+1}} \frac{\mathscr{A}_{n,j+1}(x)}{z-x} \frac{\mathrm{d}\sigma_{j+1}(x)}{Q_{n,j}(x)} \tag{2.24}$$

and

{A_njQ_nj_2}
$$T(z)\mathscr{A}_{n,0}(z) = \int_{\Delta_1} \frac{\mathscr{A}_{n,1}(x)T(x)}{z-x} \,\mathrm{d}\sigma_1(x). \tag{2.25}$$

Moreover, for $j = 1, \ldots, m-1$

rthA_njQ_nj_1}
$$\int_{\Delta_{j+1}} x^\nu \mathscr{A}_{n,j+1}(x) \frac{\mathrm{d}\sigma_{j+1}(x)}{Q_{n,j}(x)} = 0, \qquad \nu = 0, 1, \ldots, n-D-1. \tag{2.26}$$

and

rthA_njQ_nj_2}
$$\int_{\Delta_1} x^\nu \mathscr{A}_{n,1}(x) T(x) \,\mathrm{d}\sigma_1(x) = 0, \qquad \nu = 0, 1, \ldots, n-D-1. \tag{2.27}$$

Proof. Notice that $T\mathscr{A}_{n,0} = \mathscr{O}(1/z^{n-D+1}) \in \mathbf{H}(\mathbb{C} \setminus \Delta_1)$. Let Γ be a positively oriented closed Jordan curve which surrounds Δ_1 so that Δ_2 and z remain in the unbounded connected component of the complement of Γ. We have

$$\begin{aligned} T(z)\mathscr{A}_{n,0}(z) =& \frac{1}{2\pi i}\int_\Gamma \frac{(T\mathscr{A}_{n,0})(\zeta)}{z-\zeta}\,\mathrm{d}\,\zeta \\ =& \frac{1}{2\pi i}\int_\Gamma \frac{(Ta_{n,0}+\sum_{k=1}^m(-1)^k a_{n,k}Tr_k)(\zeta)}{z-\zeta}\,\mathrm{d}\,\zeta + \frac{1}{2\pi i}\int_\Gamma \frac{(T\sum_{k=1}^m(-1)^k a_{n,k}\widehat{s}_{1,k})(\zeta)}{z-\zeta}\,\mathrm{d}\,\zeta \qquad (2.28) \\ =& \int \frac{1}{2\pi i}\int_\Gamma \frac{(-Ta_{n,1}+T\sum_{k=2}^m(-1)^k a_{n,k}\widehat{s}_{2,k})(\zeta)}{(z-\zeta)(\zeta-x)}\,\mathrm{d}\,\zeta\,\mathrm{d}\,\sigma_1(x) = \int \frac{(T\mathscr{A}_{n,1})(x)}{z-x}d\sigma_1(x). \end{aligned}$$

{zeros_A_nj:1}

Indeed the first equality comes from Cauchy's integral formula for the complement of Γ. The second equality is trivial. Since $(Ta_{n,0}+\sum_{k=1}^m(-1)^k a_{n,k}Tr_k)(\zeta)/(z-\zeta)$ is analytic with respect to ζ inside Γ, the first integral in (2.28) is zero. Substituting in the second integral $\widehat{s}_{1,k}$ with its integral representation and using Fubini's theorem you get the third equality. The last equality comes from the use of Cauchy's integral formula inside Γ. Thus we obtain (2.25).

Similarly, since $z^\nu T\mathscr{A}_{n,0} = \mathscr{O}(1/z^2) \in \mathbf{H}(\mathbb{C}\setminus\Delta_1)$, $\nu = 0,\ldots,n-D-1$, we obtain

$$\begin{aligned} 0 =& \frac{1}{2\pi i}\int_\Gamma \zeta^\nu (T\mathscr{A}_{n,0})(\zeta)\,\mathrm{d}\,\zeta \\ =& \int \frac{1}{2\pi i}\int_\Gamma \frac{\zeta^\nu(-Ta_{n,1}+T\sum_{k=2}^m(-1)^k a_{n,k}\widehat{s}_{2,k})(\zeta)}{(\zeta-x)}\,\mathrm{d}\,\zeta\,\mathrm{d}\,\sigma_1(x) \\ =& \int x^\nu (T\mathscr{A}_{n,1})(x)d\sigma_1(x). \end{aligned}$$

which is (2.27).

In order to derive (2.24) and (2.26) one proceeds analogously. It is sufficient to use that $z^\nu\mathscr{A}_{n,j}/Q_{n,j} = \mathscr{O}(1/z^2) \in \mathbf{H}(\mathbb{C}\setminus\Delta_{j+1})$, $j = 1,\ldots,m-1, \nu = 0,\ldots,n-D-1$ and take Γ a positively oriented closed Jordan curve which surrounds Δ_{j+1} so that Δ_{j+2} ($\Delta_{m+1} = \emptyset$) and z remain in the unbounded connected component of the complement of Γ.

□

The previous lemma can be reformulated as follows.

Lemma 2.12:
Assume that all the zeros of T lie in the complement of $\Delta_1 \cup \Delta_m$, f has exactly D poles in $\mathbb{C}\setminus\Delta_m$, and $n > N \geq D$. For each fixed $j = 0,\ldots,m-1$,

$$\int x^\nu Q_{n,j+1}(x)\frac{\mathscr{H}_{n,j+1}(x)\,\mathrm{d}\,\sigma_{j+1}(x)}{Q_{n,j}(x)Q_{n,j+2}(x)} = 0, \qquad \nu = 0,1,\ldots,n-D-1. \tag{2.29}$$

{Orth_H_nj}

Moreover, for $j = 0,2,3,\ldots,m-1$

$$\mathscr{H}_{n,j}(z) = \int \frac{Q^2_{n,j+1}(x)}{z-x}\frac{\mathscr{H}_{n,j+1}(x)\,\mathrm{d}\,\sigma_{j+1}(x)}{Q_{n,j}(x)Q_{n,j+2}(x)}, \tag{2.30}$$

{H_mjIdentity_

and

$$\mathcal{H}_{n,1}(z) = T(z) \int \frac{Q_{n,2}^2(x)}{z-x} \frac{\mathcal{H}_{n,2}(x)\,\mathrm{d}\sigma_2(x)}{Q_{n,1}(x)Q_{n,3}(x)}. \tag{2.31}$$ {H_mjIdentity_

Recall that by convention $Q_{n,0} \equiv Q_{n,m+1} \equiv 1$.

Proof. Formula (2.29) is a restatement of (2.26) and (2.27) using the notation of the functions $\mathcal{H}_{n,j}$.

Since $\deg Q_{n,j+1} = n - D$, from (2.29) we deduce that for $j = 0, \dots, m-1$

$$\int \frac{Q_{n,j+1}(z) - Q_{n,j+1}(x)}{z-x} Q_{n,j+1}(x) \frac{\mathcal{H}_{n,j+1}(x)\,\mathrm{d}\sigma_{j+1}(x)}{Q_{n,j}(x)Q_{n,j+2}(x)} = 0.$$

This last identity can be rewritten as

$$Q_{n,j+1}(z) \int \frac{Q_{n,j+1}(x)}{z-x} \frac{\mathcal{H}_{n,j+1}(x)\,\mathrm{d}\sigma_{j+1}(x)}{Q_{n,j}(x)Q_{n,j+2}(x)} = \int \frac{Q_{n,j+1}^2(x)}{z-x} \frac{\mathcal{H}_{n,j+1}(x)\,\mathrm{d}\sigma_{j+1}(x)}{Q_{n,j}(x)Q_{n,j+2}(x)}.$$

For $j = 2, \dots, m-1$

$$\int \frac{Q_{n,j+1}(x)}{z-x} \frac{\mathcal{H}_{n,j+1}(x)\,\mathrm{d}\sigma_{j+1}(x)}{Q_{n,j}(x)Q_{n,j+2}(x)} = \int \frac{\mathcal{A}_{n,j}(x)}{z-x} \frac{\mathrm{d}\sigma_{j+1}(x)}{Q_{n,j}(x)} = \frac{\mathcal{A}_{n,j}(z)}{Q_{n,j}(z)}$$

and (2.30) immediately follows for $j = 2, \dots, m-1$. In the case $j = 1$, notice that

$$\int \frac{Q_{n,2}^2(x)}{z-x} \frac{\mathcal{H}_{n,2}(x)\,\mathrm{d}\sigma_2(x)}{Q_{n,1}(x)Q_{n,3}(x)} = \frac{Q_{n,2}(z)}{Q_{n,1}(z)} \mathcal{A}_{n,1}(z) = \frac{\mathcal{H}_{n,1}(z)}{T(z)},$$

which is equivalent to (2.31). For $j = 0$ we proceed as for (2.30), $j = 2, \dots, m-1$.

□

The previous lemma indicates that the polynomial $Q_{n,j}$, $j = 1, \dots, m$, is orthogonal with respect to the varying measure

$$\frac{\mathcal{H}_{n,j}(x)\,\mathrm{d}\sigma_j(x)}{Q_{n,j-1}(x)Q_{n,j+1}(x)}.$$

This varying measure has constant sign because $Q_{n,j-1}$ and $Q_{n,j+1}$ have constant sign on Δ_j and $\mathcal{H}_{n,j}$ also has constant sign since $Q_{n,j}$ takes away the zeros of $\mathcal{A}_{n,j}$ on $\mathring{\Delta}_j$.

2.5 Proof of general asymptotic results

For the proof of Theorem 2.3 we make use of a technique introduced in [41] for the study of the weak asymptotic of Type II multiple orthogonal polynomials associated with generalized Nikishin systems (see also [28, 32, 33]).

Proof of Theorem 2.3. The unit ball in the cone of positive Borel measures is weak star compact; therefore, it is sufficient to show that each sequence of measures $\left(\mu_{Q_{n,j}}\right)_{n \geq N}$, $j = 1, \dots, m$, has only one accumulation point which coincides with the corresponding component of the vector equilibrium measure $\vec{\lambda}$ determined by the matrix $\mathscr{C}_{\mathcal{N}}$ on the system of compact sets $\operatorname{supp}\sigma_j$, $j = 1, \ldots, m$.

Let Λ be a sequence indices such that for each $j = 1, \ldots, m$

$$* \lim_{n\in\Lambda} \mu_{Q_{n,j}} = \mu_j .$$

Notice that $\mu_j \in \mathscr{M}_1(E_j)$, $j = 1, \ldots, m$. Taking into account that all the zeros of $Q_{n,j}$ lie in Δ_j, it follows that

$$\lim_{n\in\Lambda} |Q_{n,j}(z)|^{1/n} = \exp\left(-V^{\mu_j}(z)\right), \tag{2.32}$$

{limit_nesimo_

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_j$.

The generating measures σ_j, $j = 1, \ldots, m$, have constant sign. Without loss of generality we can assume that they are positive. Notice that $\mathscr{A}_{n,m} = \pm Q_{n,m} T_n$, where $T_n \rightrightarrows T$ on compact subsets of $\mathbb{C}$ (recall that $a_{n,m}$ is monic). Hence, formula (2.29), when $j = m-1$ becomes

$$\int x^\nu Q_{n,m}(x) \frac{T_n(x)\,\mathrm{d}\,\sigma_m(x)}{Q_{n,m-1}(x)}, \qquad \nu = 0, 1, \ldots, n-D-1.$$

In order to use Lemma 1.13, write $\phi_n = T_n / Q_{n,m-1}$. Then,

$$\lim_{n\in\Lambda} \frac{1}{2n} \log \phi_n(x) = \lim_{n\in\Lambda} \left(\frac{1}{2n} \log Q_{n,m-1} - \frac{1}{2n} \log T_n \right).$$

As $T_n \rightrightarrows T$ on $\operatorname{supp}\sigma_m$, where the polynomial T has no zeros, we conclude that $0 < b \le |T_n| \le B$, and $\frac{1}{2n}\log T_n \rightrightarrows 0$ uniformly on $\operatorname{supp}\sigma_m$. According to (2.32) we get

$$\lim_{n\in\Lambda} \frac{1}{2n} \log |Q_{n,m-1}(x)| = -\frac{1}{2} V^{\mu_{m-1}}(x),$$

uniformly on $\operatorname{supp}\sigma_m$. So,

$$\lim_{n\in\Lambda} \frac{1}{2n} \log \phi_n(x) = -\frac{1}{2} V^{\mu_{m-1}}(x) > -\infty.$$

Thus, from Lemma 1.13 we deduce that μ_m is the unique solution of the extremal problem

$$V^{\mu_m}(x) - \frac{1}{2} V^{\mu_{m-1}}(x) \begin{cases} = w_m, & x \in \operatorname{supp}(\mu_m), \\ \ge w_m, & x \in \operatorname{supp}(\sigma_m), \end{cases} \tag{2.33}$$

{extremal_Qnm}

and

$$\lim_{n\in\Lambda} \left(\int \frac{Q_{n,m}^2(x)}{|Q_{n,m-1}(x)|} \,\mathrm{d}\,\sigma_m(x) \right)^{1/2n} = e^{w_m}. \tag{2.34}$$

{log_asymp_Qnm

Next, we prove by induction on decreasing values of j, that for all $j = 1, \ldots, m$

$$V^{\mu_j}(x) - \frac{1}{2} V^{\mu_{j-1}}(x) - \frac{1}{2} V^{\mu_{j+1}}(x) + w_{j+1} \begin{cases} = w_j, & x \in \operatorname{supp}\mu_j, \\ \ge w_j, & x \in \operatorname{supp}\sigma_j, \end{cases} \tag{2.35}$$

{extremal_Qnj}

where $V^{\mu_0} \equiv V^{\mu_{m+1}} \equiv 0$, $w_{m+1} = 0$, and

$$\lim_{n\in\Lambda} \left(\int Q_{n,j}^2(x) \frac{|\mathscr{H}_{n,j}(x)|\,\mathrm{d}\,\sigma_j(x)}{|Q_{n,j-1}(x) Q_{n,m+1}(x)|} \right)^{1/2n} = e^{-w_j} \tag{2.36}$$

{limit_norm2_Q

where $Q_{n,0} \equiv Q_{n,m+1} \equiv 1$.

Notice that for $j = m$ these relations are (2.34) and (2.33), and the initial step of the induction is settled. Suppose that the statement is true for $j + 1 \in \{3, \ldots, m\}$ and let us prove it for j. The step from $j = 2$ to $j = 1$ will be treated separately afterwards.

For $j = 1, \ldots, m$ the orthogonality relations (2.29) can be expressed as

{_nj_rewritten}

$$\int x^\nu Q_{n,j}(x) \frac{\mathscr{H}_{n,j}(x)\, \mathrm{d}\sigma_j(x)}{Q_{n,j-1}(x) Q_{n,j+1}(x)} = 0, \qquad \nu = 0, 1, \ldots, n - D - 1, \tag{2.37}$$

and using (2.30), $j = 2, \ldots, m$

$$\int x^\nu Q_{n,j}(x) \left(\int \frac{Q_{n,j+1}^2(t)}{x - t} \frac{\mathscr{H}_{n,j+1}(t)\, \mathrm{d}\sigma_{j+1}(t)}{Q_{n,j}(t) Q_{n,j+2}(t)} \right) \frac{\mathrm{d}\sigma_j(x)}{Q_{n,j-1}(x) Q_{n,j+1}(x)} = 0,$$

for $\nu = 0, 1, \ldots, n - D - 1$.

The limit in (2.32) gives us that

$$\lim_{n \in \Lambda} \frac{1}{2n} \log |Q_{n,j-1}(x) Q_{n,j+1}(x)| = -\frac{1}{2} V^{\mu_{j-1}}(x) - \frac{1}{2} V^{\mu_{j+1}}(x),$$

uniformly on Δ_j.

Set

$$K_{n,j+1} := \left(\int Q_{n,j+1}^2(t) \frac{|\mathscr{H}_{n,j+1}(t)|\, \mathrm{d}\sigma_{j+1}(t)}{|Q_{n,j}(t) Q_{n,j+2}(t)|} \right)^{-1/2}.$$

It follows that for $x \in \Delta_j$

$$\frac{1}{\delta_{j+1}^* K_{n,j+1}^2} \leq \int \frac{Q_{n,j+1}^2(t)}{|x - t|} \frac{|\mathscr{H}_{n,j+1}(t)|\, \mathrm{d}\sigma_{j+1}(t)}{|Q_{n,j}(t) Q_{n,j+2}(t)|} \leq \frac{1}{\delta_{j+1} K_{n,j+1}^2}$$

where $0 < \delta_{j+1} = \inf\{|x - t| : t \in \Delta_{j+1}, x \in \Delta_j\} \leq \max\{|x - t| : t \in \Delta_{j+1}, x \in \Delta_j\} = \delta_{j+1}^* < \infty$. Taking into consideration these inequalities, from the induction hypothesis, we obtain that

imit_exp_wj+1}

$$\lim_{n \in \Lambda} \left(\int \frac{Q_{n,j+1}^2(t)}{|x - t|} \frac{|\mathscr{H}_{n,j+1}(t)|\, \mathrm{d}\sigma_{j+1}(t)}{|Q_{n,j}(t) Q_{n,j+2}(t)|} \right)^{1/2n} = e^{-w_{j+1}}. \tag{2.38}$$

Taking (2.5) and (2.38) into account, Lemma 1.13 yields that μ_j is the unique solution of the extremal problem (2.35) and

$$\lim_{n \in \Lambda} \left(\iint \frac{Q_{n,j+1}^2(t)}{|x - t|} \frac{|\mathscr{H}_{n,j+1}(t)|\, \mathrm{d}\sigma_{j+1}(t)}{|Q_{n,j}(t) Q_{n,j+1}(t)|} \frac{Q_{n,j}^2(x)\, \mathrm{d}\sigma_j(x)}{|Q_{n,j-1}(x) Q_{n,j+1}(x)|} \right)^{1/2n} = e^{-w_j}.$$

As a consequence of (2.30), $j = 2, \ldots, m - 1$, the above formula reduces to (2.36).

For $j = 1$ formula (2.37) becomes

$$\int x^\nu Q_{n,1}(x) \left(T(x) \int \frac{Q_{n,2}^2(t)}{x - t} \frac{\mathscr{H}_{n,j}(t)\, \mathrm{d}\sigma_2(t)}{Q_{n,1}(t) Q_{n,3}(t)} \right) \frac{\mathrm{d}\sigma_1(x)}{Q_{n,2}(x)} = 0, \ \nu = 0, \ldots, n - D - 1.$$

From (2.32) we have $\lim_{n\in\Lambda} \frac{1}{2n} \log |Q_{n,2}(x)| = -\frac{1}{2} V^{\mu_2}(x)$ uniformly on Δ_1. Recall that $0 < b \leq T(x) \leq B$ in Δ_1, thus it follows that for $x \in \Delta_1$:

$$\frac{b}{\delta_2^* K_{n,2}^2} \leq |T(x)| \int \frac{Q_{n,2}^2(t)}{|x-t|} \frac{|\mathscr{H}_{n,j}(t)| \,\mathrm{d}\,\sigma_2(t)}{|Q_{n,1}(t) Q_{n,3}(t)|} \leq \frac{B}{\delta_2 K_{n,2}^2}.$$

From here on, all the arguments used before work as well and the induction process is completed.

We can rewrite (2.35) as

{vecequil2}
$$V^{\mu_j}(x) - \frac{1}{2} V^{\mu_{j-1}} - \frac{1}{2} V^{\mu_{j+1}}(x) \begin{cases} = w'_j, & x \in \operatorname{supp} \mu_j, \\ \geq w'_j, & x \in \operatorname{supp} \sigma_j, \end{cases} \tag{2.39}$$

for $j = 1, \ldots, m$, where

$$w'_j = w_j - w_{j+1}, \qquad w_{m+1} = 0. \tag{2.40}$$
{relations_ws}

(Recall that the terms with V^{μ_0} and $V^{\mu_{m+1}}$ do not appear when $j = 0$ and $j = m$, respectively). Now, (2.39) adopts the form of (2.5) which has only one solution. If follows that $\vec{\lambda} = (\mu_1, \ldots, \mu_m)$ is the equilibrium solution for the vector potential problem determined by the interactions matrix $\mathscr{C}_{\mathscr{N}}$ on the system of compact sets $\operatorname{supp} \sigma_j$, $j = 1, \ldots, m$ and $\omega^{\vec{\lambda}} = (w'_1, \ldots, w'_m)$ is the corresponding vector equilibrium constant. This is for any convergent subsequence; since the equilibrium problem does not depend on Λ and the solution is unique we obtain (2.6).

From the uniqueness of the vector equilibrium constant and (2.36), we get

$$\lim_{n\to\infty} \left(\int Q_{n,j}^2(x) \frac{|\mathscr{H}_{n,j}(x)| \,\mathrm{d}\,\sigma_j(x)}{|Q_{n,j-1}(x) Q_{n,j+1}(x)|} \right)^{1/2n} = e^{-w_j},$$

On the other hand, from (2.40) it follows that $w_m = \omega_m^{\vec{\lambda}}$ when $j = m$. Suppose that $w_{j+1} = \sum_{k=j+1}^m \omega_k^{\vec{\lambda}}$ where $j + 1 \in \{2, \ldots, m\}$. Then, according to (2.40)

$$w_j = w'_j + w_{j+1} = \omega_j^{\vec{\lambda}} + w_{j+1} = \sum_{k=j}^m \omega_k^{\vec{\lambda}}$$

and (2.7) immediately follows.

□

Now we are ready to give a proof of Theorem 2.4.

Proof of Theorem 2.4. Since $\mathscr{A}_{n,m} = \pm Q_{n,m} T_n$ and $T_n \rightrightarrows T$ on compact subsets of $\mathbb{C}$, (2.6) implies

$$\lim_{n\to\infty} |\mathscr{A}_{n,m}(z)|^{1/n} = \exp\left(-V^{\lambda_m}\right), \qquad z \in \mathscr{K} \subset \mathbb{C} \setminus (\Delta_m \cup Z).$$

For $j = 1, \ldots, m-1$, from (2.30) we have

$$\mathscr{A}_{n,j}(z) = \frac{Q_{n,j}(z)}{Q_{n,j+1}(z)} \int \frac{Q_{n,j+1}^2(x)}{z - x} \frac{\mathscr{H}_{n,j+1}(x) \,\mathrm{d}\,\sigma_{j+1}(x)}{Q_{n,j}(x) Q_{n,j+2}(x)}, \tag{2.41}$$
{Anj_Integral}

where $Q_{n,0} \equiv Q_{n,m+1} \equiv 1$. Now, (2.6) implies

$$\lim_{n\to\infty} \left| \frac{Q_{n,j}(z)}{Q_{n,j+1}(z)} \right|^{1/n} = \exp\left(V^{\lambda_{j+1}}(z) - V^{\lambda_j}(z) \right), \qquad \mathscr{K} \subset \mathbb{C} \setminus (\Delta_j \cup \Delta_{j+1})$$

(we also use that the zeros of $Q_{n,j}$ and $Q_{n,j+1}$ lie in Δ_j and Δ_{j+1}, respectively). It remains to find the n-th root asymptotic behavior of the integral.

Fix a compact set $\mathscr{K} \subset \mathbb{C} \setminus \Delta_{j+1}$. It is easy to verify that

unds_integral}

$$\frac{C_1}{K^2_{n,j+1}} \le \left| \int \frac{Q^2_{n,j+1}(x)}{z-x} \frac{\mathscr{H}_{n,j+1}(x)\, \mathrm{d}\,\sigma_{j+1}(x)}{Q_{n,j}(x) Q_{n,j+1}(x)} \right| \le \frac{C_2}{K^2_{n,j+1}}, \tag{2.42}$$

where

$$C_1 = \frac{\min\{\max\{|u-x|, |v| : z = u + iv\} : z \in \mathscr{K}, x \in \Delta_{j+1}\}}{\max\{|z-x|^2 : z \in \mathscr{K}, x \in \Delta_{j+1}\}}$$

and

$$C_2 = \frac{1}{\min\{|z-x| : z \in \mathscr{K}, x \in \Delta_{j+1}\}} < \infty.$$

Taking into account (2.7) we get

imit_integral}

$$\lim_{n\to\infty} \left| \int \frac{Q^2_{n,j+1}(x)}{z-x} \frac{\mathscr{H}_{n,j+1}(x)\, \mathrm{d}\,\sigma_{j+1}(x)}{Q_{n,j}(x) Q_{n,j+2}(x)} \right|^{1/n} = \exp\left(-2 \sum_{k=j+1}^{m} \omega_k^{\vec{\lambda}} \right). \tag{2.43}$$

From (2.41)-(2.43) we deduce (2.8). Finally, notice that

$$\lim_{n\to\infty} |T(z)\mathscr{A}_{n,0}(z)|^{1/n} = \lim_{n\to\infty} |\mathscr{A}_{n,0}(z)|^{1/n}$$

for all $z \in \mathbb{C} \setminus (\Delta_1 \cup Z)$, in case that the first limit exists. This last statement holds, and its proof follows easily using the same arguments as above.

□

2.6 Outcomes of Theorem 2.3

Let us find the logarithmic asymptotic of the polynomials $a_{n,j}$, $j = 0, \ldots, m$.

Corollary 2.13:

Under the assumptions of Theorem 2.3*,*

it_nrooth_anj}

$$\lim_{n\to\infty} |a_{n,j}(z)|^{1/n} = A_m(z), \qquad j = 1, \ldots, m, \tag{2.44}$$

uniformly on compact subsets of $\mathbb{C} \setminus (\Delta_m \cup Z)$.

Proof. In the proof of Theorem 2.4 we obtained (2.44) for $j = m$. Now, recall that the function $\widehat{s}_{m,j+1}$ never equals zero in $\mathbb{C} \setminus (\Delta_m \cup Z)$; therefore, for the remaining values of j, the limit (2.44) is an immediate consequence of (2.44) for $j = m$ and (2.3).

□

Regarding (2.44) for $j = 0$, aside from Z we would have to exclude from $\mathbb{C} \setminus \Delta_m$ all the points where $f = 0$.

Our next goal is to produce estimates of the rate of convergence in (2.3). First we prove {corollary_log

Corollary 2.14:

Under the assumptions of Theorem 2.3, *for* $k = 1, \dots, m$ *and* $j = 0, \dots, k-1$, *we have*

$$\limsup_{n\to\infty} \left| \frac{\mathscr{A}_{n,j}(z)}{\mathscr{A}_{n,k}(z)} \right|^{1/n} \leq \exp\left(-V^{\lambda_{k+1}}(z) + V^{\lambda_k}(z) + V^{\lambda_{j+1}}(z) - V^{\lambda_j}(z) - 2 \sum_{\ell=j+1}^{k} \omega_\ell^{\vec{\lambda}} \right), \qquad (2.45)$$ {inequality_nr

uniformly on compact subsets $\mathscr{K} \subset \mathbb{C} \setminus (\Delta_k \cup \Delta_{j+1})$, *and*

$$\lim_{n\to\infty} \left| \frac{\mathscr{A}_{n,j}(z)}{\mathscr{A}_{n,k}(z)} \right|^{1/n} = \exp\left(-V^{\lambda_{k+1}}(z) + V^{\lambda_k}(z) + V^{\lambda_{j+1}}(z) - V^{\lambda_j}(z) - 2 \sum_{\ell=j+1}^{k} \omega_\ell^{\vec{\lambda}} \right), \qquad (2.46)$$ {limit_nrooth_

uniformly on compact subsets of $\mathscr{K} \subset \mathbb{C} \setminus (\Delta_j \cup \Delta_{j+1} \cup \Delta_k \cup \Delta_{k+1})$. *If* $j = 0$ *or* $k = m$ *we must also delete from* $\mathbb{C}$ *the zeros of* T *in order that* (2.45) *and* (2.46) *remain valid. For* $k = 1, \dots, m$

$$-V^{\lambda_{k+1}}(z) + 2V^{\lambda_k}(z) - V^{\lambda_{k-1}}(z) - 2\omega_k^{\vec{\lambda}} < 0, \qquad z \in \mathbb{C} \setminus \Delta_k \qquad (2.47)$$ {potentials_in

(*by convention,* $V^{\lambda_0} \equiv V^{\lambda_{m+1}} \equiv 0$). *If* $k > j+1$

$$-V^{\lambda_{k+1}}(z) + V^{\lambda_k}(z) + V^{\lambda_{j+1}}(z) - V^{\lambda_j}(z) - 2 \sum_{\ell=j+1}^{k} \omega_\ell^{\vec{\lambda}} < 0, \qquad z \in \mathbb{C}, \qquad (2.48)$$ {inequality_su

which implies that the sequence $\{\mathscr{A}_{n,j}/\mathscr{A}_{n,k}\}$ *converges to zero with geometric rate on each compact subset of* $\mathbb{C} \setminus (\Delta_k \cup \Delta_{j+1})$ ($\mathbb{C} \setminus (\Delta_k \cup \Delta_{j+1} \cup Z$ *if* $k = m$ *or* $j = 0$).

Proof. Fix $k \in \{1, \dots, m-1\}$ and $j \in \{1, \dots, k-1\}$. Using (2.41) we get

$$\frac{\mathscr{A}_{n,j}(z)}{\mathscr{A}_{n,k}(z)} = \frac{Q_{n,j}(z) Q_{n,k+1}(z)}{Q_{n,j+1}(z) Q_{n,k}(z)} \frac{\int \frac{Q_{m,j+1}^2(z)}{z-x} \frac{\mathscr{H}_{n,j}(x)\, \mathrm{d}\sigma_{j+1}(z)}{Q_{n,j}(z) Q_{n,j+2}(z)}}{\int \frac{Q_{m,k+1}^2(z)}{z-x} \frac{\mathscr{H}_{n,k}(x)\, \mathrm{d}\sigma_{k+1}(z)}{Q_{n,k}(z) Q_{n,k+2}(z)}} \qquad (2.49)$$ {ratio_Anj_int

From (2.6) it follows that uniformly on each compact subset $\mathscr{K} \subset \mathbb{C} \setminus (\Delta_j \cup \Delta_{j+1} \cup \Delta_k \cup \Delta_{k+1})$ we have

$$\lim_{n\to\infty} \left| \frac{Q_{n,j}(z) Q_{n,k+1}(z)}{Q_{n,j+1}(z) Q_{n,k}(z)} \right|^{1/n} = \exp\left(-V^{\lambda_{k+1}}(z) + V^{\lambda_k}(z) + V^{\lambda_{j+1}}(z) - V^{\lambda_j} \right),$$

and taking into account (2.43), from (2.49) we deduce (2.46).

Now, from the principle of descent (see [92, Appendix III]), locally uniformly on $\mathbb{C}$ we have

$$\limsup_{n\to\infty} |Q_{n,j}(z) Q_{n,k+1}|^{1/n} \leq \exp\left(-V^{\lambda_{k+1}}(z) - V^{\lambda_j}(z) \right).$$

Using the lower bound in (2.42) (with j replaced by k) to estimate the integral in the denominator of (2.49) from below and the previous remarks, (2.45) readily follows.

If $k = m$ and $j = 1, \ldots, m-1$ in place of (2.49) we use the representation

$$\left|\frac{\mathscr{A}_{n,j}(z)}{\mathscr{A}_{n,m}(z)}\right| = \left|\frac{Q_{n,j}(z)}{Q_{n,j+1}(z)Q_{n,m}(z)T_n(z)}\int \frac{Q^2_{m,j+1}(z)}{z-x}\frac{\mathscr{H}_{n,j}(x)\,\mathrm{d}\,\sigma_{j+1}(z)}{Q_{n,j}(z)Q_{n,j+2}(z)}\right|,$$

where $T_n \rightrightarrows T$ and then argue as above. If $j = 0$ and $k = 1, \ldots, m$ the treatment is similar.

According to (2.35), for $k = 1, \ldots, m$ we have

$$-V^{\lambda_{k+1}}(z) + 2V^{\lambda_k}(z) - V^{\lambda_{k-1}}(z) - 2\omega_k^{\vec{\lambda}} = 0, \qquad z \in \operatorname{supp}\lambda_k. \tag{2.50}$$ {potentials_eq

Recall that all the measures λ_k are probabilities, hence for each $k = 2, \ldots, m-1$ the function $-V^{\lambda_{k+1}}(z) + 2V^{\lambda_k}(z) - V^{\lambda_{k-1}}(z) - 2\omega_k^{\vec{\lambda}}$ is harmonic at $z = \infty$, and is subharmonic in $\mathbb{C} \setminus \operatorname{supp}\lambda_k$. Using the maximum principle for subharmonic functions we obtain (2.47).

When $k = 1$, the left hand of (2.50) becomes $-V^{\lambda_2}(z) + 2V^{\lambda_1} - 2\omega_1^{\vec{\lambda}}$ which is subharmonic in $\mathbb{C} \setminus \operatorname{supp}\lambda_1$ and also subharmonic at ∞ since

$$\lim_{n\to\infty}\left(-V^{\lambda_2}(z) + 2V^{\lambda_1} - 2\omega_1^{\vec{\lambda}}\right) = -\infty.$$

Therefore, we can also use the maximum principle to derive (2.47). The case $k = m$ is completely analogous to the case $k = 1$.

When $k > j+1$ we can write

$$\begin{aligned} - V^{\lambda_{k+1}}(z) + V^{\lambda_k}(z) + V^{\lambda_{j+1}}(z) - V^{\lambda_j}(z) - 2\sum_{\ell=j+1}^{m}\omega_k^{\vec{\lambda}} = \\ \sum_{\ell=j+1}^{k}\left(-V^{\lambda_{\ell+1}}(z) + 2V^{\lambda_\ell}(z) - V^{\lambda_{\ell-1}}(z) - 2\omega_\ell^{\vec{\lambda}}\right), \end{aligned}$$

and this sum contains at least two terms because $k > j+1$. Each term is less than or equal to zero in all $\mathbb{C}$ and so too the whole sum. To prove that it is strictly negative it is sufficient to show that at each point there is at least one negative term in the sum. Let us assume that there is a $z_0 \in \mathbb{C}$ such that

$$-V^{\lambda_{\ell+1}}(z_0) + 2V^{\lambda_\ell}(z_0) - V^{\lambda_{\ell-1}}(z_0) - 2\omega_\ell^{\vec{\lambda}} = 0, \qquad \ell = j+1, \ldots, k.$$

By what was proved above, this implies that $z_0 \in \cap_{\ell=j+1}^{k}\Delta_\ell$. However, this is impossible because consecutive intervals in a Nikishin system are disjoint. From (2.45) and (2.48) the final statement is deduced.

□

rithmic_asym1} Using Corollary 2.14 we can recover the functions $\widehat{s}_{m-1,j+1}$, $j = 1, \ldots, m-2$.

Corollary 2.15:

Under the assumptions of Theorem 2.4, for each $j = 1, \ldots, m-2$ we have

over_measures} $$\lim_{n\to\infty}\frac{(a_{n,j} - a_{n,m}\widehat{s}_{m,j+1})(z)}{(a_{n,m-1} - a_{n,m}\widehat{s}_{m,m})(z)} = \widehat{s}_{m-1,j+1}(z), \tag{2.51}$$

and

er_measures_0}

$$\lim_{n\to\infty} \frac{\left(a_{n,0} + \sum_{k=1}^{m} (-1)^k a_{n,k} r_k\right) - a_{n,m}\widehat{s}_{m,1}}{(a_{n,m-1} - a_{n,m}\widehat{s}_{m,m})(z)} = \widehat{s}_{m-1,1} \tag{2.52}$$

uniformly on each compact subset of $\mathbb{C} \setminus \cup_{\ell=j+1}^{m} \Delta_\ell$.

Proof. Direct computation or [62, Lemma 2.1] allows to deduce the formula

$$\mathscr{A}_{n,j} + \sum_{k=j+1}^{m-1} (-1)^{k-j} \widehat{s}_{k,j+1} \mathscr{A}_{n,k} = (-1)^j (a_{n,j} - a_{n,m}\widehat{s}_{m,j+1}). \tag{2.53}$$

{identity_for_

The formula holds at all points where both sides are meaningful. Dividing by $\mathscr{A}_{n,m-1}$ we get

$$\frac{\mathscr{A}_{n,j}}{\mathscr{A}_{n,m-1}} + \sum_{k=j+1}^{m-2} (-1)^{k-j} \widehat{s}_{k,j+1} \frac{\mathscr{A}_{n,k}}{\mathscr{A}_{n,m-1}} + (-1)^{m-1-j} \widehat{s}_{m-1,j+1} = (-1)^{m-1+j} \frac{(a_{n,j} - a_{n,m}\widehat{s}_{m,j+1})(z)}{(a_{n,m-1} - a_{n,m}\widehat{s}_{m,m})(z)}.$$

In order to obtain (2.51), it remains to take limit on both sides and make use of the fact that the ratios $\mathscr{A}_{n,k}/\mathscr{A}_{n,m-1}$ uniformly tend to zero on compact subsets of $\mathbb{C} \setminus \cup_{\ell=j+1}^{m} \Delta_\ell$.

