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Article

f-Polynomial on Some Graph Operations

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Abstract: Given any function $f : \mathbb{Z}^+ \to \mathbb{R}^+$, let us define the f-index $I_f(G) = \sum_{u\in V(G)} f(d_u)$ and the f-polynomial $P_f(G,x) = \sum_{u\in V(G)} x^{1/f(d_u)-1}$, for $x > 0$. In addition, we define $P_f(G,0) = \lim_{x\to 0^+} P_f(G,x)$. We use the f-polynomial of a large family of topological indices in order to study mathematical relations of the inverse degree, the generalized first Zagreb, and the sum lordeg indices, among others. In this paper, using this f-polynomial, we obtain several properties of these indices of some classical graph operations that include corona product and join, line, and Mycielskian, among others.

Keywords: inverse degree index; generalized first Zagreb index; sum lordeg index; corona product; join of graphs; line graph; Mycielskian graph; polynomials in graphs

1. Introduction

A topological index is a single number that represents a chemical structure via the molecular graph, in graph theoretical terms, whenever it correlates with a molecular property. Hundreds of topological indices have been recognized to be useful tools in research, especially in chemistry. Topological indices have been used to understand physicochemical properties of compounds. They usually enclose topological properties of a molecular graph in a single real number. Several topological indices were introduced by the seminal work by Wiener [1]. They have been studied and generalized by several researchers since then. In particular, topological indices based on end-vertex degrees of edges have been studied over almost 50 years (see, e.g., [2–11]).

A graph, usually denoted $G(V(G);E(G))$, consists of a set of vertices $V(G)$ together with a set $E(G)$ of unordered pairs of vertices called edges. The number of vertices in a graph is usually denoted $n = |V(G)|$, while the number of edges is usually denoted $m = |E(G)|$; these two basic parameters are called the order and size of G, respectively. Miličević and Nikolić defined in reference [12] the first variable Zagreb index as

$$M_1^\alpha(G) = \sum_{u\in V(G)} d_u^\alpha,$$

where d_u is the degree of the vertex u and $\alpha \in \mathbb{R}$.

Note that M_1^2 is the first Zagreb index M_1, M_1^{-1} is the inverse index ID, M_1^3 is the forgotten index F, etc.

The harmonic index, defined in reference [13] as

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v},$$

has been studied in the last years (see, e.g., [14–19]).

In reference [20], the harmonic polynomial of a graph G is defined as

$$H(G,x) = \sum_{uv \in E(G)} x^{d_u + d_v - 1},$$

and the harmonic polynomials of some graphs are computed. The harmonic polynomials of the line of some graphs are computed in reference [21].

This polynomial owes its name to the fact that $2\int_0^1 H(G,x)\,dx = H(G)$.

The inverse degree index $ID(G)$ of a graph G is defined by

$$ID(G) = \sum_{u \in V(G)} \frac{1}{d_u} = \sum_{uv \in E(G)} \Big(\frac{1}{d_u^2} + \frac{1}{d_v^2} \Big).$$

The inverse degree index first attracted attention through numerous conjectures (see [13]). This index has been studied in reference [22–26].

The inverse degree polynomial of a graph G was defined in reference [27] as

$$ID(G,x) = \sum_{u \in V(G)} x^{d_u - 1}.$$

We have $\int_0^1 ID(G,x)\,dx = ID(G)$.

Given any function $f : \mathbb{Z}^+ \to \mathbb{R}^+$, let us define the f-index

$$I_f(G) = \sum_{u \in V(G)} f(d_u)$$

and the f-polynomial

$$P_f(G,x) = \sum_{u \in V(G)} x^{1/f(d_u) - 1},$$

for $x > 0$. In addition, we define $P_f(G,0) = \lim_{x \to 0^+} P_f(G,x)$. Note that $P_f(G,x) = ID(G,x)$ when $f(t) = 1/t$.

The degrees of the vertices are graph invariants, i.e., if two graphs are isomorphic, then the corresponding vertices by any isomorphism have the same degrees. Hence, the f-polynomial is also a graph invariant, i.e., two isomorphic graphs have the same ID polynomial.

Polynomials have proved to be useful in the study of several topological indices (see, e.g., [27–29]).

There are many papers studying several topological indices of graph operations (see, e.g.,[27–30]).

Throughout this paper, $G = (V(G), E(G))$ denotes a (non-oriented) finite simple (without multiple edges and loops) graph without isolated vertices. The main aim of this paper is to study mathematical relations of the inverse degree, the generalized first Zagreb, and the sum lordeg indices, among others. In order to do that, we use the f-polynomial of a large family of topological indices, introduced in reference [31]. We obtain inequalities (and even closed formulas in the case of the ID polynomial) involving the f-polynomial of many classical graph operations, which include corona product, join, line and Mycielskian, among others. These results allow us to obtain new inequalities for the inverse degree, the generalized first Zagreb, and the sum lordeg indices of these graph operations.

2. Definitions and Background

In the following sections, we obtain inequalities for the f-polynomial of many classical graph operations, which include corona product, join, line and Mycielskian, among others. The f-polynomial of other graph operations (Cartesian product, lexicographic product, and Cartesian sum) is studied in reference [31]. The different kinds of graph operations are an important research topic (see [32] and the references therein). Some large graphs are composed from some existing smaller ones by using several graph operations, and many properties of such large graphs are strongly associated with those of the corresponding smaller ones.

Let us recall the definitions of some classical products in graph theory.

The join $G_1 + G_2$ is defined as the graph obtained by taking one copy of G_1 and one copy of G_2 and joining by an edge each vertex of G_1 with each vertex of G_2.

The corona product $G_1 \circ G_2$ is the graph obtained by taking $|V(G_1)|$ copies of G_2 and joining each vertex of the i-th copy with the vertex $v_i \in V(G_1)$.

The ID polynomial is related to the polynomials associated to some topological indices.

In reference [33], Shuxian defined the following polynomial related to the first Zagreb index:

$$M_1^*(G,x) := \sum_{u \in V(G)} d_u x^{d_u}.$$

Note that $x(xID(G,x))' = M_1^*(G,x)$.

The ID polynomial is also related to other polynomials, like the harmonic polynomial (see [27] and Propositions 7–10 in this paper).

The following result states some of the main properties of P_f.

Proposition 1. *If G is a graph with order n and $f : \mathbb{Z}^+ \to \mathbb{R}^+$, then:*

- $P_f(G,x)$ *is a polynomial if and only if* $1/f(d_u) \in \mathbb{Z}^+$ *for every* $u \in V(G)$,
- $P_f(G,x)$ *is a positive* C^∞ *function on* $(0,\infty)$,
- $P_f(G,x)$ *is a continuous function on* $[0,\infty)$ *if and only if* $P_f(G,0) < \infty$,
- $P_f(G,x)$ *is a continuous function on* $[0,\infty)$ *if and only if* $f(d_u) \leqslant 1$ *for every* $u \in V(G)$,
- $P_f(G,x)$ *is an integrable function on* $[0,A]$ *for every* $A > 0$, *and* $\int_0^1 P_f(G,x)\,dx = I_f(G)$,
- $P_f(G,x)$ *is increasing on* $(0,\infty)$ *if and only if* $f(d_u) \leqslant 1$ *for every* $u \in V(G)$,
- $P_f(G,x)$ *is strictly increasing on* $(0,\infty)$ *if and only if* $f(d_u) \leqslant 1$ *for every* $u \in V(G)$, *and* $f(d_v) \neq 1$ *for some* $v \in V(G)$,
- $P_f(G,x)$ *is convex on* $(0,\infty)$ *if* $f(d_u) \in (0,1/2] \cup [1,\infty)$ *for every* $u \in V(G)$,
- $P_f(G,x)$ *is strictly convex on* $(0,\infty)$ *if* $f(d_u) \in (0,1/2] \cup [1,\infty)$ *for every* $u \in V(G)$, *and* $f(d_v) \notin \{1/2,1\}$ *for some* $v \in V(G)$,
- $P_f(G,x)$ *is concave on* $(0,\infty)$ *if* $f(d_u) \in [1/2,1]$ *for every* $u \in V(G)$,
- $P_f(G,x)$ *is strictly concave on* $(0,\infty)$ *if* $f(d_u) \in [1/2,1]$ *for every* $u \in V(G)$, *and* $f(d_v) \notin \{1/2,1\}$ *for some* $v \in V(G)$,
- $P_f(G,1) = n$.

