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On the role of axial wall conduction in mini/micro counterflow heat exchangers

Marcos Vera*, Alberto E. Quintero

Dept. de Ingeniería Térmica y de Fluidos, Escuela Politécnica Superior, Universidad Carlos III de Madrid, 28911 Leganés, Spain

Abstract

Understanding the role of multi-dimensional conjugate heat transfer on the thermal efficiency of mini/micro counterflow heat exchangers is critical for the design of high efficient equipment. This topic is addressed here by considering a simple model with exact solution: the laminar counterflow parallel-plate heat exchanger. Using as starting point the eigenfunction series solution recently obtained by the authors, a thorough parametric study is carried out to investigate the role of the two dimensionless parameters involved in multi-dimensional wall conduction: the dimensionless wall thickness, Δ_w , and the dimensionless wall thermal resistance, κ_w^{-1} . The analytical eigencondition is presented and discussed, and the resulting eigenvalue spectrum is analyzed using contour plots of the lowest-order eigenvalues in the $(\Delta_w, \kappa_w^{-1})$ plane. The complex task of determining the eigenvalues numerically is largely facilitated by approximate expressions obtained from the asymptotic analysis of the singularities that appear in the eigencondition. The fast evaluation of the eigenvalues makes then possible to obtain contour plots of the heat exchanger effectiveness in the $(\Delta_w, \kappa_w^{-1})$ plane, which exhibit distinguished regimes corresponding to limiting cases with/without axial and transverse wall conduction effects, with smooth transitions occurring for moderately small values of Δ_w and κ_w^{-1} . The analysis provides conditions for neglecting axial and transverse wall conduction, and shows that an optimum wall conductivity always exists in heat exchangers with sufficiently thin walls.

Keywords: counterflow heat exchangers, mini/micro systems, axial wall conduction, conjugate heat transfer, effectiveness, optimum wall conductivity

Nomenclature

A Expansion coefficient.
 a_i Channel half-width of fluid i .

*Corresponding author. Tel.: +34-916249987; fax: +34-916249430.
Email address: marcos.vera@uc3m.es (Alberto E. Quintero)
URL: <http://fluidos.uc3m.es/mvera.html> (Alberto E. Quintero)

C_n	Expansion coefficient corresponding to the n -th eigenfunction.
c_i	Specific heat of fluid i .
f_n	n -th eigenfunction for thermally thin walls without axial wall conduction.
h_n	n -th eigenfunction for the general problem.
h_i	Heat-transfer coefficient of fluid i .
k	Dimensionless parameter, $a_1 k_2 / (a_2 k_1)$.
k_i	Thermal conductivity of fluid i .
k_w	Thermal conductivity of the wall.
L	Heat-exchanger length.
M	Whittaker function, $M_{\kappa,\mu}(z)$.
m	Dimensionless parameter, $a_2 \text{Pe}_2 / (a_1 \text{Pe}_1)$.
Nu_i	Nusselt number of fluid i , $h_i(4a_i)/k_i$.
Pe_i	Peclet number of fluid i , $2a_i V_i / \alpha_i$.
T	Temperature.
V_i	Average flow velocity of fluid i .
W	Whittaker function, $W_{\kappa,\mu}(z)$.
X	Longitudinal distance from the inlet of fluid 1.
Y_i	Transverse distance from channel i symmetry plane.
Y_w	Transverse distance from the wall symmetry plane.
y_i	Dimensionless transverse coordinate, Y_i/a_i .
y_w	Dimensionless transverse coordinate, Y_w/δ_w .

Greek letters

α_i	Thermal diffusivity of fluid i , $k_i/(\rho_i c_i)$.
α_n	Jump of the n -th eigenfunction at the wall.
Δ_w	Dimensionless parameter, $\delta_w/(a_1 \text{Pe}_1)$.
δ_w	Wall thickness.
ε	Heat exchanger effectiveness.
Γ	Gamma function, $\Gamma(z)$.
κ	First argument of Whittaker functions.
κ_w	Dimensionless parameter, $k_w a_1 / (\delta_w k_1)$.
λ_n	n -th eigenvalue.

μ	Second argument of Whittaker functions.
$\nu_w(\xi)$	Dimensionless average wall-axial heat flux.
$\nu_i(\xi)$	Dimensionless interfacial wall-normal heat flux of fluid i .
ν_i	Kinematic viscosity of fluid i .
ρ_i	Density of fluid i .
θ_i	Normalized temperature, $(T_i - T_{1,\text{in}})/(T_{2,\text{in}} - T_{1,\text{in}})$.
ξ	Dimensionless longitudinal coordinate, $X/(a_1 \text{Pe}_1)$.
ξ_L	Dimensionless parameter, $L/(a_1 \text{Pe}_1)$.

Subscripts

i	Subscript used indistinctly for fluids 1 and 2.
in	Inlet.
L	Heat-exchanger length.
m	Bulk, or mixing-cup, temperature.
n	n -th eigenvalue/eigenfunction.
out	Outlet.
w	Heat-exchanging wall.

Superscripts

$[N]$	$2(N + 1)$ -term exact truncated solution.
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1. Introduction

The growing interest in the design and fabrication of mini/micro heat exchangers, driven by the ever-increasing miniaturization of micro-electro-mechanical Systems (MEMS), is evidenced by the large number of review articles published in the last years [1–3]. Although the performance of mini/micro heat exchangers could qualify as satisfactory, the relatively large thickness of the partition walls compared to the hydraulic diameter of the channels is known to enhance the role of axial wall conduction, which reduces the effectiveness of counterflow heat exchangers [4] to that of co-flow systems in the limit of large axial conduction [5, 6]. In addition to the enlarged wall thickness, the thermal conductivity of the wall is another physical parameter that affects axial wall conduction, and therefore the effectiveness of counterflow heat exchangers [7, 8]. Emerging technologies and cost saving strategies have led to the use of a

great variety of materials with different thermal conductivities [9–11]. As a result, a better understanding of the impact of multi-dimensional fluid-wall conjugated heat transfer on the thermal performance of mini/micro counterflow heat exchangers is still needed for the development of high efficient equipment.

The first study on the influence of axial wall conduction in balanced and unbalanced counterflow parallel-plate heat exchangers was carried out fifty years ago by Kroeger [12], who demonstrated that the effectiveness is larger for unbalanced heat exchangers, especially for small wall thermal conductivities. His results confirmed previous studies carried out for balanced heat exchangers [13]. In another seminal paper, Mori et al. [14] considered a similar configuration and compared their model predictions with those provided by the classical ε -NTU method. They concluded that when axial wall conduction becomes important the ε -NTU method does not give appropriate guidelines for the design of counterflow heat exchangers.

Pagliarini & Barozzi [15] investigated analytically and numerically the role of axial wall conduction on the thermal performance of laminar counterflow double-pipe heat exchangers. With this aim, they combined first- and second-law analyses, calculating the local entropy production rate both in the fluids and in the wall. They concluded that when axial wall conduction becomes significant, the local heat transfer rates at the fluid-wall interfaces are no longer coupled, and tend to reach an asymptotic value far from the inlets. The Nusselt numbers behave similarly, being lower bounded by the classical value for constant wall temperature, because the wall temperature becomes longitudinally uniform in the presence of strong axial wall conduction. By contrast, for negligible axial wall conduction the heat exchanger effectiveness increases monotonically with the thermal conductivity of the partition wall. When axial wall conduction is important, the overall entropy production rate exhibits a minimum, which goes hand in hand with a maximum in the effectiveness, at an optimum intermediate wall conductivity.

Bier et al. [7] observed experimentally that, under certain conditions, micro heat exchangers made of stainless steel (lower conductivity) present higher effectiveness than micro heat exchangers made of copper (higher conductivity), concluding that axial wall conduction is an important factor determining system performance. Stief et al. [8] confirmed that reducing the wall conductivity may lead to higher efficiencies in micro heat exchangers due to the detrimental effect of axial heat conduction in the channel walls. Very high wall conductivities offset the axial temperature gradients within the wall to such an extent that the heat transfer efficiency of any micro heat exchanger, independent of the flow direction specified, always approximates the lower heat transfer efficiency of co-current flow heat exchangers. On the other hand, as the wall conductivity tends to zero the insulating effect associated with the increased wall thermal resistance makes the efficiency to approach zero as well, so that an optimum intermediate

wall conductivity must exist.

Maranzana et al. [16] studied the influence of axial wall conduction on the performance of parallel-plate counterflow micro heat exchangers. Results computed ignoring axial wall conduction (1D wall model) and including both transverse and axial wall conduction effects (2D wall model) were obtained and compared. It was shown that neglecting axial conduction overestimates heat transfer, as intense axial wall conduction turns the counterflow exchanger into a mixer (a system where the outlet temperatures of both fluids tend to be equal), the consequence of axial conduction being to lower the heat exchanger effectiveness. As a result, they concluded that an optimum conductivity must exist for the wall, and that disregarding the effect of axial conduction in the wall can lead to a large bias in the experimental estimation of heat transfer coefficients, particularly for small Reynolds numbers. As main outcome of the analysis, the so-called axial conduction number M was introduced by comparing axial heat transfer by conduction in the wall to convective heat transfer in the flow, concluding that the effect of axial wall conduction could be neglected for $M < 0.01$.

Moreno et al. [17] employed a simple 1D model to investigate the steady-state heat transfer between two fluids in a microscale balanced heat exchanger with adiabatic and isothermal end-wall boundary conditions. Results obtained for the case of isothermal packaging demonstrated that the effectiveness is a weak function of the axial conduction number. As the thermal conductivity of the wall grows, conduction heat losses increase correspondingly, but the effectiveness remains high because the wall temperature is fixed by the external conditions. These results demonstrated the benefit of employing intermediate thermal conductivity wall materials to maintain rapid heat transfer while minimizing conductive heat losses via packaging. Results obtained for the case of adiabatic packaging demonstrated that heat exchanger effectiveness is significantly reduced with increasing wall thermal conductivity, when the system approaches the limit of thermal equilibrium between the two fluids and the wall, equivalent to co-current operation. By contrast, low thermal conductivity materials allow maintenance of substantial thermal gradients along the solid-phase axial length by preventing thermal equilibration.