To prove (2.52) instead of (2.53) we use the formula

$$\mathscr{A}_{n,0} + \sum_{k=1}^{m-1} (-1)^k \widehat{s}_{k,1} \mathscr{A}_{n,k} = \left(a_{n,0} + \sum_{k=1}^{m} (-1)^k a_{n,k} r_k\right) - a_{n,m}\widehat{s}_{m,1}, \tag{2.54}$$

{nth_root_An0}

which is obtained similarly. Dividing by $\mathscr{A}_{n,m-1}$ and taking limit we complete the proof.

□

We wish to mention that the convergence in (2.51)-(2.52) occurs with geometric rate, as a result of (2.45) and (2.48).

Using Corollary 2.15 we can give explicit expressions for the exact rate of convergence of the limits (2.3). {convergence_s

Theorem 2.16:
Under the assumptions of Theorem 2.3*, for each* $j = 1, \ldots, m-1$*:*

$$\lim_{n\to\infty} \left| \frac{a_{n,j}(z)}{a_{n,m}(z)} - \widehat{s}_{m,j+1}(z) \right|^{1/n} = \exp\left(2V^{\lambda_m}(z) - V^{\lambda_{m-1}}(z) - 2\omega_m^{\vec{\lambda}}\right) \tag{2.55}$$

{geometric_spe

and

$$\limsup_{n\to\infty} \left| \frac{a_{n,0}(z)}{a_{n,m}(z)} - f(z) \right|^{1/n} \le \exp\left(2V^{\lambda_m}(z) - V^{\lambda_{m-1}}(z) - 2\omega_m^{\vec{\lambda}}\right) \tag{2.56}$$

{geometric_spe

uniformly on each compact subset $\mathscr{K} \subset \mathbb{C} \setminus (\cup_{\ell=j+1}^{m} \Delta_\ell \cup Z)$.

Proof. Our starting point is (2.53), but now we divide it by $\mathscr{A}_{n,m} = (-1)^m a_{n,m}$. We get

$$\left| \frac{\mathscr{A}_{n,j}}{\mathscr{A}_{n,m}} + \sum_{k=j+1}^{m-1} (-1)^{k-j} \widehat{s}_{k,j+1} \frac{\mathscr{A}_{n,k}}{\mathscr{A}_{n,m}} \right| = \left| \frac{a_{n,j}}{a_{n,m}} - \widehat{s}_{m,j+1} \right|,$$

which is equivalent to

$$\left|\frac{\mathscr{A}_{n,m-1}}{\mathscr{A}_{n,m}}\right|\left|\frac{\mathscr{A}_{n,j}}{\mathscr{A}_{n,m-1}}+\sum_{k=j+1}^{m-1}(-1)^{k-j}\widehat{s}_{k,j+1}\frac{\mathscr{A}_{n,k}}{\mathscr{A}_{n,m-1}}\right|=\left|\frac{a_{n,j}}{a_{n,m}}-\widehat{s}_{m,j+1}\right|.$$

Now, $\widehat{s}_{m-1,j+1}(z)\neq 0$, $z\in\mathbb{C}\setminus\Delta_{m-1}$; consequently, $\lim_{n\to\infty}|\widehat{s}_{m-1,j+1}(z)|^{1/n}=1$ uniformly on compact subsets of $\mathbb{C}\setminus\Delta_{m-1}$. Therefore,

$$\lim_{n\to\infty}\left|\frac{(-1)^j\mathscr{A}_{n,j}}{\mathscr{A}_{n,m-1}}+\sum_{k=j+1}^{m-1}(-1)^k\widehat{s}_{k,j+1}\frac{\mathscr{A}_{n,k}}{\mathscr{A}_{n,m-1}}\right|^{1/n}=1,$$

uniformly on compact subsets of $\mathbb{C}\setminus\cup_{\ell=j+1}^{m}\Delta_\ell$. On the other hand, from (2.46)

$$\lim_{n\to\infty}\left|\frac{\mathscr{A}_{n,m-1}}{\mathscr{A}_{n,m}}\right|^{1/n}=\exp\left(2V^{\lambda_m}(z)-V^{\lambda_{m-1}}(z)-2\omega_m^{\vec{\lambda}}\right),$$

uniformly on compact subsets of $\mathbb{C}\setminus(\Delta_{m-1}\cup\Delta_m\cup Z)$. These relations together imply (2.55).

To estimate the speed of convergence of the quotients $a_{n,0}/a_{n,m}$ we use equality (2.54). Dividing it by $\mathscr{A}_{n,m}$ we get

$$\left|\frac{\mathscr{A}_{n,m-1}}{\mathscr{A}_{n,m}}\right|\left|\frac{\mathscr{A}_{n,0}}{\mathscr{A}_{n,m-1}}+\sum_{k=1}^{m-1}(-1)^k\widehat{s}_{k,1}\frac{\mathscr{A}_{n,k}}{\mathscr{A}_{n,m-1}}\right|=\left|\frac{a_{n,0}}{a_{n,m}}+\sum_{k=1}^{m}(-1)^k\frac{a_{n,k}}{a_{n,m}}r_k-\widehat{s}_{m,1}\right|.$$

Arguing as above this equality implies that

etric_speed-1}

$$\lim_{n\to\infty}\left|\frac{a_{n,0}(z)}{a_{n,m}(z)}+\sum_{k=1}^{m}(-1)^k\frac{a_{n,k}(z)}{a_{n,m}(z)}r_k(z)-\widehat{s}_{m,1}(z)\right|^{1/n}=\exp\left(2V^{\lambda_m}(z)-V^{\lambda_{m-1}}(z)-2\omega_m^{\vec{\lambda}}\right), \tag{2.57}$$

uniformly on each compact subset $\mathscr{K}\subset\mathbb{C}\setminus(\cup_{\ell=1}^{m}\Delta_\ell\cup Z)$.

On the other hand, using the formula for f, we obtain

$$\left|\frac{a_{n,0}}{a_{n,m}}+\sum_{k=1}^{m}(-1)^k\frac{a_{n,k}}{a_{n,m}}r_k-\widehat{s}_{m,1}\right|=\left|\left(\frac{a_{n,0}}{a_{n,m}}-f\right)+\sum_{k=1}^{m-1}(-1)^k\left(\frac{a_{n,k}}{a_{n,m}}-\widehat{s}_{m,k+1}\right)r_k\right|\geq$$
$$\left|\frac{a_{n,0}}{a_{n,m}}-f\right|-\sum_{k=1}^{m-1}\left|\left(\frac{a_{n,k}}{a_{n,m}}-\widehat{s}_{m,k+1}\right)r_k\right|;$$

that is,

$$\left|\frac{a_{n,0}(z)}{a_{n,m}(z)}-f(z)\right|\leq$$
$$\left|\frac{a_{n,0}(z)}{a_{n,m}(z)}+\sum_{k=1}^{m}(-1)^k\frac{a_{n,k}(z)}{a_{n,m}(z)}r_k(z)-\widehat{s}_{m,1}(z)\right|+\sum_{k=1}^{m-1}\left|\left(\frac{a_{n,k}(z)}{a_{n,m}(z)}-\widehat{s}_{m,k+1}(z)\right)r_k(z)\right|$$

This inequality, together with (2.55) and (2.57), implies (2.56).

□

CHAPTER

3 A generalization of multi-level Hermite-Padé polynomials

{ch:gen}

In the present chapter we are going to study Markov and Stieltjes-type theorems, but for an extension of Problem 5 introduced very recently in [66]. There, the author allowed that the interpolation conditions at infinity for the linear forms $\mathscr{A}_{n,j}$, $j = 0, 1, \ldots, m-1$ vary in certain range. With these variation he was able to prove the convergence of the method and the logarithmic asymptotic of the corresponding Hermite-Padé polynomials.

Here, we extend a little further the results obtained in [66], permitting the Nikishin systems to be generated by a wider class of measures. Furthermore, as we stated in the introduction, we study the ratio asymptotic of the associated multi-orthogonal polynomials, as well as for the corresponding multi-level Hermite-Padé polynomials.

3.1 Statement of the problem

In the sequel, unless stated differently, we use the definition given in [66, Problem A].

Let $(\mathbb{Z}_+^m)^*$ be the set of all m-dimensional vectors with non-negative integer components not identically equal to zero. For $\vec{n} = (n_1, \ldots, n_m) \in (\mathbb{Z}_+^m)^*$ we define $|\vec{n}| = n_1 + \cdots + n_m$. {df:HP:ly}

Definition 3.1:
Consider the Nikishin system $\mathscr{N}(\sigma_1, \ldots, \sigma_m)$ and $\vec{n} = (n_1, \ldots, n_m) \in (\mathbb{Z}_+^m)^*$. There exist polynomials $a_{\vec{n},0}, a_{\vec{n},1}, \ldots, a_{\vec{n},m}$, where $\deg a_{\vec{n},j} \leq |\vec{n}|-1$, $j = 0, 1, \ldots, m-1$, and $\deg a_{\vec{n},m} \leq |\vec{n}|$, not all identically equal to zero, called multi-level (ML) Hermite-Padé polynomials of $\mathscr{N}(\sigma_1, \ldots, \sigma_m)$ with respect to $\vec{n}$, that verify

$$\mathscr{A}_{\vec{n},j}(z) := \left((-1)^j a_{\vec{n},j} + \sum_{k=j+1}^{m} (-1)^k a_{\vec{n},k} \widehat{s}_{j+1,k}\right)(z) = \mathscr{O}\left(\frac{1}{z^{n_{j+1}+1}}\right), \qquad z \to \infty, \tag{3.1}$$ {df:HP:ly:1}

where $j = 0, \ldots, m-1$ (the asymptotic expansion of $\mathscr{A}_{\vec{n},j}$ at ∞ begins with $z^{-n_{j+1}-1}$, or higher). For completeness, set $\mathscr{A}_{\vec{n},m} := (-1)^m a_{\vec{n},m}$.

We warn the reader that with our terminology in [66, Problem A] the ML Hermite-Padé polynomials were defined with respect to the system $\mathscr{N}(\sigma_m, \ldots, \sigma_1)$.

When $m = 1$ the definition reduces to that of classical Padé approximation, which plays a central role in the solution of the inverse spectral problem for a discrete string with Dirichlet boundary condition, see [10, 93]. When $m = 2$ and $\vec{n} = (n, 0)$ definition 3.1 reduces to the Hermite-Padé approximation problem used in the solution of the inverse spectral problem for the discrete cubic string, see [65]. For an arbitrary m and $\vec{n} = (n, 0, \ldots, 0)$, one obtains the original definition of ML Hermite-Padé polynomials given in [62].

This scheme of approximation keeps many properties of the one originally introduced in [62]. These are: the interpolation conditions involve all the Nikishin systems of the "inner levels", i.e. $\mathcal{N}(\sigma_1, \ldots, \sigma_m)$, $\mathcal{N}(\sigma_2, \ldots, \sigma_m)$, ..., $\mathcal{N}(\sigma_m) = (s_{m,m})$; finding the polynomials $a_{\vec{n},0}, a_{\vec{n},1}, \ldots, a_{\vec{n},m}$ reduces to solving a homogeneous linear system of $|\vec{n}|(m+1)$ equations on $|\vec{n}|(m+1)+1$ unknowns, corresponding with the coefficients of the polynomials. Consequently, the system of equations has a non trivial solution.

Following Mahler's terminology [67], a multi-index $\vec{n} \in (\mathbb{Z}_+^m)^*$ is said to be normal if $\deg a_{\vec{n},j} = |\vec{n}|-1$, $j = 0, \ldots, m-1$, and $\deg a_{\vec{n},m} = |\vec{n}|$. The system of functions is said to be perfect when all the multi-indices are normal. In [66, Theorem 1.1], it was proved that the Nikishin system of functions is perfect for this approximation problem. Normality implies that the vector $(a_{\vec{n},0}, \ldots, a_{\vec{n},m})$ is uniquely determined up to a multiplicative factor. In the sequel, we normalize this vector so that $a_{\vec{n},m}$ has leading coefficient equal to one.

A sequence of multi-indices $\Lambda \subset (\mathbb{Z}_+^m)^*$ is called a ray sequence when $\lim_{\vec{n}\in\Lambda} n_j/|\vec{n}|$ exists for all $j = 1, \ldots, m$. When the Δ_j are bounded non-intersecting intervals, and $\sigma_j' \neq 0$, a.e. in Δ_j, $j = 1, \ldots, m$, in [66, Theorem 1.2] the logarithmic asymptotic of ray sequences of ML polynomials was obtained. Using that result, it was also proved [66, Proposition 1.2] that

{lim:markov} $$\lim_{\vec{n}\in\Lambda} \frac{a_{\vec{n},j}}{a_{\vec{n},m}} = \widehat{s}_{m,j+1}, \qquad j = 0, \ldots, m-1 \tag{3.2}$$

uniformly on each compact subset of $\overline{\mathbb{C}} \setminus \Delta_m$ (with geometric rate). Notice that the limits belong to the Nikishin system of functions corresponding to $\mathcal{N}(\sigma_m, \ldots, \sigma_1)$. It should be said that the proof of these results given in [66] may be adapted to the case when the measures $\sigma_j \in$ **Reg** (see Definition 1.10).

We provide a convergence result such as (3.2) in which the intervals Δ_j may be unbounded and consecutive intervals can have a common end point. This situation appears in [15] in relation with the study of the two matrix model.

When the Δ_j are bounded non-intersecting intervals, and $\sigma_j' \neq 0$ a.e. in Δ_j, we also give a result about the asymptotic of sequences of ratios of polynomials $a_{\vec{n},j}$ corresponding to consecutive multi-indices which resembles E.A. Rakhmanov's celebrated theorem on the ratio asymptotic of standard orthogonal polynomials (see [83, 84, 85, 70]). With the original definition of ML Hermite-Padé polynomials introduced in [62] the ratio asymptotic was obtained in [32].

3.1.1 Statement of the main results

Recall that a measure $s \in \mathscr{M}(\Delta)$ is said to satisfy Carleman's condition when the sequence of its moments $\{c_n\}_{n\geq 0}$ verifies

$$\sum_{n=0}^{\infty} |c_n|^{-1/2n} = \infty.$$

When Δ is either $\mathbb{R}_+$ of $\mathbb{R}_-$, this condition implies that there is only one measure whose collection of moments is $\{c_n\}_{n\geq 0}$. In turn, if the moment problem for s is determinate then the sequence of diagonal Padé approximants converges to $\widehat{s}$ on each compact subset of $\mathbb{C} \setminus \Delta$. We prove the following Carleman-Stieltjes type theorem in the context of ML Hermite-Padé approximants. {th:ML:Ly:Conv

Theorem 3.2:
Let $\Lambda \subset (\mathbb{Z}_+^m)^*$ be an infinite sequence of distinct multi-indices for which there exist $\ell \in \{0, \ldots, m-2\}$ and a (fixed) non-negative integer N such that $n_{j+1} \leq n_j + N$ for all $\ell+1 \leq j \leq m-1$ and $\vec{n} \in \Lambda$. Consider the sequence of vector polynomials $(a_{\vec{n},0}, \ldots, a_{\vec{n},m})_{\vec{n}\in\Lambda}$ associated with $\mathcal{N}(\sigma_1, \ldots, \sigma_m)$. For $j = \ell, \ldots, m-2$ the polynomial $a_{\vec{n},j}$ has at least $|\vec{n}| - 2m - N\frac{m(m+1)}{2}$ sign changes in $\mathring{\Delta}_m$. The polynomials $a_{\vec{n},m-1}$ and $a_{\vec{n},m}$ have, respectively, $|\vec{n}| - 1$ and $|\vec{n}|$ interlacing simple zeros in $\mathring{\Delta}_m$. Suppose that either the sequence of moments of σ_m satisfies Carleman's condition or Δ_{m-1} is a bounded interval which does not intersect Δ_m; then (3.2) holds uniformly on each compact subset of $\overline{\mathbb{C}} \setminus \Delta_m$ for $j = \ell, \ldots, m-1$. If $\Lambda \subset (\mathbb{Z}_+^m)^*$ is an arbitrary infinite sequence of distinct multi-indices and σ_m satisfies Carleman's condition or, Δ_{m-1} is a bounded interval which does not intersect Δ_m and $\lim_{\vec{n}\in\Lambda}(n_1 + \cdots + n_{m-1}) = \infty$, then (3.2) takes place for $j = m-1$.

If Λ is a sequence of distinct multi-indices whose components are decreasing, the (first) condition on Λ in Theorem 3.2 is verified with $\ell = 0$ and $N = 0$. More precise information regarding the zeros of the polynomials $a_{\vec{n},j}$, $j = 0, \ldots, m-2$ will be given in Section 3.2.

Let $\vec{n} \in \mathbb{Z}_+^m$ and $l \in \{1, \ldots, m\}$. Define

$$\vec{n}^l := (n_1, \ldots, n_l + 1, \ldots, n_m),$$

the multi-index obtained adding 1 to the l-th component of $\vec{n}$. {th:ratiom}

Theorem 3.3:
Consider the Nikishin system $\mathcal{N}(\sigma_1, \ldots, \sigma_m)$ where the Δ_k, $k = 1, \ldots, m$, are bounded, disjoint intervals, and $\sigma_k' \neq 0$ a.e. in Δ_k. Let $\Lambda \subset (\mathbb{Z}_+^m)^*$ be an infinite sequence of distinct multi-indices for which there exists a non-negative integer N such that $n_{j+1} \leq n_j + N$ for all $1 \leq j \leq m-1$ and $\vec{n} \in \Lambda$. Then for $k = 0, \ldots, m$

$$\lim_{\vec{n}\in\Lambda} \frac{a_{\vec{n}^l,k}(z)}{a_{\vec{n},k}(z)} = \frac{\psi_m^{(l)}(z)}{(\psi_m^{(l)})'(\infty)}, \tag{3.3}$$ {left*}

uniformly on each compact subset of $\mathbb{C} \setminus \Delta_m$, where $\psi_m^{(l)} \in \mathbf{H}(\overline{\mathbb{C}} \setminus \Delta_m)$ is defined in (3.17).

3.2 Convergence of the ML Hermite-Padé approximants

{sec:conv}

First, we study the location of the zeros of the linear forms $\mathscr{A}_{\vec{n},j}$, $j = 0,\dots,m$. Given $\vec{n} = (n_1\dots,n_m) \in \mathbb{Z}_+^m$, set

{lm:zeros}

$$\eta_{\vec{n},j} := n_1 + \cdots + n_j.$$

Lemma 3.4:

The form $\mathscr{A}_{\vec{n},0}$ has no zero in $\mathbb{C}\setminus\Delta_1$. For $j = 1,\dots,m$, $\mathscr{A}_{\vec{n},j}$ has exactly $\eta_{\vec{n},j}$ zeros in $\mathbb{C}\setminus\Delta_{j+1}$, ($\Delta_{m+1} = \emptyset$), they are all simple and lie in $\mathring{\Delta}_j$. If $w_{\vec{n},j}$, $j = 1,\dots,m-1$, denotes the monic polynomial whose roots are the simple zeros which $\mathscr{A}_{\vec{n},j}$ has in $\mathring{\Delta}_j$ then

{forma*}

$$\frac{\mathscr{A}_{\vec{n},j}}{w_{\vec{n},j}} = \mathcal{O}\left(\frac{1}{z^{\eta_{\vec{n},j+1}+1}}\right) \in \mathbf{H}(\mathbb{C}\setminus\Delta_{j+1}), \qquad z\to\infty. \tag{3.4}$$

For each $j = 0,\dots,m-1$ the order of interpolation at infinity prescribed in (3.1) *is exact.*

Proof. From (3.1) for $j = 0$, using Lemma 2.5 with $w \equiv 1$, we obtain

$$\int x^\nu \mathscr{A}_{\vec{n},1}(x)\,\mathrm{d}\,\sigma_1(x) = 0, \qquad \nu = 0,1,\dots,n_1-1.$$

Therefore, $\mathscr{A}_{\vec{n},1}$ has at least n_1 sign changes in $\mathring{\Delta}_1$.

Let $w_{\vec{n},1}$ be a polynomial whose roots lie in $\mathbb{C}\setminus\Delta_2$ and contain all the points where $\mathscr{A}_{\vec{n},1}$ changes sign in $\mathring{\Delta}_1$. Then, $\deg w_{\vec{n},1} \geq n_1$ and taking into account (3.1) for $j = 1$, we obtain

$$\frac{\mathscr{A}_{\vec{n},1}(z)}{w_{\vec{n},1}(z)} = \mathcal{O}\left(\frac{1}{z^{\eta_{\vec{n},2}+1}}\right) \in \mathbf{H}(\mathbb{C}\setminus\Delta_2).$$

Notice that $\mathscr{A}_{\vec{n},1}$ and $w_{\vec{n},1}$ satisfy the hypothesis of Lemma 2.5, so

$$\int x^\nu \mathscr{A}_{\vec{n},2}(x)\frac{\mathrm{d}\,\sigma_2(x)}{w_{\vec{n},1}(x)} = 0, \qquad \nu = 0,1,\dots,n_1+n_2-1.$$

This implies that $\mathscr{A}_{\vec{n},2}$ has, at least, n_1+n_2 sign changes in $\mathring{\Delta}_2$.

Let $w_{\vec{n},2}$ be a polynomial whose roots lie in $\mathbb{C}\setminus\Delta_3$ and contain all the points where $\mathscr{A}_{\vec{n},2}$ changes sign in $\mathring{\Delta}_2$. Then, $\deg w_{\vec{n},2} \geq n_1+n_2$ and taking into account (3.1) for $j = 2$, we obtain

$$\frac{\mathscr{A}_{\vec{n},2}(z)}{w_{\vec{n},2}(z)} = \mathcal{O}\left(\frac{1}{z^{\eta_{\vec{n},3}+1}}\right) \in \mathbf{H}(\mathbb{C}\setminus\Delta_3).$$

We have deduced analogous conclusions for $\mathscr{A}_{\vec{n},2}$ as we had for $\mathscr{A}_{\vec{n},1}$.

We can repeat these arguments inductively and obtain that for each $j = 0,\dots,m-1$ there exists a polynomial $w_{\vec{n},j}$, $\deg w_{\vec{n},j} \geq n_1+\cdots+n_j = \eta_{\vec{n},j}$ ($w_{\vec{n},0}\equiv 1$) whose roots lie in $\mathbb{C}\setminus\Delta_{j+1}$ and contain all the points where $\mathscr{A}_{\vec{n},j}$ changes sign in $\mathring{\Delta}_j$ and (3.4) takes place.

For $j = m-1$, we get

{HP_s_mm3}

$$\frac{a_{\vec{n},m}\widehat{s}_{m,m} - a_{\vec{n},m-1}}{w_{\vec{n},m-1}}(z) = \mathcal{O}\left(\frac{1}{z^{\eta_{\vec{n},m}+1}}\right) \in \mathbf{H}(\mathbb{C}\setminus\Delta_m) \tag{3.5}$$

and using again Lemma 2.5

$$\int x^{\nu} a_{\vec{n},m}(x) \frac{\mathrm{d}\, s_{m,m}(x)}{w_{\vec{n},m-1}(x)} = 0, \qquad \nu = 0, 1, \ldots, |\vec{n}| - 1.$$

This implies that $a_{\vec{n},m}$ has at least $|\vec{n}|$ sign changes in $\mathring{\Delta}_m$. Since $\deg a_{\vec{n},m} \le |\vec{n}|$ we get that $a_{\vec{n},m}$ is either identically equal to zero or it has exactly $|\vec{n}|$ simple zeros, all in $\mathring{\Delta}_m$. The first situation cannot occur since from (3.1) it would follow that $(a_{\vec{n},0} \ldots, a_{\vec{n},m}) \equiv \vec{0}$. So, only the second statement is possible.

Notice that if $\mathscr{A}_{\vec{n},0}$ has a zero in $\mathbb{C} \setminus \Delta_1$, or for some $j = 1, \ldots, m-1$, $\mathscr{A}_{\vec{n},j}$ has more than $\eta_{\vec{n},j}$ zeros in $\mathbb{C} \setminus \Delta_{j+1}$, we can get an extra order of interpolation in (3.5). This also occurs if for some $j = 0, \ldots, m-1$ the order of interpolation at ∞ in (3.1) is higher than the one imposed. This entails one more orthogonality relation for $a_{\vec{n},m}$ implying that this polynomials is identically equal to zero which is not possible. The statements of the lemma readily follow.

□

In order to prove Theorem 3.2, we need relations similar to (3.5) for $a_{\vec{n},m}\widehat{s}_{m,j+1} - a_{\vec{n},j}$, $j = 0, \ldots, m-1$. For this purpose, some transformations involving reciprocals and ratios of Cauchy transforms of measures will be employed, and were introduce in Chapter 2, see (2.11) and (2.12).

We also state a formula which connects forms of different levels of Nikishin systems. A proof appears in [62, Lemma 2.1]. Consider the linear forms with polynomial coefficients

$$\mathscr{L}_j := a_j + \sum_{k=j+1}^{m} a_k \widehat{s}_{j+1,k}, \qquad j = 0, \ldots, m-1, \qquad \mathscr{L}_m := a_m$$

where a_j are arbitrary polynomials. {lm:levels}

Lemma 3.5:
Let $(s_{1,1}, \ldots, s_{1,m}) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ be given. Then, for each $j = 0, \ldots, m-2$, and $r = j+1, \ldots, m-1$

$$\mathscr{L}_j + \sum_{k=j+1}^{r} \widehat{s}_{k,j+1} \mathscr{L}_k = a_j + (-1)^{r-j} \sum_{k=r+1}^{m} a_k \widehat{\langle s_{r+1,k}, s_{r,j+1} \rangle}. \tag{3.6}$$ {levels}

Given $\vec{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$, set

$$\chi_{\vec{n},j,k} := \min\{n_j + 1, n_{j+1} + 2, \ldots, n_k + 2\}, \quad j < k.$$

We are ready to prove {lm:asym_MP}

Lemma 3.6:
Given $\vec{n} \in (\mathbb{Z}_+^m)^*$ let $a_{\vec{n},0}, a_{\vec{n},1}, \ldots, a_{\vec{n},m}$ be the Hermite-Padé polynomials associated with the Nikishin system $\mathcal{N}(\sigma_1, \ldots, \sigma_m)$ such that (3.1) *holds. Then for each $j = 0, \ldots, m-2$*

$$\frac{a_{\vec{n},j} - a_{\vec{n},m}\widehat{s}_{m,j+1}}{w^*_{\vec{n},j}}(z) = \mathcal{O}\left(z^{-\left(\eta_{\vec{n},j+1} + \sum_{k=1}^{m-j-1} \chi_{\vec{n},j+1,j+k+1} - 2m + 2j + 3\right)}\right) \in \mathbf{H}(\mathbb{C} \setminus \Delta_m), \quad z \to \infty, \tag{3.7}$$ {orden}

where $w^*_{\vec{n},j}$ is a monic polynomial with real coefficients of degree $\sum_{k=1}^{m-j-2} \chi_{\vec{n},j+1,j+k+1} + \eta_{\vec{n},j+1} - 2m + 2j + 3$ (the sum is empty when $j = m-2$). The polynomial $a_{\vec{n},j}$, $j = 0, \ldots, m-2$, has at least $\sum_{k=1}^{m-j-1} \chi_{\vec{n},j+1,j+k+1} + \eta_{\vec{n},j+1} - 2m + 2j + 1$ sign changes in $\mathring{\Delta}_m$.

Proof. Fix $j \in \{0, \ldots, m-2\}$, using (2.11) and (2.12), we have

$$\frac{\mathscr{A}_{\vec{n},j}}{\widehat{\sigma}_{j+1}} = \left((-1)^j \ell_{j+1} a_{\vec{n},j} + \sum_{k=j+1}^{m} (-1)^k \frac{|s_{j+1,k}|}{|\sigma_{j+1}|} a_{\vec{n},k} \right) + (-1)^j a_{\vec{n},j} \widehat{\tau}_{j+1} - \sum_{k=j+2}^{m} (-1)^k a_{\vec{n},k} \widehat{\langle \tau_{j+1}, \langle s_{j+2,k}, \sigma_{j+1} \rangle \rangle}.$$

The quotient $\frac{\mathscr{A}_{\vec{n},j}}{\widehat{\sigma}_{j+1}}$ has the same structure as $\mathscr{A}$ in Lemma 2.5. Moreover, from (3.4),

$$\frac{\mathscr{A}_{\vec{n},j}(z)}{(\widehat{\sigma}_{j+1} w_{\vec{n},j})(z)} = \mathscr{O}\left(\frac{1}{z^{\eta_{\vec{n},j+1}}} \right) \in \mathbf{H}(\mathbb{C} \setminus \Delta_{j+1})$$

and, as a consequence of (2.10), for $\nu = 0, \ldots, \eta_{\vec{n},j+1} - 2$, we obtain the orthogonality relations

$$0 = \int_{\Delta_{j+1}} x^\nu \left((-1)^j a_{\vec{n},j} - \sum_{k=j+2}^{m} (-1)^k a_{\vec{n},k} \widehat{\langle s_{j+2,k}, \sigma_{j+1} \rangle} \right)(x) \frac{\mathrm{d}\,\tau_{j+1}(x)}{w_{\vec{n},j}(x)}.$$

Therefore, the expression in parentheses under the integral sign has at least $\eta_{\vec{n},j+1} - 1$ sign changes in $\mathring{\Delta}_{j+1}$. Thus, there exists a polynomial $w_{\vec{n},j,1}$ of degree $\eta_{\vec{n},j+1} - 1$ whose zeros are simple and lie in $\mathring{\Delta}_{j+1}$ such that

$$\frac{1}{w_{\vec{n},j,1}} \left((-1)^j a_{\vec{n},j} - \sum_{k=j+2}^{m} (-1)^k a_{\vec{n},k} \widehat{\langle s_{j+2,k}, \sigma_{j+1} \rangle} \right) \in \mathbf{H}(\mathbb{C} \setminus \Delta_{j+2}).$$

We can use Lemma 3.5 choosing $r = j+1$ and obtain

$$\mathscr{A}_{\vec{n},j} - \widehat{s}_{j+1,j+1} \mathscr{A}_{\vec{n},j+1} = (-1)^j a_{\vec{n},j} - \sum_{k=j+2}^{m} (-1)^k a_{\vec{n},k} \widehat{\langle s_{j+2,k}, \sigma_{j+1} \rangle}.$$

From (3.1) we know that $\mathscr{A}_{\vec{n},j} - \widehat{s}_{j+1,j+1} \mathscr{A}_{\vec{n},j+1}$ is $\mathscr{O}\left(z^{-\min\{n_{j+1}+1, n_{j+2}+2\}}\right)$, $z \to \infty$. Hence,

$$\frac{1}{w_{\vec{n},j,1}(z)} \left((-1)^j a_{\vec{n},j} - \sum_{k=j+2}^{m} (-1)^k a_{\vec{n},k} \widehat{\langle s_{j+2,k}, \sigma_{j+1} \rangle} \right)(z) = \mathscr{O}\left(\frac{1}{z^{\chi_{\vec{n},j+1,j+2} + \eta_{\vec{n},j+1} - 1}} \right).$$

Notice that if $j = m-2$ we obtain

$$\frac{a_{\vec{n},m-2} - a_{\vec{n},m} \widehat{s}_{m,m-1}}{w_{\vec{n},m-2,1}}(z) = \mathscr{O}\left(\frac{1}{z^{\chi_{\vec{n},m-1,m} + \eta_{\vec{n},m-1} - 1}} \right)$$

which is (3.7) for this value of j taking $w^*_{\vec{n},m-2} = w_{\vec{n},m-2,1}$.

Using the identity $\langle s_{j+2,k}, s_{j+1,j+1} \rangle = \langle s_{j+2,j+1}, s_{j+3,k} \rangle$ for $k = j+3, \ldots, m$, we deduce

{Prep_Elim3}

$$\begin{aligned} &(-1)^j a_{\vec{n},j} - \sum_{k=j+2}^{m} (-1)^k a_{\vec{n},k} \widehat{\langle s_{j+2,k}, \sigma_{j+1} \rangle} \\ &\qquad = (-1)^j a_{\vec{n},j} - (-1)^{j+2} a_{\vec{n},j+2} \widehat{s}_{j+2,j+1} - \sum_{k=j+3}^{m} (-1)^k a_{\vec{n},k} \widehat{\langle s_{j+2,j+1}, s_{j+3,k} \rangle}. \end{aligned} \tag{3.8}$$

We wish to eliminate the term with $\widehat{s}_{j+2,j+1}$ from the right hand side of (3.8); therefore, we divide both sides of (3.8) by $\widehat{s}_{j+2,j+1}$ and use again (2.11) and (2.12). The right hand side becomes

$$\left((-1)^j a_{\vec{n},j}\ell_{j+2,j+1} - (-1)^{j+2}a_{\vec{n},j+2} - \sum_{k=j+3}^{m}(-1)^k \frac{|\langle s_{j+2,j+1}, s_{j+3,k}\rangle|}{|s_{j+2,j+1}|}a_{\vec{n},k}\right)+$$
$$(-1)^j a_{\vec{n},j}\widehat{\tau}_{j+2,j+1} + \sum_{k=j+3}^{m}(-1)^k a_{\vec{n},k}\langle \tau_{j+2,j+1}, \widehat{\langle s_{j+3,k}, s_{j+2,j+1}\rangle\rangle}$$

which is a linear form like $\mathscr{A}$ in Lemma 2.5, and

$$\frac{1}{(w_{\vec{n},j,1}\widehat{s}_{j+2,j+1})(z)}\left((-1)^j a_{\vec{n},j} - \sum_{k=j+2}^{m}(-1)^k a_{\vec{n},k}\widehat{\langle s_{j+2,k}, \sigma_{j+1}\rangle}\right)(z) =$$
$$\mathcal{O}\left(\frac{1}{z^{\chi_{\vec{n},j+1,j+2}+\eta_{\vec{n},j+1}-2}}\right) \in \mathbf{H}(\mathbb{C}\setminus\Delta_{j+2}).$$

Therefore, for $\nu = 0,1,\ldots,\chi_{\vec{n},j+1,j+2}+\eta_{\vec{n},j+1}-4$,

$$\int x^\nu \left((-1)^j a_{\vec{n},j} + \sum_{k=j+3}^{m}(-1)^k a_{\vec{n},k}\widehat{\langle s_{j+3,k}, s_{j+2,j+1}\rangle}\right)(x)\frac{\mathrm{d}\,\tau_{j+2,j+1}(x)}{w_{\vec{n},j.1}(x)} = 0.$$

So, the expression in parenthesis under the integral sign has at least $\chi_{\vec{n},j+1,j+2}+\eta_{\vec{n},j+1}-3$ sign changes in $\mathring{\Delta}_{j+2}$, and we can guarantee the existence of a polynomial $w_{\vec{n},j,2}$, $\deg w_{\vec{n},j,2} = \chi_{\vec{n},j+1,j+2}+\eta_{\vec{n},j+1}-3$, with simple zeros located inside Δ_{j+2} such that

$$\frac{1}{w_{\vec{n},j,2}}\left((-1)^j a_{\vec{n},j} + \sum_{k=j+3}^{m}(-1)^k a_{\vec{n},k}\widehat{\langle s_{j+3,k}, s_{j+2,j+1}\rangle}\right) \in \mathbf{H}(\mathbb{C}\setminus\Delta_{j+3}).$$

On the other hand, using Lemma 3.5 with $r = j+2$ and the definition of ML Hermite-Padé approximant, we get

$$\mathscr{A}_{\vec{n},j} - \widehat{s}_{j+1,j+1}\mathscr{A}_{\vec{n},j+1} + \widehat{s}_{j+1,j+1}\mathscr{A}_{\vec{n},j+2} =$$
$$(-1)^j a_{\vec{n},j} + \sum_{k=j+3}^{m}(-1)^k a_{\vec{n},k}\widehat{\langle s_{j+3,k}, s_{j+2,j+1}\rangle} \in \mathcal{O}\left(\frac{1}{z^{\chi_{\vec{n},j+1,j+3}}}\right).$$

Thus,

$$\frac{1}{w_{\vec{n},j,2}}\left((-1)^j a_{\vec{n},j} + \sum_{k=j+3}^{m}(-1)^k a_{\vec{n},k}\widehat{\langle s_{j+3,k}, s_{j+2,j+1}\rangle}\right) \in \mathcal{O}\left(\frac{1}{z^{\chi_{\vec{n},j+1,j+3}+\chi_{\vec{n},j+1,j+2}+\eta_{\vec{n},j+1}-3}}\right).$$

In particular, if $j = m-3$, we get

$$\frac{(a_{\vec{n},m-3} - a_{n,m}\widehat{s}_{m,m-2})(z)}{w_{\vec{n},m-3,2}(z)} = \mathcal{O}\left(\frac{1}{z^{\chi_{\vec{n},m-2,m}+\chi_{\vec{n},m-2,m-1}+\eta_{m-2}-3}}\right),$$

which gives us (3.7) with $w^*_{\vec{n},m-3} = w_{\vec{n},m-3,2}$ when $j = m-3$.

