This is a postprint version of the following published document:

Quintero, A. E. & Vera, M. (2017). Laminar counterflow parallel-plate heat exchangers: An exact solution including axial and transverse wall conduction effects. *International Journal of Heat and Mass Transfer*, 104, 1229–1245.

DOI: [10.1016/j.ijheatmasstransfer.2016.09.025](https://doi.org/10.1016/j.ijheatmasstransfer.2016.09.025)

© 2016 Elsevier Ltd. All rights reserved.

This work is licensed under a [Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License](https://creativecommons.org/licenses/by-nc-nd/4.0/).

Laminar counterflow parallel-plate heat exchangers: An exact solution including axial and transverse wall conduction effects

Alberto E. Quintero, Marcos Vera*

Departamento de Ingeniería Térmica y de Fluidos, Escuela Politécnica Superior, Universidad Carlos III de Madrid, 28911 Leganés, Spain

Abstract

Multilayered, counterflow, parallel-plate heat exchangers are studied theoretically and numerically. The analysis, carried out for constant property fluids, assumes a fully developed laminar flow with moderately large Peclet numbers in the flow channels, so that longitudinal conduction can be neglected in the fluids in first approximation. By way of contrast, both axial and transverse conduction effects are fully accounted for in the heat conducting plates, making the analysis relevant to the design of mini- and micro-heat exchangers. The exact solution for the temperature field is obtained in terms of eigenfunction expansions involving infinite sets of both positive and negative eigenvalues. Based on previous results, the eigenfunctions are expressed in terms of Whittaker functions, leading to an analytical eigencondition that provides the eigenvalues numerically. Making use of a newly defined orthogonality condition, the expansion coefficients are determined through an infinite system of linear equations that must be truncated to a sufficiently large number of terms to obtain a numerical solution of the problem. The exact solution for the temperature field provides analytical expressions for the interfacial and bulk temperatures, local heat-transfer rates, overall heat-transfer coefficient, Nusselt numbers, and outlet bulk temperatures of both fluids. Numerical results evaluated using a moderate number of terms in the series are compared against finite difference solutions of the problem and against previous results reported in the literature, showing excellent agreement in both cases. The exact solution presented here may be used as a benchmark case for computational heat transfer codes, and enables future spectral and parametric studies of the problem.

Keywords: laminar, counterflow, parallel-plate, heat exchangers, axial wall conduction, eigenfunction expansions, exact solution

Nomenclature

A	Expansion coefficient.
a	Constant appearing in Eq. (60).
a_i	Channel half-width of fluid i .
b	Constant appearing in Eq. (60).
C_n	Expansion coefficient corresponding to the n -th eigenfunction.
c_i	Specific heat of fluid i .
f_n	n -th eigenfunction for thermally thin walls.
Gz_i	Graetz number of fluid i , $Pe_i a_i / L$.
h_n	n -th eigenfunction for non thermally thin walls.
h_i	Heat-transfer coefficient of fluid i .
k	Dimensionless parameter, $(a_1 k_2) / (a_2 k_1)$.
k_i	Thermal conductivity of fluid i .
k_w	Thermal conductivity of the wall.
L	Length of the heat-exchanger.
M	Whittaker's function, $M_{\kappa, \mu}(z)$.
m	Dimensionless parameter, $(a_2 Pe_2) / (a_1 Pe_1)$.
NTU	Number of transfer units.
Nu_i	Nusselt number of fluid i , $h_i (4a_i) / k_i$.
$\dot{m}_i$	Mass flow rate of fluid i , $2a_i V_i \rho_i$.

Pe_i	Peclet number of fluid i , $2a_i V_i / \alpha_i$.
Pr_i	Prandtl number of fluid i , ν_i / α_i .
$\mathcal{R}$	Residual error.
Re_i	Reynolds number of fluid i , $2a_i V_i / \nu_i$.
R	Heat capacity flow rate ratio, $(\dot{m}_i c_i)_{\min} / (\dot{m}_i c_i)_{\max}$.
T	Temperature.
$\widetilde{U}$	Dimensionless overall heat-transfer coefficient.
$u_i(Y_i)$	Longitudinal velocity of fluid i .
V_i	Average flow velocity of fluid i .
W	Whittaker's function, $W_{\kappa, \mu}(z)$.
$w(y_i)$	Weight function, $(3/4)(1 - y_i^2)$.
X	Longitudinal distance from the inlet of fluid 1.
Y_i	Transverse distance from channel i symmetry plane.
Y_w	Transverse distance from wall symmetry plane.
y_i	Dimensionless transverse coordinate, Y_i / a_i .
y_w	Dimensionless transverse coordinate, Y_w / δ_w .

Greek letters

α_i	Thermal diffusivity of fluid i , $k_i / (\rho_i c_i)$.
α_n	n -th slope for the eigenfunction of the wall.
β_n	Small parameter, $\beta_n = \Delta_w \lambda_n / 2$.
Δ_w	Dimensionless parameter, $\delta_w / a_1 Pe_1$.
δ_w	Wall thickness.
ε	Heat exchanger effectiveness.
Γ	Gamma function, $\Gamma(z)$.
κ	First argument of Whittaker functions.

*Corresponding author. Tel.: +34-916249987; fax: +34-916249430.

Email address: marcos.vera@uc3m.es (Marcos Vera*)

URL: <http://fluidosuc3m.es/people/mvcoello/> (Marcos Vera*)

κ_w	Dimensionless parameter, $k_w a_1 / \delta_w k_1$.
λ_n	n -th eigenvalue.
λ_0^0	Value of λ_0 corresponding to $\Delta_w = 0$.
λ_0^0	Approximate value of λ_0^0 , a/b .
μ	Second argument of Whittaker functions.
$\nu_w(\xi)$	Dimensionless average wall axial heat flux.
$\nu_i(\xi)$	Dimensionless interfacial heat-transfer rates.
ν_i	Kinematic viscosity of fluid i .
ρ_i	Density of fluid i .
θ_i	Dimensionless temperature, $(T_i - T_{1,\text{in}})/(T_{2,\text{in}} - T_{1,\text{in}})$.
ξ	Dimensionless longitudinal coordinate, $X/a_1 \text{Pe}_1$.
ξ_L	Dimensionless parameter, $L/a_1 \text{Pe}_1$.
$\Omega_n^\pm$	Coefficients appearing in Eq. (49).

Subscripts

i	Subscript used indistinctly for fluids 1 and 2.
in	Inlet.
L	Length of the heat-exchanger.
m	Bulk, or mixing-cup, temperature.
n	n -th eigenvalue/eigenfunction.
out	Outlet.
w	heat-exchanging wall.

Superscripts

$[N]$	$2(N + 1)$ -term truncated exact solution.
-------	--

1. Introduction

The current trend towards miniaturization of electronic devices has boosted the development of new applications in the field of mini- and micro-heat exchangers [1]. Miniaturization of heat exchanger equipment is also driven by the development of new microscale devices (e.g., power MEMS, microengines, and microcombustors) which require enhanced heat transfer performances [2–5]. However, the accompanying reduction in the channel dimensions is known to increase the importance of axial wall conduction [6]. This effect may in turn spoil the effectiveness of countercurrent heat exchangers, which reduces to that of co-current systems in the limit of large axial wall conduction [7–9].