Koyama & Asako [18] investigated the heat transfer characteristics of a gas-to-gas microchannel heat exchanger, comparing experimental results with those provided by the classical log-mean temperature difference (LMTD) method and by a simple model assuming constant wall temperature. The classical LMTD method overestimates the heat transfer rate observed in the experiments because it neglects longitudinal heat conduction in the wall, an assumption not valid when the wall is thick compared to the channel dimensions, as often occurs in micro heat exchangers. By contrast, the heat transfer rate obtained with the simple model agreed well with the experimental results, because if the thermal conductivity

of the wall is sufficiently high the wall temperature distribution becomes almost uniform. Therefore, they concluded that longitudinal heat conduction in the wall can not be neglected when discussing heat transfer characteristics of microchannel heat exchangers. More recently, Koyama et al. [19] studied the influence of wall thickness. They found that, under their particular operating conditions, a large amount of thermal energy of the hot fluid was transferred to the wall near the inlet, then conducted by the wall in the longitudinal direction, and finally transferred to the cold fluid near its inlet. They confirmed that longitudinal temperature gradients decrease as the partition wall thickness increases. This was attributed to the fact that the axial temperature distribution in the wall becomes more and more uniform as the wall thickness is increased, confirming the previous assumption of constant wall temperature [18].

An optimization process of the wall geometric parameters with respect to maximal thermal power density on parallel-plate counter-flow heat exchangers was carried out by Rogiers & Baelmans [20]. They concluded that small plate thickness and low plate conductivity yield higher thermal power densities, defined as the amount of heat transferred per unit of volume and per unit time, and that power density decreases sharply with increasing design effectiveness. Yang et al. [21] obtained experimental results in qualitative agreement with those of Stief et al. [8], in that devices with partition walls of medium thermal conductivity show larger effectiveness with respect to the case of highly conductive walls, demonstrating once more that in counterflow micro heat exchangers axial wall conduction can significantly reduce the thermal performance. The effect of axial wall conduction was evaluated for cross-flow and co-current configurations as well, obtaining that, contrary to the counterflow case, axial conduction improves the effectiveness for cross-flow and is almost negligible for co-current flow. The larger values of effectiveness were always obtained by counterflow arrangement. Like Maranzana et al. [16], a threshold value for the ratio of axial wall conduction to convective heat flux was used to define the conditions for which axial wall conduction plays a significant role on the heat exchangers effectiveness. Similar results about the better performance of counterflow heat exchangers over cross-flow configurations were also obtained by Schubert et al. [22].

The works cited above clearly show that axial wall conduction has a major impact on the thermal performance of counterflow heat exchangers. Its effect on the effectiveness of heat exchangers seems to be complex, depending on several parameters such as the thickness and thermal conductivity of the wall, the Peclet number of the flow, the heat-exchanger length, but also on whether the heat exchanger is balanced or unbalanced, and on the end-wall thermal boundary conditions. Although the number of investigations on the subject is large, there is still room for a unified mathematical theory which, emphasizing the essentials of the problem, would contribute towards the understanding of multidimensional wall con-

duction phenomenon. The research described in this paper advances in this direction, considering an idealized configuration with analytical solution, the laminar counterflow parallel-plate heat exchanger, as guiding thread. Starting with a brief overview of the theoretical framework laid out by the authors in previous work [23–26], the main non-dimensional parameters that determine axial and transverse wall conduction are identified, and the influence of these parameters on the eigenvalue spectrum and on the heat exchanger effectiveness is evaluated and discussed.

2. Problem formulation

We analyze the flow of two constant-property newtonian fluids in a multilayered counterflow parallel-plate heat exchanger composed by a relatively large number of channels separated by heat conducting plates of finite thickness, δ_w , and thermal conductivity, k_w . The conducting plates allow the exchange of heat through a section of length L . Figure 1 shows a sketch of the model problem under study, where X denotes the longitudinal coordinate measured from the inlet of fluid 1, Y_i represents the transverse coordinate measured from the symmetry plane of channel i , and Y_w is the transverse coordinate measured from wall symmetry plane, oriented from fluid 1 to fluid 2. We shall assume that the Peclet numbers of the flow in the channels, $Pe_i = 2a_i V_i / \alpha_i$, where α_i is the thermal diffusivity of fluid i , are both large compared to unity. In this limiting case we can neglect axial conduction in the fluids in first approximation, and consider that the inlet temperature of fluid i is uniform and equal to $T_{i,\text{in}}$. Further details on the problem setup can be found elsewhere [26]. With the above notation, the differential equations describing the heat exchange between the fluids and the walls in a multilayered laminar counterflow parallel-plate heat exchanger can be written in non-dimensional form using the dimensionless variables $\xi = X / (Pe_1 a_1)$, $y_i = Y_i / a_i$, $y_w = Y_w / \delta_w$, and $\theta_i = (T_i - T_{1,\text{in}}) / (T_{2,\text{in}} - T_{1,\text{in}})$, where $i = \{1, 2\}$ in the definition of y_i , and $i = \{1, 2, w\}$ in the definition of θ_i .

2.1. Thermal problem in the fluids

As previously discussed, if the Peclet numbers of the flow in the channels are sufficiently large, axial conduction in the fluids can be neglected, and the energy equation in the fluids reduces to a balance between axial convection and transverse conduction

$$\frac{3}{4}(1 - y_1^2) \frac{\partial \theta_1}{\partial \xi} = \frac{\partial^2 \theta_1}{\partial y_1^2} \quad (1)$$

$$-m \frac{3}{4}(1 - y_2^2) \frac{\partial \theta_2}{\partial \xi} = \frac{\partial^2 \theta_2}{\partial y_2^2} \quad (2)$$

in $0 < \xi < \xi_L$, $0 < y_i < 1$, where the ratio of the characteristic thermal entry lengths of fluids 2 and 1

$$m = \frac{a_2 \text{Pe}_2}{a_1 \text{Pe}_1} \quad (3)$$

and the dimensionless heat exchanger length

$$\xi_L = \frac{L}{a_1 \text{Pe}_1} \quad (4)$$

emerge as the main governing parameters, both assumed to be of order unity.

The above equations must be integrated with the no-flux boundary conditions at the channel symmetry planes

$$\frac{\partial \theta_1}{\partial y_1} = \frac{\partial \theta_2}{\partial y_2} = 0 \quad \text{at} \quad 0 < \xi < \xi_L, \quad y_i = 0, \quad (5)$$

and the specified uniform temperature profiles at the fluids inlets

$$\theta_1 = \theta_{1,\text{in}} = 0 \quad \text{at} \quad \xi = 0, \quad 0 \leq y_1 < 1, \quad (6)$$

$$\theta_2 = \theta_{2,\text{in}} = 1 \quad \text{at} \quad \xi = \xi_L, \quad 0 \leq y_2 < 1. \quad (7)$$

2.2. Thermal problem in the wall

As will be discussed below, depending on the relative values adopted by the thickness, δ_w , and the thermal conductivity, k_w , of the heat conducting wall, the relative importance of axial and transverse wall conduction may vary significantly, leading to different wall conduction models. Specifically, limiting cases of interest arising for sufficiently thin walls, where axial conduction is absent, result in simplified coupling conditions at the fluid-wall interfaces that will be discussed in Sections 2.2.1 (Model 1) and 2.2.2 (Model 2), while the more general situation where both axial and transverse conduction must be taken into account will be addressed in Section 2.2.3 (Model 3). A distinguished limit that emerges in Model 3 in which axial wall conduction is still relevant but the wall is so thin that it exhibits a nearly uniform transverse temperature profile will be addressed separately in Section 2.2.4 (Model 3').

2.2.1. Thermally thin walls without axial conduction (Model 1)

In the limit of thermally thin walls the temperature differences across the heat exchanger wall are very small and can be neglected in first approximation. This limit can be reached by allowing the wall thermal conductivity to become sufficiently large. If the wall is also sufficiently thin, the longitudinal heat flux, proportional to the product of the wall thickness and the thermal conductivity, can also be neglected. In this double limit, the wall provides perfect (but local) thermal contact between both fluid streams, hence

the thermal coupling condition reduces to the continuity of temperature and wall-normal heat fluxes at each section ξ along the heat exchanger

$$\begin{aligned} \theta_1 = \theta_2 &\equiv \theta_w(\xi) \\ \frac{\partial \theta_1}{\partial y_1} &= -k \frac{\partial \theta_2}{\partial y_2} \equiv \nu(\xi) \end{aligned} \quad \text{at } 0 < \xi < \xi_L, \quad y_i = 1, \quad (8)$$

where $\theta_w(\xi)$ denotes the (transversally uniform) wall temperature distribution and $\nu(\xi)$ the local heat transfer rate from fluid 2 to fluid 1, which are to be determined as part of the solution.

In the analysis addressed here, the absence of longitudinal heat conduction in the fluids implies that the wall temperature at both ends of the heat exchange region, $\theta_w(0)$ and $\theta_w(\xi_L)$, is imposed by the fluid entering the exchanger in that section, hence $\theta_w(0) = 0$ and $\theta_w(\xi_L) = 1$. It is interesting to note that the continuity of wall-normal heat fluxes at the heat conducting wall expressed in (8) introduces the third dimensionless parameter

$$k = \frac{a_1}{a_2} \frac{k_2}{k_1}, \quad (9)$$

representing the normalized fluid-to-fluid conductivity ratio, which is also assumed to be of order unity. For further details on the problem, including the underlying assumptions, conservation equations, and boundary conditions, as well as on the analytical solution to the problem formulated in Eqs. (1)–(9), the reader is referred to Vera & Liñán [23].

2.2.2. Thermally thick walls without axial conduction (Model 2)

In the case of heat exchange materials with lower thermal conductivities the temperature variations across the wall may become comparable to the bulk temperature difference between the fluids. However, as long as the wall remains sufficiently thin, a condition that often holds in macro heat exchanger devices, longitudinal conduction in the wall can still be neglected, introducing only noticeable effects at very short distances from the ends of the heat exchange region [16]. In this case, the wall provides imperfect local thermal contact between the fluids, and the wall temperature distribution $\theta_w(\xi, y_w)$ is given in first approximation by

$$\frac{\partial^2 \theta_w}{\partial y_w^2} = 0 \quad \text{in } 0 < \xi < \xi_L, \quad -1/2 < y_w < 1/2. \quad (10)$$

Due to the absence of the axial conduction term, the above equation can be readily integrated once to give a constant wall-normal heat flux, so that the heat released to the wall by fluid 2 at a given section ξ is absorbed by fluid 1 at the same longitudinal position.