This process can be continued inductively, and after $m-j-1$ reductions we guarantee the existence of a polynomial $w_{\vec{n},j,m-j-1}$ of degree $\sum_{k=1}^{m-j-2}\chi_{\vec{n},j+1,j+k+1}+\eta_{\vec{n},j+1}-2m+2j+3$ with simple zeros in $\mathring{\Delta}_{m-1}$ such that

$$\frac{a_{\vec{n},j}-a_{\vec{n},m}\widehat{s}_{m,j+1}}{w_{\vec{n},j,m-j-1}}(z)=\mathcal{O}\left(z^{-\left(\sum_{k=1}^{m-j-1}\chi_{\vec{n},j+1,j+k+1}+\eta_{\vec{n},j+1}-2m+2j+3\right)}\right)\in\mathbf{H}(\mathbb{C}\setminus\Delta_m),\quad z\to\infty,$$

which allows us to deduce (3.7) taking $w_{\vec{n},j}^*=w_{\vec{n},j,m-j-1}$.

As an immediate consequence we have

$$\frac{a_{\vec{n},j}-a_{\vec{n},m}\widehat{s}_{m,j+1}}{\widehat{s}_{m,j+1}w_{\vec{n},j}^*}(z)=\mathcal{O}\left(z^{-\left(\sum_{k=1}^{m-j-1}\chi_{\vec{n},j+1,j+k+1}+\eta_{\vec{n},j+1}-2m+2j+2\right)}\right)\in\mathbf{H}(\mathbb{C}\setminus\Delta_m),\qquad z\to\infty,$$

but

$$\frac{a_{\vec{n},j}-a_{\vec{n},m}\widehat{s}_{m,j+1}}{\widehat{s}_{m,j+1}}=a_{\vec{n},j}\widehat{\tau}_{m,j+1}-(a_{\vec{n},m}-\ell_{m,j+1}a_{\vec{n},j}).$$

Hence,

$$\int x^\nu a_{\vec{n},j}(x)\frac{\mathrm{d}\,\tau_{m,j+1}(x)}{w_{\vec{n},j}^*(x)}=0,\qquad \nu=0,1,\ldots,\sum_{k=1}^{m-j-1}\chi_{\vec{n},j,j+k}+\eta_{\vec{n},j}-2m+2j,$$

and the polynomial $a_{\vec{n},j}$ has at least $\sum_{k=1}^{m-j-1}\chi_{\vec{n},j+1,j+k+1}+\eta_{\vec{n},j+1}-2m+2j+1$ sign changes in $\mathring{\Delta}_m$ which is the last statement of the lemma.

□

Now we are ready to prove the convergence of the approximants associated to the ML Hermite-Padé approximation scheme.

Proof of Theorem 3.2. Let us begin with the simplest case when $j=m-1$. Let $\Lambda\subset(\mathbb{Z}_+^m)^*$ be an arbitrary sequence of multi-indices. According to (3.4) (recall that $n_1+\cdots+n_m=|\vec{n}|$)

$$\frac{a_{\vec{n},m-1}-a_{\vec{n},m}\widehat{s}_{m,m}}{w_{\vec{n},m-1}}(z)=\mathcal{O}\left(\frac{1}{z^{(|\vec{n}|+1)}}\right)\in\mathbf{H}(\mathbb{C}\setminus\Delta_m),\ z\to\infty,$$

and $\deg w_{\vec{n},m-1}=\eta_{\vec{n},m-1}<2|\vec{n}|$. Since $\deg a_{\vec{n},m}\le|\vec{n}|, \deg a_{\vec{n},m-1}\le|\vec{n}|-1$, it follows that $a_{\vec{n},m-1}/a_{\vec{n},m}$ is the standard multipoint Padé approximant of $\widehat{s}_{m,m}$ with respect to $w_{\vec{n},m-1}$ (see [50]). This implies that $a_{\vec{n},m}$ is the $|\vec{n}|$-th monic orthogonal polynomial with respect to $\mathrm{d}\,s_{m,m}/w_{\vec{n},m-1}$ and $a_{\vec{n},m-1}$ the corresponding polynomial of second kind. This implies that the zeros of these polynomials lie in $\mathring{\Delta}_m$ and interlace. Now, (3.2) for $j=m-1$ readily follows from [50, Theorem 1] (see [50, Corollary 1]) in the case that the sequence of moments of σ_m verifies Carleman's condition. When Δ_{m-1} is a compact interval bounded away from Δ_m and $\lim_{\vec{n}\in\Lambda}\eta_{\vec{n},m-1}=\infty$ then the number of interpolation conditions on Δ_{m-1} (at the zeros of $w_{\vec{n},m-1}$) suffice to guarantee the convergence of the sequence, which follows from ([50, Theorem 1, Corollary 2]).

For other values of j, there is some defect in the order of interpolation on the right hand of (3.7) and we cannot ensure that $a_{\vec{n},j}/a_{\vec{n},m}$ is an $|\vec{n}|$-th multipoint Padé approximant. That is the reason for restricting the sequence of multi-indices in that part of the statement of Theorem 3.2.

In the sequel, we assume that $\Lambda \subset (\mathbb{Z}_+^m)^*$ is an infinite sequence of distinct multi-indices such that there exist $\ell \in \{0, \ldots, m-2\}$ and a non-negative integer N such that $n_{j+1} \leq n_j + N$ for all $\ell + 1 \leq j \leq m-1$. In this case, we automatically have $\lim_{\vec{n}\in\Lambda} \eta_{\vec{n},m-1} = +\infty$. Indeed, assume that $\limsup_{\vec{n}\in\Lambda} \eta_{\vec{n},m-1} < +\infty$. In particular, this implies that there exists a constant C such that $n_{m-1} \leq C, \vec{n} \in \Lambda$. However, $\lim_{\vec{n}\in\Lambda} n_m = +\infty$ because $\lim_{\vec{n}\in\Lambda} |\vec{n}| = \infty$ since the multi-indices are distinct; therefore, it is impossible that $n_m \leq n_{m-1} + N, \vec{n} \in \Lambda$.

For $j = m-1$ the proof of (3.2) was carried out above. Fix $j \in \{\ell, \ldots, m-2\}$. We have

$$\begin{aligned}
\deg w^*_{\vec{n},j} = & \sum_{k=1}^{m-j-2} \chi_{\vec{n},j+1,j+k+1} + \eta_{\vec{n},j+1} - 2m + 2j + 3 \\
\leq & \eta_{\vec{n},j+1} - 2m + 2j + 3 + \sum_{k=1}^{m-j-2} (n_{j+k+1} + 2) \\
= & \eta_{\vec{n},m-1} - 2m + 2j + 3 + 2(m-j-2) \leq \eta_{\vec{n},m-1} - 1 < 2|\vec{n}|
\end{aligned}$$

for all $\vec{n} \in \Lambda$. Due to the assumptions imposed of the sequence Λ, we have

$$n_{j+k+1} \leq n_{j+k} + N \leq \cdots \leq n_{j+1} + kN.$$

Therefore,

$$\chi_{\vec{n},j+1,j+k+1} \geq \min\{n_{j+1}, \ldots, n_{j+k+1}\} \geq n_{j+k+1} - kN$$

Consequently,

$$\eta_{\vec{n},j+1} + \sum_{k=1}^{m-j-1} \chi_{\vec{n},j+1,j+k+1} - 2m + 2j + 1 \geq |\vec{n}| - 2m - N \sum_{k=1}^{m-j-1} k \geq |\vec{n}| - 2m - N\frac{m(m+1)}{2}. \quad (3.9) \quad \text{\{numberzeros\}}$$

Combined with the last statement of Lemma 3.6 this inequality gives the lower bound on the number of sign changes of $a_{\vec{n},j}$ on $\mathring{\Delta}_m$.

From (3.9) and (3.7), it follows that there exists a constant $\kappa \in \mathbb{Z}_+$ such that for all $\vec{n} \in \Lambda$ and $j = \ell, \ldots, m-2$

$$\frac{a_{\vec{n},j} - a_{\vec{n},m}\widehat{s}_{m,j+1}}{w^*_{\vec{n},j}}(z) = \mathcal{O}\left(\frac{1}{z^{|\vec{n}|+1-\kappa}}\right) \in \mathbf{H}(\mathbb{C} \setminus \Delta_m), \qquad z \to \infty. \quad (3.10) \quad \text{\{multiPade\}}$$

We also have $\deg a_{\vec{n},j} \leq |\vec{n}| - 1$, $\deg a_{\vec{n},m} \leq |\vec{n}|$ and $\deg w^*_{\vec{n},m} \leq 2|\vec{n}|$. This means that for each fixed j, $\ell \leq j \leq m-2$, $\left\{\frac{a_{\vec{n},j}}{a_{\vec{n}.m}}\right\}_{\vec{n}\in\Lambda}$ is a sequence of incomplete diagonal multipoint Padé approximants of $\widehat{s}_{m.j+1}$ which satisfies (3.10). It is easy to verify that if the sequence of moments of σ_m verifies Carleman's condition then for all $j, 0 \leq j \leq m-1$, the sequence of moments of $s_{m,j+1}$ also verifies Carleman's condition. Also, recall that in the present situation $\lim_{\vec{n}\in\Lambda} \eta_{\vec{n},m-1} = +\infty$ takes place. Using the assumptions imposed on the moments of σ_m or on Δ_{m-1} from [20, Lemma 2], it follows that $\left\{\frac{a_{\vec{n},j}}{a_{\vec{n}.m}}\right\}_{\vec{n}\in\Lambda}$ converges to $\widehat{s}_{m,j+1}$ in 1-Hausdorff content (Definition 1.6) on each compact subset of $\overline{\mathbb{C}} \setminus \Delta_m$. Convergence in 1-Hausdorff content means that for each compact $K \subset \mathbb{C} \setminus \Delta_m$ and for each $\varepsilon > 0$, we have

$$\lim_{\vec{n}\in\Lambda} h\left\{z \in K : \left|\frac{a_{\vec{n},j}(z)}{a_{\vec{n}.m}(z)} - \widehat{s}_{m,j+1}(z)\right| > \varepsilon\right\} = 0 \quad (3.11) \quad \text{\{convH\}}$$

The rational functions $\frac{a_{\vec{n},j}}{a_{\vec{n}.m}}$ are holomorphic in $\mathbb{C} \setminus \Delta_m$ because the zeros of $a_{\vec{n},m}$ lie in Δ_m. This together with (3.11) imply that the convergence is uniform on each compact subset of $\mathbb{C} \setminus \Delta_m$ according to Lemma 1.8.

If Δ_m is bounded, we still have to consider those compact subsets of $\overline{\mathbb{C}} \setminus \Delta_m$ which contain ∞. Due to the fact that the rational functions and $\widehat{s}_{m,j+1}$ equal zero at ∞ this situation is obtained from the general case using the maximum principle. The proof is complete.

□

The last statement of Lemma 3.6, (3.9) gives a lower bound on the number of zeros which $a_{\vec{n},j}$ has in $\mathring{\Delta}_m$ when $\vec{n} \in \Lambda$ and Λ verifies the conditions of Theorem 3.2. If we impose greater

{cor:comp} restrictions on Λ more can be said in this regard.

Theorem 3.7:
Let $\Lambda \subset (\mathbb{Z}_+^m)^*$ be an infinite sequence of distinct multi-indices for which there exists $\ell \in \{0, \ldots, m-2\}$ such that $n_j \geq n_{j+1} + 1$ for all $\ell + 1 \leq j \leq m-1$ and $\vec{n} \in \Lambda$. Consider the sequence of vector polynomials $(a_{\vec{n},0}, \ldots, a_{\vec{n},m})_{\vec{n}\in\Lambda}$ associated with $\mathcal{N}(\sigma_1, \ldots, \sigma_m)$. Then, $a_{\vec{n},j}$, $j = \ell, \ldots, m-1$, has exactly $|\vec{n}| - 1$ simple zeros which interlace the zeros of $a_{\vec{n},m}$.

Proof. We prove this by showing that for all $j = \ell, \ldots, m-1$ and $\vec{n} \in \Lambda$, the rational function $a_{\vec{n},j}/a_{\vec{n},m}$ is a diagonal multipoint Padé approximant of $\widehat{s}_{m.j+1}$. Due to (3.7) we achieve this if we show that

$$\sum_{k=1}^{m-j-1} \chi_{\vec{n},j+1,j+k+1} + \eta_{\vec{n},j+1} - 2m + 2j + 3 = |\vec{n}| - l + 1 \quad \text{with} \quad l = 0.$$

Notice that

$$\sum_{k=1}^{m-j-1} \chi_{\vec{n},j+1,j+k+1} + \eta_{\vec{n},j+1} - 2m + 2j + 3 = \sum_{k=1}^{m-j-1} \chi_{\vec{n},j+1,j+k+1} + |\vec{n}| - \sum_{i=j+2}^{m} n_i - 2m + 2j + 3.$$

Combining these two relations, canceling out common terms and making a change of parameter in the indices of the sums, we obtain the equation

$$l = 2(m-j-1) - \sum_{k=j+2}^{m} (\chi_{\vec{n},j+1,k} - n_k).$$

Taking into account that $n_j \geq n_{j+1} + 1$, $\ell + 1 \leq j \leq m-1$, it readily follows that $\chi_{\vec{n},j+1,k} = n_k + 2$. Consequently,

$$\sum_{k=j+2}^{m} (\chi_{\vec{n},j+1,k} - n_k) = 2(m-j-1)$$

and thus $l = 0$ as needed.

Hence,

$$\frac{(a_{\vec{n},j} - a_{\vec{n},m}\widehat{s}_{m,j+1})(z)}{w^*_{\vec{n},j}(z)} = \mathcal{O}\left(\frac{1}{z^{|\vec{n}|+1}}\right), \qquad j = \ell, \ldots, m-1,$$

and $\deg w^*_{\vec{n},j} \leq 2|\vec{n}|$. Consequently, $a_{\vec{n},j}/a_{\vec{n},m}$ is the $|\vec{n}|$-th diagonal multipoint Padé approximant with respect to $\widehat{s}_{m,j+1}$ with interpolation points at the zeros of $w^*_{\vec{n},j}$ and at ∞ of order $2|\vec{n}|-\deg w^*_{\vec{n},j}$.

So, the fraction $a_{\vec{n},j}/a_{\vec{n},m}$ is the $|\vec{n}|$-th diagonal multipoint Padé approximation of $\widehat{s}_{m,j+1}$. From the theory of diagonal multipoint Padé approximation (or simply using (2.10)) we know that $a_{\vec{n},m}$ is the $|\vec{n}|-th$ monic orthogonal polynomials with respect to the varying measure $\mathrm{d}\, s_{m,j+1}/w^*_{\vec{n},j}$ and $a_{\vec{n},j}$ is the corresponding polynomial of the second kind whose zeros interlace those of $a_{\vec{n},m}$. We are done.

□

3.3 Ratio asymptotic

{sec:ratio}

Throughout this section $Q_{\vec{n},j}$, $j = 1, \ldots, m$, denotes the monic polynomial whose roots coincide with the zeros of $\mathscr{A}_{\vec{n},j}$ in $\mathbb{C} \setminus \Delta_{j+1}$ ($\Delta_{m+1} = \emptyset$). In Lemma 3.4, these polynomials were denoted $w_{\vec{n},j}$. From that lemma it follows that $\deg Q_{\vec{n},j} = \eta_{\vec{n},j} = n_1 + \cdots + n_j$, its zeros are simple and lie in $\mathring{\Delta}_j$. We will show that these polynomials satisfy full orthogonality relations with respect to certain varying measures. This fact plays an important role in the study of ratio asymptotic.

3.3.1 Multi-orthogonality relations

From Lemma 2.5 and (3.4) in Lemma 3.4 it readily follows that for $j = 0, \ldots, m-1$

$$\frac{\mathscr{A}_{\vec{n},j}(z)}{Q_{\vec{n},j}(z)} = \int \frac{\mathscr{A}_{\vec{n},j+1}(x)}{z-x} \frac{\mathrm{d}\,\sigma_{j+1}(x)}{Q_{\vec{n},j}(x)}, \tag{3.12}$$ {int_Anj}

where $Q_{\vec{n},0} \equiv 1$, and

$$\int x^\nu \mathscr{A}_{\vec{n},j+1}(x) \frac{\mathrm{d}\,\sigma_{j+1}(x)}{Q_{\vec{n},j}(x)} = 0, \qquad \nu = 0, 1, \ldots \eta_{\vec{n},j+1} - 1. \tag{3.13}$$ {orth_Anj}

Set

$$\mathscr{H}_{\vec{n},j} := \frac{Q_{\vec{n},j+1}\mathscr{A}_{\vec{n},j}}{Q_{\vec{n},j}}, \qquad j = 0, \ldots, m-1, \tag{3.14}$$ {Hnj}

where $Q_{\vec{n},0} \equiv Q_{\vec{n},m+1} \equiv 1$. Since $\mathscr{A}_{\vec{n},m} = (-1)^m a_{\vec{n},m}$ and $a_{\vec{n},m}$ is monic, we take $\mathscr{H}_{\vec{n},m} = (-1)^m$. {mult_orth}

Lemma 3.8:

Consider the Nikishin system $\mathscr{N}(\sigma_1, \ldots, \sigma_m)$. For each fixed $\vec{n} \in (\mathbb{Z}_+^m)^*$ and $j = 0, \ldots, m-1$

$$\int x^\nu Q_{\vec{n},j+1}(x) \frac{\mathscr{H}_{\vec{n},j+1}(x)\,\mathrm{d}\,\sigma_{j+1}(x)}{Q_{\vec{n},j}(x) Q_{\vec{n},j+2}(x)} = 0, \qquad \nu = 0, \ldots, \eta_{\vec{n},j+1} - 1 \tag{3.15}$$ {orth_Qnj}

and

$$\mathscr{H}_{\vec{n},j}(z) = \int \frac{Q^2_{\vec{n},j+1}(x)}{z-x} \frac{\mathscr{H}_{\vec{n},j+1}(x)\,\mathrm{d}\,\sigma_{j+1}(x)}{Q_{\vec{n},j}(x) Q_{\vec{n},j+2}(x)}. \tag{3.16}$$ {Hnj_int}

Proof. Formula (3.15) is (3.13) rewritten with the new notation. Since $\deg Q_{\vec{n},j+1} = \eta_{\vec{n},j+1}$, (3.15) implies that

$$\int \frac{Q_{\vec{n},j+1}(z) - Q_{\vec{n},j+1}(x)}{z-x} Q_{\vec{n},j+1}(x) \frac{\mathcal{H}_{\vec{n},j+1}(x)\,\mathrm{d}\sigma_{j+1}(x)}{Q_{\vec{n},j}(x)Q_{\vec{n},j+2}(x)} = 0.$$

Consequently,

$$Q_{\vec{n},j+1}(z) \int \frac{Q_{\vec{n},j+1}(x)}{z-x} \frac{\mathcal{H}_{\vec{n},j+1}(x)\,\mathrm{d}\sigma_{j+1}(x)}{Q_{\vec{n},j}(x)Q_{\vec{n},j+2}(x)} = \int \frac{Q^2_{\vec{n},j+1}(x)}{z-x} \frac{\mathcal{H}_{\vec{n},j+1}(x)\,\mathrm{d}\sigma_{j+1}(x)}{Q_{\vec{n},j}(x)Q_{\vec{n},j+2}(x)}.$$

Taking into account (3.14) and (3.12) we get

$$\int \frac{Q_{\vec{n},j+1}(x)}{z-x} \frac{\mathcal{H}_{\vec{n},j+1}(x)\,\mathrm{d}\sigma_{j+1}(x)}{Q_{\vec{n},j}(x)Q_{\vec{n},j+2}(x)} = \int \frac{\mathcal{A}_{\vec{n},j+1}(x)}{z-x} \frac{\mathrm{d}\sigma_{j+1}(x)}{Q_{\vec{n},j}(x)} = \frac{\mathcal{A}_{\vec{n},j}(z)}{Q_{\vec{n},j}(z)}.$$

Therefore, (3.16) holds.

□

Given $\vec{n} \in (\mathbb{Z}^*_+)^*$ and $l \in \{1,\dots,m\}$, by $\vec{n}^l$ we denote the multi-index obtained adding 1 to the l-th component of $\vec{n}$. In the next lemma, we prove that the zeros of the polynomials $Q_{\vec{n},j}$ and $Q_{\vec{n}^l,j}$ interlace. The idea of the proof was borrowed from [6, Theorem 2.1].

{lm:interlace} **Lemma 3.9:**

Consider the Nikishin system $\mathcal{N}(\sigma_1,\dots,\sigma_m)$. For each $\vec{n} \in (\mathbb{Z}^m_+)^*$ and $j = 1,\dots,m$, the zeros of the forms $\mathcal{A}_{\vec{n},j}$ and $\mathcal{A}_{\vec{n}^l,j}$ in $\mathring{\Delta}_j$ interlace.

Proof. Fix $\vec{n} \in \mathbb{Z}^m_+$ and $j \in \{1,\dots,m\}$. Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha^2 + \beta^2 \neq 0$. Define the linear form

$$\mathcal{D}_{\vec{n},j} := \alpha\mathcal{A}_{\vec{n},j} + \beta\mathcal{A}_{\vec{n}^l,j}.$$

Repeating the arguments in the proof of Lemma 3.4 we deduce that the form $\mathcal{D}_{\vec{n},j}$ has at least $\eta_{\vec{n},j}$ sign changes in $\mathring{\Delta}_j$, and at most $\eta_{\vec{n},j} + 1$ zeros in $\mathbb{C} \setminus \Delta_{j+1}$ ($\Delta_{m+1} = \varnothing$). Consequently, all the zeros of $\mathcal{D}_{\vec{n},j}$ in $\mathbb{C} \setminus \Delta_{j+1}$ are real and simple.

From this assertion, we deduce that the forms $\mathcal{A}_{\vec{n},j}$ and $\mathcal{A}_{\vec{n}^l,j}$ cannot have common zeros. If such a point y exists, the function

$$\mathcal{D}_{\vec{n},j}(x) = \mathcal{A}_{\vec{n},j}(x) - \frac{\mathcal{A}'_{\vec{n},j}(y)}{\mathcal{A}'_{\vec{n}^l,j}(y)}\mathcal{A}_{\vec{n}^l,j}(x)$$

would have a double zero at y. But this last statement contradicts what we already know.

Fix $y \in \mathbb{R} \setminus \Delta_{j+1}$, and consider the form

$$\mathcal{D}_{\vec{n},j,y}(x) = \mathcal{A}_{\vec{n}^l,j}(y)\mathcal{A}_{\vec{n},j}(x) - \mathcal{A}_{\vec{n},j}(y)\mathcal{A}_{\vec{n}^l,j}(x).$$

By construction $\mathcal{D}_{\vec{n},j,y}(y) = 0$, and thus $\mathcal{D}'_{\vec{n},j,y}(y) \neq 0$. Take two consecutive zeros y_1, y_2 of $\mathcal{A}_{\vec{n}^l,j}$ in $\mathbb{R} \setminus \Delta_{j+1}$ and suppose that $y_1 < y_2$. The zeros of $\mathcal{A}_{\vec{n}^l,j}$ are simple; therefore, $\mathcal{A}'_{\vec{n}^l,j}(y_1) \neq 0$ and

$\mathscr{A}'_{\vec{n}^l,j}(y_2) \neq 0$. Since $\mathscr{A}_{\vec{n}^l,j}$ and $\mathscr{A}_{\vec{n},j}$ have no common zero, we also get that $\mathscr{A}_{\vec{n},j}(y_1) \neq 0$ and $\mathscr{A}_{\vec{n},j}(y_2) \neq 0$. Thus,

$$\mathscr{D}'_{\vec{n},j,y_1}(y_1) = -\mathscr{A}_{\vec{n},j}(y_1)\mathscr{A}'_{\vec{n}^l,j}(y_1) \neq 0,$$
$$\mathscr{D}'_{\vec{n},j,y_2}(y_2) = -\mathscr{A}_{\vec{n},j}(y_2)\mathscr{A}'_{\vec{n}^l,j}(y_2) \neq 0.$$

However, the function $\mathscr{D}'_{\vec{n},j,y}(y)$ preserves the same sign all along the interval $[y_1, y_2]$. Notice that $\mathscr{A}'_{\vec{n}^l,j}(y)$ changes sign when y moves from y_1 to y_2, so $\mathscr{A}_{\vec{n},j}$ must also change sign. By Bolzano's theorem $\mathscr{A}_{\vec{n},j}$ has a zero in (y_1, y_2). The proof is complete.

□

3.3.2 The Riemann surface

The ratio asymptotic of the ML multiple orthogonal polynomials is described in terms of the branches of a conformal mapping defined on a Riemann surface associated with the geometry of the problem. In the sequel, we assume that Δ_k is a closed bounded interval for all $k = 1, \ldots, m$. Let us briefly describe the Riemann surface of interest.

Let $\mathscr{R}$ denote the compact Riemann surface

$$\mathscr{R} = \overline{\bigcup_{k=0}^{m} \mathscr{R}_k}$$

formed by the $m + 1$ consecutively "glued" sheets

$$\mathscr{R}_0 := \overline{\mathbb{C}} \setminus \Delta_1, \qquad \mathscr{R}_k := \overline{\mathbb{C}} \setminus (\Delta_k \cup \Delta_{k+1}), \quad k = 1, \ldots, m, \qquad \mathscr{R}_m := \overline{\mathbb{C}} \setminus \Delta_m,$$

where the upper and lower banks of the slits of two neighboring sheets are identified. This surface is of genus zero. For this and other notions of Riemann surfaces as well as meromorphic functions defined on them we recommend [71].

Let $\pi : \mathscr{R} \longrightarrow \overline{\mathbb{C}}$ be the canonical projection from $\mathscr{R}$ to $\overline{\mathbb{C}}$ and denote by $z^{(k)}$ the point on $\mathscr{R}_k$ satisfying $\pi(z^{(k)}) = z$, $z \in \overline{\mathbb{C}}$. For a fixed $l \in \{1, \ldots, m\}$, let $\psi^{(l)} : \mathscr{R} \longrightarrow \overline{\mathbb{C}}$ denote a conformal mapping whose divisor consists of one simple zero at the point $\infty^{(0)} \in \mathscr{R}_0$ and one simple pole at $\infty^{(l)} \in \mathscr{R}_l$. This mapping exists and is uniquely determined up to a multiplicative constant. Denote the branches of $\psi^{(l)}$ by

$$\psi_k^{(l)}(z) := \psi^{(l)}(z^{(k)}), \qquad k = 0, \ldots, m, \qquad z^{(k)} \in \mathscr{R}_k. \tag{3.17}$$ {branches}

From the properties of $\psi^{(l)}$, we have

$$\psi_0^{(l)}(z) = C_{1,l}/z + O(1/z^2), \;\; z \to \infty, \qquad \psi_l^{(l)}(z) = C_{2,l}\, z + O(1), \;\; z \to \infty, \tag{3.18}$$ {divisorcond}

where $C_{1,l}$, $C_{2,l}$ are non-zero constants.

It is well known and easy to verify that the function $\prod_{k=0}^{m}\psi_k^{(l)}$ admits an analytic continuation to the whole extended plane $\overline{\mathbb{C}}$ without singularities; therefore, it is constant. Multiplying $\psi^{(l)}$ if necessary by a suitable non-zero constant, we may assume that $\psi^{(l)}$ satisfies the conditions

$$\prod_{k=0}^{m}\psi_k^{(l)} = C, \qquad |C| = 1, \qquad C_{1,l} > 0.$$

Let us show that with this normalization, C is either $+1$ or -1.

Indeed, for a point $z^{(k)} \in \mathscr{R}_k$ on the Riemann surface we define its conjugate $\overline{z^{(k)}} := \overline{z}^{(k)}$. Now, let $\overline{\psi}^{(l)} : \mathscr{R} \longrightarrow \overline{\mathbb{C}}$ be the function defined by $\overline{\psi}^{(l)}(\zeta) := \overline{\psi^{(l)}(\overline{\zeta})}$. It is easy to verify that $\overline{\psi}^{(l)}$ is a conformal mapping of $\mathscr{R}$ onto $\overline{\mathbb{C}}$ with the same divisor as $\psi^{(l)}$. Therefore, there exists a constant c such that $\overline{\psi}^{(l)} = c\psi^{(l)}$. The corresponding branches satisfy the relations

$$\overline{\psi}_k^{(l)}(z) = \overline{\psi_k^{(l)}(\overline{z})} = c\psi_k^{(l)}(z), \qquad k = 0, \ldots, m.$$

Comparing the Laurent expansions at ∞ of $\overline{\psi_0^{(l)}(\overline{z})}$ and $c\psi_0^{(l)}(z)$, using the fact that $C_{1,l} > 0$, it follows that $c = 1$. Then

$$\psi_k^{(l)}(z) = \overline{\psi_k^{(l)}(\overline{z})}, \qquad k = 0, \ldots, m.$$

This in turn implies that for each $k = 0, \ldots, m$, all the coefficients, in particular the leading one, of the Laurent expansion at infinity of $\psi_k^{(l)}$ are real numbers. Obviously, C is the product of these leading coefficients. Therefore, C is real, and $|C| = 1$ implies that C equals 1 or -1 as claimed. So, we can assume in the following that

{normconfmap}
$$\prod_{k=0}^{m}\psi_k^{(l)} \equiv e, \qquad C_{1,l} > 0, \tag{3.19}$$

where e is either 1 or -1. It is easy to see that conditions (3.18) and (3.19) determine $\psi^{(l)}$ uniquely.

{lm:BVP} We will need the following lemma. Its proof can be found in [6, Lemma 4.2]

Lemma 3.10:

Set

{boundary3}
$$F_k^{(l)} := \prod_{\nu=k}^{m}\psi_\nu^{(l)} \tag{3.20}$$

where the algebraic functions $\psi_\nu^{(l)}$ *are defined by* (3.17)-(3.19). *The collection of functions* $F_k^{(l)}$, $k = 1, \ldots, m$, *is the unique solution of the system of boundary value problems*

$$\begin{aligned}
&1)\ F_k^{(l)}, 1/F_k^{(l)} \in \mathbf{H}(\mathbb{C} \setminus \Delta_k)\\
&2a)\ F_k^{(l)}(\infty) > 0, \quad k = 1, \ldots, l-1\\
&2b)\ \left(F_k^{(l)}\right)'(\infty) > 0, \quad k = l, \ldots, m\\
&3)\ \left|F_k^{(l)}(x)\right|^2 \frac{1}{\left|\left(F_{k-1}^{(l)}F_{k+1}^{(l)}\right)(x)\right|} = 1, \quad x \in \Delta_k
\end{aligned}$$

where $F_0^{(l)} \equiv F_{m+1}^{(l)} \equiv 1$.

3.3.3 Proof of Theorem 3.3

Theorem 3.3 will be derived from Theorem 3.2 and the next result which gives the ratio asymptotic of the polynomials $Q_{\vec{n},j}$. In proving Theorem 3.11, we adapt the scheme developed in [6, Theorem 1.2] for the study of the ratio asymptotic of type II Hermite-Padé polynomials of Nikishin systems.

Given an arbitrary function $F(z)$ which has in a neighborhood of infinity a Laurent expansion of the form $F(z) = Cz^k + \mathscr{O}\left(z^{k-1}\right), C \neq 0$, and $k \in \mathbb{Z}$, we denote

$$\widetilde{F} := \frac{F}{C}.$$

{th:ratio}

Theorem 3.11:
Consider the Nikishin system $\mathcal{N}(\sigma_1, \ldots, \sigma_m)$ where the intervals Δ_k, $k = 1, \ldots, m$, are bounded and $\sigma'_k \neq 0$ a.e. in Δ_k. Let $\Lambda \subset (\mathbb{Z}_+^m)^*$ be an infinite sequence of distinct multi-indices for which there exists a non-negative integer N such that $n_{j+1} \leq n_j + N$ for all $1 \leq j \leq m-1$ and $\vec{n} \in \Lambda$. Then for $k = 1, \ldots, m$

$$\lim_{\vec{n}\in\Lambda} \frac{Q_{\vec{n}^l,k}(z)}{Q_{\vec{n},k}(z)} = \widetilde{F}_k^{(l)}(z), \tag{3.21}$$ {left}

uniformly on each compact subset of $\mathbb{C} \setminus \Delta_k$.

Proof of Theorems 3.11 and 3.3. From Lemma 3.9 we know that, for each $k = 1, \ldots, m$ the zeros of $Q_{\vec{n},k}$ and $Q_{\vec{n}^l,k}$ interlace on $\mathring{\Delta}_k$. Consequently, the family of functions $(Q_{\vec{n}^l,k}/Q_{\vec{n},k})_{\vec{n}\in\Lambda}$ is uniformly bounded on each compact subset of $\mathbb{C} \setminus \Delta_k$. Therefore, there exists $\Lambda' \subset \Lambda$ such that

$$\lim_{\vec{n}\in\Lambda'} \frac{Q_{\vec{n}^l,k}(z)}{Q_{\vec{n},k}(z)} = G_k(z), \qquad k = 1, \ldots, m, \tag{3.22}$$ {ratio:ch3}

uniformly on each compact subset of $\mathbb{C} \setminus \Delta_k$, where $G_k \in \mathbf{H}(\mathbb{C} \setminus \Delta_k)$. In principle, the limiting functions G_k may depend on Λ'. In order to prove the existence of limit along all Λ, it is sufficient to show that $G_k = \widetilde{F}_k^{(l)}$ regardless of Λ'. Our goal will be accomplished with the aid of Lemma 3.10.

First, it is obvious that the functions $G_k(z)$ and their reciprocals are analytic in $\mathbb{C} \setminus \Delta_k$. Therefore, condition 1 of Lemma 3.10 is fulfilled. On the other hand, considering the degrees of the polynomials $Q_{\vec{n},k}$ and $Q_{\vec{n}^l,k}$, for all $\vec{n} \in \Lambda$ the rational functions on the left of (3.21) at infinity are either equal to 1 when $k = 1, \ldots, l-1$, or their derivative equals 1 for $k = l, \ldots, m$; hence, the limit functions must satisfy either $2a$) or $2b$) depending on k. Thus, any normalization of these functions obtained by means of a multiplication by positive constants also satisfies 1), and $2a$) or $2b$).

Now, to prove the boundary conditions 3 it is necessary to use some tools developed for the study of ratio and relative asymptotic of polynomials orthogonal with respect to varying measures. The main sources are [24], [53] and [56].

Define the constants

$$K_{\vec{n},k-1} := \left(\int Q^2_{\vec{n},k}(x) \frac{|\mathscr{H}_{\vec{n},k}(x)|\,|\mathrm{d}\,\sigma_k(x)|}{|Q_{\vec{n},k-1}(x) Q_{\vec{n},k+1}(x)|} \right)^{-1/2}, \quad k = 1, \dots, m, \tag{3.23}$$ {K_nk}

$$K_{\vec{n},m} := 1,$$

$$\kappa_{\vec{n},k} := \frac{K_{\vec{n},k-1}}{K_{\vec{n},k}}, \qquad k = 1, \dots, m.$$

Set

{orthonormal} $$q_{\vec{n},k} := \kappa_{\vec{n},k} Q_{\vec{n},k}, \quad h_{\vec{n},k} := K^2_{\vec{n},k} \mathscr{H}_{\vec{n},k}, \quad k = 1, \dots, m, \quad h_{\vec{n},0} := K^2_{\vec{n},0} \mathscr{H}_{\vec{n},0}. \tag{3.24}$$

With this notation the expression (3.15) is equivalent to

$$\int x^\nu Q_{\vec{n},k}(x) \frac{|h_{\vec{n},k}(x)|\,|\mathrm{d}\,\sigma_k(x)|}{|Q_{\vec{n},k-1}(x) Q_{\vec{n},k+1}(x)|} = 0, \qquad \nu = 0, \dots, \eta_{\vec{n},k} - 1.$$

Recall that σ_k has constant sign and notice that $Q_{\vec{n},k}, Q_{\vec{n},k-1}$ and $\mathscr{H}_{\vec{n},k}$ have constant sign on Δ_k. Therefore, $Q_{\vec{n},k}$ is the $\eta_{\vec{n},k-1}$-th monic orthogonal polynomial with respect to the varying measure

$$\mathrm{d}\,\rho_{\vec{n},k}(x) := \frac{|h_{\vec{n},k}(x)|\,|\mathrm{d}\,\sigma_k(x)|}{|Q_{\vec{n},k-1}(x) Q_{\vec{n},k+1}(x)|},$$

and $q_{\vec{n},k}$ is the $\eta_{\vec{n},k-1}$-th orthonormal polynomial with respect to the same varying measure.