Understanding the influence of axial wall conduction on the performance of mini- and micro-heat exchangers is therefore critical for the development of high efficient equipments. Bier et al. [10] found experimentally that, for certain operating conditions, micro heat exchangers made of stainless steel (i.e., lower conductivity) present higher effectiveness than micro heat exchangers made of copper (i.e., higher conductivity), concluding that the influence of axial conduction in the wall is an important factor. Stief et al. [11] explained these results using numerical simulations to obtain an optimal conductivity for the wall: highly conductive walls offset the temperature gradients within the wall material to such an extent that the effectiveness of any micro heat exchanger, independent of the flow direction specified, will always approximate the (lower) heat transfer efficiency of a parallel-flow heat exchanger. The

existence of an optimal wall conductivity was later confirmed by Maranzana et al. [12] using the thermal quadrupole method.

Parallel-plate heat exchangers are also a key component of small cryogenic systems [13, 14]. The design of high effectiveness heat exchangers for cryogenic applications needs to consider axial heat conduction. The most severe performance deterioration is encountered for balanced heat exchangers, where the temperature difference between the fluids, which is already small throughout the exchanger, is further decreased by axial conduction [15].

Progress in the analysis of parallel-plate heat exchangers has been significant in the last decades due to their simple geometry and well established flow conditions [16, 17]. The so-called classical Graetz problem was first addressed by L. Graetz [18] in 1882, who studied heat transfer in fully developed laminar pipe flow assuming constant wall temperature, constant wall heat flux, and wall temperature proportional to heat flux boundary conditions. The solution of the resulting Sturm-Liouville problem was obtained by separation of variables as an infinite series of orthogonal eigenfunctions with their corresponding eigenvalues.

Stein [19] extend the classical Graetz problem by studying the heat exchange between two fully developed laminar streams (conjugated Graetz problem) for co-current flow in a double pipe heat exchanger. The equivalent Sturm-Liouville system consisted of two Sturm-Liouville equations coupled at a common boundary, with an orthogonality condition for the eigenfunctions defined over both regions. Nunge et al. [20–22] extended the analysis to countercurrent flows and confirmed two main results: *i*) longitudinal conduction in the fluid can be neglected for Peclet numbers of order 100 or larger (a result previously obtained by Singh [23] for the classical Graetz problem) and *ii*) the Nusselt number of each stream is affected by the presence of the other stream.

Davis and Gill [24] used hypergeometric functions to write the eigenfunctions along with the Duhamel theorem to solve the classical Graetz problem with axial conduction in the wall. Mori et al. [25] studied the influence of longitudinal conduction in the wall for a multilayered parallel plate heat exchanger with countercurrent flow. They used the numerical results previously obtained by Nunge and Gill [20] for the eigenfunctions and eigenvalues of the classical Graetz problem and the Duhamel theorem, and assumed that the interfacial temperatures could be represented by infinite power series. However, they neither obtained the eigenfunctions of their problem, nor defined an appropriate orthogonality condition for such eigenfunctions. Mori et al. [25] compared their results with those provided by the classical effectiveness-NTU method, concluding that when axial wall conduction comes into play the classical ε -NTU method does not provide appropriate guidelines for the design of heat exchangers. They also found that the effectiveness of the heat exchanger is influenced by the Graetz number, the heat capacity flow ratio, the dimensionless wall thickness, and the fluid-to-fluid and wall-to-fluid conductivity ratios.

Barozzi and Pagliarini [26, 27] carried out an experimental study of the classical Graetz problem and compared the results using a semi-numerical approach. Later they extended this

approach to the conjugated Graetz problem to study the effect of axial wall conduction in laminar counterflow double-pipe heat exchangers [28]. Preheating effects due to axial wall conduction were studied by Piva [29] in the classical Graetz problem, finding an exact series solution in terms of confluent hypergeometric functions.

Papoutsakis et al. [30–33] were the first to study the classical and conjugate Graetz problems including the effect of axial conduction in the fluid, giving birth to the so-called extended Graetz problem. The presence of axial conduction in the fluids introduces a non self-adjoint operator in the energy equation, which now includes both convective and conduction axial terms. They demonstrated that this operator can be transformed into a self-adjoint operator by means of a suitable change of variables that leads to a modified system of differential equations. The solution is expressed in terms of eigenvalues and eigenvectors like a series expansion, which can be written in terms of Kummer hypergeometric functions for cylindrical geometries.

Weigand et al. [34–37] applied the self-adjoint operator formalism introduced by Papoutsakis et al. [30, 31] to analyze different boundary conditions in the extended Graetz problem with turbulent flows and axial wall conduction, a situation of relevance for applications involving liquid metals. More recently, Pierre and Plouraboué [38] generalized the cylindrical solution to the extended Graetz problem proposed by Papoutsakis et al. [30] to non axi-symmetric geometries for any velocity profile, and also demonstrated the mathematical properties of the unbounded operator present in this method. They called this the generalized Graetz problem, which is defined by the extended and conjugated Graetz problems with non axi-symmetric geometries. Later Fehrenbach et al. [39] improved this solution for finite domains.

The aim of this paper is to apply the orthogonal eigenfunction expansion technique introduced by Stein [19] and Nunge et al. [20–22] to solve the temperature field in a multilayered laminar counterflow parallel-plate heat exchanger. The analysis neglects viscous dissipation and axial conduction in the fluids, but includes as main novelty the effect of axial and transverse heat conduction in the wall. Previous studies by Vera and Liñán [40] and Quintero et al. [41] addressed particular cases of the more general analysis presented here. Following these works, our aim is to obtain closed-form analytical expressions for the eigenfunctions and to derive an analytical eigencondition that can be used to determine the eigenvalues numerically. The exact solution for the temperature field can then be written as an infinite eigenfunction series expansion, with the expansion coefficients determined by an infinite system of linear equations derived from a properly defined orthogonality condition. This approach provides an accurate representation for the temperature field using a small number of terms in the series, thereby enabling expressions for the wall and bulk temperatures, interfacial heat-transfer rates, overall heat-transfer coefficient, Nusselt numbers, and outlet bulk temperatures of both streams. These expressions may be useful for engineering applications with relevant effects of axial wall conduction, including moderately short heat exchangers and

mini- and micro-heat exchangers. As major byproduct of the analysis, we shall discuss under which conditions can axial wall conduction be ignored when calculating the eigenvalues and eigenfunctions and the overall heat exchanger performance, comparing our criterion with that previously proposed by Maranzana et al. [12]

As in our previous works [40, 41], the purpose of this paper is not simply to present the mathematical formulation of the problem and a series of numerical integrations, which would be fairly straightforward with the current computational capabilities. On the contrary, our aim is to develop a theoretical framework which may contribute to the understanding of the basic physical mechanisms that take part in laminar counterflow heat exchangers, with particular emphasis on the effect of axial wall conduction. As usual for theoretical considerations, several hypotheses must be introduced, the most restrictive being the assumptions of fully hydrodynamically developed flow, which may break down for fluids with low Prandtl numbers, and the adiabatic lateral boundary conditions, which may not be realistic for emerging applications such as electronic cooling [42, 43]. However, as demonstrated from comparisons with a few selected numerical and experimental studies, many aspects of these systems are still correctly reproduced. Moreover, the current analysis constitutes a necessary first step towards future generalizations of the problem at hand.