Proof. The first statement is direct, since $P_f(G,x)$ is a polynomial if and only if $1/f(d_u) - 1 \in \mathbb{Z}$ for every $u \in V(G)$.

The second and third statements are direct.

The fourth statement holds since $P_f(G,x)$ is a continuous function on $[0,\infty)$ if and only if $1/f(d_u) - 1 \geqslant 0$ for every $u \in V(G)$.

Since $f > 0$, $1/f(d_u) - 1 > -1$ for every $u \in V(G)$, and $P_f(G,x)$ is an integrable function on $[0,A]$ for every $A > 0$. Thus, a simple computation gives $\int_0^1 P_f(G,x)\,dx = I_f(G)$.

If there exists $u \in V(G)$ with $f(d_u) > 1$, then $1/f(d_u) - 1 < 0$ and $\lim_{x \to 0^+} x^{1/f(d_u)-1} = \infty$; thus, $\lim_{x \to 0^+} P_f(G,x) = \infty$, and $P_f(G,x)$ is not increasing on $(0,\infty)$. If $f(d_u) \leqslant 1$ for every $u \in V(G)$, then $1/f(d_u) - 1 \geqslant 0$ for every $u \in V(G)$, and so $P_f(G,x)$ is increasing on $(0,\infty)$. If this is the case, $P_f(G,x)$

is strictly increasing on $(0,\infty)$ if $1/f(d_v)-1>0$ for some $v\in V(G)$, i.e., $f(d_v)\neq 1$ for some $v\in V(G)$; if $f(d_u)=1$ for every $u\in V(G)$, then $P_f(G,x)$ is constant, and so it is not strictly increasing on $(0,\infty)$.

If $f(d_u)\in(0,1/2]\cup[1,\infty)$ for every $u\in V(G)$, then $1/f(d_u)-1\in(-1,0]\cup[1,\infty)$ for every $u\in V(G)$, and $P_f(G,x)$ is convex on $(0,\infty)$. If this is the case, $P_f(G,x)$ is strictly convex on $(0,\infty)$ if $1/f(d_v)-1\in(-1,0)\cup(1,\infty)$ for some $v\in V(G)$, i.e., $f(d_v)\notin\{1/2,1\}$ for some $v\in V(G)$.

If $f(d_u)\in[1/2,1]$ for every $u\in V(G)$, then $1/f(d_u)-1\in[0,1]$ for every $u\in V(G)$, and $P_f(G,x)$ is concave on $(0,\infty)$. If this is the case, $P_f(G,x)$ is strictly concave on $(0,\infty)$ if $1/f(d_v)-1\in(0,1)$ for some $v\in V(G)$, i.e., $f(d_v)\notin\{1/2,1\}$ for some $v\in V(G)$.

Finally, $P_f(G,1)=\sum_{u\in V(G)}1=n$. □

In particular, if $\alpha\in\mathbb{R}$ and $f(t)=t^\alpha$, then Proposition 1 gives $\int_0^1 P_f(G,x)\,dx=M_1^\alpha(G)$.

In particular, we have the following properties.

Proposition 2. *If G is a k-regular graph with order n and* $f:\mathbb{Z}^+\to\mathbb{R}^+$, *then* $P_f(G,x)=nx^{1/f(k)-1}$.

The following result shows the polynomial P_f for well-known graphs, such as: K_n (complete graph), C_n (cycle graph), Q_n (hypercube graph), $K_{n_1;n_2}$ (complete bipartite graph), S_n (star), P_n (path graph), and W_n (wheel graph).

Proposition 3. *If* $f:\mathbb{Z}^+\to\mathbb{R}^+$, *then*

$$\begin{aligned} P_f(K_n,x)&=nx^{1/f(n-1)-1}, & P_f(C_n,x)&=nx^{1/f(2)-1},\\ P_f(Q_n,x)&=2^n x^{1/f(n)-1}, & P_f(K_{n_1,n_2},x)&=n_1x^{1/f(n_2)-1}+n_2x^{1/f(n_1)-1},\\ P_f(S_n,x)&=x^{1/f(n-1)-1}+(n-1)x^{1/f(1)-1}, & P_f(P_n,x)&=(n-2)x^{1/f(2)-1}+2x^{1/f(1)-1},\\ P_f(W_n,x)&=x^{1/f(n-1)-1}+(n-1)x^{1/f(3)-1}. \end{aligned}$$

Fix $\delta\in\mathbb{Z}^+$ and $f:\mathbb{Z}^+\to\mathbb{R}^+$. We say that f satisfies the δ-additive property 1 (and we write $f\in AP_1(\delta)$) if

$$\frac{1}{f(x+y)}\geqslant\frac{1}{f(x)}+\frac{1}{f(y)}$$

for every $x,y\in\mathbb{Z}^+$ with $x,y\geqslant\delta$.

f satisfies the δ-additive property 2 (and we write $f\in AP_2(\delta)$) if

$$\frac{1}{f(x+y)}\leqslant\frac{1}{f(x)}+\frac{1}{f(y)}$$

for every $x,y\in\mathbb{Z}^+$ with $x,y\geqslant\delta$.

Finally, f satisfies the δ-additive property 3 (and we write $f\in AP_3(\delta)$) if

$$\frac{1}{f(x+y)}\leqslant\min\left\{\frac{1}{f(x)},\frac{1}{f(y)}\right\}$$

for every $x,y\in\mathbb{Z}^+$ with $x,y\geqslant\delta$.

Remark 1. *Note that if* $f\in AP_j(\delta)$ *for some* $1\leqslant j\leqslant 3$, *then* $f\in AP_j(\delta')$ *for every* $\delta'\geqslant\delta$.

If f is an increasing function on $[\delta,\infty)$, *then* $f\in AP_3(\delta)$.

Note that $f(t)=1/t$ *satisfies the* 1*-additive properties* 1 *and* 2, *i.e.,* $f\in AP_1(1)\cap AP_2(1)$.

The following result appears in reference [31].

Theorem 1. *Let* $\alpha\in\mathbb{R}$ *and* $f(t)=t^\alpha$.

(1) *If* $\alpha\leqslant -1$, *then* $f\in AP_1(1)$.

(2) *If* $\alpha \in [-1, 0]$*, then* $f \in AP_2(1)$.
(3) *If* $\alpha \geqslant 0$*, then* $f \in AP_3(1)$.

Next, we present two useful improvements (for convex functions) of Chebyshev's inequality.