With an analogous reasoning, we have $Q_{\vec{n}^l,k}$ is the $\eta_{\vec{n}^l,k-1}$-th monic orthogonal polynomial with respect to the varying measure

rying_measure} $$\frac{|h_{\vec{n}^l,k}(x)|\,|\mathrm{d}\,\sigma_k(x)|}{|Q_{\vec{n}^l,k-1}(x) Q_{\vec{n}^l,k+1}(x)|} = \frac{|h_{\vec{n}^l,k}(x)|}{|h_{\vec{n},k}(x)|} \frac{|Q_{\vec{n},k-1}(x) Q_{\vec{n},k+1}(x)|}{|Q_{\vec{n}^l,k-1}(x) Q_{\vec{n}^l,k+1}(x)|} \mathrm{d}\,\rho_{\vec{n},k}(x). \tag{3.25}$$

Using (3.22), we deduce

{limit_Qn_Qn} $$\lim_{\vec{n} \in \Lambda'} \frac{|Q_{\vec{n},k-1}(x) Q_{\vec{n},k+1}(x)|}{|Q_{\vec{n}^l,k-1}(x) Q_{\vec{n}^l,k+1}(x)|} = \frac{1}{|G_{k-1}(z) G_{k+1}(z)|}, \qquad k = 1, \dots, m, \tag{3.26}$$

where the convergence is uniform on Δ_k. On the other hand, from (3.16) it follows that

{h_nk} $$|h_{\vec{n},k}(z)| = \left| \int \frac{q^2_{\vec{n},k+1}(x)}{z - x} \frac{|h_{\vec{n},k+1}(x)|\,|\mathrm{d}\,\sigma_{k+1}(x)|}{|Q_{\vec{n},k}(x) Q_{\vec{n},k+2}(x)|} \right|, \qquad k = 0, \dots, m-1. \tag{3.27}$$

Moreover, we have the following relation between the degrees of the polynomials $Q_{\vec{n},k}, Q_{\vec{n},k+2}$ and $q_{\vec{n},k+1}$

$$\begin{aligned} \deg Q_{\vec{n},k} Q_{\vec{n},k+2} - 2 \deg q_{\vec{n},k+1} &= \eta_{\vec{n},k-1} + \eta_{\vec{n},k+1} - 2\eta_{\vec{n},k} \\ &= n_{k+1} - n_k \leq N, \end{aligned}$$

where N is the constant given in the assumptions which is independent of $\vec{n} \in \Lambda$. Consequently, taking into account [24, Theorem 9], we obtain

{lim_hnj} $$\lim_{\vec{n} \in \Lambda} |h_{\vec{n},k}(z)| = \frac{1}{|\sqrt{(z - b_{k+1})(z - a_{k+1})}|}, \qquad k = 0, \dots, m-1, \tag{3.28}$$

uniformly on each compact subset of $\mathbb{C} \setminus \Delta_{k+1}$, where $\Delta_{k+1} = [a_{k+1}, b_{k+1}]$ (in particular on Δ_k when $k = 1, \ldots m-1$).

The proof of (3.28) is carried out by induction for decreasing values of k. Indeed, if $k = m-1$, since $h_{\vec{n},m} \equiv (-1)^m$, (3.27) reduces to

$$|h_{\vec{n},m-1}(z)| = \left| \int \frac{q^2_{\vec{n},m}(x)}{z-x} \frac{|\,\mathrm{d}\,\sigma_m(x)|}{|Q_{\vec{n},m-1}(x)|} \right|,$$

and using [24, Theorem 9], we obtain

$$\lim_{\vec{n}\in\Lambda} |h_{\vec{n},m-1}(z)| = \left| \frac{1}{\pi} \int_{a_m}^{b_m} \frac{1}{z-x} \frac{\mathrm{d}\,x}{\sqrt{(b_m-x)(x-a_m)}} \right| = \frac{1}{|\sqrt{(z-a_m)(z-b_m)}|}$$

pointwise for $z \in \mathbb{C} \setminus \Delta_m$. However, it is easy to verify that the family of functions $(h_{\vec{n},m-1})_{\vec{n}\in\Lambda}$ is uniformly bounded on compact subsets of $\mathbb{C} \setminus \Delta_m$ and uniform convergence on compact subsets of that region follows from pointwise convergence. Now, let $1 \le k+1 \le m$ and assume that (3.28) holds for $k+1$. Then, using (3.27) we can apply once more [24, Theorem 9] to obtain (3.28) for k pointwise on $\mathbb{C} \setminus \Delta_{k+1}$ and uniform convergence follows as before.

Similar arguments give

$$\lim_{\vec{n}\in\Lambda} |h_{\vec{n}^l,k}(z)| = \frac{1}{|\sqrt{(z-b_{k+1})(z-a_{k+1})}|}, \qquad k = 0, \ldots, m-1, \tag{3.29}$$ {lim_hnj2}

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_{k+1}$.

By construction $h_{\vec{n},m} \equiv h_{\vec{n}^l,m} \equiv (-1)^m$. Therefore, using (3.28) and (3.29) it follows that

$$\lim_{\vec{n}\in\Lambda} \frac{|h_{\vec{n}^l,k}(x)|}{|h_{\vec{n},k}(x)|} = 1, \qquad k = 1, \ldots, m, \tag{3.30}$$ {limitQuot_hnk

uniformly on Δ_k. Putting together (3.30) and (3.26), we have

$$\lim_{\vec{n}\in\Lambda'} \frac{|h_{\vec{n}^l,k(x)}|}{|h_{\vec{n},k}(x)|} \frac{|Q_{\vec{n},k-1}(x) Q_{\vec{n},k+1}(x)|}{|Q_{\vec{n}^l,k-1}(x) Q_{\vec{n}^l,k+1}(x)|} = \frac{1}{|G_{k-1}(x) G_{k+1}(x)|}, \qquad k = 1, \ldots, m, \tag{3.31}$$ {limit_varying

uniformly on the interval Δ_k. The function on the right hand side of the previous expression is different from zero on Δ_k.

Fix $k = 1, \ldots, m$. We distinguish two cases. If $k = 1, \ldots, l-1$ ($l \ge 2$), then $\deg Q_{\vec{n}^l,k} = \deg Q_{\vec{n},k} = \eta_{\vec{n},k}$. Using (3.25) and (3.31), the result on relative asymptotic of orthogonal polynomials with respect to varying measures which appears in [9, Theorem 2] implies that

$$\lim_{\vec{n}\in\Lambda'} \frac{Q_{\vec{n}^l,k}(z)}{Q_{\vec{n},k}(z)} = G_k(z) = \frac{\mathsf{S}_k(z)}{\mathsf{S}_k(\infty)}, \qquad k = 1, \ldots, l-1, \tag{3.32}$$ {ratio_S}

where S_k is the Szegő function on $\overline{\mathbb{C}} \setminus \Delta_k$ with respect to the weight function

$$|G_{k-1}(z) G_{k+1}(z)|^{-1}, \qquad x \in \Delta_k.$$

Consequently,

$$|\mathsf{S}_k(x)|^2 |G_{k-1}(z) G_{k+1}(z)|^{-1} = 1, \qquad x \in \Delta_k. \tag{3.33}$$ {Sk_Gk}

and for $x \in \Delta_k$

$$\frac{|G_k(x)|^2}{|G_{k-1}(x)G_{k+1}(x)|} = \frac{1}{\mathsf{S}_k^2(\infty)}, \qquad k = 1, \ldots, l-1. \tag{3.34}$$ {boundary1}

Now, if $k = l, \ldots, m$, then $\deg Q_{\vec{n}^l,k} = \deg Q_{\vec{n},k} + 1 = \eta_{\vec{n},k} + 1$. Let $Q^*_{\vec{n},k}$ be the $\eta_{\vec{n},k}$-th monic orthogonal polynomial with respect to the varying measure (3.25). Take

$$\frac{Q_{\vec{n}^l,k}}{Q_{\vec{n},k}} = \frac{Q_{\vec{n}^l,k}}{Q^*_{\vec{n},k}} \frac{Q^*_{\vec{n},k}}{Q_{\vec{n},k}}.$$

For the second factor, reasoning as above, we get

{ratio_S^*} $$\lim_{\vec{n}\in\Lambda'} \frac{Q^*_{\vec{n},k}(z)}{Q_{\vec{n},k}(z)} = \frac{\mathsf{S}_k(z)}{\mathsf{S}_k(\infty)}, \tag{3.35}$$

where S_k is the same Szegő function we had before. In the first factor, we have the ratio of two monic polynomials of consecutive degrees orthogonal with respect to the same varying measure and with the help of the theorem on ratio asymptotic of orthogonal polynomials with respect to varying measures [24, Theorem 6] we deduce

{ratio_phi} $$\lim_{\vec{n}\in\Lambda'} \frac{Q_{\vec{n}^l,k}}{Q^*_{\vec{n},k}}(z) = \frac{\varphi_k(z)}{\varphi_k'(\infty)}, \tag{3.36}$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_j$, where φ_k is the conformal representation of $\overline{\mathbb{C}} \setminus \Delta_k$ onto the exterior of the unit disc such that $\varphi_k(\infty) = \infty$ and $\varphi_k'(\infty) > 0$. Combining (3.22) with (3.36) and (3.35) we have

{Qnk_Gk_SK} $$\lim_{\vec{n}\in\Lambda'} \frac{Q_{\vec{n}^l,k}(z)}{Q_{\vec{n},k}(z)} = G_k(z) = \frac{\mathsf{S}_k(z)\varphi_k(z)}{\mathsf{S}_k(\infty)\varphi_k'(\infty)}, \qquad k = l, \ldots, m, \tag{3.37}$$

and using (3.33) it follows that for $x \in \Delta_k$,

{boundary2} $$\frac{|G_k(x)|^2}{|G_{k-1}(x)G_{k+1}(x)|} = \frac{1}{(\mathsf{S}_k(\infty)\varphi_k'(\infty))^2}, \quad k = l, \ldots, m. \tag{3.38}$$

Putting together (3.34) and (3.38) we have proved that the collection of functions $(G_k)_{k=1}^m$ satisfies the conditions of Lemma 3.10, where the right hand side of 3) is $1/w_k$

{wk} $$w_k = \begin{cases} (\mathsf{S}_k(\infty))^2, & k = 1, \ldots, l-1, \\ (\mathsf{S}_k(\infty)\varphi_k'(\infty))^2, & k = l, \ldots, m, \end{cases} \tag{3.39}$$

(instead of 1).

Let $\widetilde{G}_k = c_k G_k$, where c_k, $k = 1, \ldots, m$, are constants chosen appropriately so that

$$\frac{c_k^2}{w_k c_{k-1} c_{k+1}} = 1, \qquad k = 1, \ldots, m \qquad (c_0 = c_{m+1} = 1).$$

Such constants exist. Indeed, taking logarithm we obtain the linear system of equations (in $\ln c_k$)

{linear_sys} $$2 \ln c_k - \ln c_{k-1} - \ln c_{k+1} = \ln w_k, \qquad k = 1, \ldots, m, \tag{3.40}$$

which has a solution because the determinant of the system is different from zero. It is easy to verify that the collection of functions $(\widetilde{G}_k)_{k=1}^m$ satisfies all the conditions of Lemma 3.10. Since that system of boundary value problems has only one solution, it follows that

$$\widetilde{G}_k = c_k G_k = F_k^{(l)}, \qquad k = 1, \ldots, m.$$

Now, $G_k(\infty) = 1$ when $k = 1, \ldots, l-1$ ($l \geq 2$), and $G_k'(\infty) = 1$ when $k = l, \ldots, m$; therefore, taking limit as $z \to \infty$ it follows that

$$c_k = \begin{cases} F_k^{(l)}(\infty), & k = 1, \ldots, l-1, \\ (F_k^{(l)})'(\infty), & k = l, \ldots, m. \end{cases} \tag{3.41}$$ {ck}

In any case, we have shown that independent of the subsequence $\Lambda' \subset \Lambda$ taken such that (3.22) takes place the limiting functions are

$$G_k = \widetilde{F}_k^{(l)}, \qquad k = 1, \ldots, m,$$

and (3.21) follows. With this we conclude the proof of Theorem 3.11.

Since $a_{\vec{n},m} = Q_{\vec{n},m}$ for all $\vec{n}$, (3.3) is a direct consequence of (3.21) and (3.20) when $k = m$. Now, fix $k \in \{0, \ldots, m-1\}$ and $\varepsilon > 0$. Consider the positively oriented closed curve Γ which surrounds Δ_m at distance ε. From (3.2) and the argument principle it follows that

$$\lim_{\vec{n} \in \Lambda} \frac{1}{2\pi i} \int_\Gamma \frac{(a_{\vec{n},k}/a_{\vec{n},m})'(\zeta)}{(a_{\vec{n},k}/a_{\vec{n},m})(\zeta)} \, \mathrm{d}\zeta = \frac{1}{2\pi i} \int_\Gamma \frac{\widehat{s}'_{m,k+1}(\zeta)}{\widehat{s}_{m,k+1}(\zeta)} \, \mathrm{d}\zeta = 1$$

because $\widehat{s}_{m,k+1}$ has a simple zero at ∞ and no other zero or pole in all $\mathbb{C} \setminus \Delta_m$. The integrals on the left hand side only take integer values so they must be constantly equal to 1 for all $\vec{n} \in \Lambda$ such that $|\vec{n}|$ is sufficiently large. Now, $\deg a_{\vec{n},m} = |\vec{n}|$ and its zeros lie on Δ_m and $\deg a_{\vec{n},k} \leq |\vec{n}| - 1$, $k = 0, \ldots, m-1$. It readily follows that for all $\vec{n} \in \Lambda$ with $|\vec{n}|$ sufficiently large, $\deg a_{\vec{n},k} = |\vec{n}| - 1$ and $a_{\vec{n},k}$ has no zeros in the unbounded connected component of $\mathbb{C} \setminus \Gamma$. Since $\varepsilon > 0$ is arbitrary, we also obtain that the zeros of $a_{\vec{n},k}$ accumulate on Δ_m.

Now, using (3.2) and (3.3) (for $k = m$) it follows that

$$\lim_{\vec{n} \in \Lambda} \frac{a_{\vec{n}^l,k}(z)}{a_{\vec{n},k}(z)} = \lim_{\vec{n} \in \Lambda} \frac{a_{\vec{n}^l,k}(z)}{a_{\vec{n}^l,m}(z)} \frac{a_{\vec{n},m}(z)}{a_{\vec{n},k}(z)} \frac{a_{\vec{n}^l,m}(z)}{a_{\vec{n},m}(z)} = \frac{\psi_m^{(l)}(z)}{(\psi_m^{(l)})'(\infty)},$$

uniformly on each compact subset of $\mathbb{C} \setminus \Delta_m$ and (3.3) follows for $k = 0, \ldots, m-1$.

□

The next result complements Theorem 3.11.

Corollary 3.12:
Assume that the conditions of Theorem 3.11 *hold. Let* $(q_{\vec{n},k} = \kappa_{\vec{n},k} Q_{\vec{n},k})_{k=1}^m$, $\vec{n} \in \Lambda$, *be the system of orthonormal polynomials defined in* (3.24) *and* $(K_{\vec{n},k})_{k=1}^m$, $\vec{n} \in \Lambda$, *the values given in* (3.23).

Then, for each fixed $k = 1, \ldots, m$ we have

$$\lim_{\vec{n}\in\Lambda} \frac{\kappa_{\vec{n}^l,k}}{\kappa_{\vec{n},k}} = \kappa_k, \tag{3.42}$$ {kappa}

$$\lim_{\vec{n}\in\Lambda} \frac{K_{\vec{n}^l,k-1}}{K_{\vec{n},k-1}} = \kappa_k \cdots \kappa_m, \tag{3.43}$$ {Knk_asym}

and

$$\lim_{\vec{n}\in\Lambda} \frac{q_{\vec{n}^l,k}(z)}{q_{\vec{n},k}(z)} = \kappa_k \widetilde{F}_k^{(l)}(z), \tag{3.44}$$ {q_kappa_F}

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_k$, where

{def_ck}
$$\kappa_k = \frac{c_k}{\sqrt{c_{k-1}c_{k+1}}}, \qquad c_k = \begin{cases} F_k^{(l)}(\infty), & k = 1, \ldots, l-1, \\ (F_k^{(l)})'(\infty), & k = l, \ldots, m, \end{cases} \tag{3.45}$$

and $c_0 = c_m = 1$. We also have

{ratio_Anj}
$$\lim_{n} \left| \frac{\mathscr{A}_{\vec{n}^l,k}(z)}{\mathscr{A}_{\vec{n},k}(z)} \right| = \frac{1}{\kappa_{k+1}^2 \cdots \kappa_m^2} \left| \frac{\widetilde{F}_k^{(l)}(z)}{\widetilde{F}_{k+1}^{(l)}(z)} \right|, \qquad k = 0, \ldots, m-1, \tag{3.46}$$

uniformly on compact subsets of $\mathbb{C} \setminus (\Delta_k \cup \Delta_{k+1})$. When $k = 0$, $\Delta_0 = \emptyset$.

Proof. From (3.21) it follows that in place of (3.31) we can write

$$\lim_{\vec{n}\in\Lambda} \frac{|h_{\vec{n}^l,k}(x)|}{|h_{\vec{n},k}(x)|} \frac{|Q_{\vec{n},k-1}(x)Q_{\vec{n},k+1}(x)|}{|Q_{\vec{n}^l,k-1}(x)Q_{\vec{n}^l,k+1}(x)|} = \frac{1}{|\widetilde{F}_{k-1}^{(l)}(x)\widetilde{F}_{k+1}^{(l)}(x)|}, \qquad k = 1, \ldots, m.$$

By the same token, (3.32) and (3.37) hold with the limit taken along all Λ.

With the same arguments that led to (3.32) and (3.37), but in connection with orthonormal polynomials (see [9] and [24]) it follows that

{ortonormal}
$$\lim_{\vec{n}\in\Lambda} \frac{q_{\vec{n}^l,k}(z)}{q_{\vec{n},k}(z)} = \begin{cases} \mathsf{S}_k(z), & k = 1, \ldots, l-1, \\ \mathsf{S}_k(z)\varphi_k(z), & k = l, \ldots, m. \end{cases} \tag{3.47}$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_j$. Now, dividing (3.47) by (3.32) or (3.37), we obtain

$$\lim_{n} \frac{\kappa_{\vec{n}^l,k}}{\kappa_{\vec{n},k}} = \sqrt{w_k} = \frac{c_k}{\sqrt{c_{k-1}c_{k+1}}} := \kappa_k,$$

where w_k is given by (3.39) and the c_k are the normalizing constants found solving the linear system of equations (3.40) whose values were given in (3.41). Therefore, formulas (3.42) and (3.45) take place. Now, (3.43) follows from (3.42) because

$$\frac{K_{\vec{n}^l,k-1}}{K_{\vec{n},k-1}} = \frac{\kappa_{\vec{n}^l,k} \cdots \kappa_{\vec{n}^l,m}}{\kappa_{\vec{n},k} \cdots \kappa_{\vec{n},m}}$$

and (3.44) from (3.42) and (3.21) since

$$\frac{q_{\vec{n}^l,k}}{q_{\vec{n},k}} = \frac{\kappa_{\vec{n}^l,k}Q_{\vec{n}^l,k}}{\kappa_{\vec{n},k}Q_{\vec{n},k}}.$$

From (3.14), (3.16), and (3.24), we deduce

$$\mathscr{A}_{\vec{n},k}(z) = \frac{1}{K_{\vec{n},k}^2} \frac{Q_{\vec{n},k}(z)}{Q_{\vec{n},k+1}(z)} \int \frac{q_{\vec{n},k+1}^2}{z-x} \frac{h_{\vec{n},k+1}(x)\,\mathrm{d}\,\sigma_{k+1}(x)}{Q_{\vec{n},k}(x) Q_{\vec{n},k+2}(x)}, \qquad k = 0, \ldots, m-1,$$

and similarly

$$\mathscr{A}_{\vec{n}^l,k}(z) = \frac{1}{K_{\vec{n}^l,k}^2} \frac{Q_{\vec{n}^l,k}(z)}{Q_{\vec{n}^l,k+1}(z)} \int \frac{q_{\vec{n}^l,k+1}^2}{z-x} \frac{h_{\vec{n}^l,k+1}(x)\,\mathrm{d}\,\sigma_{k+1}(x)}{Q_{\vec{n}^l,k}(x) Q_{\vec{n}^l,k+2}(x)}, \qquad k = 0, \ldots, m-1,$$

Dividing the second expression by the first, taking absolute values, and the limit over $\vec{n} \in \Lambda$ from (3.21), (3.43), (3.27), and (3.30), formula (3.46) readily follows. □

CHAPTER

4 Strong asymptotic of Cauchy biorthogonal polynomials

{ch:strong}

In [14] pairs of polynomials $\{(P_m, Q_n)\}_{m,n\in\mathbb{Z}_+}$ were introduced which satisfy certain biorthogonality relations (that we will discuss in the following pages). These kind of polynomials have a particular interest in the study of partial differential equations and the two matrix model.

Our approach to study their strong asymptotic was to exploit their relationship with the multi-level Hermite-Padé polynomials. Once in the context of Hermite-Padé polynomials we attack the problem using ideas introduced by A.I. Aptekarev to study the strong asymptotic of Type II multi-orthogonal polynomials with respect to Angelesco and Nikishin systems.

4.1 Statement of the main results

4.1.1 Cauchy biorthogonal polynomials

Let $\mathbf{\Delta} = (\Delta_1, \Delta_2)$ be a pair of intervals, contained in the real line $\mathbb{R}$, which have at most one common point. By $\mathscr{M}(\mathbf{\Delta})$ we denote the cone of all pairs (σ_1, σ_2) of Borel measures with constant sign and finite moments whose supports verify $\operatorname{supp}\sigma_k \subset \Delta_k$ and

$$\int\int \frac{\mathrm{d}\,\sigma_1(x)\,\mathrm{d}\,\sigma_2(y)}{|x-y|} < \infty.$$

Fix $(\sigma_1, \sigma_2) \in \mathscr{M}(\mathbf{\Delta})$. For each pair of non negative integers $(m, n) \in \mathbb{Z}^2_{\geq 0}$ there exists a pair (P_m, Q_n) of monic polynomials whose degrees verify $\deg P_m \leq m, \deg Q_n \leq n$, and

{biort}

$$\int_{\Delta_1}\int_{\Delta_2} P_m(x)Q_n(y)\frac{\mathrm{d}\,\sigma_1(x)\,\mathrm{d}\,\sigma_2(y)}{x-y} = C_n\delta_{m,n}, \quad C_n \neq 0. \tag{4.1}$$

(As usual, $\delta_{m,n} = 0, m \neq n, \delta_{n,n} = 1$.) These polynomials were introduced in [14] and called Cauchy biorthogonal polynomials. The original definition uses the kernel $(x+y)^{-1}$ (and measures supported in the positive real line to avoid singularities in the kernel except when $x = y = 0$), but we find it more convenient to employ $(x-y)^{-1}$ instead, since it adapts better to our presentation. Some interesting properties were revealed. In particular, it was shown that $\deg P_n = n$, its zeros are simple, interlace for consecutive values of n, and lie in $\mathring{\Delta}_1$ (the interior of Δ_1 with the Euclidean topology of $\mathbb{R}$). The same goes for the Q_n on Δ_2.

Cauchy biorthogonal polynomials appear in the analysis of the two matrix model [13, 15] and were used to find discrete solutions of the Degasperis-Procesi equation [14] through a Hermite-Padé approximation problem for two discrete measures. In [15], the authors apply the nonlinear steepest descent method to a class of 3×3 Riemann-Hilbert problems introduced in connection with the Cauchy two-matrix random model, solve the Riemann-Hilbert problem, and establish strong asymptotic results for the Cauchy biorthogonal polynomials for a class of measures given by weights with exponential decay at infinity (of Laguerre type). The results obtained in [15] were later extended in [16].

Our goal is to prove strong asymptotic results for Cauchy biorthogonal polynomials when the intervals $\Delta_k = [a_k, b_k]$, $k = 1, 2$ are bounded non intersecting intervals and the measures σ_1, σ_2 verify Szegő's condition

$$\int_{\Delta_k} \ln \sigma_k'(x) \, \mathrm{d}\, \eta_{\Delta_k}(x) > -\infty, \qquad k = 1, 2, \tag{4.2}$$ {szego}

where σ' denotes the Radon-Nikodym derivative of σ with respect to the Lebesgue measure and $\mathrm{d}\,\eta_\Delta(x)$ the Chebyshev measure on the interval $\Delta = [a, b]$ (see (1.16)). In this case we write $(\sigma_1, \sigma_2) \in \mathscr{S}(\boldsymbol{\Delta})$. Therefore, we extend Szegő's theory on the strong asymptotic of orthogonal polynomials supported on a bounded interval of the real line to the context of Cauchy biorthogonal polynomials. In the sequel, the intervals Δ_1, Δ_2 are bounded and do not intersect. {mainbiort}

Theorem 4.1:
Let $(\sigma_1, \sigma_2) \in \mathscr{S}(\boldsymbol{\Delta})$ and $\{P_n\}_{n\geq 0}, \{Q_n\}_{n\geq 0}$ be the sequences of monic polynomials determined by (4.1)*. Then*

$$\lim_n \frac{P_n(z)}{\Phi_1^n(z)} = \frac{\mathsf{G}_1^*(z)}{\mathsf{G}_1^*(\infty)}, \qquad \lim_n \frac{Q_n(z)}{\Phi_2^n(z)} = \frac{\mathsf{G}_2(z)}{\mathsf{G}_2(\infty)}, \tag{4.3}$$ {asinbio}

uniformly on each compact subset of $\Omega_1 = \overline{\mathbb{C}} \setminus \Delta_1$ and $\Omega_2 = \overline{\mathbb{C}} \setminus \Delta_2$, respectively, where $\Phi_k \in \mathbf{H}(\Omega_k)$, $k = 1, 2$, (holomorphic in Ω_k) is the exponential of a complex potential constructed from a vector equilibrium problem (see (4.40) *and* (4.42)*), and $\mathsf{G}_1^*, \mathsf{G}_2$ are Szegő functions obtained as components of fixed points of the maps $T_{\mathbf{w}_P}$ and $T_{\mathbf{w}_Q}$, respectively (see Def.* 4.12*,* (4.72) *and* (4.66)*).*

The logarithmic and ratio asymptotic of biorthogonal polynomials were obtained in [32] for more general Cauchy type kernels involving $m \geq 2$ measures. As in [32], we reduce the study of the strong asymptotic of Cauchy biorthogonal polynomials to that of polynomials arising from an associated mixed type Hermite-Padé approximation problem. The Hermite-Padé polynomials turn out to be orthogonal with respect to varying measures. So, the strong asymptotic of such sequences of orthogonal polynomials play a central role in our discussion.

4.1.2 Orthogonal polynomials with varying measures

Let $\Delta = [a, b] \subset \mathbb{R}$. Consider a sequence $\{\mathrm{d}\,\mu_n / w_{2n}\}_{n\geq 0}$ where μ_n is a finite positive Borel measure supported on Δ and w_{2n} is a polynomial with real coefficients, $\deg w_{2n} = i_n \leq 2n$, whose zeros $\{x_{2n,i}\}_{i=2n-i_n+1}^{2n}$ lie in $\mathbb{C} \setminus \Delta$. This is called a sequence of varying measures. Let $L_n(x) = x^n + \cdots$ be the n-th monic orthogonal polynomial satisfying

{def:Ln}
$$\int x^\nu L_n(x) \frac{\mathrm{d}\,\mu_n(x)}{|w_{2n}(x)|} = 0, \qquad \nu = 0, 1, \ldots, n-1. \tag{4.4}$$

The sequence $\{L_n\}_{n\geq 0}$ is called the sequence of monic orthogonal polynomials with respect to the given varying measures. A common normalization is to take

$$\tau_n := \left(\int L_n^2(x) \frac{\mathrm{d}\,\mu_n(x)}{|w_{2n}(x)|} \right)^{-1/2},$$

and define $l_n(x) := \tau_n L_n(x)$ as the orthonormal polynomial of degree n.

In the context of multipoint Padé and Hermite-Padé approximation, orthogonal polynomials with respect to varying measures arise naturally (see, for example, [5, 8, 20, 41, 56]). Recall that depending on the type of asymptotic one wishes to obtain for the sequence $\{L_n\}_{n\geq 0}$ (or $(l_n)_{n\geq 0}$), some conditions must be imposed on the varying measures. In this chapter we will use combinations of (S1)-(S4) (see Subsection 1.5).

In many applications, $\mathrm{d}\,\mu_n = h_n\,\mathrm{d}\,\overline{\mu}, \overline{\mu}' > 0$ a.e. on Δ, where $\{h_n\}_{n\geq 0}$ is a sequence of positive continuous functions which converges uniformly on Δ to a positive continuous function h, and the zeros of the polynomials $\{w_{2n}\}_{n\geq 0}$ are uniformly bounded away from Δ in which case (S1) and (S3) are immediate, and (S2) holds if $\overline{\mu}$ verifies Szegő's condition.

Conditions (S1)-(S3) are sufficient to prove strong asymptotic for $\{L_n\}_{n\geq 0}$. The first result in this direction appeared in [54] and was later improved in [24] and [9]. An alternative proof of the main result in [54] may be found in [91]. The answer in [54] is given in terms of a Szegő function associated with μ and a Blaschke product in which the zeros of w_{2n} intervene (see (4.13) below). We wish to replace the Blaschke product in the asymptotic formula by the n-th power of a fixed function (as in Szegő's classical result). In order to achieve this, some knowledge of the asymptotic behavior of the polynomials w_{2n} is required and condition (S4) comes in.

Let φ be a positive continuous function on Δ. Let λ_φ be the (unitary) equilibrium measure supported on Δ which solves the equilibrium problem for the logarithmic potential with external field $-\frac{1}{2}\ln\varphi$. It is well known that λ_φ is uniquely determined by the equilibrium conditions on Δ (see [88, Theorem I.3])

{equilibrium}
$$V_{\lambda_\varphi}(x) - \frac{1}{2}\ln\varphi(x) \begin{cases} \leq \gamma, & x \in \operatorname{supp}\lambda_\varphi, \\ \geq \gamma, & x \in \Delta \setminus e,\ \operatorname{cap}(e) = 0, \end{cases} \tag{4.5}$$

where γ is a constant, $\operatorname{cap}(e)$ denotes the logarithmic capacity of e, and

$$V_{\lambda_\varphi}(u) = \int \ln\frac{1}{|u - x|}\,\mathrm{d}\,\lambda_\varphi(x)$$

denotes the logarithmic potential of λ_φ. We will assume that φ is such that $\operatorname{supp}\lambda_\varphi = \Delta$. (This is true, for example, if $\frac{1}{2}\ln\varphi = V_\rho$ is the logarithmic potential of a measure ρ, of total mass $c \le 1$, supported on an interval disjoint from Δ. In this case, λ_φ is the balayage of ρ on Δ plus $(1-c)$ times $\mathrm{d}\,\eta_\Delta/\pi$, and equality is attained in (4.5) on all Δ due to the regularity of the interval Δ with respect to the Dirichlet problem.)

Set

$$\Phi(u) := e^{-v_\varphi(u)}, \qquad v_\varphi := V_{\lambda_\varphi} + i\widetilde{V}_{\lambda_\varphi}, \qquad C := e^\gamma, \tag{4.6}$$
{PhiC}

where $\widetilde{V}_{\lambda_\varphi}$ denotes the harmonic conjugate of V_{λ_φ} in $\mathbb{C}\setminus\Delta$ (which equals zero when $u > b, \Delta = [a,b]$). Though $\widetilde{V}_{\lambda_\varphi}$ is multi-valued, it has an increment of 2π if we surround once the interval Δ in the positive direction; consequently, Φ is a single-valued analytic function in $\mathbb{C}\setminus\Delta$ with a simple pole at ∞ since $\Phi(u) = u + \mathcal{O}(1)$, $u \to \infty$.

We write $\mu \in \mathscr{S}(\Delta)$ when μ verifies Szegő's condition on Δ. Recall that the Szegő function of μ is defined as

$$\mathsf{G}(\mu,u) := \exp\left[\frac{\sqrt{(u-b)(u-a)}}{2\pi}\int_\Delta \frac{\ln(\sqrt{(b-x)(x-a)}\,\mu'(x))}{x-u}\,\mathrm{d}\,\eta_\Delta(x)\right].$$

The square root outside the integral is taken to be positive for $u > b$ and those inside the integral are positive when $x \in (a,b)$. The Szegő function is characterized in terms of a boundary value problem (for details see Section 1.4.1). {main_th}

Theorem 4.2:
Assume that $\{(\mu_n, w_{2n})\}_{n\ge0}$ *verifies* $(S1)$-$(S4)$ *and* $\operatorname{supp}\lambda_\varphi = \Delta$ *(see* (4.5)*). Then,*

$$\lim_n \frac{l_n(u)}{C^n\Phi^n(u)} = \frac{1}{\sqrt{2\pi}}\mathsf{G}(\psi\mu, u), \tag{4.7}$$
{strong}

uniformly on compact subsets of Ω*,* $\psi\mu$ *is the measure with differential expression* $\psi\,\mathrm{d}\,\mu$*, and* ψ *is given by* (1.28)*. Moreover,*

$$\lim_n \frac{\tau_n}{C^n} = \frac{1}{\sqrt{2\pi}}\mathsf{G}(\psi\mu, \infty), \tag{4.8}$$
{constante}

and

$$\lim_n \frac{L_n(u)}{\Phi^n(u)} = \frac{\mathsf{G}(\psi\mu, u)}{\mathsf{G}(\psi\mu, \infty)} \tag{4.9}$$
{asintc}

uniformly on compact subsets of Ω*.*

The following result is obtained from Theorem 4.2. It is in the spirit of [95, Theorem 14.3]. The assumptions have points in common but they are not the same. In some regards the conditions in [95] are more general, in others our assumptions are weaker. The most notable difference is that in [95, Theorem 14.3] the measure μ is required to be absolutely continuous with respect to the Lebesgue measure whereas we do not need this restriction. {cor:a}

Theorem 4.3:
Let $\{\mu_n\}_{n\ge0}$ *be a sequence of measures verifying* $(S1)$-$(S2)$*. Let* $\tau, \mathrm{d}\,\tau = v\,\mathrm{d}\,x$*, be a probability*

measure on Δ *such that* $\operatorname{supp}\tau = \Delta$, v *is continuous on* Δ, *and let there be constants* $A, \beta > -1$, *and* β_0 *such that*

$$A^{-1}((b-x)(x-a))^{\beta_0} \le v(x) \le A((b-x)(x-a))^{\beta}, \qquad x \in (a,b). \tag{4.10}$$ {cond:a}

Set

$$\Phi_\tau(u) = \exp\left(-V_\tau(u) - i\widetilde{V}_\tau(u)\right),$$

where $\widetilde{V}_\tau$ *is the harmonic conjugate in* $\mathbb{C} \setminus \Delta$ *of the logarithmic potential* V_τ. *Then*

{strong:a} $$\lim_n \frac{p_n(u)}{\Phi_\tau^n(u)} = \frac{1}{\sqrt{2\pi}} \mathsf{G}(\mu, u), \tag{4.11}$$

uniformly on compact subsets of Ω, *where* p_n *is the n-th orthonormal polynomial verifying*

$$\int p_m(x) p_n(x) \frac{\mathrm{d}\,\mu_n(x)}{|\Phi_\tau^{2n}(x)|} = \begin{cases} 0, & m < n, \\ 1, & m = n. \end{cases}$$

Due to the extension and technical difficulties of the proof of the main results, we begin with a brief description of the present chapter.

4.1.3 Outline and structure of the proofs

Section 4.2 is dedicated to the proof of Theorems 4.2 and 4.3. These results are used in Section 4.3 in the proof of Theorem 4.1 but they have independent interest and may be employed to obtain exact estimates of the rate of convergence of multipoint Padé and Hermite-Padé approximations.

Section 4.3 is devoted to the study of the strong asymptotic of a sequence of Hermite-Padé polynomials intimately connected with the Cauchy biorthogonal polynomials defined above. The proof of Theorem 4.1 is not simple because it requires several steps some of which are quite technical. A brief description of the idea of the proof is helpful for a better understanding of it.

In [62] the authors noticed the connection between Cauchy biorthogonal polynomials and a so called multilevel Hermite-Padé approximation problem. For convenience of the reader, we summarize this relationship in subsection 4.3.1. In subsections 4.3.2 and 4.3.3 we prove some useful formulas verified by these approximants and their associated polynomials. In particular, the biorthogonal polynomials $\{(P_n, Q_n)\}_{n\ge 0}$ are identified with certain Hermite-Padé polynomials which turn out to be orthogonal with respect to varying measures. So the initial problem is reduced to finding the strong asymptotic of the associated Hermite-Padé polynomials.