The linear temperature distribution established in the wall transverse direction as a result of the non-zero thermal resistance of the wall introduces now a finite temperature jump between both fluids. As a result, the thermal coupling conditions reduce to the continuity of wall-normal heat fluxes at each section ξ of the heat exchanger, accompanied by an interfacial wall temperature jump proportional to the local heat transfer rate, $\nu(\xi)$, which must be determined as part of the solution

$$\begin{aligned} \theta_2 - \theta_1 &= \frac{1}{\kappa_w} \frac{\partial \theta_1}{\partial y_1} = \frac{\nu(\xi)}{\kappa_w} \\ \frac{\partial \theta_1}{\partial y_1} &= -k \frac{\partial \theta_2}{\partial y_2} \equiv \nu(\xi) \end{aligned} \quad \text{at } 0 < \xi < \xi_L, \quad y_i = 1. \quad (11)$$

Note that the absence of longitudinal heat conduction in the fluids implies once again that at both ends of the heat exchange region the interfacial wall temperature is imposed by the fluid entering the heat exchanger in that section, hence $\theta_w(0, -1/2) = \theta_1(0, 1) = 0$ and $\theta_w(\xi_L, 1/2) = \theta_2(\xi_L, 1) = 1$. Note also that the expression for the temperature jump at the heat conducting wall given in (11) introduces the fourth dimensionless parameter

$$\kappa_w = \frac{a_1}{\delta_w} \frac{k_w}{k_1}, \quad (12)$$

representing the normalized wall-to-fluid conductivity ratio. Interestingly enough, the inverse of this parameter represents the dimensionless wall thermal resistance, $\kappa_w^{-1} = (\delta_w/k_w)/(a_1/k_1)$, which is directly proportional to the wall thickness and inversely proportional to the wall thermal conductivity. In this work, the discussion of results will be carried out indistinctly in terms of κ_w or κ_w^{-1} .

As implied by the first equation in (11), the limit of thermally thin walls addressed in Section 2.2.1 should be recovered as $\kappa_w \rightarrow \infty$, i.e., as the wall thermal resistance goes to zero, $\kappa_w^{-1} \rightarrow 0$. However, a more refined condition can be established by requiring that the normalized wall temperature jump be small compared to unity. According to Eq. (11), this condition can be written as

$$\frac{\nu(\xi)}{\kappa_w} \ll 1. \quad (13)$$

Estimating the local heat transfer rate $\nu(\xi)$ from the zeroth-order two-term solution for nearly-balanced heat exchangers obtained by Vera & Liñán [23, Eq. (87)]

$$\nu(\xi) \simeq \nu^{(0)}(\xi) \simeq \frac{1}{2} \left(\xi_L + \frac{17}{70} \frac{k+1}{k} \right)^{-1}, \quad (14)$$

and requiring that the wall temperature jump be less than, say, 1% of the overall temperature difference, the condition (13) takes the form

$$\kappa_w^{-1} < 0.01 \times 2 \left(\xi_L + \frac{17}{70} \frac{k+1}{k} \right) \quad (15)$$

written in terms of the dimensionless parameters k , κ_w , and ξ_L .

A thorough analysis of laminar counterflow parallel-plate heat exchanger with thermally thick walls without axial wall conduction, including the analytical solution to the problem stated in Eqs. (1)–(7) and (10)–(12), can be found in Quintero et al. [25].

2.2.3. Thermally thick walls with axial conduction (Model 3)

Regardless of the value of the wall conductivity, if the thickness of the heat conducting wall becomes sufficiently large there is a point where the effect of axial wall conduction can no longer be neglected. In this case, the wall energy equation must also incorporate the longitudinal conduction term

$$\Delta_w^2 \frac{\partial^2 \theta_w}{\partial \xi^2} + \frac{\partial^2 \theta_w}{\partial y_w^2} = 0 \quad \text{in} \quad 0 < \xi < \xi_L, \quad -1/2 < y_w < 1/2, \quad (16)$$

which introduces the dimensionless wall thickness

$$\Delta_w = \frac{\delta_w}{a_1} \frac{1}{\text{Pe}_1}, \quad (17)$$

as fifth dimensionless parameter measuring the relative importance between the axial and transverse conduction terms. This new parameter is defined as the ratio of the wall thickness, δ_w , to the characteristic thermal entry length of fluid 1, $a_1 \text{Pe}_1$, the same scale used to define the dimensionless coordinate ξ and the dimensionless heat exchanger length ξ_L . As a result, in this work the dimensionless wall thickness will be assumed to take moderately small values, ranging roughly from 10^{-4} to 1. It should be noted that the ratio $\xi_L/\Delta_w = L/\delta_w$ represents the aspect ratio of the heat exchanger wall, which is typically large in practical applications, even in mini/micro systems. Although this could restrict the dimensionless wall thickness to be moderately small compared to unity [26], the results presented below extend to values up to $\Delta_w \sim 1$ in order to give a broader parametric perspective.

Inspection of Eq. (16) shows that the limit of thermally thick walls with negligible axial wall conduction, addressed in Section 2.2.2, must be recovered as $\Delta_w \rightarrow 0$, although local effects of axial wall conduction may still appear in small boundary-layer like regions at very short distances from the ends of the heat exchange region even for vanishingly small values of Δ_w .

The presence of the axial wall conduction term in (16) requires in this case integration of the heat equation in the wall (16) subject to the compatibility conditions of continuity of temperatures

$$\begin{aligned} \theta_1(\xi, 1) = \theta_w(\xi, -1/2) &\equiv \theta_{w1}(\xi) \\ \theta_2(\xi, 1) = \theta_w(\xi, 1/2) &\equiv \theta_{w2}(\xi) \end{aligned} \quad \text{at} \quad 0 < \xi < \xi_L, \quad (18)$$

and wall-normal heat fluxes

$$\begin{aligned} \left. \frac{\partial \theta_1}{\partial y_1} \right|_{y_1=1} &= \kappa_w \left. \frac{\partial \theta_w}{\partial y_w} \right|_{y_w=-1/2} \equiv v_1(\xi) \\ -k \left. \frac{\partial \theta_2}{\partial y_2} \right|_{y_2=1} &= \kappa_w \left. \frac{\partial \theta_w}{\partial y_w} \right|_{y_w=1/2} \equiv v_2(\xi) \end{aligned} \quad \text{at } 0 < \xi < \xi_L, \quad (19)$$

at the fluid-wall interfaces, where the interfacial fluid-wall temperatures, $\theta_{wi}(\xi)$, and heat fluxes, $v_i(\xi)$, must be obtained as part of the solution.

The longitudinal conduction term forces also to set appropriate boundary conditions at both ends of the heat exchanging wall. One possibility, compatible with the presence of adiabatic regions at both ends of the heat exchange region, is to impose insulated end-wall (i.e., homogeneous Neumann) boundary conditions

$$\begin{aligned} \frac{\partial \theta_w}{\partial \xi} &= 0 \quad \text{at } \xi = 0, \quad -1/2 < y_w < 1/2, \\ \frac{\partial \theta_w}{\partial \xi} &= 0 \quad \text{at } \xi = \xi_L, \quad -1/2 < y_w < 1/2. \end{aligned} \quad (20)$$

However, one may alternatively prescribe the temperature at both ends of the heat conducting wall, resulting in the specified end-wall temperature (i.e., Dirichlet) conditions

$$\begin{aligned} \theta_w &= 0 \quad \text{at } \xi = 0, \quad -1/2 < y_w < 1/2, \\ \theta_w &= 1 \quad \text{at } \xi = \xi_L, \quad -1/2 < y_w < 1/2, \end{aligned} \quad (21)$$

where the end-wall temperature is set equal to that of the fluid entering the heat exchanger in that section.

It should be noted that end-wall boundary conditions depend of the way the channels are connected to the outside. In this sense, the best way to isolate the exchanger from the outside is to specify adiabatic boundary conditions at both ends of the wall. In the real world, and particularly in mini/micro systems, heat losses or gains from the environment could hardly be avoided. However, including this effect would require to analyze the whole system, not only the unitary heat exchange cell studied here. As first approximation, one may specify end-wall temperatures as imposed by the fluids entering the heat exchanger, as originally proposed by Peterson [4]. This is a limiting case that could be used to assess the effect of conductive heat losses or gains from the environment at both ends of micro heat exchangers, and is included here for completeness. However, the analytical and numerical results to be presented in Section 4 will be mainly concerned with the adiabatic end-wall case. It should be noted that the use of adiabatic or specified temperature end-wall boundary conditions introduces only small changes in the heat exchanger effectiveness for intermediate values of the dimensionless parameters Δ_w and κ_w^{-1} [26], hence no qualitative changes are anticipated to occur for end-wall boundary conditions other than the adiabatic case considered here.

In order to clarify the limits of applicability of the simplified formulations presented in Sections 2.2.1 and 2.2.2 (Models 1 and 2) with respect to the more general model presented in this section (Model 3), Quintero & Vera [26] resorted to the concept of axial conduction number successfully employed by Maranzana et al. [16] to derive an analytical condition for neglecting axial wall conduction. In particular, the numerical results to be presented in Section 4 confirm that axial conduction has a negligible effect in laminar counterflow parallel-plate heat-exchanger performance ($< 1 - 5\%$ effectiveness reduction due to axial wall conduction) as long as

$$\frac{\kappa_w \Delta_w^2}{\xi_L + \left[\frac{17}{70} \frac{k+1}{k} + \frac{1}{2\kappa_w} \right]} < 0.01. \quad (22)$$

The assumptions underlying this condition (i.e., nearly-balanced heat exchangers that are sufficiently long to be correctly described by the first non-trivial eigenmode) make it work particularly well for moderately long heat exchangers ($\xi_L \gtrsim 1$), while the simpler criterion established by Maranzana et al. [16], written in our nomenclature as

$$\frac{\kappa_w \Delta_w^2}{\xi_L} < 0.01, \quad (23)$$

performs better for short heat exchangers ($\xi_L \lesssim 1$) as will be seen later.

2.2.4. Thermally thin walls with axial conduction (Model 3')

In connection with the above result, it is interesting to note that, in the thermally thin wall limit, $\kappa_w \rightarrow \infty$, the criterion (22) can only be fulfilled if Δ_w tends to zero faster than $\kappa_w^{1/2}$ tends to infinity. If, on the other hand, Δ_w is kept constant while κ_w is allowed to grow indefinitely, the inequality in (22) would eventually be violated at some point. This would bring axial heat conduction into play, something that is known to have a significant impact on heat-exchanger performance [14]. In this particular limit, in which the wall behaves as thermally thin, $\theta_w(\xi, y_w) \simeq \theta_w(\xi)$, but axial wall conduction still plays a role, the condition of continuity of temperatures at the fluid-wall interfaces given in (18) can be substituted by the continuity of temperatures across the wall

$$\theta_1(\xi, 1) = \theta_2(\xi, 1) \equiv \theta_w(\xi) \quad \text{in} \quad 0 < \xi < \xi_L. \quad (24)$$

In addition, the energy equation in the solid (16) can be integrated across the wall to give

$$\Delta_w^2 \kappa_w \frac{d^2 \theta_w}{d\xi^2} + \nu_1(\xi) - \nu_2(\xi) = 0 \quad \text{in} \quad 0 < \xi < \xi_L, \quad (25)$$

where the interfacial wall-normal heat fluxes $\nu_i(\xi)$ introduced in (19) appear when evaluating the upper and lower limits of the integral of the transverse conduction term. Note that in the limit $\Delta_w^2 \kappa_w \rightarrow 0$

this equation reduces identically to the condition of continuity of heat fluxes, $v_1(\xi) = v_2(\xi) \equiv v_w(\xi)$, characterizing the performance of heat exchangers with negligible effects of axial wall conduction.