2. Problem Formulation

In this paper we analyze the flow of two constant-property newtonian fluids in a multilayered counterflow parallel-plate heat exchanger composed by a relatively large number of channels separated by plates of finite thickness, δ_w , and thermal conductivity, k_w . The conducting plates allow the exchange of heat through a section of length L , presenting insulated regions at both ends of the heat exchange section where no heat transfer is allowed [25]. In the configuration considered here, the two fluids, denoted by 1 and 2 (the subscript $i = 1, 2$ denoting indistinctly both fluids), flow in opposite directions in adjacent channels. Then, if the characteristic dimension of the heat exchanger is large compared with the channel widths, $2a_i$, the temperature field, as seen with this scale, appears as periodic in the transverse direction, with period $2(a_1 + a_2)$. Accordingly, when describing the temperature field in the unitary cell of the heat exchanger we can use symmetry boundary conditions at the channel symmetry planes, where no energy transfer is allowed.

Figure 1 shows a sketch of the physical model under study, where X denotes the longitudinal coordinate measured from the inlet of fluid 1, Y_i represents the transverse coordinate measured from the symmetry plane of channel i , and Y_w is the transverse coordinate measured from wall symmetry plane, oriented from fluid 1 to fluid 2. The Reynolds numbers of the flow in the channels, $Re_i = 2a_i V_i / \nu_i$, based on the channel width, $2a_i$, the average flow velocity, V_i , and the kinematic viscosity, ν_i , of fluid i , will be considered to be sufficiently small for the flow to remain laminar and steady [44]. In addition, the Prandtl number, $Pr_i = \nu_i / \alpha_i$, defined as the ratio of the kinematic

viscosity, ν_i , and the thermal diffusivity, $\alpha_i = k_i/(\rho_i c_i)$, of fluid i , is assumed to be large compared to unity, so that the thermal entry length is large compared to the hydrodynamic entry length, a good approximation for most non-metallic liquids in applications. Then, the flow will be assumed to be a fully developed Poiseuille flow which, due to the constant property assumption, is independent of the temperature field, $u_i(Y_i) = \pm(3/2)V_i[1 - (Y_i/a_i)^2]$, the $+$ ($-$) sign holding for fluid 1 (2).

In addition, we shall assume here that the Peclet numbers of the flow in the channels, $Pe_i = Re_i Pr_i = 2a_i V_i / \alpha_i$, are both large compared to unity. Axial heat conduction in the fluids is then confined to small regions located at the end sections of the heat exchanger, close to the heat conducting plates, whose characteristic size, of order $Pe_i^{-1/2} a_i$, is small compared to a_i when $Pe_i \gg 1$. In this limiting case we can neglect axial conduction in the fluids in first approximation, and assume that the inlet temperature of fluid i is uniform and equal to $T_{i,in}$. As previously discussed, several analyses of counterflow heat exchangers have concluded that this is a good approximation for Peclet numbers of order 100 or larger [22, 23].

To write the problem in non-dimensional form, we use the dimensionless variables $\xi = X/(Pe_1 a_1)$, $y_i = Y_i/a_i$, $y_w = Y_w/\delta_w$, and $\theta_i = (T_i - T_{1,in})/(T_{2,in} - T_{1,in})$, where $i = \{1, 2\}$ in the definition of y_i and $\{1, 2, w\}$ in the definition of θ_i . Then, the energy equation in the fluids reduces, in first approximation, to a balance between axial convection and transverse conduction

$$\frac{3}{4}(1 - y_1^2) \frac{\partial \theta_1}{\partial \xi} = \frac{\partial^2 \theta_1}{\partial y_1^2} \quad (1)$$

$$-m \frac{3}{4}(1 - y_2^2) \frac{\partial \theta_2}{\partial \xi} = \frac{\partial^2 \theta_2}{\partial y_2^2} \quad (2)$$

in $0 < \xi < \xi_L$, $0 < y_i < 1$, while in the heat conducting wall it takes the form

$$\Delta_w^2 \frac{\partial^2 \theta_w}{\partial \xi^2} + \frac{\partial^2 \theta_w}{\partial y_w^2} = 0 \quad (3)$$

in $0 < \xi < \xi_L$, $-\frac{1}{2} < y_w < \frac{1}{2}$, where the first term accounts for the effect of axial wall conduction, assumed to be negligibly small ($\Delta_w \rightarrow 0$) in the previous work by Quintero et al. [41]. In the above equations, the dimensionless parameters

$$m = \frac{a_2 Pe_2}{a_1 Pe_1} \quad \text{and} \quad \xi_L = \frac{L}{a_1 Pe_1} \quad (4)$$

are assumed to be of order unity, whereas

$$\Delta_w = \frac{\delta_w}{a_1 Pe_1}, \quad (5)$$

which measures the relative importance between the axial and transverse wall conduction terms in Eq. (3), is assumed to be moderately small, with values ranging from 10^{-3} to 10^{-1} . Moreover, the aspect ratio of the heat exchanger wall will be assumed to be moderately large, $L/\delta_w \gg 1$, as typically occurs in applications, so that, in general, we have $\Delta_w \ll \xi_L \sim 1$.

The above equations have to be integrated with the no-flux boundary conditions at the channel symmetry planes

$$\frac{\partial \theta_1}{\partial y_1} = \frac{\partial \theta_2}{\partial y_2} = 0 \quad \text{at} \quad 0 < \xi < \xi_L, \quad y_i = 0, \quad (6)$$

and the conditions of continuity of temperatures

$$\begin{aligned} \theta_1(\xi, 1) &= \theta_w(\xi, -1/2) \equiv \theta_{w1}(\xi) \\ \theta_2(\xi, 1) &= \theta_w(\xi, 1/2) \equiv \theta_{w2}(\xi) \end{aligned} \quad \text{at} \quad 0 < \xi < \xi_L, \quad (7)$$

and wall-normal heat fluxes

$$\begin{aligned} \left. \frac{\partial \theta_1}{\partial y_1} \right|_{y_1=1} &= \kappa_w \left. \frac{\partial \theta_w}{\partial y_w} \right|_{y_w=-1/2} \equiv \nu_1(\xi) \\ -k \left. \frac{\partial \theta_2}{\partial y_2} \right|_{y_2=1} &= \kappa_w \left. \frac{\partial \theta_w}{\partial y_w} \right|_{y_w=1/2} \equiv \nu_2(\xi) \end{aligned} \quad \text{at} \quad 0 < \xi < \xi_L, \quad (8)$$

at the fluid-wall interfaces. Two additional parameters appear when imposing the continuity of wall-normal heat fluxes, namely

$$k = \frac{a_1 k_2}{a_2 k_1} \quad \text{and} \quad \kappa_w = \frac{a_1 k_w}{\delta_w k_1}, \quad (9)$$

which represent the normalized fluid-to-fluid and wall-to-fluid heat conductivity ratios, respectively.