The results in [62] clearly indicate which functions Φ_1, Φ_2 must be taken to compare the Hermite-Padé polynomials to establish their strong asymptotic behavior. This is explained in detail in subsection 4.3.4. These functions are the exponentials of the complex potentials associated with the equilibrium measures of the vector equilibrium problem used to describe the logarithmic asymptotic of the same polynomials.

Because of the definition of biorthogonality, if the role of the measures σ_1, σ_2 is interchanged then the polynomials P_n, Q_n are also interchanged; therefore, if the strong asymptotic of the sequence $\{Q_n\}_{n\geq 0}$ is obtained then that of the sequence $\{P_n\}_{n\geq 0}$ readily follows. Thus, we focus on $\{Q_n\}_{n\geq 0}$; more precisely, on their associated Hermite-Padé polynomials.

To obtain their strong asymptotic we adapt a very clever method devised by A.I. Aptekarev to obtain the strong asymptotic of Type II Hermite-Padé polynomials for Angelesco [4] and Nikishin [5] systems of measures using fixed point theorems. To understand what this is about, we need to advance some formulas.

We show that for each $n \geq 1$ there exist polynomials $Q_{n,1}, Q_{n,2}$, with $Q_{n,2} = Q_n$, and a continuous function $h_{n,1}$ on Δ_1 such that

$$0 = \int x^\nu Q_{n,2}(x) \frac{\mathrm{d}\sigma_2(x)}{|Q_{n,1}(x)|}, \qquad 0 = \int x^\nu Q_{n,1}(x) \frac{|h_{n,1}(x)|\mathrm{d}\sigma_1(x)}{|Q_{n,2}(x)|}, \qquad \nu = 0, \ldots, n-1$$

where $\lim_n |h_{n,1}(x)| = (\sqrt{x-a_2)(b_2-x})^{-1}$ uniformly on Δ_1. Theorem 4.2 cannot be used directly because the polynomial which in one relation is orthogonal in the other relation appears in the denominator of the varying part of the measure. To handle this, we (temporarily) unlink this inter dependence. For that purpose, for each $n \geq 0$ a (non-linear) operator $\widetilde{T}_n$ is introduced, defined on the set of all pairs $(\widehat{Q}_1, \widehat{Q}_2)$ of monic polynomials with real coefficients of degree n with zeros in the complement of Δ_2 and Δ_1, respectively, such that $\tilde{T}_n(\widehat{Q}_1, \widehat{Q}_2) = (Q_1^*, Q_2^*)$ verifies

$$0 = \int x^\nu Q_2^*(x) \frac{\mathrm{d}\sigma_2(x)}{|\widehat{Q}_1(x)|}, \qquad 0 = \int x^\nu Q_1^*(x) \frac{|h_{n,1}(x)|\mathrm{d}\sigma_1(x)}{|\widehat{Q}_2(x)|}, \qquad \nu = 0, \ldots, n-1. \tag{4.12}$$

{relort}

Notice that $(Q_{n,1}, Q_{n,2})$ is a fixed point of $\tilde{T}_n$. (Indeed, in subsection 4.3.6 a more general operator is defined where it is only required that the sequence $(|h_{n,1}|)_{n\geq 0}$ converges uniformly to a positive continuous function on Δ_1. This extension allows to cover other possible applications we have in mind.)

Take sequences of denominators $(\widehat{Q}_{n,1})_{n\geq 1}, (\widehat{Q}_{n,2})_{n\geq 1}$, and their associated by (4.12) orthogonal polynomials $(Q^*_{n,1})_{n\geq 0}, (Q^*_{n,2})_{n\geq 1}$. If we suppose that $\widehat{g}_1, \widehat{g}_2$ are the uniform limits on Δ_2 and Δ_1, respectively, of the sequences $(\widehat{Q}_{n,1}/\Phi_1^n)_{n\geq 0}, (\widehat{Q}_{n.2}/\Phi_2^n)_{n\geq 0}$, using Theorem 4.2 in subsection 4.3.6 we obtain the strong asymptotic (g_1^*, g_2^*) of $(Q^*_{n,1}/\Phi_1^n)_{n\geq 0}, (Q^*_{n.2}/\Phi_2^n)_{n\geq 0}$. Previously, in subsection 4.3.5 using Theorem 4.3 we show that any pair $(\widehat{g}_1, \widehat{g}_2)$ of Szegő functions on $\mathbb{C}\setminus\Delta_1, \mathbb{C}\setminus\Delta_2$, respectively, can be obtained as strong limits of the initial pair of sequences. From the boundary properties verified by Szegő functions it turns out that $(\widehat{g}_1, \widehat{g}_2)$ and (g_1^*, g_2^*) are connected by the boundary value equations

$$\begin{aligned} |g_1^*(x)|^2 =& \frac{\widehat{g}_2(x)\sqrt{(b_2-x)(x-a_2)}}{\sqrt{(b_1-x)(x-a_1)}\sigma_1'(x)}, \quad \text{a.e. on} \quad [a_1, b_1] = \Delta_1, \\ |g_2^*(x)|^2 =& \frac{\widehat{g}_1(x)}{\sqrt{(b_2-x)(x-a_2)}\sigma_2'(x)}, \quad \text{a.e. on} \quad [a_2, b_2] = \Delta_2 \end{aligned}$$

where $(\sigma_1, \sigma_2) \in S(\mathbf{\Delta})$. This motivates the introduction of another operator in subsection 4.3.7 $T(\widehat{g}_1, \widehat{g}_2) = (g_1^*, g_2^*)$ which on an appropriate metric space of functions is contractive and due to

Banach's fixed point theorem it has a unique fixed point. (Indeed in subsection 4.3.7 a more general situation is considered but we limit ourselves here to the operator which is relevant in the case of multi level Hermite-Padé polynomials.) The final step consists in showing that any neighborhood of the fixed point of the operator T contains fixed points of the operators $\tilde{T}_n$ for all sufficiently large n. This is done in Theorem 4.15 of subsection 4.3.8 using Brouwer's fixed point theorem. Theorem 4.16 is a simple corollary of Theorem 4.15 applied to multi level Hermite Padé polynomials.

In subsection 4.3.10 we return to the biorthogonal polynomials. Since $Q_n = Q_{n,2}$, Theorem 4.16 gives directly the asymptotic of the Q_n. Then, we briefly discuss what needs to be done for the polynomials P_n. In subsection 4.3.10 we derive the strong asymptotic of other functions related with the multi level Hermite Padé approximation problem and the final section 4.3.11 contains a different approach for defining the comparison functions Φ_1, Φ_2 on the basis of a three sheeted Riemann surface of genus zero.

4.2 Strong asymptotic of orthogonal polynomials with varying measures

{sec:str:var}

As mentioned above our goal here is to prove Theorems 4.2 and 4.3. They are essential in the proof of Theorem 4.1, but have independent interest and may find other applications. We begin explaining our choice of Szegő function for measures supported on an interval of the real line.

4.2.1 A starting point

Let $x_{2n,i}, 2n - i_n + 1 \leq i \leq 2n$, denote the zeros of w_{2n}. If $i_n < 2n$ we define $x_{2n,i} = \infty, 1 \leq i \leq 2n - i_n$. Set

$$B_{2n}(u) := \prod_{i=1}^{2n} \frac{\Psi(u) - \Psi(x_{2n,i})}{1 - \overline{\Psi(x_{2n,i})}\Psi(u)}.$$

When $x_{2n,i} = \infty$ the corresponding factor in the Blaschke product is replaced by $1/\Psi(u)$.

In [24, Theorem 4] a strong asymptotic result is given. We state it as a lemma for convenience

{LB_Blaschke} of the reader and further reference.

Lemma 4.4:
Assume that $\{(\mu_n, w_{2n}\}_{n\geq 0}$ *verifies* $(S1)$-$(S3)$ *and* l_n *is the* n*-th orthonormal polynomial associated with* (4.4). *Then*

{asintb}
$$\lim_n \frac{l_n^2(u)}{w_{2n}(u)} B_{2n}(u) = \frac{1}{2\pi} \mathsf{G}^2(\mu, u), \tag{4.13}$$

uniformly on compact subsets of Ω.

We wish to point out that in [24, Theorem 4] there is a typo when writing the condition (S2). There, it appears in terms of the Lebesgue measure $\mathrm{d}x$ instead of the Chebyshev measure $\mathrm{d}\eta_\Delta$. Except for that, the proof given is correct.

When $\mu_n = \mu$ is fixed and $w_{2n} \equiv 1$ (so that $B_{2n} \equiv 1/\Psi^{2n}$) we retrieve the standard result for the strong asymptotic of orthogonal polynomials with respect to $\mu \in \mathscr{S}(\Delta)$. The drawback of Lemma 4.4 is the appearance of the Blaschke product on the left hand side of (4.13), but nothing can be done to simplify the expression unless some restriction is imposed on the asymptotic behavior of the sequence of polynomials $\{w_{2n}\}_{n\geq 0}$.

If $\mathrm{d}\,\mu_n = h_n\,\mathrm{d}\,\overline{\mu}$, where $\overline{\mu}$ is a fixed measure satisfying Szegő's condition on Δ, $\{h_n\}_{n\geq 0}$ is a sequence of positive continuous functions such that $\lim_n h_n = h$, and $\lim_{n\to\infty} |w_{2n}(x)|\varphi^n(x) = 1/\psi(x) > 0$ uniformly on Δ, the right hand side of (4.7) becomes $\frac{1}{\sqrt{2\pi}}\mathsf{G}(\psi h\overline{\mu}, u)$.

4.2.2 Proof of Theorem 4.2

We begin with an auxiliary lemma. {l1}

Lemma 4.5:
Assume that the sequence of polynomials $\{w_{2n}\}_{n\geq 0}$ *verifies* $(S4)$. *Then*

$$\lim_{n\to\infty} C^{2n}\Phi^{2n}(u)\frac{B_{2n}(u)}{w_{2n}(u)} = \mathsf{G}^{-2}(\psi, u) \tag{4.14}$$ {eq:a}

uniformly on compact subsets of $\overline{\mathbb{C}}\setminus\Delta$, *where* Φ *and* C *are defined as in* (4.6).

Proof. Notice that

$$C^{2n}\Phi^{2n}(u)\frac{B_{2n}(u)}{w_{2n}(u)} = \left(\frac{C^{2n}\Phi^{2n}(u)}{\Psi^{2n}(u)}\right)\left(\frac{\Psi^{2n}(u)B_{2n}(u)}{w_{2n}(u)}\right) \tag{4.15}$$ {eq:b}

and consider each factor in parentheses on the right hand side separately.

Define the function

$$f_{2n,i}(u) := \frac{\Psi(u)}{u - x_{2n,i}}\frac{\Psi(u) - \Psi(x_{2n,i})}{1 - \overline{\Psi(x_{2n,i})}\Psi(u)}.$$

It is easy to verify that this function is holomorphic and never vanishes in $\overline{\mathbb{C}}\setminus\Delta$. Also, $|f_{2n,i}|$ can be extended continuously to Δ with boundary values $|f_{2n,i}(x)| = |x - x_{2n,i}|^{-1}, x \in \Delta$. Moreover,

$$f_{2n,i}(u) = \frac{\Psi(u)}{u}\frac{1}{1 - x_{2n,i}u^{-1}}\frac{1 - \Psi(x_{2n,i})\Psi^{-1}(u)}{\Psi^{-1}(u) - \overline{\Psi(x_{2n,i})}},$$

thus $f_{2n,i}(\infty) = -\Psi'(\infty)/\overline{\Psi(x_{2n,i})}$. As $|f_{2n,i}|$ is continuous and different from zero in $\overline{\mathbb{C}}$, it follows that $f_{2n,i}$ and $f_{2n,i}^{-1}$ are in $H_1(\overline{\mathbb{C}}\setminus\Delta)$ with respect to the Chebyshev measure on Δ; consequently, $f_{2n,i}$ is an outer function (see [87, Chap. 17, Ex. 19]). Then,

$$f_{2n,i}(u) = c_i \exp\left[\frac{\sqrt{(u-a)(u-b)}}{\pi}\int_\Delta \frac{\ln|x - x_{2n,i}|}{x - u}\,\mathrm{d}\,\eta_\Delta(x)\right],$$

(see (1.17)) where c_i is a constant, $|c_i| = 1$. Should w_{2n} be monic, an easy consequence of this representation is

$$\frac{(\Psi^{2n}B_{2n})(u)}{w_{2n}(u)} = \prod_{i=1}^{2n} f_{2n,i}(u) = \exp\left[\frac{\sqrt{(u-a)(u-b)}}{\pi}\int_\Delta \frac{\ln|w_{2n}(x)|}{x - u}\,\mathrm{d}\,\eta_\Delta(x)\right]. \tag{4.16}$$ {eq:c}

(The product of all the constants c_i gives 1.) If w_{2n} is not monic then the same representation holds due to the fact that for any positive constant κ

$$\exp\left[\frac{\sqrt{(u-a)(u-b)}}{\pi}\int_\Delta \frac{\ln\kappa}{x-u}\,\mathrm{d}\,\eta_\Delta(x)\right] = \frac{1}{\kappa}.$$

On the other hand, $(C\Phi)^2/\Psi^2$ is analytic and different from zero in $\overline{\mathbb{C}}\setminus\Delta$. Moreover, $|C\Phi(x)/\Psi(x)|^2 = \exp(2\gamma - 2V_{\lambda_\varphi}(x)), x \in \Delta$, and using the equilibrium condition $|C\Phi(x)/\Psi(x)|^2 = \exp(-\ln\varphi(x)) = 1/\varphi(x), x \in \Delta$. Consequently, $C^2\Phi^2/\Psi^2$ is an outer function and we have

{asymptotic3}
$$\frac{C^{2n}\Phi^{2n}(u)}{\Psi^{2n}(u)} = \exp\left(\frac{\sqrt{(u-a)(u-b)}}{\pi}\int_\Delta \frac{n\ln\varphi(x)}{x-u}\,\mathrm{d}\,\eta_\Delta(x)\right). \tag{4.17}$$

Putting together (4.15), (4.16), and (4.17), we have

$$C^{2n}\Phi^{2n}(u)\frac{B_{2n}(u)}{w_{2n}(u)} = \exp\left(\frac{\sqrt{(u-a)(u-b)}}{\pi}\int_\Delta \frac{\ln(|w_{2n}(x)|\varphi^n(x))}{x-u}\,\mathrm{d}\,\eta_\Delta(x)\right).$$

To deduce (4.14) it remains to use (1.29) and the definition of $\mathsf{G}(\psi, u)$. □

With Lemma 4.5 at hand Theorem 4.2 is easy to derive.

Proof of Theorem 4.2. Note that

$$\frac{l_n^2(u)}{C^{2n}\Phi^{2n}(u)} = \frac{l_n^2(u)B_{2n}(u)}{w_{2n}(u)}\frac{w_{2n}(u)}{C^{2n}\Phi^{2n}(u)B_{2n}(u)}.$$

As $n\to\infty$, the limit of the first factor on the right is given by Lemma 4.4 and that of the second one by Lemma 4.5. The proof of (4.7) has been concluded.

Next, we deduce the asymptotic behavior of the monic orthogonal polynomials L_n and the leading coefficients τ_n of l_n. It is easy to see that

$$\Phi(u) = e^{-v_{\lambda_\varphi}(u)} = u + \mathscr{O}(1), \qquad u\to\infty.$$

Using (4.7) at $u=\infty$, we obtain (4.8). Then, (4.9) follows directly from (4.7) and (4.8). □

4.2.3 Proof of Theorem 4.3

We wish to express the orthogonality relations of the polynomials p_n in such a way that we can apply Theorem 4.2.

Notice that $|\Phi_\tau(x)|^{-1} = \exp V_\tau(x)$. According to [95, Theorem 10.2] (see also [95, Lemma 9.1]), there exists a sequence of polynomials $\{H_{n-1}\}_{n\geq 0}, \deg H_{n-1}\leq n-1$, which do not vanish on Δ whose zeros verify condition (S3) (see assertion on page 94 in [95]) such that

{eq:d}
$$|H_{n-1}(x)/\Phi_\tau^n(x)| \leq 1, \qquad x\in(a,b), \tag{4.18}$$

{eq:e} $$\lim_n |H_{n-1}(x)/\Phi_\tau^n(x)| = 1, \tag{4.19}$$

uniformly on compact subsets of (a, b), and

{eq:f} $$\lim_n \int_a^b \ln(|H_{n-1}(x)/\Phi_\tau^n(x)|)\, \mathrm{d}\,\eta_\Delta(x) = 0. \tag{4.20}$$

Now, the orthogonality relations satisfied by the polynomials p_n can be rewritten as

$$\int p_m(x)p_n(x)\frac{|H_{n-1}^2(x)|}{|\Phi_\tau^{2n}(x)|}\frac{\mathrm{d}\,\mu_n(x)}{|H_{n-1}^2(x)|} = \begin{cases} 0, & m < n, \\ 1, & m = n. \end{cases}$$

Let us check that the sequence

$$\left\{\frac{|H_{n-1}^2(x)|\,\mathrm{d}\,\mu_n}{|\Phi_\tau^{2n}(x)|}, H_{n-1}^2(x)\right\}_{n\ge 0}$$

verifies (S1)-(S4). Indeed, the zeros of the polynomials H_{n-1}, and thus of the polynomials H_{n-1}^2, $\deg H_{n-1}^2 \le 2n$, verify condition (S3). On the other hand, (4.18), (4.19), and condition (S1) for the sequence of measures $\{\mu_n\}_{n\ge 0}$ imply condition (S1) for the sequence of measures $\{(|H_{n-1}^2|\,\mathrm{d}\,\mu_n/|\Phi_\tau^{2n}|)\}_{n\ge 0}$ and

$$\liminf_n \int \ln\left(\frac{|H_{n-1}^2(x)|}{|\Phi_\tau^{2n}(x)|}\mu_n'(x)\right)\mathrm{d}\,\eta_\Delta(x) \ge \int \ln \mu'(x)\,\mathrm{d}\,\eta_\Delta(x);$$

therefore, (S2) takes place. Take $w_{2n} = H_{n-1}^2$ and $\varphi = e^{2V_\tau}$. Using (4.20) we obtain (S4) with $\psi \equiv 1$. The equilibrium condition corresponding to this case is the trivial one

$$V_\tau(x) - V_\tau(x) \equiv 0$$

and the equilibrium constant is $\gamma = 0$; therefore $C = 1$. Applying Theorem 4.2 the thesis of Theorem 4.3 readily follows. □

4.2.4 Applications to rational approximation

Let μ be a positive measure with $\operatorname{supp}\mu = \Delta$ that satisfies Szegő's condition and its Markov function $\widehat{\mu}$, as in (1.2). (In this section we take $h_n \equiv 1, n \ge 0$.) Consider a sequence of polynomials $\{w_{2n}\}_{n\ge 0}$ as above, positive on Δ.

From Problem 2 it is known that for each $n \ge 1$, there exists a rational function $R_n = \frac{L_{n-1}^*}{L_n}$, $\deg L_{n-1}^* \le n-1$ and $\deg L_n \le n$ such that

$$\frac{(L_n\widehat{\mu} - L_{n-1}^*)(z)}{w_{2n}(z)} = \frac{A_n}{z^{n+1}} + \cdots, \qquad z \to \infty$$

where the function on the left hand side is analytic in $\overline{\mathbb{C}} \setminus \Delta$. Recall that R_n is called the n-th multi-point Padé approximant of $\widehat{\mu}$ with respect to w_{2n}.

Moreover, we have already seen that L_n is an n-th orthogonal polynomial with respect to the varying measure μ/w_{2n} and it can be taken to be monic. The remainder of $\widehat{\mu} - R_n$ has the integral expression (see 1.3)

$$(\widehat{\mu} - R_n)(z) = \frac{w_{2n}(z)}{L_n^2(z)} \int_\Delta \frac{L_n^2(x)\,\mathrm{d}\,\mu(x)}{w_{2n}(x)(z-x)} = \frac{w_{2n}(z)}{l_n^2(z)} \int_\Delta \frac{l_n^2(x)\,\mathrm{d}\,\mu(x)}{w_{2n}(x)(z-x)},$$

where l_n denotes the corresponding orthonormal polynomial.

Taking into account [23, Theorem 8], we know that

$$\lim_n \int_\Delta \frac{l_n^2(x)\,\mathrm{d}\,\mu(x)}{|w_{2n}(x)|(z-x)} = \frac{1}{\sqrt{(z-b)(z-a)}},$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \Delta$, where the square root is chosen to be positive when $z > b$. So, a direct consequence of Lemma 4.4 and Theorem 4.2 is the next result.

Corollary 4.6:
Assume that $(S1)$-$(S3)$ *take place where* $h_n \equiv 1, n \geq 0$. *We have*

$$\lim_n \frac{(\widehat{\mu} - R_n)(z)}{B_{2n}(z)} = \frac{2\pi \mathsf{G}^{-2}(\mu, z)}{\sqrt{(z-a)(z-b)}}.$$

If, additionally, $(S4)$ *holds and* $\operatorname{supp} \lambda_\varphi = \Delta$, *then*

$$\lim_n \frac{(C\Phi)^{2n}(z)(\widehat{\mu} - R_n)(z)}{w_{2n}(z)} = \frac{2\pi \mathsf{G}^{-2}(\psi\mu, z)}{\sqrt{(z-a)(z-b)}}.$$

The limits are uniform on compact subsets of Ω.

4.3 Biorthogonal polynomials and multi level Hermite Padé polynomials

{sec:bio:ML}

4.3.1 Multilevel HP polynomials

{subsec:ML_HP}

Let Δ_1, Δ_2 be non-intersecting closed intervals of the real line. Let $(\sigma_1, \sigma_2) \in \mathscr{M}(\mathbf{\Delta})$ where $\mathbf{\Delta} = (\Delta_1, \Delta_2)$. Consider the Nikishin system $\mathscr{N}(\sigma_1, \sigma_2) := (s_{1,1}, s_{1,2})$, and recall that $s_{1,1} = \sigma_1$ and $\mathrm{d}\, s_{1,2}(x) = \widehat{\sigma}_2(x)\,\mathrm{d}\,\sigma_1(x)$ (see Definition 1.15 and (1.2)). Inverting the role of the measures we define similarly $s_{2,1}, \mathrm{d}\, s_{2,1}(x) = \widehat{\sigma}_1(x)\,\mathrm{d}\,\sigma_2(x)$, and keep in mind that $\mathscr{N}(\sigma_1, \sigma_2) \neq \mathscr{N}(\sigma_2, \sigma_1)$.

Nikishin system have found numerous applications in different areas of mathematics. In particular, the ones generated by two measures appear in the analysis of the two matrix model [13, 15] and in finding discrete solutions of the Degasperis-Procesi equation [14] through a Hermite-Padé approximation problem for two discrete measures. Motivated in [14], the approximation problem was extended in [62] for arbitrary $m \geq 2$ and general measures proving the convergence of the method. We will focus on the case of two measures.

Though some elements appearing next have been discussed in previous chapters, we esteem convenient to repeat them in the particular case of two measures. For each $n \in \mathbb{N}$, there exists a vector polynomial $(a_{n,0}, a_{n,1}, a_{n,2})$, not identically equal to zero, with $\deg a_{n,0} \le n-1$, $\deg a_{n,1} \le n-1$, and $\deg a_{n,2} \le n$, that satisfies

{JLS1}
$$\mathscr{A}_{n,0}(z) := \left(a_{n,0} - a_{n,1}\widehat{s}_{1,1} + a_{n,2}\widehat{s}_{1,2}\right)(z) = \mathscr{O}(1/z^{n+1}), \tag{4.21}$$

{JLS2}
$$\mathscr{A}_{n,1}(z) := \left(-a_{n,1} + a_{n,2}\widehat{s}_{2,2}\right)(z) = \mathscr{O}(1/z). \tag{4.22}$$

Here and below, the symbol $\mathscr{O}(\cdot)$ is taken as $z \to \infty$. By extension we take $\mathscr{A}_{n,2} \equiv a_{n,2}$. The polynomials $a_{n,0}, a_{n,1}, a_{n,2}$ are called multilevel Hermite-Padé polynomials.

It can be shown that $\deg a_{n,2} = n$ and the vector polynomial can be normalized taking $a_{n,2}$ monic. With this normalization $(a_{n,0}, a_{n,1}, a_{n,2})$ is unique. Moreover, all the zeros of $a_{n,2}$ are simple and lie in the interior $\mathring{\Delta}_2$ (with the Euclidean topology of $\mathbb{R}$) of the interval Δ_2. For more details, see [62, Theorem 1.4] and Lemma 4.7 below.

Combining Cauchy's theorem, Fubini's theorem, and Cauchy's integral formula, from (4.21) it follows that

$$\int x^{\nu} \mathscr{A}_{n,1}(x)\, \mathrm{d}\sigma_1(x) = 0, \qquad \nu = 0, \ldots, n-1,$$

and from (4.22) we get the integral representation

$$\mathscr{A}_{n,1}(x) = \int \frac{a_{n,2}(y)\, \mathrm{d}\sigma_2(y)}{x-y}.$$

Therefore,

$$\int\int \frac{x^{\nu} a_{n,2}(y)}{x-y}\, \mathrm{d}\sigma_1(x)\, \mathrm{d}\sigma_2(y) = 0, \qquad \nu = 0, \ldots, n-1.$$

Consequently $a_{n,2}$, normalized to be monic, verifies the same orthogonality relations as the biorthogonal polynomial Q_n (see (4.1)) and coincides with it.

Analogously, for each $n \in \mathbb{N}$, there exists a vector polynomial $(b_{n,0}, b_{n,1}, b_{n,2})$, not identically equal to zero, with $\deg b_{n,0} \le n-1$, $\deg b_{n,1} \le n-1$, and $\deg b_{n,2} \le n$, that satisfies

$$\mathscr{B}_{n,0}(z) := \left(b_{n,0} - b_{n,1}\widehat{s}_{2,2} + b_{n,2}\widehat{s}_{2,1}\right)(z) = \mathscr{O}(1/z^{n+1}), \tag{4.23}$$
{JLS1*}

$$\mathscr{B}_{n,1}(z) := \left(-b_{n,1} + b_{n,2}\widehat{s}_{1,1}\right)(z) = \mathscr{O}(1/z). \tag{4.24}$$
{JLS2*}

By extension we take $\mathscr{B}_{n,2} \equiv b_{n,2}$. Normalizing $b_{n,2}$ to be monic, we have $b_{n,2} = P_n$ (the other biorthogonal polynomial in (4.1)).

Therefore, in order to prove Theorem 4.1, we need to find the strong asymptotic of the sequences of polynomials $\{a_{n,2}\}_{n\ge 0}$ and $\{b_{n,2}\}_{n\ge 0}$. Because of the symmetry of the problem, it suffices to analyze the first sequence and the results for the second one are immediate.

Indeed, we will give the strong asymptotic of the forms $\mathscr{A}_{n,0}, \mathscr{A}_{n,1}$ and the polynomials $a_{n,0}, a_{n,1}, a_{n,2}$, as $n \to \infty$, under the assumption that the generating measures σ_1, σ_2 are in the Szegő class; that is, $(\sigma_1, \sigma_2) \in \mathscr{S}(\mathbf{\Delta})$ (see (4.2)). For general Nikishin systems of $m \ge 2$ measures, the logarithmic and ratio asymptotic of ML Hermite-Padé polynomials was studied in [32] (see also [66]).

4.3.2 Some useful properties

{subsec:use}

The forms $\mathscr{A}_{n,k}$, $k = 0, 1, 2$, are interlinked and satisfy interesting orthogonality relations which will be of great use. The following result, is a special case ($m = 2$) of [32, Lemma 2.4]. It is stated {l2} here for convenience of the reader.

Lemma 4.7:

Consider the Nikishin system $\mathcal{N}(\sigma_1, \sigma_2)$. For each fixed $n \in \mathbb{Z}_+$ and $j = 1, 2$, $\mathscr{A}_{n,j}$ has exactly n zeros in $\mathbb{C} \setminus \Delta_{j+1}$ they are all simple and lie in $\mathring{\Delta}_j$ ($\Delta_3 = \emptyset$). $\mathscr{A}_{n,0}$ has no zero in $\mathbb{C} \setminus \Delta_1$. Let $Q_{n,j}$, $j = 1, 2$, denote the monic polynomial of degree n whose zeros are those of $\mathscr{A}_{n,j}$ in Δ_j. For $j = 0, 1$,

{int1}
$$\frac{\mathscr{A}_{n,j}(z)}{Q_{n,j}(z)} = \int \frac{\mathscr{A}_{n,j+1}(x)}{z - x} \frac{\mathrm{d}\,\sigma_{j+1}(x)}{Q_{n,j}(x)}, \tag{4.25}$$

where $Q_{n,0} \equiv 1$, and

{int2}
$$\int x^\nu \mathscr{A}_{n,j+1}(x) \frac{\mathrm{d}\sigma_{j+1}(x)}{Q_{n,j}(x)} = 0, \qquad \nu = 0, \ldots, n - 1. \tag{4.26}$$

The orthogonality relations involving the linear forms $\mathscr{A}_{n,j}$ stated in (4.26) can be rewritten in terms of orthogonal polynomials with varying measures. That is

{int3}
$$0 = \int x^\nu Q_{n,2}(x) \frac{\mathrm{d}\sigma_2(x)}{Q_{n,1}(x)}, \qquad \nu = 0, \ldots, n - 1. \tag{4.27}$$

and

{int4}
$$0 = \int x^\nu Q_{n,1}(x) \mathscr{H}_{n,1}(x) \frac{\mathrm{d}\sigma_1(x)}{Q_{n,2}(x)}, \qquad \nu = 0, \ldots, n - 1. \tag{4.28}$$

where, using (4.25) with $j = 0$ and (4.27)

{hn1}
$$\mathscr{H}_{n,1}(z) := \frac{Q_{n,2}(z)\mathscr{A}_{n,1}(z)}{Q_{n,1}(z)} = Q_{n,2}(z) \int \frac{Q_{n,2}(x)}{z - x} \frac{\mathrm{d}\sigma_2(x)}{Q_{n,1}(x)} = \int \frac{Q_{n,2}^2(x)}{z - x} \frac{\mathrm{d}\,\sigma_2(x)}{Q_{n,1}(x)}. \tag{4.29}$$

{prop1}

Proposition 4.8:

There is a unique pair of monic polynomials with real coefficients $(Q_{n,1}, Q_{n,2})$ each one of degree n, whose zeros lie in $\mathbb{C} \setminus \Delta_2$ and $\mathbb{C} \setminus \Delta_1$, respectively, satisfying (4.27)-(4.28) *with*

$$\mathscr{H}_{n,1}(z) = \int \frac{Q_{n,2}^2(x)}{z - x} \frac{\mathrm{d}\,\sigma_2(x)}{Q_{n,1}(x)}.$$

Proof. The existence of such polynomials is guaranteed by Lemma 4.7. We must show that if $(Q_{n,1}, Q_{n,2})$ is a pair of monic polynomials of degree n which satisfy (4.27)-(4.28) with $\mathscr{H}_{n,1}$ as indicated then we can construct forms $\mathscr{A}_{n,0}, \mathscr{A}_{n,1}, \mathscr{A}_{n,2}$ verifying (4.21)-(4.22) whose zeros are those of the polynomials $Q_{n,1}, Q_{n,2}$.

So, let $(Q_{n,1}, Q_{n,2})$ be an arbitrary pair of monic polynomials of degree n which satisfy (4.27)-(4.28). Take $\mathscr{A}_{n,2} = a_{n,2} := Q_{n,2}$ and

$$a_{n,1}(z) := \int \frac{Q_{n,2}(z) - Q_{n,2}(x)}{z - x} \mathrm{d}\,\sigma_2(x).$$

Obviously, $a_{n,1}$ is a polynomial of degree $\leq n-1$. Rearranging this equality and using (4.27), we get

$$\mathscr{A}_{n,1}(z) := (-a_{n,1} + a_{n,2}\widehat{s}_{2,2})(z) = \int \frac{(Q_{n.1}Q_{n,2})(x)}{z-x}\frac{\mathrm{d}\,\sigma_2(x)}{Q_{n,1}(x)} = Q_{n,1}(z)\int \frac{Q_{n,2}(x)}{z-x}\frac{\mathrm{d}\,\sigma_2(x)}{Q_{n,1}(x)}.$$

The first equality tells us that $\mathscr{A}_{n,1}(z) = \mathscr{O}(1/z)$, so that (4.22) takes place, and the last equality implies that the zeros of $\mathscr{A}_{n,1}$ in $\mathbb{C}\setminus\Delta_2$ coincide with the simple roots that $Q_{n,1}$ has in the interior of Δ_1. Moreover, these relations together with (4.27)-(4.28) imply that for each $\nu = 0,1,\ldots,n-1$

$$\int x^\nu \mathscr{A}_{n,1}(x)\,\mathrm{d}\,\sigma_1(x) = \int x^\nu Q_{n,1}(x)\int \frac{Q_{n,2}(t)}{x-t}\frac{\mathrm{d}\,\sigma_2(t)}{Q_{n,1}(t)}\,\mathrm{d}\,\sigma_1(x) =$$

$$\int x^\nu Q_{n,1}(x)\int \frac{Q^2_{n,2}(t)}{x-t}\frac{\mathrm{d}\,\sigma_2(t)}{Q_{n,1}(t)}\frac{\mathrm{d}\,\sigma_1(x)}{Q_{n,2}(x)} = \int x^\nu Q_{n,1}(x)\mathscr{H}_{n,1}(x)\frac{\mathrm{d}\,\sigma_1(x)}{Q_{n,2}(x)} = 0.$$

These orthogonality relations verified by $\mathscr{A}_{n,1}$ in turn imply that

$$\int \frac{\mathscr{A}_{n,1}(x)}{z-x}\,\mathrm{d}\,\sigma_1(x) = \frac{1}{z^n}\int \frac{x^n\mathscr{A}_{n,1}(x)}{z-x}\,\mathrm{d}\,\sigma_1(x) = \mathscr{O}(1/z^{n+1}). \tag{4.30}$$ {orderAn0}

Using the definition of $\mathscr{A}_{n,1}(x)$, we get

$$a_{n,0}(z) := a_{n,1}(z)\widehat{s}_{1,1}(z) - a_{n,2}(z)\widehat{s}_{1,2}(z) + \int \frac{\mathscr{A}_{n.1}(x)}{z-x}\,\mathrm{d}\,\sigma_1(x) =$$

$$\int \frac{a_{n,1}(z)-a_{n,1}(x)}{z-x}\,\mathrm{d}\,\sigma_1(x) - \int \frac{a_{n,2}(z)-a_{n,2}(x)}{z-x}\,\mathrm{d}\,s_{1,2}(x),$$

which is obviously a polynomial of degree $\leq n-1$. Rearranging this equality and taking account of (4.30), it follows that

$$\mathscr{A}_{n,0}(z) := a_{n,0}(z) - a_{n,1}(z)\widehat{s}_{1,1}(z) + a_{n,2}(z)\widehat{s}_{1,2}(z) = \int \frac{\mathscr{A}_{n.1}(x)}{z-x}\,\mathrm{d}\,\sigma_1(x) = \mathscr{O}(1/z^{n+1}).$$

Thus, $\mathscr{A}_{n,0}$ verifies (4.21).

From our findings, we deduce that the vector polynomial $(a_{n,0}, a_{n,1}, a_{n,2})$ defined previously is the unique solution of (4.21)-(4.22). In particular, $a_{n,2} = Q_{n,2}$ is uniquely determined and by (4.28) so is $Q_{n,1}$ since the measure $(\mathscr{H}_{n,1}\,\mathrm{d}\,\sigma_1)/Q_{n,2}$ has constant sign on Δ_1. We are done. □

4.3.3 Normalization

{subsec:norm}

Set

$$\kappa_{n,2}^{-2} := \int Q^2_{n,2}(x)\frac{\mathrm{d}\,\sigma_2(x)}{|Q_{n,1}(x)|}, \qquad (\kappa_{n,1}\kappa_{n,2})^{-2} := \int Q^2_{n,1}(x)\frac{|\mathscr{H}_{n,1}(x)|\,\mathrm{d}\,\sigma_1(x)}{|Q_{n,2}(x)|}. \tag{4.31}$$ {kappa:4}

Take

$$q_{n,1} := \kappa_{n,1}Q_{n,1}, \qquad q_{n,2} := \kappa_{n,2}Q_{n,2}, \qquad h_{n,1} := \kappa^2_{n,2}\mathscr{H}_{n,1}. \tag{4.32}$$ {Qs}

Notice that

$$\kappa_{n,1}^{-2} := \int Q^2_{n,1}(x)\frac{|h_{n,1}(x)|\,\mathrm{d}\,\sigma_1(x)}{|Q_{n,2}(x)|}.$$

We can rewrite (4.27)-(4.28) as

$$0 = \int x^\nu q_{n,2}(x)\frac{\mathrm{d}\,\sigma_2(x)}{|Q_{n,1}(x)|} = 0, \qquad \nu = 0,\ldots,n-1, \tag{4.33}$$ {int5}

and

$$0 = \int x^\nu q_{n,1}(x)\frac{|h_{n,1}(x)|\,\mathrm{d}\,\sigma_1(x)}{|Q_{n,2}(x|)}, \qquad \nu = 0,\ldots,n-1. \tag{4.34}$$ {int6}

We also have

{int7} $$\int q_{n,2}^2(x)\frac{\mathrm{d}\sigma_2(x)}{|Q_{n,1}(x)|} = 1, \tag{4.35}$$

and

{int8} $$\int q_{n,1}^2(x)\frac{|h_{n,1}(x)|\,\mathrm{d}\,\sigma_1(x)}{|Q_{n,2}(x)|} = 1. \tag{4.36}$$

Consequently, $q_{n,1}$ and $q_{n,2}$ are the n-th orthonormal polynomials with respect to the varying measures $\frac{|h_{n,1}|\,\mathrm{d}\,\sigma_1}{|Q_{n,2}|}$ and $\frac{\mathrm{d}\,\sigma_2}{|Q_{n,1}|}$, respectively. Recall that the zeros of $Q_{n,j}$ lie in $\mathring{\Delta}_j = (a_j, b_j)$, $j = 1, 2$.

