Study of the Gromov hyperbolicity constant on graphs

by

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Abstract

The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space and Riemannian manifolds of negative sectional curvature. It is remarkable that a simple concept leads to such a rich general theory. The study of hyperbolic graphs is an interesting topic since the hyperbolicity of any geodesic metric space is equivalent to the hyperbolicity of a graph related to it.

In this Ph. D. Thesis we characterize the hyperbolicity constant of interval graphs and circular-arc graphs. Likewise, we provide relationships between dominant sets and the hyperbolicity constant. Finally, we study the invariance of the hyperbolicity constant when the graphs are transformed by several operators.

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Introduction

Hyperbolicity in Gromov's sense is a simple concept, but at the same time it leads to a very rich general theory. This concept captures the qualities of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature, and discrete spaces like trees and Cayley graphs of many finitely generated groups. Hyperbolicity is a very useful tool that helps us in understanding the relationships between graphs and manifolds. It is known from the works of Gromov and Kanai that graphs can model manifolds and many metric spaces (this is an interesting result, since it allows to go from working with a continuous structure that could be complicated to deal with a discrete structure, see, e.g., [43, 47]). In [93, 102, 108] the equivalence between the hyperbolicity of many surfaces and the hyperbolicity of simple graphs is proved.

One of the first applications of the Gromov hyperbolic spaces was the analysis of finitely generated groups (see, for example, [87]). Today, the mathematical properties and the applications of Gromov hyperbolicity are subjects of study with a growing interest within graph theory (see [11, 12, 13, 14, 20, 22, 25, 41, 42, 61, 63, 64, 71, 80, 81, 92, 93, 94, 95, 100, 101, 108, 112] and the references therein).

Computer science is another area in which the hyperbolicity of Gromov has been applied in subjects such as automatic groups (see, for example, [87]), networks and algorithms (see [72] and its references), random graphs (see, for example, [103, 104, 105]), etc. For example, when considering the graph that models the routing of the Internet, it adapts better to a hyperbolic space instead of a Euclidean one (see [106, 109]); in fact, many real networks are hyperbolic (see, for example, [1, 2, 31, 74, 82]). Recently, algorithmic problems have been considered in hyperbolic spaces. Hyperbolicity has also been used in issues such as the secure transmission of information through the network (see [60]) as well as the spread of viruses (see [62], [63]). Other problems that have been addressed are sensor networks, distance estimation, traffic flow, congestion minimization (see [7], [65], [66], [84], [106]). This tool has been used even in topics such as large-scale data analysis (see [83]) and DNA study (see [20]).

There are several definitions of Gromov hyperbolicity which are are equivalent, in the sense that if X is δ-hyperbolic with respect to the definition A, then it is δ'-hyperbolic with respect to the definition B for some δ' (see, e.g., [19, 43]). We will work mainly with the definition of Gromov hyperbolicity given by the Rips condition (see Definition 1.1.1) for its geometric meaning, but in some sections we will also use the definition given by the Gromov product (see Definition 1.1.3).

Throughout this work we will study the hyperbolicity of metric graphs. Graph theory is an interesting field in Discrete Mathematics. Despite being a relatively recent area, it is growing very rapidly with many results discovered in the last three decades.

This theory allows to treat, in a simpler way, any problem where there is a binary relationship between certain objects, so its uses are quite broad. Proof of this is that we can find applications to areas within the same Mathematics, Engineering, Biology, Sociology, Administration, etc.

Three main problems on Gromov hyperbolic graphs are the following:

I. To characterize the hyperbolicity for some classes of graphs.

II To obtain inequalities that relate the hyperbolicity constant and other parameters of the graph.

III To study the invariance of the hyperbolicity of graphs under appropriate transformations.

In this work, attending to item I, we characterize the hyperbolicity for interval graphs and circular-arc graphs. To attend item II, we find relationships between the hyperbolicity constant and several domination numbers associated with different kinds of domination, such as total-domination number, distance k-domination number, etc. And finally we deal with item III when we study the hyperbolicity on graphs by considering operators acting on them.

The structure of this work is as follows.

In Chapter 1 we include some definitions, a brief introduction to hyperbolic spaces and some previous results.

In Chapter 2, we prove some bounds for the hyperbolicity constant of interval graphs.

In Chapter 3, we prove some bounds for the hyperbolicity constant of circular-arc graphs.

In Chapter 4, we obtain several inequalities between the hyperbolicity constant and some domination numbers, such as k-domination number, total k-domination number, distance k-domination number, etc. Two of the main results are Theorems 4.2.6 and 4.2.8, since these results deal with simplest domination parameters.

In Chapter 5, we prove inequalities relating the hyperbolicity constants of a graph G and its graph operators $\mathcal{L}(G)$, $S(G)$, $T(G)$, $R(G)$ and $Q(G)$.

Chapter 1

Background and previous results

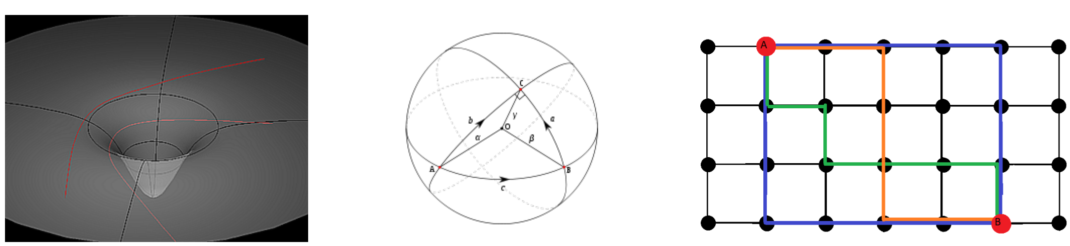
When we work in traditional Euclidean spaces, the use of straight lines is essential to understand them. In metric spaces it is necessary to consider the "shortest" possible paths, which are called geodesics (see Figure 1.1). Next, we give the formal definition.

Definition 1.0.1. *If $\gamma : [a, b] \longrightarrow X$ is a continuous curve in a metric space (X, d), we can define the* length *of γ as*

$$L(\gamma) := \sup \Big\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : \, a = t_0 < t_1 < \cdots < t_n = b \Big\} .$$

We say that γ is a geodesic *if it is an isometry, i.e., $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$ (then γ is equipped with an arc-length parametrization).*

The metric space X is said *geodesic* if for every couple of points in X there exists a geodesic joining them; we denote by $[xy]$ any geodesic joining x and y; this notation is ambiguous, since in general we do not have uniqueness of geodesics, but it is very convenient. Consequently, any geodesic metric space is connected. If the metric space X is a graph, then the edge joining the vertices u and v will be denoted by uv.



(a) Two geodesics (red) on a curved surface corresponding to a spherical gravitational field. Image taken from wikipedia.

(b) Geodesics on a sphere. Image taken from wikipedia.

(c) Geodesics joining two vertices in a metric graph with edges of length 1.

Figure 1.1: Geodesic spaces.

1.1 Definitions of Gromov hyperbolicity and examples

A hyperbolic metric space satisfies certain metric relationships between its points. These spaces, introduced by Mikhael Gromov (see [3, 47]), generalize the metric properties of classical hyperbolic geometry and trees.

As we mentioned before, there are several definitions for hyperbolic spaces. We give here a few and some equivalences.

The following definition is attributed to Eliyahu Rips.

Rips condition

Definition 1.1.1. *If X is a geodesic metric space and $J = \{J_1, J_2, \dots, J_n\}$, with $J_j \subseteq X$, we say that J is* δ-thin *if for every $x \in J_i$ we have that $d(x, \cup_{j\neq i} J_j) \le \delta$. We denote by $\delta(J)$ the sharpest thin constant of J, i.e., $\delta(J) := \inf\{\delta \ge 0 :\ J \text{ is } \delta\text{-thin}\}$. If $x_1, x_2, x_3 \in X$, a geodesic triangle $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$; it is usual to write also $T = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$ and we will say that x_1, x_2, x_3 are the vertices of the triangle. The space X is* δ-hyperbolic (*or satisfies the* Rips condition *with constant δ*) *if every geodesic triangle in X is δ-thin. We define the hyperbolicity constant of X as*

$$\delta(X) := \sup\{\delta(T) :\ T \text{ is a geodesic triangle in } X\}.$$

We say that X is hyperbolic *if X is δ-hyperbolic for some $\delta \ge 0$ or, equivalently, if $\delta(X) < \infty$. If X is hyperbolic, then $\delta(X) = \inf\{\delta \ge 0 :\ X \text{ is } \delta\text{-hyperbolic}\}$. If X has connected components $\{X_i\}_{i\in I}$, then we define $\delta(X) := \sup_{i\in I} \delta(X_i)$, and we say that X is hyperbolic if $\delta(X) < \infty$.*

Note that a geodesic bigon (a geodesic triangle such that two of its vertices are the same point) in a δ-hyperbolic space is δ-thin. Note also that every geodesic polygon with n sides in a δ-hyperbolic space is $(n-2)\delta$-thin.

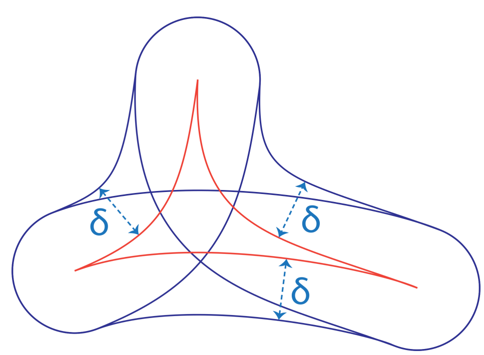


Figure 1.2: T is δ-thin, if any side of T is contained in a δ-neighborhood of the union of the two other sides.

Fine definition

Definition 1.1.2. *Let X be a geodesic metric space, and $T = \{x, y, z\} \subset X$ a geodesic triangle; we can construct in the Euclidean plane a triangle $\overline{T} = \{\overline{x}, \overline{y}, \overline{z}\}$ with sides of the same length than T. Let $f : T \mapsto \overline{T}$ be the map such that $f([xy]) = [\overline{x}\,\overline{y}]$, $f([xz]) = [\overline{x}\,\overline{z}]$, $f([yz]) = [\overline{y}\,\overline{z}]$ and the restriction of f to each side of T is an isometry. The maximum inscribed circle in $\overline{T}$ meets the sides $[\overline{x}\,\overline{y}], [\overline{y}\,\overline{z}]$ and $[\overline{x}\,\overline{z}]$ in $\overline{p}_z$, $\overline{p}_x$ and $\overline{p}_y$ respectively such that $d(\overline{x}, \overline{p}_y) = d(\overline{x}, \overline{p}_z)$, $d(\overline{y}, \overline{p}_x) = d(\overline{y}, \overline{p}_z)$ and $d(\overline{z}, \overline{p}_x) = d(\overline{z}, \overline{p}_y)$. We define $p_x = f^{-1}(\overline{p}_x)$, $p_y = f^{-1}(\overline{p}_y)$ and $p_z = f^{-1}(\overline{p}_z)$. There exists a map g from $\overline{T}$ onto a tripod T_0 (a star graph with one vertex p of degree 3, and three vertices x', y', z' of degree one), such that $d(x', p) = d(\overline{x}, \overline{p}_y) = d(\overline{x}, \overline{p}_z)$, $d(y', p) = d(\overline{y}, \overline{p}_x) = d(\overline{y}, \overline{p}_z)$ and $d(z', p) = d(\overline{z}, \overline{p}_x) = d(\overline{z}, \overline{p}_y)$. Thus, the restriction of g to each side of $\overline{T}$ is an isometry.*

Consider the map $h = g \circ f$ from T onto T_0. We say that the triangle $T = \{x, y, z\}$ is δ-fine *if for every couple of points q and r in T with $h(q) = h(r)$ we have $d(q, r) \leq \delta$. We say that the space X is* δ-fine *if every geodesic triangle is δ-fine.*

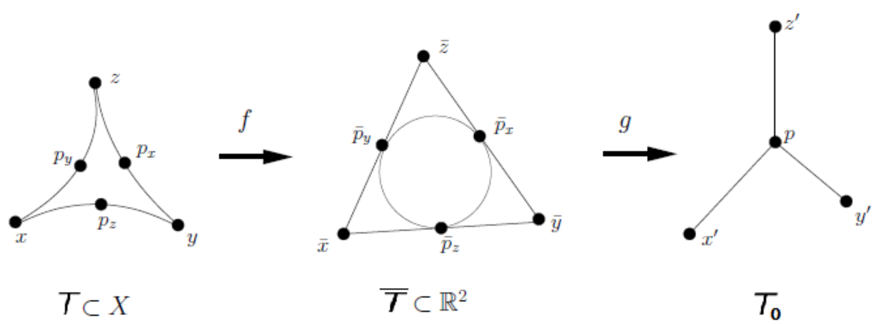


Figure 1.3: Given a geodesic triangle $T \subset X$ there exists a map from T onto a tripod T_0, such that its restriction to each side of T is an isometry.

Gromov product definition

Definition 1.1.3. *Let (X, d) be a metric space and $x, y \in X$. The* Gromov product *between x and y, with base point $w \in X$, is defined as*

$$(x, y)_w := \frac{1}{2}\big(d(x, w) + d(y, w) - d(x, y)\big) \geq 0.$$

We say that the metric space (X, d) is δ-hyperbolic *with respect to the Gromov product, for some constant $\delta \geq 0$, if*

$$(x, z)_w \geq \min\big\{(x, y)_w, (y, z)_w\big\} - \delta \tag{1.1}$$

for every $x, y, z, w \in X$.

This definition will be very useful in Chapters 4 and 5.

The following result states the equivalence between Rips condition (Definition 1.1.1), fine property (Definition 1.1.2) and hyperbolicity with respect to the Gromov product (Definition 1.1.3):

Theorem 1.1.4 ([3] and [43])**.** *Let X be a geodesic metric space:*

1. *If X is δ-hyperbolic with respect to the Gromov product, then X is 3δ-hyperbolic and 4δ-fine.*

2. *If X is δ-hyperbolic, then X is 4δ-hyperbolic with respect to the Gromov product and 4δ-fine.*

3. *If X is δ-fine, then X is 2δ-hyperbolic with respect to the Gromov product and δ-hyperbolic.*

Insize definition

Definition 1.1.5. *Let X be a geodesic metric space, $T = \{x, y, z\} \subset X$ a geodesic triangle, and p_x, p_y, p_z the internal points in T given in Definition 1.1.2. We define the* insize *of the geodesic triangle T to be*

$$insize(T) := \operatorname{diam}\{p_x, p_y, p_z\} = \max\{d(p_x, p_y), d(p_x, p_z), d(p_y, p_z)\}. \tag{1.2}$$

*The space X is δ-*insize *if every geodesic triangle in X has insize at most δ.*

This definition of insize is also equivalent to hyperbolicity. Besides, we have the following quantitative result.

Theorem 1.1.6. *[43, Proposition 2.21] Let us consider a geodesic metric space X.*
(1) *If X is δ-hyperbolic, then it is 4δ-insize.*
(2) *If X is δ-insize, then it is 2δ-hyperbolic.*

Minsize definition

Definition 1.1.7. *Let X be a geodesic metric space and $T = \{x, y, z\} \subset X$ a geodesic triangle. We define the* minsize *of the geodesic triangle T as*

$$minsize(T) := \min\big\{ \operatorname{diam}\{x', y', z'\} : x' \in [yz], y' \in [zx], z' \in [xy]\big\}. \tag{1.3}$$

*The space X is δ-*minsize *if every geodesic triangle in X has minsize at most δ.*

This definition is also equivalent to hyperbolicity:

Theorem 1.1.8. *[43, Proposition 2.21] Let us consider a geodesic metric space X.*
(1) *If X is δ-hyperbolic, then it is 4δ-minsize.*
(2) *If X is δ-minsize, then it is 8δ-hyperbolic.*

Geodesics diverge

As usual, we denote by $B_k(x)$ the open ball in a metric space, i.e.,

$$B_k(x) := \{y \in X : d(x,y) < k\} \quad \text{for any } x \in X \text{ and } k > 0.$$

Definition 1.1.9. *Given a geodesic metric space X, we say that $e : [0,\infty) \to (0,\infty)$ is a* divergence function *for X, if for every point $x \in X$ and all geodesics $\gamma = [xy]$, $\gamma' = [xz]$, the function e satisfies the following condition:*

For every $R, r > 0$ such that $R + r \le \min\{L([xy]), L([xz])\}$, if $d(\gamma(R), \gamma'(R)) \ge e(0)$, and α is a path in $X \setminus B_{R+r}(x)$ from $\gamma(R+r)$ to $\gamma'(R+r)$, then we have $L(\alpha) > e(r)$ (see Figure 1.4*).*

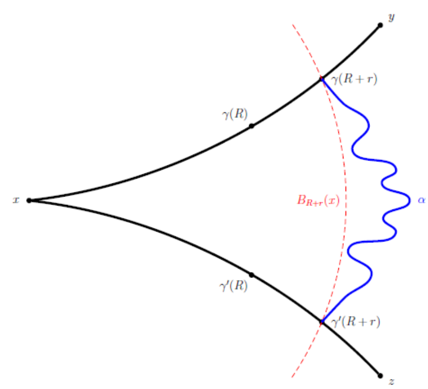


Figure 1.4: Geodesics diverge.

We say that *geodesics diverge* in X if there is a divergence function $e(r)$ such that

$$\lim_{r \to \infty} e(r) = \infty.$$

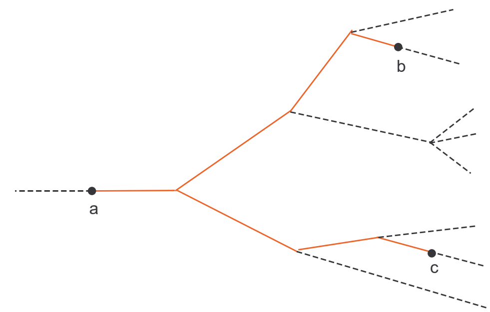
We say that *geodesics diverge exponentially* in X if there is an exponential divergence function. Theorem 1.1 in [91] shows that in a geodesic metric space X, geodesics diverge in X if and only if geodesics diverge exponentially in X.

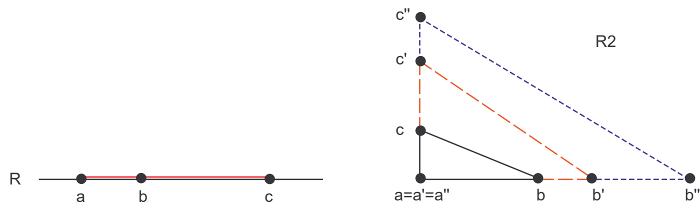
It is known that Definition 1.1.9 is also equivalent to hyperbolicity (see [3, 91]).

Examples of hyperbolic spaces

The following are interesting examples of hyperbolic spaces.

- Every bounded metric space X is $(\frac{1}{2} \operatorname{diam} X)$-hyperbolic.
- The real line $\mathbb{R}$ is 0-hyperbolic: In fact, any point of a geodesic triangle in the real line belongs to two sides of the triangle simultaneously, and therefore any geodesic triangle in $\mathbb{R}$ is 0-thin.



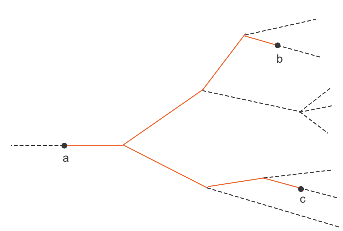


- The Euclidean plane $\mathbb{R}^2$ is not hyperbolic, since the midpoint of a side on a large equilateral triangle is far from all points on the other two sides.

  These arguments can be applied to higher dimensions:

- A normed real vector space is hyperbolic if and only if it has dimension 1.

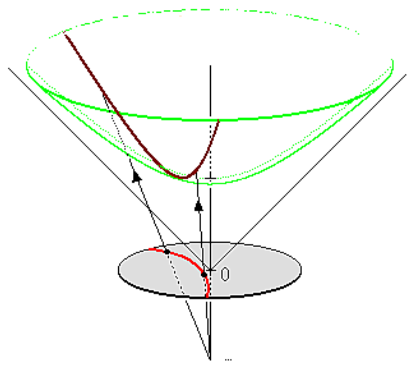
- Every metric tree with arbitrary edge lengths is 0-hyperbolic, by the same reason that the real line.



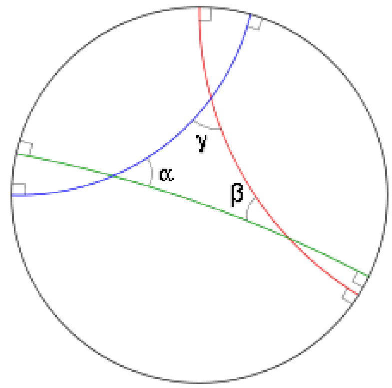
- The unit disk $\partial\mathbb{D}$ (with its Poincaré metric) is $\log(1+\sqrt{2}\,)$-thin: Consider any geodesic triangle T in $\mathbb{D}$. It is clear that T is contained in an ideal triangle T' with three vertices in $\partial\mathbb{D}$ (and so, the sides of T' have infinite length), with $\delta(T) \leq \delta(T')$. Since all ideal triangles are isometric, we can consider just one fixed T'. Then, a computation gives $\delta(T') = \log(1+\sqrt{2}\,)$.

- Every simply connected complete Riemannian manifold with sectional curvatures verifying $K \leq -c^2 < 0$, for some constant c, is hyperbolic (see, e.g., [43, p. 52]).

- The graph Γ of the routing infrastructure of the Internet is also empirically shown to be hyperbolic (see [6]). One can think that this is a trivial (and then a non-useful) fact, since



(a) Obtaining the Poincaré disk model. Image taken from wikipedia.



(b) Geodesic triangle in the Poincaré model. Image taken from wikipedia.

Figure 1.5: Poincaré disk.

every bounded metric space X is $(\frac{1}{2}\operatorname{diam} X)$-hyperbolic. The point is that the quotient

$$\frac{\delta(\Gamma)}{\operatorname{diam}\Gamma}$$

is very small, and this makes the tools of hyperbolic spaces applicable to the graph Γ (see, e.g., [26]).

Finally, note that the hyperbolicity constant $\delta(X)$ of a geodesic metric space can be viewed as a measure of how "tree-like" the space is, since those spaces with $\delta(X) = 0$ are precisely the metric trees. This is an interesting subject since, in many applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see, e.g., [27]).

We would like to point out that deciding whether or not a space is hyperbolic is usually very difficult. The main problem is that, usually, we do not know the location of geodesics in the space.

Since the hyperbolicity of many geodesic metric spaces is equivalent to the hyperbolicity of some graphs related to them (see, e.g., [19]), the study of hyperbolic graphs becomes an interesting topic.

1.2 Previous results on hyperbolic graphs

We state now some of the main facts about hyperbolic spaces.

First of all, we present some important maps in the theory of hyperbolic spaces, since they preserve hyperbolicity.

Definition 1.2.1. *Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \longrightarrow Y$ is said to be an* (α, β)-quasi-isometric embedding, *with constants $\alpha \geq 1$ and $\beta \geq 0$ if for every $x, y \in X$:*

$$\alpha^{-1} d_X(x, y) - \beta \leq d_Y(f(x), f(y)) \leq \alpha d_X(x, y) + \beta.$$

The function f is ε-full *if for each $y \in Y$ there exists $x \in X$ with $d_Y(f(x), y) \leq \varepsilon$.*

Definition 1.2.2. *A map $f : X \longrightarrow Y$ is said to be a* quasi-isometry, *if there exist constants $\alpha \geq 1$, $\beta, \varepsilon \geq 0$ such that f is an ε-full (α, β)-quasi-isometric embedding.*

Definition 1.2.3. *An* (α, β)-quasigeodesic *in X is an (α, β)-quasi-isometric embedding between an interval of $\mathbb{R}$ and X.*

Example 1.2.4. *Every isometric embedding is a $(1, 0)$-quasi-isometry embedding. Every isometry is a 0-full $(1, 0)$-quasi-isometry.*

Example 1.2.5. *The integer part $f : \mathbb{R} \longrightarrow \mathbb{Z}$ defined by $f(x) = [x]$ is a $(1, 1)$-quasi-isometry, although f is not continuous at the points in $\mathbb{Z}$.*

Remark 1.2.6. *These functions are very flexible (since as we saw in the Example 1.2.5, the continuity is not requested), and they are also a fundamental tool to determine the hyperbolicity of a space.*

In the study of any mathematical property, the class of maps which preserve that property plays a central role in the theory. The following result shows that quasi-isometries preserve hyperbolicity.

Theorem 1.2.7 (Invariance of hyperbolicity)**.** *Let $f : X \longrightarrow Y$ be an (α, β)-quasi-isometric embedding between the geodesic metric spaces X and Y. If Y is hyperbolic, then X is hyperbolic.*

Besides, if f is ε-full for some $\varepsilon \geq 0$ (a quasi-isometry), then X is hyperbolic if and only if Y is hyperbolic.

We next discuss the connection between hyperbolicity and *geodesic stability*. In the complex plane (with its Euclidean distance), there is only one optimal way of joining two points: a straight line segment. However if we allow "limited suboptimality", the set of "reasonably efficient paths" (quasigeodesics) are well spread. For instance, if we split the circle $\partial D(0, R) \subset \mathbb{C}$ into its two semicircles between the points R and $-R$, then we have two reasonably efficient paths (two $(\pi/2, 0)$-quasigeodesics) between these endpoints such that the point Ri on one of the semicircles is far from all points on the other semicircle provided that R is large. Even an additive suboptimality can lead to paths that fail to stay close together. For instance, the union of the two line segments in $\mathbb{C}$ given by $[0, R + i\sqrt{R}]$ and $[R + i\sqrt{R}, 2R]$ gives a path of length less than $2R + 1$ (since $2\sqrt{R^2 + R} \leq 2R + 1$), and so is "additively inefficient" by less than 1 (it is a $(1, 1)$-quasigeodesic).

However, its corner point is very far from all points on the line segment $[0, 2R]$ when R is very large.

The situation in Gromov hyperbolic spaces is very different, since all such reasonably efficient paths ((α, β)-quasigeodesics for fixed α, β) stay within a bounded distance of each other:

Definition 1.2.8. *Let X be a metric space, Y a non-empty subset of X and ε a positive number. We call* ε-neighborhood *of Y in X, denoted by $V_\varepsilon(Y)$, to the set $\{x \in X : d_X(x, Y) \leq \varepsilon\}$. The* Hausdorff distance *between two subsets Y and Z of X, denoted by $\mathcal{H}(Y, Z)$, is the number defined by:*

$$\inf\{\varepsilon > 0 : Y \subset V_\varepsilon(Z) \text{ and } Z \subset V_\varepsilon(Y)\}.$$

Theorem 1.2.9 (Geodesic stability)**.** *For any constants $\alpha \geq 1$ and $\beta, \delta \geq 0$, there exists a constant $H = H(\delta, \alpha, \beta)$ such that for every δ-hyperbolic geodesic metric space and for every pair of (α, β)-quasigeodesics g, h with the same endpoints, $\mathcal{H}(g, h) \leq H$.*

The geodesic stability is not just a useful property of hyperbolic spaces; in fact, M. Bonk proved in [17] that the geodesic stability is equivalent to the hyperbolicity:

Theorem 1.2.10. (*[17, p.286]*) *Let X be a geodesic metric space with the following property: For each $a \geq 1$ there exists a constant H such that for every $x, y \in X$ and any $(a, 0)$-quasigeodesic g in X starting in x and finishing in y there exists a geodesic γ joining x and y satisfy $\mathcal{H}(g, \gamma) \leq H$. Then X is hyperbolic.*

Throughout this work, and unless otherwise is specified, $G = (V, E) = (V(G), E(G))$ denotes a (finite or infinite) simple (without loops and multiple edges) graph (not necessarily connected) such that $V \neq \emptyset$ and we have defined a length function, denoted by L_G or L, on the edges $L_G : E(G) \to \mathbb{R}_+$; the length of a path $\eta = \{e_1, e_2, \dots, e_k\}$ is defined as $L_G(\eta) = \sum_{i=1}^{k} L_G(e_i)$. We assume that $\ell(G) := \sup\big\{L_G(e) \mid e \in E(G)\big\} < \infty$. In order to consider a graph G as a geodesic metric space, identify (by an isometry $\mathcal{I}$) any edge $uv \in E(G)$ with the interval $[0, L_G(uv)]$ in the real line; then the edge uv (considered as a graph with just one edge) is isometric to the interval $[0, L_G(uv)]$. If $x, y \in uv$ and η_{xy} denotes the segment contained in uv joining x and y, we define the length of η_{xy} as $L_G(\eta_{xy}) = |\mathcal{I}(x) - \mathcal{I}(y)|$. Thus, the points in G are the vertices and, also, the points in the interior of any edge of G.