Previous investigations pertaining to the simplified wall conduction model represented by Eqs. (24) and (25), hereafter denoted as Model 3', have been carried out, for instance, by Koyama et al. [18, 19]. Although a separated analysis of this problem could be of interest by itself, it would require a different approach in order to obtain an exact analytical solution in the distinguished limit $\kappa_w \rightarrow \infty$ with $\Delta_w^2 \kappa_w$ of order unity. Nevertheless, as for Models 1 and 2, numerical results corresponding to Model 3' can be readily obtained from the general solution of Model 3 using appropriate values of the dimensionless parameters, and therefore it will not be addressed separately below.

2.3. Summary of problems under study

Table 1 summarizes the differential equations, boundary conditions, and dimensionless parameters involved in the three wall conduction models discussed in the previous sections. Moving from top to bottom, the mathematical complexity of the problem increases, as it does the number of parameters involved. The table also includes the thermal coupling conditions imposed at the fluid-wall interfaces in the three models under study. As previously discussed, all the results presented in this work as pertaining to Models 1, 2, and 3' have been computed from the solution of Model 3 with the dimensionless parameters evaluated in the corresponding domain of validity. Thus, in the plots presented below the labels Model 1, 2, 3, and 3' serve only to indicate the domains of validity of the different models, but do not mean that the corresponding results have been obtained with the labeled model. Adiabatic end-wall boundary conditions were considered in all cases.

2.4. Analytical solution to Model 3

The analytical solution to the mathematical problem corresponding to Model 3, given by Eqs. (1)–(7) and (16)–(20), has been recently obtained by the authors [26] and is summarized here for completeness. It provides the temperature field of both fluids, $\theta_i(\xi, y_i)$, along with the wall temperature distribution, $\theta_w(\xi, y_w)$, as infinite series of eigenfunctions and eigenvalues for given values of m , k , κ_w , Δ_w , and ξ_L . The temperature distribution for fluid i can be written as

$$\theta_i(\xi, y_i) = A + \sum_{n=-\infty}^{+\infty} C_n e^{-\lambda_n \xi} h_{n,i}(y_i), \quad (26)$$

while the temperature distribution at the wall is given by

$$\theta_w(\xi, y_w) = A + \sum_{n=-\infty}^{+\infty} C_n e^{-\lambda_n \xi} h_{n,w}(y_w). \quad (27)$$

The eigenvalues λ_n are determined by the eigencondition

$$\begin{aligned} & \left[f'_{n,1}(1; \lambda_n) - k f'_{n,2}(1; m \lambda_n) \right]^2 \\ &= \left\{ \left[f'_{n,1}(1; \lambda_n) + k f'_{n,2}(1; m \lambda_n) \right] \tan(\Delta_w \lambda_n / 2) + 2 \kappa_w \Delta_w \lambda_n \right\} \\ & \quad \times \left\{ \frac{f'_{n,1}(1; \lambda_n) + k f'_{n,2}(1; m \lambda_n)}{\tan(\Delta_w \lambda_n / 2)} - 2 \kappa_w \Delta_w \lambda_n \right\}, \quad (28) \end{aligned}$$

which restricts the values of λ_n to an infinite numerable set of both positive and negative eigenvalues that can be obtained numerically for specified values of m , k , κ_w , and Δ_w .

As shown in [26], the eigenfunctions $h_{n,i}(y_i)$ and $h_{n,w}$ appearing in (26) and (27) can be expressed in terms of the eigenfunctions obtained by Vera & Liñán [23] in the limit of thermally thin walls without axial conduction (Model 1)

$$f_{n,i}(y_i) = \frac{1}{\sqrt{y_i}} \left\{ \frac{2 \sqrt{\pi} M_{\kappa_{n,i}, \frac{1}{4}}(4 \kappa_{n,i} y_i^2) + \Gamma\left(\frac{1}{4} - \kappa_{n,i}\right) W_{\kappa_{n,i}, \frac{1}{4}}(4 \kappa_{n,i} y_i^2)}{2 \sqrt{\pi} M_{\kappa_{n,i}, \frac{1}{4}}(4 \kappa_{n,i}) + \Gamma\left(\frac{1}{4} - \kappa_{n,i}\right) W_{\kappa_{n,i}, \frac{1}{4}}(4 \kappa_{n,i})} \right\}, \quad (29)$$

and their derivatives at the wall

$$\begin{aligned} f'_{n,i}(1) \equiv \left. \frac{df_{n,i}}{dy_i} \right|_{y_i=1} &= -\frac{1}{2} + 2 \left\{ \kappa_{n,i} \right. \\ & \quad \left. + \frac{2 \sqrt{\pi} (3/4 + \kappa_{n,i}) M_{\kappa_{n,i}+1, \frac{1}{4}}(4 \kappa_{n,i}) - \Gamma\left(\frac{1}{4} - \kappa_{n,i}\right) W_{\kappa_{n,i}+1, \frac{1}{4}}(4 \kappa_{n,i})}{2 \sqrt{\pi} M_{\kappa_{n,i}, \frac{1}{4}}(4 \kappa_{n,i}) + \Gamma\left(\frac{1}{4} - \kappa_{n,i}\right) W_{\kappa_{n,i}, \frac{1}{4}}(4 \kappa_{n,i})} \right\}, \quad (30) \end{aligned}$$

where $\kappa_{n,1} = \sqrt{3 \lambda_n} / 8$, $\kappa_{n,2} = \mathbf{i} \sqrt{3 m \lambda_n} / 8$, $M_{\kappa, \mu}(z)$ and $W_{\kappa, \mu}(z)$ are the Whittaker functions [28, 29], $\Gamma(z)$ is the Gamma function, and $\mathbf{i}$ is the imaginary unit. Note that the eigenvalue λ_n appears both in the argument and in the first index of the Whittaker functions through $\kappa_{n,i}$ [30]. After obtaining a sufficiently large number of eigenvalues from Eq. (28), the solution of the problem can be evaluated by calculating the expansion constants A and C_n appearing in Eqs. (26) and (27) as indicated by Quintero & Vera [26, Eqs. 47–49].

3. Analysis of the eigenvalue spectrum

To better understand the role of axial and transverse wall conduction on the thermal performance of laminar counterflow parallel-plate heat exchangers, it is convenient to start with a detailed analysis of the eigenvalue spectrum $\lambda_n(m, k, \Delta_w, \kappa_w^{-1})$. The eigencondition that determines the eigenvalues in the more general case (Model 3) is given by Eq. (28), with $f'_{n,i}(1)$ evaluated from Eq. (30). Given the mathematical complexity of the eigencondition, initial guesses for the eigenvalues λ_n are highly desirable when trying to solve it numerically. To this end, accurate estimates can be obtained from the asymptotic analysis of the singularities exhibited by (28), which is to be presented below.

3.1. Singularities of the eigencondition and initial guesses for the eigenvalues

As discussed by Quintero & Vera [26], the eigencondition (28) of the more general problem (Model 3) reduces to simpler forms (Models 1 and 2) when axial wall conduction is negligible. This can be used to obtain initial guesses for the eigenvalues for small values of Δ_w . The condition for axial wall conduction to be negligible (say, below 1% error) in the determination of λ_n is given by Quintero & Vera [26, Eq. 58] as

$$\kappa_w \Delta_w^2 \lambda_n^2 < 0.01 \quad \text{or} \quad |\lambda_n| < \frac{0.1}{\sqrt{\kappa_w \Delta_w}}. \quad (31)$$

When this condition is fulfilled, an initial approximation for λ_n can be obtained from the simplified eigencondition obtained by Quintero et al. [25, Eq. 30] in the limit of thermally thick walls without axial wall conduction (Model 2). Note that the simpler eigencondition of Model 1, obtained by Vera & Liñán [23, Eq. 25], emerges as a particular case of Model 2 in the limit of vanishingly small wall thermal resistance, $\kappa_w^{-1} \rightarrow 0$.

However, condition (31) is not strictly applicable to the lowest order eigenvalue, λ_0 , which may be affected by axial wall conduction earlier than implied by this condition. The following approximate expression for λ_0 was obtained by Quintero & Vera [26, Eq. 63]

$$\lambda_0 \simeq \frac{\lambda_0^0}{\left[1 + \frac{2\kappa_w \Delta_w^2}{mk-1} \lambda_0^0\right]}, \quad (32)$$

where λ_0^0 is the numerical value of λ_0 obtained as $\Delta_w \rightarrow 0$, provided, e.g., from the eigencondition of Quintero et al. [25, Eq. 30]). The previous expression shows that the condition for neglecting axial wall conduction in the determination of λ_0 (again, below 1% error) can be written as

$$\frac{2\kappa_w \Delta_w^2}{mk-1} \lambda_0^0 < 0.01. \quad (33)$$

When conditions (31) and (33) are not fulfilled, the eigenvalues depend both on the wall thermal resistance, κ_w^{-1} , and the dimensionless wall thickness, Δ_w . Concerning the lowest order eigenvalue, λ_0 , a good initial guess can be obtained from Eq. (32), valid for all Δ_w . By contrast, the dependence of the eigenvalues of higher order on κ_w^{-1} and Δ_w is not that simple. A better understanding of this complex behavior can be obtained by rearranging eigencondition (28) in the alternative form

$$\begin{aligned} & k f'_{n,1}(1; \lambda_n) f'_{n,2}(1; m, \lambda_n) - 4\kappa_w^2 \left(\frac{\Delta_w \lambda_n}{2} \right)^2 \\ & + \kappa_w \left(\frac{\Delta_w \lambda_n}{2} \right) \left[f'_{n,1}(1; \lambda_n) + k f'_{n,2}(1; m, \lambda_n) \right] \\ & \times \left[\frac{1}{\tan(\Delta_w \lambda_n / 2)} - \tan(\Delta_w \lambda_n / 2) \right] = 0. \end{aligned} \quad (34)$$

The fact is that in this equation there are several terms that become infinitely large in certain regions of the $(\Delta_w, \kappa_w^{-1})$ plane. As those terms become large, others must become small for the equation to remain fulfilled. As will be shown below, this simple idea can be used to obtain simplified forms of the eigenfunction in the vicinity of the singularities.