Lemma 1 ([34]). *If* $f_1, \ldots, f_k$ *are non-negative convex functions on* $[a, b]$*, then*

$$\frac{1}{b-a}\int_a^b \prod_{i=1}^k f_i(x)\,dx \geqslant \frac{2^k}{k+1}\prod_{i=1}^k \frac{1}{b-a}\int_a^b f_i(x)\,dx.$$

Lemma 2 ([35], Corollary 5.2). *If* $f_1, \ldots, f_k$ *are non-negative convex functions on* $[a, b]$*, then*

$$\int_a^b \prod_{i=1}^k f_i(x)\,dx \leqslant \frac{2}{k+1}\left(\prod_{i=1}^k \int_a^b f_i(x)\,dx\right)^{1/k}\left(\prod_{i=1}^k \big(f_i(a)+f_i(b)\big)\right)^{1-1/k}.$$

3. Join of Graphs

Theorem 2. *Let* $\delta \in \mathbb{Z}^+$ *and* G_1 *and* G_2 *be two graphs with order* n_1 *and* n_2*, respectively, and minimum degree of at least* δ*, and* $f : \mathbb{Z}^+ \to \mathbb{R}^+$*. The* f*-polynomial of the join* $G_1 + G_2$ *satisfies the following inequalities for* $x \in (0, 1]$.

(1) *If* $f \in AP_1(\delta)$*, then*

$$P_f(G_1+G_2, x) \leqslant x^{1/f(n_2)} P_f(G_1, x) + x^{1/f(n_1)} P_f(G_2, x).$$

(2) *If* $f \in AP_2(\delta)$*, then*

$$P_f(G_1+G_2, x) \geqslant x^{1/f(n_2)} P_f(G_1, x) + x^{1/f(n_1)} P_f(G_2, x).$$

(3) *If* $f \in AP_3(\delta)$*, then*

$$P_f(G_1+G_2, x) \geqslant P_f(G_1, x) + P_f(G_2, x).$$

Proof. If $u \in V(G_1)$ (respectively, $u \in V(G_2)$), then its degree in $G_1 + G_2$ is $d_u + n_2$ (respectively, $d_u + n_1$).

Assume first that $f \in AP_1(\delta)$. Since $d_u \geqslant \delta$ for every $u \in V(G_1) \cup V(G_2)$, $f \in AP_1(\delta)$, and $x \in (0, 1]$,

$$\begin{aligned} P_f(G_1+G_2, x) &= \sum_{u \in V(G_1)} x^{1/f(d_u+n_2)-1} + \sum_{v \in V(G_2)} x^{1/f(d_v+n_1)-1} \\ &\leqslant \sum_{u \in V(G_1)} x^{1/f(d_u)-1} x^{1/f(n_2)} + \sum_{v \in V(G_2)} x^{1/f(d_v)-1} x^{1/f(n_1)} \\ &= x^{1/f(n_2)} P_f(G_1, x) + x^{1/f(n_1)} P_f(G_2, x). \end{aligned}$$

If $f \in AP_2(\delta)$, then a similar argument allows us to obtain the corresponding inequality.

Assume now that $f \in AP_3(\delta)$. We have

$$\begin{aligned} P_f(G_1+G_2, x) &= \sum_{u \in V(G_1)} x^{1/f(d_u+n_2)-1} + \sum_{v \in V(G_2)} x^{1/f(d_v+n_1)-1} \\ &\geqslant \sum_{u \in V(G_1)} x^{1/f(d_u)-1} + \sum_{v \in V(G_2)} x^{1/f(d_v)-1} = P_f(G_1, x) + P_f(G_2, x). \end{aligned}$$

□

Theorems 1 and 2 have the following consequence.

Theorem 3. *Let G_1 and G_2 be two graphs with order n_1 and n_2, respectively, $\alpha \in \mathbb{R}$, and $f(t) = t^\alpha$. The f-polynomial of the join $G_1 + G_2$ satisfies the following inequalities for $x \in (0,1]$.*

(1) *If $\alpha \leqslant -1$, then*

$$P_f(G_1 + G_2, x) \leqslant x^{n_2^{-\alpha}} P_f(G_1, x) + x^{n_1^{-\alpha}} P_f(G_2, x).$$

(2) *If $\alpha \in [-1, 0]$, then*

$$P_f(G_1 + G_2, x) \geqslant x^{n_2^{-\alpha}} P_f(G_1, x) + x^{n_1^{-\alpha}} P_f(G_2, x).$$

(3) *If $\alpha \geqslant 0$, then*

$$P_f(G_1 + G_2, x) \geqslant P_f(G_1, x) + P_f(G_2, x).$$

Theorem 3 has the following consequence.

Corollary 1. *Given two graphs G_1 and G_2, with order n_1 and n_2, respectively, the ID polynomial of the join $G_1 + G_2$ is*

$$ID(G_1 + G_2, x) = x^{n_2} ID(G_1, x) + x^{n_1} ID(G_2, x).$$

Since $f(t) = t\sqrt{\log t} \in AP_3(2)$, Theorem 2 has the following consequence.

Corollary 2. *Let G_1 and G_2 be two graphs without pendant vertices and with order n_1 and n_2, respectively. If $f(t) = t\sqrt{\log t}$, then the f-polynomial of the join $G_1 + G_2$ satisfies for $x \in (0,1]$*

$$P_f(G_1 + G_2, x) \geqslant P_f(G_1, x) + P_f(G_2, x).$$

Next, we obtain bounds for $I_f(G_1 + G_2)$ by using the previous inequalities for $P_f(G_1 + G_2, x)$.

Proposition 4. *Let $\delta \in \mathbb{Z}^+$ and let G_1 and G_2 be two graphs with order n_1 and n_2, respectively, and a minimum degree of at least δ. If $f \in AP_3(\delta)$, then*

$$I_f(G_1 + G_2) \geqslant I_f(G_1) + I_f(G_2).$$

Proof. Theorem 2 gives

$$P_f(G_1 + G_2, x) \geqslant P_f(G_1, x) + P_f(G_2, x)$$

for every $x \in (0,1]$. Thus, Proposition 1 gives

$$I_f(G_1 + G_2) = \int_0^1 P_f(G_1 + G_2, x)\, dx \geqslant \int_0^1 P_f(G_1, x)\, dx + \int_0^1 P_f(G_2, x)\, dx = I_f(G_1) + I_f(G_2).$$

□

Theorem 4. *Let $\delta \in \mathbb{Z}^+$, and let G_1 and G_2 be two graphs with order n_1 and n_2, respectively, and a minimum degree of at least δ, and $a > 0$. If $f : \mathbb{Z}^+ \cap [\delta, \infty) \to (0, a/2]$, then the following inequalities hold.*

(1) *If $f \in AP_1(\delta)$, then*

$$I_f(G_1 + G_2) \leqslant \frac{2}{3}\Big(\frac{a\, n_1 f(n_2)}{a + f(n_2)}\, I_f(G_1)\Big)^{1/2} + \frac{2}{3}\Big(\frac{a\, n_2 f(n_1)}{a + f(n_1)}\, I_f(G_2)\Big)^{1/2}.$$

(2) *If $f \in AP_2(\delta)$, then*

$$I_f(G_1 + G_2) \geqslant \frac{4}{3}\Big(\frac{f(n_2)}{a + f(n_2)}\, I_f(G_1) + \frac{f(n_1)}{a + f(n_1)}\, I_f(G_2)\Big).$$

Proof. Let us define the function $g = f/a$. Thus, $g : \mathbb{Z}^+ \cap [\delta, \infty) \to (0, 1/2]$ and Proposition 1 gives that $P_g(G_1, x)$ and $P_g(G_2, x)$ are convex functions on $(0, \infty)$ and continuous on $[0, \infty)$; hence, they are convex on $[0, 1]$.