Note that, instead of m and k , one may alternatively choose the ratio of the heat-capacity flow rates $mk = (\dot{m}_2 c_2)/(\dot{m}_1 c_1)$ among the dimensionless parameters of the problem, together with either m or k . This represents a more convenient choice for the analysis of nearly-balanced heat exchangers, for which $1 - (mk)^{-1} = \epsilon \ll 1$ emerges as a suitable small parameter for the analysis [40]. The ratio of heat-capacity flow rates is also the main parameter that appears in lumped models concerned only with the bulk flow temperatures.

As previously discussed, uniform temperature profiles are specified at the inlet sections of both fluids

$$\theta_1 = \theta_{1,in} = 0 \quad \text{at} \quad \xi = 0, \quad 0 \leq y_1 < 1, \quad (10)$$

$$\theta_2 = \theta_{2,in} = 1 \quad \text{at} \quad \xi = \xi_L, \quad 0 \leq y_2 < 1, \quad (11)$$

whereas insulated (i.e., Neumann) boundary conditions are imposed at both ends of the heat-exchanging wall

$$\frac{\partial \theta_w}{\partial \xi} = 0 \quad \text{at} \quad \xi = 0, \quad -1/2 < y_w < 1/2, \quad (12)$$

$$\frac{\partial \theta_w}{\partial \xi} = 0 \quad \text{at} \quad \xi = \xi_L, \quad -1/2 < y_w < 1/2. \quad (13)$$

One may also prescribe the temperature at both ends of the heat conducting wall, resulting in the alternative (i.e., Dirichlet) boundary conditions

$$\theta_w = 0 \quad \text{at} \quad \xi = 0, \quad -1/2 < y_w < 1/2, \quad (12')$$

$$\theta_w = 1 \quad \text{at} \quad \xi = \xi_L, \quad -1/2 < y_w < 1/2, \quad (13')$$

where the end-wall temperature is set equal to that of the inlet fluid. This alternative boundary condition has been considered in the literature to assess the effect of conductive heat losses at both ends of micro heat exchangers [6], and is included here for completeness, although the numerical results to be presented below are mainly concerned with the insulated case.

The solution to the problem stated by Eqs. (1)–(13) provides the temperature field for both fluids, $\theta_i(\xi, y_i)$, as well as the

wall temperature distribution, $\theta_w(\xi, y_w)$, for given values of m , k , κ_w , Δ_w , and ξ_L . It also provides the interfacial wall-fluid temperature distributions, $\theta_{wi}(\xi)$, the interfacial heat-transfer rate from fluid 2 to the wall, $v_2(\xi)$, and from the wall to fluid 1, $v_1(\xi)$, and the fluid temperature profiles at the outlets

$$\theta_1 = \theta_{1,\text{out}}(y_1) \quad \text{at} \quad \xi = \xi_L, \quad 0 \leq y_1 < 1, \quad (14)$$

$$\theta_2 = \theta_{2,\text{out}}(y_2) \quad \text{at} \quad \xi = 0, \quad 0 \leq y_2 < 1. \quad (15)$$

Another interesting result is the distribution of the bulk—or mixing-cup—temperatures of both fluids

$$\theta_{mi}(\xi) = \int_0^1 \frac{3}{2} (1 - y_i^2) \theta_i(\xi, y_i) dy_i, \quad (16)$$

defined as the uniform temperature that would eventually be attained if the fluid i at a particular section ξ was allowed to evolve adiabatically. In particular, one of the main outcomes of the analysis is the value of the outlet bulk temperatures of both streams, $\theta_{m1,\text{out}} \equiv \theta_{m1}(\xi_L)$ and $\theta_{m2,\text{out}} \equiv \theta_{m2}(0)$, closely related to the heat exchanger effectiveness [40, 41]

$$\varepsilon = \begin{cases} \theta_{m1,\text{out}} & \text{for } mk \geq 1, \\ 1 - \theta_{m2,\text{out}} & \text{for } mk \leq 1. \end{cases} \quad (17)$$

Of special relevance for the study of axial wall conduction is the average axial heat flux conducted in the negative ξ -direction by the heat exchanging wall

$$v_w(\xi) = \Delta_w \kappa_w \int_{-1/2}^{1/2} \frac{\partial \theta_w}{\partial \xi} dy_w. \quad (18)$$

Just like the local heat-transfer rates, $v_i(\xi)$, the average axial heat flux, $v_w(\xi)$, is nondimensionalized with the characteristic scale $k_1(T_{2,\text{in}} - T_{1,\text{in}})/a_1$.

For the case of adiabatic end-wall boundary conditions, an steady-state energy balance applied to the wall portion between 0 and ξ establishes that the heat flux conducted from fluid 2 to the wall, plus the heat flux conducted through the wall in the negative ξ -direction at the arbitrary axial station ξ , must equal the heat flux conducted from the wall to fluid 1, that is,

$$\int_0^\xi v_2(\xi') d\xi' + \Delta_w v_w(\xi) = \int_0^\xi v_1(\xi') d\xi', \quad (19)$$

an integral relation that can be solved for the average axial heat flux to give

$$v_w(\xi) = \frac{1}{\Delta_w} \int_0^\xi [v_1(\xi') - v_2(\xi')] d\xi'. \quad (20)$$

Integration of Eqs. (1) and (2) across the channel width making use of the boundary conditions (6) and (8) and the definition of the mixing cup temperature (16), leads to

$$v_1(\xi) = \frac{1}{2} \frac{\partial \theta_{m1}}{\partial \xi}, \quad v_2(\xi) = \frac{mk}{2} \frac{\partial \theta_{m2}}{\partial \xi}, \quad (21)$$

which, upon substitution in (20), provides the average axial wall heat flux in terms of the local bulk temperatures of both streams

$$v_w(\xi) = \frac{\theta_{m1}(\xi) - mk [\theta_{m2}(\xi) - \theta_{m2,\text{out}}]}{2\Delta_w}. \quad (22)$$

Particularizing this expression at $\xi = \xi_L$, where $v_w = 0$, $\theta_{m2} = 1$, and $\theta_{m1} = \theta_{m1,\text{out}}$, one obtains the following relation between the outlet bulk temperatures of fluids 1 and 2

$$\theta_{m2,\text{out}} = 1 - (mk)^{-1} \theta_{m1,\text{out}}. \quad (23)$$

Similar expressions can be derived for the case of prescribed wall temperature at both ends of the heat conducting plate

$$v_w(\xi) - v_w(0) = \frac{\theta_{m1}(\xi) - mk [\theta_{m2}(\xi) - \theta_{m2,\text{out}}]}{2\Delta_w}, \quad (22')$$

$$\theta_{m2,\text{out}} = 1 - (mk)^{-1} \{\theta_{m1,\text{out}} + 2\Delta_w [v_w(0) - v_w(\xi_L)]\}, \quad (23')$$

which now involve the non-zero axial wall heat fluxes at both ends of the heat-exchanging wall, $v_w(0)$ and $v_w(\xi_L)$, which must be determined as part of the solution of the problem.