In this way, any connected graph G has a natural distance defined on its points, induced by taking shortest paths in G, and we can see G as a metric graph. We denote by d_G or d this distance. If x, y are in different connected components of G, we define $d_G(x, y) = \infty$. Otherwise, if a graph has edges with different lengths, then we also assume that it is locally finite (i.e., in each metric ball there are just a finite number of edges). These properties guarantee that any connected component of any graph is a geodesic metric space.

Now, we collect some important results which will be useful for the development of our work.

As usual, by *cycle* we mean a simple closed curve, i.e., a path with different vertices, unless the last one, which is equal to the first vertex.

Lemma 1.2.11. *[102, Lemma 2.1] Let us consider a geodesic metric space X. If every geodesic triangle in X that is a cycle is δ-thin, then X is δ-hyperbolic.*

This lemma has the following direct consequence.

Corollary 1.2.12. *In any geodesic metric space X,*

$$\delta(X) = \sup\big\{\delta(T) \mid T \ \textit{is a geodesic triangle in } X \textit{ that is a cycle}\big\}.$$

In particular, for every graph G, it is satisfied

$$\delta(G) = \sup\{\delta(T) : T \ \text{is a geodesic triangle in } G \text{ that is a cycle}\}.$$

The union of the set of the midpoints of the edges of a graph G and the set of vertices, $V(G)$, will be denoted by $J(G)$. Let $\mathbb{T}_1$ be the set of geodesic triangles T in G that are cycles and such that each vertex of T belongs to $J(G)$. Let us define

$$\delta_1(G) := \inf\{\lambda : \text{every geodesic triangle in } \mathbb{T}_1 \text{ is } \lambda\text{-thin}\}.$$

The following result will be used throughout the work.

Theorem 1.2.13. *[11, Theorems 2.5 and 2.7] For every graph G we have $\delta_1(G) = \delta(G)$. Furthermore, if G is hyperbolic, then there exists $T \in \mathbb{T}_1$ with $\delta(T) = \delta(G)$.*

The previous theorem allows to reduce the study of the hyperbolicity constant of a graph G to study only the geodesic triangles of G, whose vertices are vertices of G (i.e., belong to $V(G)$) or midpoints of the edges of G.

Theorem 1.2.14. *[11, Theorem 2.6] For every hyperbolic graph G, $\delta(G)$ is a multiple of $\frac{1}{4}$.*

Given a graph G, we define

$$\operatorname{diam} V(G) := \sup\big\{d_G(v,w) \mid v,w \in V(G)\big\},$$
$$\operatorname{diam} G := \sup\big\{d_G(x,y) \mid x,y \in G\big\},$$

i.e, $\operatorname{diam} V(G)$ is the diameter of the set of vertices of G, and $\operatorname{diam} G$ is the diameter of the whole graph G (recall that in order to have a geodesic metric space, G must contain both the vertices and the points in the interior of any edge of G).

It is clear that $\operatorname{diam} V(G) \le \operatorname{diam} G \le \operatorname{diam} V(G) + 1$.

The following result is well-known. Since the proof is short, we include it for the sake of completeness.

Lemma 1.2.15. *For any geodesic triangle T in a graph G we have $\delta(T) \le (\operatorname{diam} T)/2 \le L(T)/4$.*

Proof. Let T be a geodesic triangle $T = \{x,y,z\}$ and $p \in [xy]$. Consider $s,t \in T$ with $d(s,t) = \operatorname{diam} T$. Since T contains two curves joining s and t, we have $\operatorname{diam} T \le L(T)/2$ and

$$d(p,[xz] \cup [zy]) \le d(p,\{x,y\}) \le \frac{1}{2}\operatorname{diam} T \le \frac{1}{4} L(T),$$

for any $p \in [xy]$. Since we can rename the vertices of T, we conclude $\delta(T) \le (\operatorname{diam} T)/2 \le L(T)/4$. □

We have the following direct consequence.

Corollary 1.2.16. *The inequalities*

$$\delta(G) \le \frac{1}{2} \operatorname{diam} G \le \frac{1}{2} \big(\operatorname{diam} V(G) + \ell(G) \big)$$

hold for every graph G.

Throughout this work, and unless otherwise specified, we will consider graphs with edge length 1. Thus the previous result provides the following corollary:

Corollary 1.2.17. *In any graph G with edges of length 1 the inequality*

$$\delta(G) \le \frac{1}{2} \operatorname{diam} G \le \frac{1}{2} (\operatorname{diam} V(G) + 1)$$

holds.

If H is a subgraph of G, we always have $d_H(x,y) \ge d_G(x,y)$ for every $x,y \in H$. A subgraph H of G is said *isometric* if $d_H(x,y) = d_G(x,y)$ for every $x,y \in H$. Note that this condition is equivalent to $d_H(u,v) = d_G(u,v)$ for every vertices $u,v \in V(H)$.

The following results appear in [12, Lemma 9] and [101, Theorem 11].

Lemma 1.2.18. *If H is an isometric subgraph of G, then $\delta(H) \le \delta(G)$.*

From [101, Theorem 11] we have the following result:

Theorem 1.2.19. *The following graphs with n vertices have these precise values of δ.*
- *If P_n is the* path graph*, then $\delta(P_n) = 0$ for all $n \ge 1$.*
- *If C_n is the* cycle graph*, then $\delta(C_n) = \frac{1}{4} L(C_n) = \frac{n}{4}$ for all $n \ge 3$.*
- *If K_n is the* complete graph*, then $\delta(K_1) = \delta(K_2) = 0$, $\delta(K_3) = 3/4$ and $\delta(K_n) = 1$ for all $n \ge 4$.*

By [80, Proposition 5 and Theorem 7], we have the following result.

Lemma 1.2.20. *If G is any graph with a cycle g with length $L(g) \ge 3$, then $\delta(G) \ge 3/4$. If there exists a cycle g in G with length $L(g) \ge 4$, then $\delta(G) \ge 1$.*

In [10, Theorem 3.2] appears the following result.

Theorem 1.2.21. *Given any graph G, we have $\delta(G) \ge 5/4$ if and only if there exist a cycle g in G with length $L(g) \ge 5$ and a vertex $w \in g$ such that* $\deg_g(w) = 2$.

The following result appears in [57, Theorem 4.9].

Theorem 1.2.22. *If G is a graph with n vertices and minimum degree $n-3$, then $\delta(G) \le 5/4$.*

If H is a subgraph of G and $w \in V(H)$, we denote by $\deg_H(w)$ the degree of the vertex w in the subgraph induced by $V(H)$. Note that, if C is a cycle, $v \in V(G) \cap C$ and Γ is the subgraph induced by $V(C)$ then Γ could contain edges that are not contained in C, and thus it is possible to have $\deg_C(v) \ge 2$.

T-decomposition of a graph

We say that a vertex v in a connected graph G is a *cut-vertex* if $G \setminus v$ is not connected. A graph is *biconnected* if it is connected and it does not contain cut-vertices. Given a graph G, we say that a family of subgraphs $\{G_s\}_s$ of G is a *T-decomposition* of G if $\cup_s G_s = G$ and $G_s \cap G_r$ is either a *cut-vertex* or the empty set for each $s \neq r$. The well-known biconnected decomposition of any graph is an example of T-decomposition.

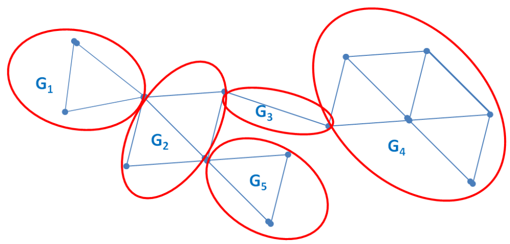


Figure 1.6: T-decomposition of a graph.

It is known that the hyperbolicity constant of a graph is the supremum of the hyperbolicity constants of its biconnected components [47]. One can check that the following result also holds (see, e.g., [12, Theorem 3] for a proof).

Proposition 1.2.23. *If G is a graph and $\{G_s\}_s$ is any T-decomposition of G, then*

$$\delta(G) = \sup_s \delta(G_s).$$

Chapter 2

Interval graphs

For a finite graph with n vertices it is possible to compute $\delta(G)$ in time $O(n^{3.69})$ [40] (this is improved in [18, 31, 33]). Given a Cayley graph (of a presentation with solvable word problem) there is an algorithm which allows to decide if it is hyperbolic [90]. However, deciding whether or not a general infinite graph is hyperbolic is usually very difficult. Therefore, it is interesting to relate hyperbolicity with other properties of graphs. The papers [20, 112, 9, 24] prove, respectively, that chordal, k-chordal, edge-chordal and join graphs are hyperbolic. Moreover, in [9] it is shown that hyperbolic graphs are path-chordal graphs. These results relating chordality and hyperbolicity are improved in [78]. Some other authors have obtained results on hyperbolicity for particular classes of graphs: vertex-symmetric graphs, bipartite and intersection graphs, bridged graphs and expanders [21, 34, 71, 77].

A geometric graph is a graph in which the vertices or edges are associated with geometric objects. Two of the main classes of geometric graphs are Euclidean graphs and intersection graphs. A graph is *Euclidean* if the vertices are points in $\mathbb{R}^n$ and the length of each edge connecting two vertices is the Euclidean distance between them (this makes a lot of sense with the cities and roads analogy commonly used to describe graphs). An *intersection graph* is a graph in which each vertex is associated with a set and in which vertices are connected by edges whenever the corresponding sets have a nonempty intersection. In this chapter we work with interval graphs (a class of intersection graphs) and indifference graphs (a class of Euclidean graphs).

An *interval graph* is the intersection graph of a family of intervals on the real line. It has one vertex for each interval in the family, and an edge between every pair of corresponding vertices to intervals that intersect. Usually, we consider that every edge of an interval graph has length 1, but we also consider interval graphs whose edges have different lengths. Interval graphs are chordal graphs and hence perfect graphs [38, 45]. Their complements belong to the class of comparability graphs [44], and the comparability relations are precisely the interval orders [38]. The mathematical theory of interval graphs was developed with a view towards applications by researchers at the RAND Corporation's mathematics department [30, pp. ix-10]. In particular, Cohen applied interval graphs to mathematical models of population biology, specifically food webs [30, pp. 12–33].

Interval graphs are used to represent resource allocation problems in operations research and scheduling theory. In these applications, each interval represents a request for a resource (such as a processing unit of a distributed computing system or a room for a class) for a specific period of time.

The maximum weight independent set problem for the graph represents the problem of finding the best subset of requests that can be satisfied without conflicts [5]. An optimal graph coloring of the interval graph represents an assignment of resources that covers all of the requests with as few resources as possible; it can be found in polynomial time by a greedy coloring algorithm that colors the intervals in sorted order by their left endpoints [32]. Other applications include genetics, bioinformatics, and computer science. Finding a set of intervals that represent an interval graph can also be used as a way of assembling contiguous subsequences in DNA mapping [114]. Interval graphs also play an important role in temporal reasoning [46].

An *indifference graph* is an interval graph that has a set of unit interval graphs on the real line as vertex set, with two intervals adjacent if and only if they intersect, and the length of the corresponding edge is the distance between the midpoints of the two intervals. Also, we can see an indifference graph as a Euclidean graph in $\mathbb{R}$ constructed by assigning a real number to each vertex and connecting two vertices by an edge when their corresponding numbers are within one unit from each other. Since it is a Euclidean graph, the length of each edge connecting two vertices is the Euclidean distance between them. Indifference graphs possess several interesting properties. Every connected indifference graph has a Hamiltonian path [15]. An indifference graph has a Hamiltonian cycle if and only if it is biconnected [89]. Indifference graphs obey the reconstruction conjecture: they are uniquely determined by their vertex-deleted subgraphs [110]. In the same direction, we consider indifference graphs since for these graphs we can remove one of the hypothesis of a main theorem on interval graphs (compare Theorem 2.1.9 and Corollary 2.1.10).

We would like to mention that the paper [76] collects very rich results, especially those concerning path properties, about interval graphs and unit interval graphs.

It is well-known that interval graphs (with a very weak hypothesis) and indifference graphs are hyperbolic. One of the main results in this chapter is Theorem 2.1.9, which provides a sharp upper bound of the hyperbolicity constant of interval graphs verifying a very weak hypothesis. This result allows to obtain bounds for the hyperbolicity constant of every indifference graph (Corollary 2.1.10) and the hyperbolicity constant of every interval graph with edges of length 1 (Corollary 2.1.11). Moreover, Theorem 2.2.2 provides sharp bounds for the hyperbolicity constant of the complement of any interval graph with edges of length 1. Note that it is not usual to obtain such precise bounds for large classes of graphs. The main result in this chapter is Theorem 2.1.16, which allows to compute the hyperbolicity constant of every interval graph with edges of length 1, by using geometric criteria.

2.1 Interval graphs and hyperbolicity

Recall that a *chordal graph* is one in which all cycles of four or more vertices have a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle. Equivalently, every induced cycle in the graph should have three vertices.

Lemma 2.1.1. *[20, Lemma 2.2] Consider a chordal graph G and a cycle C in G with $a, v, b \in C \cap V(G)$ and $av, vb \in E(G)$. If $ab \notin E(G)$, then* $\deg_C(v) \geq 3$.

Lemma 2.1.1 has the following direct consequence.

Corollary 2.1.2. *Consider a cycle C in a chordal graph G and v_1, v_2, v_3 consecutive vertices in C. If $\deg_C(v_2) = 2$, then $v_1v_3 \in E(G)$. Consequently, if C has at least 4 vertices, then $\deg_C(v_1) \geq 3$ and $\deg_C(v_3) \geq 3$.*

The following result in [80, Theorem 11] will be useful.

Theorem 2.1.3. *Let G be a graph. If $\delta(G) < 1$, then we have either $\delta(G) = 0$ or $\delta(G) = 3/4$. Furthermore,*

- *$\delta(G) = 0$ if and only if G is a tree.*
- *$\delta(G) = 3/4$ if and only if G is not a tree and every cycle in G has length 3.*

Corollary 2.1.4. *A graph G with edges of length 1 satisfies $\delta(G) \geq 1$ if and only if there exists a cycle in G with length at least 4.*

In order to characterize from a geometric viewpoint the interval graphs with hyperbolicity constant 1, we need the following result, which is a direct consequence of Theorems 1.2.14 and 2.1.3, and [10, Theorem 4.14].

Theorem 2.1.5. *Let G be any graph with edges of length 1. We have $\delta(G) = 1$ if and only if $\delta(G) \notin \{0, 3/4\}$ and for every cycle C in G and every $x, y \in C \cap J(G)$ we have $d(x, y) \leq 2$.*

Theorems 2.1.3 and 2.1.5 have the following consequence.

Corollary 2.1.6. *Let G be any graph with edges of length 1. We have $\delta(G) \leq 1$ if and only if for every cycle C in G and every $x, y \in C \cap J(G)$ we have $d(x, y) \leq 2$.*

The following result is a direct consequence of Theorems 1.2.14 and 2.1.3, and [10, Theorems 4.14 and 4.21].

Theorem 2.1.7. *Let G be any graph with edges of length 1. If there exists a cycle in G with $p, q \in V(G)$ and $d(p, q) \geq 3$, then $\delta(G) \geq 3/2$.*

We will need also this last result.

Theorem 2.1.8. *[80, Theorem 30] If G is any graph with edges of length 1 and n vertices, then $\delta(G) \leq n/4$.*

Given a cycle C in an interval graph G, let $\{v_1, \dots, v_k\}$ be the vertices in G with

$$C = v_1v_2 \cup \cdots \cup v_{k-1}v_k \cup v_kv_1.$$

Denote by $\{I_1, \dots, I_k\}$ the corresponding intervals to $\{v_1, \dots, v_k\}$. If $I_j = [a_j, b_j]$, then let us define the *minimal interval of C* as the interval $I_{j_1} = [a_{j_1}, b_{j_1}]$ with $a_{j_1} \leq a_j$ for every $1 \leq j \leq k$ and $b_{j_1} > b_j$ if $a_j = a_{j_1}$ with $1 \leq j \leq k$ and $j \neq j_1$, and the *maximal interval of C* as the interval $I_{j_2} = [a_{j_2}, b_{j_2}]$ with $b_{j_2} \geq b_j$ for every $1 \leq j \leq k$ and $a_{j_2} < a_j$ if $b_j = b_{j_2}$ with $1 \leq j \leq k$ and $j \neq j_2$. If $i \in \mathbb{Z} \setminus \{1, 2, \dots, k\}$, $1 \leq j \leq k$ and $i = j \pmod{k}$, then we define $v_i := v_j$ and $I_i := I_j$.

We say that a graph G is *length-proper* if every edge is a geodesic. A large class of length-proper graphs are the graphs with edges of length 1. Another important class of length-proper graphs are the following geometric graphs: Consider a discrete set V in an Euclidean space (or in a metric space) where we consider two points connected by an edge if some criterium is satisfied. If we define the length of an edge as the distance between its vertices, then we obtain a length-proper graph.

It is well-known that every interval graph is chordal. Hence, every length-proper interval graph is hyperbolic. The following result is one of the main theorems in this chapter, since it provides a sharp inequality for the hyperbolicity constant of any length-proper interval graph. Recall that $\ell(G) := \sup\{L_G(e) \mid e \in E(G)\}$.

Theorem 2.1.9. *Every length-proper interval graph G satisfies the sharp inequality*

$$\delta(G) \le \frac{3}{2}\,\ell(G).$$

Proof. Consider a geodesic triangle $T = \{x, y, z\}$ that is a cycle in G and $p \in [xy]$. Assume first that T satisfies the following property:

$$\text{if } a, b \in V(G) \cap [xy] \text{ and } ab \in E(G), \text{ then } ab \subseteq [xy]. \tag{2.1}$$

Consider the consecutive vertices $\{v_1, \dots, v_k\}$ in the cycle T, and their corresponding intervals $\{I_1, \dots, I_k\}$. As before, we denote by I_{j_1} and I_{j_2} the minimal and maximal intervals, respectively.

If $k < 4$, then $L(T) \le 3\ell(G)$ and Lemma 1.2.15 gives

$$d(p, [xz] \cup [zy]) \le \frac{1}{4}\,L(T) \le \frac{3}{4}\,\ell(G). \tag{2.2}$$

Assume now that $k \ge 4$.

Case (A). Assume that $p \in V(G)$. Let $a, b \in V(G)$ with $ap, bp \in E(G)$ and $ap \cup bp \subset T$.

Case $(A.1)$. If $ab \notin E(G)$, then Lemma 2.1.1 gives $\deg_T(p) \ge 3$, and there exists $q \in V(G) \cap T$ with $pq \in E(G)$ such that pq is not contained in T. By (2.1), $q \in [xz] \cup [zy]$ and so

$$d(p, [xz] \cup [zy]) \le d(p, q) = L(pq) \le \ell(G). \tag{2.3}$$

Case $(A.2)$. If $ab \in E(G)$, then ab is not contained in T, since T is a cycle and $k \ge 4$. By (2.1), $\{a, b\}$ is not contained in $[xy]$, and

$$d(p, [xz] \cup [zy]) \le \max\{d(p, a),\, d(p, b)\} = \max\{L(pa),\, L(pb)\} \le \ell(G). \tag{2.4}$$

Case (B). Assume that $p \notin V(G)$. Let $a, b \in V(G)$ with $p \in ab \subset T$ and $d(p, a) \le L(ab)/2 \le \ell(G)/2$. Corollary 2.1.2 gives that we have $\deg_T(a) \ge 3$ or $\deg_T(b) \ge 3$.

Case $(B.1)$. Assume that $\deg_T(a) \ge 3$.

Case $(B.1.1)$. If $a \notin [xy]$, then

$$d(p, [xz] \cup [zy]) \le d(p, a) \le \frac{1}{2}\,\ell(G). \tag{2.5}$$

Case $(B.1.2)$. Assume that $a \in [xy]$. Since $\deg_T(a) \geq 3$, there exists $q \in V(G) \cap T$ with $aq \in E(G)$ such that aq is not contained in T. By (2.1), $q \in [xz] \cup [zy]$ and so

$$\begin{aligned} d(p, [xz] \cup [zy]) &\leq d(p, a) + d(a, [xz] \cup [zy]) \leq d(p, a) + d(a, q) \\ &= d(p, a) + L(aq) \leq \frac{1}{2}\,\ell(G) + \ell(G) = \frac{3}{2}\,\ell(G). \end{aligned} \tag{2.6}$$

Case $(B.2)$. Assume that $\deg_T(a) = 2$ and $\deg_T(b) \geq 3$. Let $\alpha \neq b$ with $\alpha \in V(G)$, $\alpha a \in E(G)$ and $\alpha a \subset T$. Corollary 2.1.2 gives that we have $\alpha b \in E(G)$. By (2.1), we have that $\{\alpha, b\}$ is not contained in $[xy]$, and

$$\begin{aligned} d(p, [xz] \cup [zy]) &\leq \max\big\{d(p, \alpha),\, d(p, b)\big\} \leq \max\big\{d(p, a) + d(a, \alpha),\, d(p, b)\big\} \\ &\leq \max\Big\{\frac{1}{2}\,\ell(G) + \ell(G),\, \ell(G)\Big\} = \frac{3}{2}\,\ell(G). \end{aligned} \tag{2.7}$$

Inequalities (2.2), (2.3), (2.4), (2.5), (2.6) and (2.7) give in every case $d(p, [xz] \cup [zy]) \leq 3\ell(G)/2$.

Consider now a geodesic triangle $T = \{x, y, z\} = \{\,[xy], [xz], [yz]\}$ that does not satisfy Property (2.1). We are going to obtain a new geodesic γ joining x and y such that the geodesic triangle $T' = \{\gamma, [xz], [yz]\}$ satisfies (2.1).

Let us define inductively a finite sequence of geodesics $\{g_0, g_1, g_2, \dots, g_r\}$ joining x and y in the following way:

If $j = 0$, then $g_0 := [xy]$.

Assume that $j \geq 1$. If the geodesic triangle $\{g_{j-1}, [xz], [yz]\}$ satisfies (2.1), then $r = j - 1$ and the sequence stops. If $\{g_{j-1}, [xz], [yz]\}$ does not satisfy (2.1), then there exist $a, b \in V(G) \cap [xy]$ such that $ab \in E(G)$ and ab is not contained in $[xy]$. Denote by $[ab]$ the geodesic joining a and b contained in g_{j-1}. Let us define $g_j := (g_{j-1} \setminus [ab]) \cup ab$. Note that $g_j \cap V(G) \subset g_{j-1} \cap V(G)$ and $|g_j \cap V(G)| < |g_{j-1} \cap V(G)|$.

Since $|g_j \cap V(G)| < |g_{j-1} \cap V(G)|$ for any $j \geq 1$, this sequence must finish with some geodesic g_r such that the geodesic triangle $T' := \{g_r, [xz], [yz]\}$ satisfies (2.1). Thus define $\gamma := g_r$. Hence,

$$g_r \cap V(G) \subset g_{r-1} \cap V(G) \subset \cdots \subset g_1 \cap V(G) \subset g_0 \cap V(G),$$

and so $\gamma \cap V(G) \subset [xy] \cap V(G)$.

Let us consider $p \in [xy] \subset T$.

If $p \in \gamma \subset T'$, then by applying the previous argument to the geodesic triangle T' we obtain $d(p, [xz] \cup [zy]) \leq 3\ell(G)/2$. Assume that $p \notin \gamma$.

Since $\gamma \cap V(G) \subset [xy] \cap V(G)$, there exist $v, w \in \gamma \cap V(G)$ with $vw \in E(G)$ such that if $[vw]$ denotes the geodesic joining v and w contained in $[xy]$, then

$$p \in [vw], \qquad [vw] \cap vw = \{v, w\}.$$

Since vw and $[vw]$ are geodesics, we have $L(vw) = L([vw])$. Thus we can define $p' \in \gamma$ as the point in vw with $d(p', v) = d(p, v)$ and $d(p', w) = d(p, w)$. By applying the previous argument to p' and T', we obtain $d(p', [xz] \cup [zy]) \leq 3\ell(G)/2$. Since p' belongs to the edge vw, we have $d(p', [xz] \cup [zy]) = d(p', v) + d(v, [xz] \cup [zy])$ or $d(p', [xz] \cup [zy]) = d(p', w) + d(w, [xz] \cup [zy])$. By symmetry, we can assume that $d(p', [xz] \cup [zy]) = d(p', v) + d(v, [xz] \cup [zy])$. Since $d(p', v) = d(p, v)$, we have

$$d(p, [xz] \cup [zy]) \leq d(p, v) + d(v, [xz] \cup [zy]) = d(p', v) + d(v, [xz] \cup [zy]) = d(p', [xz] \cup [zy]) \leq \frac{3}{2}\, \ell(G).$$

Finally, Corollary 1.2.12 gives $\delta(G) \leq 3\ell(G)/2$.

Proposition 2.1.12 below shows that the inequality is sharp. $\square$

Note that if we remove the hypothesis $\ell(G) < \infty$, then there are non-hyperbolic length-proper interval graphs: If Γ is any graph such that every cycle in Γ has exactly 3 vertices and $\sup\{L(C) \mid C$ is a cycle in $\Gamma\} = \infty$, then Γ is a non-hyperbolic chordal graph. Some of these graphs Γ are length-proper interval graphs.

An *indifference graph* is an interval graph that has a set of unit interval graphs on the real line as vertex set, with two intervals adjacent if and only if they intersect, and the length of the corresponding edge is the distance between the midpoints of the two intervals. Also, we can see an indifference graph as a Euclidean graph in $\mathbb{R}$ constructed by assigning a real number to each vertex and connecting two vertices by an edge when their corresponding numbers are within one unit from each other. As an Euclidean graph, the length of an edge is the Euclidean distance between the corresponding numbers to the vertices in the edge. Hence, every indifference graph G is a length-proper graph and $\ell(G) \leq 1$.

Theorem 2.1.9 has the following direct consequence.

Corollary 2.1.10. *Every indifference graph G satisfies the inequality*

$$\delta(G) \leq \frac{3}{2}\, \ell(G) \leq \frac{3}{2}\,.$$

In this part we just consider graphs with edges of length 1. This is a very usual class of graphs. Note that every graph G with edges of length 1 is a length-proper graph with $\ell(G) = 1$.

The goal of part of the section is to compute the precise value of the hyperbolicity constant of every interval graph with edges of length 1 (see Theorem 2.1.16). We wish to emphasize that it is unusual to be able to compute the hyperbolicity constant of every graph in a large class of graphs. Let us start with a direct consequence of Theorem 2.1.9.

Corollary 2.1.11. *Every interval graph G with edges of length* 1 *satisfies the inequality*

$$\delta(G) \leq \frac{3}{2}\,.$$

First of all we characterize the interval graphs with edges of length 1 and $\delta(G) = 3/2$ in Proposition 2.1.12 below. Furthermore, Proposition 2.1.12 shows that the inequality in Theorem 2.1.9 is sharp.

Let G be an interval graph. We say that G has the (3/2)-*intersection property* if there exists two disjoint intervals I' and I'' corresponding to vertices in a cycle C in G such that there is no corresponding interval I to a vertex in G with $I \cap I' \neq \emptyset$ and $I \cap I'' \neq \emptyset$.

Proposition 2.1.12. *An interval graph G with edges of length 1 satisfies $\delta(G) = 3/2$ if and only if G has the $(3/2)$-intersection property.*

Proof. Assume that G has the $(3/2)$-intersection property. Thus there exist two disjoint corresponding intervals I' and I'' to vertices in a cycle C in G such that there is no corresponding interval I to a vertex in G with $I \cap I' \neq \emptyset$ and $I \cap I'' \neq \emptyset$. If v' and v'' are the corresponding vertices to I' and I'', respectively, then $v', v'' \in C$ and $d(v', v'') \geq 3$. Thus Theorem 2.1.7 gives $\delta(G) \geq 3/2$ and, since $\delta(G) \leq 3/2$ by Corollary 2.1.11, we conclude $\delta(G) = 3/2$.