First, Quintero & Vera [26] noticed the existence of divergences of the function $f'_{n,i}(1; \lambda_n)$ as λ_n approaches an infinitely numerable set of singular values. Thus, for the positive eigenfunctions, $n > 0$, the singularities arise as $\lambda_n \rightarrow \lambda_p^\infty$, whereas for the negative eigenfunctions, $n < 0$, they arise as $-\lambda_n \rightarrow \lambda_p^\infty$, where λ_p^∞ , $p \in \mathbb{N}$, represents the set of singular values

$$\lambda_p^\infty \simeq \{3.77, 42.86, 124.63, 249.07, 416.18, 625.96, \dots\}. \quad (35)$$

In addition, the tangents appearing in (34) can go either to zero or to infinity according to

$$\tan(\Delta_w \lambda_n / 2) \rightarrow \begin{cases} 0 & \text{as } \Delta_w \lambda_n / 2 \rightarrow q\pi \\ \infty & \text{as } \Delta_w \lambda_n / 2 \rightarrow (q + 1/2)\pi \end{cases} \quad (36)$$

with $q \in \mathbb{N}$, which introduces additional singularities in the last term of Eq. (34).

One can take advantage of these singularities to extract relevant information from the eigencondition that would be very complicated to obtain otherwise (e.g., numerically). For instance, grouping the terms involving a $f'_{n,1}$ factor in Eq. (34) and rearranging gives the following expression

$$\begin{aligned} k f'_{n,2}(1; m, \lambda_n) + \kappa_w \left(\frac{\Delta_w \lambda_n}{2} \right) \left[\frac{1}{\tan(\Delta_w \lambda_n / 2)} - \tan(\Delta_w \lambda_n / 2) \right] = \\ \frac{1}{f'_{n,1}(1; \lambda_n)} \left\{ 4 \kappa_w^2 \left(\frac{\Delta_w \lambda_n}{2} \right)^2 - \kappa_w \left(\frac{\Delta_w \lambda_n}{2} \right) k f'_{n,2}(1; m, \lambda_n) \right. \\ \left. \times \left[\frac{1}{\tan(\Delta_w \lambda_n / 2)} - \tan(\Delta_w \lambda_n / 2) \right] \right\}, \quad (37) \end{aligned}$$

whose right hand side vanishes as $\lambda_n \rightarrow \lambda_p^\infty$, $n > 0$, when $|f'_{n,1}(1; \lambda_n)| \rightarrow \infty$. As a result, the asymptotic form of the eigencondition in this particular limit,

$$\kappa_w^{-1} k f'_{n,2}(1; m, \lambda_p^\infty) + \left(\frac{\Delta_w \lambda_p^\infty}{2} \right) \left[\frac{1}{\tan(\Delta_w \lambda_p^\infty / 2)} - \tan(\Delta_w \lambda_p^\infty / 2) \right] \simeq 0, \quad (38)$$

can be used to determine the pairs $(\Delta_w, \kappa_w^{-1})$ where the original eigenfunction (34) becomes singular due to the divergence of $f'_{n,1}$. A similar expression is obtained in the limiting case $\lambda_n \rightarrow -\lambda_p^\infty / m$, $n < 0$, when

$f'_{n,2}$ is the term that diverges, namely

$$\kappa_w^{-1} f'_{n,1}(1; -\lambda_p^\infty/m) + \left(\frac{\Delta_w \lambda_p^\infty}{2m} \right) \left[\frac{1}{\tan(\Delta_w \lambda_p^\infty/2m)} - \tan(\Delta_w \lambda_p^\infty/2m) \right] \simeq 0. \quad (39)$$

The above expressions can be simplified further in the limit of small wall thermal resistance, $\kappa_w^{-1} \rightarrow 0$, when they reduce respectively to $\tan^2(\Delta_w \lambda_p^\infty/2) \simeq 1$ and $\tan^2(\Delta_w \lambda_p^\infty/2m) \simeq 1$, giving

$$\Delta_w \simeq \begin{cases} \frac{(q+1/2)\pi/2}{\lambda_p^\infty/2} & \lambda_n \simeq \lambda_p^\infty \\ \frac{(q+1/2)\pi/2}{\lambda_p^\infty/2m} & \lambda_n \simeq -\lambda_p^\infty/m \end{cases} \quad n, p \in \mathbb{N}, \quad q \in \mathbb{N}_0 \quad (40)$$

expressions that provide appropriate initial guesses when Eqs. (38) and (39) are used to determine the singular value of Δ_w corresponding to a specified value of κ_w^{-1} and a certain singular value λ_p^∞ .

As previously discussed, another type of divergence occurs when the tangents in (34) go either to zero or to infinity, as indicated by Eq. (36). For the left hand side of Eq. (34) to remain bounded, the factor multiplying the tangents must then go to zero

$$f'_{n,1}(1; \lambda_n) + k f'_{n,2}(1; m, \lambda_n) \simeq 0, \quad (41)$$

which corresponds exactly to the eigencondition obtained by Vera & Liñán [23, Eq. 25] in the limit of thermally thin walls without wall thermal resistance (Model 1). That is, when the tangents become singular the eigenvalues reduce to those obtained by Vera & Liñán [23], λ_n^{VL} , with the associated singular value of Δ_w given by

$$\Delta_w = \frac{r\pi}{\lambda_n^{\text{VL}}} \quad n \in \{0, 1, 2, \dots, r-1\}, \quad r \in \mathbb{N}, \quad (42)$$

obtained directly from (36), with r odd (even) for $\tan(\Delta_w \lambda_n/2) \rightarrow \infty (\rightarrow 0)$. When $\tan(\Delta_w \lambda_n/2)$ is neither infinite nor zero, but it is sufficiently large or small, the last equation can be inverted to get an approximate expression for the eigenvalues

$$\lambda_n \simeq \frac{r\pi}{\Delta_w} \quad r \in \mathbb{N}, \quad (43)$$

corresponding to the final asymptotes shown on the left panel of Fig. 2, to be discussed below. Once the singularities of the eigenfunction have been unveiled it is much easier to evaluate the eigenvalues in an efficient way. But before computing the thermal response of the system, illustrative results concerning the eigenvalue spectrum will be presented and discussed in the next subsection.

3.2. The eigenvalue spectrum

Figure 2 shows the variation of λ_0 , $\lambda_{\pm 1}$, and $\lambda_{\pm 2}$ with the dimensionless heat exchanger wall Δ_w for the particular case $m = 2$, $k = 1$, and $\kappa_w = 20$. As can be seen, all the eigenvalues present a qualitatively similar dependence with Δ_w , except perhaps λ_0 , which behaves differently. This dependence includes an initial plateau for very small values of Δ_w , when axial wall conduction is absent; a smooth decrease down to a second narrower plateau (whose width increases with κ_w , although this behavior is not appreciated in the figure); and a final stair-shaped descent where the λ_n follow successively the horizontal and diagonal asymptotes defined by Eqs. (40) and (43), which are approached asymptotically as $\kappa_w \rightarrow \infty$. In order to check that the eigenvalues are correctly computed from (28), the plot also shows the singular points (symbols) and asymptotic trends (dashed lines) obtained in the previous section, as clearly indicated in the caption. The figure also shows the variation of $f'_{n,1}(1; \lambda_n)$ with λ_n as given by Eq. (30), highlighting the divergences of $f'_{n,1}(1; \lambda_n)$ at the singular values λ_p^∞ listed in (35), which are also shown on top of the plot for reference purposes. The terms responsible for the singularities appearing in eigencondition (28) are shown in different colors, blue for $f'_{n,1}(1; \lambda_n)$, red for $f'_{n,2}(1; \lambda_n)$, and green for $\tan(\Delta_w \lambda_n/2)$. As shown in the figure, the singularities act as local attractors for the eigenvalues, whose calculation becomes largely benefited by the a-priori knowledge of the asymptotic behavior of the eigencondition (28) in the vicinity of those singularities discussed in the previous section.

Figure 3 shows contour plots of λ_0 , $\lambda_{\pm 1}$, $\lambda_{\pm 2}$, $\lambda_{\pm 3}$ and $\lambda_{\pm 4}$ in the $(\Delta_w, \kappa_w^{-1})$ plane for $m = 2$ and $k = 1$. Three different regions can be distinguished in the plots. In the first region, to the left of the solid blue line, conditions (31) and (33) are fulfilled, so that there is no influence of axial wall conduction on the eigenvalues, which depend solely on the dimensionless wall thermal resistance, κ_w^{-1} , but not on the dimensionless wall thickness, Δ_w . The horizontal dashed blue lines represented in the different plots were obtained from the eigencondition obtained by Quintero et al. [25, Eq. 30], conveniently rewritten in the form

$$\kappa_w = - \frac{f'_{n,1}(1; \lambda_n)}{\frac{f'_{n,1}(1; \lambda_n)}{k f'_{n,2}(1; m, \lambda_n)} + 1}, \quad (44)$$

to explicitly obtain the value of κ_w corresponding to given values of m , k , and λ_n .

In the second region, between the solid blue and red lines, the eigenvalues depend both on κ_w and Δ_w , meaning that both axial and transverse wall conduction are equally important. This region widens for small values of κ_w^{-1} , when the eigencondition reduces in first approximation to

$$f'_{n,1}(1) + k f'_{n,2}(1) = \kappa_w \Delta_w^2 \lambda_n^2, \quad (45)$$

as obtained by Quintero & Vera [26, Eq. 56] by retaining the leading order term accounting for the effect of axial wall conduction in the limit of small wall thermal resistance, $\kappa_w^{-1} \ll 1$. This equation can be rewritten in the form

$$\kappa_w \Delta_w^2 = \frac{f'_{n,1}(1) + k f'_{n,2}(1)}{\lambda_n^2}, \quad (46)$$

which provides explicitly the value of $\kappa_w \Delta_w^2$ corresponding to given values of m , k , and λ_n . The red dashed lines plotted in the intermediate region of Fig. 3 are obtained from Eq. (46) and correspond to straight diagonal lines with slope 2 in the $(\Delta_w, \kappa_w^{-1})$ plane when plotted in log-log scale.

The third region appears to the right of the red solid line, where the eigenvalues depend strongly on Δ_w but only marginally on κ_w^{-1} . The behavior in this region is mainly dictated by the singular values λ_p^∞ given in (35) and by the final asymptotes (43). For instance, the solid red line corresponds to $\Delta_w = \pi/\lambda_{n-1}^\infty$ for $n > 0$ (or $m\pi/\lambda_{n+1}^\infty$ for $n < 0$), that is, the value of Δ_w where the second plateau of λ_n intercepts the first asymptote, $|\lambda_n| \simeq \pi/\Delta_w$. To the right of this line the eigenvalue decreases almost linearly with Δ_w until it reaches a small third plateau, located around λ_{n-1}^∞ , where it remains approximately constant until reaching the second asymptote in $\Delta_w \simeq 2\pi/\lambda_{n-2}^\infty$ for $n > 0$ (or $m\pi/\lambda_{n+2}^\infty$ for $n < 0$). This sequence is repeated until the n -th eigenvalue reaches its final asymptote $|\lambda_n| \simeq |n|\pi/\Delta_w$ at around $\Delta_w \simeq |n|\pi/\lambda_1^\infty$ for $n > 0$ (or $m|n|\pi/\lambda_1^\infty$ for $n < 0$), along which it finally decays to zero as $\Delta_w \rightarrow \infty$.