Apart from their purely mathematical interest, the identities (22) and (23) find also useful application in the assessment of the numerical accuracy of the finite-difference solutions to be presented in Section 4.

3. Theoretical analysis by separation of variables

3.1. General formulation

Following standard practice, the solution of the problem will be sought as an eigenfunction expansion extending to fluids and the solid wall coupled by compatibility conditions at the fluid-wall interfaces. Assuming separation of variables in the form $\theta_i(\xi, y_i) = \phi_n(\xi) h_{n,i}(y_i)$, where $i = \{1, 2, w\}$, yields the axial dependence $\phi_n(\xi) = e^{-\lambda_n \xi}$ in $0 \leq \xi \leq \xi_L$. Therefore, the solution for fluid i can be written as

$$\theta_i(\xi, y_i) = A + \sum_{n=-\infty}^{+\infty} C_n e^{-\lambda_n \xi} h_{n,i}(y_i), \quad (24a)$$

while the temperature distribution at the wall takes the form

$$\theta_w(\xi, y_w) = A + \sum_{n=-\infty}^{+\infty} C_n e^{-\lambda_n \xi} h_{n,w}(y_w). \quad (24b)$$

Satisfaction of the inlet boundary conditions (10)–(13) at both ends of the heat exchanger forces us to use a set of both positive ($n > 0$) and negative ($n < 0$) eigenvalues, λ_n , with their corresponding eigenfunctions, $h_{n,i}(y_i)$ and $h_{n,w}(y_w)$ [20]. Here we shall use $n = 0$ to denote the eigenvalue with the lowest absolute value, λ_0 , which determines the behavior of the solution away from the thermal entrance regions, where the contribution of the remaining (i.e., higher-order) eigenvalues dies out exponentially. This eigenvalue is positive for $mk > 1$, negative for $mk < 1$, and takes small values in the limit of nearly-balanced heat exchangers, $\lambda_0 \sim 1 - (mk)^{-1} \ll 1$ [40, 41].

Use of expansion (24) in Eqs. (1)–(3) leads to the following problem for the n -th eigenfunction in the fluids

$$-\lambda_n \frac{3}{4} (1 - y_1^2) h_{n,1} = \frac{d^2 h_{n,1}}{dy_1^2}, \quad (25)$$

$$\lambda_n m \frac{3}{4} (1 - y_2^2) h_{n,2} = \frac{d^2 h_{n,2}}{dy_2^2}, \quad (26)$$

for $0 < y_i < 1$, and in the wall

$$0 = (\Delta_w \lambda_n)^2 h_{n,w} + \frac{d^2 h_{n,w}}{dy_w^2}. \quad (27)$$

for $-1/2 < y_w < 1/2$. The above equations must be integrated subject to the symmetry boundary conditions at the channel mid-planes

$$\frac{dh_{n,1}}{dy_1} = \frac{dh_{n,2}}{dy_2} = 0 \quad \text{at} \quad y_i = 0, \quad (28)$$

and the coupling conditions at the fluid-wall interfaces

$$\begin{aligned} h_{n,1}(1) &= h_{n,w}(-1/2), \\ h_{n,2}(1) &= h_{n,w}(1/2), \end{aligned} \quad (29)$$

and

$$\begin{aligned} \left. \frac{dh_{n,1}}{dy_1} \right|_{y_1=1} &= \kappa_w \left. \frac{dh_{n,w}}{dy_w} \right|_{y_w=-1/2} \equiv h'_{n,1}(1) \\ -k \left. \frac{dh_{n,2}}{dy_2} \right|_{y_2=1} &= \kappa_w \left. \frac{dh_{n,w}}{dy_w} \right|_{y_w=1/2} \equiv -kh'_{n,2}(1) \end{aligned} \quad (30)$$

following from (7) and (8), respectively. Due to the linearity of the problem we must also specify a normalization condition; thus, without loss of generality, we set

$$\frac{h_{n,1}(1) + h_{n,2}(1)}{2} = \frac{h_{n,w}(-1/2) + h_{n,w}(1/2)}{2} = 1. \quad (31)$$

3.2. Determination of the eigenfunctions

3.2.1. Case of thermally thin walls without axial conduction

As shown by Vera and Liñán [40], in the limit of thermally thin walls, $\kappa_w^{-1} \rightarrow 0$, without axial wall conduction, $\Delta_w \rightarrow 0$, the problem (25)–(31) admits the following analytical solution

$$f_{n,i}(y_i) = \frac{1}{\sqrt{y_i}} \left\{ \frac{2\sqrt{\pi} M_{\kappa_{n,i}, \frac{1}{4}}(4\kappa_{n,i} y_i^2) + A_{n,i} W_{\kappa_{n,i}, \frac{1}{4}}(4\kappa_{n,i} y_i^2)}{2\sqrt{\pi} M_{\kappa_{n,i}, \frac{1}{4}}(4\kappa_{n,i}) + A_{n,i} W_{\kappa_{n,i}, \frac{1}{4}}(4\kappa_{n,i})} \right\} \quad (32)$$

where $i = \{1, 2\}$, with

$$A_{n,i} = \Gamma\left(\frac{1}{4} - \kappa_{n,i}\right), \quad \kappa_{n,1} = \frac{\sqrt{3\lambda_n}}{8}, \quad \kappa_{n,2} = i \frac{\sqrt{3m\lambda_n}}{8}, \quad (33)$$

where $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ are the Whittaker functions [45, 46], $\Gamma(z)$ is the Gamma function, and i denotes the imaginary unit. Note that the eigenvalue λ_n appears both in the argument and in the first index of the Whittaker functions through $\kappa_{n,i}$ [47], and that the eigenfunctions verify the normalization condition $f_{n,i}(1) = 1$, which results from (31) in the limit $\kappa_w^{-1} \rightarrow 0$, when the wall eigenfunctions become constant and, therefore, equal to unity, $f_{n,w}(y_w) = 1$.

Applying symbolic derivation to (32) and evaluating the derivatives at the fluid-wall interfaces yields

$$\begin{aligned} f'_{n,i}(1) \equiv \left. \frac{df_{n,i}}{dy_i} \right|_{y_i=1} &= -\frac{1}{2} + 2 \left\{ \kappa_{n,i} \right. \\ &\quad \left. + \frac{2\sqrt{\pi}(3/4 + \kappa_{n,i})M_{\kappa_{n,i}+1, \frac{1}{4}}(4\kappa_{n,i}) - A_{n,i}W_{\kappa_{n,i}+1, \frac{1}{4}}(4\kappa_{n,i})}{2\sqrt{\pi}M_{\kappa_{n,i}, \frac{1}{4}}(4\kappa_{n,i}) + A_{n,i}W_{\kappa_{n,i}, \frac{1}{4}}(4\kappa_{n,i})} \right\}, \end{aligned} \quad (34)$$

with $A_{n,i}$ and $\kappa_{n,i}$ given in terms of m and λ_n by (33). The above expressions will be used below as building blocks for the determination of the eigenvalues and eigenfunctions in the general case of non-thermally thin walls, $\kappa_w^{-1} > 0$, with axial conduction, $\Delta_w > 0$.