If $f \in AP_1(\delta)$, then $f/a \in AP_1(\delta)$ and Theorem 2 gives

$$P_{f/a}(G_1 + G_2, x) \leqslant x^{a/f(n_2)} P_{f/a}(G_1, x) + x^{a/f(n_1)} P_{f/a}(G_2, x).$$

Note that $f/a \leqslant 1/2$ gives $a/f - 1 \geqslant 1$, and so $P_{f/a}(G_i, 0) = 0$ and $P_{f/a}(G_i, 1) = n_i$ for $i = 1, 2$. Since $a/f(n_2) \geqslant 2$, we have that $x^{a/f(n_2)}$ is also a convex function on $[0, 1]$, and Lemma 2 gives

$$\begin{aligned}\int_0^1 x^{a/f(n_2)} P_{f/a}(G_1, x)\, dx &\leqslant \frac{2}{3}\Big(\int_0^1 x^{a/f(n_2)} dx \int_0^1 P_{f/a}(G_1, x)\, dx \Big)^{1/2} \\ &\qquad \cdot \Big((0+1)\big(P_{f/a}(G_1, 0) + P_{f/a}(G_1, 1)\big) \Big)^{1/2} \\ &= \frac{2}{3}\Big(n_1 \frac{f(n_2)}{a + f(n_2)}\, I_{f/a}(G_1) \Big)^{1/2} \\ &= \frac{2}{3}\Big(\frac{n_1 f(n_2)}{a + f(n_2)}\, \frac{1}{a}\, I_f(G_1) \Big)^{1/2}.\end{aligned}$$

We obtain in a similar way

$$\int_0^1 x^{a/f(n_1)} P_{f/a}(G_2, x)\, dx \leqslant \frac{2}{3}\Big(\frac{n_2 f(n_1)}{a + f(n_1)}\, \frac{1}{a}\, I_f(G_2) \Big)^{1/2}.$$

Hence,

$$\begin{aligned}\frac{1}{a}\, I_f(G_1 + G_2) = I_{f/a}(G_1 + G_2) &= \int_0^1 P_{f/a}(G_1 + G_2, x)\, dx \\ &\leqslant \frac{2}{3}\Big(\frac{n_1 f(n_2)}{a + f(n_2)}\, \frac{1}{a}\, I_f(G_1) \Big)^{1/2} + \frac{2}{3}\Big(\frac{n_2 f(n_1)}{a + f(n_1)}\, \frac{1}{a}\, I_f(G_2) \Big)^{1/2}.\end{aligned}$$

If $f \in AP_2(\delta)$, then $f/a \in AP_2(\delta)$ and Theorem 2 gives

$$P_{f/a}(G_1 + G_2, x) \geqslant x^{a/f(n_2)} P_{f/a}(G_1, x) + x^{a/f(n_1)} P_{f/a}(G_2, x).$$

Thus, Lemma 1 gives

$$\begin{aligned}\frac{1}{a}\, I_f(G_1 + G_2) = I_{f/a}(G_1 + G_2) &= \int_0^1 P_{f/a}(G_1 + G_2, x)\, dx \\ &\geqslant \frac{4}{3} \int_0^1 x^{a/f(n_2)} dx \int_0^1 P_{f/a}(G_1, x)\, dx + \frac{4}{3} \int_0^1 x^{a/f(n_1)} dx \int_0^1 P_{f/a}(G_2, x)\, dx \\ &= \frac{4}{3}\, \frac{f(n_2)}{a + f(n_2)}\, \frac{1}{a}\, I_f(G_1) + \frac{4}{3}\, \frac{f(n_1)}{a + f(n_1)}\, \frac{1}{a}\, I_f(G_2).\end{aligned}$$

□

Corollary 3. *Let $\delta \in \mathbb{Z}^+$, and let G_1 and G_2 be two graphs with order n_1 and n_2, respectively, and a minimum degree of at least δ, and $a > 0$. If $f : \mathbb{Z}^+ \cap [\delta, \infty) \to (0, a/2]$ and $f \in AP_1(\delta)$, then*

$$I_f(G_1 + G_2) \leqslant \frac{2}{3}\Big(\frac{n_1 a}{3}\, I_f(G_1) \Big)^{1/2} + \frac{2}{3}\Big(\frac{n_2 a}{3}\, I_f(G_2) \Big)^{1/2}.$$

Proof. Since $F(t) = at/(a+t)$ is an increasing function on $t \in [0, \infty)$ and $f \leqslant a/2$,

$$n_i \frac{af(n_j)}{a + f(n_j)} \leqslant n_i \frac{a\,\frac{a}{2}}{a + \frac{a}{2}} = \frac{n_i a}{3},$$

and Theorem 4 gives the desired inequality. □

Next, we obtain inequalities for several topological indices of joins of graphs.

Theorems 1 and 4 (with $a = 2$) and Proposition 4 have the following consequence.

Corollary 4. *Let G_1 and G_2 be two graphs with order n_1 and n_2, respectively, and $\alpha \in \mathbb{R}$.*

(1) *If $\alpha \leqslant -1$, then*

$$M_1^\alpha(G_1 + G_2) \leqslant \frac{2}{3}\Big(\frac{2n_1 n_2^\alpha}{2 + n_2^\alpha} M_1^\alpha(G_1)\Big)^{1/2} + \frac{2}{3}\Big(\frac{2n_2 n_1^\alpha}{2 + n_1^\alpha} M_1^\alpha(G_2)\Big)^{1/2}.$$

(2) *If $\alpha \in [-1, 0]$, then*

$$M_1^\alpha(G_1 + G_2) \geqslant \frac{4}{3}\Big(\frac{n_2^\alpha}{2 + n_2^\alpha} M_1^\alpha(G_1) + \frac{n_1^\alpha}{2 + n_1^\alpha} M_1^\alpha(G_2)\Big).$$

(3) *If $\alpha \geqslant 0$, then*

$$M_1^\alpha(G_1 + G_2) \geqslant M_1^\alpha(G_1) + M_1^\alpha(G_2).$$

Corollary 4 gives the following result.

Corollary 5. *If G_1 and G_2 are two graphs with order n_1 and n_2, respectively, then*

$$\frac{4}{3}\Big(\frac{1}{2n_2 + 1} ID(G_1) + \frac{1}{2n_1 + 1} ID(G_2)\Big) \leqslant ID(G_1 + G_2)$$
$$\leqslant \frac{2}{3}\Big(\frac{2n_1}{2n_2 + 1} ID(G_1)\Big)^{1/2} + \frac{2}{3}\Big(\frac{2n_2}{2n_1 + 1} ID(G_2)\Big)^{1/2}.$$

The sum lordeg index

$$SL(G) = \sum_{u \in V(G)} d_u \sqrt{\log d_u}$$

is one of the Adriatic indices introduced in reference [36].

Since $f(t) = t\sqrt{\log t}$ is an increasing function on $[1, \infty)$, $f \in AP_3(2)$. Thus, we have the following:

Lemma 3. *If $f(t) = t\sqrt{\log t}$, then $f \in AP_3(2)$.*

Lemma 3 and Proposition 4 have the following consequence.

