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ON ASYMMETRIC BEHAVIORS IF VOTING IS COSTLY*

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Abstract

Most of the voting models restrict themselves to the analysis of symmetric equilibria, i.e. equilibria in which "similar" voters make "similar" voting decisions. In this paper we investigate this assumption under costly plurality voting. In any pure strategy equilibrium, if two active voters have the same preference order over candidates, they do vote for the same candidate. However, as an example shows, this type of result cannot be hoped for mixed strategies equilibria.

Keywords: Strategic Voting, Symmetric Equilibria

JEL Classification: C72, D72

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1 Introduction

Since mid-eighties non-cooperative game theory has emerged as one of the dominant formal approaches to model institutions. The common denominator of this approach is that it views political interaction as a non-cooperative game. Non-cooperative voting games, in spite of the described relevance in political economy and their peculiar features that make them attractive also from a purely game theoretical point of view, have not been sufficiently analyzed. As a matter of fact, the characterization of solutions in political models of elections is based either on the assumption that voters do not behave strategically, or, when strategic voting is allowed, on ad hoc assumptions, such as only two candidates, or limiting the analysis to symmetric equilibria, i.e. equilibria in which "similar" voters take "similar" decisions. A crucial question is to understand at which level the conclusions drawn from such models are sensitive to the way problems are defined and to the assumptions that are imposed[1].

In this paper we focus on one of the most widely used assumption in the theoretical literature of elections, which is to restrict the analysis to symmetric equilibria. Palfrey and Rosenthal (1985), in a very famous paper on voter participation in plurality election when voting is costly, motivate it by explaining that "...it is natural to assume that voters in the same group (i.e. voters facing a similar decision problem) use the same decision rule in equilibrium. We restrict the analysis to this kind of equilibrium, which we refer to as a symmetric equilibrium. This approach simplifies the problem considerably..." (Palfrey and Rosenthal, 1985, p.67). Many models, with various objectives and motivations, make use of this assumption. Among many others, see Palfrey (1989), Feddersen and Pesendorfer (1996), and Börgers (2004). One can observe that such an assumption was so extensively used that is taken nowadays as a given of the problem.

The objective of this paper is to disentangle this assumption for plurality games with positive cost of voting. There are two main reasons to focus on this class of games. First of all, plurality games with positive cost of voting are among the most studied voting games.[2] Another important reason is that the simple Nash equilibrium concept seems to be appropriate for solving this type of games (see De Sinopoli and Iannantuoni, 2005), whereas without cost of voting even quite strong refinements as perfection or properness do not appear sufficiently restrictive.

The strength of the assumption that "similar" voters make "similar" voting decisions depends on what it is meant by "similar". For example, by "similar voters" can be meant either voters who simply have the same preference order over candidates, or voters who have exactly the same utility function. Analogously, for similar voting decision can be meant either that they use exactly the same strategy, or that their strategies have the same support. Combining

[1] Various authors study the use of game theory in political economy, see, among many others, Hanson (2003).

[2] The analysis has been usually limited to two parties, while our result are of some interest for three or more candidates

the various interpretations we can obtain a stronger or a weaker assumption. If the strongest assumption is irrelevant, than the weaker is. Viceversa a counterexample to the neutrality of the weakest will be a counterexample for the neutrality of the strongest. In the following, we will prove that the strongest assumption is irrelevant in the case of pure strategy equilibria, while not even the weakest assumption is neutral for mixed strategy equilibria.

More precisely, we first show that in any pure strategy equilibrium, if two active voters (i.e., two voters who do not abstain) have the same preference order over candidates, they do vote for the same candidate. Then we obtain, via an example, a mixed strategy equilibrium in which two identical voters (i.e., with exactly the same utility function) use pure strategy but they vote for different candidates!

2 The model

Let $K = (1, ..., k)$ be the finite set of candidates and $N = (1, ..., n)$ the finite set of voters. Under plurality rule every voter has $k+1$ pure strategies, namely voting for each candidate or abstaining. Given a pure strategy vector, the candidate receiving the largest amount of votes is elected, while in case of a tie we assume an equal probability lottery among the winners. Given N and K, a plurality game with positive cost of voting is identified by the utility vectors $\left\{u^i\right\}_{i\in N}$, where $u^i = \left(u_1^i, ..., u_k^i\right)$ and u_c^i is the player i's utility when candidate c is elected, and by the vector of costs of voting $\delta = (\delta^1, ..., \delta^n)$. In other words, a plurality game with positive cost of voting with n voters and k candidates can be seen as a point $(u, \delta) \in \Re^{nk} \times \Re^n_{++}$. The pure strategy space of each player is $S = K \cup \{\phi\}$, where ϕ denotes the abstention. As usual, (σ_{-i}, s) denotes the strategy combination where player i uses with probability 1 the pure strategy s and the others play accordingly to σ_{-i}.