It is interesting to note that Fig. 2 represents a horizontal slide at $\kappa_w^{-1} = 1/20 = 5 \times 10^{-2}$ of various of the subplots shown in Fig. 3, and thus it also shows the three different regions discussed above: the initial plateau, the second plateau, and the final stair-shaped descent along the final asymptotes.

As shown in the center plot of Fig. 3, the behavior of the lowest order eigenvalue is simpler, and exhibits also a faster (in this case quadratic) decay rate for large values of Δ_w , given asymptotically by $\lambda_0 \simeq (mk - 1)/(2\kappa_w \Delta_w^2)$, as implied by Eq. (32) of 3.1. The numerical results show that for a fixed value of Δ_w there is always an optimum value of κ_w that maximizes λ_0 , which indicates the possibility of an optimum wall thermal conductivity that maximizes heat exchanger performance. An approximate expression for this optimum conductivity can be obtained using the relation (32) in 3.1, with λ_0^0 given approximately by Quintero et al. [25, Appendix A, Eq. A.1], which yields

$$\kappa_w^{\text{opt}, \lambda_0} \simeq \frac{1}{2} \frac{\sqrt{mk}}{\Delta_w}. \quad (47)$$

The dashed black lines shown in the contour plots of λ_0 indicate that (47) provides a very good approximation to the actual value of $\kappa_w^{\text{opt}, \lambda_0}$. In this regard, note that although the value of λ_0^0 obtained by Quintero et al. [25, Appendix A, Eq. (A.1)] was obtained for nearly-balanced heat exchangers ($mk \simeq 1$), it seems to perform perfectly for moderately unbalanced heat exchangers as well. The relevance of Eq. (47) lies

in that for sufficiently long heat exchangers, $\xi_L \geq 1$, the influence of higher order eigenvalues is limited to small regions close to the fluid inlets, and the behavior of the heat exchanger is mainly determined by the lowest-order eigenmode. In this case, higher heat exchanger effectiveness are obtained for larger values of λ_0 . To finish the discussion, it is worth noting that the differences between the positive and negative eigenvalues observed in Fig. 3 are due to the strong dependence of the negative eigenvalues on the value of m , as already noted by Quintero & Vera [26, Eq. 42] and, previously, by Vera & Liñán [23].

4. Parametric study of heat exchanger performance

Relevant results for the discussion of mini/micro counterflow heat exchanger performance include the local heat-transfer rates from fluid 2 to the wall, $v_2(\xi)$, and from the wall to fluid 1, $v_1(\xi)$, the interfacial wall-fluid temperature distributions

$$\theta_{w1}(\xi) = \theta_1(\xi, 1) = \theta_w(\xi, -1/2) \quad \text{at} \quad 0 < \xi < \xi_L, \quad (48)$$

$$\theta_{w2}(\xi) = \theta_2(\xi, 1) = \theta_w(\xi, 1/2) \quad \text{at} \quad 0 < \xi < \xi_L, \quad (49)$$

and the bulk—or mixing-cup—temperature distributions

$$\theta_{mi}(\xi) = \int_0^1 \frac{3}{2} (1 - y_i^2) \theta_i(\xi, y_i) dy_i, \quad i = 1, 2. \quad (50)$$

The main outcome of the analysis is, in particular, the outlet bulk temperature of the fluid streams, $\theta_{m1,\text{out}} \equiv \theta_{m1}(\xi_L)$ and $\theta_{m2,\text{out}} \equiv \theta_{m2}(0)$, closely related to the heat exchanger effectiveness [23, 25]

$$\varepsilon = \begin{cases} \theta_{m1,\text{out}} & \text{for } mk \geq 1, \\ 1 - \theta_{m2,\text{out}} & \text{for } mk \leq 1. \end{cases} \quad (51)$$

Other relevant results for the discussion of axial wall conduction effects are the average longitudinal heat flux conducted by the separating wall in the negative ξ -direction, given in dimensionless form by

$$v_w(\xi) = \Delta_w k_w \int_{-1/2}^{1/2} \frac{\partial \theta_w}{\partial \xi} dy_w, \quad (52)$$

along with the local Nusselt numbers, defined as follows

$$\text{Nu}_1(\xi) = \frac{4v_1(\xi)}{\theta_{w1}(\xi) - \theta_{m1}(\xi)}, \quad (53)$$

$$\text{Nu}_2(\xi) = \frac{4v_2(\xi)}{k(\theta_{m2}(\xi) - \theta_{w2}(\xi))}. \quad (54)$$

Exact and approximate expressions for these output data can be found in our previous investigations devoted to the detailed analysis of the different wall conduction models [23, 25, 26].

4.1. Domains of validity of the wall conduction models

The discussion on the applicability of the different wall conduction models presented in Sections 2.2.1 to 2.2.4 results in two conditions that divide the $(\Delta_w, \kappa_w^{-1})$ plane in four regions, roughly one for each of the models considered in this study: 1, 2, 3, and 3'. The transverse wall temperature jump can be neglected as long as the dimensionless wall thermal resistance is sufficiently small, as indicated by condition (15), whereas the effect of axial wall conduction can be neglected as long as the dimensionless wall thickness is sufficiently small, according to condition (22).

To illustrate the domains of validity of the different wall conduction models, Fig. 4 shows contour plots of the normalized effectiveness $\varepsilon/\varepsilon_{\max}$ for fixed values of m , k , and $\xi_L \gtrsim 1$. In this and subsequent figures $\varepsilon_{\max}$ represents the maximum effectiveness of the system, achieved in the limit of thermally thin walls without axial wall conduction (Model 1) located on the lower left corner of the plots, where therefore $\varepsilon/\varepsilon_{\max}$ always tends to unity. The horizontal solid line obtained from condition (15) divides the $(\Delta_w, \kappa_w^{-1})$ plane into a lower half without significant wall thermal resistance effects, where the temperature is continuous across the wall, and an upper half where the wall temperature jump becomes significant. The oblique solid line that divides the $(\Delta_w, \kappa_w^{-1})$ plane represents, on the other hand, the limit for neglecting axial wall conduction. To the left of this line axial wall conduction can be safely neglected, which defines the limit of applicability of Models 1 and 2 (thermally thin/thick walls without axial conduction, Sections 2.2.1 and 2.2.2), while to the right of this line the more general Model 3 (thermally thick walls with axial conduction, Section 2.2.3) should be used instead. Model 3 reduces to the simpler Model 3' (thermally thin walls with axial conduction, Section 2.2.4) in the distinguished limit $\kappa_w \rightarrow \infty$, $\Delta_w^2 \kappa_w \sim 1$, which shows up as the diagonal strip labeled 3' in Fig. 4. It should be noted that to the right of this strip, where $\Delta_w^2 \kappa_w \gg 1$, the problem is still described by Model 3.

Table 2 gives typical values of the geometrical, physical, and dimensionless parameters characterizing the two sample macro and micro heat exchangers represented in Fig. 4 with markers A and B, respectively. Interestingly enough, heat exchanger performance in regions 1 and 2 is seen to correspond to classical macro-heat exchanger analysis [32], whereas regions 3 and 3' are more representative of mini/micro heat exchangers, on which the channel size is reduced but the partition wall between the flow passages can not be made excessively thin in order to maintain an adequate mechanical strength [21], resulting in significant axial wall conduction effects.

Before proceeding further, the reader should note that increasing the wall conductivity, k_w , while fixing the wall thickness, δ_w , implies moving in Fig. 4 from top to bottom along a vertical line defined by $\Delta_w \propto \delta_w = \text{constant}$. By contrast, increasing the wall conductivity, k_w , and the wall thickness, δ_w , by

the same factor, so that the wall thermal resistance remains constant, implies moving from left to right along a horizontal line defined by $\kappa_w^{-1} \propto k_w/\delta_w = \text{constant}$. A third possibility corresponds to fixing the wall conductivity, k_w , while increasing the wall thickness, δ_w , which implies moving up in Fig. 4 along a straight diagonal line with slope 1 in the log-log plot defined by $\Delta_w \kappa_w \propto k_w = \text{constant}$. Finally, the last possibility implies increasing the wall conductivity, k_w , while decreasing the wall thickness, δ_w , by the same factor, so that the product $\kappa_w \Delta_w^2 \propto k_w \delta_w$ remains constant. This implies moving along a straight diagonal line with slope 2, which in the limit of small wall thermal resistance represents cases with quantitatively similar axial wall conduction effects, as the relevant dimensionless parameter in Model 3' was precisely the product $\kappa_w \Delta_w^2$.

4.2. Discussion of heat exchanger performance

Figure 5 shows contour plots of the normalized effectiveness $\varepsilon/\varepsilon_{\max}$ for the same values of m and k considered in Fig. 4, but different heat exchanger lengths. The effectiveness of the heat exchanger is clearly affected by the heat exchanger length, as reflected in the figure by the growth of $\varepsilon_{\max}$ with ξ_L . For short heat exchangers, three different regimes are seen to exist depending on the value of the dimensionless wall thickness, Δ_w . The first regime prevails for very small values of Δ_w , where there is no influence of axial wall conduction. In this regime the effectiveness of the heat exchanger increases monotonically when decreasing the wall thermal resistance (i.e., increasing the dimensionless wall conductivity), as occurs in the classical domain of macro-heat exchanger analysis. The second regime exhibits an optimum wall conductivity, $\kappa_w^{\text{opt},\varepsilon}$, that maximizes the effectiveness of the heat exchanger for a given value of Δ_w . This regime has been addressed in previous investigations [8, 16] concerned with the effects of axial wall conduction. The third regime, which prevails for moderately large values of Δ_w , is dominated by heat conduction in the wall. In this regime, as in the first one, the effectiveness increases monotonically when decreasing the wall thermal resistance, although for highly conductive walls the effectiveness is significantly reduced compared to the first regime. This third regime has also been addressed in previous investigations [18, 19]. It is interesting to note that for long heat exchangers the third regime is displaced to unrealistically large values of Δ_w , so that only the first two regimes can be observed.

All the subplots shown in Fig. 5 exhibit the two lines dividing the $(\Delta_w, \kappa_w^{-1})$ plane in the four regions discussed above. As can be seen, as ξ_L grows both lines shift up and to the right, i.e., the range of κ_w^{-1} with negligible wall temperature and the range of Δ_w with negligible axial wall conduction become larger simultaneously. For comparative purposes, the criterion for neglecting axial wall conduction proposed by Maranzana et al. [16], given by Eq. (23), has also been included. As can be seen, Maranzana's criterion is more robust for short heat exchangers than the more refined criterion proposed by Quintero

& Vera [26], given by Eq. (22), which was essentially derived for long heat exchangers dominated by the first non-trivial eigenmode. As a result, although criterion (22) performs marginally better for long heat exchangers with low wall conductivity, we encourage the use of the simpler and more robust criterion of Maranzana, Eq. (23), which provides accurate guidelines about the relevance of axial wall conduction effects in a wider parameter range, particularly for small ξ_L .