Assume now that G does not have the $(3/2)$-intersection property. Seeking for a contradiction assume that $\delta(G) = 3/2$. By Theorem 1.2.13, there exist a geodesic triangle $T = \{x, y, z\}$ that is a cycle in G and $p \in [xy]$ such that $d(p, [xz] \cup [zy]) = \delta(T) = \delta(G) = 3/2$ and $x, y, z \in J(G)$. Since $d(p, \{x, y\}) \geq d(p, [xz] \cup [zy]) = 3/2$, we have $d(x, y) \geq 3$. Since G does not have the $(3/2)$-intersection property, for each two disjoint corresponding intervals I' and I'' to vertices in the cycle T there exists a corresponding interval I to a vertex in G with $I \cap I' \neq \emptyset$ and $I \cap I'' \neq \emptyset$. If v' and v'' are the corresponding vertices to I' and I'', respectively, then $v', v'' \in T$ and $d(v', v'') = 2$. We conclude that $\operatorname{diam}(T \cap V(G)) \leq 2$ and $\operatorname{diam} T \leq 3$. Since $d(x, y) \geq 3$ with $x, y \in J(G)$, we have $\operatorname{diam}(T \cap V(G)) = 2$, $\operatorname{diam} T = 3$, $d(x, y) = 3$, $L([xy])/2 = d(p, x) = d(p, y) = d(p, [xz] \cup [zy]) = \delta(T) = \delta(G) = 3/2$ and p is the midpoint of $[xy]$. Thus $x, y \in J(G) \setminus V(G)$ and $p \in V(G)$. If $x \in x_1x_2 \in E(G)$ and $y \in y_1y_2 \in E(G)$, then $d(\{x_1, x_2\}, \{y_1, y_2\}) = 2$. Let $I_{x_1}, I_{x_2}, I_{y_1}, I_{y_2}, I_p$ be the corresponding intervals to the vertices x_1, x_2, y_1, y_2, p, respectively. We can assume that $x_1, y_1 \in [xy]$ and thus $I_{x_1} \cap I_p \neq \emptyset$ and $I_{y_1} \cap I_p \neq \emptyset$. Since $d(x_1, y_1) = 2$, $I_{x_1} \cap I_{y_1} = \emptyset$. Thus there exists $\zeta \in I_p \setminus (I_{x_1} \cup I_{y_1})$. Since $[xy] \cap V(G) = \{x_1, p, y_1\}$ and T is a cycle containing x_1, p, y_1, by continuity, there exists a corresponding interval J to a vertex $v \in ([xz] \cup [zy]) \cap V(G)$ with $\zeta \in J$. Thus $pv \in E(G)$ and $3/2 = d(p, [xz] \cup [zy]) \leq d(p, v) = 1$, which is a contradiction. Hence, $\delta(G) \neq 3/2$. □

Corollary 2.1.11 and Theorems 1.2.14 and 2.1.3 give that $\delta(G) \in \{0, 3/4, 1, 5/4, 3/2\}$ for every interval graph G with edges of length 1. Proposition 2.1.12 characterizes the interval graphs with edges of length 1 and $\delta(G) = 3/2$. In order to characterize the interval graphs with the other values of the hyperbolicity constant, we need some definitions.

Let G be an interval graph.

We say that G has the 0-*intersection property* if for every three corresponding intervals I', I'' and I''' to vertices in G we have $I' \cap I'' \cap I''' = \emptyset$.

G has the $(3/4)$-*intersection property* if it does not have the 0-intersection property and for every four corresponding intervals I', I'', I''' and I'''' to vertices in G we have $I' \cap I'' \cap I''' = \emptyset$ or $I' \cap I'' \cap I'''' = \emptyset$.

By a *couple* of intervals in a cycle C of G we mean the union of two non-disjoint intervals whose corresponding vertices belong to C. We say that G has the 1-*intersection property* if it does not have the 0 and $(3/4)$-intersection properties and for every cycle C in G each interval and couple of corresponding intervals to vertices in C are not disjoint.

One can check that every chordal graph that has a cycle with length at least four has a cycle with length four and, since this cycle has a chord, it also has a cycle with length three.

Next we provide a characterization of the interval graphs with hyperbolicity constant 0. It is well-known that these are the caterpillar trees (the trees for which removing the leaves and incident edges produces a path graph), see [67], but we prefer to characterize them by the 0-intersection property in Proposition 2.1.13 below, since it looks similar to the other intersection properties.

Proposition 2.1.13. *An interval graph G with edges of length* 1 *satisfies* $\delta(G) = 0$ *if and only if G has the* 0*-intersection property.*

Proof. By Theorem 2.1.3, $\delta(G) = 0$ if and only if G is a tree. Since every interval graph is chordal, G is not a tree if and only if it contains a cycle with length 3, and this last condition holds if and only if there exist three corresponding intervals I', I'' and I''' to vertices in G with $I' \cap I'' \cap I''' \neq \emptyset$. Hence, G has a cycle if and only if it does not have the 0-intersection property. □

Proposition 2.1.14. *An interval graph G with edges of length* 1 *satisfies* $\delta(G) = 3/4$ *if and only if G has the* $(3/4)$*-intersection property.*

Proof. By Theorem 2.1.3, $\delta(G) = 3/4$ if and only if G is not a tree and every cycle in G has length 3. Proposition 2.1.13 gives that G is not a tree if and only if G does not have the 0-intersection property. Therefore, it suffices to show that every cycle in G has length 3 if and only if for every four corresponding intervals I', I'', I''' and I'''' to vertices in G we have $I' \cap I'' \cap I''' = \emptyset$ or $I' \cap I'' \cap I'''' = \emptyset$.

Since every interval graph is chordal, G has a cycle with length at least 4 if and only if it has a cycle C with length 4 and this cycle has at least a chord.

Assume first that there exists such a cycle C. If I', I'', I''', I'''' are the corresponding intervals to the vertices in C and I', I'' correspond to vertices with a chord, then $I' \cap I'' \cap I''' \neq \emptyset$ and $I' \cap I'' \cap I'''' \neq \emptyset$.

Assume now that there are corresponding intervals I', I'', I''', I'''' to the vertices v', v'', v''', v'''' in G with $I' \cap I'' \cap I''' \neq \emptyset$ and $I' \cap I'' \cap I'''' \neq \emptyset$. Thus $v'v''', v''v''' \in E(G)$ and $v'v''', v''v'''' \in E(G)$, and so $v'v''' \cup v'''v'' \cup v''v'''' \cup v''''v'$ is a cycle in G with length 4. □

Proposition 2.1.15. *An interval graph G with edges of length* 1 *satisfies* $\delta(G) = 1$ *if and only if G has the* 1*-intersection property.*

Proof. By Theorem 2.1.5, $\delta(G) = 1$ if and only if $\delta(G) \notin \{0, 3/4\}$ and for every cycle C in G and every $x, y \in C \cap J(G)$ we have $d(x, y) \leq 2$. Propositions 2.1.13 and 2.1.14 give that $\delta(G) \notin \{0, 3/4\}$ if and only if G does not have the 0 and $(3/4)$-intersection properties. Therefore, it suffices to show that for every cycle C in G, we have $d(x, y) \leq 2$ for every $x, y \in C \cap J(G)$ if and only if each interval and couple of corresponding intervals to vertices in C are not disjoint.

Fix a cycle C in G. Each interval and couple of corresponding intervals to vertices in C are not disjoint if and only if $d(x, y) \leq 3/2$ for every $x \in C \cap V(G)$ and $y \in C \cap (J(G) \setminus V(G))$. Since every point in $C \cap (J(G) \setminus V(G))$ has a point in $C \cap V(G)$ at distance $1/2$, this last condition is equivalent to $d(x, y) \leq 2$ for every $x, y \in C \cap J(G)$. □

Finally, we collect the previous geometric characterizations in the following theorem. Note that the characterization of $\delta(G) = 5/4$ in Theorem 2.1.16 is much simpler that the one in [10]. Recall that to characterize the graphs with hyperbolicity $3/2$ is a very difficult task, as it was showed in [10, Remark 4.19].

Theorem 2.1.16. *Every interval graph G with edges of length 1 is hyperbolic and $\delta(G) \in \{0, 3/4, 1, 5/4, 3/2\}$. Furthermore,*

- $\delta(G) = 0$ *if and only if G has the 0-intersection property.*
- $\delta(G) = 3/4$ *if and only if G has the (3/4)-intersection property.*
- $\delta(G) = 1$ *if and only if G has the 1-intersection property.*
- $\delta(G) = 5/4$ *if and only if G does not have the 0, 3/4, 1 and (3/2)-intersection properties.*
- $\delta(G) = 3/2$ *if and only if G has the (3/2)-intersection property.*

2.2 Complement of interval graphs

As usual, the *complement* $\overline{G}$ of the graph G is defined as the graph with $V\big(\overline{G}\big) = V(G)$ and such that $e \in E\big(\overline{G}\big)$ if and only if $e \notin E(G)$. Note that that $\overline{G}$ could be disconnected in general, and recall that for every disconnected graph G we define $\delta(G)$ as the supremum of $\delta(G_i)$ where G_i varies in the set of connected components of G.

We consider that the length of the edges of every complement graph is 1.

If Γ is a subgraph of G, we consider in Γ the *inner metric* obtained by the restriction of the metric in G, that is

$$d_\Gamma(v, w) := \inf \big\{L(\gamma) \mid \gamma \subset \Gamma \text{ is a continuous curve joining } v \text{ and } w\big\} \geq d_G(v, w)\,.$$

Note that the inner metric d_Γ is the usual metric if we consider the subgraph Γ as a graph.

Since the complements of interval graphs belong to the class of comparability graphs [44], it is natural to study also the hyperbolicity constant of complements of interval graphs. In order to do it we need some preliminary results and the following technical lemma.

Lemma 2.2.1. *Let G be an interval graph with edges of length 1, $V(G) = \{v_1, \dots, v_r\}$ and corresponding intervals $\{I_1, \dots, I_r\}$. We have $\operatorname{diam} V(G) = 2$ if and only if there exists an interval I_i with $I_j \cap I_i \neq \emptyset$ for every $1 \leq j \leq r$ and $\operatorname{diam} V(G') \geq 2$, where G' is the corresponding interval graph to $\{I_1, \dots, I_r\} \setminus I_i$. Furthermore, if this is the case, then $\delta\big(\overline{G}\big) = \delta\big(\overline{G'}\big)$.*

Proof. Assume first that $\operatorname{diam} V(G) = 2$. Let $[a_j, b_j] = I_j$ for $1 \leq j \leq r$. Consider integers $1 \leq i_1, i_2 \leq r$ satisfying

$$b_{i_1} \leq b_j, \qquad a_j \leq a_{i_2}, \qquad \text{for every } 1 \leq j \leq r. \tag{2.8}$$

Since $\operatorname{diam} V(G) = 2$, we have $b_{i_1} < a_{i_2}$. Thus $d_G(v_{i_1}, v_{i_2}) = 2$ and there exists i with $v_i v_{i_1}, v_i v_{i_2} \in E(G)$. Hence, $I_{i_1} \cap I_i \neq \emptyset$ and $I_{i_2} \cap I_i \neq \emptyset$. Thus (2.8) gives $I_j \cap I_i \neq \emptyset$ for every $1 \leq j \leq r$, and we deduce $d_G(v_j, v_i) \leq 1$ for every $1 \leq j \leq r$.

Seeking for a contradiction assume that $\operatorname{diam} V(G') \leq 1$. Thus $d_G(v_j, v_{j'}) \leq d_{G'}(v_j, v_{j'}) \leq 1$ for every $1 \leq j, j' \leq r$ with $j, j' \neq i$. Furthermore, we have proved $d_G(v_j, v_i) \leq 1$ for every $1 \leq j \leq r$.

Therefore, $d_G(v_j, v_{j'}) \le 1$ for every $1 \le j, j' \le r$ and we conclude $\operatorname{diam} V(G) \le 1$, which is a contradiction. Hence, $\operatorname{diam} V(G') \ge 2$.

The converse implication is well-known.

Finally, since $v_j v_i \in E(G)$ for every $1 \le j \le r$ with $j \ne i$, we have $\overline{G} = \{v_i\} \cup \overline{G'}$ and

$$\delta\big(\,\overline{G}\,\big) = \max\big\{\delta\big(\{v_i\}\big),\, \delta\big(\,\overline{G'}\,\big)\big\} = \max\big\{0,\, \delta\big(\,\overline{G'}\,\big)\big\} = \delta\big(\,\overline{G'}\,\big).$$

□

The next theorem provides good bounds for the hyperbolicity constant of the complement of every interval graph. Note that it is not usual to obtain such close lower and upper bounds for a large class of graphs. Some inequalities are not difficult to prove; the most difficult cases are the upper bound when $\operatorname{diam} V(G) = 2$ (recall that this is the more difficult case in the study of the complement of a graph), and the lower bound when $\operatorname{diam} V(G) \ge 4$.

Theorem 2.2.2. *Let G be any interval graph.*

- *If* $\operatorname{diam} V(G) = 1$*, then* $\delta\big(\,\overline{G}\,\big) = 0$.
- *If* $2 \le \operatorname{diam} V(G) \le 3$*, then* $0 \le \delta\big(\,\overline{G}\,\big) \le 2$.
- *If* $\operatorname{diam} V(G) \ge 4$*, then* $5/4 \le \delta\big(\,\overline{G}\,\big) \le 3/2$.

Furthermore, the lower bounds on $\delta\big(\,\overline{G}\,\big)$ are sharp.

Proof. If $\operatorname{diam} V(G) = 1$, then G is a complete graph. Thus $\overline{G}$ is a union of isolated vertices and $\delta\big(\,\overline{G}\,\big) = 0$.

Let us prove now the upper bounds.

It is well-known that if $\operatorname{diam} V(G) \ge 3$, then $\overline{G}$ is connected and $\operatorname{diam} V\big(\,\overline{G}\,\big) \le 3$. Therefore, Corollary 1.2.16 gives $\delta\big(\,\overline{G}\,\big) \le 2$.

If $\operatorname{diam} V(G) \ge 4$, then [59, Theorem 2.14] gives $\delta\big(\,\overline{G}\,\big) \le 3/2$.

Assume now that $\operatorname{diam} V(G) = 2$. By Lemma 2.2.1, there exists an interval graph G' with $|V(G')| = |V(G)| - 1$, $\operatorname{diam} V(G') \ge 2$ and $\delta\big(\,\overline{G}\,\big) = \delta\big(\,\overline{G'}\,\big)$. Let us define inductively a finite sequence of interval graphs $\{G^{(0)}, G^{(1)}, G^{(2)}, \dots, G^{(k)}\}$ with

$$\delta\big(\,\overline{G^{(0)}}\,\big) = \delta\big(\,\overline{G^{(1)}}\,\big) = \delta\big(\,\overline{G^{(2)}}\,\big) = \cdots = \delta\big(\,\overline{G^{(k)}}\,\big),$$

$$|V(G^{(j)})| = |V(G^{(j-1)})| - 1, \qquad \text{for } 0 < j \le k,$$

$$\operatorname{diam} V(G^{(j)}) \ge 2, \qquad \text{for } 0 \le j \le k,$$

in the following way:

If $j = 0$, then $G^{(0)} := G$.

If $j = 1$, then $G^{(1)} := G'$.

Assume that $j > 1$. If $\operatorname{diam} V(G^{(j-1)}) \ge 3$, then $k = j - 1$ and the sequence stops. If $\operatorname{diam} V(G^{(j-1)}) = 2$, then Lemma 2.2.1 provides an interval graph $(G^{(j-1)})'$ with

$$|V((G^{(j-1)})')| = |V(G^{(j-1)})| - 1, \quad \operatorname{diam} V((G^{(j-1)})') \ge 2, \quad \delta\big(\,\overline{G^{(j-1)}}\,\big) = \delta\big(\,\overline{(G^{(j-1)})'}\,\big),$$

and we define $G^{(j)} := (G^{(j-1)})'$.

Since $|V(G^{(j)})| = |V(G^{(j-1)})| - 1$ for $0 < j \le k$ and the diameter of a graph with just a vertex is 0, this sequence must finish with some graph $G^{(k)}$ satisfying $\operatorname{diam} V(G^{(k)}) \ge 3$. Thus

$$\delta\big(\overline{G}\big) = \delta\big(\overline{G^{(0)}}\big) = \delta\big(\overline{G^{(1)}}\big) = \cdots = \delta\big(\overline{G^{(k)}}\big) \le 2.$$

We prove now that $\delta\big(\overline{G}\big) \ge 5/4$ if $\operatorname{diam} V(G) \ge 4$. Let us fix any graph G with $\operatorname{diam} V(G) \ge 4$. Thus, there exists a geodesic $[v_0v_4] = v_0v_1 \cup v_1v_2 \cup v_2v_3 \cup v_3v_4$ in G. If Γ is the subgraph of $\overline{G}$ induced by $\{v_0, v_1, v_2, v_3, v_3, v_4\}$, then $E(\Gamma) = \{v_0v_2, v_0v_3, v_0v_4, v_1v_3, v_1v_4, v_2v_4\}$. Consider the cycle $C := v_0v_2 \cup v_2v_4 \cup v_4v_1 \cup v_1v_3 \cup v_3v_0$ in Γ. If p is the midpoint of v_0v_2, then $d_\Gamma(v_1, p) = 5/2$ and so Corollary 2.1.6 gives $\delta(\Gamma) > 1$. Therefore, Theorem 1.2.14 gives $\delta(\Gamma) \ge 5/4$. Since Γ is an induced subgraph of $\overline{G}$, if g is a path in $\overline{G}$ joining v_i and v_j $(0 \le i, j \le 4)$ and g is not contained in Γ, then $L_{\overline{G}}(g) \ge 2$. Since $\operatorname{diam}_{\overline{G}} V(\Gamma) = 2$, we have $d_\Gamma(v_j, v_j) = d_{\overline{G}}(v_j, v_j)$ for every $0 \le i, j \le 4$; consequently, $d_\Gamma(x, y) = d_{\overline{G}}(x, y)$ for every $x, y \in \Gamma$, i.e., Γ is an isometric subgraph of $\overline{G}$. Hence, the geodesic triangles in Γ are also geodesic triangles in $\overline{G}$, and we have $\delta\big(\overline{G}\big) \ge \delta(\Gamma) \ge 5/4$.

Let us show now that the lower bounds on $\delta\big(\overline{G}\big)$ are sharp. Recall that the path graph with n vertices P_n is a graph with $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $E(P_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}$.

Consider the path graph with four vertices $G = P_4$. Since $\overline{G} = P_4$, we have $\operatorname{diam} V(G) = 3$ and $\delta\big(\overline{G}\big) = 0$.

Consider the path graph with five vertices $G = P_5$. Since $\operatorname{diam} V(G) = 4$, we have $\delta\big(\overline{G}\big) \ge 5/4$. Note that $\overline{G}$ has 5 vertices and thus Theorem 2.1.8 gives $\delta\big(\overline{G}\big) \le 5/4$. Hence, we conclude $\delta\big(\overline{G}\big) = 5/4$. $\square$

In 1956, Nordhaus and Gaddum gave lower and upper bounds on the sum and the product of the chromatic number of a graph and its complement in [85]. Since then, relations of a similar type have been proposed for many other graph invariants, in several hundred papers. Corollary 2.1.11 and Theorem 2.2.2 provide some Nordhaus-Gaddum type results for the hyperbolicity constant.

Corollary 2.2.3. *If G is any interval graph with edges of length 1, then*

$$\delta(G)\,\delta\big(\overline{G}\big) \le \begin{cases} 0 & \text{if } \operatorname{diam} V(G) = 1, \\ 3 & \text{if } 2 \le \operatorname{diam} V(G) \le 3, \\ 9/4 & \text{if } \operatorname{diam} V(G) \ge 4. \end{cases}$$

Note that we can not improve the trivial lower bound $\delta(G)\delta(\overline{G}) \ge 0$, since it is attained if G is any tree.

Corollary 2.2.4. *If G is any interval graph with edges of length 1, then*

$$\delta(G) + \delta\big(\overline{G}\big) \le \begin{cases} 3/2 & \text{if } \operatorname{diam} V(G) = 1, \\ 7/2 & \text{if } 2 \le \operatorname{diam} V(G) \le 3, \\ 3 & \text{if } \operatorname{diam} V(G) \ge 4. \end{cases}$$

Besides, $\delta(G) + \delta\big(\overline{G}\big) \ge 5/4$ for every graph G with $\operatorname{diam} V(G) \ge 4$.

Chapter 3

Circular-arc graphs

In the previous chapter we study the hyperbolicity constant of interval graphs. In this chapter we work with circular-arc graphs (another important class of intersection graphs). Circular-arc graphs are useful in modeling periodic resource allocation problems in operations research (each arc represents a request for a resource for a specific period repeated in time). They also have applications in different fields such as genetic research, traffic control, computer compiler design and statistics (see, e.g., [88]).

Of course, every interval graph can be viewed as a circular-arc graph; if a representation of a circular-arc graph G leaves some point of the unit circle uncovered, it is topologically the same as an interval representation of G (by cutting the circle and straighten it out to a straight line); we will use this identification along the chapter.

In this chapter we give sharp bounds for the hyperbolicity constant of circular-arc graphs (see Theorem 3.1.1). These bounds are improved in Theorem 3.1.2 for proper circular-arc graphs. Theorem 3.1.5 gives a sufficient condition in order to attain the lower bound of $\delta(G)$ in Theorem 3.1.1; in particular, it shows that this bound is sharp. Propositions 3.1.3 and 3.1.4 characterize the circular-arc graphs with small hyperbolicity constant. In Section 3.2, we obtain bounds in Theorems 3.2.7 and 3.2.16 for the hyperbolicity constant of the complement and line of a circular-arc graph, respectively. These theorems improve, for circular-arc graphs, the general bounds for the hyperbolicity constant of the complement and line graphs. In order to do that, we obtain new results about regular, chordal and line graphs which are interesting by themselves (see Theorems 3.2.5 and 3.2.15).

3.1 Circular-arc graphs and hyperbolicity

Given a set of arcs of a circle, we obtain its *circular-arc graph*, as the intersection graph of the set, i.e., we associate a vertex with each arc and add an edge between two vertices if and only if the corresponding arcs intersect.

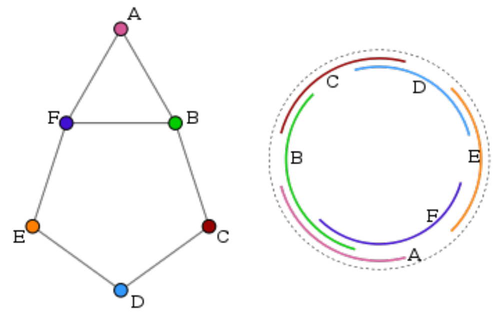


Figure 3.1: Circular-arc graph. Image taken from Wikipedia.

If G is a circular-arc graph, then a set of vertices $K = \{v_1, \dots, v_r\}$ and corresponding arcs $\{I_1, \dots, I_r\}$ is said *total* if $I_1 \cup \cdots \cup I_r = \mathbb{S}^1$, and we say that r is the *size* of K.

We say that a circular-arc graph G is NI if it has a total set of vertices. If either G is a finite circular-arc graph or every arc is open, then G is NI if and only if the union of the corresponding arcs to vertices in G is $\mathbb{S}^1$. Note that a circular-arc graph G is also an interval graph if and only if it is not NI. In [58] the authors study the hyperbolicity constant of interval graphs.

For any NI circular-arc graph G, let us define the *minimum size* of G as

$$\varrho(G) := \min \left\{ \, size(K) \,|\, K \text{ is a total set of vertices in } G \, \right\}.$$

If G is an interval graph, then we define $\varrho(G) := 0$. Hence, a circular-arc graph G is NI if and only if $\varrho(G) \geq 1$. Note that the minimum size is 1 if and only if an arc is the whole unit circle $\mathbb{S}^1$.

The minimum size $\varrho(G)$ plays an important role in the study of the hyperbolicity of circular-arc graphs, as the following result shows. Recall that $\lfloor t \rfloor$ denotes the lower integer part of the real number t, i.e., the greatest integer less than or equal to t.

Since any NI circular-arc graph is a bounded set, we have that it is hyperbolic. The following result provides sharp inequalities for the hyperbolicity constant of any circular-arc graph.

Theorem 3.1.1. *Let G be a circular-arc graph. If $\varrho(G) \neq 1, 2$, then G satisfies the sharp inequalities*

$$\frac{1}{4}\,\varrho(G) \leq \delta(G) \leq \frac{1}{2}\left\lfloor \frac{1}{2}\,\varrho(G) \right\rfloor + \frac{3}{2}\,.$$

If $\varrho(G) = 1$, then G satisfies the sharp inequalities

$$0 \leq \delta(G) \leq \frac{3}{2}\,.$$

If $\varrho(G) = 2$, then G satisfies the sharp inequalities

$$0 \leq \delta(G) \leq 2.$$

Proof. The result is known if G is an interval graph (i.e., if $\varrho(G) = 0$), see [58, Corollary 6].

Assume now that $\varrho(G) \geq 1$. Let us prove the upper bound of $\delta(G)$. Fix any set of vertices $K = \{v_1, \dots, v_{\varrho(G)}\}$ and corresponding arcs $\{I_1, \dots, I_{\varrho(G)}\}$ with $I_1 \cup \cdots \cup I_{\varrho(G)} = \mathbb{S}^1$. Thus, every arc in $\mathbb{S}^1$ intersects some arc in $\{I_1, \dots, I_{\varrho(G)}\}$. Hence,

$$\operatorname{diam} V(G) \le 1 + \operatorname{diam} K + 1 = \Big\lfloor \frac{1}{2}\,\varrho(G) \Big\rfloor + 2,$$
$$\operatorname{diam} G \le \frac{1}{2} + \operatorname{diam} V(G) + \frac{1}{2} \le \Big\lfloor \frac{1}{2}\,\varrho(G) \Big\rfloor + 3,$$

and Corollary 1.2.17 gives the upper bound.

Let us prove now that this bound is sharp. Given $\theta_1 < \theta_2$, denote by $[e^{i\theta_1}, e^{i\theta_2}]$ the arc

$$[e^{i\theta_1}, e^{i\theta_2}] := \big\{ e^{i\theta} \mid \theta \in [\theta_1, \theta_2] \big\}.$$

Fix any even integer $\varrho \ge 6$ with $\varrho \equiv 2 \,(\bmod\ 4)$ and consider the family of arcs

$$\big\{ [e^{2\pi i (j-1)/\varrho}, e^{2\pi i j/\varrho}] \big\}_{j=1}^{\varrho}$$

Denote by I_j the arc $[e^{2\pi i (j-1)/\varrho}, e^{2\pi i j/\varrho}]$. Let z_1, z_2, z_3 be the points $e^{\pi i/(2\varrho)}, e^{2\pi i/(2\varrho)}, e^{3\pi i/(2\varrho)}$ in I_1, respectively, z_4, z_5 the points $e^{2\pi i/\varrho + 2\pi i/(3\varrho)}, e^{2\pi i/\varrho + 4\pi i/(3\varrho)}$ in I_2, respectively, and z_j the midpoint of I_{j-3} with $6 \le j \le \varrho/2 + 3$. Let $z_{\varrho/2+4}, z_{\varrho/2+5}, z_{\varrho/2+6}$ be the points $-e^{\pi i/(2\varrho)}, -e^{2\pi i/(2\varrho)}, -e^{3\pi i/(2\varrho)}$ in $I_{\varrho/2+1}$, respectively, $z_{\varrho/2+7}, z_{\varrho/2+8}$ the points $-e^{2\pi i/\varrho + 2\pi i/(3\varrho)}, -e^{2\pi i/\varrho + 4\pi i/(3\varrho)}$ in $I_{\varrho/2+2}$, respectively, and z_k the midpoint of I_{k-6} with $\varrho/2 + 9 \le k \le \varrho + 6$.

Consider the circular-arc graph G_ϱ defined as the intersection graph of the family of arcs

$$\big\{ [e^{2\pi i (j-1)/\varrho}, e^{2\pi i j/\varrho}] \big\}_{j=1}^{\varrho} \cup \big\{ [z_j, z_{j+1}] \big\}_{j=1}^{\varrho+6} \cup \big\{ [z_{\varrho+6}, z_1] \big\}.$$

Let x (respectively, y) be the midpoint of the edge of G_ϱ with endpoints corresponding to the arcs $[z_1, z_2]$ and $[z_2, z_3]$ (respectively, $[z_{\varrho/2+4}, z_{\varrho/2+5}]$ and $[z_{\varrho/2+5}, z_{\varrho/2+6}]$).

We have $d_{G_\varrho}(x, y) = 3/2 + \varrho/2 + 3/2 = \varrho/2 + 3$. Let γ_1 and γ_2 be two geodesics in G_ϱ joining x and y such that γ_1 contains the corresponding vertices to the arcs $\big\{ [z_j, z_{j+1}] \big\}_{j=2}^{\varrho/2+4}$ and γ_2 contains the corresponding vertices to the arcs $\big\{ [z_j, z_{j+1}] \big\}_{j=\varrho/2+5}^{\varrho+6} \cup \big\{ [z_{\varrho+6}, z_1] \big\} \cup \big\{ [z_1, z_2] \big\}$. Consider the geodesic bigon $\{\gamma_1, \gamma_2\}$. If p is the midpoint of γ_1, then $d_{G_\varrho}(p, \gamma_2) = d_{G_\varrho}(x, y)/2 = \varrho/4 + 3/2$. Hence, $\varrho/4 + 3/2 = d_{G_\varrho}(p, \gamma_2) \le \delta(G_\varrho) \le \varrho/4 + 3/2$, and we conclude $\delta(G_\varrho) = \varrho/4 + 3/2$.

If $\varrho(G) = 1$, then this upper bound is also attained. The wheel graph with seven vertices W_7 is a circular-arc graph with $\varrho(G) = 1$, and [101, Theorem 11] gives that $\delta(W_7) = 3/2$.