2.1 Results

We first investigate the symmetry assumption for pure strategy equilibria. A very simple example shows that not even players with the same utility function need to use the same strategy. Consider a situation in which there are only two voters, say i and j, and only two alternatives, say k and l. Suppose that both players strictly prefer k over l. Clearly, player i voting for k and player j abstaining is a pure strategy Nash equilibrium, if the cost of voting is sufficiently low. The main feature of this example is given by the fact that one player abstains. It is immediate that if two players with the same preference order vote, they vote for the same candidate:

Proposition 1 *If players i and j have the same preference order over candidates, and if they vote, they vote for the same candidate.*

Proof. Assume that they vote differently. Let W be the set of winning candidates. Because they vote, they both vote for candidate in W and they are not completely indifferent among the candidates in W (if not, abstaining, and saving the cost, will be strictly preferred). Either player i or player j can assure the election of one of their preferred candidates in W, simply switching his vote by voting for it. Hence the result. ■

Despite of its triviality, the above Proposition is quite significant. It tells us that, limiting to the active voters, even the strongest symmetry assumption is completely neutral. Indeed, in any pure strategy Nash equilibrium all the players with the same preference order vote for the same candidate.

Unfortunately, this type of conclusion cannot be obtained for mixed strategies. The weakest possible assumption that can be made in this context is that two players with the same utility function, if they use a pure voting strategy, they vote for the same candidate.

The following example shows that there are cases in which not even this holds.

There are nine voters $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and three candidates $\{A, B, C\}$. The utility each voter gets from the election of any candidate is:

$$\begin{aligned}
u^1 &= u^2 = \left(u_A^1, u_B^1, u_C^1\right) = (13, 10, 0) \\
u^3 &= u^4 = \left(u_A^3, u_B^3, u_C^3\right) = (65, 0, 10) \\
u^5 &= u^6 = \left(u_A^5, u_B^5, u_C^5\right) = (0, 100, 10) \\
u^7 &= u^8 = \left(u_A^7, u_B^7, u_C^7\right) = (10, 0, 10000) \\
u^9 &= \left(u_A^9, u_B^9, u_C^9\right) = (10, 0, 14) .
\end{aligned}$$

The costs of voting are: $\delta^1 = \delta^2 = \delta^5 = \delta^6 = \delta^7 = \delta^8 = \frac{1}{100}$, $\delta^3 = \delta^4 = \frac{259}{40}$, $\delta^9 = \frac{171}{25}$. A Nash equilibrium of the above described game is the following:

$$e = (A^1, B^2, \frac{4}{5}\phi^3 + \frac{1}{5}A^3, \frac{4}{5}\phi^4 + \frac{1}{5}A^4, B^5, B^6, C^7, C^8, \frac{99}{100}\phi^9 + \frac{1}{100}C^9),$$

where k^i denotes player i's pure strategy of voting for candidate k, and ϕ^i his abstention.

Notice that e is a regular equilibrium[3] (see the Appendix for a complete proof of this claim) and, hence, the above example is not a pathological one, because the equilibrium e survives all the usual refinements based either on perturbation of strategies or on perturbation of utilities, and, moreover, a similar result holds in a complete neighborhood of the described game.[4]

3 Conclusions

We have analyzed, under costly plurality voting, if it is the case that "similar" voters make "similar" voting decisions. We have proved that in any pure strategy equilibrium, if two active voters (i.e., two voters who do not abstain) have

[3] For a definition of regularity see van Damme (1991).

[4] Remember that, fixed N and K, a game is a point in $\Re^{nk} \times \Re^n_{++}$.

the same preference order over candidates, they do vote for the same candidate. However, an example shows how this type of result cannot be hoped for mixed strategies equilibria.

It would be interesting to develop a similar analysis for plurality games when the cost of voting is zero. The main issue, in this class of games, is represented by the fact that a concept as Nash equilibrium is completely inadequate.[5] We highlight that the example for mixed strategies above constructed has been obtained working extensively with the abstention of voters using mixed strategies. The fact that abstention is a weakly dominated strategy for plurality games without cost of voting, leaves open the question if a result such *"similar" voters make "similar" voting decisions* can be hoped also for mixed strategies when the cost of voting is zero, at least if an appropriate solution concept is used.

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Appendix 1

[5] For an analysis of equilibrium solution concepts under plurality rule see De Sinopoli (2000).

There are nine voters $\{1,2,3,4,5,6,7,8,9\}$, and three candidates $\{A,B,C\}$. The utility each voter gets from the election of any candidate is:

$$\begin{aligned} u^1 &= u^2 = \left(u^1_A, u^1_B, u^1_C\right) = (13, 10, 0) \\ u^3 &= u^4 = \left(u^3_A, u^3_B, u^3_C\right) = (65, 0, 10) \\ u^5 &= u^6 = \left(u^5_A, u^5_B, u^5_C\right) = (0, 100, 10) \\ u^7 &= u^8 = \left(u^7_A, u^7_B, u^7_C\right) = (10, 0, 10000) \\ u^9 &= \left(u^9_A, u^9_B, u^9_C\right) = (10, 0, 14)\,. \end{aligned}$$

The costs of voting are: $\delta^1 = \delta^2 = \delta^5 = \delta^6 = \delta^7 = \delta^8 = \frac{1}{100}$, $\delta^3 = \delta^4 = \frac{259}{40}$, $\delta^9 = \frac{171}{25}$.