The oblique dashed lines shown in Fig. 5 represent the value of $\kappa_w^{\text{opt},\varepsilon}$ that maximizes the effectiveness of the heat exchanger for a fixed value of Δ_w . Also shown in the figure is the value of $\kappa_w^{\text{opt},\lambda_0}$ given by Eq. (47), which maximizes the lowest order eigenvalue λ_0 for fixed Δ_w . It is interesting to note that for long heat exchangers with sufficiently thin walls the locus of maximum effectiveness coincides with the locus of maximum λ_0 , which highlights the dominant role of the lowest-order eigenmode on heat exchanger performance, and confirms that Eq. (47) provides a good estimation of the optimum wall conductivity in this particular case. However, for short devices, or even for long heat exchangers with sufficiently thick walls, the influence of higher-order modes starts to affect the value of the optimum conductivity $\kappa_w^{\text{opt},\varepsilon}$ which may differ significantly from the value given by (47). Interestingly enough, the results here provided show that the optimum wall conductivity first decreases with Δ_w roughly as predicted by Eq. (47), then reaches a minimum, and finally increases again, diverging at a finite value of Δ_w that grows with ξ_L and beyond which the optimum conductivity no longer exists.

Figures 6 and 7 represent bulk temperature profiles, $\theta_{mi}(\xi)$, and interfacial wall temperature profiles, $\theta_{wi}(\xi)$, for two heat exchanger lengths, $\xi_L = 0.125$ and 1, and selected values of m , k , κ_w , and Δ_w . Figure 6 is representative of short heat exchangers and Fig. 7 of long heat exchangers. The nine subplots in each figure correspond to points A, B, C, ..., H, I in the respective subplots of Fig. 5, and are located in the same relative positions in order to facilitate comparisons. As can be seen, when decreasing the wall thermal resistance (i.e., increasing the dimensionless wall conductivity) the wall temperature jump is seen to decrease, so that for highly conductive walls there is no noticeable temperature jump at all. On the other hand, for the specific cases of thick walls (subplots C, F, I) the temperature gradient along the wall is significantly reduced as a result of axial wall conduction, reaching an almost constant temperature over the entire wall in subplot I in agreement with the arguments put forward by Koyama et al. [19]. In this case the bulk temperature profiles are similar to those found in co-current flow configurations, a behavior that was anticipated by Stief et al. [8]. Moving diagonally up in Figs. 6 and 7 by simultaneously increasing the values of κ_w^{-1} and Δ_w is equivalent to increase the wall thickness while its conductivity remains constant. In this case the bulk and interfacial temperature profiles evolve monotonically in the sense of increasing the wall temperature jump and reducing the effectiveness of the device.

In a recent study, Ciofalo [31] showed that small-scale, or local, longitudinal heat conduction may enhance overall heat transfer rates in systems with corrugated walls. This effect can be generalized also to large-scale axial wall conduction of moderate intensity by comparing the bulk temperature profiles of Figs. 6 and 7 with the local heat transfer rates shown in Figs. 8 and 9. In these figures an increase of the wall thermal conductivity for a fixed wall thickness increases the heat exchanger effectiveness, but also the average wall-axial heat flux. Specifically, Figs. 8 and 9 show that around the optimum conductivity there is a change in the relative magnitude of the axial and normal wall fluxes: for wall conductivities below the optimum the average wall-axial heat flux is smaller than the wall-normal heat fluxes, at about the optimum wall conductivity they are comparable, and for higher wall conductivities the wall-axial heat flux is significantly larger than the wall-normal heat fluxes. The same behavior is obtained for short heat exchangers with high wall thermal conductivity but varying the thickness of the wall, as it is shown in subplots F and H of Fig. 8 where the line of the optimum wall thermal conductivity drawn in Fig. 5 for $\xi_L = 0.125$ is crossed and, once again, there is a change between the amount of longitudinal and interfacial heat fluxes. A physical justification of why the optimum conductivity leads to comparable wall-axial and wall-normal heat fluxes is still lacking, although it may be explained using entropy generation arguments [15].

Regarding the local Nusselt numbers, shown in Figs. 10 and 11 for $\xi_L = 0.125$ and 1, respectively, it is evident that the classical approach of assuming constant Nusselt numbers is only valid for long heat exchangers (Fig. 11) where the flow is thermally developed over most of the heat exchanger length, while for short heat exchangers (Fig. 10) the Nusselt numbers are seen to vary by a significant amount along the heat exchanger. The dependence of the local Nusselt number distributions with the parameter κ_w is almost negligible, whereas the influence of Δ_w is larger. In particular, in the thermally developed region the Nusselt number of the fluid with the smaller heat capacity flow rate (fluid 1 in this case) is seen to vary between the value for uniform heat flux (UHF) in the case of thin walls, and the value corresponding to uniform wall temperature (UWT) for thick walls. These results are in accordance with the fluid-wall interfacial temperature profiles presented in Figs. 6 and 7. The Nusselt number distributions also agree well with those obtained by Nunge & Gill [33], who concluded that for the counterflow case the uniform wall temperature condition represents a lower bound, whereas, contrary to the co-current case, the uniform wall heat flux condition does not represent an upper bound.

5. Conclusions

The influence of the different dimensionless parameters determining the performance of mini/micro counterflow heat exchangers has been studied based on an exact analytical solution previously obtained by the authors. The parametric study, focused on the effect of the dimensionless wall thickness, Δ_w , and wall thermal resistance, κ_w^{-1} , has benefited from a meticulous analysis of the eigenvalue spectrum that has identified and described different types of singularities arising in the analytical eigencondition. Once the intricate behavior of the eigencondition has been unveiled the lowest order eigenvalues have been calculated and represented in the form of contour plots in a 4-decade square log-log domain of the $(\Delta_w, \kappa_w^{-1})$ plane. Simplified asymptotic expressions of the analytical eigencondition for limiting values of the parameters that have been used to check that the obtained plots were meaningful.

Regarding the parametric study of heat exchanger performance, different regions have been identified in the $(\Delta_w, \kappa_w^{-1})$ plane depending on the presence or not of axial or transverse wall conduction: a region where the performance depends only on the dimensionless wall conductivity (for $\Delta_w \rightarrow 0$), a region where both dimensionless wall conductivity and dimensionless wall thickness have an influence on the performance (for $\Delta_w \ll 1$), and a region where the dimensionless wall thickness plays the most important role (for $\Delta_w \sim 1$). It has been shown that axial wall conduction can not be neglected in the study of heat exchangers with highly conductive walls of comparable size to the channel half-widths, and that in highly conductive walls with significant axial wall conduction the longitudinal temperature distribution becomes almost constant, so that the performance of the counterflow heat exchanger tends to that of a co-current flow system.

The result obtained in previous studies [8] about the existence of an optimum wall conductivity that maximizes the thermal performance of counterflow heat exchangers have been put into context in this study. Specifically, it has been shown that the optimum wall conductivity ceases to exist beyond a finite value of Δ_w that grows with ξ_L , so that in sufficiently short heat exchangers with moderately thick walls the optimum wall conductivity may not be defined. Moreover, the criterion introduced by Maranzana et al. [16] for neglecting axial wall conduction has proven to be surprisingly robust, providing a simple and efficient rule-of-thumb for determining whether axial wall conduction should be retained or not in a given system. The analysis has also shown that, when the optimum wall conductivity exists, it is such that the wall-axial and wall-normal heat fluxes are roughly of the same order, an aspect that deserves further attention to be better understood.

As a final remark, the results have shown that the Nusselt numbers of both flows are affected by the presence of the other, and that a constant Nusselt number assumption is not valid in short heat exchangers.

Finally, it has been confirmed that the Nusselt number for the uniform wall temperature (UWT) boundary condition acts as a lower bound for the fluid with the smaller heat capacity flow rate, while there is not an upper bound, because the Nusselt numbers can largely exceed the value corresponding to the uniform heat flux (UHF) boundary condition.

Summarizing, the analysis presented in this paper i) unifies the different wall conduction models used to describe counterflow heat exchangers, ii) provides guidelines indicating which is the most appropriate model based on five design and operational dimensionless parameters, iii) explains and confirms several previous findings reported in the literature, and iv) will hopefully contribute to a better understanding of the heat transfer performance of mini/micro counterflow heat exchangers.

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Figure 1: Unitary cell of a multilayered laminar counterflow parallel-plate heat exchanger showing the geometrical dimensions, the coordinate system, the velocity field, and the physical properties of the fluids and the heat conducting wall. The domain under study is shaded gray.

Table 1: Summary of differential equations, boundary conditions, and dimensionless parameters involved in the different models discussed in Section 2.