Corollary 6. *If G_1 and G_2 are graphs without pendant vertices and with order n_1 and n_2, respectively, then*

$$SL(G_1 + G_2) \geqslant SL(G_1) + SL(G_2).$$

4. Corona Products

Theorem 5. *Let G_1 and G_2 be two graphs with order n_1 and n_2, respectively, and $f : \mathbb{Z}^+ \to \mathbb{R}^+$. The f-polynomial of the corona product $G_1 \circ G_2$ satisfies the following inequalities for $x \in (0, 1]$.*

(1) *If $f \in AP_1(1)$, then*

$$P_f(G_1 \circ G_2, x) \leqslant x^{1/f(n_2)} P_f(G_1, x) + n_1 x^{1/f(1)} P_f(G_2, x).$$

(2) *If $f \in AP_2(1)$, then*

$$P_f(G_1 \circ G_2, x) \geqslant x^{1/f(n_2)} P_f(G_1, x) + n_1 x^{1/f(1)} P_f(G_2, x).$$

(3) *If $\delta \in \mathbb{Z}^+$, G_1 and G_2 have a minimum degree of at least δ, and f is increasing on $\mathbb{Z}^+ \cap [\delta, \infty)$, then*

$$P_f(G_1 \circ G_2, x) \geqslant P_f(G_1, x) + n_1 P_f(G_2, x).$$

Proof. If $u \in V(G_1)$, then its degree in $G_1 \circ G_2$ is $d_u + n_2$. If $G_2^{(i)}$ is a copy of G_2 in $G_1 \circ G_2$ and $v \in V(G_2^{(i)})$, then its degree in $G_1 \circ G_2$ is $d_v + 1$.

Assume first that $f \in AP_1(1)$. Since $f \in AP_1(1)$ and $x \in (0, 1]$,

$$\begin{aligned} P_f(G_1 \circ G_2, x) &= \sum_{u \in V(G_1)} x^{1/f(d_u+n_2)-1} + n_1 \sum_{v \in V(G_2)} x^{1/f(d_v+1)-1} \\ &\leqslant \sum_{u \in V(G_1)} x^{1/f(d_u)-1} x^{1/f(n_2)} + n_1 \sum_{v \in V(G_2)} x^{1/f(d_v)-1} x^{1/f(1)} \\ &= x^{1/f(n_2)} P_f(G_1, x) + n_1 x^{1/f(1)} P_f(G_2, x). \end{aligned}$$

If $f \in AP_2(1)$, then a similar argument allows us to obtain the corresponding inequality.

Finally, assume that G_1 and G_2 have a minimum degree of at least δ, and f is increasing on $\mathbb{Z}^+ \cap [\delta, \infty)$. We have

$$\begin{aligned} P_f(G_1 \circ G_2, x) &= \sum_{u \in V(G_1)} x^{1/f(d_u+n_2)-1} + n_1 \sum_{v \in V(G_2)} x^{1/f(d_v+1)-1} \\ &\geqslant \sum_{u \in V(G_1)} x^{1/f(d_u)-1} + n_1 \sum_{v \in V(G_2)} x^{1/f(d_v)-1} = P_f(G_1, x) + n_1 P_f(G_2, x). \end{aligned}$$

□

Theorems 1 and 5 have the following consequence.

Corollary 7. *Let G_1 and G_2 be two graphs with order n_1 and n_2, respectively, and let $\alpha \in \mathbb{R}$ and $f(t) = t^\alpha$. The f-polynomial of the corona product $G_1 \circ G_2$ satisfies the following inequalities for $x \in (0, 1]$.*

(1) *If $\alpha \leqslant -1$, then*

$$P_f(G_1 \circ G_2, x) \leqslant x^{n_2^{-\alpha}} P_f(G_1, x) + n_1 x P_f(G_2, x).$$

(2) *If $\alpha \in [-1, 0]$, then*

$$P_f(G_1 \circ G_2, x) \geqslant x^{n_2^{-\alpha}} P_f(G_1, x) + n_1 x P_f(G_2, x).$$

(3) *If $\alpha \geqslant 0$, then*

$$P_f(G_1 \circ G_2, x) \geqslant P_f(G_1, x) + n_1 P_f(G_2, x).$$

Corollary 7 has the following consequence.

Corollary 8. *Given two graphs G_1 and G_2 with order n_1 and n_2, respectively, the ID polynomial of the corona product $G_1 \circ G_2$ is*

$$ID(G_1 \circ G_2, x) = x^{n_2} ID(G_1, x) + n_1 x ID(G_2, x).$$

Theorem 5 has the following consequence.

Corollary 9. *Let G_1 and G_2 be two graphs without pendant vertices and with order n_1 and n_2, respectively. If $f(t) = t\sqrt{\log t}$, then the f-polynomial of the corona product $G_1 \circ G_2$ satisfies for $x \in (0,1]$*

$$P_f(G_1 \circ G_2, x) \geqslant P_f(G_1, x) + n_1 P_f(G_2, x).$$

Next, we obtain bounds for $I_f(G_1 \circ G_2)$ by using the previous inequalities for $P_f(G_1 \circ G_2, x)$.

Proposition 5. *Let $\delta \in \mathbb{Z}^+$, and let G_1 and G_2 be two graphs with order n_1 and n_2, respectively, and a minimum degree of at least δ. If f is increasing on $\mathbb{Z}^+ \cap [\delta, \infty)$, then*

$$I_f(G_1 \circ G_2) \geqslant I_f(G_1) + n_1 I_f(G_2).$$

Proof. Theorem 5 gives

$$P_f(G_1 \circ G_2, x) \geqslant P_f(G_1, x) + n_1 P_f(G_2, x)$$

for every $x \in (0,1]$. Thus, Proposition 1 gives the desired inequality. □

Theorem 6. *Let G_1 and G_2 be two graphs with order n_1 and n_2, respectively, and $a > 0$. If $f : \mathbb{Z}^+ \to (0, a/2]$, then the following inequalities hold.*

(1) *If $f \in AP_1(1)$, then*

$$I_f(G_1 \circ G_2) \leqslant \frac{2}{3}\Big(\frac{a n_1 f(n_2)}{a + f(n_2)}\, I_f(G_1)\Big)^{1/2} + \frac{2n_1}{3}\Big(\frac{a n_2 f(1)}{a + f(1)}\, I_f(G_2)\Big)^{1/2}.$$

(2) *If $f \in AP_2(1)$, then*

$$I_f(G_1 \circ G_2) \geqslant \frac{4}{3}\Big(\frac{f(n_2)}{a + f(n_2)}\, I_f(G_1) + \frac{n_1 f(1)}{a + f(1)}\, I_f(G_2)\Big).$$

Proof. Let us define the function $g = f/a$. Thus, $g : \mathbb{Z}^+ \to (0, 1/2]$ and Proposition 1 gives that $P_g(G_1, x)$ and $P_g(G_2, x)$ are convex functions on $(0, \infty)$ and continuous on $[0, \infty)$; hence, they are convex on $[0,1]$.

If $f \in AP_1(1)$, then $f/a \in AP_1(1)$ and Theorem 5 gives

$$P_{f/a}(G_1 \circ G_2, x) \leqslant x^{a/f(n_2)} P_{f/a}(G_1, x) + n_1 x^{a/f(1)} P_{f/a}(G_2, x).$$

Note that $f/a \leqslant 1/2$ gives $a/f - 1 \geqslant 1$, and so $P_{f/a}(G_i, 0) = 0$ and $P_{f/a}(G_i, 1) = n_i$ for $i = 1, 2$. Since $a/f(n_2) \geqslant 2$, we have that $x^{a/f(n_2)}$ is also a convex function on $[0,1]$, and Lemma 2 gives

$$\begin{aligned} \int_0^1 x^{a/f(n_2)} P_{f/a}(G_1, x)\, dx &\leqslant \frac{2}{3}\Big(\int_0^1 x^{a/f(n_2)} dx \int_0^1 P_{f/a}(G_1, x)\, dx\Big)^{1/2} \\ &\quad \cdot \Big((0+1)\big(P_{f/a}(G_1, 0) + P_{f/a}(G_1, 1)\big)\Big)^{1/2} \\ &= \frac{2}{3}\Big(n_1 \frac{f(n_2)}{a + f(n_2)}\, \frac{1}{a}\, I_f(G_1)\Big)^{1/2}. \end{aligned}$$