If $\varrho(G) = 2$, then this upper bound is attained by the circular-arc graph G corresponding to the arcs

$$[e^{0i}, e^{\pi i}] \cup [e^{\pi i}, e^{2\pi i}] \cup [e^{0i}, e^{\pi i/4}] \cup [e^{\pi i/4}, e^{\pi i/2}] \cup [e^{\pi i/2}, e^{3\pi i/4}] \cup [e^{3\pi i/4}, e^{\pi i}]$$
$$\cup [e^{\pi i}, e^{5\pi i/4}] \cup [e^{5\pi i/4}, e^{3\pi i/2}] \cup [e^{3\pi i/2}, e^{7\pi i/4}] \cup [e^{7\pi i/4}, e^{2\pi i}].$$

In order to prove $\delta(G) = 2$, let x (respectively, y) be the midpoint of the edge in G with endpoints $[e^{\pi i/4}, e^{\pi i/2}]$ and $[e^{\pi i/2}, e^{3\pi i/4}]$ (respectively, $[e^{5\pi i/4}, e^{3\pi i/2}]$ and $[e^{3\pi i/2}, e^{7\pi i/4}]$). One can check that there are two geodesics γ_1 and γ_2 such that the midpoint p of γ_1 satisfies $d_G(p, \gamma_2) = 2$. If we consider the geodesic bigon $\{\gamma_1, \gamma_2\}$, then $2 = d_G(p, \gamma_2) \le \delta(G) \le 2$, and we conclude $\delta(G) = 2$.

In order to prove the lower bound of $\delta(G)$, we deal first with the case $\varrho(G) \geq 3$. As above, fix any set of vertices $K = \{v_1, \dots, v_{\varrho(G)}\}$ and corresponding arcs $\{I_1, \dots, I_{\varrho(G)}\}$ with $I_1 \cup \cdots \cup I_{\varrho(G)} = \mathbb{S}^1$. The definition of $\varrho(G)$ gives that the subgraph Γ_K of G induced by K is an isometric subgraph of G. Since $\varrho(G) \geq 3$, the subgraph Γ_K is isomorphic to the cycle graph $C_{\varrho(G)}$. Therefore, Lemma 1.2.18 and Theorem 1.2.19 give $\delta(G) \geq \delta(\Gamma_K) = \varrho(G)/4$.

Any circular-arc graph isomorphic to the cycle graph C_ϱ attains this lower bound.

Finally, the lower bounds for the cases $\varrho(G) = 1, 2$ are trivial, and they are attained by the graphs G_1 with just a vertex and G_2 with just an edge, respectively. $\square$

An important subset of circular-arc graphs are proper circular-arc graphs. A circular-arc graph G is said *proper* if there is a representation of G where none of the arcs contains another. The following result improves Theorem 3.1.1 for this kind of graphs.

Theorem 3.1.2. *Let G be a proper circular-arc graph. If $\varrho(G) \geq 3$, then G satisfies the sharp inequalities*

$$\frac{1}{4}\,\varrho(G) \leq \delta(G) \leq \frac{1}{2}\Big\lfloor \frac{1}{2}\,\varrho(G)\Big\rfloor + 1.$$

If $\varrho(G) = 1$, then $\delta(G) = 0$. If $\varrho(G) = 2$, then G satisfies the sharp inequalities $0 \leq \delta(G) \leq 5/4$.

Proof. Assume first that the minimum size of G is 1. Since $\mathbb{S}^1$ is a corresponding arc to a vertex of G and G is a proper circular-arc graph, we have that it has just a vertex, and $\delta(G) = 0$.

Assume now that $\varrho(G) \geq 2$. The lower bounds are a consequence of Theorem 3.1.1 (note that the examples in the proof of Theorem 3.1.1 attaining the lower bounds are proper circular-arc graphs). Let us prove the upper bound of $\delta(G)$. Fix any set of vertices $K = \{v_1, \dots, v_{\varrho(G)}\}$ and corresponding arcs $\{I_1, \dots, I_{\varrho(G)}\}$ with $I_1 \cup \cdots \cup I_{\varrho(G)} = \mathbb{S}^1$. Thus, every arc in $\mathbb{S}^1$ intersects two arcs in $\{I_1, \dots, I_{\varrho(G)}\}$. Given any $u, w \in V(G) \setminus K$, there are $v_{1,u}, v_{2,u}, v_{1,w}, v_{2,w} \in K$ with $uv_{1,u}, uv_{2,u}, wv_{1,w}, wv_{2,w}, v_{1,u}v_{2,u}, v_{1,w}v_{2,w} \in E(G)$. Thus,

$$d_G(v_{1,u}v_{2,u}, v_{1,w}v_{2,w}) \leq \Big\lfloor \frac{1}{2}\,(\varrho(G) - 2)\Big\rfloor = \Big\lfloor \frac{1}{2}\,\varrho(G)\Big\rfloor - 1.$$

Hence,

$$\operatorname{diam} V(G) \leq 1 + \Big(\Big\lfloor \frac{1}{2}\,\varrho(G)\Big\rfloor - 1\Big) + 1 = \Big\lfloor \frac{1}{2}\,\varrho(G)\Big\rfloor + 1,$$
$$\operatorname{diam} G \leq \frac{1}{2} + \operatorname{diam} V(G) + \frac{1}{2} \leq \Big\lfloor \frac{1}{2}\,\varrho(G)\Big\rfloor + 2,$$

and Corollary 1.2.17 gives the upper bound for $\varrho(G) \geq 3$, and $\delta(G) \leq 3/2$ if $\varrho(G) = 2$.

Let us prove now that the upper bound for $\varrho(G) \geq 3$ is sharp. Fix any even integer $\varrho \geq 4$ and $0 < \varepsilon < \pi/\varrho$, and consider the proper circular-arc graph Γ_ϱ defined as the intersection graph of the family of arcs

$$\begin{aligned}\big\{[e^{2\pi i(j-1)/\varrho}, e^{2\pi ij/\varrho}]\big\}_{j=1}^{\varrho} \cup \big\{[e^{\varepsilon i+2\pi i(j-1)/\varrho}, e^{\varepsilon i+2\pi ij/\varrho}]\big\}_{j=1}^{\varrho/2-1} \cup \big\{[e^{\varepsilon i+\pi i+2\pi i(j-1)/\varrho}, e^{\varepsilon i+\pi i+2\pi ij/\varrho}]\big\}_{j=1}^{\varrho/2-1}\\ \cup [e^{-\varepsilon i+\pi i-2\pi i/\varrho}, e^{-\varepsilon i+\pi i}] \cup [e^{-\varepsilon i+2\pi i-2\pi i/\varrho}, e^{-\varepsilon i+2\pi i}]\\ \cup [e^{-\varepsilon i/2}, e^{\varepsilon i}] \cup [e^{-\varepsilon i/2+\pi i}, e^{\varepsilon i+\pi i}]\\ \cup [e^{-\varepsilon i+\pi i}, e^{\varepsilon i/2+\pi i}] \cup [e^{-\varepsilon i+2\pi i}, e^{\varepsilon i/2+2\pi i}].\end{aligned}$$

Let x (respectively, y) be the midpoint of the edge of Γ_ϱ with endpoints corresponding to the arcs $[e^{-\varepsilon i/2}, e^{\varepsilon i}]$ and $[e^{-\varepsilon i+2\pi i}, e^{\varepsilon i/2+2\pi i}]$ (respectively, $[e^{-\varepsilon i/2+\pi i}, e^{\varepsilon i+\pi i}]$ and $[e^{-\varepsilon i+\pi i}, e^{\varepsilon i/2+\pi i}]$). We have $d_{\Gamma_\varrho}(x,y) = \varrho/2+2$. Let γ_1 and γ_2 be two geodesics in Γ_ϱ joining x and y such that γ_1 contains the corresponding vertices to the arcs

$$[e^{-\varepsilon i/2}, e^{\varepsilon i}] \cup \big\{[e^{\varepsilon i+2\pi i(j-1)/\varrho}, e^{\varepsilon i+2\pi ij/\varrho}]\big\}_{j=1}^{\varrho/2-1} \cup [e^{-\varepsilon i+\pi i-2\pi i/\varrho}, e^{-\varepsilon i+\pi i}] \cup [e^{-\varepsilon i+\pi i}, e^{\varepsilon i/2+\pi i}]$$

and γ_2 contains the corresponding vertices to the arcs

$$[e^{-\varepsilon i/2+\pi i}, e^{\varepsilon i+\pi i}] \cup \big\{[e^{\varepsilon i+\pi i+2\pi i(j-1)/\varrho}, e^{\varepsilon i+\pi i+2\pi ij/\varrho}]\big\}_{j=1}^{\varrho/2-1} \cup [e^{-\varepsilon i+2\pi i-2\pi i/\varrho}, e^{-\varepsilon i+2\pi i}] \\ \cup [e^{-\varepsilon i+2\pi i}, e^{\varepsilon i/2+2\pi i}].$$

Consider the geodesic bigon $\{\gamma_1, \gamma_2\}$. If p is the midpoint of γ_1, then $d_{\Gamma_\varrho}(p,\gamma_2) = d_{\Gamma_\varrho}(x,y)/2 = \varrho/4+1$. Hence, $\varrho/4+1 = d_{\Gamma_\varrho}(p,\gamma_2) \le \delta(\Gamma_\varrho) \le \varrho/4+1$, and we conclude $\delta(\Gamma_\varrho) = \varrho/4+1$.

Assume that $\varrho(G) = 2$. We have proved $\delta(G) \le 3/2$. Seeking for a contradiction assume that $\delta(G) = 3/2$. By Theorem 1.2.13, there exist $T = \{x,y,z\} \in \mathbb{T}_1$ and $p \in [xy]$ with $d_G(p, [xz]\cup[yz]) = \delta(G) = 3/2$. Since we have proved $\operatorname{diam} V(G) \le 2$ and $\operatorname{diam} G \le 3$, we have $d_G(x,y) = 3$, $d_G(p,\{x,y\}) = d_G(p,[xz]\cup[yz]) = 3/2$, $x,y \in J(G)\setminus V(G)$ and $p \in V(G)$. Hence, x (respectively, y) is the midpoint of $u_1u_2 \in E(G)$ with $u_1,u_2 \in V(G)\setminus K$ and corresponding arcs H_1,H_2 (respectively, the midpoint of $w_1w_2 \in E(G)$ with $w_1,w_2 \in V(G)\setminus K$ and corresponding arcs J_1,J_2). Note that each arc H_1,H_2,J_1,J_2 intersects $I_1\cap I_2$ and it is different from I_1 and I_2. Since G is a proper circular-arc graph, we have that both H_1 and H_2 contain the same connected component Λ of $I_1\cap I_2$; also, both J_1 and J_2 contain the other connected component Λ' of $I_1\cap I_2$. Denote by I the corresponding arc to p. Since G is a proper circular-arc graph, we have that I contains either Λ or Λ'. Assume that I contains Λ (if I contains Λ', then the argument is similar). Thus, $d_G(p,u_1) = d_G(p,u_2) = 1$. Without loss of generality we can assume that $u_1 \in [xy]$. Therefore, $u_2 \in [xz]\cup[yz]$ and we conclude $3/2 = d_G(p,[xz]\cup[yz]) \le d_G(p,u_2) = 1$, a contradiction. Hence, $\delta(G) < 3/2$ and Theorem 1.2.13 gives $\delta(G) \le 5/4$.

Finally, we show that the proper circular-arc graph Γ (with $\varrho(G) = 2$) corresponding to the arcs

$$[e^{0i}, e^{\pi i}] \cup [e^{\pi i}, e^{2\pi i}] \cup [e^{-\pi i/8}, e^{\pi i/4}] \cup [e^{-\pi i/4}, e^{\pi i/8}] \cup [e^{\pi i-\pi i/8}, e^{\pi i+\pi i/4}] \cup [e^{\pi i-\pi i/4}, e^{\pi i+\pi i/8}],$$

satisfies $\delta(\Gamma) = 5/4$. Let x (respectively, y) be the midpoint of the edge in Γ with endpoints $[e^{-\pi i/8}, e^{\pi i/4}]$ and $[e^{-\pi i/4}, e^{\pi i/8}]$ (respectively, $[e^{\pi i-\pi i/8}, e^{\pi i+\pi i/4}]$ and $[e^{\pi i-\pi i/4}, e^{\pi i+\pi i/8}]$). We have $d_\Gamma(x,y) = 3$. One can check that there are two geodesics γ_1 and γ_2 such that the midpoint q of γ_1 is a vertex of Γ and $d_\Gamma(q,\gamma_2) = 1$. If p is a point in γ_1 with $d_\Gamma(p,q) = 1/4$, then $d_\Gamma(p,\gamma_2) = 5/4$. If we consider the geodesic bigon $\{\gamma_1,\gamma_2\}$, then $5/4 = d_\Gamma(p,\gamma_2) \le \delta(\Gamma) \le 5/4$, and we conclude $\delta(\Gamma) = 5/4$. □

Note that Theorem 3.1.5 below gives a sufficient condition in order to attain the lower bound of $\delta(G)$ in Theorem 3.1.1. This sufficient condition is, in fact, a characterization when $3 \le \varrho(G) \le 4$.

Next, we are going to characterize the circular-arc graphs with the two smallest possible values for the hyperbolicity constant: 0 and 3/4.

We say that a circular-arc graph G has the 0-*property* if we have either:

(1) G is an interval graph with the 0-intersection property.

(2) $\varrho(G) = 1$ and given two corresponding arcs I, J to vertices in G with $I, J \neq \mathbb{S}^1$, we have $I \cap J = \emptyset$.

(3) $\varrho(G) = 2$ and there exist two corresponding arcs I_1, I_2 to vertices in G with $I_1 \cup I_2 = \mathbb{S}^1$ such any other corresponding arc to some vertex in G intersects just one of the arcs I_1, I_2, and if G_j is the interval graph corresponding to the arcs intersecting I_j then G_j has the 0-intersection property for $j = 1, 2$.

Proposition 3.1.3. *A circular-arc graph G satisfies $\delta(G) = 0$ if and only if G has the* 0*-property.*

Proof. If G is an interval graph, then Theorem 2.1.16 gives the result. Assume now that G is a NI circular-arc graph.

If G satisfies (2) in the definition of 0-property, then G is a tree (in fact, it is a star graph) and we have $\delta(G) = 0$.

If G satisfies (3) in the definition of 0-property, then G is a tree and we have $\delta(G) = 0$.

Assume that $\delta(G) = 0$. Theorem 3.1.1 gives that $\varrho(G) \leq 2$.

Assume that $\varrho(G) = 1$. Seeking for a contradiction assume that there exist two corresponding arcs I, J to vertices in G with $I, J \neq \mathbb{S}^1$ and $I \cap J \neq \emptyset$. Therefore, there exists a cycle with length three corresponding to the arcs $I, J, \mathbb{S}^1$, and Lemma 1.2.20 gives $0 = \delta(G) \geq 3/4$, a contradiction. Thus, we have $I \cap J = \emptyset$ and G has the 0-property.

Assume that $\varrho(G) = 2$. Thus, there exist two corresponding arcs I_1, I_2 to vertices in G with $I_1 \cup I_2 = \mathbb{S}^1$. Seeking for a contradiction assume that there exists a corresponding arc I to some vertex in G intersecting both arcs I_1 and I_2. Therefore, there is a cycle of length 3 in G corresponding to I, I_1, I_2, and we have $\delta(G) \geq 3/4$ by Lemma 1.2.20, which is a contradiction. So, any other corresponding arc to some vertex in G intersects just one of the arcs I_1, I_2. Let G_j be the interval graph corresponding to the arcs intersecting I_j for $j = 1, 2$. Since $\delta(G) = 0$, Proposition 1.2.23 gives that $\delta(G_1) = \delta(G_2) = 0$. Thus, Theorem 2.1.16 gives that G_1 and G_2 have the 0-intersection property. $\square$

We say that a circular-arc graph G has the (3/4)-*property* if we have either:

(1) G is an interval graph with the (3/4)-intersection property.

(2) $\varrho(G) = 1$, there exist two corresponding arcs $I', I'' \neq \mathbb{S}^1$ to vertices in G with $I' \cap I'' \neq \emptyset$, and for every three corresponding arcs $I, J, K \neq \mathbb{S}^1$ to vertices in G, we have either $I \cap J = \emptyset$ or $I \cap K = \emptyset$.

(3) $\varrho(G) = 2$ and there exist two corresponding arcs I_1, I_2 to vertices in G with $I_1 \cup I_2 = \mathbb{S}^1$ such any other corresponding arc to some vertex in G intersects just one of the arcs I_1, I_2, and if G_j is the interval graph corresponding to the arcs intersecting I_j then G_1 has the (3/4)-intersection property and G_2 has either the 0- or the (3/4)-intersection property.

(4) $\varrho(G) = 2$ and there exist three corresponding arcs I, I_1, I_2 to vertices in G with $I_1 \cup I_2 = \mathbb{S}^1$ and $I \cap I_j \neq \emptyset$ for $j = 1, 2$, such that any other arc corresponding to some vertex in G intersects just one of the arcs I_1, I_2 and does not intersect I, and if G_j is the interval graph corresponding to the arcs intersecting I_j except for I then G_j has either the 0- or the (3/4)-intersection property for each $j = 1, 2$.

(5) $\varrho(G) = 3$ and there exist three corresponding arcs I_1, I_2, I_3 to vertices in G with $I_1 \cup I_2 \cup I_3 = \mathbb{S}^1$ such any other corresponding arc to some vertex in G intersects just one of the arcs I_1, I_2, I_3, and if G_j is the interval graph corresponding to the arcs intersecting I_j then G_j has either the 0- or the $(3/4)$-intersection property for each $j = 1, 2, 3$.

Proposition 3.1.4. *A circular-arc graph G satisfies $\delta(G) = 3/4$ if and only if G has the $(3/4)$-property.*

Proof. If G is an interval graph, then Theorem 2.1.16 gives the result. Assume now that G is a NI circular-arc graph.

If G satisfies either (2), (3), (4) or (5) in the definition of $(3/4)$-property, then Theorems 2.1.3 and 2.1.16 and Proposition 1.2.23 give that $\delta(G) = 3/4$.

Assume that $\delta(G) = 3/4$. Theorem 3.1.1 gives $\varrho(G) \leq 3$.

Assume that $\varrho(G) = 3$. Thus, there exist three corresponding arcs I_1, I_2, I_3 to vertices in G with $I_1 \cup I_2 \cup I_3 = \mathbb{S}^1$ Seeking for a contradiction assume that there exists a corresponding arc I to some vertex in G intersecting at least two arcs in $\{I_1, I_2, I_3\}$. Therefore, there is a cycle of length 4 in G corresponding to I, I_1, I_2, I_3 and we have $\delta(G) \geq 1$ by Lemma 1.2.20, which is a contradiction. So, any other corresponding arc to some vertex in G intersects just one of the arcs I_1, I_2, I_3. Let G_j be the interval graph corresponding to the arcs intersecting I_j for $j = 1, 2, 3$. Let us denote by G_0 the subgraph of G induced by the corresponding vertices to I_1, I_2, I_3 (G_0 is a cycle graph with three vertices). Note that $\{G_0, G_1, G_2, G_3\}$ is a T-decomposition of G. Since $\delta(G) = 3/4$, Proposition 1.2.23 gives that $\delta(G_j) \leq 3/4$ for $j = 1, 2, 3$. Thus, Theorem 2.1.16 gives that G_j has either the 0- or the $(3/4)$-intersection property for each $j = 1, 2, 3$, and we obtain condition (5).

Assume that $\varrho(G) = 2$. Thus, there exist two corresponding arcs I_1, I_2 to vertices in G with $I_1 \cup I_2 = \mathbb{S}^1$.

Assume that any other corresponding arc to some vertex in G intersects just one of the arcs I_1, I_2. Let G_j be the interval graph corresponding to the arcs intersecting I_j for $j = 1, 2$, and let G_0 be the subgraph of G induced by the corresponding vertices to I_1, I_2 (G_0 has just an edge). Since $\{G_0, G_1, G_2\}$ is a T-decomposition of G, Proposition 1.2.23 gives

$$\frac{3}{4} = \delta(G) = \max\big\{\delta(G_0), \delta(G_1), \delta(G_2)\big\} = \max\big\{\delta(G_1), \delta(G_2)\big\}.$$

Hence, a subgraph, say G_1, has hyperbolicity constant $3/4$ and $\delta(G_2) \leq 3/4$. Thus, Theorems 2.1.3 and 2.1.16 give that G_1 has the $(3/4)$-intersection property and G_2 has either the 0- or the $(3/4)$-intersection property, and we obtain condition (3).

Assume that there exist corresponding arcs I, I_1, I_2 to vertices in G with $I \cap I_j \neq \emptyset$ for $j = 1, 2$. Seeking for a contradiction assume that there exists another corresponding arc J to some vertex in G intersecting both arcs I_1 and I_2. Hence, there is a cycle of length four in G corresponding to I, I_1, I_2, J and we have $\delta(G) \geq 1$ by Lemma 1.2.20, which is a contradiction. Thus, any other corresponding arc to some vertex in G intersects just one of the arcs I_1, I_2. A similar argument gives that any other corresponding arc to some vertex in G does not intersect I. Let G_j be the interval graph corresponding to the arcs intersecting I_j except for I. Let us denote by G_0 the subgraph of G induced by the corresponding vertices to I, I_1, I_2 (G_0 is a cycle graph with three vertices). Since $\{G_0, G_1, G_2\}$ is a T-decomposition of G, Proposition 1.2.23 gives

$$\frac{3}{4} = \delta(G) = \max\big\{\delta(G_0), \delta(G_1), \delta(G_2)\big\} = \max\Big\{\frac{3}{4}, \delta(G_1), \delta(G_2)\Big\}.$$

This equation holds if and only if $\delta(G_j) \le 3/4$ for $j = 1, 2$. Thus, Theorems 2.1.3 and 2.1.16 give that G_j has either the 0- or the (3/4)-intersection property for each $j = 1, 2$, and we obtain condition (4).

Finally, assume that $\varrho(G) = 1$.

Seeking for a contradiction assume that for every two corresponding arcs $I', I'' \neq \mathbb{S}^1$ to vertices in G, we have $I' \cap I'' = \emptyset$. Thus, G is a star graph and $\delta(G) = 0$, a contradiction. Hence, there exist two arcs $I', I'' \neq \mathbb{S}^1$ with $I' \cap I'' \neq \emptyset$.

Seeking for a contradiction assume that there exist three corresponding arcs $I, J, K \neq \mathbb{S}^1$ to vertices in G with $I \cap J \neq \emptyset$ and $I \cap K \neq \emptyset$. Therefore, there exists a cycle with length four corresponding to the arcs $I, J, K, \mathbb{S}^1$, and Lemma 1.2.20 gives $\delta(G) \ge 1$, a contradiction. Thus, for every three arcs $I, J, K \neq \mathbb{S}^1$ we have either $I \cap J = \emptyset$ or $I \cap K = \emptyset$, and we obtain condition (2). □

We say that a circular-arc graph G with minimum size at least 3 has the *$\varrho(G)$-property* if there exist $\varrho(G)$ corresponding arcs $I_1, \dots, I_{\varrho(G)}$ to vertices in G with $I_1 \cup \dots \cup I_{\varrho(G)} = \mathbb{S}^1$ such any other corresponding arc to some vertex in G intersects just one of the arcs $I_1, \dots, I_{\varrho(G)}$, and if G_j is the interval graph corresponding to the arcs intersecting I_j for $1 \le j \le \varrho(G)$ and $\varrho(G) \le 5$ then:

(1) G_j has either the 0- or (3/4)-intersection property for $1 \le j \le \varrho(G)$ if $\varrho(G) = 3$.

(2) G_j has either the 0-, (3/4)- or 1-intersection property for $1 \le j \le \varrho(G)$ if $\varrho(G) = 4$.

(3) G_j does not have the (3/2)-intersection property for $1 \le j \le \varrho(G)$ if $\varrho(G) = 5$.

The next result gives a sufficient condition in order to attain the lower bound of $\delta(G)$ in Theorem 3.1.1. We also prove that this sufficient condition is, in fact, a characterization when $3 \le \varrho(G) \le 4$.

Theorem 3.1.5. *Let G be a circular-arc graph with $\varrho(G) \ge 3$. If G has the $\varrho(G)$-property, then G satisfies $\delta(G) = \varrho(G)/4$. Furthermore, if $\delta(G) = \varrho(G)/4$ with $3 \le \varrho(G) \le 4$, then G has the $\varrho(G)$-property.*

Proof. Assume first that G has the $\varrho(G)$-property. Let us denote by G_0 the subgraph of G induced by the corresponding vertices to $I_1, \dots, I_{\varrho(G)}$ (G_0 is a cycle graph with $\varrho(G)$ vertices). Since $\{G_0, G_1, \dots, G_{\varrho(G)}\}$ is a T-decomposition of G, Theorems 3.1.1 and 1.2.19 and Proposition 1.2.23 give

$$\frac{\varrho(G)}{4} \le \delta(G) = \max\big\{\delta(G_0), \delta(G_1), \dots, \delta(G_{\varrho(G)})\big\} = \max\Big\{\frac{\varrho(G)}{4}, \delta(G_1), \dots, \delta(G_{\varrho(G)})\Big\}. \qquad (3.1)$$

Since G_j is an interval graph for $1 \le j \le \varrho(G)$, if $\varrho(G) \ge 6$, then Theorem 2.1.16 gives $\delta(G_j) \le 3/2 \le \varrho(G)/4$ for $1 \le j \le \varrho(G)$.

If $\varrho(G) = 3$, then Theorem 2.1.16 gives $\delta(G_j) \le 3/4 = \varrho(G)/4$ for $1 \le j \le \varrho(G)$.

If $\varrho(G) = 4$, then Theorem 2.1.16 gives $\delta(G_j) \le 1 = \varrho(G)/4$ for $1 \le j \le \varrho(G)$.

If $\varrho(G) = 5$, then Theorem 2.1.16 gives $\delta(G_j) \le 5/4 = \varrho(G)/4$ for $1 \le j \le \varrho(G)$.

These inequalities and (3.1) give $\delta(G) = \varrho(G)/4$ in every case.

Assume now that $\delta(G) = \varrho(G)/4$ with $3 \le \varrho(G) \le 4$.

Seeking for a contradiction assume that there exists a corresponding arc I to some vertex in G intersecting at least two arcs in $\{I_1, \dots, I_{\varrho(G)}\}$. Denote by $\{v_1, \dots, v_{\varrho(G)}\}$ their corresponding vertices in G, and by C the cycle in G with vertices $\{v_1, \dots, v_{\varrho(G)}\}$. Let v_I be the corresponding vertex in G to I.

If $\varrho(G) = 3$, then there is a cycle in G with vertices $\{v_1, v_2, v_3, v_I\}$, and Lemma 1.2.20 gives $\delta(G) \ge 1$, a contradiction.

If $\varrho(G) = 4$, then we show now that there is a cycle in G with vertices $\{v_1, v_2, v_3, v_4, v_I\}$. The definition of $\varrho(G)$ gives that v_I is neighbor of at most three vertices in $\{v_1, v_2, v_3, v_4\}$. Without loss of generality we can assume that $v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_I, v_2v_I \in E(G)$ and $v_4v_I \notin E(G)$. Consider the cycle $g := v_1v_I \cup v_Iv_2 \cup v_2v_3 \cup v_3v_4 \cup v_4v_1$ in G. Since $L(g) = 5$ and $\deg_g(v_4) = 2$, Theorem 1.2.21 gives $\delta(G) \ge 5/4$, a contradiction.

Thus, any corresponding arc to some vertex in G intersects just one of the arcs $\{I_1, \dots, I_{\varrho(G)}\}$. Let G_j be the interval graph corresponding to the arcs intersecting I_j for $1 \le j \le \varrho(G)$. Let us denote by G_0 the subgraph of G induced by the corresponding vertices to $I_1, \dots, I_{\varrho(G)}$ for $1 \le j \le \varrho(G)$ (G_0 is a cycle graph with $\varrho(G)$ vertices). Since $\{G_0, G_1, \dots, G_{\varrho(G)}\}$ is a T-decomposition of G, Proposition 1.2.23 and Theorem 1.2.19 give

$$\frac{\varrho(G)}{4} = \delta(G) = \max\left\{\delta(G_0), \delta(G_1), \dots, \delta(G_{\varrho(G)})\right\} = \max\left\{\frac{\varrho(G)}{4}, \delta(G_1), \dots, \delta(G_{\varrho(G)})\right\}.$$

This equation holds if and only if $\delta(G_j) \le \varrho(G)/4$ for $1 \le j \le \varrho(G)$.

If $\varrho(G) = 3$, then Theorem 2.1.16 gives $\delta(G_j) \le 3/4 = \varrho(G)/4$ for $1 \le j \le \varrho(G)$ if and only if (1) holds.