Models	Differential equations	Boundary conditions	Parameters
Thermal problem in the fluids (Section 2.1)			
Models 1 2 3 & 3'	$\frac{3}{4}(1-y_1^2)\frac{\partial\theta_1}{\partial\xi} = \frac{\partial^2\theta_1}{\partial y_1^2} \quad \text{in} \quad 0 < \xi < \xi_L$ $-m\frac{3}{4}(1-y_2^2)\frac{\partial\theta_2}{\partial\xi} = \frac{\partial^2\theta_2}{\partial y_2^2} \quad \text{in} \quad 0 < y_i < 1$	$\frac{\partial\theta_1}{\partial y_1} = \frac{\partial\theta_2}{\partial y_2} = 0 \quad \text{at} \quad \begin{matrix} 0 < \xi < \xi_L \\ y_i = 0 \end{matrix}$ $\theta_1(0, y_1) = 0 \quad \text{at} \quad 0 \leq y_1 < 1$ $\theta_2(\xi_L, y_2) = 1 \quad \text{at} \quad 0 \leq y_2 < 1$	m ξ_L
Thermal problem in the wall (Section 2.2)			
Model 1 [23]		$\theta_1 = \theta_2 \equiv \theta_w(\xi) \quad \text{at} \quad 0 < \xi < \xi_L$	
Thermally thin walls w/out axial wall conduction		$\frac{\partial\theta_1}{\partial y_1} = -k\frac{\partial\theta_2}{\partial y_2} \equiv \nu(\xi) \quad \text{at} \quad y_i = 1$ $\theta_w(0) = 0$ $\theta_w(\xi_L) = 1$	k
Model 2 [25]		$\theta_2 - \theta_1 = \frac{1}{\kappa_w} \frac{\partial\theta_1}{\partial y_1} = \frac{\nu(\xi)}{\kappa_w} \quad \text{at} \quad 0 < \xi < \xi_L$	
Thermally thick walls w/out axial wall conduction	$\frac{\partial^2\theta_w}{\partial y_w^2} = 0 \quad \text{in} \quad \begin{matrix} 0 < \xi < \xi_L \\ -1/2 < y_w < 1/2 \end{matrix}$	$\frac{\partial\theta_1}{\partial y_1} = -k\frac{\partial\theta_2}{\partial y_2} \equiv \nu(\xi) \quad \text{at} \quad y_i = 1$ $\theta_w(0, -1/2) = 0$ $\theta_w(\xi_L, 1/2) = 1$	k κ_w
Model 3 [26]		$\theta_1(\xi, 1) = \theta_w(\xi, -1/2) \quad \text{at} \quad 0 < \xi < \xi_L$ $\theta_2(\xi, 1) = \theta_w(\xi, 1/2)$	
Thermally thick walls with axial wall conduction	$\Delta_w^2 \frac{\partial^2\theta_w}{\partial \xi^2} + \frac{\partial^2\theta_w}{\partial y_w^2} = 0 \quad \text{in} \quad \begin{matrix} 0 < \xi < \xi_L \\ -1/2 < y_w < 1/2 \end{matrix}$	$\frac{\partial\theta_1}{\partial y_1} \Big _{y_1=1} = \kappa_w \frac{\partial\theta_w}{\partial y_w} \Big _{y_w=-1/2} \quad \text{at} \quad 0 < \xi < \xi_L$ $-k \frac{\partial\theta_2}{\partial y_2} \Big _{y_2=1} = \kappa_w \frac{\partial\theta_w}{\partial y_w} \Big _{y_w=1/2}$ $\frac{\partial\theta_w}{\partial \xi} \Big _{\xi=0} = 0$ $\frac{\partial\theta_w}{\partial \xi} \Big _{\xi=\xi_L} = 0 \quad \text{at} \quad -1/2 < y_w < 1/2 \quad (\dagger)$	k κ_w Δ_w
Model 3'		$\theta_1 = \theta_2 \equiv \theta_w(\xi)$	
Thermally thin walls with axial wall conduction	$\Delta_w^2 \kappa_w \frac{d^2\theta_w}{d\xi^2} + \nu_1(\xi) - \nu_2(\xi) = 0 \quad \text{in} \quad 0 < \xi < \xi_L$	$\frac{\partial\theta_1}{\partial y_1} = \nu_1(\xi) \quad \text{at} \quad 0 < \xi < \xi_L$ $-k \frac{\partial\theta_2}{\partial y_2} = \nu_2(\xi) \quad \text{at} \quad y_i = 1$ $\frac{d\theta_w}{d\xi} \Big _{\xi=0} = \frac{d\theta_w}{d\xi} \Big _{\xi=\xi_L} = 0 \quad (\dagger)$	k $\Delta_w^2 \kappa_w$

(†) Alternatively, one may use the specified end-wall temperature (i.e., Dirichlet) boundary conditions presented in Eq. (21).

$$\left[f'_{n,1}(1; \lambda_n) - k f'_{n,2}(1; m\lambda_n) \right]^2 = \left\{ \left[f'_{n,1}(1; \lambda_n) + k f'_{n,2}(1; m\lambda_n) \right] \tan(\Delta_w \lambda_n / 2) + 2\kappa_w \Delta_w \lambda_n \right\} \left\{ \frac{f'_{n,1}(1; \lambda_n) + k f'_{n,2}(1; m\lambda_n)}{\tan(\Delta_w \lambda_n / 2)} - 2\kappa_w \Delta_w \lambda_n \right\}$$

Figure 2: Left: Variation of the eigenvalues λ_0 , $\lambda_{\pm 1}$, and $\lambda_{\pm 2}$ with Δ_w for $m = 2$, $k = 1$, and $\kappa_w = 20$, with the positive (negative) eigenvalues shown in blue (red) and the lowest order eigenvalue shown in black. Oblique dashed lines represent the conditions indicated by the boxed texts. Dots ($\cdot$): Eq (32); circle ($\circ$): Eq (33); triangles (Δ): singular points given by the solution of (38) and (39); squares ($\square$): singular points given by (42). Right: Variation of the derivative of the eigenfunction of fluid 1 at the λ_n wall as given by Eq. (30), showing the divergences appearing at $\lambda_n = \lambda_p^\infty$. Bottom: general eigencondition, Eq. (28), highlighting in color the terms responsible for the different singularities, shown as dashed color lines on the upper left plot.

Figure 3: Contour plots of λ_0 , $\lambda_{\pm 1}$, $\lambda_{\pm 2}$, $\lambda_{\pm 3}$, and $\lambda_{\pm 4}$ in the $(\Delta_w, \kappa_w^{-1})$ plane for $m = 2$ and $k = 1$. Solid blue line: entrance of the axial conduction effects, Eq. (31) (or (33) for λ_0). Solid red line: intercept of λ_n with the first final asymptote, Eq. (42). Dashed blue lines: Eq. (44). Dashed red lines: Eq. (46). The dashed black line in the contour plot of λ_0 shows the value of κ_w^{-1} that maximizes λ_0 for a given Δ_w as obtained from Eq. (47). The horizontal dash-dotted lines shown in the subplots of λ_0 , $\lambda_{\pm 1}$, and $\lambda_{\pm 2}$ indicate the location of the data represented in Fig. 3

Figure 4: Iso-contours of normalized effectiveness, $\varepsilon/\varepsilon^{\max}$, as a function of the dimensionless wall thermal resistance, κ_w^{-1} , and the dimensionless wall thickness, Δ_w , extracted from Fig. 5, corresponding to $m = 2$, $k = 1$, and $\xi_L = 2$. The regions of application of the different wall conduction models are clearly indicated. Solid black line: criterion for neglecting axial wall conduction, Eq. (22). Dash-dotted line: criterion for thermally thin walls (i.e., negligible wall temperature jump), Eq. (15). Markers A and B indicate the position in parameter space of the two representative macro and micro heat exchangers presented in Table 2.

Table 2: Geometrical, physical, and dimensionless parameters corresponding to the two representative macro and micro heat exchangers shown in Fig. 4 as markers A and B, respectively.

		Macro heat exchanger (<i>A</i>)	Micro heat exchanger (<i>B</i>)
Geometrical parameters	a_i	1.5 mm	$1.0 \times 10^2 \mu\text{m}$
	δ_w	$6.0 \times 10^{-1} \text{ mm}$	$4.0 \times 10^3 \mu\text{m}$
	L	1.2 m	$1.6 \times 10^2 \text{ mm}$
Physical parameters	k_i	$3.0 \times 10^{-2} \text{ W/(m} \cdot \text{K)}$	$3.0 \times 10^{-2} \text{ W/(m} \cdot \text{K)}$
	k_w	$1.2 \times 10^{-1} \text{ W/(m} \cdot \text{K)}$	$1.2 \times 10^2 \text{ W/(m} \cdot \text{K)}$
Dimensionless parameters	Pe_l	410	800
	m	2	2
	k	1	1
	κ_w	5	100
	Δ_w	1×10^{-3}	5×10^{-2}
	ξ_L	2	2

Figure 5: Isocontours of normalized effectiveness, $\varepsilon/\varepsilon^{\max}$, in the $(\Delta_w, \kappa_w^{-1})$ plane for $m = 2$ and $k = 1$ and different heat exchanger lengths. Solid black/white lines: criterion for neglecting axial wall conduction, Eq. (22/23). Dash-dotted line: criterion for thermally thin walls (i.e., negligible wall temperature jump), Eq. (15). Dashed line: optimum wall conductivity for maximum effectiveness, $\kappa_w^{\text{opt},\varepsilon}$, obtained numerically for fixed wall thickness, $\Delta_w \propto \delta_w$. Dotted line: optimum wall conductivity for maximum λ_0 , $\kappa_w^{\text{opt},\lambda_0}$, given by Eq. (47).

Figure 6: Bulk temperature of the fluids given by Eq. (50) with $\theta_i(\xi, y_i)$ in Eq. (26) truncated to $n = 10$. Temperature of the interfaces wall-f fluids given by Eq. (27) truncated to $n = 10$ particularized for $y_w = -1/2$ and $y_w = 1/2$. Parameters of the heat exchanger correspond to $m = 2$, $k = 1$, $\kappa_w = [2, 20, 200]$, $\Delta_w = [0.0015, 0.015, 0.15]$ and $\xi_L = 0.125$ and length of the heat exchanger normalized with the total length.

Figure 7: Bulk temperature of the fluids given by Eq. (50) with $\theta_i(\xi, y_i)$ in Eq. (26) truncated to $n = 10$. Temperature of the interfaces wall-fluids given by Eq. (27) particularized for $y_w = -1/2$ and $y_w = 1/2$. Parameters of the heat exchanger correspond to $m = 2$, $k = 1$, $\kappa_w = [2, 20, 200]$, $\Delta_w = [0.0015, 0.015, 0.15]$ and $\xi_L = 1$ for $N = 10$ and length of the heat exchanger normalized with the total length.

Figure 8: Heat fluxes between fluids and wall given by Eq. (19) and heat flux across the wall given by Eq. (52), for a heat exchanger corresponding to $m = 2$, $k = 1$, $\kappa_w = [2, 20, 200]$, $\Delta_w = [0.0015, 0.015, 0.15]$ and $\xi_L = 0.125$ for $N = 10$ and length of the heat exchanger normalized with the total length.

Figure 9: Heat fluxes between fluids and wall given by Eq. (19) and heat flux across the wall given by Eq. (52), for a heat exchanger corresponding to $m = 2$, $k = 1$, $\kappa_w = [2, 20, 200]$, $\Delta_w = [0.0015, 0.015, 0.15]$ and $\xi_L = 1$ for $N = 10$ and length of the heat exchanger normalized with the total length.

Figure 10: Nusselt number of fluids 1 and 2 given by (53) and (54) for a heat exchanger with $m = 2$, $k = 1$, $\kappa_w = [2, 20, 200]$ (rows), $\Delta_w = [0.0015, 0.015, 0.15]$ (columns) and $\xi_L = 0.125$. Dash-dotted lines correspond to the Nusselt number for fully developed laminar flow with uniform heat flux (UHF) at the wall-fluid interface, $Nu = 8.23$, and uniform wall temperature (UWT), $Nu = 7.54$.

Figure 11: Nusselt number of fluids 1 and 2 given by (53) and (54) for a heat exchanger with $m = 2$, $k = 1$, $\kappa_w = [2, 20, 200]$ (rows), $\Delta_w = [0.0015, 0.015, 0.15]$ (columns) and $\xi_L = 1$. Dash-dotted lines correspond to the Nusselt number for fully developed laminar flow with uniform heat flux (UHF) at the wall-fluid interface, $Nu = 8.23$, and uniform wall temperature (UWT), $Nu = 7.54$.