3.2.2. Case of non-thermally thin walls with axial conduction

Following Quintero et al. [41], the solution of the problem stated in (25)–(31) can be written in the form

$$h_{n,i}(y_i) = \left(1 \pm \frac{\alpha_n}{2}\right) f_{n,i}(y_i), \quad (35)$$

where in this case the $+$ ($-$) sign holds for fluid 2 (1), and α_n is a constant to be determined as part of the solution (see Eq. (39) below). Note in particular that (35) satisfies the normalization condition (31) regardless of the value of α_n . The corresponding eigenfunction at the wall, $h_{n,w}$, is obtained by direct integration of Eq. (27) to give

$$h_{n,w}(y_w) = a_n \sin(\Delta_w \lambda_n y_w) + b_n \cos(\Delta_w \lambda_n y_w), \quad (36)$$

with the constants

$$a_n = \frac{\alpha_n/2}{\sin(\Delta_w \lambda_n/2)} \quad \text{and} \quad b_n = \frac{1}{\cos(\Delta_w \lambda_n/2)} \quad (37)$$

determined by the coupling conditions

$$\begin{aligned} -a_n \sin(\Delta_w \lambda_n/2) + b_n \cos(\Delta_w \lambda_n/2) &= 1 - \frac{\alpha_n}{2}, \\ a_n \sin(\Delta_w \lambda_n/2) + b_n \cos(\Delta_w \lambda_n/2) &= 1 + \frac{\alpha_n}{2}, \end{aligned} \quad (38)$$

resulting from (29) and (35).

Using (35)–(37) in (30) we get two alternative expressions for α_n , namely

$$\begin{aligned} \frac{\alpha_n}{2} &= \frac{f'_{n,1}(1) - kf'_{n,2}(1)}{f'_{n,1}(1) + kf'_{n,2}(1) + \frac{2\kappa_w \Delta_w \lambda_n}{\tan(\Delta_w \lambda_n/2)}} \\ &= \frac{f'_{n,1}(1) + kf'_{n,2}(1) - 2\kappa_w \Delta_w \lambda_n \tan(\Delta_w \lambda_n/2)}{f'_{n,1}(1) - kf'_{n,2}(1)} \end{aligned} \quad (39)$$

where, according to (35), we have written

$$h'_{n,i}(1) = \left(1 \pm \frac{\alpha_n}{2}\right) f'_{n,i}(1), \quad (40)$$

with $f'_{n,i}(1)$ given analytically by (34).

3.3. Determination of the eigenvalues

The second equation in (39) provides the eigencondition that must be satisfied by the eigenvalues λ_n . After some algebra, it can be conveniently rewritten in the form

$$\begin{aligned} [f'_{n,1}(1; \lambda_n) - kf'_{n,2}(1; m\lambda_n)]^2 \\ = \left\{ [f'_{n,1}(1; \lambda_n) + kf'_{n,2}(1; m\lambda_n)] \tan(\Delta_w \lambda_n/2) + 2\kappa_w \Delta_w \lambda_n \right\} \\ \times \left\{ \frac{f'_{n,1}(1; \lambda_n) + kf'_{n,2}(1; m\lambda_n)}{\tan(\Delta_w \lambda_n/2)} - 2\kappa_w \Delta_w \lambda_n \right\}, \end{aligned} \quad (41)$$

where, for clarity in the presentation, we have explicitly stated the dependence of $f'_{n,1}(1)$ on λ_n and $f'_{n,2}(1)$ on $m\lambda_n$, which was previously left out in (34). This equation restricts the values of λ_n to an infinite numerable set of both positive and negative eigenvalues which can be obtained numerically for specified values of m, k, κ_w , and Δ_w . As a final remark, it is interesting to note that positive and negative eigenvalues satisfy the relation

$$\lambda_{-n}(m, k, \kappa_w, \Delta_w) = -m^{-1} \lambda_n(m^{-1}, k^{-1}, k^{-1} \kappa_w, m^{-1} \Delta_w) \quad (42)$$

resulting from the mathematical symmetries of the problem.

Table 1 presents numerical values of λ_n , $n = 0, \pm 1, \dots, \pm 4$, for $m = 2, k = 1$, and different values of κ_w and Δ_w . The results are to be compared with those reported by Vera and Liñán [40] for thermally thin walls without axial wall conduction ($\kappa_w^{-1} = \Delta_w = 0$) and by Quintero et al. [41] for walls with finite thermal resistance (finite κ_w) but negligible axial conduction ($\Delta_w = 0$) shown in the top rows of the Table. The computed results indicate that the effect of increasing Δ_w for a fixed value of κ_w (that is, of increasing the importance of axial wall conduction while keeping constant the transverse wall thermal resistance) is always to reduce the absolute value of λ_n regardless of the order n , thereby leading to a deterioration of the heat exchanger performance.

By way of contrast, the effect of increasing the dimensionless wall conductivity κ_w for a fixed value of Δ_w is less trivial. In the absence of axial wall conduction, $\Delta_w = 0$, all eigenvalues grow in absolute value when the wall thermal resistance κ_w^{-1} is reduced (i.e., $|\lambda_n|$ grows with κ_w). But this trend changes as soon as axial conduction becomes important, i.e., for $\Delta_w > 0$. For instance, for $\Delta_w = 10^{-3}$ a nonmonotonic dependence of λ_n with κ_w is first observed at $n = 3$, whereas for $\Delta_w = 10^{-2}$ it appears at $n = \pm 1$. In general, the nontrivial dependence of λ_n on κ_w reaches lower order eigenvalues as Δ_w becomes larger, the case $\Delta_w = 10^{-1}$ exhibiting such nonmonotonic dependence in the lowest order eigenvalue, which is larger for $\kappa_w = 10$ ($\lambda_0 \approx 0.78$) than for $\kappa_w = 2$ (0.66) or $\kappa_w = 50$ (0.48). We anticipate that the local maxima reached by $\lambda_0, \lambda_{\pm 1}, \dots, \lambda_{\pm n}$ at intermediate values of κ_w must be somehow related to the existence of an optimum value for the wall heat conductivity, a concept previously discussed by several authors [11, 12, 25].