Claim 2 *A regular equilibrium of the above described game is the following:*

$$e = (A^1, B^2, x\phi^3 + (1-x)\,A^3, x\phi^4 + (1-x)\,A^4, B^5, B^6, C^7, C^8, y\phi^9 + (1-y)\,C^9).$$

where $x = \frac{4}{5}$, *and* $y = \frac{99}{100}$.

Proof: First of all, we prove that e is a quasi-strict Nash equilibrium. To this end, notice that, since voting is costly, voting for the worst alternative is strictly dominated by abstention. Hence, in order to check for best replies, we have to consider only three strategies for each player.

Player 1

Let's start with player 1. We have to check that, for the above strategy combination of the others, voting for A gives an higher utility than voting for B or abstaining. We first calculate the probabilities of any possible event (ω_{-1}), i.e. the number of votes for any of the three candidates without the vote of player 1:

$\Pr\{\omega_{-1} = (0,3,2)\} = x^2 y$

$\Pr\{\omega_{-1} = (1,3,2)\} = 2x\,(1-x)\,y$

$\Pr\{\omega_{-1} = (2,3,2)\} = (1-x)^2\,y$

$\Pr\{\omega_{-1} = (0,3,3)\} = x^2\,(1-y)$

$\Pr\{\omega_{-1} = (1,3,3)\} = 2x\,(1-x)\,(1-y)$

$\Pr\{\omega_{-1} = (2,3,3)\} = (1-x)^2\,(1-y)\,.$

Now we can compute the utility player 1 gets by voting for A:

$$u^1(\sigma_{-1}, A) = x^2 y u^1_B + 2x\,(1-x)\,y u^1_B + (1-x)^2\,y\,\frac{u^1_A + u^1_B}{2} + x^2\,(1-y)\,\frac{u^1_B + u^1_C}{2} +$$
$$+2x\,(1-x)\,(1-y)\,\frac{u^1_B + u^1_C}{2} + (1-x)^2\,(1-y)\,\frac{u^1_A + u^1_B + u^1_C}{3} - \delta^1 =$$

$= \frac{396}{625}10 + \frac{198}{625}10 + \frac{99}{2500}\frac{23}{2} + \frac{4}{625}5 + \frac{2}{625}5 + \frac{1}{2500}\frac{23}{3} - \frac{1}{100} = \frac{150\,007}{15\,000}.$

Let's also compute the utility player 1 gets by voting for B :

$$u^1(\sigma_{-1}, B) = x^2 y u_B^1 + 2x(1-x) y u_B^1 + (1-x)^2 y u_B^1 + x^2(1-y) u_B^1 + $$
$$+ 2x(1-x)(1-y) u_B^1 + (1-x)^2 (1-y) u_B^1 - \delta^1 =$$
$$= \frac{396}{625}10 + \frac{198}{625}10 + \frac{99}{2500}10 + \frac{4}{625}10 + \frac{2}{625}10 + \frac{1}{2500}10 - \frac{1}{100} = \frac{999}{100}.$$

Let's compute also the utility player 1 gets by abstaining:

$$u^1(\sigma_{-1}, \phi) = x^2 y u_B^1 + 2x(1-x) y u_B^1 + (1-x)^2 y u_B^1 + + x^2(1-y)\frac{u_B^1 + u_C^1}{2} +$$
$$+ 2x(1-x)(1-y)\frac{u_B^1 + u_C^1}{2} + (1-x)^2(1-y)\frac{u_B^1 + u_C^1}{2} =$$
$$= \frac{396}{625}10 + \frac{198}{625}10 + \frac{99}{2500}10 + \frac{4}{625}5 + \frac{2}{625}5 + \frac{1}{2500}5 = \frac{199}{20}.$$

Clearly, $u^1(\sigma_{-1}, A) > u^1(\sigma_{-1}, B)$, as well as $u^1(\sigma_{-1}, A) > u^1(\sigma_{-1}, \phi)$.