We obtain, in a similar way,

$$n_1 \int_0^1 x^{a/f(1)} P_{f/a}(G_2, x)\, dx \leqslant \frac{2n_1}{3}\Big(\frac{n_2 f(1)}{a + f(1)}\, \frac{1}{a}\, I_f(G_2)\Big)^{1/2}.$$

Hence,

$$\begin{aligned}\frac{1}{a}\, I_f(G_1 \circ G_2) = I_{f/a}(G_1 \circ G_2) &= \int_0^1 P_{f/a}(G_1 \circ G_2, x)\, dx \\ &\leqslant \frac{2}{3}\Big(\frac{n_1 f(n_2)}{a+f(n_2)}\, \frac{1}{a}\, I_f(G_1)\Big)^{1/2} + \frac{2n_1}{3}\Big(\frac{n_2 f(1)}{a+f(1)}\, \frac{1}{a}\, I_f(G_2)\Big)^{1/2}.\end{aligned}$$

If $f \in AP_2(\delta)$, then $f/a \in AP_2(\delta)$ and Theorem 5 gives

$$P_{f/a}(G_1 \circ G_2, x) \geqslant x^{a/f(n_2)} P_{f/a}(G_1, x) + n_1 x^{a/f(1)} P_{f/a}(G_2, x).$$

Thus, Lemma 1 gives

$$\begin{aligned}\frac{1}{a}\, I_f(G_1 \circ G_2) = I_{f/a}(G_1 \circ G_2) &= \int_0^1 P_{f/a}(G_1 \circ G_2, x)\, dx \\ &\geqslant \frac{4}{3} \int_0^1 x^{a/f(n_2)} dx \int_0^1 P_{f/a}(G_1, x)\, dx + \frac{4n_1}{3} \int_0^1 x^{a/f(1)} dx \int_0^1 P_{f/a}(G_2, x)\, dx \\ &= \frac{4}{3}\, \frac{f(n_2)}{a+f(n_2)}\, \frac{1}{a}\, I_f(G_1) + \frac{4}{3}\, \frac{n_1 f(1)}{a+f(1)}\, \frac{1}{a}\, I_f(G_2).\end{aligned}$$

□

Next, we obtain inequalities for several topological indices of the corona product of graphs. Theorems 1 and 6 (with $a = 2$) and Proposition 5 have the following consequence.

Corollary 10. *Let G_1 and G_2 be two graphs with order n_1 and n_2, respectively, and $\alpha \in \mathbb{R}$.*

(1) *If $\alpha \leqslant -1$, then*

$$M_1^\alpha(G_1 \circ G_2) \leqslant \frac{2}{3}\Big(\frac{2n_1 n_2^\alpha}{2+n_2^\alpha}\, M_1^\alpha(G_1)\Big)^{1/2} + \frac{2n_1}{3}\Big(\frac{2n_2}{3}\, M_1^\alpha(G_2)\Big)^{1/2}.$$

(2) *If $\alpha \in [-1, 0]$, then*

$$M_1^\alpha(G_1 \circ G_2) \geqslant \frac{4}{3}\Big(\frac{n_2^\alpha}{2+n_2^\alpha}\, M_1^\alpha(G_1) + \frac{n_1}{3}\, M_1^\alpha(G_2)\Big).$$

(3) *If $\alpha \geqslant 0$, then*

$$M_1^\alpha(G_1 \circ G_2) \geqslant M_1^\alpha(G_1) + n_1 M_1^\alpha(G_2).$$

Corollary 10 gives the following result.

Corollary 11. *If G_1 and G_2 are two graphs with order n_1 and n_2, respectively, then*

$$\begin{aligned}\frac{4}{3}\Big(\frac{1}{2n_2+1}\, ID(G_1) + \frac{n_1}{3}\, ID(G_2)\Big) &\leqslant ID(G_1 \circ G_2) \\ &\leqslant \frac{2}{3}\Big(\frac{2n_1}{2n_2+1}\, ID(G_1)\Big)^{1/2} + \frac{2n_1}{3}\Big(\frac{2n_2}{3}\, ID(G_2)\Big)^{1/2}.\end{aligned}$$

Proposition 5 has the following consequence.

Corollary 12. *If G_1 and G_2 are graphs without pendant vertices and with order n_1 and n_2, respectively, then*

$$SL(G_1 \circ G_2) \geqslant SL(G_1) + n_1 SL(G_2).$$

5. Mycielskian Graphs

Given a graph G with $V(G) = \{v_1, \ldots, v_n\}$, its Mycielskian graph $\mu(G)$ contains G itself as a subgraph, together with $n+1$ additional vertices $\{u_1, \ldots, u_n, w\}$. Each vertex u_i is connected by an edge to w. In addition, for each edge $v_i v_j$ of G, the Mycielskian graph includes two edges, $u_i v_j$ and $v_i u_j$. Thus, if G has order n and size m, then $\mu(G)$ has $2n+1$ vertices and $3m+n$ edges. In addition,

$$d_{v_i,\mu(G)} = 2\, d_{v_i}, \qquad d_{u_i,\mu(G)} = d_{v_i} + 1, \qquad d_{w,\mu(G)} = n,$$

for each $i = 1, \ldots, n$, where d_v and $d_{v,\mu(G)}$ denote the degree of the vertex v in G and $\mu(G)$, respectively.

Mycielskian graphs are a construction for embedding any graph into a larger graph with a higher chromatic number while avoiding the creation of additional triangles (see [37]). Mycielskian graphs have been used also in mathematical chemistry; see, e.g., [38–41].

Theorem 7. *Let G be a graph with order n, $a > 0$, $\alpha \in \mathbb{R}$, $f(t) = t^\alpha / a$, and $x \in (0,1]$.*

(1) *If $\alpha \leqslant -1$, then*

$$P_{f/a}(\mu(G), x) \leqslant x^{a2^{-\alpha}-1}\, P_f(G, x^{a2^{-\alpha}}) + x^a P_{f/a}(G, x) + x^{an^{-\alpha}-1}.$$

(2) *If $\alpha \in [-1, 0]$, then*

$$P_{f/a}(\mu(G), x) \geqslant x^{a2^{-\alpha}-1}\, P_f(G, x^{a2^{-\alpha}}) + x^a P_{f/a}(G, x) + x^{an^{-\alpha}-1}.$$

(3) *If $\alpha \geqslant 0$, then*

$$P_{f/a}(\mu(G), x) \geqslant x^{a2^{-\alpha}-1}\, P_f(G, x^{a2^{-\alpha}}) + P_{f/a}(G, x) + x^{an^{-\alpha}-1}.$$

Proof. Assume first that $\alpha \leqslant -1$. Thus, $-\alpha \geqslant 1$ and $(d_{v_i} + 1)^{-\alpha} \geqslant d_{v_i}^{-\alpha} + 1$. Since $x \in (0,1]$, we have

$$\begin{aligned} P_{f/a}(\mu(G), x) &= \sum_{i=1}^{n} x^{a(d_{v_i,\mu(G)})^{-\alpha}-1} + \sum_{i=1}^{n} x^{a(d_{u_i,\mu(G)})^{-\alpha}-1} + x^{a(d_{w,\mu(G)})^{-\alpha}-1} \\ &= \sum_{i=1}^{n} x^{a2^{-\alpha}(d_{v_i}^{-\alpha}-1)+a2^{-\alpha}-1} + \sum_{i=1}^{n} x^{a(d_{v_i}+1)^{-\alpha}-1} + x^{an^{-\alpha}-1} \\ &\leqslant x^{a2^{-\alpha}-1} \sum_{i=1}^{n} \left(x^{a2^{-\alpha}}\right)^{d_{v_i}^{-\alpha}-1} + \sum_{i=1}^{n} x^{ad_{v_i}^{-\alpha}+a-1} + x^{an^{-\alpha}-1} \\ &= x^{a2^{-\alpha}-1}\, P_f(G, x^{a2^{-\alpha}}) + x^a P_{f/a}(G, x) + x^{an^{-\alpha}-1}. \end{aligned}$$