The eigenvalues reported in Table 1 can be used to evaluate the eigenfunctions using Eqs. (32)–(37). Figure 2 shows the eigenfunctions $h_{n,i}(y_i)$, $n = 0, \pm 1, \dots, \pm 4$, for $m = 2, k = 1, \kappa_w = (50, 10, 2)$ and $\Delta_w = (10^{-3}, 10^{-2})$. As can be seen, the effect of axial wall conduction (which is more prominent for $\Delta_w = 10^{-2}$) has a clear impact on the shape of the eigenfunctions. On one hand, the eigenfunctions are no longer linear in the heat conducting wall, as occurs when axial wall conduction is absent. On the other hand, there is a noticeable amplification of the higher order eigenfunctions for walls with low thermal resistance (large κ_w) and non-negligible axial conduction (e.g., for $\Delta_w = 10^{-2}$). This unexpected behavior indicates that the mathematical analysis of the limit $\kappa_w \gg 1$ with fixed Δ_w may require an alternative scaling for the eigenfunctions, although preliminar results obtained with even larger values of Δ_w indicate that the present formulation is able to deal with such amplified eigenfunctions without difficulties.

The origin of the amplification of certain eigenfunctions in the eigenfunction spectrum can be traced back to the existence of singularities in $f'_{n,i}(1; \lambda_n)$, which is seen to diverge as λ_n approaches certain singular values. Thus, for the positive eigenfunctions, $n > 0$, the singularities arise as $\lambda_n \rightarrow \lambda_p^\infty$, whereas for the negative eigenfunctions, $n < 0$, they arise as $-m\lambda_n \rightarrow \lambda_p^\infty$, where λ_p^∞ , $p \in \mathbb{N}$, represents the set of singular values

$$\lambda_p^\infty \approx \{3.77, 42.86, 124.63, 249.07, 416.18, 625.96, \dots\} \quad (43)$$

This explains, for instance, the significant amplification of the eigenfunctions $f'_{3,i}, f'_{4,i}$, and $f'_{-4,i}$ displayed in the upper right panel of Fig. 2, corresponding to $m = 2, k = 1, \Delta_w = 10^{-2}$ and $\kappa_w = 50$. In this particular case, the eigenvalues $\lambda_3 \approx 125.44$, $\lambda_4 \approx 247.19$, and $\lambda_{-4} \approx -125.23$, given by Table 1, are very close to the singular values $\lambda_3^\infty \approx 124.63$, $\lambda_4^\infty \approx 249.07$ and $-\lambda_4^\infty/m \approx -124.54$, making the corresponding eigenfunctions to grow well above the others, which remain of order unity everywhere. It is interesting to note that the set of singular values λ_p^∞ given in (43) is universal, in that they are independent on the dimensionless parameters of the problem.

3.4. Orthogonality condition for the eigenfunctions

As thoroughly discussed in Appendix A, the eigenfunctions of the problem satisfy the orthogonality condition

$$\langle h_n, h_q \rangle = 0 \quad \text{for } n \neq q, \quad n, q = 0, \pm 1, \pm 2, \dots \quad (44)$$

with respect to the inner product

$$\begin{aligned} \langle h_n, h_q \rangle = & \int_0^1 w(y_1) h_{n,1}(y_1) h_{q,1}(y_1) dy_1 \\ & - mk \int_0^1 w(y_2) h_{n,2}(y_2) h_{q,2}(y_2) dy_2 \\ & + (\lambda_n + \lambda_q) \kappa_w \Delta_w^2 \int_{-1/2}^{1/2} h_{n,w}(y_w) h_{q,w}(y_w) dy_w, \end{aligned} \quad (45)$$

where $w(y_i) = (3/4)(1 - y_i^2)$ is the weight function and h_n is the eigenfunction defined by $h_{n,1}(y_1)$ for $0 < y_1 < 1$, $h_{n,2}(y_2)$ for $0 < y_2 < 1$, and $h_{n,w}(y_w)$ for $-1/2 < y_w < 1/2$.

It is also interesting to note that all eigenfunctions are orthogonal to unity

$$\begin{aligned} \langle 1, h_q \rangle = & \int_0^1 w(y_1) h_{q,1}(y_1) dy_1 - mk \int_0^1 w(y_2) h_{q,2}(y_2) dy_2 \\ & + \lambda_q \kappa_w \Delta_w^2 \int_{-1/2}^{1/2} h_{q,w}(y_w) dy_w = 0. \end{aligned} \quad (46)$$

This orthogonality condition holds because zero is always an eigenvalue of the problem, the associated eigenfunction being constant and equal to unity due to the normalization condition (31). This trivial eigenmode appears explicitly in the expansion coefficient (24) as the constant A , which will be obtained below, along with the remaining expansion coefficients C_n , using conditions (44) and (46).

3.5. Determination of the expansion coefficients

The orthogonality conditions derived in the previous section may be used, together with the inlet conditions (10)–(11) and the adiabatic end-wall boundary conditions (12)–(13), to calculate the expansion coefficients A and C_n [40] (see Appendix B.1 for details). Omitting the lengthy algebra, this procedure leads, for $mk \neq 1$, to the following infinite set of linear equations

$$mk = A(1 + mk) - \sum_{q=-\infty}^{+\infty} C_q \left(e^{-\lambda_q \xi_L} h'_{q,1}(1) - kh'_{q,2}(1) \right) 2\lambda_q^{-1} \quad (47)$$

and

$$\begin{aligned} -e^{\lambda_n \xi_L} \lambda_n^{-1} kh'_{n,2}(1) &= A \left(h'_{n,1}(1) - e^{\lambda_n \xi_L} kh'_{n,2}(1) \right) \lambda_n^{-1} \\ &+ C_n \left[\Omega_n^- - \frac{\alpha_n}{2} \Omega_n^+ + \left(\frac{kh'_{n,2}(1) + h'_{n,1}(1)}{2} \right) \frac{\partial \alpha_n}{\partial \lambda_n} \right] \\ &+ \sum_{\substack{q=-\infty \\ q \neq n}}^{+\infty} C_q \left\{ \left(\frac{1 + e^{(\lambda_n - \lambda_q) \xi_L}}{\lambda_n - \lambda_q} \right) \left[\left(1 - \frac{\alpha_q}{2} \right) h'_{n,1}(1) - \left(1 - \frac{\alpha_n}{2} \right) h'_{q,1}(1) \right] \right. \\ &\left. + \left(\frac{\lambda_q + \lambda_n e^{(\lambda_n - \lambda_q) \xi_L}}{\lambda_n^2 - \lambda_q^2} \right) \left[\left(1 - \frac{\alpha_n}{2} \right) h'_{q,1}(1) - \left(1 - \frac{\alpha_q}{2} \right) h'_{n,1}(1) \right] \right. \\ &\left. + \left(1 + \frac{\alpha_n}{2} \right) kh'_{q,2}(1) - \left(1 + \frac{\alpha_q}{2} \right) kh'_{n,2}(1) \right\} \quad (48) \end{aligned}$$

for $n = 0, \pm 1, \pm 2, \dots$, where the coefficients

$$\Omega_n^\pm = \frac{\partial}{\partial \lambda_n} \left[h'_{n,1}(1) \pm kh'_{n,2}(1) \right] \quad (49)$$

must be evaluated numerically, since the derivatives of $h'_{n,i}(1)$ with respect to λ_n can not be written, in general, in analytical form.