Player 2

$\Pr\{\omega_{-2} = (1,2,2)\} = x^2 y$

$\Pr\{\omega_{-2} = (2,2,2)\} = 2x(1-x) y$

$\Pr\{\omega_{-2} = (3,2,2)\} = (1-x)^2 y$

$\Pr\{\omega_{-2} = (1,2,3)\} = x^2(1-y)$

$\Pr\{\omega_{-2} = (2,2,3)\} = 2x(1-x)(1-y)$

$\Pr\{\omega_{-2} = (3,2,3)\} = (1-x)^2(1-y).$

Now we can compute the utility player 2 gets by voting for B:

$$u^2(\sigma_{-2}, B) = x^2 y u_B^2 + 2x(1-x) y u_B^2 + (1-x)^2 y \frac{u_A^2 + u_B^2}{2} + x^2(1-y)\frac{u_B^2 + u_C^2}{2} +$$
$$+ 2x(1-x)(1-y)\frac{u_B^2 + u_C^2}{2} + (1-x)^2(1-y)\frac{u_A^2 + u_B^2 + u_C^2}{3} - \delta^2 =$$
$$= \frac{396}{625}10 + \frac{198}{625}10 + \frac{99}{2500}\frac{23}{2} + \frac{4}{625}5 + \frac{2}{625}5 + \frac{1}{2500}\frac{23}{3} - \frac{1}{100} = \frac{150\,007}{15\,000}.$$

Let's also compute the utility player 1 gets by voting for A :

$$u^2(\sigma_{-2}, A) = x^2 y \frac{u_A^2 + u_B^2 + u_C^2}{3} + 2x(1-x) y u_A^2 + (1-x)^2 y u_A^2 + x^2(1-y) u_C^2 +$$
$$+ 2x(1-x)(1-y)\frac{u_A^2 + u_C^2}{2} + (1-x)^2(1-y) u_A^2 - \delta^2 =$$
$$= \frac{396}{625}\frac{23}{3} + \frac{198}{625}13 + \frac{99}{2500}13 + \frac{4}{625}0 + \frac{2}{625}\frac{13}{2} + \frac{1}{2500}13 - \frac{1}{100} = \frac{23\,767}{2500}.$$

Let's compute the utility player 1 gets by abstaining:

$$u^2(\sigma_{-2},\phi) = x^2 y \frac{u_B^2+u_C^2}{2} + 2x(1-x)\,y\frac{u_A^2+u_B^2+u_C^2}{3} + (1-x)^2\, y u_A^2 + x^2(1-y)\,u_C^2 +$$
$$+2x(1-x)(1-y)\,u_C^2 + (1-x)^2(1-y)\,\frac{u_A^2+u_C^2}{2} =$$
$$= \frac{396}{625}5 + \frac{198}{625}\frac{23}{3} + \frac{99}{2500}13 + \frac{4}{625}0 + \frac{2}{625}0 + \frac{1}{2500}\frac{13}{2} = \frac{30\,571}{5000}.$$

Clearly, $u^2(\sigma_{-2},B) > u^2(\sigma_{-2},A)$, as well as $u^2(\sigma_{-2},B) > u^2(\sigma_{-2},\phi)$.

Player 3 and 4

$\Pr\{\omega_{-3} = (1,3,2)\} = xy$
$\Pr\{\omega_{-3} = (2,3,2)\} = (1-x)\,y$
$\Pr\{\omega_{-3} = (1,3,3)\} = x(1-y)$
$\Pr\{\omega_{-3} = (2,3,3)\} = (1-x)(1-y)\,.$

Now we can compute the utility player 3 gets by voting for A :

$$u^3(\sigma_{-3},A) = xyu_B^3 + (1-x)\,y\frac{u_A^3+u_B^3}{2} + x(1-y)\,\frac{u_B^3+u_C^3}{2} +$$
$$+(1-x)(1-y)\,\frac{u_A^3+u_B^3+u_C^3}{3} - \delta^3 =$$
$$= \frac{99}{125}0 + \frac{99}{500}\frac{65}{2} + \frac{1}{125}5 + \frac{1}{500}\frac{75}{3} - \frac{259}{40} = \frac{1}{20}.$$

Let's compute now the utility player 3 gets by abstaining:

$$u^3(\sigma_{-3},\phi) = xyu_B^3 + (1-x)\,yu_B^3 + x(1-y)\,\frac{u_B^3+u_C^3}{2} +$$
$$+(1-x)(1-y)\,\frac{u_B^3+u_C^3}{2} =$$
$$= \frac{99}{125}0 + \frac{99}{500}0 + \frac{1}{125}5 + \frac{1}{500}5 = \frac{1}{20}.$$

Let's compute now the utility player 3 gets by voting for C:

$$u^3(\sigma_{-3},C) = xy\frac{u_B^3+u_C^3}{2} + (1-x)\,y\frac{u_B^3+u_C^3}{2} + x(1-y)\,u_C^3 +$$
$$+(1-x)(1-y)\,u_C^3 - \delta^3 =$$
$$= \frac{99}{125}5 + \frac{99}{500}5 + \frac{1}{125}10 + \frac{1}{500}10 - \frac{259}{40} = -\frac{57}{40}.$$

Clearly, $u^3(\sigma_{-3},A) = u^3(\sigma_{-3},\phi) > u^3(\sigma_{-3},C)\,.$

Player 5 and 6

$\Pr\{\omega_{-5} = (1,2,2)\} = x^2 y$

$\Pr\{\omega_{-5} = (2,2,2)\} = 2x(1-x)y$

$\Pr\{\omega_{-5} = (3,2,2)\} = (1-x)^2 y$

$\Pr\{\omega_{-5} = (1,2,3)\} = x^2 (1-y)$

$\Pr\{\omega_{-5} = (2,2,3)\} = 2x(1-x)(1-y)$

$\Pr\{\omega_{-5} = (3,2,3)\} = (1-x)^2 (1-y)$.