If $\alpha \in [-1, 0]$, then $(d_{v_i} + 1)^{-\alpha} \leqslant d_{v_i}^{-\alpha} + 1$, and so

$$\begin{aligned} P_{f/a}(\mu(G), x) &= \sum_{i=1}^{n} x^{a2^{-\alpha}(d_{v_i}^{-\alpha}-1)+a2^{-\alpha}-1} + \sum_{i=1}^{n} x^{a(d_{v_i}+1)^{-\alpha}-1} + x^{an^{-\alpha}-1} \\ &\geqslant x^{a2^{-\alpha}-1} \sum_{i=1}^{n} \left(x^{a2^{-\alpha}}\right)^{d_{v_i}^{-\alpha}-1} + \sum_{i=1}^{n} x^{ad_{v_i}^{-\alpha}+a-1} + x^{an^{-\alpha}-1} \\ &= x^{a2^{-\alpha}-1}\, P_f(G, x^{a2^{-\alpha}}) + x^a P_{f/a}(G, x) + x^{an^{-\alpha}-1}. \end{aligned}$$

If $\alpha \geqslant 0$, then $(d_{v_i}+1)^{-\alpha} \leqslant d_{v_i}^{-\alpha}$, and so

$$\begin{aligned} P_{f/a}(\mu(G),x) &= \sum_{i=1}^{n} x^{a2^{-\alpha}(d_{v_i}^{-\alpha}-1)+a2^{-\alpha}-1} + \sum_{i=1}^{n} x^{a(d_{v_i}+1)^{-\alpha}-1} + x^{an^{-\alpha}-1} \\ &\geqslant x^{a2^{-\alpha}-1} \sum_{i=1}^{n} \left(x^{a2^{-\alpha}}\right)^{d_{v_i}^{-\alpha}-1} + \sum_{i=1}^{n} x^{ad_{v_i}^{-\alpha}-1} + x^{an^{-\alpha}-1} \\ &= x^{a2^{-\alpha}-1} P_f(G, x^{a2^{-\alpha}}) + P_{f/a}(G,x) + x^{an^{-\alpha}-1}. \end{aligned}$$

□

Theorem 7 has the following consequences.

Corollary 13. *If G is a graph with order n, then*

$$ID(\mu(G),x) = x\, ID(G,x^2) + x\, ID(G,x) + x^{n-1}.$$

Next, we obtain bounds for $M_1^{\alpha}(\mu(G))$ by using the previous inequalities for $P_f(\mu(G),x)$.

Theorem 8. *Let G be a graph with order n and $\alpha \in \mathbb{R}$.*

(1) *If $\alpha \leqslant -1$, then*

$$M_1^{\alpha}(\mu(G)) \leqslant 2^{\alpha} M_1^{\alpha}(G) + \frac{2}{3}\Big(\frac{2}{3}\, n\, M_1^{\alpha}(G)\Big)^{1/2} + n^{\alpha}.$$

(2) *If $\alpha \in [-1,0]$, then*

$$M_1^{\alpha}(\mu(G)) \geqslant \Big(2^{\alpha} + \frac{4}{9}\Big) M_1^{\alpha}(G) + n^{\alpha}.$$

(3) *If $\alpha \geqslant 0$, then*

$$M_1^{\alpha}(\mu(G)) \geqslant (2^{\alpha}+1) M_1^{\alpha}(G) + n^{\alpha}.$$

Proof. Let us consider the function $f(t) = t^{\alpha}$. Assume first that $\alpha \geqslant 0$. Theorem 7 with $a = 1$ gives

$$P_f(\mu(G),x) \geqslant x^{2^{-\alpha}-1} P_f(G, x^{2^{-\alpha}}) + P_f(G,x) + x^{n^{-\alpha}-1}$$

for $x \in (0,1]$. Thus,

$$\begin{aligned} M_1^{\alpha}(\mu(G)) &= \int_0^1 P_f(\mu(G),x)\,dx \\ &\geqslant \int_0^1 x^{2^{-\alpha}-1} P_f(G,x^{2^{-\alpha}})\,dx + \int_0^1 P_f(G,x)\,dx + \int_0^1 x^{n^{-\alpha}-1}dx \\ &= \int_0^1 \frac{1}{2^{-\alpha}} P_f(G,t)\,dt + M_1^{\alpha}(G) + \left[\frac{x^{n^{-\alpha}}}{n^{-\alpha}}\right]_0^1 \\ &= (2^{\alpha}+1) M_1^{\alpha}(G) + n^{\alpha}. \end{aligned}$$

Note that $f(t) \leqslant 1$ for every $t \in \mathbb{Z}^+$ if $\alpha \leqslant 0$. Thus, $2/f - 1 \geqslant 1$, and $P_{f/2}(\mu(G),x)$ is a convex function on $[0,1]$ with $P_{f/2}(\mu(G),0) = 0$.

Assume now that $\alpha \leqslant -1$. Theorem 7 with $a = 2$ gives

$$P_{f/2}(\mu(G),x) \leqslant x^{2^{1-\alpha}-1} P_f(G, x^{2^{1-\alpha}}) + x^2 P_{f/2}(G,x) + x^{2n^{-\alpha}-1}$$

for $x \in (0,1]$. Thus,

$$\begin{aligned} M_1^\alpha(\mu(G)) &= 2\, I_{f/2}(\mu(G)) = 2\int_0^1 P_{f/2}(\mu(G),x)\,dx \\ &\leqslant 2\int_0^1 x^{2^{1-\alpha}-1}\,P_f(G,x^{2^{1-\alpha}})\,dx + 2\int_0^1 x^2 P_{f/2}(G,x)\,dx + 2\int_0^1 x^{2n^{-\alpha}-1}dx \\ &= \int_0^1 \frac{1}{2^{-\alpha}}\,P_f(G,t)\,dt + 2\int_0^1 x^2 P_{f/2}(G,x)\,dx + \left[\frac{x^{2n^{-\alpha}}}{n^{-\alpha}}\right]_0^1 \\ &= 2^\alpha M_1^\alpha(G) + 2\int_0^1 x^2 P_{f/2}(G,x)\,dx + n^\alpha. \end{aligned}$$

Since x^2 and $P_{f/2}(\mu(G),x)$ are non-negative convex functions on $[0,1]$, Lemma 2 gives

$$\begin{aligned} \int_0^1 x^2 P_{f/2}(G,x)\,dx &\leqslant \frac{2}{3}\left(\int_0^1 x^2 dx \int_0^1 P_{f/2}(G,x)\,dx\right)^{1/2} \left(P_{f/2}(G,0)+P_{f/2}(G,1)\right)^{1/2} \\ &= \frac{2}{3}\Big(\frac{1}{3}\cdot\frac{1}{2}\,M_1^\alpha(G)\,n\Big)^{1/2} = \frac{1}{3}\Big(\frac{2}{3}\,n\,M_1^\alpha(G)\Big)^{1/2}. \end{aligned}$$

Finally, assume that $\alpha \in [-1,0]$. Theorem 7 with $a = 2$ gives

$$P_{f/2}(\mu(G),x) \geqslant x^{2^{1-\alpha}-1}\,P_f(G,x^{2^{1-\alpha}}) + x^2 P_{f/2}(G,x) + x^{2n^{-\alpha}-1}$$

for $x \in (0,1]$. Hence, the previous argument gives

$$M_1^\alpha(\mu(G)) \geqslant 2^\alpha M_1^\alpha(G) + 2\int_0^1 x^2 P_{f/2}(G,x)\,dx + n^\alpha.$$

Since x^2 and $P_{f/2}(\mu(G),x)$ are non-negative and convex on $[0,1]$, Lemma 1 gives

$$2\int_0^1 x^2 P_{f/2}(G,x)\,dx \geqslant \frac{8}{3}\int_0^1 x^2 dx \int_0^1 P_{f/2}(G,x)\,dx = \frac{4}{9}\,M_1^\alpha(G).$$

□

Theorem 8 has the following consequence for the inverse degree index.