If $\varrho(G) = 4$, then Theorem 2.1.16 gives $\delta(G_j) \le 1 = \varrho(G)/4$ for $1 \le j \le \varrho(G)$ if and only if (2) holds.

Hence, G has the $\varrho(G)$-property. □

Example 3.1.6. *The second statement in Theorem 3.1.5 does not hold for $\varrho \ge 5$, as the following example shows. Consider the graph G_ϱ obtained from the cycle graph C_ϱ with ϱ vertices and an additional vertex v_I connected by an edge with just three consecutive vertices in C_ϱ. We have that G_ϱ is a circular-arc graph without the ϱ-property. [80, Theorem 30] gives that the hyperbolicity constant of any graph with n vertices is at most $n/4$. Hence, $\varrho/4 \le \delta(G_\varrho) \le (\varrho+1)/4$. By using the characterization in [80, Theorem 30] of the graphs with n vertices and hyperbolicity constant $n/4$, we obtain $\delta(G_\varrho) < (\varrho+1)/4$. Since $\delta(G)$ is a multiple of* $1/4$ *by Theorem 1.2.13, we conclude $\delta(G_\varrho) = \varrho/4$.*

3.2 Complement and line graph

In this section we obtain bounds for the hyperbolicity constant of the complement and line of a circular-arc graph, respectively. These theorems improve, for circular-arc graphs, the general bounds for the hyperbolicity constant of the complement and line graphs.

The following result in [14, Theorem 2.2] gives a sharp bound for the hyperbolicity constant of the complement of a graph.

Theorem 3.2.1. *If G is a graph with* $\operatorname{diam} V(G) \geq 3$*, then its complement graph $\overline{G}$ satisfies* $0 \leq \delta(\overline{G}) \leq 2$*.*

We will need the following result in [58, Theorem 10] that improves Theorem 3.2.1 for interval graphs (recall that the most difficult case in the study of the complement of a graph is the set of graphs G with $\operatorname{diam} V(G) = 2$).

Theorem 3.2.2. *If G is an interval graph, then* $0 \leq \delta(\overline{G}) \leq 2$*.*

Now, we are interested in the hyperbolicity of the complement of circular-arc graphs. Let us start with two technical results.

Lemma 3.2.3. *Let G be a circular-arc graph with $\varrho(G) \geq 1$. If two vertices u and v are not neighbors and have two common neighbors v_1, v_2, such that v_1 and v_2 are not neighbors, then their corresponding arcs satisfy $I_u \cup I_v \cup I_{v_1} \cup I_{v_2} = \mathbb{S}^1$.*

Proof. Since $\varrho(G) \geq 1$, $I_u \cap I_v = \emptyset$, $I_{v_1} \cap I_{v_2} = \emptyset$, $I_u \cap I_{v_j} \neq \emptyset$ and $I_v \cap I_{v_j} \neq \emptyset$ for $j = 1, 2$, we have $I_u \cup I_v \cup I_{v_1} \cup I_{v_2} = \mathbb{S}^1$. □

Lemma 3.2.4. *Let G be a circular-arc graph with $\varrho(G) > 4$. If two vertices u and v have two common neighbors v_1, v_2, such that v_1 and v_2 are not neighbors, then u and v are neighbors.*

Proof. Seeking for a contradiction, assume that u and v are not neighbors. Lemma 3.2.3 gives $I_u \cup I_v \cup I_{v_1} \cup I_{v_2} = \mathbb{S}^1$, and thus $\varrho(G) \leq 4$, a contradiction. So, u and v are neighbors. □

Recall that a graph is s-regular if every vertex has degree s, i.e., has s neighbors. In order to prove Theorem 3.2.7 below we need the following surprising result about regular graphs which is interesting by itself.

Theorem 3.2.5. *Let G be a $(n-3)$-regular graph with $n \geq 5$ vertices. Then $\delta(G) = 1$ if $\overline{G}$ is a union of cycle graphs with three vertices, and $\delta(G) = 5/4$ otherwise.*

Proof. Assume first that $\overline{G}$ is a union of cycle graphs with three vertices (thus, $n \geq 6$). [56, Lemma 5.7] gives that $\delta(G) \leq 1$. Since $n \geq 6$, we have $n - 3 \geq n/2$ and there exists a Hamiltonian cycle with $n \geq 6$ vertices; thus, Lemma 1.2.20 gives that $\delta(G) \geq 1$.

Assume now that $\overline{G}$ is not a union of cycle graphs with three vertices. If $n = 5$, then G is a cycle graph with five vertices and $\delta(G) = 5/4$. Assume that $n \geq 6$. Hence, there exists $v \in V(G)$ such that the connected component of $\overline{G}$ containing v is not a cycle graph with three vertices. Let v_1, v_2 be the vertices with $v_1v, v_2v \notin E(G)$. Seeking for a contradiction assume that $v_1v_2 \notin E(G)$. Thus, the connected component of $\overline{G}$ containing v is the cycle graph with vertices v, v_1, v_2, a contradiction. Hence, $v_1v_2 \in E(G)$. Since $n \geq 6$, we have $2(n-3) \geq n$ and there are at least two common neighbors of v and v_j for each $j = 1, 2$. Therefore, there exist two different vertices v_3, v_4 with $v_3v, v_4v, v_3v_1, v_4v_2 \in E(G)$, and we have the cycle g given by v, v_3, v_1, v_2, v_4, v in G. Since $L(g) = 5$ and $\deg_g(v) = 2$ (recall that $v_1v, v_2v \notin E(G)$), Theorem 1.2.21 gives $\delta(G) \geq 5/4$. Finally, Theorem 1.2.22 gives $\delta(G) \leq 5/4$. □

Theorem 3.2.5 has the following direct consequence.

Corollary 3.2.6. *If G is a $(n-3)$-regular graph with $n \geq 5$ vertices and n is not a multiple of 3, then $\delta(G) = 5/4$.*

The following result provides sharp bounds for the hyperbolicity constant of the complement of any circular-arc graph (even the circular-arc graphs G with $\operatorname{diam} V(G) = 2$). Note that it improves Theorem 3.2.1 for circular-arc graphs; recall that the most difficult case in the study of the complement of a graph are the graphs G with $\operatorname{diam} V(G) = 2$ (this is the case if $\varrho(G) = 4$ or $\varrho(G) = 5$), and that Theorem 3.2.1 does not deal with these graphs.

Theorem 3.2.7. *Let G be a circular-arc graph. If $\varrho(G) = 0$, then $0 \leq \delta(\overline{G}) \leq 2$. If $\varrho(G) > 4$, then $5/4 \leq \delta(\overline{G}) \leq 3/2$. If $\varrho(G) = 4$, then $0 \leq \delta(\overline{G}) \leq 7/2$. Furthermore, the lower bounds are sharp; in particular, they are attained by the cycle graphs for $\varrho(G) \geq 4$.*

Proof. If $\varrho(G) = 0$, then Theorem 3.2.2 gives the result.

Assume now that $\varrho(G) > 4$. We are going to prove that $\operatorname{diam} \overline{G} \leq 3$ (note that it is possible to have $\operatorname{diam} V(G) = 2$, and that the inequality $\operatorname{diam} \overline{G} \leq 3$ is stronger than $\operatorname{diam} V(\overline{G}) \leq 3$). Seeking for a contradiction assume that $\operatorname{diam} \overline{G} > 3$.

Assume first that $\operatorname{diam} V(\overline{G}) \geq 4$. Thus, there exist $v, w \in V(\overline{G})$ with $d_{\overline{G}}(v,w) = 4$. Let $v_0 = v, v_1, v_2, v_3, v_4 = w \in V(\overline{G})$ such that $v_{j-1}v_j \in E(\overline{G})$ for $1 \leq j \leq 4$. Therefore, v_0 and v_1 have two common neighbors v_3, v_4 in G with $v_3v_4 \notin E(G)$, and Lemma 3.2.4 gives that v_0 and v_1 are neighbors in G. This contradicts $v_0v_1 \in E(\overline{G})$.

Assume that $\operatorname{diam} V(\overline{G}) = 3$. Thus, there exist $v \in V(\overline{G})$ and a midpoint x of an edge v_3v_3' in $\overline{G}$ with $d_{\overline{G}}(v,x) = 7/2$. Hence, there exist $v_0 = v, v_1, v_2 \in V(\overline{G})$ such that $v_{j-1}v_j \in E(\overline{G})$ for $1 \leq j \leq 3$. Therefore, v_3 and v_3' have two common neighbors v_0, v_1 in G with $v_0v_1 \notin E(G)$, and Lemma 3.2.4 gives that v_3 and v_3' are neighbors in G. This contradicts $v_3v_3' \in E(\overline{G})$.

Hence, $\operatorname{diam} \overline{G} \leq 3$ and Corollary 1.2.17 gives $\delta(\overline{G}) \leq 3/2$.

In order to prove the lower bound, consider a cycle C in G given by $v_1, v_2, \dots, v_{\varrho(G)}, v_1$ such that the subgraph induced by this vertices is C. Consider the cycle g in $\overline{G}$ given by $v_3, v_5, v_2, v_4, v_1, v_3$. Since $L(g) = 5$ and $\deg_g(v_3) = 2$ (recall that $v_3v_2, v_3v_4 \notin E(\overline{G})$), Theorem 1.2.21 gives $\delta(\overline{G}) \geq 5/4$.

Consider now the cycle graph with ϱ vertices C_ϱ. Since $\varrho > 4$, $\overline{C}_\varrho$ is $(\varrho - 3)$-regular and its complement is C_ϱ, Theorem 3.2.5 gives $\delta\big(\overline{C}_\varrho\big) = 5/4$ and the bound is attained.

Finally, assume that $\varrho(G) = 4$ and consider a geodesic triangle $T = \{x, y, z\}$ in $\overline{G}$ and $p \in [xy]$. By Theorem 1.2.13, we can assume that $x, y, z \in J(\overline{G})$. If $d_{\overline{G}}(x,y) \leq 4$, then $d_{\overline{G}}(p, [xz] \cup [yz]) \leq 2 < 7/2$. Assume that $d_{\overline{G}}(x,y) > 4$. Since $x, y, z \in J(\overline{G})$, we have $d_{\overline{G}}(x,y) \geq 9/2$. Thus, there exist $u, v \in V(\overline{G}) \cap [xy]$ with $d_{\overline{G}}(u,v) = 4$ and vertices $v_0 = u, v_1, v_2, v_3, v_4 = v \in V(\overline{G}) \cap [xy]$ such that $v_{j-1}v_j \in E(\overline{G})$ for $1 \leq j \leq 4$ and $d_{\overline{G}}(p, v_2) \leq 1/2$. Therefore, v_0 and v_1 have two common neighbors v_3, v_4 in G, and $v_0v_1, v_3v_4 \notin E(G)$. Hence, Lemma 3.2.3 gives that their corresponding arcs satisfy $I_{v_0} \cup I_{v_1} \cup I_{v_3} \cup I_{v_4} = \mathbb{S}^1$. Seeking for a contradiction assume that there exists a vertex $w_0 \in V(\overline{G})$ with corresponding arc I_{w_0} such that $I_{w_0} \cap I_{v_j} \neq \emptyset$ for $j = 0, 1, 3, 4$. Since $I_{v_0} \cup I_{v_1} \cup I_{v_3} \cup I_{v_4} = \mathbb{S}^1$, there exist $i, j \in \{0, 1, 3, 4\}$ with $I_{w_0} \cup I_{v_i} \cup I_{v_j} = \mathbb{S}^1$. This contradicts $\varrho(G) = 4$, and so every vertex $w_0 \in V(\overline{G})$ has at most three neighbors in $\{v_0, v_1, v_3v_4\}$ in G. Thus, given any vertex $w_0 \in V(\overline{G}) \cap ([xz] \cup [yz])$, there exists $k \in \{0, 1, 3, 4\}$ with $w_0v_k \notin E(G)$, and

$$d_{\overline{G}}(p, [xz] \cup [yz]) \leq d_{\overline{G}}(p, w_0) \leq d_{\overline{G}}(p, v_2) + d_{\overline{G}}(v_2, v_k) + d_{\overline{G}}(v_k, w_0) \leq \frac{1}{2} + 2 + 1 = \frac{7}{2}.$$

So, $\delta(\overline{G}) \le 7/2$.

The lower bound $\delta(\overline{G}) \ge 0$ trivially holds. If we consider the cycle graph $G = C_4$, then $\overline{G}$ is the union of two disjoint edges and $\delta(\overline{G}) = 0$. Hence, the lower bound is attained. $\square$

Remark 3.2.8. *Note that Theorems 1.2.13 and 3.2.7 give that if G is a circular-arc graph with $\varrho(G) > 4$, then we have either $\delta(\overline{G}) = 5/4$ or $\delta(\overline{G}) = 3/2$.*

Theorems 3.1.1 and 3.2.7 have the following consequence.

Corollary 3.2.9. *If G is a circular-arc graph with $\varrho(G) \ge 7$, then $\delta(\overline{G}) < \delta(G)$.*

In 1956, Nordhaus and Gaddum gave lower and upper bounds on the sum and the product of the chromatic number of a graph and its complement in [85]. Since then, relations of a similar type have been proposed for many other graph invariants, in several hundred papers (see, e.g., [4]).

Also, Theorems 3.1.1 and 3.2.7 provide some Nordhaus-Gaddum type results.

Corollary 3.2.10. *If G is a circular-arc graph, then*

$$\begin{aligned} \frac{5\varrho(G)}{16} \le \delta(G)\delta(\overline{G}) &\le \frac{3\varrho(G)}{8} + \frac{9}{4}, & \frac{\varrho(G)+5}{4} \le \delta(G)+\delta(\overline{G}) &\le \frac{\varrho(G)}{4} + 3, & \text{if } \varrho(G) > 4, \\ 0 \le \delta(G)\delta(\overline{G}) &\le \frac{7\varrho(G)}{8} + \frac{21}{4}, & \frac{\varrho(G)}{4} \le \delta(G)+\delta(\overline{G}) &\le \frac{\varrho(G)}{4} + 5, & \text{if } \varrho(G) = 4, \\ 0 \le \delta(G)\delta(\overline{G}) &\le \frac{\varrho(G)}{2} + 3, & \frac{\varrho(G)}{4} \le \delta(G)+\delta(\overline{G}) &\le \frac{\varrho(G)}{4} + \frac{7}{2}, & \text{if } \varrho(G) = 0. \end{aligned}$$

If G is a graph with edges $E(G) = \{e_i\}_{i\in\mathcal{I}}$, the *line graph* $\mathcal{L}(G)$ of G is a graph which has a vertex $v_{e_i} \in V(\mathcal{L}(G))$ for each edge e_i of G, and an edge joining v_{e_i} and v_{e_j} when $e_i \cap e_j \neq \emptyset$. The line graph of G is interesting in the theory of geometric graphs, since it is the intersection graph of $E(G)$.

A graph is *chordal* if all cycles of length at least four have a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle (i.e., it does not have induced cycles of length greater than three).

The following result appears in [20, Lemma 2.2].

Lemma 3.2.11. *Suppose that G is chordal, and that $x_1, x_2, \dots, x_n, x_1$ is a cycle in G, where $n \ge 4$. If $d(x_1, x_3) = 2$, then there exists $i \in \{4, 5, \dots, n\}$ such that $x_i x_2 \in E(G)$.*

We want to prove a similar result for $\mathcal{L}(G)$. In order to do it we need some background.

Given $v_e \in V(\mathcal{L}(G))$, let us define $h(v_e)$ as the midpoint of the edge $e \in E(G)$ and $H(v_e) = e$. Thus, h and H are maps with $h : V(\mathcal{L}(G)) \to G$ and $H : V(\mathcal{L}(G)) \to E(G)$.

[23, Remark 3.3] gives that the map h is an isometry:

Lemma 3.2.12. *For every $x, y \in V(\mathcal{L}(G))$, we have*

$$d_{\mathcal{L}(G)}(x, y) = d_G(h(x), h(y)).$$

Lemma 3.2.13. *Suppose that G is chordal, and that $u_1, u_2, \dots, u_n, u_1$ is a cycle in $\mathcal{L}(G)$, where $n \ge 6$. If $d_{\mathcal{L}(G)}(u_1, u_4) = 3$, then there exists $i \in \{4, 5, \dots, n\}$ and $u \in V(\mathcal{L}(G))$ such that $u_2u, u_3u, u_iu, u_{i+1}u \in E(\mathcal{L}(G))$, where $u_{n+1} = u_1$.*

Proof. Denote by C the cycle $u_1, u_2, \dots, u_n, u_1$ and by C_0 its corresponding cycle in G. Lemma 3.2.12 gives $d_G(h(u_1), h(u_4)) = d_{\mathcal{L}(G)}(u_1, u_4) = 3$. Thus, the vertices $H(u_1) \cap H(u_2)$ and $H(u_3) \cap H(u_4)$ in C_0 satisfy $d_G(H(u_1) \cap H(u_2), H(u_3) \cap H(u_4)) = 2$, and Lemma 3.2.11 gives that there exists $i \in \{4, 5, \dots, n\}$ with $d_G(H(u_2) \cap H(u_3), H(u_i) \cap H(u_{i+1})) = 1$. If we denote by u the corresponding vertex in $\mathcal{L}(G)$ to the edge in G with endpoints $H(u_2) \cap H(u_3)$ and $H(u_i) \cap H(u_{i+1})$, then $u_2u, u_3u, u_iu, u_{i+1}u \in E(\mathcal{L}(G))$. □

The following result in [23, Corollary 3.12] relates the hyperbolicity constants of G and $\mathcal{L}(G)$.

Theorem 3.2.14. *For any graph G we have*

$$\delta(G) \le \delta(\mathcal{L}(G)) \le 5\delta(G) + \frac{5}{2}\,.$$

In order to study the line of circular-arc graphs (see Theorem 3.2.16 below) we need the following result about the line of chordal graphs which is interesting by itself, and improves the upper bound $\delta(\mathcal{L}(G)) \le 5\delta(G) + 5/2$ in Theorem 3.2.14 for chordal graphs (since $\delta(G) \le 3/2$ for every chordal graph G, see [20]).

Theorem 3.2.15. *If G is a chordal graph, then*

$$\delta(\mathcal{L}(G)) \le \frac{5}{2}\,.$$

Proof. Let us consider a geodesic triangle $T = \{x, y, z\}$ in $\mathcal{L}(G)$ and $p \in [xy]$. By Theorem 1.2.13, we can assume that T is a cycle.

We are going to prove $d_{\mathcal{L}(G)}(p, [xz] \cup [yz]) \le 3$. Without loss of generality we can assume that $d_{\mathcal{L}(G)}(p, [xz] \cup [yz]) \ge 2$. If we denote the vertices of the cycle T by $u_1, u_2, \dots, u_n, u_1$, then we can assume that $p \in u_2u_3$. Since $d_{\mathcal{L}(G)}(p, \{x, y\}) \ge d_{\mathcal{L}(G)}(p, [xz] \cup [yz]) \ge 2$, we have $u_1u_2 \cup u_2u_3 \cup u_3u_4 \subset [xy]$ and so, $d_{\mathcal{L}(G)}(u_1, u_4) = 3$. By Lemma 3.2.13, there exist $j \in \{2, 3\}$, $k \in \{4, 5, \dots, n\}$ and $u \in V(\mathcal{L}(G))$ such that $u_2u, u_3u, u_ku \in E(\mathcal{L}(G))$ and $d_T(u_j, u_k) \ge 3$. Since $u_j \in [xy]$, $d_{\mathcal{L}(G)}(u_j, u_k) = 2$ and $d_T(u_j, u_k) \ge 3$, we have $u_k \in [xz] \cup [yz]$ and

$$d_{\mathcal{L}(G)}(p, [xz] \cup [yz]) \le d_{\mathcal{L}(G)}(p, u_k) \le d_{\mathcal{L}(G)}(p, \{u_2, u_3\}) + d_{\mathcal{L}(G)}(\{u_2, u_3\}, u) + d_{\mathcal{L}(G)}(u, u_k) \le \frac{5}{2}\,.$$

Hence, $\delta(\mathcal{L}(G)) \le 5/2$. □

The following result improves the upper bound in Theorem 3.2.14 for circular-arc graphs.

Theorem 3.2.16. *Let G be a circular-arc graph. If $\varrho(G) \ge 3$, then*

$$\frac{1}{4}\,\varrho(G) \le \delta(\mathcal{L}(G)) \le \frac{1}{2}\left\lfloor \frac{1}{2}\,\varrho(G) \right\rfloor + \frac{5}{2}\,.$$

If $\varrho(G) = 0, 2$, then

$$0 \le \delta(\mathcal{L}(G)) \le \frac{5}{2}\,.$$

If $\varrho(G) = 1$, then

$$0 \le \delta(\mathcal{L}(G)) \le 2.$$

Proof. Theorems 3.1.1 and 3.2.14 give the lower bounds.

If $\varrho(G) = 0$, then Theorem 3.2.15 gives the upper bound, since every interval graph is chordal.

Assume that $\varrho(G) > 0$ and let us prove the upper bounds of $\delta(\mathcal{L}(G))$. Fix any set of vertices $K = \{v_1, \dots, v_{\varrho(G)}\}$ and corresponding arcs $\{I_1, \dots, I_{\varrho(G)}\}$ with $I_1 \cup \dots \cup I_{\varrho(G)} = \mathbb{S}^1$.

Assume first $\varrho(G) \geq 3$, and denote by C the cycle in G with $V(C) = K$ and by C' the corresponding cycle in $\mathcal{L}(G)$ to C. Since every vertex in G is at distance at most 1 from C, every vertex in $\mathcal{L}(G)$ is at distance at most 2 from C', and

$$\operatorname{diam} V(\mathcal{L}(G)) \leq 2 + \operatorname{diam} V(C') + 2 = \left\lfloor \frac{1}{2}\, \varrho(G) \right\rfloor + 4,$$
$$\operatorname{diam} \mathcal{L}(G) \leq \left\lfloor \frac{1}{2}\, \varrho(G) \right\rfloor + 5,$$

and Corollary 1.2.17 gives the upper bound.

If $\varrho(G) = 2$, then the previous argument gives the desired upper bound, by taking $v_{v_1 v_2}$ (with diameter zero) instead of C'.

If $\varrho(G) = 1$, then every vertex in G is a neighbor of v_1 and the set of edges in G incident on v_1 corresponds with a complete graph in $\mathcal{L}(G)$. Hence, $\operatorname{diam} V(\mathcal{L}(G)) \leq 3$, $\operatorname{diam} \mathcal{L}(G) \leq 4$ and $\delta(\mathcal{L}(G)) \leq 2$. $\square$

Chapter 4

Domination and hyperbolicity

The idea of domination in graphs was mathematically formalized by Berge [8] and Ore [86] in 1962. Currently, this topic has been detailed in the two, well-known, books by Haynes, Hedetniemi, and Slater (see [52]). The theory of domination in graphs is an area of increasing interest in discrete mathematics and combinatorial computing. Besides of the mathematical and combinatorial importance of the theory, it has been applied successfully in different practical problems such as: analysis of social networks [68], efficient identification of web communities [39], bioinformatics [51], foodwebs [69]. Another application of the concept of domination is the study of the transmission of information in the network associated with defense systems [96].

4.1 Domination on graphs

In 1998 Haynes, Hedetniemi and Slater published a book about domination in graphs (see [52]) which list 1222 papers in this area. There are several variations for domination, usually imposing additional conditions on the set of vertices, the dominant set or the set difference from these. At least 75 such variations are mentioned in [52].

A historical root of domination is the following chess problem. Consider an 8×8 chessboard on which we can move a queen vertically, horizontally or diagonally. We are interested in finding the minimum number of queens with which we can dominate all squares, i.e., every square is either occupied or can be attacked by a queen. To model this problem, consider a graph whose vertices represent the squares of the chess board and two vertices are adjacent if and only if the corresponding squares are separated by a number of squares vertically, horizontally or diagonally. The vertices that correspond to the squares in which these queens would be placed represent a dominating set.

Dominating set and k-dominating set

Given a graph $G = (V, E)$, we say that a subset S of vertices is *dominating* if every vertex in $V(G) \setminus S$ has a neighbour in S. A graph can have several dominating sets, we are interested in

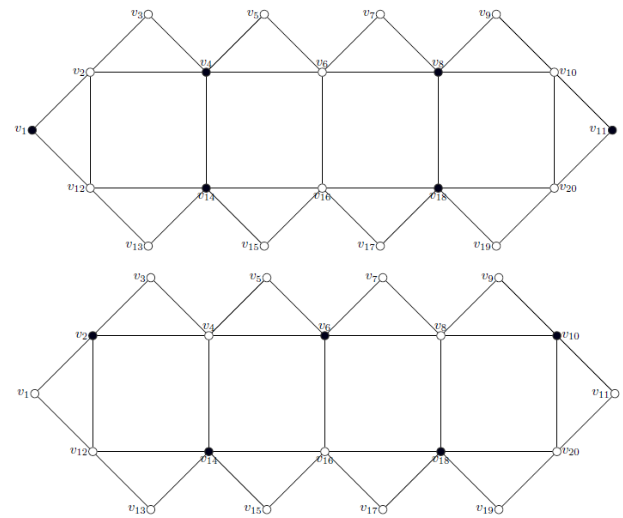


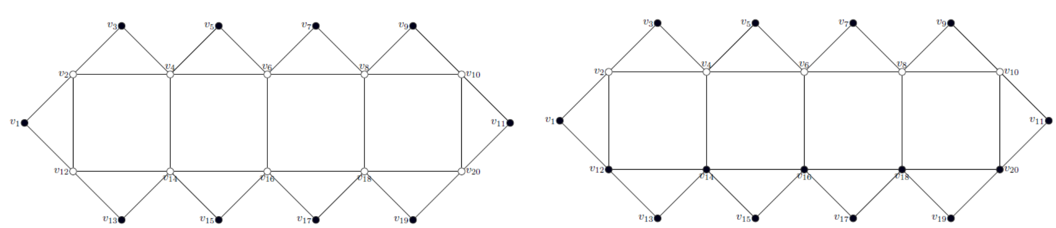
Figure 4.1: Two dominating sets for the same graph G. $\gamma(G) = 5$.

those with minimal cardinality. We define the *domination number* of G as

$$\gamma(G) := \min\left\{\, |S| \;:\; S \text{ is a dominating set of } G \,\right\}.$$

To determine the size of the minimum dominant set is NP-complete, even though we restrict ourselves to certain kinds of graphs such as bipartite graphs and chordal graphs. However, there are certain families of graphs such as trees or interval graphs, for which $\gamma(G)$ can be calculated in polynomial time.

In [29], Cockayne, Gamble and Shepherd defined a generalization of domination in graphs as follows: given a graph G, a set $S \subseteq V$ is a *k-dominating set* if every vertex $v \in V \setminus S$ satisfies $\delta_S(v) \geq k$. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality among all k-dominating sets.



(a) 2-dominating set

(b) 3-dominating set

Figure 4.2: k-dominating sets of the same graph.

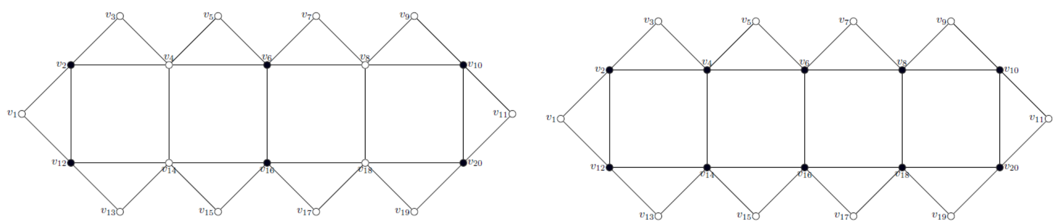
Total dominating set and Total k-dominating set

Given a graph G, we say that a subset of vertices $S \subset V(G)$ is *total dominating* if every vertex in $V(G)$ has a neighbour in S. We define the *total domination number* of G as

$$\gamma_t(G) := \min \{ |S| \,:\, S \text{ is a total dominating set of } G \}.$$

Again the problem of calculating $\gamma_t(G)$ is NP-complete.

A set $S \subseteq V(G)$ is a *total k-dominating set* if every vertex $v \in V(G)$ satisfies $\delta_S(v) \geq k$.
The *total k-domination number* $\gamma_{kt}(G)$ is the minimum cardinality among all total k-dominating sets (see [37, 49, 53]). Note that, $\gamma_t(G) = \gamma_{1t}(G)$ (see [28, 37]).



(a) Total dominating set

(b) Total 2-dominating set.

Figure 4.3: Total k-dominating sets of the same graph.