Now we can compute the utility player 5 gets by voting for B :

$$u^5(\sigma_{-5}, B) = x^5 y u_B^5 + 2x(1-x) y u_B^5 + (1-x)^2 y \frac{u_A^5 + u_B^5}{2} + x^2 (1-y) \frac{u_B^5 + u_C^5}{2} + \\ + 2x(1-x)(1-y) \frac{u_B^5 + u_C^5}{2} + (1-x)^2 (1-y) \frac{u_A^5 + u_B^5 + u_C^5}{3} - \delta^5 = \\ = \tfrac{396}{625} 100 + \tfrac{198}{625} 100 + \tfrac{99}{2500} 50 + \tfrac{4}{625} 55 + \tfrac{2}{625} 55 + \tfrac{1}{2500} \tfrac{110}{3} - \tfrac{1}{100} = \tfrac{146\,329}{1500}.$$

Let's compute the utility player 5 gets by voting for C :

$$u^5(\sigma_{-5}, C) = x^2 y u_C^5 + 2x(1-x) y u_C^5 + (1-x)^2 y \frac{u_A^5 + u_C^5}{2} + x^2 (1-y) u_C^5 + \\ + 2x(1-x)(1-y) u_C^5 + (1-x)^2 (1-y) u_C^5 - \delta^5 = \\ = \tfrac{396}{625} 10 + \tfrac{198}{625} 10 + \tfrac{99}{2500} 5 + \tfrac{4}{625} 10 + \tfrac{2}{625} 10 + \tfrac{1}{2500} 10 - \tfrac{1}{100} = \tfrac{1224}{125}.$$

Let's compute the utility player 5 gets by abstaining:

$$u^5(\sigma_{-5}, \phi) = x^2 y \frac{u_B^5 + u_C^5}{2} + 2x(1-x) y \frac{u_A^5 + u_B^5 + u_C^5}{3} + (1-x)^2 y u_A^5 + x^2 (1-y) u_C^5 + \\ + 2x(1-x)(1-y) u_C^2 + (1-x)^2 (1-y) \frac{u_A^5 + u_C^5}{2} = \\ = \tfrac{396}{625} 55 + \tfrac{198}{625} \tfrac{110}{3} + \tfrac{99}{2500} 0 + \tfrac{4}{625} 10 + \tfrac{2}{625} 10 + \tfrac{1}{2500} 5 = \tfrac{23\,281}{500}.$$

Clearly, $u^5(\sigma_{-5}, B) > u^5(\sigma_{-5}, C)$, as well as $u^5(\sigma_{-5}, B) > u^5(\sigma_{-5}, \phi)$.

Player 7 and 8

$\Pr\{\omega_{-7} = (1,3,1)\} = x^2 y$

$\Pr\{\omega_{-7} = (2,3,1)\} = 2x(1-x)y$

$\Pr\{\omega_{-7} = (3,3,1)\} = (1-x)^2 y$

$\Pr\{\omega_{-7} = (1,3,2)\} = x^2 (1-y)$

$\Pr\{\omega_{-7} = (2,3,2)\} = 2x(1-x)(1-y)$

$\Pr\{\omega_{-7} = (3,3,2)\} = (1-x)^2 (1-y)$.

Now we can compute the utility player 7 gets by voting for C :

$$u^7(\sigma_{-7}, C) = x^2 y u_B^7 + 2x(1-x) y u_B^7 + (1-x)^2 y \frac{u_A^7 + u_B^7}{2} + x^2 (1-y) \frac{u_B^7 + u_C^7}{2} +$$
$$+2x(1-x)(1-y) \frac{u_B^7 + u_C^7}{2} + (1-x)^2 (1-y) \frac{u_A^7 + u_B^7 + u_C^7}{3} - \delta^7 =$$
$$= \frac{396}{625} 0 + \frac{198}{625} 0 + \frac{99}{2500} 5 + \frac{4}{625} 5000 + \frac{2}{625} 5000 + \frac{1}{2500} \frac{10010}{3} - \frac{1}{100} = \frac{18\,571}{375}.$$