Corollary 14. *If G is a graph with order n, then*

$$\frac{17}{18}\,ID(G) + \frac{1}{n} \leqslant ID(\mu(G)) \leqslant \frac{1}{2}\,ID(G) + \frac{2}{3}\Big(\frac{2}{3}\,n\,ID(G)\Big)^{1/2} + \frac{1}{n}.$$

6. ID Polynomials of Other Graph Operations

Let us recall the definition of other graph operations.

Let G be a graph. Given an edge $e = uv$ of G, let $V(e) = \{u, v\}$. Now we can define the following five graph operations.

The line graph, denoted by $\mathcal{L}(G)$, is the graph whose vertices correspond to the edges of G, and two vertices are adjacent if and only if the corresponding edges in G share a vertex.

The subdivision graph, denoted by $S(G)$, is the graph obtained from G by replacing each of its edges by a path of length two, or equivalently, by inserting an additional vertex into each edge of G.

The total graph, denoted by $T(G)$, has as its vertices the edges and vertices of G. Adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G.

The graph $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G and then joining each new vertex to the end vertices of the corresponding edge. Another way to describe $R(G)$ is to replace each edge of G by a triangle.

The graph $Q(G)$ is the graph obtained from G by inserting a new vertex into each edge of G and by joining edges the pairs of these new vertices that lie on adjacent edges of G.

Given a graph G, we may define the following sets:

$$\begin{aligned} EE(G) &:= \{\{e,e'\} : e,e' \in E(G), e \neq e', |V(e) \cap V(e')| = 1\}, \\ EV(G) &:= \{\{e,v\} : e \in E(G), v \in V(e)\}. \end{aligned}$$

We may then write these five graph operations as follows:

$$\begin{aligned} \mathcal{L}(G) &:= (E(G), EE(G)), \\ S(G) &:= (V(G) \cup E(G), EV(G)), \\ T(G) &:= (V(G) \cup E(G), E(G) \cup EV(G) \cup EE(G)), \\ R(G) &:= (V(G) \cup E(G), E(G) \cup EV(G)), \\ Q(G) &:= (V(G) \cup E(G), EV(G) \cup EE(G)). \end{aligned}$$

The following result is elementary.

Proposition 6. *If G is a graph with m edges, then*

$$ID(S(G),x) = ID(G,x) + mx, \qquad ID(S(G)) = ID(G) + \frac{1}{2}\,m.$$

The computation of the ID polynomials of other graph operations involves harmonic polynomials.

As we have seen, the line graph $\mathcal{L}(G)$ of G is a graph that has a vertex $w_e \in V(\mathcal{L}(G))$ for each edge $e \in E(G)$, and an edge joining w_{e_i} and w_{e_j} when e_i and e_j share a vertex (i.e., $\mathcal{L}(G)$ is the intersection graph of $E(G)$). It is easy to check that if $uv \in E(G)$, then the degree of $w_{uv} \in V(\mathcal{L}(G))$ is $d_u + d_v - 2$.

Line graphs were initially introduced in the papers [1,42], although the terminology of line graph was used in reference [43] for the first time. They are an active topic of research at this moment. In particular, several papers study some topological indices on line graphs (see, e.g., [44,45]).

Proposition 7. *If G is a graph, then*

$$H(G,x) = x^2 ID(\mathcal{L}(G),x), \qquad ID(\mathcal{L}(G)) = \int_0^1 \frac{H(G,x)}{x^2}\,dx.$$

Proof. We have

$$\begin{aligned} H(G,x) &= \sum_{uv\in E(G)} x^{d_u+d_v-1} = \sum_{w\in V(\mathcal{L}(G))} x^{d_w+1} = x^2 \sum_{w\in V(\mathcal{L}(G))} x^{d_w-1} = x^2 ID(\mathcal{L}(G),x), \\ ID(\mathcal{L}(G)) &= \int_0^1 ID(\mathcal{L}(G),x)\,dx = \int_0^1 \frac{H(G,x)}{x^2}\,dx. \end{aligned}$$

□

Proposition 8. *If G is a graph, then*

$$ID(T(G),x) = xID(G,x^2) + H(G,x), \qquad ID(T(G)) = \frac{1}{2}\,ID(G) + \frac{1}{2}\,H(G).$$

Proof. We have

$$
\begin{aligned}
ID(T(G),x) &= \sum_{u\in V(G)} x^{2d_u-1} + \sum_{uv\in E(G)} x^{d_u+d_v-1} = x \sum_{u\in V(G)} (x^2)^{d_u-1} + H(G,x) \\
&= xID(G,x^2) + H(G,x), \\
ID(T(G)) &= \frac{1}{2}\int_0^1 2xID(G,x^2)\,dx + \int_0^1 H(G,x)\,dx = \frac{1}{2}\int_0^1 ID(G,t)\,dt + \frac{1}{2}\,H(G) \\
&= \frac{1}{2}\,ID(G) + \frac{1}{2}\,H(G).
\end{aligned}
$$

□

Proposition 9. *If G is a graph with m edges, then*

$$
ID(R(G),x) = xID(G,x^2) + mx, \qquad ID(R(G)) = \frac{1}{2}\,ID(G) + \frac{1}{2}\,m.
$$

Proof. We have

$$
\begin{aligned}
ID(R(G),x) &= \sum_{u\in V(G)} x^{2d_u-1} + \sum_{uv\in E(G)} x^{2-1} = xID(G,x^2) + mx, \\
ID(R(G)) &= \frac{1}{2}\int_0^1 2xID(G,x^2)\,dx + \int_0^1 mx\,dx = \frac{1}{2}\,ID(G) + \frac{1}{2}\,m.
\end{aligned}
$$

□

Proposition 10. *If G is a graph, then*

$$
ID(Q(G),x) = ID(G,x) + H(G,x), \qquad ID(Q(G)) = ID(G) + \frac{1}{2}\,H(G).
$$

Proof. We have

$$
\begin{aligned}
ID(Q(G),x) &= \sum_{u\in V(G)} x^{d_u-1} + \sum_{uv\in E(G)} x^{d_u+d_v-1} = ID(G,x) + H(G,x), \\
ID(Q(G)) &= \int_0^1 ID(G,x)\,dx + \int_0^1 H(G,x)\,dx = ID(G) + \frac{1}{2}\,H(G).
\end{aligned}
$$

□

Corollary 15. *If G is a graph with m edges, then*

$$
\begin{aligned}
ID(S(G)) &= ID(R(G)) + \frac{1}{2}\,ID(G) = 2\,ID(R(G)) - \frac{1}{2}\,m, \\
ID(Q(G)) &= ID(T(G)) + \frac{1}{2}\,ID(G).
\end{aligned}
$$

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