Distance k-dominating set

In [55] is introduced the concept of distance domination (see also [48], [54], [79]). Given a graph G and $k \geq 1$, we say that a subset of vertices $S \subset V(G)$ is *distance k-dominating* if for any vertex $v \in V(G)$ there is $w \in S$ with $d_G(v,w) \leq k$. Since $d_G(w,w) = 0 \leq k$, we can replace the condition "$d_G(v,w) \leq k$ for any $v \in V(G)$" by "$d_G(v,w) \leq k$ for any $v \in V(G) \setminus S$". Given a graph G, we define the *distance k-domination number* of G as

$$\gamma^k(G) := \min \{ |S| \,:\, S \text{ is a distance } k\text{-dominating set of } G \}.$$

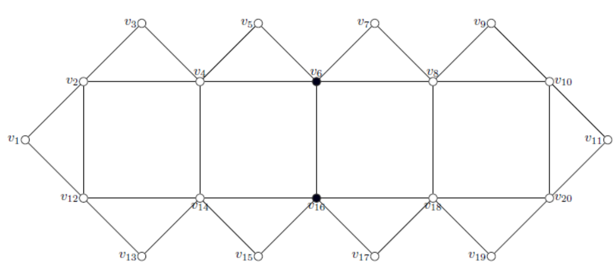


Figure 4.4: A distance 3-dominating set.

Dominating cycles

Given a cycle C in a graph G, we say that C is a *dominating cycle* if every vertex is in the neighbourhood of some vertex on the cycle C.

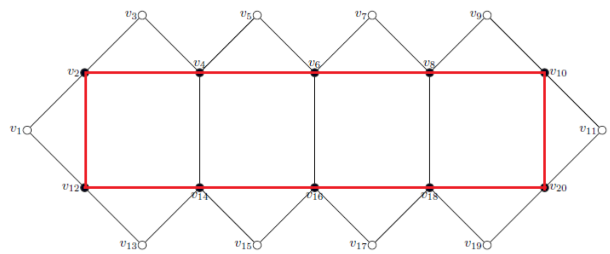


Figure 4.5: Dominating cycle of a graph.

The concept of dominant cycle was introduced by Lesniak and Williamson (see [75]). To find a dominant cycle is a NP-complete problem, even reducing it to planar graphs; however, for a few graph classes, such as circular-arc graphs, there are polynomial time algorithms. Not all graphs have dominating cycles.

4.2 Domination and hyperbolicity

In Section 1.1 and in the classical references on this work (see, e.g., [19, 43]) appear several different definitions of Gromov hyperbolicity, which are equivalent in the sense that if X is δ-hyperbolic with respect to one definition, then it is δ'-hyperbolic with respect to another definition (for some δ' related to δ). We have chosen the Rips definition by its deep geometric meaning [43], however, it is sometimes very useful to use another definition.

Let's recall that if G is a Gromov hyperbolic graph and $(x, z)_w$ denote the Gromov product of $x, y \in G$ with base point $w \in G$ (see Definition 1.1.3), it holds

$$(x, z)_w \geq \min \big\{ (x, y)_w, (y, z)_w \big\} - \delta$$

for every $x, y, z, w \in G$ and some constant $\delta \geq 0$ (see e.g. [3, 43]). Let us denote by $\delta^*(G)$ the sharp constant for this inequality, *i.e.*,

$$\delta^*(G) := \sup \big\{ \min \big\{ (x, y)_w, (y, z)_w \big\} - (x, z)_w : x, y, z, w \in G \big\}. \tag{4.1}$$

From Theorem 1.1.4 (see too [3, 43]) we known that (1.1) is, in fact, equivalent to our definition of Gromov hyperbolicity; furthermore, we have $\delta^*(G) \leq 4\,\delta(G)$ and $\delta(G) \leq 3\,\delta^*(G)$. In [107, Proposition II.20] we found the following improvement of the previous inequality $\delta^*(G) \leq 2\,\delta(G)$.

The following result is elementary (the proof follows the same argument as the Lemma 1.2.18).

Lemma 4.2.1. *If Γ is an isometric subgraph of G, then $\delta^*(\Gamma) \leq \delta^*(G)$.*

In particular, every isometric graph is connected.

We say that a subgraph Γ of G is *distance k-dominating* if $V(\Gamma)$ is distance k-dominating.

Theorem 4.2.2. *Let G be a graph, $k \geq 1$ and Γ an isometric distance k-dominating subgraph of G. Then*

$$\delta^*(\Gamma) \leq \delta^*(G) \leq \delta^*(\Gamma) + 6k + 3.$$

Proof. Lemma 4.2.1 gives the first inequality.

Let f be a projection map $f : G \to \Gamma$, i.e., a map such that $d_G(x, f(x)) = d_G(x, \Gamma)$ for every $x \in G$ (in particular, $f|_\Gamma$ is the identity map). Since Γ an isometric distance k-dominating subgraph, we have $d_G(x, f(x)) \leq k + 1/2$ and

$$\begin{aligned}
(f(x), f(y))_{f(w)} &= \frac{1}{2}\Big(d_\Gamma(f(x), f(w)) + d_\Gamma(f(y), f(w)) - d_\Gamma(f(x), f(y))\Big) \\
&= \frac{1}{2}\Big(d_G(f(x), f(w)) + d_G(f(y), f(w)) - d_G(f(x), f(y))\Big) \\
&\leq \frac{1}{2}\Big(d_G(x, w) + 2k + 1 + d_G(y, w) + 2k + 1 - d_G(x, y) + 2k + 1\Big) \\
&= (x, y)_w + 3k + \frac{3}{2}\,.
\end{aligned}$$

We obtain in a similar way

$$(f(x), f(y))_{f(w)} \geq (x, y)_w - 3k - \frac{3}{2}\,,$$

and thus

$$\begin{aligned}
(x, z)_w &\geq (f(x), f(z))_{f(w)} - 3k - \frac{3}{2} \\
&\geq \min\big\{(f(x), f(y))_{f(w)}, (f(y), f(z))_{f(w)}\big\} - \delta^*(\Gamma) - 3k - \frac{3}{2} \\
&\geq \min\Big\{(x, y)_w - 3k - \frac{3}{2}\,,\ (y, z)_w - 3k - \frac{3}{2}\Big\} - \delta^*(\Gamma) - 3k - \frac{3}{2} \\
&= \min\big\{(x, y)_w, (y, z)_w\big\} - \delta^*(\Gamma) - 6k - 3.
\end{aligned}$$

Hence, we conclude

$$\delta^*(G) \leq \delta^*(\Gamma) + 6k + 3.$$

$\square$

Theorem 4.2.2 has the following consequence.

Theorem 4.2.3. *Let G be a graph, $k \geq 1$ and Γ an isometric distance k-dominating subgraph of G. Then*

$$\delta(\Gamma) \leq \delta(G) \leq 6\,\delta(\Gamma) + 18k + 9.$$

Proof. Lemma 1.2.18 gives the first inequality.

Using the inequalities relating $\delta^*(G)$ and $\delta(G)$ and Theorem 4.2.2, we conclude

$$\delta(G) \leq 3\,\delta^*(G) \leq 3\big(\delta^*(\Gamma) + 6k + 3\big) \leq 6\,\delta(G) + 18k + 9.$$

□

The following example shows that it is not possible to have the inequality

$$\delta(G) \le \Psi\big(\delta(\Gamma)\big),$$

for every graph G and distance k-dominating subgraph Γ (not necessarily isometric) and some function Ψ. For each integer $n > 2k$ consider the cycle graph C_n with vertices $V(C_n) = \{v_1, v_2, \dots, v_{n-1}, v_n\}$ and edges $E(C_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$ and the subgraph Γ_n induced by $\{v_1, \dots, v_{n-2k}\}$. It is clear that $V(\Gamma_n)$ is a distance k-dominating set. Since Γ_n is a tree, $\delta(\Gamma_n) = 0$. However, $\delta(C_n) = n/4$.

It is well-known (see [36, Theorem 4]) that

$$\gamma_t(G) \ge \frac{\operatorname{diam} V(G) + 1}{2}\,.$$

Thus, Corollary 1.2.17 gives the following result.

Proposition 4.2.4. *If G is a graph, then*

$$\delta(G) \le \gamma_t(G)\,.$$

Proposition 4.2.4 can be improved for graphs with small maximum degree.

Theorem 4.2.5. *If G is a graph with maximum degree Δ, then*

$$\delta(G) \le \frac{\Delta}{4}\,\gamma_t(G)\,.$$

Proof. Let $S \subseteq V(G)$ be a total dominating set with $|S| = \gamma_t(G)$, and $n := |V(G)|$. Denote by $\overline{S}$ the complement $\overline{S} := V(G) \setminus S$ of the set S, and by $E_{S,\overline{S}}$ the set of edges joining a vertex in S with a vertex in $\overline{S}$. Since S is a dominating set, $\big|\overline{S}\big| \le \big|E_{S,\overline{S}}\big|$. Since S is a total dominating set, $\big|E_{S,\overline{S}}\big| \le (\Delta - 1)|S|$, and we conclude

$$n - |S| = \big|\overline{S}\big| \le \big|E_{S,\overline{S}}\big| \le (\Delta - 1)|S|, \qquad n \le \Delta\gamma_t(G).$$

The inequality $\delta(G) \le n/4$ (see [80, Theorem 30]) gives $\delta(G) \le \Delta\gamma_t(G)/4$. □

We have similar results for $\gamma^k(G)$.

Theorem 4.2.6. *Let G be a graph and $k \ge 1$. Then*

$$\gamma^k(G) \ge \frac{\operatorname{diam} V(G) + 1}{2k + 1}\,, \qquad \gamma^k(G) \ge \frac{2\delta(G)}{2k + 1}\,.$$

Proof. Let S be a distance k-dominating set of G with $|S| = \gamma^k(G)$, and $\sigma = [uv]$ a geodesic in G with $u, v \in V(G)$ and $d_G(u, v) = \operatorname{diam} V(G)$. Since S is distance k-dominating, there exists $s_1 \in S$ with $d_G(u, s_1) \le k$.

Let $\{u_1, u_2, \dots, u_r, u_{r+1}\} = V(G) \cap \sigma$ with $u_1 = u$, $u_{r+1} = v$, $r = \operatorname{diam} V(G)$ and $u_i u_{i+1} \in E(G)$ for $1 \le i \le r$. Define

$$t_1 := \max \big\{ 1 \le t \le r+1 \;:\; d_G(u_t, s_1) \le k \big\}.$$

Since σ is a geodesic and the diameter of the closed ball $\overline{B_G}(u_1, k)$ is at most $2k$, we have $t_1 \le 2k+1$.

If $r+1 > 2k+1$, then there exists $s_2 \in S$ with $d_G(u_{t_1+1}, s_2) \le k$. Define

$$t_2 := \max \big\{ t_1 + 1 \le t \le r+1 \;:\; d_G(u_t, s_2) \le k \big\}.$$

Thus, $t_2 \le 4k+2$.

If $r+1 > 4k+2$, then we can repeat this process obtaining two finite sequences $\{s_1, \dots, s_j\} \subseteq S$ and $1 \le t_1 < t_2 < \cdots < t_j \le r+1$ with $r+1 \le (2k+1)j$. Hence, we obtain

$$\frac{\operatorname{diam} V(G) + 1}{2k+1} = \frac{r+1}{2k+1} \le j = \big|\{s_1, \dots, s_j\}\big| \le |S| = \gamma^k(G),$$

and Corollary 1.2.17 gives the second inequality. $\square$

We recall that given a graph G, we say that a subset of vertices $S \subset V(G)$ is k-total-dominating ($k \ge 1$) if every vertex $v \in V(G)$ has k neighbors in S. Denote by $\langle S \rangle$ the subgraph of G induced by S. We say that S is *k-total-connected-dominating* if it is k-total-dominating and $\langle S \rangle$ is connected. We define the *k-total-connected-domination number* of G as

$$\gamma_{tc}^k(G) := \min \big\{ |S| : \; S \text{ is a } k\text{-total-connected-dominating set of } G \big\}.$$

As usual, we denote by $\lfloor t \rfloor$ the lower integer part of t, i.e., the largest integer least than or equal to t.

Theorem 4.2.7. *If G is a graph and $k \ge 2$, then*

$$\delta(G) \le \frac{1}{2} \max \left\{ 5, \left\lfloor \frac{3\,\gamma_{tc}^k(G) - 2}{k+1} \right\rfloor + 1 \right\}.$$

Proof. Given a graph G, fix a k-total-connected-dominating set S with $|S| = \gamma_{tc}^k(G)$. Define $s := \operatorname{diam}_{\langle S \rangle} S$ and choose $u, v \in V(S)$ with $d_S(u,v) = s$. For each $0 \le j \le s$, let $n_j := |S_j|$ with $S_j := \{w \in S \;:\; d_S(w,u) = j\}$. Note that a vertex of S_j and a vertex of $S_0 \cup S_1 \cup \cdots \cup S_{j-2}$ can not be neighbors for $2 \le j \le s$. Since $\langle S \rangle$ is connected, we have $\sum_{j=0}^{s} n_j = |S| = \gamma_{tc}^k(G)$, $n_0 = 1$, $n_1 \ge k$ and $n_j \ge 1$ for each $2 \le j \le s$.

Since S is a k-total-connected-dominating set S, if $s < 3$, then $\operatorname{diam}_G V(G) \le s + 2 \le 4$ and Corollary 1.2.17 gives $\delta(G) \le 5/2$. Hence, we can assume that $s \ge 3$.

Define $n_{s+1} := 0$ and

$$a_s := \sum_{j=3}^{s} \big(n_{j-1} + n_j + n_{j+1} \big) = \begin{cases} n_2 + 2n_3 + 3\sum_{j=4}^{s-1} n_j + 2n_s, & \text{if } s > 4, \\ n_2 + 2n_3 + 2n_4, & \text{if } s = 4, \\ n_2 + n_3, & \text{if } s = 3. \end{cases}$$

Note that for any $3 \leq j \leq s$, we have $n_{j-1} + n_j + n_{j+1} \geq k+1$ and so, $a_s \geq (s-2)(k+1)$. Thus,

$$\begin{aligned}
3|S| &= 3\sum_{j=0}^{s} n_j = 3 + 3n_1 + 2n_2 + n_3 + n_s + a_s \\
&\geq 3 + 3k + 2 + 1 + 1 + (s-2)(k+1) = (s+1)(k+1) + 4, \\
\frac{3|S|-4}{k+1} &\geq s+1, \\
\operatorname{diam}_{\langle S\rangle} S &\leq \left\lfloor \frac{3|S|-4}{k+1} \right\rfloor - 1 = \left\lfloor \frac{3\,\gamma_{tc}^k(G)-4}{k+1} \right\rfloor - 1.
\end{aligned}$$

Since S is a k-total-connected-dominating set, we have that $\operatorname{diam}_G V(G) \leq \operatorname{diam}_{\langle S\rangle} S + 2$. Let us assume that $\operatorname{diam}_G V(G) = \operatorname{diam}_{\langle S\rangle} S + 2$. Hence, there exist $u', v' \in V(G) \setminus S$ and $u, v \in S$ with $uu', vv' \in E(G)$ and $\operatorname{diam}_G V(G) = d_G(u', v') = d_S(u, v) + 2$.

For each $-1 \leq j \leq s+1$, let $n_j := |S_{j+1}|$ with $S_j := \{w \in S : d_S(w, u') = j\}$. Using the previous argument, since S is a k-total-connected-dominating set, we have in this case $n_0, n_s \geq k$ and $n_1, n_2, n_3 \geq 1$. Therefore, we deduce

$$\begin{aligned}
3|S| &= 3n_0 + 3n_1 + 2n_2 + n_3 + n_s + a_s \\
&\geq 3k + 3 + 2 + 1 + k + (s-2)(k+1) = (s+2)(k+1) + 2, \\
\operatorname{diam}_{\langle S\rangle} S &\leq \left\lfloor \frac{3\,\gamma_{tc}^k(G)-2}{k+1} \right\rfloor - 2, \\
\operatorname{diam}_G V(G) &\leq \left\lfloor \frac{3\,\gamma_{tc}^k(G)-2}{k+1} \right\rfloor.
\end{aligned}$$

If $\operatorname{diam}_G V(G) < \operatorname{diam}_{\langle S\rangle} S + 2$, then

$$\operatorname{diam}_G V(G) \leq \operatorname{diam}_{\langle S\rangle} S + 1 \leq \left\lfloor \frac{3\,\gamma_{tc}^k(G)-4}{k+1} \right\rfloor \leq \left\lfloor \frac{3\,\gamma_{tc}^k(G)-2}{k+1} \right\rfloor.$$

Hence, we have

$$\operatorname{diam}_G V(G) \leq \max\left\{4, \left\lfloor \frac{3\,\gamma_{tc}^k(G)-2}{k+1} \right\rfloor\right\},$$

and Corollary 1.2.17 gives

$$\delta(G) \leq \frac{1}{2}\max\left\{5, \left\lfloor \frac{3\,\gamma_{tc}^k(G)-2}{k+1} \right\rfloor + 1\right\}.$$

□

The two following results improve Proposition 4.2.4.

Given $s \in \mathbb{R}$, denote by $\lceil s \rceil$ the upper integer part of s, i.e., the smallest integer greater than or equal to s.

Theorem 4.2.8. *If G is a graph, then*

$$\delta(G) \le \begin{cases} \frac{1}{2}\,\gamma_t(G)+1, & \text{if } \gamma_t(G) \le 3, \\ \frac{1}{2}\,\gamma_t(G)+3, & \text{if } \gamma_t(G) \ge 4. \end{cases}$$

Proof. Fix a total dominating set $S \subset V(G)$ with $|S| = \gamma_t(G)$.

Assume first that $\gamma_t(G) \le 3$. Thus, S is a connected set, and we deduce $\operatorname{diam}_G S \le \gamma_t(G)-1$ and $\operatorname{diam}_G V(G) \le \gamma_t(G)+1$. Thus, Corollary 1.2.17 gives $\delta(G) \le \gamma_t(G)/2+1$.

Assume now that $\gamma_t(G) \ge 4$.

By Theorem 1.2.13, there exist a triangle $T=\{x,y,z\}$ that is a cycle with $x,y,z \in J(G)$ and $p \in [xy]$ such that $d_G(p,[xz]\cup[zy]) = \delta(G)$. Let $V(G)\cap[xy] = \{a_1,a_2,\dots,a_r\}$ with $a_ja_{j+1} \in E(G)\cap[xy]$ for $1 \le j < r$, $d_G(a_1,x) \le 1/2$ and $d_G(a_r,y) \le 1/2$. Let $V(G)\cap([xz]\cup[zy]) = \{b_1,b_2,\dots,b_\beta\}$ with $b_jb_{j+1} \in E(G)\cap([xz]\cup[zy])$ for $1 \le j < \beta$, $d_G(b_1,x) \le 1/2$ and $d_G(b_\beta,y) \le 1/2$ (note that $r \le \beta$, since $[xy]$ is a geodesic and $x,y \in J(G)$). Let $1 \le \alpha \le \alpha' \le \beta$ be such that $V(G)\cap[xz] = \{b_1,b_2,\dots,b_\alpha\}$ and $V(G)\cap[zy] = \{b_{\alpha'},b_{\alpha'+1},\dots,b_\beta\}$ (note that $\alpha=\alpha'$ if and only if $z \in V(G)$; otherwise, $\alpha'=\alpha+1$).

If $a_j \in S$, then we define $s_j := a_j$; since S is a total dominating set, if $a_j \notin S$, then there exists $s_j \in N(a_j)\cap S$. If $b_j \in S$, then we define $\overline{s}_j := b_j$; since S is a total dominating set, if $b_j \notin S$, then there exists $\overline{s}_j \in N(b_j)\cap S$.

We are going to define subsets $S_1,S_2 \subset S$ associated to $[xy]$ and $[xz]\cup[zy]$, respectively.

Since $[xy]$ is a geodesic, if $s_i = s_j$, then $|i-j| \le 2$. Let $\mathfrak{I}$ be the set

$$\mathfrak{I} := \big\{1 \le i \le r-2 :\ s_i = s_{i+1} = s_{i+2}\big\}.$$

If $\mathfrak{I} = \emptyset$, then

$$\big|\{s_1,s_2,\dots,s_r\}\big| \ge \left\lceil \frac{r}{2} \right\rceil.$$

Hence, the set $S_1 := \{s_1,s_2,\dots,s_r\}$ satisfies $|S_1| \ge \lceil r/2 \rceil$.

Since S is a total dominating set, if $\mathfrak{I} \ne \emptyset$ and $i \in \mathfrak{I}$, then there exists $s_i' \in N(s_i)\cap S$. Assume that $i,j \in \mathfrak{I}$ with $i \ne j$ (without loss of generality we can assume that $i<j$, and thus $i+3 \le j$); then $s_i' \ne s_j'$, since otherwise $5 = i+3+2-i \le j+2-i = d_G(a_i,a_{j+2}) \le d_G(a_i,s_i) + d_G(s_i,s_i') + d_G(s_j',s_{j+2}) + d_G(s_{j+2},a_{j+2}) \le 4$, a contradiction. Note that $s_i' \notin \{a_i,a_{i+1},a_{i+2}\}$; also, $s_i' \notin \{a_1,a_2,\dots,a_r\}$, since otherwise $s_i \in N(a_i)\cap N(a_{i+1})\cap N(a_{i+2})\cap N(s_i')$, a contradiction. Besides, $s_i' \ne s_j$ if $s_j = s_{j+1}$ and $\{i,i+1,i+2\}\cap\{j,j+1\} = \emptyset$. Furthermore, there exists at most one j with $s_i' = s_j$, $j \notin \{i,i+1,i+2\}$ and $s_{j-1} \ne s_j \ne s_{j+1}$. Thus,

$$\big|\cup_{i\in\mathfrak{I}}\{s_i'\}\cup\{s_1,s_2,\dots,s_r\}\big| \ge \left\lceil \frac{r}{2} \right\rceil.$$

Therefore, the set $S_1 := \cup_{i\in\mathfrak{I}}\{s_i'\}\cup\{s_1,s_2,\dots,s_r\}$ satisfies $|S_1| \ge \lceil r/2 \rceil$ in both cases.

Next, we define a similar set associated to $[xz]\cup[zy]$.

Given $v_1,v_2,\dots,v_k \in V(G)$ such that for each $1 \le j < k$ we have either $v_jv_{j+1} \in E(G)$ or $v_j = v_{j+1}$, we denote by $v_1v_2\cdots v_k$ the path containing the edges (or vertices) v_jv_{j+1} for $1 \le j < k$.

Let us consider the sets

$$\begin{aligned}
\Gamma_0 &:= \big\{\gamma \text{ path } \subset G:\ \gamma = b_1 b_2 \cdots b_\beta \big\},\\
\Gamma_1 &:= \big\{\gamma \text{ path } \subset G:\ \gamma = b_1 b_2 \cdots b_i \overline{s}_i b_j \cdots b_\beta \text{ if } \overline{s}_i = \overline{s}_j \big\},\\
\Gamma_2 &:= \big\{\gamma \text{ path } \subset G:\ \gamma = b_1 b_2 \cdots b_i \overline{s}_i \overline{s}_j b_j \cdots b_\beta \text{ if } \overline{s}_i \overline{s}_j \in E(G) \big\},\\
\Gamma_3 &:= \big\{\gamma \text{ path } \subset G:\ \gamma = b_1 b_2 \cdots b_i \overline{s}_i \overline{s}_0 \overline{s}_j b_j \cdots b_\beta \text{ if } \exists\, \overline{s}_0 \in S \cap N(\overline{s}_i) \cap N(\overline{s}_j) \big\},\\
\Gamma &:= \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3.
\end{aligned}$$

Let us choose $\sigma \in \Gamma$ with

$$L(\sigma) = \min \big\{ L(\gamma) :\ \gamma \in \Gamma \big\}.$$

Since σ joins b_1 and b_β, we have that $|\sigma \cap V(G)| \geq r$. Let i_0, j_0 be the integers such that $1 \leq i_0 \leq \alpha \leq \alpha' \leq j_0 \leq \beta$, $b_1, \ldots, b_{i_0}, b_{j_0}, \ldots, b_\beta \in \sigma$, $b_{i_0+1} \notin \sigma \cap [xz]$ and $b_{j_0-1} \notin \sigma \cap [zy]$.

Let us define the set

$$\overline{\mathfrak{I}} := \big\{ 1 \leq i \leq i_0 - 2 :\ \overline{s}_i = \overline{s}_{i+1} = \overline{s}_{i+2} \big\} \cup \big\{ j_0 \leq i \leq \beta - 2 :\ \overline{s}_i = \overline{s}_{i+1} = \overline{s}_{i+2} \big\}. \tag{4.2}$$

Since S is a total dominating set, if $i \in \overline{\mathfrak{I}}$, then there exists $\overline{s}_i' \in N(\overline{s}_i)$.

Case A. Assume that $\sigma \notin \Gamma_0$.

Since $\sigma \notin \Gamma_0$, the minimality of σ gives $\overline{s}_{i_0} \neq \overline{s}_i$ for every $1 \leq i < i_0$ and $\overline{s}_{j_0} \neq \overline{s}_j$ for every $j_0 < j \leq \beta$; in particular, this gives $i_0 - 2, j_0 \notin \overline{\mathfrak{I}}$, and we can write

$$\overline{\mathfrak{I}} = \big\{ 1 \leq i < i_0 - 2 :\ \overline{s}_i = \overline{s}_{i+1} = \overline{s}_{i+2} \big\} \cup \big\{ j_0 < i \leq \beta - 2 :\ \overline{s}_i = \overline{s}_{i+1} = \overline{s}_{i+2} \big\}. \tag{4.3}$$

If $i, j \in \overline{\mathfrak{I}}$ with $i \neq j$ and either $1 \leq i, j < i_0 - 2$ or $j_0 < i, j \leq \beta - 2$, then the argument in the case of S_1 gives $\overline{s}_i' \neq \overline{s}_j'$. If $1 \leq i < i_0 - 2$ and $j_0 < j \leq \beta - 2$, then the minimality of σ gives $\overline{s}_i' \neq \overline{s}_j'$.

Also, the minimality of σ gives $\overline{s}_i' \neq \overline{s}_j$ if $i \in \overline{\mathfrak{I}}$ with $1 \leq i \leq i_0 - 2$ and $j_0 \leq j \leq \beta$, and $\overline{s}_i \neq \overline{s}_j'$ if $j \in \overline{\mathfrak{I}}$ with $1 \leq i \leq i_0$ and $j_0 \leq j \leq \beta - 2$.

If $1 \leq i < i_0$ and $j_0 < j \leq \beta$, then the minimality of σ also gives $\overline{s}_i \neq \overline{s}_j$, $\overline{s}_{i_0} \neq \overline{s}_j$ and $\overline{s}_i \neq \overline{s}_{j_0}$. Note that in the paths $b_i \overline{s}_i b_j$ (if $\sigma \in \Gamma_1$ and $\overline{s}_i = \overline{s}_j$), $b_i \overline{s}_i \overline{s}_j b_j$ (if $\sigma \in \Gamma_2$ and $\overline{s}_i \overline{s}_j \in E(G)$), and $b_i \overline{s}_i \overline{s}_0 \overline{s}_j b_j$ (if $\sigma \in \Gamma_3$ and there exists $\overline{s}_0 \in S \cap N(\overline{s}_i) \cap N(\overline{s}_j)$), the cardinal of the vertices in S plus 1 is greater than or equal to the cardinal of the points in $V(G) \setminus S$. Thus, the set $S_2 := \cup_{i \in \overline{\mathfrak{I}}} \{\overline{s}_i'\} \cup \{\overline{s}_1, \overline{s}_2, \ldots, \overline{s}_r\}$ satisfies $|S_2| \geq \lceil (r-1)/2 \rceil$.

Case B. Assume that $\sigma \in \Gamma_0$.

Note that $\overline{\mathfrak{I}}$ is defined by (4.2); since $\sigma \in \Gamma_0$, (4.3) can be false.

As in Case A, let us define $S_2 := \cup_{i \in \overline{\mathfrak{I}}} \{\overline{s}_i'\} \cup \{\overline{s}_1, \overline{s}_2, \ldots, \overline{s}_r\}$.

Assume that $z \notin V(G)$, since the argument when $z \in V(G)$ is analogous. Thus, $i_0 = \alpha$ and $j_0 = \alpha' = \alpha + 1$.

The minimality of σ gives the following six facts:

$\overline{s}_i \neq \overline{s}_j$ for every $1 \leq i < i_0$ and $j_0 < j \leq \beta$.

$\overline{s}_{i_0} \neq \overline{s}_j$ for every $j_0 + 2 \leq j \leq \beta$ and $\overline{s}_{j_0} \neq \overline{s}_i$ for every $1 \leq i \leq i_0 - 2$.

$\overline{s}_i' \neq \overline{s}_j'$ if $i, j \in \overline{\mathfrak{I}}$ with $i \neq j$ and either $1 \leq i, j \leq i_0 - 2$ or $j_0 \leq i, j \leq \beta - 2$.

$\overline{s}_i' \neq \overline{s}_j'$ if $i, j \in \overline{\mathfrak{I}}$ with $1 \leq i \leq i_0 - 2$ and $j_0 \leq j \leq \beta - 2$.