Now we can compute the utility player 7 gets by voting for A :

$$u^7(\sigma_{-7}, A) = x^2 y u_B^7 + 2x(1-x) y \frac{u_A^7 + u_B^7}{2} + (1-x)^2 y u_A^7 + x^2 (1-y) u_B^7 +$$
$$+2x(1-x)(1-y) \frac{u_A^7 + u_B^7}{2} + (1-x)^2 (1-y) u_A^7 - \delta^7 =$$
$$= \frac{396}{625} 0 + \frac{198}{625} 5 + \frac{99}{2500} 10 + \frac{4}{625} 0 + \frac{2}{625} 5 + \frac{1}{2500} 10 - \frac{1}{100} = \frac{199}{100}.$$

Now we can compute the utility player 7 gets by abstaining:

$$u^7(\sigma_{-7}, \phi) = x^2 y u_B^7 + 2x(1-x) y u_B^7 + (1-x)^2 y \frac{u_A^7 + u_B^7}{2} + x^2 (1-y) u_B^7 +$$
$$+2x(1-x)(1-y) u_B^7 + (1-x)^2 (1-y) \frac{u_A^7 + u_B^7}{2} =$$
$$= \frac{396}{625} 0 + \frac{198}{625} 0 + \frac{99}{2500} 5 + \frac{4}{625} 0 + \frac{2}{625} 0 + \frac{1}{2500} 5 = \frac{1}{5}.$$

Clearly, $u^7(\sigma_{-7}, C) > u^7(\sigma_{-7}, A)$, as well as $u^7(\sigma_{-7}, C) > u^7(\sigma_{-7}, \phi)$.

Player 9

$\Pr\{\omega_{-9} = (1,3,2)\} = x^2$

$\Pr\{\omega_{-9} = (2,3,2)\} = 2x(1-x)$

$\Pr\{\omega_{-9} = (3,3,2)\} = (1-x)^2$.

Now we can compute the utility player 9 gets by voting for C :

$$u^9(\sigma_{-9}, C) = x^2 \frac{u_B^9 + u_C^9}{2} + 2x(1-x) \frac{u_B^9 + u_C^9}{2} + (1-x)^2 \frac{u_A^9 + u_B^9 + u_C^9}{3} - \delta^9 =$$
$$= \frac{16}{25} 7 + \frac{8}{25} 7 + \frac{1}{25} \frac{24}{3} - \frac{171}{25} = \frac{1}{5}.$$

Now we can compute the utility player 9 gets by abstaining:

$$u^9(\sigma_{-9}, \phi) = x^2 u_B^9 + 2x(1-x) u_B^9 + (1-x)^2 \frac{u_A^9 + u_B^9}{2}$$
$$= \frac{16}{25} 0 + \frac{8}{25} 0 + \frac{1}{25} 5 = \frac{1}{5}.$$

Now we can compute the utility player 9 gets by voting for A :

$$u^9(\sigma_{-9}, A) = x^2 u_B^9 + 2x(1-x) \frac{u_A^9 + u_B^9}{2} + (1-x)^2 u_A^9 - \delta^9 =$$
$$= \frac{16}{25} 0 + \frac{8}{25} 5 + \frac{1}{25} 10 - \frac{171}{25} = -\frac{121}{25}.$$

Clearly, $u^9(\sigma_{-9}, C) = u^9(\sigma_{-9}, \phi) > u^9(\sigma_{-9}, A)$.

This completes the proof that

$$e = (A^1, B^2, \frac{4}{5}\phi^3 + \frac{1}{5}A^3, \frac{4}{5}\phi^4 + \frac{1}{5}A^4, B^5, B^6, C^7, C^8, \frac{99}{100}\phi^9 + \frac{1}{100}C^9).$$

is a quasi-strict equilibrium.

Before proceeding, we review the definition of regular equilibrium and some basic properties we are going to use.

Let $\Gamma = (N, \{P_i\}_{i\in N}, \{U_i\}_{i\in N})$ be a normal form games, i.e. N is the set of players, P_i the pure strategy of player i, and $U_i : \prod_i P_i \to \Re$ the payoff functions. Let $m_i = \#P_i$, $X_i = \Re^{m_i}$ and $X \equiv \prod_i X_i$. Let the payoff functions U_i be extended to X in the obvious way[6], i.e. $U_i(x) = \sum x(p)U(p)$.

Let fix a pure strategy vector $\bar{p} \in P$, where $P = \prod_i P_i$ and consider the system:

$$\begin{aligned} x_p^i \left[U_i(x,p) - U_i(x,\bar{p}_i)\right] &= 0 \quad \forall i \in N,\ \forall p \in P_i, p \neq \bar{p}_i \\ \sum_{p\in P_i} x_p^i - 1 &= 0 \quad \forall i \in N. \end{aligned} \tag{1}$$

Let $F(\cdot \mid \bar{p})$ be the mapping defined by the left hand side of (1), more precisely

$$\begin{aligned} F_i^p(x \mid \bar{p}) &= x_p^i \left[U_i(x,p) - U_i(x,\bar{p}_i)\right] \quad \forall i \in N,\ \forall p \in P_i, p \neq \bar{p}_i \\ F_i^{\bar{p}}(x \mid \bar{p}) &= \sum_{p\in P_i} x_p^i - 1 \quad \forall i \in N. \end{aligned} \tag{2}$$