From [23, Theorem 8] it follows that if $\sigma_2' > 0$ a.e. on Δ_2, then for any bounded measurable function g_2 on Δ_2,

{int9} $$\lim_n \int \frac{g_2(x)q_{n,2}^2(x)\,\mathrm{d}\,\sigma_2(x)}{|Q_{n,1}(x)|} = \frac{1}{\pi}\int_{a_2}^{b_2} g_2(x)\,\mathrm{d}\,\eta_{\Delta_2}(x). \tag{4.37}$$

Taking into account (4.29) and using (4.37) with $g_2(x) = |t-x|^{-1}, t \in \Delta_1$, and (4.29), it follows that

{int12} $$\lim_n |h_{n,1}(t)| = \lim_n \int \frac{q_{n,2}^2(x)\,\mathrm{d}\,\sigma_2(x)}{|t-x||Q_{n,1}(x)|} = \tag{4.38}$$
$$= \frac{1}{\pi}\int_{a_2}^{b_2} \frac{\mathrm{d}\,\eta_{\Delta_2}(x)}{|t-x|} = \frac{1}{\sqrt{|t-a_2||t-b_2|}} =: h(t),$$

uniformly for $t \in \Delta_1$. Then (4.36), (4.38), and [23, Theorem 8] imply that if $\sigma_1' > 0$ a.e. on Δ_1, then for any bounded Borel measurable function g_1 on Δ_1 we have

{int10} $$\lim_n \int \frac{g_1(x)q_{n,1}^2(x)|h_{n,1}(x)|\mathrm{d}\sigma_1(x)}{|Q_{n,2}(x)|} = \frac{1}{\pi}\int_{a_1}^{b_1} g_1(x)\,\mathrm{d}\,\eta_{\Delta_1}(x). \tag{4.39}$$

4.3.4 The comparison functions

{subsec:comp}

The logarithmic asymptotic of general ML Hermite-Padé polynomials was studied in [32, Section 3]. In particular, it was proved that this asymptotic behavior can be described in terms of the solution of a vector equilibrium problem which, in the case we are dealing with, reduces to finding a pair of probability measures (λ_1, λ_2), $\operatorname{supp}\lambda_1 \subset \Delta_1$, $\operatorname{supp}\lambda_2 \subset \Delta_2$, and a pair of constants (γ_1, γ_2) such that

{vector_equil} $$\begin{cases} V_{\lambda_1}(x) - \frac{1}{2}V_{\lambda_2}(x) \equiv \gamma_1, & x \in \Delta_1, \\ V_{\lambda_2}(x) - \frac{1}{2}V_{\lambda_1}(x) \equiv \gamma_2, & x \in \Delta_2. \end{cases} \tag{4.40}$$

It is well known that this problem has a unique solution. From [32, Theorem 3.4] it follows that if σ_1 and σ_2 are regular measures (for the definition and properties of regular measures, see [92, Chap. 3]) then for $k = 1, 2$ we have

{weak}
$$\lim_n |Q_{n,k}|^{1/n} = \exp(-V_{\lambda_k}), \qquad \lim_n \kappa_{n,k}^{1/n} = \gamma_k, \tag{4.41}$$

where the first limit is uniform on compact subsets of $\mathbb{C} \setminus \Delta_k$.

Since strong asymptotic implies weak asymptotic, (4.41) reveals that the functions with which one must compare the polynomials $q_{n,1}, q_{n,2}$ in order to have strong asymptotic (should it exist) are tightly connected with the potentials $V_{\lambda_1}, V_{\lambda_2}$ and the constants γ_1, γ_2. With this in mind (see (4.6)), we define

$$\Phi_k(z) := e^{-(V_{\lambda_k} + i\widetilde{V}_{\lambda_k})(z)}, \qquad C_k := e^{\gamma_k}, \qquad k = 1, 2, \tag{4.42}$$
{compfunc}

where $\widetilde{V}_{\lambda_k}$ denotes the harmonic conjugate of V_{λ_k} in $\mathbb{C} \setminus \Delta_k$. For a different expression of the comparison functions see (4.85).

In (4.27)-(4.28) we see that the orthogonality relations verified by the polynomials $Q_{n,1}, Q_{n,2}$ are interconnected. This prevents the direct use of Theorem 4.2 to obtain their asymptotic because to give the asymptotic of one of the sequences one must know that of the second, and vice versa. So, as indicated in the introduction, we will follow an indirect approach devised by A.I. Aptekarev to attack analogous problems in [4] and [5].

4.3.5 Prescribed asymptotic behavior

{subsec:pres}

An important ingredient of the method consists in being capable of producing a sequence of functions of the form $P_{n,k}/\Phi_k^n$, $k = 1, 2$, where $P_{n,k}$ is a polynomial of degree n, whose limit is a predetermined Szegő function.

Let (λ_1, λ_2) be the solution of the vector equilibrium problem (4.40). From [25, Theorem 1.34] it follows that $\mathrm{d}\,\lambda_k = v_k \,\mathrm{d}\,x$ on Δ_k, $k = 1, 2$, and the weights v_1, v_2 verify the assumptions relative to v in Theorem 4.3 on the intervals Δ_1, Δ_2, respectively. In the sequel

$$\Omega_k := \overline{\mathbb{C}} \setminus \Delta_k, \qquad k = 1, 2.$$

{prop2}

Proposition 4.9:
Assume that $(\mu_1, \mu_2) \in \mathscr{S}(\boldsymbol{\Delta})$ and for each $n \geq 0$, $(\tilde{q}_{n,1}, \tilde{q}_{n,2})$ is the pair of polynomials of degree n such that

$$\int \tilde{q}_{n,k}(x)\tilde{q}_{m,k}(x)\frac{\mathrm{d}\,\mu_k(x)}{|\Phi_k^{2n}(x)|} = \begin{cases} 0, & 0 \leq m < n, \\ 1, & \quad m = n, \end{cases} \qquad k = 1, 2. \tag{4.43}$$
{poltilde}

Then

$$\lim_{n\to\infty} \frac{\tilde{q}_{n,k}(z)}{\Phi_k^n(z)} = \frac{\mathsf{G}(\mu_k, z)}{\sqrt{2\pi}}, \tag{4.44}$$
{predet}

and

$$\lim_{n\to\infty}\frac{\tilde{Q}_{n,k}(z)}{\Phi_k^n(z)}=\frac{\mathsf{G}(\mu_k,z)}{\mathsf{G}(\mu_k,\infty)},\tag{4.45}$$ {predet*}

uniformly on each compact subset of Ω_k, $k=1,2$, *where the* Φ_k *were introduced in* (4.42) *and* $\tilde{Q}_{n,k}$, $k=1,2$ *is* $\tilde{q}_{n,k}$ *renormalized to be monic.*

Proof. As was mentioned above, [25, Theorem 1.34] guarantees that the components of the equilibrium measures (λ_1,λ_2) are absolutely continuous with respect to the Lebesgue measure on the corresponding intervals and their weights v_1,v_2 verify (4.10) with parameters $\beta=\beta_0=-1/2$ on the intervals Δ_1,Δ_2, respectively. The assumptions of Theorem 4.3 are verified and (4.44) follows directly from (4.11). If $\tilde{\kappa}_{n,k}$ is the leading coefficient of $\tilde{q}_{n,k}$, applying (4.44) at $z=\infty$ we get

$$\lim_n \tilde{\kappa}_{n,k}=\frac{\mathsf{G}(\mu_k,\infty)}{\sqrt{2\pi}}$$

and (4.45) follows at once. □

4.3.6 The operator $\tilde{T}_n$

{subsec:operTn}

Definition 4.10:

Let $(\sigma_1,\sigma_2)\in\mathscr{M}(\mathbf{\Delta})$. *Let* $\mathscr{P}_{n,k}$, $k=1,2$, *be the set of all monic polynomials with real coefficients of degree n whose zeros lie in* $\mathbb{C}\setminus\Delta_2$ *when* $k=1$ *and in* $\mathbb{C}\setminus\Delta_1$ *when* $k=2$. *Define an operator*

$$\tilde{T}_n:\mathscr{P}_{n,1}\times\mathscr{P}_{n,2}\longrightarrow\mathscr{P}_{n,1}\times\mathscr{P}_{n,2}$$

where, for every $(\widehat{Q}_{n,1},\widehat{Q}_{n,2})\in\mathscr{P}_{n,1}\times\mathscr{P}_{n,2}$

{Tn}
$$\tilde{T}_n(\widehat{Q}_{n,1},\widehat{Q}_{n,2}):=(Q^*_{n,1},Q^*_{n,2}),\tag{4.46}$$

being $(Q^*_{n,1},Q^*_{n,2})$ *the unique pair of monic polynomials of degree n constructed recursively as follows. First, find the polynomial* $Q^*_{n,2}$ *verifying*

$$\int x^\nu Q^*_{n,2}(x)\frac{\mathrm{d}\,\sigma_2(x)}{\widehat{Q}_{n,1}(x)}=0,\quad \nu=0,1,\ldots,n-1.$$

Second, define $\mathscr{H}^*_{n,2}\equiv 1$ *and*

$$\mathscr{H}^*_{n,1}(z):=\int\frac{(Q^*_{n,2}(x))^2}{z-x}\frac{\mathrm{d}\,\sigma_2(x)}{\widehat{Q}_{n,1}(x)}.$$

Third, find the polynomial $Q^*_{n,1}$ *verifying*

$$\int x^\nu Q^*_{n,1}(x)\frac{\mathscr{H}^*_{n,1}(x)\,\mathrm{d}\,\sigma_1(x)}{\widehat{Q}_{n,2}(x)}=0,\quad \nu=0,1,\ldots,n-1.$$

Finally, set

{kappa*}
$$K^*_{n,2}:=1,\ K^*_{n,k-1}:=\left(\int(Q^*_{n,2}(x))^2\frac{\mathrm{d}\,\sigma_2(x)}{|\tilde{Q}_{n,1}(x)|}\right)^{-1/2},\ \kappa^*_{n,k}:=\frac{K^*_{n,k-1}}{K^*_{n,k}},\ k=1,2;\tag{4.47}$$

and take

{hn}
$$q^*_{n,k}:=\kappa^*_{n,k}Q^*_{n,k},\quad h^*_{n,k}:=(K^*_{n,k})^2\mathscr{H}^*_{n,k},\quad k=1,2.\tag{4.48}$$

Remark: Due to [23, Theorem 8] an easy consequence of the definition of $h^*_{n,1}$ is that

$$\lim_{n\to\infty} |h^*_{n,1}(z)| = \frac{1}{|\sqrt{(z-a_{k+1})(z-b_{k+1})}|}, \tag{4.49}$$

uniformly on compact subsets of $\mathbb{C}\setminus\Delta_{k+1}$.

Let us prove the continuity of the operator $\tilde{T}_n$, for each fixed $n\geq 0$. Given $\mathbf{Q}_1=(Q_{1,1},Q_{1,2})$, $\mathbf{Q}_2=(Q_{2,1},Q_{2,2})\in\mathscr{P}_{n,1}\times\mathscr{P}_{n,2}$ define the metric d_n as follows

$$d_n(\mathbf{Q}_1,\mathbf{Q}_2)=\max\{\|Q_{1,1}-Q_{1,2}\|_{\Delta_2},\|Q_{2,1}-Q_{2,2}\|_{\Delta_1}\}$$

where $\|\cdot\|_\Delta$ denotes the sup-norm on the interval Δ. Suppose that

$$\lim_{m\to\infty} d_n(\mathbf{Q}_m,\mathbf{Q})=0$$

and $\mathbf{Q}_m=(Q_{m,1},Q_{m,2}),\mathbf{Q}=(Q_1,Q_2)\in\mathscr{P}_{n,1}\times\mathscr{P}_{n,2}$. From the location of the zeros of the polynomials it readily follows that

$$\lim_{m\to\infty}\|Q_{m,1}^{-1}-Q_1^{-1}\|_{\Delta_2}=0,\qquad \lim_{m\to\infty}\|Q_{m,2}^{-1}-Q_2^{-1}\|_{\Delta_1}=0.$$

Therefore, for each fixed $\nu\geq 0$

$$\lim_{m\to\infty}\int x^\nu\frac{\mathrm{d}\,\sigma_1}{|Q_{m.2}|}=\int x^\nu\frac{\mathrm{d}\,\sigma_1}{|Q_2|}=:c_\nu \tag{4.50}$$ {moments}

and similarly for the other sequence of polynomials. If Q_1^* is the n-th monic orthogonal polynomial with respect to $\frac{\mathrm{d}\,\sigma_1}{|Q_2|}$, the determinantal formula for the orthogonal polynomials allows to write

$$Q_1^*(x)=C_n^{-1}\begin{vmatrix} c_0 & c_1 & \cdots & c_n\\ c_1 & c_2 & \cdots & c_{n+1}\\ \vdots & \vdots & \ddots & \vdots\\ c_{n-1} & c_n & \cdots & c_{2n-1}\\ 1 & x & \cdots & x^n\end{vmatrix},\qquad C_n=\begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1}\\ c_1 & c_2 & \cdots & c_n\\ \vdots & \vdots & \ddots & \vdots\\ c_{n-1} & c_n & \cdots & c_{2n-2}\end{vmatrix}\neq 0,$$

and similarly for the other n-th orthogonal polynomials $Q^*_{m,1},Q^*_{m,2},Q^*_2$. The determinantal formula shows that the orthogonal polynomials depend continuously on the moments of the corresponding measure. This together with (4.50) clearly imply that

$$\lim_{m\to\infty} d_n(\mathbf{Q}^*_m,\mathbf{Q}^*)=0.$$

From Proposition 4.8 it follows that

$$\tilde{T}_n(Q_{n,1},Q_{n,2})=(Q_{n,1},Q_{n,2}),$$

where $(Q_{n,1},Q_{n,2})$ is the unique pair of polynomials of degree n verifying (4.33)-(4.36). Therefore, $(Q_{n,1},Q_{n,2})$ is a fixed point of the operator $\tilde{T}_n$. In the case of arbitrary $\tilde{h}_n$ it is not difficult to prove that $\tilde{T}_n$ also has fixed points. (In general, it may not be unique.)

Indeed, given n if $(\tilde{Q}_{n,1}, \tilde{Q}_{n,2})$ is a fixed point then the n zeros of $\tilde{Q}_{n,1}$ must lie in Δ_1 and the n zeros of $\tilde{Q}_{n,2}$ must be in Δ_2. Consequently, it is sufficient to restrict the operator $\tilde{T}_n$ to the class $\tilde{\mathscr{P}}_{n,1} \times \tilde{\mathscr{P}}_{n,2}$ of all pairs of monic polynomials whose first component has all its zeros on Δ_1 and the second has its zeros on Δ_2. Suppose that

$$\tilde{Q}_{n,1}(x) = \prod_{j=1}^{n}(x - x_{n,j}), \qquad \tilde{Q}_{n,2}(x) = \prod_{j=1}^{n}(x - y_{n,j}).$$

Assume that the zeros are indexed in such a way that

$$a_1 \leq x_{n,1} \leq \cdots \leq x_{n,n} \leq b_1, \qquad a_2 \leq y_{n,1} \leq \cdots \leq y_{n,n} \leq b_2.$$

There is a canonical homeomorphism between $\tilde{\mathscr{P}}_{n,1} \times \tilde{\mathscr{P}}_{n,2}$ and $\tilde{\Delta}_1 \times \tilde{\Delta}_2$, where $\tilde{\Delta}_k, k = 1,2$, is the subset of Δ_k^n made up of all points whose coordinates are increasing, given by

$$(\tilde{Q}_{n,1}, \tilde{Q}_{n,2}) \longrightarrow ((x_{n,1}, \ldots, x_{n,n}), (y_{n,1}, \ldots, y_{n,n}))$$

The operator $\tilde{T}_n$ induces an operator from $\tilde{\Delta}_1 \times \tilde{\Delta}_2$ into itself, where the image is determined by the zeros of $(Q^*_{n,1}, Q^*_{n,2}) = \tilde{T}_n(\tilde{Q}_{n,1}, \tilde{Q}_{n,2})$. The induced operator is continuous and $\tilde{\Delta}_1 \times \tilde{\Delta}_2$ is a convex compact subset of $\mathbb{R}^n \times \mathbb{R}^n$; therefore, by Brouwer's fixed point theorem [42, Th. 7.2 (3)] the induced operator has at least one fixed point. Consequently, so does $\tilde{T}_n$.

{prop3} We are ready to use Theorem 4.2.

Proposition 4.11:
Assume that $(\mu_1, \mu_2) \in \mathscr{S}(\mathbf{\Delta})$ and for each $n \geq 0$, $(\tilde{Q}_{n,1}, \tilde{Q}_{n,2})$ is the pair of monic polynomials of degree n which satisfies (4.45). *Let $(\sigma_1, \sigma_2) \in \mathscr{S}(\mathbf{\Delta})$ and let $(Q^*_{n,1}, Q^*_{n,2}) = \tilde{T}_n(\tilde{Q}_{n,1}, \tilde{Q}_{n,2})$ where $\{\tilde{h}_n\}_{n\geq 0}$ fulfills* (4.48). *Then*

{limfund1}
$$\lim_n \frac{q^*_{n,1}(z)}{C_1^n \Phi_1^n(z)} = \frac{1}{\sqrt{2\pi}} \mathsf{G}(f_2^{-1}\tilde{h}\sigma_1, z), \qquad \lim_n \frac{q^*_{n,2}(z)}{C_2^n \Phi_2^n(z)} = \frac{1}{\sqrt{2\pi}} \mathsf{G}(f_1^{-1}\sigma_2, z), \tag{4.51}$$

uniformly on compact subsets of Ω_1 and Ω_2, respectively, $f_k = \mathsf{G}(\mu_k, \cdot)/\mathsf{G}(\mu_k, \infty), k = 1,2,$ and $q^*_{n,k} = \kappa^*_{n,k} Q^*_{n,k}$ is the corresponding orthonormal polynomial of degree n (see (4.47)). *Additionally,*

{conductor1}
$$\lim_n \frac{\kappa^*_{n,1}}{C_1^n} = \frac{1}{\sqrt{2\pi}} \mathsf{G}(f_2^{-1}\tilde{h}\sigma_1, \infty), \qquad \lim_n \frac{\kappa^*_{n,2}}{C_2^n} = \frac{1}{\sqrt{2\pi}} \mathsf{G}(f_1^{-1}\sigma_2, \infty), \qquad k = 1,2. \tag{4.52}$$

Consequently,

{limfund*1}
$$\lim_n \frac{Q^*_{n,1}(z)}{\Phi_1^n(z)} = \frac{\mathsf{G}(f_2^{-1}\tilde{h}\sigma_1, z)}{\mathsf{G}(f_2^{-1}\tilde{h}\sigma_1, \infty)}, \qquad \lim_n \frac{Q^*_{n,2}(z)}{\Phi_2^n(z)} = \frac{\mathsf{G}(f_1^{-1}\sigma_2, z)}{\mathsf{G}(f_1^{-1}\sigma_2, \infty)}. \tag{4.53}$$

Proof. It is easy to see that the sequences $(\tilde{h}_n \, d\,\sigma_1, \tilde{Q}_{n,2})_{n\geq 0}, (d\,\sigma_2, \tilde{Q}_{n,1})_{n\geq 0}$ verify (S1)-(S4) on the intervals Δ_1 and Δ_2. Therefore, the assumptions of Theorem 4.2 are fulfilled. Consequently, (4.51)-(4.53) follow directly from Proposition 4.9 and Theorem 4.2 taking into account the equilibrium equations (4.40) verified by the equilibrium measures and the defining formulas (4.42) for the functions Φ_k and constants C_k, $k = 1,2$. □

Formulas (4.51)-(4.53) describe the strong asymptotic behavior of the components of the image of $\tilde{T}_n$. In the next section, we give an operator approach to interpret the limiting functions appearing in these relations.

4.3.7 The operator $T_{\mathbf{w}}$

{subsec:op:T}

The Szegő functions which describe the limits (4.51)-(4.53) verify the boundary equations (see (1.17)-(1.19))

$$|\mathsf{G}(f_2^{-1}\tilde{h}\sigma_1, x)|^2 = \frac{|f_2(x)|}{\sqrt{(b_1-x)(x-a_1)}(\tilde{h}\sigma_1')(x)}, \quad \text{a.e. on} \quad [a_1,b_1] = \Delta_1, \tag{4.54}$$ {boundeq1}

and

$$|\mathsf{G}(f_1^{-1}\sigma_2, x)|^2 = \frac{|f_1(x)|}{\sqrt{(b_2-x)(x-a_2)}\sigma_2'(x)}, \quad \text{a.e. on} \quad [a_2,b_2] = \Delta_2. \tag{4.55}$$ {boundeq2}

The functions f_1, f_2 themselves are expressed in terms of Szegő functions and $\tilde{h}$ is a positive continuous function on Δ_1. The Szegő functions above are symmetric with respect to the real line, never equal zero, and are positive at infinity. Consequently, on the real line, outside of the intervals supporting their defining measures, they are positive. Relations (4.54)-(4.55) suggest the definition of an operator.

Let $\mathbf{\Delta} = (\Delta_1, \Delta_2)$. We denote by $\mathbf{C}_{\mathbf{\Delta}}$ the space of all pairs $\mathbf{g} = (g_1, g_2)$ of real valued functions such that g_1 is continuous on Δ_2 and g_2 is continuous on Δ_1. The functions g_1 and g_2 could be defined on certain supersets of Δ_2 and Δ_1, respectively, but for the time being we only need to specify their analytic properties as indicated. Set

$$\|\mathbf{g}\|_{\mathbf{C}_{\mathbf{\Delta}}} := \max\{\|g_1\|_{\Delta_2}, \|g_2\|_{\Delta_1}\},$$

where $\|\cdot\|_X$ denotes the sup norm on X. Obviously $(\mathbf{C}_{\mathbf{\Delta}}, \|\cdot\|_{\mathbf{C}_{\mathbf{\Delta}}})$ is a Banach space. Consider the cone $\mathbf{C}_{\mathbf{\Delta}}^+$ of all the vectors in $\mathbf{C}_{\mathbf{\Delta}}$ such that g_1 is positive on Δ_2 and g_2 is positive on Δ_1. The application $(g_1, g_2) \mapsto (\ln g_1, \ln g_2)$ establishes a homeomorphism between $\mathbf{C}_{\mathbf{\Delta}}^+$ and $\mathbf{C}_{\mathbf{\Delta}}$. Given $\mathbf{g}^{(1)} = (g_1^{(1)}, g_2^{(1)}), \mathbf{g}^{(2)} = (g_1^{(2)}, g_2^{(2)}) \in \mathbf{C}_{\mathbf{\Delta}}^+$, set

$$d(\mathbf{g}^{(1)}, \mathbf{g}^{(2)}) := \max\{\|\ln(g_1^{(1)}/g_1^{(2)})\|_{\Delta_2}, \|\ln(g_2^{(1)}/g_2^{(2)})\|_{\Delta_1}\}.$$

It is easy to check that $(\mathbf{C}_{\mathbf{\Delta}}^+, d)$ is a complete metric space. Certainly, on $\mathbf{C}_{\mathbf{\Delta}}^+$ we can also consider the norm $\|\cdot\|_{\mathbf{C}_{\mathbf{\Delta}}}$ but $\mathbf{C}_{\mathbf{\Delta}}^+$ is not complete with that norm; however, given a sequence $(\mathbf{g}^{(n)})_{n\geq 0} \subset \mathbf{C}_{\mathbf{\Delta}}^+$ and $\mathbf{g} \in \mathbf{C}_{\mathbf{\Delta}}^+$, we have

$$\lim_n \|\mathbf{g}^{(n)} - \mathbf{g}\|_{\mathbf{C}_{\mathbf{\Delta}}} = 0 \quad \Leftrightarrow \quad \lim_n d(\mathbf{g}^{(n)}, \mathbf{g}) = 0. \tag{4.56}$$ {equiv}

Now we are in a position to give a precise definition of the operator hinted by relations (4.54)-(4.55).

{oper_T}

Definition 4.12:

Let w_1 and w_2 be two integrable functions satisfying Szegő's condition on Δ_1 and Δ_2, respectively, and write $\mathbf{w} = (w_1, w_2)$. Define the operator

$$T_{\mathbf{w}} : \mathbf{C}^+_{\mathbf{\Delta}} \longrightarrow \mathbf{C}^+_{\mathbf{\Delta}},$$

where $T_{\mathbf{w}}(g_1, g_2) = (g_1^*, g_2^*)$ is the pair of Szegő functions, $g_k^* \in \mathbf{H}(\Omega_k)$, $k = 1, 2$, verifying

$$|g_1^*(x)|^2 = \frac{g_2(x)}{w_1(x)}, \quad \text{a.e. on} \quad [a_1, b_1] = \Delta_1,$$
$$|g_2^*(x)|^2 = \frac{g_1(x)}{w_2(x)}, \quad \text{a.e. on} \quad [a_2, b_2] = \Delta_2.$$

From the definition of the Szegő function it readily follows that g_1^* is positive and continuous on $\mathbb{R} \setminus \Delta_1 \supset \Delta_2$ and g_2^* is positive and continuous on $\mathbb{R} \setminus \Delta_2 \supset \Delta_1$.

Finding g_k^*, $k = 1, 2$, reduces to solving the Dirichlet problems for a harmonic function u_k in Ω_k, with boundary values integrable on Δ_k and equal to $\frac{1}{2}\ln(g_2/w_1)$ a.e. on Δ_1 in the case of u_1 and $\frac{1}{2}\ln(g_1/w_2)$ a.e. on Δ_2 in the case of u_2, and the subsequent problem of finding their harmonic conjugates $\widetilde{u}_k, \widetilde{u}_k(\infty) = 0$, in Ω_k. Then

$$g_k^* = \exp(u_k + i\,\widetilde{u}_k), \qquad k = 1, 2.$$

Set $\mathbf{\Omega} = (\Omega_1, \Omega_2)$. Let $\mathbf{h}_{\mathbf{\Omega}}$ be the set of pairs of harmonic functions in Ω_1 and Ω_2, respectively, with integrable boundary values. Given $\mathbf{g} = (g_1, g_2) \in \mathbf{C}^+_{\mathbf{\Delta}}$ let $\chi = (\chi_1, \chi_2) = (\ln g_1, \ln g_2) \in \mathbf{C}_{\mathbf{\Delta}}$. The map $T_{\mathbf{w}}$ induces the map

$$t_{\mathbf{w}} : \mathbf{C}_{\mathbf{\Delta}} \longrightarrow \mathbf{h}_{\mathbf{\Omega}} \subset \mathbf{C}_{\mathbf{\Delta}},$$

where

$$t_{\mathbf{w}}(\chi) := \frac{1}{2}(P(\chi) + \beta), \qquad \chi = (\chi_1, \chi_2)^t \in \mathbf{C}_{\mathbf{\Delta}},$$

$\beta = (\beta_1, \beta_2)^t$ is the (column) vector made up of harmonic functions with boundary values

$$\beta_k(x) = -\ln w_k(x), \qquad \text{a.e. on} \quad \Delta_k,\ k = 1, 2$$

and P is the linear operator

$$P := \begin{pmatrix} 0 & P_{1,2} \\ P_{2,1} & 0 \end{pmatrix},$$

such that $P_{1,2}(\chi_2)$ is the harmonic function on Ω_1 with boundary values equal to χ_2 on Δ_1, and $P_{2,1}(\chi_1)$ is the harmonic function on Ω_2 with boundary values equal to χ_1 on Δ_2.

{prop4} The following result is contained in [5, Proposition 1.1].

Proposition 4.13:

The operator $T_{\mathbf{w}}$ (see Def. 4.12) is a contraction in $\mathbf{C}^+_{\mathbf{\Delta}}$ with respect to the metric d. More precisely

$$d(T_{\mathbf{w}}(\mathbf{g}^{(1)}), T_{\mathbf{w}}(\mathbf{g}^{(2)})) \leq \frac{1}{2} d(\mathbf{g}^{(1)}, \mathbf{g}^{(2)}), \qquad \mathbf{g}^{(1)}, \mathbf{g}^{(2)} \in \mathbf{C}^+_{\mathbf{\Delta}}.$$

Therefore, the map $T_{\mathbf{w}}$ has a unique fixed point in $\mathbf{C}^+_{\mathbf{\Delta}}$.

Proof. The proof is simple so, for completeness, we include it. As mentioned above, $(\mathbf{C}^+_{\mathbf{\Delta}}, d)$ is a complete metric space so the second statement follows from the first.

Set $\chi^{(k)} := (\ln g_1^{(k)}, \ln g_2^{(k)})$, $k = 1, 2$. From the definitions of $d, T_{\mathbf{w}}$, and $t_{\mathbf{w}}$ it follows that

$$d(T_{\mathbf{w}}(\mathbf{g}^{(1)}), T_{\mathbf{w}}(\mathbf{g}^{(2)})) = \|t_{\mathbf{w}}(\chi^{(1)}) - t_{\mathbf{w}}(\chi^{(2)})\|_{\mathbb{C}_{\mathbf{\Delta}}} = \frac{1}{2}\|P(\chi^{(1)}) - P(\chi^{(2)})\|_{\mathbb{C}_{\mathbf{\Delta}}} \leq$$

$$\frac{1}{2}\|P\|\|\chi^{(1)} - \chi^{(2)}\|_{\mathbb{C}_{\mathbf{\Delta}}} = \frac{1}{2}d(\mathbf{g}^{(1)}, \mathbf{g}^{(2)}),$$

where in the last inequality the maximum principle is used to establish that $\|P\| = 1$. □

Consider now the particular case $T_{\widetilde{\mathbf{w}}}$ where

$$\widetilde{\mathbf{w}} := \left(\sqrt{(b_1 - x)(x - a_1)}(\tilde{h}\sigma_1')(x),\ \sqrt{(b_2 - x)(x - a_2)}\sigma_2'(x)\right). \tag{4.57}$$ {oper_T_tilde}

Here $\tilde{h}, \sigma_1, \sigma_2$ are the same as in (4.48), (4.54) and (4.55). Then, let $\mathsf{G} = (\mathsf{G}_1, \mathsf{G}_2)$ be the unique fixed point of the operator $T_{\widetilde{\mathbf{w}}}$. That is

$$T_{\widetilde{\mathbf{w}}}(\mathsf{G}) = \mathsf{G}$$

and the components of G are characterized by the system of boundary values

$$|\mathsf{G}_1(x)|^2 = \frac{\mathsf{G}_2(x)}{\sqrt{(b_1 - x)(x - a_1)}(\tilde{h}\sigma_1')(x)}, \quad \text{a.e. on} \quad [a_1, b_1] = \Delta_1,$$

and

$$|\mathsf{G}_2(x)|^2 = \frac{\mathsf{G}_1(x)}{\sqrt{(b_2 - x)(x - a_2)}\sigma_2'(x)}, \quad \text{a.e. on} \quad [a_2, b_2] = \Delta_2.$$

Obviously, the components of G are Szegő functions in Ω_1 and Ω_2, respectively.

Now, we must show that any neighborhood of a fixed point of the operator $T_{\widetilde{\mathbf{w}}}$ determines fixed points of the operators $\tilde{T}_n$ for all sufficiently large n. By Proposition 4.8, when we take $\tilde{h}_n = |h_{n,1}|$ as in (4.32), the operator $\tilde{T}_n$ has only one fixed point. We need one last ingredient. {prop5}

Proposition 4.14:
Let $(\tilde{Q}_{n,1}, \tilde{Q}_{n,2})_{n\geq 0}$ be an arbitrary sequence of vector polynomials such that $(\tilde{Q}_{n,1}, \tilde{Q}_{n,2}) \in \mathscr{P}_{n,1} \times \mathscr{P}_{n,2}$. Set

$$\mathbf{f}^{(n)} = \left(\frac{\tilde{Q}_{n,1}}{\Phi_1^n}, \frac{\tilde{Q}_{n,2}}{\Phi_2^n}\right).$$

Assume that there exists $\mathbf{f} = (f_1, f_2) \in \mathbf{C}^+_{\mathbf{\Delta}}$ and a sequence of non-negative integers Λ such that

$$\lim_{n\in\Lambda} \|\mathbf{f}^{(n)} - \mathbf{f}\|_{\mathbf{C}_{\mathbf{\Delta}}} = 0. \tag{4.58}$$ {limden}

Let $(\sigma_1, \sigma_2) \in \mathscr{S}(\mathbf{\Delta})$ and let $(Q^*_{n,1}, Q^*_{n,2}) = \tilde{T}_n(\tilde{Q}_{n,1}, \tilde{Q}_{n,2})$ where $(\tilde{h}_n)_{n\geq 0}$ fulfills (4.48). *Then*

$$\lim_{n\in\Lambda} \frac{q^*_{n,1}(z)}{C_1^n \Phi_1^n(z)} = \frac{1}{\sqrt{2\pi}}\mathsf{G}(f_2^{-1}\tilde{h}\sigma_1, z), \qquad \lim_{n\in\Lambda} \frac{q^*_{n,2}(z)}{C_2^n \Phi_2^n(z)} = \frac{1}{\sqrt{2\pi}}\mathsf{G}(f_1^{-1}\sigma_2, z), \tag{4.59}$$ {limfund}

uniformly on compact subsets of Ω_1 and Ω_2, respectively. Additionally,

$$\lim_{n\in\Lambda} \frac{\kappa^*_{n,1}}{C_1^n} = \frac{1}{\sqrt{2\pi}}\mathsf{G}(f_2^{-1}\tilde{h}\sigma_1, \infty), \qquad \lim_{n\in\Lambda} \frac{\kappa^*_{n,2}}{C_2^n} = \frac{1}{\sqrt{2\pi}}\mathsf{G}(f_1^{-1}\sigma_2, \infty), \qquad k = 1, 2.$$

Consequently,

$$\lim_{n\in\Lambda}\frac{Q^*_{n,1}(z)}{\Phi_1^n(z)}=\frac{\mathsf{G}(f_2^{-1}\tilde{h}\sigma_1,z)}{\mathsf{G}(f_2^{-1}\tilde{h}\sigma_1,\infty)},\qquad \lim_{n\in\Lambda}\frac{Q^*_{n,2}(z)}{\Phi_2^n(z)}=\frac{\mathsf{G}(f_1^{-1}\sigma_2,z)}{\mathsf{G}(f_1^{-1}\sigma_2,\infty)}.\tag{4.60}$$ {limfund*}

Proof. The proof is identical to that of Proposition 4.11. In that proof, it is not used that the full sequences of indices is considered and the result only depends on the asymptotic behavior of the sequences of denominators of the varying part of the measures of orthogonality on the intervals Δ_1 and Δ_2, respectively, for which assumption (4.58) was included. The details are omitted. □

4.3.8 Proof of Theorem 4.15

{subsec:3.8}

Let $\mathsf{G}=(\mathsf{G}_1,\mathsf{G}_2)$ be the fixed point of the operator $T_{\tilde{\mathbf{w}}}$ (see Def. 4.12 and (4.57)). The function G_k, $k=1,2$ is a Szegő function in Ω_k; therefore, the value $\mathsf{G}_k(\infty)$ is well defined. Set

$$H^{+,n}:=\left\{\left(\frac{\mathsf{G}_1(\infty)P_{n,1}}{\Phi_1^n},\frac{\mathsf{G}_2(\infty)P_{n,2}}{\Phi_2^n}\right):P_{n,k}\in\mathscr{P}_{n,k},k=1,2\right\}.$$

(Recall that $\mathscr{P}_{n,k}$ is the set of all monic polynomials of degree n with real coefficients whose zeros lie in Ω_j, $j\neq k$, $j,k=1,2$.)