$\overline{s}_i' \neq \overline{s}_j$ if $i \in \overline{\mathfrak{I}}$ with $1 \leq i \leq i_0 - 2$ and $j_0 < j \leq \beta$.

$\overline{s}_i \neq \overline{s}'_j$ if $j \in \overline{\mathfrak{I}}$ with $1 \leq i < i_0$ and $j_0 \leq j \leq \beta - 2$.

Case B.1. If $\overline{s}_i \neq \overline{s}_j$ for every $1 \leq i \leq i_0$ and $j_0 \leq j \leq \beta$, $\overline{s}'_i \neq \overline{s}_j$ for every $i \in \overline{\mathfrak{I}}$ with $1 \leq i \leq i_0 - 2$ and $j_0 \leq j \leq \beta$, and $\overline{s}_i \neq \overline{s}'_j$ for every $j \in \overline{\mathfrak{I}}$ with $1 \leq i \leq i_0$ and $j_0 \leq j \leq \beta - 2$, then $|S_2| \geq \lceil r/2 \rceil$.

Case B.2. Assume that we are not in Case B.1. We have five different cases:

Case B.2.1. $i_0 - 2 \in \overline{\mathfrak{I}}$ and $\overline{s}'_{i_0} = \overline{s}_{j_0}$. The minimality of σ gives $\overline{s}_i \neq \overline{s}_j$ for every $1 \leq i \leq i_0$ and $j_0 \leq j \leq \beta$, and $\overline{s}_i \neq \overline{s}'_j$ for every $j \in \overline{\mathfrak{I}}$ with $1 \leq i \leq i_0$ and $j_0 \leq j \leq \beta - 2$. Besides, the two vertices $\overline{s}_{i_0}$ and $\overline{s}'_{i_0} = \overline{s}_{j_0}$ in S_2 are associated to the four vertices $b_{i_0-2}, b_{i_0-1}, b_{i_0}, b_{j_0}$. Hence, we also conclude that $|S_2| \geq \lceil r/2 \rceil$.

Case B.2.2. $j_0 \in \overline{\mathfrak{I}}$ and $\overline{s}'_{j_0} = \overline{s}_{i_0}$. A symmetric argument to the one in the previous case also gives $|S_2| \geq \lceil r/2 \rceil$.

Case B.2.3. $\overline{s}_{i_0} = \overline{s}_{j_0+1}$ and $\overline{s}_{i_0-1} \neq \overline{s}_{j_0}$. The minimality of σ gives $\overline{s}_i \neq \overline{s}_{i_0}$ for every $1 \leq i < i_0$ and $\overline{s}_{j_0+1} \neq \overline{s}_j$ for every $j_0 + 2 \leq j \leq \beta$. Thus, $i_0 - 2, j_0 \notin \overline{\mathfrak{I}}$ and we conclude $\overline{s}'_i \neq \overline{s}_j$ if $i \in \overline{\mathfrak{I}}$ with $1 \leq i \leq i_0 - 2$ and $j_0 \leq j \leq \beta$, and $\overline{s}_i \neq \overline{s}'_j$ if $j \in \overline{\mathfrak{I}}$ with $1 \leq i \leq i_0$ and $j_0 \leq j \leq \beta - 2$. The vertex $\overline{s}_{i_0} = \overline{s}_{j_0+1} \in S_2$ is associated to the three vertices $b_{i_0}, b_{j_0}, b_{j_0+1}$, and so, twice the cardinal of the vertices in S_2 plus 1 is greater than or equal to the cardinal of the points in $\{b_1, b_2, \dots, b_\beta\}$. Hence, $|S_2| \geq \lceil (r-1)/2 \rceil$.

Case B.2.4. $\overline{s}_{i_0-1} = \overline{s}_{j_0}$ and $\overline{s}_{i_0} \neq \overline{s}_{j_0+1}$. A symmetric argument to the one in the previous case also gives $|S_2| \geq \lceil (r-1)/2 \rceil$.

Case B.2.5. $\overline{s}_{i_0-1} = \overline{s}_{j_0}$ and $\overline{s}_{i_0} = \overline{s}_{j_0+1}$. The minimality of σ gives $\overline{s}_{i_0-1} \neq \overline{s}_{j_0+1}$. A similar argument to the one in Case B.2.3 (now, with the two vertices $\overline{s}_{i_0-1} = \overline{s}_{j_0}, \overline{s}_{i_0} = \overline{s}_{j_0+1} \in S_2$ associated to the four vertices $b_{i_0-1}, b_{i_0}, b_{j_0}, b_{j_0+1}$) gives $|S_2| \geq \lceil r/2 \rceil$.

Hence, we have in every case $|S_2| \geq \lceil (r-1)/2 \rceil$.

We consider several cases.

(1) Assume first that $S_1 \cap S_2 = \emptyset$. Thus,

$$\gamma_t(G) = |S| \geq |S_1| + |S_2| \geq \left\lceil \frac{r}{2} \right\rceil + \left\lceil \frac{r-1}{2} \right\rceil = r.$$

Since $x, y \in J(G)$ and $|\,[xy] \cap V(G)| = r$, we conclude $L([xy]) \leq r \leq \gamma_t(G)$, and

$$\delta(G) = d_G(p, [xz] \cup [zy]) \leq d_G(p, \{x, y\}) \leq \frac{1}{2}\, L([xy]) \leq \frac{1}{2}\, \gamma_t(G)\,.$$

(2) Assume now that $S_1 \cap S_2 \neq \emptyset$.

(2.1) Assume that $d_G(p, [xz] \cup [zy]) \leq 5$. Thus,

$$\delta(G) = d_G(p, [xz] \cup [zy]) \leq \frac{4}{2} + 3 \leq \frac{1}{2}\, \gamma_t(G) + 3,$$

since $\gamma_t(G) \geq 4$.

(2.2) Assume that $d_G(p, [xz] \cup [zy]) > 5$. If $p = a_l \in V(G)$, then S_2 does not intersect the subset of S_1 associated to $\{a_l\}$ (i.e., s_l and perhaps s'_l); and if $p \notin V(G)$, then $p \in a_l a_{l+1} \in E(G)$ and

S_2 does not intersect the subset of S_1 associated to $\{a_l, a_{l+1}\}$ (i.e., s_l, s_{l+1} and perhaps s'_l and/or s'_{l+1}). Thus, there exists a maximal connected subset $\mathcal{A} := \{a_{i_1}, a_{i_1+1}, \dots, a_{i_2-1}, a_{i_2}\}$ of $[xy] \cap V(G)$ (with respect to the inclusion) such that $p \in [a_{i_1} a_{i_2}]$ and $S_1(\mathcal{A}) \cap S_2 = \emptyset$, where $S_1(\mathcal{A})$ is the subset of S_1 associated to $\mathcal{A}$.

Fix a positive integer u.

(2.2.1) If $i_1 \ge u+1$ and $i_2 \le r-u$, then $|\sigma \cap V(G)| \ge r \ge |\mathcal{A}| + 2u$ and

$$\begin{aligned}\gamma_t(G) = |S| \ge |S_1(\mathcal{A})| + |S_2| &\ge \left\lceil \frac{1}{2}\, |\mathcal{A}| \right\rceil + \left\lceil \frac{1}{2} \left(|\mathcal{A}| + 2u - 1 \right) \right\rceil \\ &\ge \left\lceil \frac{1}{2}\, |\mathcal{A}| \right\rceil + \left\lceil \frac{1}{2} \left(|\mathcal{A}| - 1 \right) \right\rceil + u = |\mathcal{A}| + u.\end{aligned}$$

The maximality of $\mathcal{A}$ gives $d_G(a_{i_1-1}, [xz] \cup [zy]) \le 4$ and $d_G(a_{i_2+1}, [xz] \cup [zy]) \le 4$. Let g_1 (respectively, g_2) be a geodesic in G joining a_{i_1-1} (respectively, a_{i_2+1}) and $[xz] \cup [zy]$, and ρ the curve

$$\rho := g_1 \cup a_{i_1-1} a_{i_1} \cdots a_{i_2} a_{i_2+1} \cup g_2.$$

Since ρ joins two points in $[xz] \cup [zy]$, $p \in \rho$ and $L(\rho) \le 4 + |\mathcal{A}| + 1 + 4$, we have

$$\delta(G) = d_G(p, [xz] \cup [zy]) \le \frac{1}{2}\, L(\rho) \le \frac{1}{2}\, |\mathcal{A}| + \frac{9}{2} \le \frac{1}{2}\, \gamma_t(G) + \frac{9-u}{2}\,.$$

(2.2.2) If $i_1 \le u$ and $i_2 \ge r-u+1$, then $|\sigma \cap V(G)| \ge r \ge |\mathcal{A}| + 1$ (since $S_1 \cap S_2 \neq \emptyset$) and

$$\gamma_t(G) = |S| \ge |S_1(\mathcal{A})| + |S_2| \ge \left\lceil \frac{1}{2}\, |\mathcal{A}| \right\rceil + \left\lceil \frac{1}{2}\, |\mathcal{A}| \right\rceil \ge |\mathcal{A}|.$$

We also have

$$\begin{aligned} d_G(a_{i_1}, x) &\le d_G(a_{i_1}, a_1) + d_G(a_1, x) \le u - 1 + \frac{1}{2}\,, \\ d_G(a_{i_2}, y) &\le d_G(a_{i_2}, a_r) + d_G(a_r, y) \le u - 1 + \frac{1}{2}\,, \\ L([xy]) &= d_G(x, a_{i_1}) + |\mathcal{A}| - 1 + d_G(a_{i_2}, y) \le \gamma_t(G) + 2u - 2\,, \\ \delta(G) = d_G(p, [xz] \cup [zy]) &\le d_G(p, \{x, y\}) \le \frac{1}{2}\, L([xy]) \le \frac{1}{2}\, \gamma_t(G) + u - 1. \end{aligned}$$

(2.2.3) If $i_1 \le u$ and $i_2 \le r-u$, then $|\sigma \cap V(G)| \ge r \ge |\mathcal{A}| + u$ and

$$\begin{aligned}\gamma_t(G) = |S| \ge |S_1(\mathcal{A})| + |S_2| &\ge \left\lceil \frac{1}{2}\, |\mathcal{A}| \right\rceil + \left\lceil \frac{1}{2} \left(|\mathcal{A}| + u - 1 \right) \right\rceil \\ &\ge \left\lceil \frac{1}{2}\, |\mathcal{A}| \right\rceil + \left\lceil \frac{1}{2} \left(|\mathcal{A}| - 1 \right) \right\rceil + \left\lfloor \frac{u}{2} \right\rfloor = |\mathcal{A}| + \left\lfloor \frac{u}{2} \right\rfloor.\end{aligned}$$

The maximality of $\mathcal{A}$ gives $d_G(a_{i_2+1}, [xz] \cup [zy]) \le 4$. Let g be a geodesic in G joining a_{i_2+1} and $[xz] \cup [zy]$, and ρ the curve

$$\rho := [x a_{i_1}] \cup a_{i_1} \cdots a_{i_2} a_{i_2+1} \cup g.$$

Thus,

$$d_G(a_{i_1}, x) \le d_G(a_{i_1}, a_1) + d_G(a_1, x) \le u - 1 + \frac{1}{2}\,,$$
$$L(\rho) \le u - 1 + \frac{1}{2} + |\mathcal{A}| + 4 = u + \frac{7}{2} + |\mathcal{A}|.$$

Since ρ joins two points in $[xz] \cup [zy]$ and $p \in \rho$, we have

$$\delta(G) = d_G(p, [xz] \cup [zy]) \le \frac{1}{2}\, L(\rho) \le \frac{1}{2}\Big(u + \frac{7}{2} + |\mathcal{A}|\Big) \le \frac{1}{2}\,\gamma_t(G) + \frac{1}{2}\Big(\frac{7}{2} + u - \Big\lfloor \frac{u}{2} \Big\rfloor\Big).$$

(2.2.4) If $i_1 \ge u+1$ and $i_2 \ge r - u + 1$, then a similar argument to the previous one in (2.2.3) gives the same inequality for $\delta(G)$.

Since the function

$$F(u) := \max\Big\{\, \frac{9-u}{2}\,,\, u-1,\, \frac{1}{2}\Big(\frac{7}{2} + u - \Big\lfloor \frac{u}{2} \Big\rfloor\Big)\Big\},$$

with $u \in \mathbb{Z}^+$, attains its minimum value 3 for $u = 3$ and $u = 4$, we have

$$\delta(G) \le \frac{1}{2}\,\gamma_t(G) + 3.$$

$\square$

The following example shows that Theorem 4.2.8 is asymptotically sharp.

For each integer $k \ge 1$ consider the cycle graph C_{4k} with vertices $V(C_{4k}) = \{v_1, v_2, \dots, v_{4k-1}, v_{4k}\}$ and edges $E(C_{4k}) = \{v_1v_2, v_2v_3, \dots, v_{4k-1}v_{4k}, v_{4k}v_1\}$. Given points $x, y \notin V(C_{4k})$, let G_k be the graph with

$$V(G_k) = \{x, y\} \cup V(C_{4k}),$$
$$E(G_k) = \{xv_1, xv_{4k}, yv_{2k}, yv_{2k+1}\} \cup E(C_{4k}).$$

Consider the geodesics g_1, g_2 in G_k joining x and y with $g_1 \cap g_2 = \{x, y\}$. If p is the midpoint of g_1, then Corollary 1.2.17 gives

$$\frac{1}{2}\operatorname{diam} G_k \ge \delta(G_k) \ge d_{G_k}(p, g_2) = d_{G_k}(p, \{x, y\}) = \frac{1}{2}\, L(g_1) = k + \frac{1}{2} = \frac{1}{2}\operatorname{diam} G_k,$$

and we conclude $\delta(G_k) = k + 1/2$. [70] gives $\gamma_t(C_{4k}) = 2k$, and one can check that $\gamma_t(G_k) = \gamma_t(C_{4k}) = 2k$. Hence, $\delta(G_k) = k + 1/2 = \gamma_t(G_k)/2 + 1/2$.

One can think that perhaps it is possible to obtain an upper bound of $\gamma_t(G)$ in terms of $\delta(G)$, i.e., the inequality

$$\gamma_t(G) \le \Psi\big(\delta(G)\big), \tag{4.4}$$

for every graph G and some function Ψ. However, this is not possible, as the following example shows. For each integer $n \ge 2$ consider the path graph P_n. Since P_n is a tree, $\delta(P_n) = 0$, but $\lim_{n\to\infty} \gamma_t(P_n) = \infty$.

However, we can obtain (4.4) for a kind of graphs.

Theorem 4.2.9. *If G is a graph with an isometric dominating cycle C, then*

$$\gamma_t(G) \leq 4\delta(G).$$

Proof. Since C is a dominating cycle, $C \cap V(G)$ is a total dominating set and $\gamma_t(G) \leq |\,C \cap V(G)| = L(C) = 4\delta(C)$. Since C is an isometric subgraph of G, Lemma 1.2.18 gives the inequality. $\square$

Theorem 4.2.10. *If G is a graph with a dominating cycle C, then*

$$\delta(G) \leq \frac{1}{2}\left\lfloor \frac{L(C)}{2} \right\rfloor + \frac{3}{2},$$

and the inequality is sharp.

Proof. Since C is a dominating cycle, we have

$$\operatorname{diam} V(G) \leq \operatorname{diam} V(C) + 2 = \left\lfloor \frac{L(C)}{2} \right\rfloor + 2,$$

and Corollary 1.2.17 gives the inequality. [99, Theorem 3.1] gives that the inequality is sharp. $\square$

Proposition 4.2.11. *If G is a graph with no induced C_4 or P_4, then*

$$\delta(G) \leq \frac{3}{2}.$$

Proof. Since G is a graph with no induced C_4 or P_4, [111] (see also [35, Theorem 1]) gives that G has a dominating vertex. Thus, $\operatorname{diam} V(G) \leq 2$ and Corollary 1.2.17 gives the inequality. $\square$

This result can be improved as follows.

Theorem 4.2.12. *If G is a graph with no induced P_4, then*

$$\delta(G) \leq \frac{5}{4},$$

and the inequality is sharp.

Proof. Seeking for a contradiction assume that $\operatorname{diam} V(G) > 2$. Thus, there exist $u, v \in V(G)$ with $d_G(u, v) = 3$. Let $u', v' \in V(G)$ with $uu', u'v', v'v \in E(G)$. Since $uu'v'v$ is a P_4 on G, it is not induced and so, $d_G(u, v) < 3$, a contradiction. Hence, $\operatorname{diam} V(G) \leq 2$, $\operatorname{diam} G \leq 3$ and Corollary 1.2.17 gives $\delta(G) \leq 3/2$.

Seeking for a contradiction assume that $\delta(G) > 5/4$. Thus, Theorem 1.2.14 gives $\delta(G) = 3/2$. By Theorem 1.2.13, there exists a geodesic triangle $T = \{x, y, z\}$ that is a cycle with $x, y, z \in J(G)$ and $\delta(T) = 3/2 = d_G(p, [yz] \cup [zx])$ for some $p \in [xy]$. Then $d_G(p, \{x, y\}) \geq d_G(p, [yz] \cup [zx]) = 3/2$ and $d_G(x, y) \geq 3$. Therefore, $\operatorname{diam} G = 3$, $\operatorname{diam} V(G) = 2$, $x, y \in J(G) \setminus V(G)$ and $p \in V(G)$. Thus, $x \in u_x v_x \in E(G)$ and $y \in u_y v_y \in E(G)$, with $u_x, u_y \in [xy]$ and $d_G(u_y, \{u_x, v_x\}) = 2$, and so, $u_y u_x, u_y v_x \notin E(G)$. Since $v_x u_x p\, u_y$ is a P_4 on G, it is not induced and so, $v_x p \in E(G)$ (recall that $u_y u_x, u_y v_x \notin E(G)$); thus, $3/2 = d_G(p, [yz] \cup [zx]) \leq d_G(p, v_x) = 1$, a contradiction. Hence, $\delta(G) \leq 5/4$.

Let K_4 be a complete graph with vertices $\{v_1, v_2, v_3, v_4\}$. We denote by G the graph obtained from K_4 by adding a new vertex v_5 and two edges v_5v_1, v_5v_2. Denote by y the midpoint of v_3v_4. Let us consider the geodesic bigon $\{v_5, y\}$ which is the union of the geodesics $\gamma_1 = v_5v_1 \cup v_1v_4 \cup [v_4y]$ and $\gamma_2 = v_5v_2 \cup v_2v_3 \cup [v_3y]$. If p is the midpoint of γ_1, then we have $\delta(G) \geq d_G(p, \gamma_2) = 5/4$. Since $\operatorname{diam} G = 5/2$, Corollary 1.2.17 gives $\delta(G) \leq 5/4$, and we conclude $\delta(G) = 5/4$. $\square$

Chapter 5

Operators on graphs and hyperbolicity

In [73], J. Krausz introduced the concept graph operators. A graph operator is a mapping $F : \Gamma \to \Gamma'$, where Γ and Γ' are families of graphs. The different kinds of graph operators are an important research topic in Discrete Mathematics and its applications. In particular, in the studies on graph dynamics (see [50, 97]) and topological indices (see [16, 98, 113]). Some large graphs are composed from some existing smaller ones by using graph operators, and many properties of such large graphs are strongly associated with that of the corresponding smaller ones. Motivated from the above works, we study here the hyperbolicity constant of some graph operators.

Given an edge $e = uv \in E(G)$ with endpoints u and v, we write $V(e) = \{u, v\}$. Next, we recall the definition of some of the main graph operators.

The *line graph*, denoted by $\mathcal{L}(G)$, is the graph whose vertices correspond to the edges of G with two vertices being adjacent if and only if the corresponding edges in G have a vertex in common.

The *subdivision graph*, denoted by $S(G)$, is the graph obtained from G by replacing each of its edge by a path of length two, or equivalently, by inserting an additional vertex into each edge of G.

The *total graph*, denoted by $T(G)$, has as its vertices the edges and vertices of G. Adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G.

The graph $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge. Another way to describe $R(G)$ is to replace each edge of G by a triangle.

The graph $Q(G)$ is the graph obtained from G by inserting a new vertex into each edge of G and by joining edges those pairs of these new vertices which lie on adjacent edges of G.

Given $G = (V(G), E(G))$, we may define two other sets that we use frequently:

$$EE(G) := \{\{e, e'\} : e, e' \in E(G), e \neq e', |V(e) \cap V(e')| = 1\}.$$

$$EV(G) := \{\{e, v\} : e \in E(G), v \in V(e)\}.$$

We may then write the subdivision operators above as follows:
$\mathcal{L}(G) := (E(G), EE(G))$.
$S(G) := (V(G) \cup E(G), EV(G))$.
$T(G) := (V(G) \cup E(G), E(G) \cup EV(G) \cup EE(G))$.
$R(G) := (V(G) \cup E(G), E(G) \cup EV(G))$.
$Q(G) := (V(G) \cup E(G), EV(G) \cup EE(G))$.

In this chapter we prove inequalities relating the hyperbolicity constants of a graph G and its graph operators $\mathcal{L}(G)$, $S(G)$, $T(G)$, $R(G)$ and $Q(G)$.

5.1 Hyperbolicity on subdivision operator

Let us consider hyperbolicity with the Gromov product(see Definition 1.1.3). In Chapter 4 we denote by $\delta^*(G)$ the sharp constant for the Inequality 1.1, *i.e.*,

$$\delta^*(G) := \sup\big\{\min\big\{(x,y)_w, (y,z)_w\big\} - (x,z)_w : x,y,z,w \in G\big\}.$$

Theorem 1.1.4 gives $\delta^*(G) \le 4\delta(G)$ and $\delta(G) \le 3\delta^*(G)$. In [107, Proposition II.20] we found the following improvement of the previous inequality: $\delta^*(G) \le 2\delta(G)$.

We denote by $\delta_v^*(G)$ the constant of hyperbolicity of the Gromov product restricted to the vertices of G, i.e.,

$$\delta_v^*(G) := \sup\big\{\min\big\{(x,y)_w, (y,z)_w\big\} - (x,z)_w : x,y,z,w \in V(G)\big\}.$$

The following result is immediate from the definition of $S(G)$.

Proposition 5.1.1. *Let G be a graph. Then*

$$\delta(S(G)) = 2\delta(G), \qquad \delta^*(S(G)) = 2\delta^*(G).$$

We remark that the equality is not true for $\delta_v^*(G)$ (e.g., $S(C_5) = C_{10}$ but $2\delta_v^*(C_5) = 1 \ne 2 = \delta_v^*(S(G))$), but there are inequalities. In order to obtain these inequalities, we need the following result [34, Theorem 4].

Theorem 5.1.2. *Let $B = (V_0 \cup V_1, E)$ be a bipartite graph. We have $\delta_B(V_i) \le \delta_v^*(B) \le \delta_B(V_i) + 2$, where*

$$\delta_B(V_i) = \sup\{\min\big\{(x,y)_w, (y,z)_w\big\} - (x,z)_w : x,y,z,w \in V_i\}$$

for every $i \in \{1, 2\}$.

Corollary 5.1.3. *Let G be a graph. Then*

$$2\delta_v^*(G) \le \delta_v^*(S(G)) \le 2\delta_v^*(G) + 2.$$

Proof. Observe that $S(G)$ can be considered as a bipartite graph, where $V(S(G)) = V(G) \cup V(\mathcal{L}(G))$. Theorem 5.1.2 gives $\delta_{S(G)}(V(G)) \le \delta_v^*(S(G)) \le \delta_{S(G)}(V(G)) + 2$. Since $\delta_{S(G)}(V(G)) = 2\delta_v^*(G)$, the desired inequalities hold. □

Proposition 5.1.4. *Let G be a graph. Then*

$$\delta_v^*(G) \le \delta^*(G) \le \delta_v^*(G) + 3.$$

Proof. The inequality $\delta_v^*(G) \le \delta^*(G)$ is direct. Let us prove the other inequality.

For every $x_0, x_1, x_2 \in G$ there are $x_0', x_1', x_2' \in V(G)$ such that $d(x_i, x_i') \le 1/2$ for $i = 0, 1, 2$. Then

$$\begin{aligned}
\big|(x_1, x_2)_{x_0} - (x_1', x_2')_{x_0'}\big| &= \frac{1}{2}\big|d(x_0, x_1) + d(x_0, x_2) - d(x_1, x_2) - d(x_0', x_1') - d(x_0', x_2') + d(x_1', x_2')\big| \\
&\le \frac{1}{2}\big|d(x_0, x_1) - d(x_0', x_1')\big| + \frac{1}{2}\big|d(x_0, x_2) - d(x_0', x_2')\big| + \frac{1}{2}\big|d(x_1, x_2) - d(x_1', x_2')\big| \\
&\le \frac{3}{2}.
\end{aligned}$$

Given $x_0, x_1, x_2, x_3 \in G$, let $x_0', x_1', x_2', x_3' \in V(G)$, with $d(x_i, x_i') \le 1/2$ for $i = 0, 1, 2, 3$. We have

$$\begin{aligned}
(x_1, x_3)_{x_0} \ge (x_1', x_3')_{x_0'} - \frac{3}{2} &\ge \min\Big\{(x_1', x_2')_{x_0'}, (x_2', x_3')_{x_0'}\Big\} - \delta_v^*(G) - \frac{3}{2} \\
&\ge \min\Big\{(x_1, x_2)_{x_0} - \frac{3}{2}, (x_2, x_3)_{x_0} - \frac{3}{2}\Big\} - \delta_v^*(G) - \frac{3}{2} \\
&= \min\{(x_1, x_2)_{x_0}, (x_2, x_3)_{x_0}\} - \delta_v^*(G) - 3,
\end{aligned}$$

and we conclude $\delta^*(G) \le \delta_v^*(G) + 3$. □

5.2 Hyperbolicity on operators Q, R and T

We will need the following well-know result. We include a proof for the sake of completeness.

Lemma 5.2.1. *If Γ is an isometric subgraph of G, then $\delta(\Gamma) \le \delta(G)$, $\delta^*(\Gamma) \le \delta^*(G)$ and $\delta_v^*(\Gamma) \le \delta_v^*(G)$.*

Proof. Note that by hypothesis $d_\Gamma(x, y) = d_G(x, y)$ for every $x, y \in \Gamma$; therefore, every geodesic triangle in Γ is a geodesic triangle in G. Hence, $\delta(\Gamma) \le \delta(G)$. Similarly, we have $\delta^*(\Gamma) \le \delta^*(G)$ and $\delta_v^*(\Gamma) \le \delta_v^*(G)$. □

Since G is an isometric subgraph of $T(G)$ and $R(G)$, and $\mathcal{L}(G)$ is an isometric subgraph of $T(G)$ and $Q(G)$, we have the following consequence of Lemma 5.2.1.

Corollary 5.2.2. *For any graph G, we have*

$$\begin{aligned}
\delta(G) &\le \delta(T(G)), & \delta^*(G) &\le \delta^*(T(G)), & \delta_v^*(G) &\le \delta_v^*(T(G)), \\
\delta(G) &\le \delta(R(G)), & \delta^*(G) &\le \delta^*(R(G)), & \delta_v^*(G) &\le \delta_v^*(R(G)), \\
\delta(\mathcal{L}(G)) &\le \delta(T(G)), & \delta^*(\mathcal{L}(G)) &\le \delta^*(T(G)), & \delta_v^*(\mathcal{L}(G)) &\le \delta_v^*(T(G)), \\
\delta(\mathcal{L}(G)) &\le \delta(Q(G)), & \delta^*(\mathcal{L}(G)) &\le \delta^*(Q(G)), & \delta_v^*(\mathcal{L}(G)) &\le \delta_v^*(Q(G)).
\end{aligned}$$

The hyperbolicity of the line graph has been studied previously (see [23, 25, 34]). We have the following results.

Theorem 5.2.3. *[23, Corollary 3.12] Let G be a graph. Then*

$$\delta(G) \le \delta(\mathcal{L}(G)) \le 5\delta(G) + 5/2.$$

Furthermore, the first inequality is sharp: the equality is attained by every cycle graph.

Theorem 5.2.4. *[34, Theorem 6] Let G be a graph. Then*

$$\delta_v^*(G) - 1 \le \delta_v^*(\mathcal{L}(G)) \le \delta_v^*(G) + 1.$$

Theorem 5.2.5. *Let G be a graph. Then*

$$\delta^*(G) - 4 \le \delta^*(\mathcal{L}(G)) \le \delta^*(G) + 4.$$

Proof. Theorem 5.2.4 and Proposition 5.1.4 give $\delta^*(G) \le \delta_v^*(G)+3 \le \delta_v^*(\mathcal{L}(G))+4 \le \delta^*(\mathcal{L}(G))+4$, and $\delta^*(\mathcal{L}(G)) \le \delta_v^*(\mathcal{L}(G)) + 3 \le \delta_v^*(G) + 4 \le \delta^*(G) + 4$. □

Proposition 5.1.1, and Theorems 5.2.3 and 5.2.5 have the following consequence.