Note that if $\bar{\sigma}$ is an equilibrium of Γ with $\bar{p} \in C(\bar{\sigma})$,[7] then $F(\bar{\sigma} \mid \bar{p}) = 0$. Let $J(\bar{\sigma} \mid \bar{p})$ be the Jacobian of $F(\cdot \mid \bar{p})$ evaluated at $\bar{\sigma}$, i.e.

$$J(\bar{\sigma} \mid \bar{p}) = \left.\frac{\partial F(x \mid \bar{p})}{\partial x}\right|_{x=\bar{\sigma}}.$$

With this background, we can now define:

Definition 3 *A Nash equilibrium $\bar{\sigma}$ is regular if $J(\bar{\sigma} \mid \bar{p})$ is nonsingular for some $\bar{p} \in C(\bar{\sigma})$.*

[6]This extension corresponds to the usual one made for mixed strategies, but, here, we do not ask these to belong to the simplex.

[7]With $C(\bar{\sigma})$ it is denoted the carrier of $\bar{\sigma}$.

In the next Lemma we list some well known properties of regular equilibria that we will need (see van Damme, 1991, lemmas 2.5.4 and 2.5.2).

Lemma 4 *Let $\bar{\sigma}$ be an equilibrium and let $\bar{p} \in C(\bar{\sigma})$. Then: (i) $J(\bar{\sigma} \mid \bar{p})$ is nonsingular if and only if $J(\bar{\sigma} \mid \bar{p}')$ is nonsingular for every $\bar{p}' \in C(\bar{\sigma})$; (ii) If $\bar{\sigma}$ is quasi-strict then $J(\bar{\sigma} \mid \bar{p})$ is not singular if and only if $\tilde{J}(\bar{\sigma} \mid \bar{p})$ is not singular, where $\tilde{J}(\bar{\sigma} \mid \bar{p})$ denotes the Jacobian obtained crossing out rows and columns corresponding to strategies that do not belong to $C(\bar{\sigma})$*

The above lemma shows that the strategy used as "reference point" does not affect the definition of regularity, and that moreover, if an equilibrium is quasi-strict (as our equilibrium e), we can limit the analysis to the Jacobian associated to the map $\tilde{F}\left(\cdot \mid \bar{C}, \bar{p}\right)$ defined by eliminating strategies which do not belongs to $C(e)$. In other words, to check the regularity of the quasi-strict equilibrium

$$e = (A^1, B^2, \frac{4}{5}\phi^3 + \frac{1}{5}A^3, \frac{4}{5}\phi^4 + \frac{1}{5}A^4, B^5, B^6, C^7, C^8, \frac{99}{100}\phi^9 + \frac{1}{100}C^9).$$

we have just to consider the Jacobian of the map $F : \Re^6 \to \Re^6$ defined by:

$$\begin{aligned}
F_3^\phi &= x_\phi^3 \left[u^3\left(s, x_\phi^4, x_\phi^9, x_A^4, x_C^9, \phi\right) - u^3\left(s, x_\phi^4, x_\phi^9, x_A^4, x_C^9, A\right)\right] \\
F_4^\phi &= x_\phi^4 \left[u^4\left(s, x_\phi^4, x_\phi^9, x_A^4, x_C^9, \phi\right) - u^4\left(s, x_\phi^4, x_\phi^9, x_A^4, x_C^9, A\right)\right] \\
F_9^\phi &= x_\phi^9 \left[u^9\left(s, x_\phi^3, x_\phi^4, x_A^3, x_A^4, \phi\right) - u^9\left(s, x_\phi^3, x_\phi^4, x_A^3, x_A^4, C\right)\right] \\
F_3^A &= x_\phi^3 + x_A^3 - 1 \\
F_4^A &= x_\phi^4 + x_A^4 - 1 \\
F_9^C &= x_\phi^9 + x_C^9 - 1
\end{aligned}$$

where $s = (A^1, B^2, B^5, B^6, C^7, C^8)$, and evaluate this Jacobian at $(x_\phi^3, x_A^3, x_\phi^4, x_A^4, x_\phi^9, x_C^9, s) = e$, that is to say at $x_\phi^3 = x_\phi^4 = \frac{4}{5}, x_A^3 = x_A^4 = \frac{1}{5}, x_\phi^9 = \frac{99}{100}$ and $x_C^9 = \frac{1}{100}$.