Let $\tilde{T}_{n,1}$ and $\tilde{T}_{n,2}$ be the operators defined on $\mathscr{P}_{n,1}\times\mathscr{P}_{n,2}$ which determine the components of $\tilde{T}_n$; that is, $\tilde{T}_n=(\tilde{T}_{n,1},\tilde{T}_{n,2})$ (see (4.46)). Define

$$T_n:H^{+,n}\longrightarrow H^{+,n}$$

where

$$T_n\left(\frac{\mathsf{G}_1(\infty)P_{n,1}}{\Phi_1^n},\frac{\mathsf{G}_2(\infty)P_{n,2}}{\Phi_2^n}\right)=\left(\frac{\mathsf{G}_1(\infty)\tilde{T}_{n,1}(P_{n,1},P_{n,2})}{\Phi_1^n},\frac{\mathsf{G}_1(\infty)\tilde{T}_{n,2}(P_{n,1},P_{n,2})}{\Phi_2^n}\right).$$

Notice that any fixed point of T_n generates a fixed point of $\tilde{T}_n$. The continuity of $\tilde{T}_n$ implies the
{main} continuity of T_n.

Theorem 4.15:
Let $(\sigma_1,\sigma_2)\in\mathscr{S}(\mathbf{\Delta})$ and $\{\tilde{h}_n\}_{n\geq 0}$ fulfills (4.48)*. Then, there exists a sequence $\{(Q_{n,1},Q_{n,2})\}_{n\geq 0}$, where $(Q_{n,1},Q_{n,2})$ is a fixed point of $\tilde{T}_n$, such that*

{limfund***}
$$\lim_{n}\frac{Q_{n,k}(z)}{\Phi_k^n(z)}=\frac{\mathsf{G}_k(z)}{\mathsf{G}_k(\infty)},\qquad k=1,2,\tag{4.61}$$

uniformly on compact subsets of Ω_k, where $\mathsf{G}=(\mathsf{G}_1,\mathsf{G}_2)$ is the fixed point of $T_{\tilde{\mathbf{w}}}$ and Φ_k are as in (4.42)*. Additionally, if*

$$(\kappa_{n,2})^{-2}:=\int Q_{n,2}^2(x)\frac{\mathrm{d}\sigma_2(x)}{|Q_{n,1}(x)|},\qquad (\kappa_{n,1})^{-2}:=\int Q_{n,1}^2(x)\frac{\tilde{h}_n(x)\mathrm{d}\sigma_1(x)}{|Q_{n,2}(x)|},$$

and $q_{n,k}=\kappa_{n,k}Q_{n,k}$, $k=1,2$, then

$$\lim_n\frac{\kappa_{n,k}}{C_k^n}=\frac{1}{\sqrt{2\pi}}\mathsf{G}_k(\infty),$$

and

{limfund2} $$\lim_n \frac{q_{n,k}(z)}{C_k^n \Phi_k^n(z)} = \frac{1}{\sqrt{2\pi}} \mathsf{G}_k(z), \qquad k = 1, 2, \tag{4.62}$$

uniformly on compact subsets of Ω_k.

Proof. Due to the way in which $H^{+,n}, T_n$, and $T_{\widetilde{\mathbf{w}}}$ were defined, the statements (4.61)-(4.62) follow directly from Proposition 4.14 if we show that there exists a sequence $\{\mathbf{g}^{(n)}\}_{n \geq n_0}$, where $\mathbf{g}^{(n)}$ is a fixed point of T_n, such that

$$\lim_{n \geq n_0} \|\mathbf{g}^{(n)} - \mathsf{G}\|_{\mathbf{C}_\Delta} = 0. \tag{4.63}$$ {fixed}

The components of G are Szegő functions in Ω_k, $k = 1, 2$, respectively. Let $\mathbf{K} = (K_1, K_2)$ be a pair of non intersecting closed disks, symmetric with respect to $\mathbb{R}$, whose interior in the Euclidean topology of $\mathbb{C}$ verify

$$\Delta_2 \subset \mathring{K}_1, \qquad \Delta_1 \subset \mathring{K}_2.$$

By $H^+(\mathbf{K})$ we denote the cone of all pairs (g_1, g_2) of functions such that g_k is holomorphic and different from zero in $\mathring{K}_k$, and positive on $\mathring{K}_k \cap \mathbb{R}$. For $\mathbf{g} = (g_1, g_2) \in H^+(\mathbf{K})$ we define

$$\|\mathbf{g}\|_{\mathbf{K}} := \max\{\{\sup |g_k(z)| : z \in \mathring{K}_k\} : k = 1, 2\} \ (\leq \infty),$$

and

$$\min_{\mathbf{\Delta}} \mathbf{g} := \min\{\min_{x \in \Delta_2} g_1(x), \min_{x \in \Delta_1} g_2(x)\}.$$

Fix a constant $C > 0$. Define

$$H^+(\mathbf{K}, C) := \{\mathbf{g} \in H^+(\mathbf{K}) : \|\mathbf{g}\|_{\mathbf{K}} \leq C, \min_{\mathbf{\Delta}} \mathbf{g} \geq C^{-1}\}.$$

Take C sufficiently large so that $\mathsf{G} \in H^+(\mathbf{K}, C)$.

Let $\mathbf{g}^{(1)}, \mathbf{g}^{(2)} \in H^+(\mathbf{K}, C)$ and let $0 \leq \beta \leq 1$, then

$$\|\beta \mathbf{g}^{(1)} + (1 - \beta) \mathbf{g}^{(2)}\|_{\mathbf{K}} \leq \beta \|\mathbf{g}^{(1)}\|_{\mathbf{K}} + (1 - \beta) \|\mathbf{g}^{(2)}\|_{\mathbf{K}} \leq C,$$

and

$$\min_{\mathbf{\Delta}} \left(\beta \mathbf{g}^{(1)} + (1 - \beta) \mathbf{g}^{(2)}\right) \geq \beta \min_{\mathbf{\Delta}} \mathbf{g}^{(1)} + (1 - \beta) \min_{\mathbf{\Delta}} \mathbf{g}^{(2)} \geq C^{-1}.$$

Therefore, $\beta \mathbf{g}^{(1)} + (1 - \beta) \mathbf{g}^{(2)} \in H^+(\mathbf{K}, C)$. This shows that $H^+(\mathbf{K}, C)$ is convex.

On the other hand, if $(\mathbf{g}^{(n)})_{n \geq 0}$ is an arbitrary sequence of elements in $H^+(\mathbf{K}, C)$. Then the components form normal families in $\mathring{K}_1$ and $\mathring{K}_2$, respectively. Therefore, there exists a sequence of indices Λ such that $(\mathbf{g}^{(n)})_{n \in \Lambda}$ converges componentwise to some vector function $\mathbf{g}$ uniformly on each compact subset of $\mathring{K}_1$ and $\mathring{K}_2$, respectively. The components of $\mathbf{g}$ are, therefore, analytic on $\mathring{K}_1$ and $\mathring{K}_2$, respectively. The uniform limit of holomorphic functions which never equal zero must either be identically equal to zero or never zero. The first case is not possible because

$$\min_{\mathbf{\Delta}} \mathbf{g} = \lim_{n \in \Lambda} \min_{\mathbf{\Delta}} \mathbf{g}^{(n)} \geq C^{-1}.$$

Also

$$\|\mathbf{g}\|_{\mathbf{K}} = \lim_{n\in\Lambda} \|\mathbf{g}^{(n)}\|_{\mathbf{K}} \leq C.$$

Consequently, $\mathbf{g} \in H^+(\mathbf{K}, C)$. We conclude that $H^+(\mathbf{K}, C)$ is compact.

Fix an arbitrary $\theta > 0$. Let

$$\omega(\theta) = \{\mathbf{g} \in H^+(\mathbf{K}, C) : \|\mathbf{g} - \mathsf{G}\|_{\mathbf{C}_\Delta} \leq \theta\}.$$

This is a closed subset of $H^+(\mathbf{K}, C)$ and, therefore, it is compact. Obviously, it is convex. Analogously, for every $\varepsilon > 0$, set

$$\omega_\varepsilon := \{\mathbf{g} \in H^+(\mathbf{K}, C) : d(\mathbf{g}, \mathsf{G}) \leq \varepsilon\}.$$

There exists ε_0 such that

$$\omega_\varepsilon \subset \omega(\theta), \qquad 0 < \varepsilon \leq \varepsilon_0,$$

for, otherwise, we could find a sequence of vector functions in $H^+(\mathbf{K}, C) \subset \mathbf{C}^+_\Delta$ which converges to G in the d metric but not in the $\|\cdot\|_{\mathbf{C}_\Delta}$ norm which would contradict (4.56).

Take

$$\omega_{\varepsilon,n} = \omega_\varepsilon \cap H^{+,n}.$$

For each fixed n the set $\omega_{\varepsilon,n}$ is a closed, bounded subset of a finite dimensional space, therefore it is compact.

Let μ_k be the representing measure of G_k so that $\mathsf{G}_k(z) = \mathsf{G}(\mu_k, z), z \in \Omega_k$. From Proposition 4.9, using (4.44) and (4.56) it follows that

$$\lim_n d(\mathbf{g}^{(n)}, \mathsf{G}) = 0,$$

where

$$\mathbf{g}^{(n)} := \left(\frac{\mathsf{G}_1(\infty)\tilde{Q}_{n,1}}{\Phi_1^n}, \frac{\mathsf{G}_2(\infty)\tilde{Q}_{n,2}}{\Phi_2^n}\right)$$

and $\tilde{Q}_{n,k}, k = 1, 2$, is given by (4.43). Consequently, for every fixed $\varepsilon, 0 < \varepsilon \leq \varepsilon_0$, there exists n_0 such that $\omega_{\varepsilon,n} \neq \emptyset$ for all $n \geq n_0$. Using the structure of the elements of $H^{+,n}$, the definition of the metric d, and the monotonicity of the logarithm, it is easy to verify that $\omega_{\varepsilon,n}$ is convex.

Let us show that $T_n(\omega_{\varepsilon,n}) \subset \omega_{\varepsilon,n}$ for all sufficiently large n. We claim that there exists n_0 such that for all $n \geq n_0$ and $\mathbf{g} \in \omega_{\varepsilon,n}$, we have

{desig} $$d(T_n(\mathbf{g}), T_{\tilde{\mathbf{w}}}(\mathbf{g})) < \varepsilon/2. \tag{4.64}$$

Should this not occur, we could find a sequence $\{\mathbf{g}^{(n)}\}_{n\in\Lambda}, \mathbf{g}^{(n)} \in \omega_{\varepsilon,n}$, such that

{contra} $$d(T_n(\mathbf{g}^{(n)}), T_{\tilde{\mathbf{w}}}(\mathbf{g}^{(n)})) \geq \varepsilon/2 \tag{4.65}$$

The elements of $\omega_{\varepsilon,n}$ belong to $H^+(K, C)$; therefore, $\{\mathbf{g}^{(n)}\}_{n\in\Lambda}$ is uniformly bounded in the $\|\cdot\|_{\mathbf{K}}$ norm. Consequently, there exists $\mathbf{g} \in H^+(K, C)$ and a subsequence of indices $\Lambda' \subset \Lambda$ such that

$$\lim_{n\in\Lambda'} \|\mathbf{g}^{(n)} - \mathbf{g}\|_{\mathbf{K}} = 0.$$

In particular,

$$\lim_{n\in\Lambda'} \|\mathbf{g}^{(n)} - \mathbf{g}\|_{\mathbf{C}_\Delta} = 0.$$

Then, according to (4.60) in Proposition 4.14

$$\lim_{n\in\Lambda'} \|T_n(\mathbf{g}^{(n)}) - T_{\widetilde{\mathbf{w}}}(\mathbf{g})\|_{\mathbf{C}_\Delta} = 0$$

which contradicts (4.65) due to (4.56).

Using the triangle inequality and (4.64) for every $n \geq n_0$ and $\mathbf{g} \in \omega_{\varepsilon,n}$

$$d(\mathsf{G}, T_n(\mathbf{g})) \leq d(T_{\widetilde{\mathbf{w}}}(\mathsf{G}), T_{\widetilde{\mathbf{w}}}(\mathbf{g})) + d(T_{\widetilde{\mathbf{w}}}(\mathsf{g}), T_n(\mathbf{g})) < \frac{1}{2}d(\mathsf{G},\mathbf{g}) + \frac{1}{2}\varepsilon \leq \varepsilon.$$

Consequently, $T_n(\omega_{\varepsilon,n}) \subset \omega_{\varepsilon,n}$ as claimed. Now, using Brouwer's fixed point Theorem [42, Th. 7.2 (3)] we obtain that for all $n \geq n_0$ the operator T_n has a fixed point in $\omega_{\varepsilon,n}$.

Since $\theta > 0$ is arbitrary, we have shown that (4.63) is true and we conclude the proof. □

To apply Theorem 4.15 to the case of ML Hermite-Padé polynomials we need to select $\tilde{h}_n = |h_{n,1}|$, $\tilde{h} = h$ (see (4.32) and (4.38)), and

$$\tilde{\mathbf{w}} = \mathbf{w}_Q := (\sqrt{(b_1 - x)(x-a_1)}(h\sigma_1')(x),\ \sqrt{(b_2-x)(x-a_2)}\sigma_2'(x)), \tag{4.66}$$ {oper_TQ}

to determine the operator $T_{\mathbf{w}_Q}$ (see Def. 4.12). {ML}

Theorem 4.16:

Let $(\sigma_1, \sigma_2) \in \mathscr{S}(\mathbf{\Delta})$. Let $(Q_{n,1}, Q_{n,2})_{n\geq 0}$ be the sequence of ML Hermite-Padé polynomials defined by (4.27)-(4.29) *and let $\kappa_{n,1}, \kappa_{n,2}, q_{n,1}, q_{n,2}$, and $h_{n,1}$ be defined as in* (4.31)-(4.32). *Finally, let $\mathsf{G} = (\mathsf{G}_1, \mathsf{G}_2)$ be the fixed point of the operator $T_{\mathbf{w}_Q}$, as in Definition* 4.12 *and* (4.66). *Then*

$$\lim_n \frac{Q_{n,k}(z)}{\Phi_k^n(z)} = \frac{\mathsf{G}_k(z)}{\mathsf{G}_k(\infty)}, \qquad k = 1,2,$$

uniformly on compact subsets of Ω_k. Additionally,

$$\lim_n \frac{\kappa_{n,k}}{C_k^n} = \frac{1}{\sqrt{2\pi}}\mathsf{G}_k(\infty), \tag{4.67}$$ {conductor*}

and

$$\lim_n \frac{q_{n,k}(z)}{C_k^n \Phi_k^n(z)} = \frac{1}{\sqrt{2\pi}}\mathsf{G}_k(z), \qquad k = 1,2, \tag{4.68}$$ {limfund**}

uniformly on compact subsets of Ω_k.

Proof. It is sufficient to apply Theorem 4.15 with $\tilde{h}_n = |h_{n,1}|$, taking note that (4.38) takes place and that according to Proposition 4.8 the operator $\tilde{T}_n$ has a unique fixed point in all of $\mathscr{P}_{n,1} \times \mathscr{P}_{n,2}$ which coincides with $(Q_{n,1}, Q_{n,2})$ □

Our method differs from Aptekarev's in two aspects. Proposition 4.9, which plays a key role, is derived using arguments from complex function theory. The corresponding result in [5, Theorem 2] uses a quite intricate approximative construction on a Riemann surface which is not very transparent. Secondly, in [5], Widom's approach introduced in [99] is followed closely to obtain L_2 estimates, on segments of the real line, of the asymptotic behavior of the multiple orthogonal polynomials. Thus, the results are obtained for measures in the Szegő class which are absolutely continuous with respect to the Lebesgue measure. We use instead the results obtained in Section 2 on orthogonal polynomials with respect to varying measures and do not need to restrict to absolutely continuous measures. In consequence, we only give the asymptotic in the complement of the intervals.

4.3.9 Proof of Theorem 4.1

Recall that the biorthogonal polynomial Q_n coincides with $Q_{n,2}$. Consequently, the second relation in (4.3) follows directly from (4.60).

To obtain the asymptotic of the biorthogonal polynomials P_n we need a result similar to Theorem 4.16 working with the definition (4.23)-(4.24) corresponding to the Nikishin system $\mathcal{N}(\sigma_2, \sigma_1)$. We outline the main ingredients.

From Lemma 4.7 and Proposition 4.8 it follows that there exist a unique pair $(P_{n,1}, P_{n,2})$ of monic polynomials of degree n, where $P_{n,2} = b_{n,2} = P_n$, such that

{orth_P2}

$$\int x^\nu P_{n,2}(x) \frac{\mathrm{d}\,\sigma_1(x)}{P_{n,1}(x)} = 0, \qquad \nu = 0, 1, \ldots, n-1, \tag{4.69}$$

{orth_P1}

$$\int x^\nu P_{n,1}(x) \frac{\mathcal{L}_{n,1}(x)\,\mathrm{d}\,\sigma_2(x)}{P_{n,2}(x)} = 0, \qquad \nu = 0, 1, \ldots, n-1, \tag{4.70}$$

where

$$\mathcal{L}_{n,1}(z) := \frac{\mathcal{B}_{n,1}(z) P_{n,2}(z)}{P_{n,1}(z)} = P_{n,2}(z) \int \frac{P_{n,2}(x)}{z-x} \frac{\mathrm{d}\,\sigma_1(x)}{P_{n,1}(x)} = \int \frac{P^2_{n,2}(x)}{z-x} \frac{\mathrm{d}\,\sigma_1(x)}{P_{n,1}(x)}.$$

The normalization in this case is

$$\xi_{n,2}^{-2} = \int P^2_{n,2}(x) \frac{\mathrm{d}\,\sigma_1(x)}{|P_{n,1}(x)|}, \qquad (\xi_{n,1}\xi_{n,2})^{-2} = \int P^2_{n,1}(x) \frac{|\mathcal{L}_{n,1}(x)|\,\mathrm{d}\,\sigma_2(x)}{|P_{n,2}(x)|}.$$

Take,

$$p_{n,1} = \xi_{n,1} P_{n,1}, \qquad p_{n,2} = \xi_{n,2} P_{n,2}, \qquad \ell_{n,1} = \xi^2_{n,2} \mathcal{L}_{n,1}.$$

Then, the orthogonality relation (4.69) and (4.70) can be restated as

$$\int x^\nu p_{n,2}(x) \frac{\mathrm{d}\,\sigma_1(x)}{|P_{n,1}(x)|} = 0, \qquad \nu = 0, 1, \ldots, n-1,$$

$$\int x^\nu p_{n,1}(x) \frac{|\ell_{n,1}(x)|\,\mathrm{d}\,\sigma_2(x)}{|P_{n,2}(x)|} = 0, \qquad \nu = 0, 1, \ldots, n-1,$$

and the polynomials $p_{n,2}$ and $p_{n,1}$ are orthonormal with respect to the corresponding varying measures. Following the same arguments that led us to (4.38), we obtain

{ell}

$$\lim_n |\ell_{n,1}(t)| = \frac{1}{\sqrt{|t-a_1||t-b_1|}} =: \ell(t) \tag{4.71}$$

uniformly for $t \in \Delta_2$.

The operator $T_{\mathbf{w}_P} : \mathbf{C}^+_{\mathbf{\Delta}} \longrightarrow \mathbf{C}^+_{\mathbf{\Delta}}$ which is relevant to describe the strong asymptotic of the polynomials $P_{n,2}, P_{n,1}$ and their orthonormal versions $p_{n,2}, p_{n,1}$ is the one determined by

{oper_TP}

$$\mathbf{w}_P := (\sqrt{(b_1-x)(x-a_1)}\sigma_1'(x),\ \sqrt{(b_2-x)(x-a_2)}(\ell\sigma_2')(x)). \tag{4.72}$$

In other words, $T_{\mathbf{w}_P}(g_1, g_2) = (g_1^*, g_2^*)$ is the pair of Szegő functions, $g_k^* \in \mathscr{H}(\Omega_k), k = 1,2$, verifying

$$|g_1^*(x)|^2 = \frac{g_2(x)}{\sqrt{(b_1-x)(x-a_1)}\sigma_1'(x)}, \quad \text{a.e. on} \quad [a_1,b_1] = \Delta_1, \tag{4.73}$$

{boundeq3*}

and

$$|g_2^*(x)|^2 = \frac{g_1(x)}{\sqrt{(b_2-x)(x-a_2)}(\ell\sigma_2')(x)}, \quad \text{a.e. on} \quad [a_2,b_2] = \Delta_2, \tag{4.74}$$

{boundeq4*}

where ℓ is given in (4.71).

Following the same reasoning as before, if $(\mathsf{G}_1^*, \mathsf{G}_2^*)$ is the fixed point of the operator $T_{\mathbf{w}_P}$ defined through (4.73)-(4.74), we have

$$\lim_n \frac{P_{n,1}(z)}{\Phi_2^n(z)} = \frac{\mathsf{G}_2^*(z)}{\mathsf{G}_2^*(\infty)}, \qquad \lim_n \frac{P_{n,2}(z)}{\Phi_1^n(z)} = \frac{\mathsf{G}_1^*(z)}{\mathsf{G}_1^*(\infty)} \tag{4.75}$$

{lim_P*}

uniformly on compact subsets of Ω_2 and Ω_1, respectively. Since $P_n = P_{n,2}$, the second limit in (4.75) gives us the strong asymptotic of the sequence $(P_n)_{n\geq 0}$, We are done. □

Naturally, the asymptotic of the sequence of normalizing coefficients and of the orthonormal polynomials $p_{n,1}, p_{n,2}$ can also be given. We leave the details to the reader.

4.3.10 Asymptotic of ML Hermite-Padé polynomials

{subsec:asym:M

In this subsection, as an easy consequence of the strong asymptotic of the polynomials $Q_{n,1}$ and $Q_{n,2}$, we obtain the strong asymptotic of $\mathscr{A}_{n,j}$, and $a_{n,j}$, $j = 0,1$. Recall that $Q_{n,2} \equiv a_{n,2} \equiv \mathscr{A}_{n,2}$.

Let f be a function which has a constant sign on some interval $\Delta \subset \mathbb{R}$. We define

$$\mathrm{sg}_\Delta(f) := \begin{cases} 1, & f > 0 \ \text{ on } \ \Delta, \\ -1, & f < 0 \ \text{ on } \ \Delta. \end{cases}$$

{asym_forms}

Corollary 4.17:
Let $(\sigma_1, \sigma_2) \in \mathscr{S}(\mathbf{\Delta})$ and $\mathscr{A}_{n,j}$, $j = 0,1$, is defined by (4.21)–(4.22). Then,

$$\lim_n \mathrm{sg}_{\Delta_2}(Q_{n,1}) \frac{\kappa_{n,2}^2 \mathscr{A}_{n,1}(z)}{(\Phi_1/\Phi_2)^n(z)} = \frac{\mathsf{G}_2(\infty)}{\mathsf{G}_1(\infty)} \frac{\mathsf{G}_1(z)}{\mathsf{G}_2(z)} \frac{1}{\sqrt{(z-b_2)(z-a_2)}}, \tag{4.76}$$

{asym_An1}

and

$$\lim_n \mathrm{sg}_{\Delta_1}\left(\frac{h_{n,1}}{Q_{n,2}}\right) \frac{(\kappa_{n,1}\kappa_{n,2})^2 \mathscr{A}_{n,0}(z)}{\Phi_1^{-n}(z)} = \frac{\mathsf{G}_1(z)}{\mathsf{G}_1(\infty)} \frac{1}{\sqrt{(z-a_1)(z-b_1)}}, \tag{4.77}$$

{asym_An0}

where the limits are uniform on compact subsets of $\overline{\mathbb{C}} \setminus (\Delta_1 \cup \Delta_2)$ and $\overline{\mathbb{C}} \setminus \Delta_1$, respectively.

Proof. Formula (4.29) can be rewritten as

$$\mathscr{A}_{n,1}(z) = \frac{Q_{n,1}(z)}{Q_{n,2}(z)} \int \frac{Q^2_{n,2}(x)}{z-x} \frac{\mathrm{d}\,\sigma_2(x)}{Q_{n,1}(x)},$$

where the equality holds in Ω_2. Then,

{asym_An1_1} $$\mathrm{sg}_{\Delta_2}(Q_{n,1}) \frac{\kappa^2_{n,2}\mathscr{A}_{n,1}(z)}{(\Phi_1/\Phi_2)^n(z)} = \frac{Q_{n,1}(z)}{\Phi_1^n(z)} \frac{\Phi_2^n(z)}{Q_{n,2}(z)} \int \frac{q^2_{n,2}(x)}{z-x} \frac{\mathrm{d}\,\sigma_2(x)}{|Q_{n,1}(x)|}, \quad z \in \overline{\mathbb{C}} \setminus (\Delta_1 \cup \Delta_2). \tag{4.78}$$

From (4.37) with $g_2(x) = (z-x)^{-1}$, we have

{asym_An1_3} $$\lim_n \int \frac{q^2_{n,2}(x)}{z-x} \frac{\mathrm{d}\,\sigma_2(x)}{|Q_{n,1}(x)|} = \frac{1}{\sqrt{(z-b_2)(z-a_2)}}, \tag{4.79}$$

uniformly on compact subsets of Ω_2. This, together with (4.60) and (4.78), gives us (4.76).

Combining (4.25) for $j=0$ with (4.29) we get

$$\mathscr{A}_{n,0}(z) = \int \frac{Q_{n,1}(x)}{z-x} \frac{\mathscr{H}_{n,1}(x)\,\mathrm{d}\,\sigma_1(x)}{Q_{n,2}(x)}.$$

By orthogonality, we have

$$Q_{n,1}(z) \int \frac{Q_{n,1}(x)}{z-x} \frac{\mathscr{H}_{n,1}(x)\,\mathrm{d}\,\sigma_1(x)}{Q_{n,2}(x)} = \int \frac{Q^2_{n,1}(x)}{z-x} \frac{\mathscr{H}_{n,1}(x)\,\mathrm{d}\,\sigma_1(x)}{Q_{n,2}(x)}.$$

Therefore,

$$\mathscr{A}_{n,0}(z) = \frac{1}{Q_{n,1}(z)} \int \frac{Q^2_{n,1}(x)}{z-x} \frac{\mathscr{H}_{n,1}(x)\,\mathrm{d}\,\sigma_1(x)}{Q_{n,2}(x)},$$

where the equality holds for $z \in \Omega_1$. So,

{asym_An0_1} $$\mathrm{sg}_{\Delta_1}\left(\frac{h_{n,1}}{Q_{n,2}}\right) \frac{(\kappa_{n,1}\kappa_{n,2})^2 \mathscr{A}_{n,0}(z)}{\Phi_1^{-n}(z)} = \frac{\Phi_1^n(z)}{Q_{n,1}(z)} \int \frac{q^2_{n,1}(x)}{z-x} \frac{|h_{n,1}(x)|\,\mathrm{d}\,\sigma_1(x)}{|Q_{n,2}(x)|}. \tag{4.80}$$

From (4.39) with $g_1(x) = (z-x)^{-1}$, it follows that

$$\lim_n \int \frac{q^2_{n,1}(x)}{z-x} \frac{|h_{n,1}(x)|\,\mathrm{d}\,\sigma_1(x)}{|Q_{n,2}(x)|} = \frac{1}{\sqrt{(z-a_1)(z-b_1)}},$$

uniformly on compact subsets of Ω_1. This formula combined with (4.80) and (4.60) gives us (4.77). We have completed the proof. □

Regarding the sign functions in (4.76) and (4.77) it is easy to deduce the following (see (4.29))

$$\mathrm{sg}_{\Delta_2}(Q_{n,1}) = \begin{cases} 1, & n \text{ is even}, \\ 1, & n \text{ is odd}, \quad b_1 < a_2, \\ -1, & n \text{ is odd}, \quad b_2 < a_1, \end{cases} \qquad \mathrm{sg}_{\Delta_1}\left(\frac{h_{n,1}}{Q_{n,2}}\right) = \begin{cases} -1, & n \text{ is even}, \quad b_1 < a_2, \\ 1, & n \text{ is even}, \quad b_2 < a_1, \\ 1, & n \text{ is odd}, \quad b_1 < a_2, \\ -1, & n \text{ is odd}, \quad b_2 < a_1. \end{cases}$$

Notice that

$$\widehat{\sigma}_2(z) - \frac{a_{n,1}(z)}{a_{n,2}(z)} = \frac{\mathscr{A}_{n,1}(z)}{Q_{n,2}(z)} = \frac{Q_{n,1}(z)}{q^2_{n,2}(z)} \int \frac{q^2_{n,2}(x)}{z-x} \frac{\mathrm{d}\,\sigma_2(x)}{Q_{n,1}(x)}.$$

Therefore, using (4.60), (4.68), and (4.79), we obtain

{cero}
$$\lim_n \left|\frac{C_2^2\Phi_2^2(z)}{\Phi_1(z)}\right|^n \left|\widehat{\sigma}_2(z) - \frac{a_{n,1}(z)}{a_{n,2}(z)}\right| = \frac{2\pi|\mathsf{G}_1(z)|}{\mathsf{G}_1(\infty)|\mathsf{G}_2^2(z)|\sqrt{|z-a_2||z-b_2|}}, \tag{4.81}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus (\Delta_1 \cup \Delta_2)$. The definition of Φ_1, Φ_2, and C_2 imply

$$\left|\frac{C_2^2\Phi_2^2(z)}{\Phi_1(z)}\right| = \exp\left(-(2V_{\lambda_2}(z) - V_{\lambda_1}(z) - 2\gamma_2)\right).$$

The second equilibrium equation in (4.40) and the maximum principle for subharmonic function entail

$$2V_{\lambda_2}(z) - V_{\lambda_1}(z) - 2\gamma_2 < 0, \qquad z \in \Omega_2.$$

Consequently, (4.81) gives a precise description of the rate with which $(a_{n,1}/a_{n,2})_{n\geq 0}$ converges to $\widehat{\sigma}_2$. Exactly the same formula can be obtained substituting in (4.81) $\widehat{\sigma}_2 - a_{n,1}/a_{n,2}$ by $\widehat{s}_{2,1} - a_{n,0}/a_{n,2}$. However, we will not dwell into this because it requires the introduction of new transformations which drive us off track.

From [62, Th. 1.6], we know that, for $j = 0, 1$

$$\lim_n \frac{a_{n,j}(z)}{a_{n,2}(z)} = \widehat{s}_{2,j+1}(z),$$

where the limit is uniform on compact subsets of Ω_2. {asym_pol_anj}

Corollary 4.18:
Let $(\sigma_1, \sigma_2) \in \mathscr{S}(\mathbf{\Delta})$. Then, for $j = 0, 1$

$$\lim_n \frac{a_{n,j}(z)}{\Phi_2^n(z)} = \frac{\mathsf{G}_2(z)}{\mathsf{G}_2(\infty)} \widehat{s}_{2,j+1}(z),$$

where the limit is uniform on compact subsets of Ω_2.

Proof. Since $a_{n,2} \equiv Q_{n,2}$, we have

$$\lim_n \frac{a_{n,j}(z)}{\Phi_2^n(z)} \frac{\Phi_2^n(z)}{Q_{n,2}(z)} = \widehat{s}_{2,j+1}(z).$$

Taking into account (4.68) the proof readily follows. □

Results analogous to Corollaries 4.17 and 4.18 for the forms $\mathscr{B}_{n,0}, \mathscr{B}_{n,1}$, and the polynomials $b_{n,0}, b_{n,1}$, follow immediately considering the Nikishin system $\mathscr{N}(\sigma_2, \sigma_1)$. The details are left to the reader.

4.3.11 A different expression for the functions Φ_1, Φ_2 and the constants C_1, C_2

{subsec:otro}

In [32, Theorem 4.2] the ratio asymptotic of general ML Hermite-Padé polynomials was given. The limit was expressed in terms of the branches of a conformal map of a certain Riemann surface. Since strong asymptotic implies ratio asymptotic, we can use that result to interpret the comparison functions Φ_1, Φ_2 and the constants C_1, C_2 in a different way. (Which coincides with the form in which they were defined in [5].)

We introduce the Riemann surface which is relevant in our case of two measures. Let $\mathscr{R}$ denote the compact Riemann surface

$$\mathscr{R} = \overline{\bigcup_{k=0}^{2} \mathscr{R}_k}$$

formed by 3 consecutively "glued" copies of the extended complex plane

$$\mathscr{R}_0 := \overline{\mathbb{C}} \setminus \Delta_1, \qquad \mathscr{R}_1 := \overline{\mathbb{C}} \setminus (\Delta_1 \cup \Delta_2), \quad \mathscr{R}_2 := \overline{\mathbb{C}} \setminus \Delta_2.$$

The upper and lower banks of the slits of two neighboring sheets are identified.

Let $\pi : \mathscr{R} \longrightarrow \overline{\mathbb{C}}$ be the canonical projection from $\mathscr{R}$ to $\overline{\mathbb{C}}$ and denote by $z^{(k)}$ the point on $\mathscr{R}_k$ verifying $\pi(z^{(k)}) = z, z \in \overline{\mathbb{C}}$. Let $\phi : \mathscr{R} \longrightarrow \overline{\mathbb{C}}$ denote a conformal mapping whose divisor consists of one simple zero at $\infty^{(0)} \in \mathscr{R}_0$ and one simple pole at $\infty^{(2)} \in \mathscr{R}_2$. This mapping exists and is uniquely determined up to a multiplicative constant. Denote the branches of ϕ by

$$\phi_k(z) := \phi(z^{(k)}), \qquad k = 0, 1, 2, \qquad z^{(k)} \in \mathscr{R}_k.$$

From the properties of ϕ, we have

divisorcond:4}

$$\phi_0(z) = c_1/z + \mathscr{O}(1/z^2), \qquad \phi_2(z) = c_2\, z + \mathscr{O}(1), \qquad z \to \infty, \tag{4.82}$$

where c_1, c_2 are non-zero constants.

Let $x \in \Delta_k, k = 1, 2$. We write $z \to x_+$ when $z \in \mathbb{C}$ approaches x from above the real line. Analogously, $z \to x_-$ means that z approaches x from below the real line. Let us define

$$\phi_k(x_+) := \lim_{z \to x_+} \phi_k(z) = \lim_{z \to x_+} \phi(z^{(k)})$$

and

$$\phi_k(x_-) := \lim_{z \to x_-} \phi_k(z) = \lim_{z \to x_-} \phi(z^{(k)}).$$

Except when x is an end point of Δ_k, these limits are different due to the fact that $\lim_{z \to x_+} z^{(k)} \neq \lim_{z \to x_-} z^{(k)}$ on $\mathscr{R}$. However, due to the identification made of the points on the slits it is easy to verify that

{identif}

$$\phi_k(x_+) = \phi_{k+1}(x_-), \qquad \phi_k(x_-) = \phi_{k+1}(x_+), \qquad k = 0, 1, \tag{4.83}$$

because

$$\lim_{z \to x_+} z^{(k)} = \lim_{z \to x_-} z^{(k+1)}, \qquad \lim_{z \to x_-} z^{(k)} = \lim_{z \to x_+} z^{(k+1)}.$$

Taking account of the way in which the functions ϕ_k were extended to Δ_k and (4.83) it follows that $\prod_{k=0}^{2} \phi_k$ is a single-valued analytic function on $\overline{\mathbb{C}}$ without singularities; therefore, by Liouville's theorem, it is constant. We normalize ϕ so that

$$\prod_{k=0}^{2} \phi_k = c, \qquad |c| = 1, \qquad c_1 > 0.$$

Let us show that with this normalization c is $+1$.

Indeed, for a point $z^{(k)} \in \mathscr{R}_k$ on the Riemann surface we define its conjugate $\overline{z^{(k)}} := \overline{z}^{(k)}$. For a points $z^{(k)}$ on the upper bank of the slit Δ_k the conjugate is the one corresponding to the lower bank. Now, we define $\phi^* : \mathscr{R} \longrightarrow \overline{\mathbb{C}}$ as follows $\phi^*(\zeta) := \overline{\phi(\overline{\zeta})}$. It is easy to verify that ϕ^* is a conformal mapping of $\mathscr{R}$ onto $\overline{\mathbb{C}}$ with the same divisor as ϕ. Therefore, there exists a constant κ such that $\phi^* = \kappa\phi$. The corresponding branches satisfy the relations

$$\phi_k^*(z) = \overline{\phi_k(\overline{z})} = \kappa\phi_k(z), \qquad k = 0, 1, 2.$$

Comparing the Laurent expansions at ∞ of $\overline{\phi_0(\overline{z})}$ and $\kappa\phi_0(z)$, using the fact that $c_1 > 0$, it follows that $\kappa = 1$. Then

$$\phi_k(z) = \overline{\phi_k(\overline{z})}, \qquad k = 0, 1, 2.$$

This in turn implies that for each $k = 0, 1, 2$ all the coefficients, in particular the leading one, of the Laurent expansion at infinity of ϕ_k are real numbers. Obviously, c is the product of these leading coefficients. Since they are real numbers c is real and since it is of module 1, it has to be either 1 or -1. Analyzing the Laurent expansion of the branches at ∞ one easily concludes that indeed $c = 1$. So, we can assume in the following that

$$\prod_{k=0}^{2} \phi_k \equiv 1, \qquad c_1 > 0. \tag{4.84}$$ {normconfmap:4

It is easy to see that conditions (4.82) and (4.84) determine ϕ uniquely.