Corollary 5.2.6. *Let G be a graph. Then*

$$\delta(S(G)) \le 2\delta(\mathcal{L}(G)) \le 5\delta(S(G)) + 5,$$
$$\delta^*(S(G)) - 8 \le 2\delta^*(\mathcal{L}(G)) \le \delta^*(S(G)) + 8.$$

Corollary 5.2.2 and Theorems 5.2.3, 5.2.4 and 5.2.5 have the following consequence.

Corollary 5.2.7. *Let G be a graph. Then*

$$\delta(G) \le \delta(Q(G)),$$
$$\delta_v^*(G) \le \delta_v^*(Q(G)) + 1,$$
$$\delta^*(G) \le \delta^*(Q(G)) + 4.$$

Note that Theorem 5.2.5 improves the inequality $\delta^*(\mathcal{L}(G)) \le \delta^*(G) + 6$ in [25].

Given a graph G with multiple edges, we define $B(G)$ as the graph (without multiple edges) obtained from G by replacing each multiple edge by a single edge with the minimum length of the edges corresponding to that multiple edge. From [12, Theorem 8] we have the following result.

Theorem 5.2.8. *If G is a graph with multiple edges, then G is hyperbolic if and only if $B(G)$ is hyperbolic and $J := \sup\{L(\beta) : \beta$ is an edge contained in a multiple edge of $G\}$ is finite. Besides, if $j := \inf\{d(x,y) : x, y$ are joined by a multiple edge of $G\}$, then*

$$\max\left\{\delta(B(G)), \frac{J+j}{4}\right\} \le \delta(G) \le \max\left\{\delta(B(G)) + \frac{J-j}{2}, J\right\}.$$

Remark 5.2.9. *The argument in the proof of [12, Theorem 8] has the following direct consequence: If in each multiple edge there is at most one edge with length greater than j, then* $\delta(G) \le \max\left\{\delta(B(G)) + \frac{J-j}{2}, \frac{J+j}{4}\right\}$.

Corollary 5.2.10. *Let G be a graph. Then*

$$\max\left\{\delta(G), \frac{3}{4}\right\} \le \delta(R(G)) \le \max\left\{\delta(G) + \frac{1}{2}, \frac{3}{4}\right\}.$$

Proof. Note that $R(G)$ can be obtained by adding an edge of length 2 to each pair of adjacent vertices in G, so the graph becomes a graph with multiple edges, with $j = 1$ and $J = 2$. Then Theorem 5.2.8 and Remark 5.2.9 give the result. □

Given any graph G which is not a tree, we define its *girth* $g(G)$ as the infimum of the lengths of the cycles in G.

From [80, Theorem 17] we have the following result.

Theorem 5.2.11. *Let G be a graph which is not a tree. Then*

$$\delta(G) \ge \frac{g(G)}{4}.$$

We have the following consequence.

Corollary 5.2.12. *Let G be a graph which is not a tree. Then*

$$\delta(G) \ge \frac{3}{4}.$$

Corollary 5.2.13. *Let G be a graph which is not a tree. Then*

$$\delta(G) \le \delta(R(G)) \le \delta(G) + \frac{1}{2}.$$

Proof. Since G is not a tree, Corollary 5.2.12 gives $\delta(G) \ge 3/4$, and so

$$\max\left\{\delta(G), \frac{3}{4}\right\} = \delta(G), \qquad \max\left\{\delta(G) + \frac{1}{2}, \frac{3}{4}\right\} = \delta(G) + \frac{1}{2},$$

and Corollary 5.2.10 gives the inequalities. □

Theorem 5.2.3 and Corollary 5.2.13 have the following consequence.

Corollary 5.2.14. *Let G be a graph which is not a tree. Then*

$$\delta(R(G)) - \frac{1}{2} \le \delta(\mathcal{L}(G)) \le 5\delta(R(G)) + \frac{5}{2}.$$

Proposition 5.1.1 and Corollary 5.2.13 have the following consequence.

Corollary 5.2.15. *Let G be a graph which is not a tree. Then*

$$\delta(S(G)) \le 2\delta(R(G)) \le \delta(S(G)) + 1.$$

Theorem 5.2.16. *Let G be a graph. Then*

$$\begin{aligned}
\delta^*(\mathcal{L}(G)) &\le \delta^*(Q(G)) \le \delta_v^*(\mathcal{L}(G)) + 6 \le \delta^*(\mathcal{L}(G)) + 6,\\
\delta_v^*(\mathcal{L}(G)) &\le \delta_v^*(Q(G)) \le \delta_v^*(\mathcal{L}(G)) + 6,\\
\delta^*(\mathcal{L}(G)) &\le \delta^*(T(G)) \le \delta_v^*(\mathcal{L}(G)) + 9 \le \delta^*(\mathcal{L}(G)) + 9,\\
\delta_v^*(\mathcal{L}(G)) &\le \delta_v^*(T(G)) \le \delta_v^*(\mathcal{L}(G)) + 6,\\
\delta^*(G) &\le \delta^*(R(G)) \le \delta_v^*(G) + 6 \le \delta^*(G) + 6,\\
\delta_v^*(G) &\le \delta_v^*(R(G)) \le \delta_v^*(G) + 6,\\
\delta^*(G) &\le \delta^*(T(G)) \le \delta_v^*(G) + 9 \le \delta^*(G) + 9,\\
\delta_v^*(G) &\le \delta_v^*(T(G)) \le \delta_v^*(G) + 6.
\end{aligned}$$

Proof. The lower bounds follow from Corollary 5.2.2. We consider the map $P : Q(G) \to \mathcal{L}(G)$ such that $P(x) = x$ if $x \in \mathcal{L}(G)$, $P(x) = v_x$ if $x \notin \mathcal{L}(G)$, where $v_x \in V(\mathcal{L}(G))$ and $d_{Q(G)}(x, v_x) \le 1$. If $x_0, x_1, x_2, x_3 \in Q(G)$, then

$$\big|d_{Q(G)}(x_i, x_j) - d_{\mathcal{L}(G)}(P(x_i), P(x_j))\big| = \big|d_{Q(G)}(x_i, x_j) - d_{Q(G)}(P(x_i), P(x_j))\big| \le 2,$$

since $\mathcal{L}(G)$ is an isometric subgraph of $Q(G))$ and

$$\begin{aligned}
&\big|(x_i, x_j)_{x_0} - (P(x_i), P(x_j))_{P(x_0)}\big|\\
&\quad = \frac{1}{2}\big|d_{Q(G)}(x_0, x_i) + d_{Q(G)}(x_0, x_j) - d_{Q(G)}(x_i, x_j)\\
&\qquad - d_{\mathcal{L}(G)}(P(x_0), P(x_i)) - d_{\mathcal{L}(G)}(P(x_0), P(x_j)) + d_{\mathcal{L}(G)}(P(x_i), P(x_j))\big| \le 3,
\end{aligned}$$

for $i, j \in \{1, 2, 3\}$. Thus,

$$\begin{aligned}
(x_1, x_3)_{x_0} &\ge (P(x_1), P(x_3))_{P(x_0)} - 3\\
&\ge \min\{(P(x_1), P(x_2))_{P(x_0)}, (P(x_2), P(x_3))_{P(x_0)}\} - \delta_v^*(\mathcal{L}(G)) - 3\\
&\ge \min\{(x_1, x_2)_{x_0} - 3, (x_2, x_3)_{x_0} - 3\} - \delta_v^*(\mathcal{L}(G)) - 3\\
&= \min\{(x_1, x_2)_{x_0}, (x_2, x_3)_{x_0}\} - \delta_v^*(\mathcal{L}(G)) - 6.
\end{aligned}$$

Therefore,

$$\delta^*(\mathcal{L}(G)) + 6 \ge \delta_v^*(\mathcal{L}(G)) + 6 \ge \delta^*(Q(G)) \ge \delta_v^*(Q(G)).$$

These inequalities give the upper bounds of $\delta^*(Q(G))$ and $\delta_v^*(Q(G))$. We obtain the other upper bounds in a similar way. □

Corollary 5.2.7 and Theorems 5.2.4 and 5.2.16 have the following consequence.

Corollary 5.2.17. *Let G be a graph. Then*

$$\delta_v^*(G) - 1 \le \delta_v^*(Q(G)) \le \delta_v^*(G) + 7,$$
$$\delta^*(G) - 4 \le \delta^*(Q(G)) \le \delta_v^*(G) + 7 \le \delta^*(G) + 7.$$

The inequalities $\delta(G) \le 3\delta^*(G)$ and $\delta^*(G) \le 2\delta(G)$, Theorem 5.2.16 and Corollaries 5.2.2, 5.2.7 and 5.2.17 have the following consequence.

Corollary 5.2.18. *Let G be a graph. Then*

$$\delta(\mathcal{L}(G)) \le \delta(Q(G)) \le 6\delta(\mathcal{L}(G)) + 18,$$
$$\delta(\mathcal{L}(G)) \le \delta(T(G)) \le 6\delta(\mathcal{L}(G)) + 27,$$
$$\delta(G) \le \delta(T(G)) \le 6\delta(G) + 27,$$
$$\delta(G) \le \delta(Q(G)) \le 6\delta(G) + 21.$$

Proof. Corollaries 5.2.2 and 5.2.7 give the lower bounds. On the other hand, Theorem 5.2.16 gives $\delta(Q(G)) \le 3\delta^*(Q(G)) \le 3\delta^*(\mathcal{L}(G)) + 18 \le 6\delta(\mathcal{L}(G)) + 18$, $\delta(T(G)) \le 3\delta^*(T(G)) \le 3(\delta^*(\mathcal{L}(G)) + 9) \le 6\delta(\mathcal{L}(G)) + 27$; we obtain the third upper bound in a similar way. Corollary 5.2.17 gives $3\delta^*(Q(G)) \le 3(\delta^*(G) + 7) \le 6\delta(G) + 21$, obtaining the last upper bound. □

The following results improve the inequality $\delta(Q(G)) \le \delta(\mathcal{L}(G)) + 18$ in Corollary 5.2.18.

Theorem 5.2.19. *Let G be a graph. If G is a path graph, then*

$$0 = \delta(\mathcal{L}(G)) \le \delta(Q(G)) \le 3/4.$$

Proof. Since G is a path graph, $\mathcal{L}(G)$ is also a path graph, and so $0 = \delta(\mathcal{L}(G)) \le \delta(Q(G))$.

Consider the T-decomposition $\{G_n\}$ of $Q(G)$. Since each connected component G_n is either a cycle C_3 or a path of length 1, we have $\delta(Q(G)) = \sup_n\{\delta(G_n)\} \le 3/4$, by Theorem 1.2.19 and Proposition 1.2.23. □

Theorem 5.2.20. *Let G be a graph. If G is not a path graph, then*

$$\delta(\mathcal{L}(G)) \le \delta(Q(G)) \le \delta(\mathcal{L}(G)) + 1/2.$$

Proof. Corollary 5.2.2 gives the first inequality. Let us prove the second one. If $\delta(Q(G)) = \infty$, then Theorem 5.2.16 gives $\delta(\mathcal{L}(G)) = \infty$, and the second inequality holds. Assume now that $\delta(Q(G)) < \infty$ (and so, $\delta(\mathcal{L}(G)) < \infty$ by Theorem 5.2.16). Since G is not a path graph, $\mathcal{L}(G)$ is not a tree and Corollary 5.2.12 gives $\delta(\mathcal{L}(G)) \ge 3/4$.

For each $v \in V(G)$, let us define $V_v := \{u \in V(Q(G)) : uv \in E(Q(G))\} = \{u \in V(\mathcal{L}(G)) : uv \in E(Q(G))\}$. Denote by G_v and G_v^* the subgraphs of $Q(G)$ induced by the sets $V_v \cup \{v\}$ and V_v, respectively. Note that both G_v and G_v^* are complete graphs for every $v \in V(G)$, and if G^* is a complete graph with r vertices, then G_v is a complete graph with $r+1$ vertices. Also, $Q(G) = \mathcal{L}(G) \cup (\cup_{v\in V(G)} G_v)$.

By Theorem 1.2.13 there exists a geodesic triangle $T \in \mathbb{T}_1$ in $Q(G)$ with $\delta(T) = \delta(Q(G))$. Denote by $\gamma_1, \gamma_2, \gamma_3$ the sides of T. Without loss of generality we can assume that there exists $p \in \gamma_1$ with $d_{Q(G)}(p, \gamma_2 \cup \gamma_3) = \delta(T) = \delta(Q(G))$. Thus, T is a cycle and each vertex of T is either a vertex of $Q(G)$ or the midpoint of some edge of $Q(G)$.

If T is contained in G_v for some $v \in V(G)$, then $\delta(Q(G)) = \delta(T) \le \delta(G_v) \le 1 < 3/4 + 1/2 \le \delta(\mathcal{L}(G)) + 1/2$ by Theorem 1.2.19, since G_v is an isometric subgraph of $Q(G)$.

If T is contained in $\mathcal{L}(G)$, then $\delta(Q(G)) = \delta(T) \le \delta(\mathcal{L}(G))$ by Lemma 5.2.1, since $\mathcal{L}(G)$ is an isometric subgraph of $Q(G)$.

Assume that T is not contained neither in $\mathcal{L}(G)$ nor G_v with $v \in V(G)$.

Note that if $T \cap (G_v \setminus G_v^*) \neq \emptyset$ for some $v \in V(G)$, then there exists at least one vertex of T in $G_v \setminus \mathcal{L}(G)$. We are going to construct a triangle $T^* \subset \mathcal{L}(G)$ from T.

We define $\gamma_i^* := \gamma_i \cap \mathcal{L}(G)$. Note that, for $i \in \{1, 2, 3\}$, γ_i^* is a geodesic, since $\mathcal{L}(G)$ is a isometric subgraph of $Q(G)$.

We denote by $x_{i,j}$ the common vertex of γ_i and γ_j and by u_i and u_j the other vertices of γ_i and γ_j respectively.

We consider the following cases:

Case A. We assume that exactly one vertex of T belongs to $Q(G) \setminus \mathcal{L}(G)$.

Without loss of generality we can assume that $x_{i,j} \in T \setminus \mathcal{L}(G)$. Denote by v the vertex $v \in V(G)$ with $x_{i,j} \in G_v \setminus \mathcal{L}(G)$. Let x_i (respectively, x_j) be the closest point of γ_i^* (respectively, γ_j^*) to $x_{i,j}$. Thus, $x_i x_j \in E(\mathcal{L}(G))$. Let us define v^* as the midpoint of the edge $x_i x_j$. Let us denote by T_1 the connected component of $T \setminus \mathcal{L}(G)$ joining x_i and x_j. Note that $L(T_1) = 2$. We have two possibilities:

Case A1. Assume that $x_{i,j} \in V(Q(G))$. Let us define $\sigma_i := \gamma_i^* \cup [x_i v^*]$ and $\sigma_j := \gamma_j^* \cup [x_j v^*]$. We are going to prove that σ_i and σ_j are geodesics in $\mathcal{L}(G)$. In fact, we prove now that if $\gamma_j^* = [z_j x_j]$, then $d_{Q(G)}(z_j, x_j) \le d_{Q(G)}(z_j, x_i)$. Seeking for a contradiction assume that $d_{Q(G)}(z_j, x_j) > d_{Q(G)}(z_j, x_i)$. Thus,

$$d_{Q(G)}(z_j, x_i) + d_{Q(G)}(x_i, x_{i,j}) = d_{Q(G)}(z_j, x_i) + 1 \le d_{Q(G)}(z_j, x_j) + d_{Q(G)}(x_j, x_{i,j})$$

and this implies that γ_j is not a geodesic. This is the contradiction we were looking for, and we conclude $d_{Q(G)}(z_j, x_j) \le d_{Q(G)}(z_j, x_i)$. Hence, σ_i is a geodesic in $\mathcal{L}(G)$.

Case A2. If $x_{i,j}$ is the midpoint of some edge of $E(Q(G)) \setminus E(\mathcal{L}(G))$, then without loss of generality we can assume that it is the midpoint of $x_i v$, and we define $\sigma_i := \gamma_i^*$ and $\sigma_j := \gamma_j^* \cup x_j x_i$. Thus, σ_i is a geodesic in $\mathcal{L}(G)$.

Note that $\gamma_j^* \cup x_j v \cup [v x_{i,j}]$ and $\sigma_j \cup [x_i x_{i,j}] = \gamma_j^* \cup x_j x_i \cup [x_i x_{i,j}]$ have the same endpoints and length; therefore, σ_j is also a geodesic in $\mathcal{L}(G)$.

Case B. Assume that there are two vertices of T in some connected component of $T \setminus \mathcal{L}(G)$. Without loss of generality we can assume that $u_i, u_j \in G_v \setminus G_v^*$ for some v. We denote by x_i' (respectively, x_j') the closest point in γ_i^* (respectively, γ_j^*) to u_i (respectively, u_j); then $x_i' x_j' \in E(\mathcal{L}(G))$. Let us define v' as the midpoint of the edge $x_i' x_j'$. Let us denote by T_2 the connected component of $T \setminus \mathcal{L}(G)$ joining x_i' and x_j'. Note that $L(T_2) = 2$.

Since each vertex of T is a vertex of $V(Q(G))$ or the midpoint of an edge of $E(Q(G))$, we have two possibilities again:

Case B1. The vertices u_i, u_j of T are the midpoints of $x_i'v$ and $x_j'v$. Thus, $\sigma_i := \gamma_i^*$, $\sigma_j := \gamma_j^*$ and $\sigma_k := x_i'x_j'$ are geodesics in $\mathcal{L}(G)$.

Case B2. Otherwise, we can assume without loss of generality that $u_j = v$ and u_i is the midpoint of x_iv. We have $d_{Q(G)}(u_i, x_j) = d_{Q(G)}(u_i, x_i) + 1$ and so, $\sigma_i := \gamma_i^*$ and $\sigma_j := \gamma_j^* \cup x_j'x_i'$ are geodesics in $\mathcal{L}(G)$. In this case we define $\sigma_k := \{x_i'\}$.

By repeating this process at most three times we obtain a geodesic triangle T^* in $\mathcal{L}(G)$ with sides γ_1', γ_2' and γ_3' containing γ_1^*, γ_2^* and γ_3^*, respectively.

If $p \in \mathcal{L}(G)$, then one can check that $\delta(Q(G)) = d_{Q(G)}(p, \gamma_2 \cup \gamma_3) \leq d_{Q(G)}(p, \gamma_2' \cup \gamma_3') + 1/2 \leq \delta(\mathcal{L}(G)) + 1/2$. If $p \notin \mathcal{L}(G)$, then $\delta(Q(G)) = d_{Q(G)}(p, \gamma_2 \cup \gamma_3) \leq 5/4$; since $\delta(\mathcal{L}(G)) \geq 3/4$, we have $\delta(\mathcal{L}(G)) + 1/2 \geq 5/4 \geq \delta(Q(G))$. This finishes the proof. □

Proposition 5.1.1, Theorems 5.2.3 and 5.2.20, and Corollary 5.2.6 have the following consequence.

Corollary 5.2.21. *Let G be a graph. If G is not a path graph, then*

$$\delta(S(G)) \leq 2\delta(Q(G)) \leq 5\delta(S(G)) + 6.$$

Conclusions

In this work the hyperbolicity constant was studied. Bounds and characterizations were given for families of graphs. Relationships between the hyperbolic constant and some domination numbers were provided and, finally, the hyperbolicity constant was studied by considering operators in graphs.

Interval graphs

In Chapter 2 bounds and characterizations were provided for interval graphs.

By Corollary 2.1.11, we have that every interval graph G with edges of length 1 satisfies the inequality

$$\delta(G) \le \frac{3}{2}\,.$$

Let G be an interval graph, the following properties was defined for G:

- We say that G has the 0-*intersection property* if for every three corresponding intervals I', I'' and I''' to vertices in G we have $I' \cap I'' \cap I''' = \emptyset$.

- G has the (3/4)-*intersection property* if it does not have the 0-intersection property and for every four corresponding intervals I', I'', I''' and I'''' to vertices in G we have $I' \cap I'' \cap I''' = \emptyset$ or $I' \cap I'' \cap I'''' = \emptyset$.

- By a *couple* of intervals in a cycle C of G we mean the union of two non-disjoint intervals whose corresponding vertices belong to C. We say that G has the 1-*intersection property* if it does not have the 0 and (3/4)-intersection properties and for every cycle C in G each interval and couple of corresponding intervals to vertices in C are not disjoint.

- Let G be an interval graph. We say that G has the (3/2)-*intersection property* if there exists two disjoint corresponding intervals I' and I'' to vertices in a cycle C in G such that there is no interval I (corresponding to a vertex in G) with $I \cap I' \neq \emptyset$ and $I \cap I'' \neq \emptyset$.

The characterizations for interval graphs are given in the Theorem 2.1.16: Every interval graph G is hyperbolic and $\delta(G) \in \{0, 3/4, 1, 5/4, 3/2\}$. Furthermore,

- $\delta(G) = 0$ if and only if G has the 0-intersection property.

- $\delta(G) = 3/4$ if and only if G has the $(3/4)$-intersection property.
- $\delta(G) = 1$ if and only if G has the 1-intersection property.
- $\delta(G) = 5/4$ if and only if G does not have the 0, 3/4, 1 and $(3/2)$-intersection properties.
- $\delta(G) = 3/2$ if and only if G has the $(3/2)$-intersection property.

Circular-arc graphs.

In Chapter 3, we study the hyperbolicity constant of circular-arc graphs, and we obtained bounds and characterizations. Some inequalities are:

- If G is a circular-arc graph and $\varrho(G) \neq 1, 2$, then

$$\frac{1}{4}\,\varrho(G) \le \delta(G) \le \frac{1}{2}\Big\lfloor \frac{1}{2}\,\varrho(G) \Big\rfloor + \frac{3}{2}\,.$$

- If G is a proper circular-arc graph and $\varrho(G) = 1$, then

$$\delta(G) = 0.$$

- If G is a proper circular-arc graph and $\varrho(G) = 2$, then

$$0 \le \delta(G) \le 5/4.$$

- If G is a proper circular-arc graph and $\varrho(G) \ge 3$, then

$$\frac{1}{4}\,\varrho(G) \le \delta(G) \le \frac{1}{2}\Big\lfloor \frac{1}{2}\,\varrho(G) \Big\rfloor + 1.$$

Here, the parameter $\varrho(G)$ is defined as

$$\varrho(G) := \min\big\{\, size(K) \mid K \text{ is a total set of vertices in } G \,\big\}.$$

In addition, we characterize the circular-arc graphs with the two smallest possible values for the hyperbolicity constant: 0 and 3/4.

Nordhaus and Gaddum type results were obtained, such as

$$\begin{aligned}
\frac{5\varrho(G)}{16} \le \delta(G)\delta(\overline{G}) \le \frac{3\varrho(G)}{8} + \frac{9}{4}\,, \qquad & \frac{\varrho(G)+5}{4} \le \delta(G) + \delta(\overline{G}) \le \frac{\varrho(G)}{4} + 3, \qquad \text{if } \varrho(G) > 4,\\
0 \le \delta(G)\delta(\overline{G}) \le \frac{7\varrho(G)}{8} + \frac{21}{4}\,, \qquad & \frac{\varrho(G)}{4} \le \delta(G) + \delta(\overline{G}) \le \frac{\varrho(G)}{4} + 5, \qquad \text{if } \varrho(G) = 4,\\
0 \le \delta(G)\delta(\overline{G}) \le \frac{\varrho(G)}{2} + 3, \qquad & \frac{\varrho(G)}{4} \le \delta(G) + \delta(\overline{G}) \le \frac{\varrho(G)}{4} + \frac{7}{2}\,, \qquad \text{if } \varrho(G) = 0.
\end{aligned}$$

Finally there are relations for the line graph

- If $\varrho(G) \geq 3$, then
$$\frac{1}{4}\,\varrho(G) \leq \delta(\mathcal{L}(G)) \leq \frac{1}{2}\left\lfloor \frac{1}{2}\,\varrho(G) \right\rfloor + \frac{5}{2}\,.$$
- If $\varrho(G) = 0, 2$, then
$$0 \leq \delta(\mathcal{L}(G)) \leq \frac{5}{2}\,.$$
- If $\varrho(G) = 1$, then
$$0 \leq \delta(\mathcal{L}(G)) \leq 2.$$

Domination and hyperbolicity

In Chapter 4, we study the relationship of hyperbolicity with some types of domination on graphs, and we obtain inequalities relating the hyperbolicity constant and the total-domination number, distance k-domination number, and other parameters. Some of these results are the following.

- If G is a graph with maximum degree Δ, then
$$\delta(G) \leq \frac{\Delta}{4}\,\gamma_t(G)\,.$$
- Let G be a graph and $k \geq 1$. Then
$$\gamma^k(G) \geq \frac{\operatorname{diam} V(G) + 1}{2k+1}\,, \qquad \gamma^k(G) \geq \frac{2\delta(G)}{2k+1}\,.$$
- If G is a graph and $k \geq 2$, then
$$\delta(G) \leq \frac{1}{2}\max\left\{5, \left\lfloor \frac{3\,\gamma_{tc}^k(G) - 2}{k+1} \right\rfloor + 1\right\}.$$
- If G is a graph, then
$$\delta(G) \leq \begin{cases} \frac{1}{2}\,\gamma_t(G) + 1, & \text{if } \gamma_t(G) \leq 3, \\ \frac{1}{2}\,\gamma_t(G) + 3, & \text{if } \gamma_t(G) \geq 4. \end{cases}$$
- If G is a graph with an isometric dominating cycle C, then
$$\gamma_t(G) \leq 4\delta(G).$$
- If G is a graph with an isometric dominating cycle C, then
$$\gamma_t(G) \leq 4\delta(G).$$

- If G is a graph with no induced C_4 or P_4, then

$$\delta(G) \leq \frac{3}{2}\,.$$

- If G is a graph with no induced P_4, then

$$\delta(G) \leq \frac{5}{4}\,,$$

  and the inequality is sharp.

The importance of the results lies in the fact that it is interesting to obtain relations between the hyperbolic constant of a graph and its domination parameters. Since both parameters are computationally complicated problems, to calculate one of their allows to obtain information about the other one.

Operators on graphs and hyperbolicity

In Chapter 5, we studied the hyperbolicity constant of a graph G and how it was related to the hyperbolic constant of the graph obtained by applying an operator T to G, i. e., we found relationships between $\delta(G)$ and $\delta(T(G))$. The operators studied were: $\mathcal{L}(G)$, $S(G)$, $T(G)$, $R(G)$ and $Q(G)$. Some relationships are the following:

- For any graph G, we have

$$\begin{aligned}
\delta(G) &= \delta(S(G))/2,\\
\delta(G) &\leq \delta(T(G)),\\
\delta(G) &\leq \delta(R(G)),\\
\delta(G) &\leq \delta(Q(G)),\\
\delta(\mathcal{L}(G)) &\leq \delta(T(G)),\\
\delta(\mathcal{L}(G)) &\leq \delta(Q(G)),\\
\delta(S(G)) \leq 2\delta(\mathcal{L}(G)) &\leq 5\delta(S(G)) + 5.
\end{aligned}$$

- Let G be a graph. Then

$$\max\left\{\delta(G), \frac{3}{4}\right\} \leq \delta(R(G)) \leq \max\left\{\delta(G) + \frac{1}{2}, \frac{3}{4}\right\}.$$

- Let G be a graph. Then

$$\begin{aligned}\delta(\mathcal{L}(G)) &\leq \delta(Q(G)) \leq 6\delta(\mathcal{L}(G)) + 18,\\ \delta(\mathcal{L}(G)) &\leq \delta(T(G)) \leq 6\delta(\mathcal{L}(G)) + 27,\\ \delta(G) &\leq \delta(T(G)) \leq 6\delta(G) + 27,\\ \delta(G) &\leq \delta(Q(G)) \leq 6\delta(G) + 21.\end{aligned}$$

- Let G be a graph which is not a tree. Then

$$\begin{aligned}\delta(G) \leq& \delta(R(G)) \leq \delta(G) + \frac{1}{2},\\ \delta(R(G)) - \frac{1}{2} \leq& \delta(\mathcal{L}(G)) \leq 5\delta(R(G)) + \frac{5}{2},\\ \delta(S(G)) &\leq 2\delta(R(G)) \leq \delta(S(G)) + 1.\end{aligned}$$

- Let G be a graph. If G is a path graph, then

$$0 = \delta(\mathcal{L}(G)) \leq \delta(Q(G)) \leq 3/4.$$

  If G is not a path graph, then:

$$\delta(\mathcal{L}(G)) \leq \delta(Q(G)) \leq \delta(\mathcal{L}(G)) + 1/2.$$

$$\delta(S(G)) \leq 2\delta(Q(G)) \leq 5\delta(S(G)) + 6.$$

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