It is quite easy to compute F_3^ϕ, F_4^ϕ, and F_9^ϕ:

$$\begin{aligned}
F_3^\phi &= x_\phi^3 \left[u^3\left(s, x_\phi^4, x_\phi^9, x_A^4, x_C^9, \phi\right) - u^3\left(s, x_\phi^4, x_\phi^9, x_A^4, x_C^9, A\right)\right] = \\
&= x_\phi^3 \left[5x_\phi^4 x_C^9 + 5x_A^4 x_C^9 - \tfrac{65}{2} x_A^4 x_\phi^9 - 5x_\phi^4 x_C^9 - 25 x_A^4 x_C^9 + \tfrac{259}{40}\right] = \\
&= x_\phi^3 \left[-20 x_A^4 x_C^9 - \tfrac{65}{2} x_A^4 x_\phi^9 + \tfrac{259}{40}\right]
\end{aligned}$$

It's immediate to calculate that:
$\left.\frac{\partial F_3^\phi}{\partial x_\phi^3}\right|_{(x_\phi^3, x_A^3, x_\phi^4, x_A^4, x_\phi^9, x_C^9, s) = e} = 0,$

$$\left.\frac{\partial F_3^\phi}{\partial x_\phi^4}\right|_{(x_\phi^3,x_A^3,x_\phi^4,x_A^4,x_\phi^9,x_C^9,s)=e}=0,$$

$$\left.\frac{\partial F_3^\phi}{\partial x_\phi^9}\right|_{(x_\phi^3,x_A^3,x_\phi^4,x_A^4,x_\phi^9,x_C^9,s)=e}=-\frac{26}{5},$$

$$\left.\frac{\partial F_3^\phi}{\partial x_A^3}\right|_{(x_\phi^3,x_A^3,x_\phi^4,x_A^4,x_\phi^9,x_C^9,s)=e}=0,$$

$$\left.\frac{\partial F_3^\phi}{\partial x_A^4}\right|_{(x_\phi^3,x_A^3,x_\phi^4,x_A^4,x_\phi^9,x_C^9,s)=e}=-\frac{259}{10},$$

$$\left.\frac{\partial F_3^\phi}{\partial x_C^9}\right|_{(x_\phi^3,x_A^3,x_\phi^4,x_A^4,x_\phi^9,x_C^9,s)=e}=-\frac{16}{5}.$$

We have now to analyze F_9^ϕ:

$$F_9^\phi = x_\phi^9\left[u^9\left(s,x_\phi^3,x_\phi^4,x_A^3,x_A^4,\phi\right)-u^9\left(s,x_\phi^3,x_\phi^4,x_A^3,x_A^4,C\right)\right]=$$

$$=x_\phi^9\left[5x_A^3x_A^4-7x_\phi^3x_\phi^4-7\left(x_\phi^3x_A^4+x_A^3x_\phi^4\right)-8x_A^3x_A^4+\frac{171}{25}\right]=$$

$$=x_\phi^9\left[-3x_A^3x_A^4-7x_\phi^3x_\phi^4-7x_\phi^3x_A^4-7x_A^3x_\phi^4+\frac{171}{25}\right]$$

It's immediate to calculate also in this case the following:

$$\left.\frac{\partial F_9^\phi}{\partial x_\phi^3}\right|_{(x_\phi^3,x_A^3,x_\phi^4,x_A^4,x_\phi^9,x_C^9,s)=e}=-\frac{693}{100},$$

$$\left.\frac{\partial F_3^\phi}{\partial x_\phi^4}\right|_{(x_\phi^3,x_A^3,x_\phi^4,x_A^4,x_\phi^9,x_C^9,s)=e}=-\frac{693}{100},$$

$$\left.\frac{\partial F_3^\phi}{\partial x_\phi^9}\right|_{(x_\phi^3,x_A^3,x_\phi^4,x_A^4,x_\phi^9,x_C^9,s)=e}=0,$$

$$\left.\frac{\partial F_3^\phi}{\partial x_A^3}\right|_{(x_\phi^3,x_A^3,x_\phi^4,x_A^4,x_\phi^9,x_C^9,s)=e}=-\frac{3069}{500},$$

$$\left.\frac{\partial F_3^\phi}{\partial x_A^4}\right|_{(x_\phi^3,x_A^3,x_\phi^4,x_A^4,x_\phi^9,x_C^9,s)=e}=-\frac{3069}{500},$$

$$\left.\frac{\partial F_3^\phi}{\partial x_C^9}\right|_{(x_\phi^3,x_A^3,x_\phi^4,x_A^4,x_\phi^9,x_C^9,s)=e}=0.$$

Then,

$$\left.\frac{\partial F}{\partial x}\right|_{(x_\phi^3,x_A^3,x_\phi^4,x_A^4,x_\phi^9,x_C^9,s)=e}=\begin{bmatrix} 0 & 0 & \frac{-26}{5} & 0 & \frac{-259}{10} & \frac{-16}{5}\\ 0 & 0 & \frac{-26}{5} & \frac{-259}{10} & 0 & \frac{-16}{5}\\ \frac{-693}{100} & \frac{-693}{100} & 0 & \frac{-3069}{500} & \frac{-3069}{500} & 0\\ 1 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

that is not singular, being its determinant equal to $\frac{51\,282}{625}$ and this completes the proof that e is a regular equilibrium.