The question of finding explicit expressions for conformal representations of three sheeted Riemann surfaces of genus zero depending on the values of the end points of the intervals Δ_1, Δ_2 was considered in [63]. This problem is not solvable in closed form. [63, Theorem 3.1] gives an expression in terms of the solution of a system of two nonlinear equations of higher order. Already the simpler case of two intervals of equal length requires the solution of a bicuartic equation [63, Theorem 3.3]. A numerical method for solving the system of equations is given in [63, Theorem 6.1].

In [6, Lemma 4.2] the authors proved the following result. {uniqueness}

Lemma 4.19:
Their exists a unique pair of functions (F_1, F_2) *such that for* $k = 1, 2$

1. $F_k, 1/F_k \in \mathbf{H}(\mathbb{C} \setminus \Delta_k)$,
2. $F_k'(\infty) > 0$,
3. $\frac{|F_k(x)|^2}{|F_{k-1}(x)F_{k+1}(x)|} = 1$, $x \in \Delta_k$,

$(F_0 \equiv F_3 \equiv 1)$. *The functions may be expressed by the formulas*

$$F_k := \prod_{\nu=k}^{2} \phi_\nu, \qquad k = 1, 2.$$

The boundary conditions for the functions F_k, $k = 1, 2$ are

$$\begin{aligned} |F_1(x)|^2 =& |F_2(x)|, \qquad x \in \Delta_1, \\ |F_2(x)|^2 =& |F_1(x)|, \qquad x \in \Delta_2. \end{aligned}$$

Compare with (4.40) after taking logarithm.

From [32, Theorem 4.2, Corollary 4.3] we know that for $k = 1, 2$,

$$\lim_{n\to\infty} \frac{Q_{n+1,k}(z)}{Q_{n,k}(z)} = \frac{F_k(z)}{F_k'(\infty)},$$

uniformly on compact subsets of Ω_k and

$$\lim_{n\to\infty} \frac{\kappa_{n+1,k}}{\kappa_{n,k}} = \frac{F_k'(\infty)}{\sqrt{F_{k-1}'(\infty)F_{k+1}'(\infty)}},$$

where by definition we take $F_0'(\infty) = F_3'(\infty) = 1$. On the other hand, (4.60) and (4.67) imply that

$$\lim_{n\to\infty} \frac{Q_{n+1,k}(z)}{\Phi_k^{n+1}(z)} \frac{\Phi_k^n(z)}{Q_{n,k}(z)} = \frac{1}{\Phi_k(z)} \lim_{n\to\infty} \frac{Q_{n+1,k}(z)}{Q_{n,k}(z)} = 1,$$

uniformly on compact subsets of Ω_k and

$$\lim_{n\to\infty} \frac{\kappa_{n+1,k}}{C_k^{n+1}} \frac{C_k^n}{\kappa_{n,k}} = \frac{1}{C_k} \lim_{n\to\infty} \frac{\kappa_{n+1,k}}{\kappa_{n,k}} = 1.$$

Consequently

{otro} $$\Phi_k(z) \equiv \frac{F_k(z)}{F_k'(\infty)}, \qquad C_k = \frac{F_k'(\infty)}{\sqrt{F_{k-1}'(\infty)F_{k+1}'(\infty)}}, \qquad k = 1, 2. \tag{4.85}$$

CHAPTER

5 Conclusions and Future Research

{ch:future}

If the 1970's marked the rebirth of the interest in rational approximation, in particular of Padé type, nowadays we witness its good health. The connection of rational approximation with other branches of mathematical research and its very own development keep it as a living an attractive subject.

In the present dissertation we have focused on simultaneous rational approximation, in particular we have studied a mixed-type Hermite-Padé approximation problem which is known in the literature as multi-level Hermite-Padé. In the following pages we summarize the main results presented in the previous chapters. Furthermore, we discuss briefly some problems we consider interesting and might be attractive for future research as well.

5.1 Conclusions

Let $\Delta_j \subset \mathbb{R}$, $j = 1, \ldots, m$ be a collection of intervals such that $\Delta_j \cap \Delta_{j+1} = \varnothing$, $j = 1, \ldots, m-1$. Consider a vector of measures $(\sigma_1, \ldots, \sigma_m)$ with $\mathrm{Co}(\operatorname{supp} \sigma_j) = \Delta_j$ and $\sigma_j \in \mathscr{M}(\Delta_j)$ (the family of Borel measures with constant sign and finite moments supported on Δ_j). With this we construct the Nikishin system of measures $(s_{1,1}, s_{1,2}, \ldots, s_{1,m}) = \mathscr{N}(\sigma_1, \ldots, \sigma_m)$ (see Definition 1.15), and the Nikishin system of functions $(\widehat{s}_{1,1}, \ldots, \widehat{s}_{1,m})$ defined as the Markov functions of the measures $s_{i,j}$ (see (1.2)).

Since their introduction in [76], Nikishin systems have received great attention, because they are very well suited to extend "naturally" the results of classical orthogonality to multi-orthogonal polynomials. Moreover, Nikishin system have proved to be nice systems of functions to study the convergence properties of Hermite-Padé simultaneous approximants.

Recently, in [62] was introduced a mixed-type Hermite-Padé approximation problem (see Problem 5 for details). Given a Nikishin system $\mathscr{N}(\sigma_1, \ldots, \sigma_m)$, for each $n \in \mathbb{N}$, there exist polynomials $a_{n,0}, a_{n,1}, \ldots, a_{n,m}$, with $\deg a_{n,j} \leq n-1$, $j = 0, 1, \ldots, m-1$, $\deg a_{n,m} \leq n$, not all identically equal to zero, called multi-level (ML) Hermite-Padé polynomials that verify:

$$\mathscr{A}_{n,0} := \left[a_{n,0} + \sum_{k=1}^{m}(-1)^k a_{n,k}\widehat{s}_{1,k}\right] \in \mathcal{O}\left(\frac{1}{z^{n+1}}\right)$$
$$\mathscr{A}_{n,j} := \left[(-1)^j a_{n,j} + \sum_{k=j+1}^{m}(-1)^k a_{n,k}\widehat{s}_{j+1,k}\right] \in \mathcal{O}\left(\frac{1}{z}\right), j = 1, \ldots, m-1.$$

The study of different properties of multi-level Hermite-Padé polynomials, specially convergence of the approximants and asymptotic, has served as the backbone of the present thesis.

5.1.1 Chapter 2. Rational perturbation of multi-level Hermite-Padé polynomials

The goal of Chapter 2 was to obtain a result in the spirit of Gonchar's theorem ([36]) on the convergence of the Padé approximants to meromorphic functions of the form $\widehat{\mu} + r$, where μ is a finite Borel measure with constant sign and compact support on the real line, while r is a rational fraction with real coefficients, $r(\infty) = 0$ and poles outside of $\operatorname{supp}\mu$. The natural precedents in the case of simultaneous approximation are [60, 61].

The starting point was to consider a perturbed multi-level Hermite-Padé approximation problem in the following way:

$$\mathscr{A}_{n,0} := \left[a_{n,0} + \sum_{k=1}^{m}(-1)^k a_{n,k}(\widehat{s}_{1,k} + r_k)\right] \in \mathcal{O}\left(\frac{1}{z^{n+1}}\right)$$
$$\mathscr{A}_{n,j} := \left[(-1)^j a_{n,j} + \sum_{k=j+1}^{m}(-1)^k a_{n,k}\widehat{s}_{j+1,k}\right] \in \mathcal{O}\left(\frac{1}{z}\right), j = 1, \ldots, m-1.$$

Here, each $r_k = v_k/t_k$ is an irreducible rational fraction with real coefficients, $r_k(\infty) = 0$ and its poles are in the complement of Δ_m. Notice that we only introduced the perturbation in the first level, this is, solely the linear form $\mathscr{A}_{n,0}$ has been modified with respect to the original formulation of the problem. Recall that $T = \operatorname{lcm}(t_1, \ldots, t_m)$, $D = \deg T$.

1. Firstly, we study some general properties of the zeros of the linear forms $\mathscr{A}_{n,j}$, $j = 0, 1, \ldots, m$ in Lemma 2.6. We could prove that each $\mathscr{A}_{n,j}$ has, at least, $n - 2D$ zeros in the interval $\mathring{\Delta}_j$, $j = 1, \ldots, m$. If T has its zeros away of Δ_1, then $\mathscr{A}_{n,j}$ has $n - D$ sign changes inside Δ_j. Regarding $\mathscr{A}_{n,0}$ we can say that it has, at most, $2D$ zeros in $\mathbb{C} \setminus \Delta_1$. This amount of "wild" zeros reduces to D if the zeros of T are outside Δ_1.

2. Theorem 2.7 establishes the convergence in Hausdorff content of $\{a_{n,j}/a_{n,m}\}_{n\in\mathbb{N}}$, $j = 0, 1, \ldots, m-1$ in compact subsets of $\mathbb{C} \setminus \Delta_m$. Moreover, we proved in the same theorem that each ML polynomial $a_{n,j}$ has at least $n - 2D - m + j$ sign changes in Δ_m. The bound on the amount of sign changes of $a_{n,j}$ improves if the zeros of T are outside Δ_1. In this case the $a_{n,j}$ has at least $n - D - m + j$ sign changes in Δ_m.

3. Finally, assuming that the zeros of T lie outside $\Delta_1 \cup \Delta_m$, Theorem 2.2 is an easy consequence of the previous results together with Gonchar's lemma (see Lemma 1.8). In this Stieltjes-type theorem we obtain the uniform convergence of the sequence $\left\{a_{n,j}/a_{n,m}\right\}_{n\in\mathbb{N}}$, $j = 0, 1, \ldots, m-1$ in compact subsets of $\mathbb{C} \setminus (\Delta_m \cup \{z : T(z) = 0\})$. Furthermore, we also got that for large n the polynomials $a_{n,j}$, $j = 0, 1, \ldots, m$ have maximal degree, together with the location of their zeros.

4. The final part of the chapter was devoted to finding the logarithmic asymptotic of the ML Hermite-Padé polynomials, and to obtain better estimates for the rate of convergence of the approximants. Having accomplished these tasks, we study the multi-orthogonal polynomials associated to the approximation problem at hand, and after that we state a vector equilibrium problem, whose solutions allow us to describe the general asymptotic behavior of the polynomials $a_{n,j}$, $j = 0, 1, \ldots, m$ as well as of the linear forms $\mathscr{A}_{n,j}$, $j = 0, 1, \ldots, m-1$.

5.1.2 Chapter 3. A generalization of multi-level Hermite-Padé polynomials

Very recently, V.G. Lysov proposed a generalization on the multi-level Hermite-Padé approximation problem [66]. He proposed to consider more general interpolation conditions at infinity. Given a multi-index $\vec{n} = (n_1, \ldots, n_m) \in (\mathbb{Z}_+^m)^*$ (the set of all m-dimensional vectors with non-negative integer components not identically equal to zero), let

$$\mathscr{A}_{\vec{n},j}(z) := \left((-1)^j a_{\vec{n},j} + \sum_{k=j+1}^{m} (-1)^k a_{\vec{n},k} \widehat{s}_{j+1,k}\right)(z) = \mathscr{O}\left(\frac{1}{z^{n_{j+1}+1}}\right), \qquad z \to \infty.$$

for $j = 0, \ldots, m-1$.

In Chapter 3 we extended Lysov's result on the convergence of the approximants for a wider class of measures, and we complemented Lysov's asymptotic study by finding the ratio asymptotic of the polynomials $a_{\vec{n},j}$ and the linear forms $\mathscr{A}_{\vec{n},j}$. So, we also extended the theorems previously proven for the original definition of the ML Hermite-Padé polynomials in [62, 32].

1. Firstly, we studied the general properties of the zeros of the linear forms $\mathscr{A}_{\vec{n},j}$, $j = 0, \ldots, m$. Here we proved that each form has $n_1 + \cdots + n_j$ simple zeros in the interval Δ_j and the order of interpolation at infinity is exact (see Lemma 3.4). Taking ray sequences of multi-indices we got the convergence of the approximants (see Theorem 3.2). This part constitutes a natural generalization of [62, Th. 1.6] and [66, Prop. 1.2].

2. As intermediate step was to prove that the zeros of the forms $\mathscr{A}_{\vec{n},j}$ and $\mathscr{A}_{\vec{n}^l,j}$ interlace $j = 1, \ldots, m$ (see Lemma 3.9), and constructed a Riemann surface which is essential to describe the ratio asymptotic of the associated multi-orthogonal polynomials. In this way, we extend [32, Lemma 2.7].

3. Finally, we have adapted the proofs of [32, Th. 4.2] and [6, Th. 2.1] to obtain the ratio asymptotic of the polynomials $Q_{\vec{n},j}$, $j = 1, \ldots, m$ (see Theorem 3.3). This result generalizes [32, Th. 4.2].

5.1.3 Chapter 4. Strong asymptotic of Cauchy biorthogonal polynomials

Previously, we have mentioned how Cauchy biorthogonal polynomials have found countless applications recently. Their strong asymptotic behavior had been studied for measures of Laguerre type supported on a half line (see [14]). We studied the strong asymptotic of Cauchy biorthogonal polynomials for measures with compact support adapting a previous idea developed by A.I. Aptekarev in [4, 5].

1. Firstly, we studied the strong asymptotic of orthogonal polynomials with respect to varying measures. With Theorem 4.2 we refined its natural precedent, [24, Th. 4]. On the other hand, Theorem 4.3 gives the asymptotic for orthogonal polynomials associated to varying measures of a particular kind. Though in the spirit of [95, Th. 14.3], our theorem covers a more general class of measures.

2. Secondly, we discuss the connection between Cauchy biorthogonal polynomials and multilevel Hermite-Padé polynomials associated to Nikishin systems generated by two measures (Section 4.3). We exploit this link in order to use Aptekarev's methodology. Among other interesting properties, we remark the fact that multilevel Hermite-Padé polynomials are uniquely determined by the recursive nature of multi-orthogonality relations associated to Nikishin systems and Hermite-Padé approximation (see Proposition 4.8).

3. After we give a convenient normalization, we found a pair of sequences of orthogonal polynomials with varying measures with a prescribed asymptotic behavior. This is, given certain Szegő type functions we can give a pair of sequences whose strong asymptotic is described by those Szegő functions (Proposition 4.9). This step is fundamental to simplify Aptekarev's methodology, because his approach relies heavily in an intricate construction of a sequence of polynomials over a Riemann surface [5, Sec. 2.2].

4. Thanks to a topological reasoning we are able to give the strong asymptotic of the multiple orthogonal polynomials associated to multilevel Hermite-Padé approximation to a Nikishin system of two measures. The method is based on Banach's and Brouwer's fixed point theorems. An immediate consequence is the strong asymptotic of Cauchy biorthogonal polynomials (see Theorem 4.1).

5.2 Some open problems

As one can expect, during the preparation of the present dissertation we have found some interesting problems, which are connected with the ones we solved. In the present section we discuss some open questions that we consider attractive enough to bring further attention.

- A natural extension of Theorem 2.2 is to consider rational perturbations r_j, $j = 1, \ldots, m$ with complex coefficients. That is, given a Nikishin system $\mathcal{N}(\sigma_1, \ldots, \sigma_m)$ study a multilevel Hermite-Padé approximation problem like the one in Definition 2.1, but the rational fractions r_j, $j = 1, \ldots, m$ with complex coefficients instead. This question is in the line initiated by A.A. Gonchar in [36] and continued by G. López Lagomasino in [56]. We can conjecture that first it is necessary to solve the same problem but for Type I approximants.

- On the other hand, an obvious question is to analyse the strong asymptotic of the Cauchy biorthogonal polynomials when the sequence is generated by m measures (see [32, Sec. 1]). The problem here is that the associated operator $T_{\mathbf{w}}$ is not contractive but non-expansive. So, it is impossible to extend directly the ideas discussed in Chapter 4. Anyway, we think that with a suitable modification Aptekarev's method still works.

- Another interesting direction is to study the relative asymptotic of multilevel Hermite-Padé polynomials when the generating measures are modified by a "nice" rational function. More precisely, given the Nikishin systems $\mathcal{N}(\sigma_1, \ldots, \sigma_m)$ and $\mathcal{N}(r_1\sigma_1, \ldots, r_m\sigma_m)$, where r_j, $j = 1, \ldots, m$ are rational fractions with real coefficients and whose zeros and poles are outside Δ_j. A similar problem was studied in [48] for Type II polynomials.

- The results obtained in Chapter 4 can be refined strengthening the restrictions over the measures and using other techniques. In this case, an interesting path is to make a Riemann-Hilbert analysis of multilevel Hermite-Padé polynomials. This would allow to describe the asymptotic of the polynomials around the endpoints of the intervals.

Bibliography

[1] M.A. Angelesco. Sur l'approximatión simultaneé de plusieurs integrales definies. *C.R. Acad. Sci. Paris* **167** (1918), 629–631.

[2] M.A. Angelesco. Sur deux extensions des fractions continues algébraique. *C.R. Acad. Sci. Paris* **168** (1919), 262–265.

[3] A.I. Aptekarev and H. Stahl. "Asymptotics of Hermite-Padé polynomials". In: *Progress in Approximation Theory. An International Perspective*. Ed. by A.A. Gonchar and E.B. Saff. Springer Series in Computational Mathematics, 19. Springer-Verlag, 1992, 127–167. DOI: 10/fgvwg9.

[4] A.I. Aptekarev. Asymptotics of simultaneous orthogonal polynomials in the Angelesco case. *Math. USSR Sb.* **64** (1989), 57–84. DOI: 10/ftczrk.

[5] A.I. Aptekarev. Strong asymptotics of multiply orthogonal polynomials for Nikishin systems. *Sb. Math.* **190** (1999), 631–669. DOI: 10/cxv2tp.

[6] A.I. Aptekarev, G. López Lagomasino, and I.A. Rocha. Ratio asymptotics of Hermite-Padé polynomials for Nikishin systems. *Sb. Math.* **196** (2005), 1089–1107. DOI: 10/fn7ndw.

[7] G.A. Baker and P. Graves-Morris. *Padé Approximants. Part II: Extensions and Applications*. 1st ed. Encyclopedia of Mathematics and its Applications, 14. Addison Wesley Publishing Company, Reading, MA, 1981.

[8] L. Baratchart, H. Stahl, and F. Wielonsky. Asymptotic error estimates for L^2 best rational approximants to Markov functions. *J. of Approx. Theory* **108** (2001), 53–96. DOI: 10/cgmgkh.

[9] D. Barrios Rolanía, B. de la Calle Ysern, and G. López Lagomasino. Ratio and relative asymptotic of polynomials orthogonal to varying Denisov-type measures. *J. Approx. Theory* **139** (2006), 223–256. DOI: 10/crjfdn

[10] R. Beal, D.H. Sattinger, and J. Szmigielski. Multipeakons and the classical moment problem. *Adv. Math.* **154** (2000), 229–257. DOI: 10/c2wphd.

[11] M. Bello Hernández, G. López Lagomasino, and J. Mínguez Ceniceros. Fourier-Padé approximants of Angelesco systems. *Const. Approx.* **26** (2007), 339–359. DOI: 10/c7jd9j.

[12] C. Berg. Markov's theorem revisited. *J. Approx. Theory* **78** (1994), 260–275. DOI: 10/cvjcp6.

[13] M. Bertola. Two matrix models and biorthogonal polynomials. In: *The Oxford handbook of random matrix theory*. Ed. by G. Akemann, J. Baik and P. Di Francesco. Oxford University Press, Oxford, 2011, 310–328. DOI: 10/g8z3.

[14] M. Bertola, M. Gekhtman, and J. Szmigielski. Cauchy biorthogonal polynomials. *J. Approx. Theory* **162** (2010), 832-867. DOI: 10/bj56g3.

[15] M. Bertola, M. Gekhtman, and J. Szmigielski. Strong asymptotics for Cauchy biorthogonal polynomials with application to the Cauchy two–matrix model. *J. Math. Physics.* **54** (2013), 043517. DOI: 10/gbc3x8.

[16] M. Bertola and T. Bothner. Universality conjecture and results for a model of several coupled positive-definite matrices. *Commun. Math. Phys.* **337** (2015), 1107-1141. DOI: 10/gkpd2x.

[17] F. Beukers. "Padé-approximations in number theory". In: *Padé Approximation and its Applications, Amsterdam 1980*. Ed. M.G. de bruin and H. van Rossum. Lecture Notes in Mathematics, 888. Springer-Verlag, Berlin, 1981, 90–99. DOI: 10/br8h3f3.

[18] A. Branquinho, J. Bustamante A. Foulquié Moreno, and G. López Lagomasino. Normal indices in Nikishin system. *J. Approx Theory* **124** (2003), 254–262. DOI: 10/cbcx76.

[19] C. Brezinski. *History of Continued Fractions and Padé Approximants*. 1st ed. Springer Series in Computational Mathematics, 12. Springer-Verlag, Berlin, 1991. DOI: 10/dcz59d.

[20] J. Bustamante and G. López Lagomasino. Hermite-Padé approximation for Nikishin systems of analytic functions. *Russian Acad. Sci. Sb. Math* **77** (1994), 367–384. DOI: 10/cq24sh.

[21] T. Carleman. *Les Fonctions quasi analytiques: leçons professées au College de France*. 1st ed. Gauthier-Villars, Paris, 1926.

[22] J. Coates. On the algebraic approximation of functions. I, II, III. *Indag. Math.* **28** (1966), 421–461.

[23] B. de la Calle Ysern and G. López Lagomasino. Weak convergence of varying measures and Hermite-Padé orthogonal polynomials. *Const. Approx.* **15** (1999), 553–575. DOI: 10/dkkcxs.

[24] B. de la Calle Ysern and G. López Lagomasino. Strong asymptotics of orthogonal polynomials with varying measures and Hermite-Padé approximation. *J. Comp. Appl. Math.* **99** (1998), 91–103. DOI: 10/bfxr9n.

[25] P. Deift, T. Krieckerbauer and K.T.-R. McLaughlin. New results on the equilibrium measure for logarithmic potentials in the presence of an external field. *J. of Approx. Theory* **95** (1998), 388-475. DOI: 10/cmt29d.

[26] K. Driver and H. Stahl. Normality on Nikishin systems. *Indag. Math.* **5** (1994), 161–187. DOI: 10/bxk9bb.

[27] P. Erdös and P. Turán. On interpolation III. Interpolatory theory of polynomials. *Ann. Math.* **41** (1940), 510–553. DOI: 10/fsvzk9.

[28] U. Fidalgo Prieto and G. López Lagomasino. Rate of convergence of generalized Hermite-Padé approximants of Nikishin systems. *Const. Approx.* **23** (2006), 165–196. DOI: 10/fhxk9c.

[29] U. Fidalgo Prieto and G. López Lagomasino. General results on the convergence of multipoint Hermite-Padé approximants of Nikishin systems. *Const. Approx.* **25** (2007), 89–107. DOI: 10/fbm89b.

[30] U. Fidalgo Prieto and G. López Lagomasino. Nikishin systems are perfect. *Const. Approx* **34** (2011), 297–356. DOI: 10/bmqcpt.

[31] U. Fidalgo Prieto and G. López Lagomasino. Nikishin systems are perfect. Case of unbounded and touching supports. *J. Approx. Theory* **163** (2011), 779–811. DOI: 10/fqw6dd.

[32] U. Fidalgo Prieto, G. López Lagomasino, and S. Medina Peralta. Asymptotic of Cauchy biorthogonal polynomials. *Mediterr. J. Math.* **17** (2020), 22. DOI: 10/g39d.

[33] U. Fidalgo Prieto, A. López García, G. López Lagomasino and V.N. Sorokin. Mixed type multiple orthogonal polynomials for two Nikishin systems. *Const. Approx.* **32** (2010), 255–306. DOI: 10/b7tmxd.

[34] G. Freud. *Orthogonal Polynomials*. 1st ed. Akadémiai Kiadó, Budapest, 1971.

[35] L.Ya. Geronimus. *Orthogonal Polynomials*. 1st ed. Consultants Bureau, New York, NY, 1961.

[36] A.A. Gonchar. On convergence of Padé approximants for some classes of meromorphic functions. *Math. USSR Sb.* **26** (1975), 555–575. DOI: 10/dz2vmg.

[37] A.A. Gonchar. On the convergence of generalized Padé approximants of meromorphic functions. *Math. USSR Sb.* **27** (1975), 503–514. DOI: 10/cjpwgg.

[38] A.A. Gonchar. On the speed of rational approximation of some analytic functions. *Math. USSR Sb.* **34** (1978), 131–145. DOI: 10/chs94f.

[39] A.A. Gonchar and E.A. Rakhmanov. On the convergence of simultaneous Padé approximants for a system of functions of Markov type. *Proc. Steklov Inst. Math.* **157** (1983), 31–50.

[40] A.A. Gonchar and E.A. Rakhmanov. Equilibrium measure and the distribution of zeros of extremal polynomials. *Math. USSR Sb.* **53** (1986), 117–127. DOI: 10/cndq4h.

[41] A.A. Gonchar, E.A. Rakhmanov, and V.N. Sorokin. Hermite-Padé approximants for systems of Markov-type functions. *Sb. Math.* **188** (1997), 671–696. DOI: 10/chskzg.

[42] A. Granas and J. Dugundji. *Fixed Point Theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2003.

[43] H. Hamburger. Über eine Erweiterung des Stieltjesschen Momentenproblems. *Math. Ann.* **81** (1920), 235–319.

[44] C. Hermite. *Sur la fonction exponentielle*. Gauthier-Villars, Paris, 1874. URL: eu-dml.org/doc/203956.

[45] H. Jager. A simultaneous generalization of the Padé table I-VI. *Indag. Math.* **26** (1964), 193–249.

[46] M.G. Krein and A.A. Nudel'man. *The Markov Moment Problem and Extremal Problems*. Translation of Mathematical Monographs, 50. Amer. Math. Soc., Providence, RI, 1997.

[47] A. López García and G. López Lagomasino. Ratio asymptotic of Hermite-Padé orthogonal polynomials for Nikishin systems. II. *Adv. Math.* **218** (2008), 1081–1106. DOI: 10/fj3fn6.

[48] A. López García and G. López Lagomasino. Relative asymptotic of multiple orthogonal polynomials for Nikishin systems. *J. Approx. Theory* **158** (2009), 214-241. DOI: 10/dqsw23

[49] G. López Lagomasino. On the convergence of multipoint Padé approximants for functions of Stieltjes type. *Soviet Math. Dokl.* **19** (1978), 425–428.

[50] G. López Lagomasino. Conditions for convergence of multipoint Padé approximants for functions of Stieltjes type. *Math. USSR Sb.* **35** (1979), 363–376. DOI: 10/c7df49.

[51] G. López Lagomasino. Sobre la convergencia de los aproximantes multipuntuales de Padé a funciones meromorfas de tipo Markov. *Rev. Ciencias Matemáticas* **1** (1980), 19–28.

[52] G. López Lagomasino. On the convergence of Padé approximants for Stieltjes type meromorphic functions. *Math. USSR Sb.* **39** (1981), 281–287. DOI: 10/bwtvx2.

[53] G. López Lagomasino. On the asymptotic of the ratio of orthogonal polynomials and convergence of multipoint Padé approximants. *Math. USSR Sb.* **56** (1987), 207–219. DOI: 10/dj65j6.

[54] G. López Lagomasino. "Szegő's theorem for polynomials orthogonal with respect to varying measures". In: *Orthogonal Polynomials and their Applications*. Ed. by M. Alfaro; J.S. Dehesa; F. Marcellán; J.L. Rubio de Francia and J. Vinuesa. Lecture Notes in Mathematics, 1329. Springer-Verlag, Berlin, 1988, 225-260. DOI: 10/dgr5x3.

[55] G. López Lagomasino. Asymptotics of polynomials orthogonal with respect to varying measures. *Const. Approx* **5** (1989), 199-229. DOI: 10/bhx6t9.

[56] G. López Lagomasino. Convergence of Padé approximants of Stieltjes type meromorphic functions and comparative asymptotics of orthogonal polynomials. *Math. USSR Sb.* **64** (1989), 207–229. DOI: 10/c7dh7x

[57] G. López Lagomasino. An introduction to multiple orthogonal polynomials and Hermite-Padé approximation. In: *Orthogonal Polynomials: Current Trends and Applications, Proc. of the 7-th EIBPOA Conference*. Ed. by F. Marcellán and E.J. Huertas. SEMA SIMAI Springer Series, 22. Springer-Verlag, Berlin, 2021. DOI: 10/g4f9.

[58] G. López Lagomasino and A.A. Gonchar. On Markov's theorem for multipoint Padé approximants, *Math. USSR Sb.* **34** (1978), 449-459. DOI: 10/d4gzm2.

[59] G. López Lagomasino and S. Medina Peralta. On the convergence of type I Hermite-Padé approximants. *Adv. Math.* **273** 2015, 124–148. DOI: 10/f7fcq4.

[60] G. López Lagomasino and S. Medina Peralta. On the convergence of type I Hermite-Padé approximants for rational perturbations of a Nikishin system. *J. Comput. Appl. Math.* **284** 2015, 216–227. DOI: 10/g355.

[61] G. López Lagomasino, S. Medina Peralta, and U. Fidalgo Prieto. Hermite-Padé approximation for certain systems of meromorphic functions. *Sb. Math.* **206** (2015), 57–76. DOI: 10/g356.

[62] G. López Lagomasino, S. Medina Peralta, and J. Szmigielski. Mixed type Hermite-Padé approximation inspired by the Degasperis-Procesi equation. *Adv. Math.* **349** (2019), 813–838. DOI: 10/g36b.

[63] G. López Lagomasino, D. Pestana, J. M. Rodríguez, and D. Yakubovich. Computation of conformal representations of compact Riemann surfaces. *Math. of Comp.* **79** (2010), 365–382. DOI: 10/dzsqm5.

[64] D.S. Lubinsky. A survey of general orthogonal polynomials for weights on finite and infinite intervals. *Acta Appl. Math.* **10** (1987), 237–296.

[65] H. Lundmark and J. Szmigielski. Multi-peakon solutions of the Degasperis-Procesi equation. *Int. Math. Res. Pap.* **2** (2005), 53–116. DOI: 10/btfnn9.

[66] V.G. Lysov. Mixed type Hermite–Padé approximants for a Nikishin system. *Proc. Steklov Inst. Math.* **311** (2020), 199–213. DOI: 10/g4cf.

[67] K. Mahler. Perfect systems. *Compos. Math.* **19** (1968), 95–116. URL: eudml.org/doc/88959.

[68] A.A. Markov. Deux démonstrations de la convergence de certaines fractions continues. *Acta Math.* **19** (1895), 93–104.

[69] A. Matté and P. Nevai. Remarks on E. A. Rahmanov's paper "On the asymptotics of the ratio of orthogonal polynomials". *J. Approx Theory* **36** (1982), 64–72. DOI: 10/dwh9w3.

[70] A. Matté, P. Nevai, and V. Totik. Asymptotics for the ratio of leading coefficients of orthonormal polynomials on the unit circle. *Const. Approx.* **1** (1985), 63–69. DOI: 10/b59p8p.

[71] R. Miranda. *Algebraic Curves and Riemann Surfaces*. Graduate Studies in Mathematics, 5. Amer. Math. Soc., Providence, RI, 1995.

[72] R. Montessus de Ballore. Sur les fractions continues algébriques. *Bull. Soc. Math. France* **30** (1902), 28–36.

[73] P. Nevai. *Orthogonal Polynomials*. 1st ed. Memoirs of the American Mathematical Society, 213. Amer. Math. Soc., Providence, RI, 1979.

[74] P. Nevai. Weakly convergent sequences of functions and orthogonal polynomials. *J. Approx Theory* **65** (1991), 322–340. DOI: 10/cj9zzh.

[75] E.M. Nikishin. A system of Markov functions. *Moscow Univ. Math. Bull.* **34** (1979), 60–63.

[76] E.M. Nikishin. On simultaneous Padé approximants. *Math. USSR Sb.* **41** (1982), 409–425. DOI: 10/bvqd2t.

[77] E.M. Nikishin and V.N. Sorokin. *Rational Approximation and Orthogonality*. 1st ed. Translations of Mathematical Monographs 92. Amer. Math. Soc., Providence, RI, 1991.

[78] J. Nuttal. Padé approximants of meromorphic functions. *J. Math. Anal. Appl.* **31** (1970), 147–153. DOI: 10/fg4t7v

[79] J. Nuttal. Asymptotics of diagonal Hermite-Padé polynomials. *J. Approx. Theory* **42** (1984), 299–386. DOI: 10/bwm47x.

[80] H. Padé. Sur la représentation approché d'une fonction par des fractions rationnelles. *Annales scientifiques de l'École Normal Superiore 3 série* **9** (1892), 3–93.

[81] C. Pommerenke. Padé approximants and convergence in capacity. *J. Math. Anal. Appl.* **41** (1973), 775–780. DOI: 10/czgnxf.

[82] E.A. Rahkmanov. Convergence of diagonal Padé approximants. *Math. USSR Sb.* **33** (1977), 243–260. DOI: 10/bx5rb6.

[83] E.A. Rahkmanov. On the asymptotics of the ratio of orthogonal polynomials. *Math. USSR Sb.* **32** (1977), 199–213. DOI: 10/bdp53c.

[84] E.A. Rahkmanov. On the asymptotics of the ratio of orthogonal polynomials II. *Math. USSR Sb.* **46** (1983), 105–117. DOI: 10/cd7x5k.

[85] E.A. Rahkmanov. On asymptotic properties of orthogonal polynomials on the unit circle with weights not satisfying Szegő's condition. *Math. USSR Sb.* **58** (1983), 149–167. DOI: 10/dx5d6h.

[86] T. Ransford. *Potential Theory on the Complex Plane*. 1st ed. London Mathematical Society Student Text, 28. Cambridge University Press, New York, NY, 1995.

[87] W. Rudin. *Real and Complex Analysis*. 1st ed. McGraw-Hill Series in Higher Mathematics, 6. McGraw-Hill, New York, NY, 1966.

[88] E.B. Saff and V. Totik. *Logarithmic Potentials with External Fields*. 1st ed. Grundlehren der mathematischen Wissenschaften, 316. Springer-Verlag, Berlin, 1997.

[89] J.A. Shohat and J.D. Tamarkin. *The Problem of Moments*. 4th ed. Mathematical Surveys, 1. Amer. Math. Soc., New York, NY, 1970.

[90] B. Simon. *Orthogonal Polynomials in the Unit Circle. Part 1: Classical Theory*. 1st ed. Colloquium Publications, 54. Amer. Math. Soc., Providence, RI, 2005.

[91] H. Stahl. Strong asymptotic for orthogonal polynomials with varying weights. *Acta Sci. Math. (Szeged)* **66** (2000), 147-192.

[92] H. Stahl and V. Totik. *General Orthogonal Polynomials*. Encyclopedia of Mathematics and its Applications, 43. Cambridge University Press, Cambridge, UK, 1992.

[93] T.J. Stieltjes. Recherches sur les fractions continues. *Ann. Fac. Sci. Univ. Toulouse 1re série* **8** (1894), J1–J122.

[94] G. Szegő. *Orthogonal Polynomials*. 4th ed. Colloquium Publications 23. Amer. Math. Soc., Providence, RI, 1975.

[95] V. Totik. *Weighted Approximation with Varying Weights*. 1st ed. Lecture Notes in Mathematics, 1569. Springer-Verlag, Berlin, 1994.

[96] W. van Assche. The impact of Stieltjes' work on continued fractions and orthogonal polynomials: additional material. *J. Comp. App. Math.* **65** (1995), 419–447. DOI: 10/bhr8bb.

[97] W. van Assche. "Analytic number theory and rational approximation". In: *Coimbra Lectures Notes on Orthogonal Polynomials*. Ed. by A. Branquinho and A. Foulquié. Nova Science Pu., 2008, 197–229.

[98] W. van Assche. *Asymptotic for Orthogonal Polynomials*. 1st ed. Lecture Notes in Mathematics, 1265. Springer-Verlag, Berlin, 2006.

[99] H. Widom. Extremal polynomials associated with a system of curves in the complex plane. *Adv. Math.* **3** (1969), 127–232. DOI: 10/c442m7.