Following the same procedure, but using now the specified end-wall temperature conditions (12')–(13') instead of the adiabatic conditions (12)–(13), an alternative infinite set of linear equations is obtained (see Appendix B.2 for details)

$$mk = A(1 + mk) - \sum_{q=-\infty}^{+\infty} C_q \left(h'_{q,1}(1) - e^{-\lambda_q \xi_L} kh'_{q,2}(1) \right) 2\lambda_q^{-1} \quad (47')$$

and

$$\begin{aligned} -e^{\lambda_n \xi_L} \lambda_n^{-1} h'_{n,1}(1) &= A \left(kh'_{n,2}(1) - e^{\lambda_n \xi_L} h'_{n,1}(1) \right) \lambda_n^{-1} \\ &- C_n \left[\Omega_n^- - \frac{\alpha_n}{2} \Omega_n^+ + \left(\frac{kh'_{n,2}(1) + h'_{n,1}(1)}{2} \right) \frac{\partial \alpha_n}{\partial \lambda_n} \right] \\ &- \sum_{\substack{q=-\infty \\ q \neq n}}^{+\infty} C_q \left\{ \left(\frac{1 + e^{(\lambda_n - \lambda_q) \xi_L}}{\lambda_n - \lambda_q} \right) \left[\left(1 - \frac{\alpha_q}{2} \right) h'_{n,1}(1) - \left(1 - \frac{\alpha_n}{2} \right) h'_{q,1}(1) \right] \right. \\ &\left. + \left(\frac{\lambda_n + \lambda_q e^{(\lambda_n - \lambda_q) \xi_L}}{\lambda_n^2 - \lambda_q^2} \right) \left[\left(1 - \frac{\alpha_n}{2} \right) h'_{q,1}(1) - \left(1 - \frac{\alpha_q}{2} \right) h'_{n,1}(1) \right] \right. \\ &\left. + \left(1 + \frac{\alpha_n}{2} \right) kh'_{q,2}(1) - \left(1 + \frac{\alpha_q}{2} \right) kh'_{n,2}(1) \right\} \quad (48') \end{aligned}$$

for $n = 0, \pm 1, \pm 2, \dots$, with $\Omega_n^\pm$ given again by Eq. (49).

3.5.1. Solution procedure

The infinite systems (47)–(48) or (47')–(48'), corresponding to adiabatic end-wall boundary conditions or specified end-wall temperature conditions, respectively, must be truncated to a finite number of terms in order to get a numerical solution of the problem. The particularity of counterflow heat exchangers is that the addition of new terms to the solution has an effect on the preceding expansion coefficients, which can not be calculated successively as often occurs in single-stream or cocurrent flow system [20, 21]. Moreover, the convergence of the series becomes very slow at small axial distances, where a large number of terms may be required to get an accurate description of the solution [20, 33]. However, the lack of pointwise convergence near the inlets does not affect the accuracy of global results, such as the heat exchanger effectiveness, which may be evaluated with high precision using only a moderate number of terms.

4. Validation and discussion

The problem under study has been investigated by different authors using a wide variety of semi-analytical methods. This includes, for instance, the use of Duhamel's theorem upon the exact eigenfunction solution of the classical Graetz problem for flow between parallel plates [25], or the application of the thermal quadrupole method combined with a Fourier-cosinus transform description for heat conduction in the wall [12]. The aim of this section is to validate the exact solution proposed here against those previous results, and to compare the semi-analytical temperature field against finite difference solutions obtained for selected parameter values.

Figure 3 shows the variation of the outlet bulk temperature of fluid 1 with the dimensionless heat exchanger length ξ_L provided by the truncated exact solution

$$\theta_{m1,\text{out}}^{[N]}(\xi_L) = A - \sum_{n=-N}^{+N} 2C_n e^{-\lambda_n \xi_L} \lambda_n^{-1} h'_{n,1}(1) \quad (50)$$

with $N = 5$ (12-term solution). The different curves correspond to different values of m and κ_w , a constant ratio $\Delta_w/\xi_L = 0.01$, chosen so as to mimic the dimensionless parameters used in Fig. 6 of Mori et al. [25], and $k = 1$. Note that according to (17) $\theta_{m1,\text{out}}$ coincides here with the heat exchanger effectiveness, since $mk > 1$ in all cases shown in the figure. As can be seen, the truncated exact solution requires only a moderate number of terms to accurately reproduce the results of Mori et al. [25], predicting a nonmonotonic variation of $\theta_{m1,\text{out}}$ with ξ_L that shows the existence of an optimum heat exchanger length that maximizes the thermal performance of the system.

It should be noted that Mori et al. [25] employed a slightly different set of dimensionless parameters, so that appropriate conversion relations were required to determine the values of m , k , κ_w , Δ_w , and ξ_L corresponding to each point captured from Fig. 6 of [25] (shown as hollow circles in Fig. 3). In particular, instead of using the thermal entry length $a_1 \text{Pe}_1$ they scaled the wall thickness using the heat exchanger length L , which results in the alternative dimensionless group $\delta_w/L (= \Delta_w/\xi_L)$.

This subtle difference in the choice of dimensionless parameters enabled us to use separation of variables and to expand the solution in series of eigenfunctions, since by defining the dimensionless wall thickness as $\Delta_w = \delta_w/(a_1 \text{Pe}_1)$ the problem (25)–(31) for the transverse eigenfunctions becomes free of any longitudinal (or axial) input, the dimensionless heat exchanger length ξ_L appearing only in the system of equations (47)–(48) that determines the expansion coefficients.

To further verify the validity of the proposed solution, finite-difference integrations were carried out for selected values of the parameters m , k , κ_w , Δ_w , and ξ_L . Equations (1)–(3) were discretized using second-order centered finite differences on a distorted grid with points clustered near the inlet, outlet and fluid-wall interfaces. Second-order non-centered schemes were used to discretize the derivatives appearing in the boundary conditions (6)–(13). The steady solution was obtained using a time-marching method with local time stepping until the convergence criteria $|\mathcal{R}| < 10^{-5}$ was achieved, $|\mathcal{R}|$ being a suitable norm of the residuals (e.g. the infinite norm). Further details of the numerical method can be found elsewhere [40].

Table 2 compares the values of $\theta_{m1,\text{out}}$ obtained numerically with those predicted by the truncated exact solution with $N = 5$ and 10. It also shows the relative errors of the finite-difference results and the truncated exact solution with $N = 5$ compared to the truncated exact solution with $N = 10$, taken as benchmark solution here. For adiabatic end-wall boundary conditions the relative error associated with the truncated exact solution for $N = 5$ is too small to be appreciable, whereas finite-difference solutions show larger errors (up to 0.3%). Interestingly enough, for specified end-wall temperature conditions the relative errors are significantly larger (up to 1-2%), indicating that this case must be somewhat harder to solve.

As a further validation, Fig. 4 shows the variation of $\theta_{m1,\text{out}}$ with κ_w reported by Maranzana et al. [12] ($\Delta_w = 0.3$) along with that provided by the truncated exact solution for $N = 10$ (22-term solution). As can be seen, the truncated exact solution is able to capture with high precision this dependence, showing the existence of an optimum wall conductivity κ_w^{opt} that maximizes heat exchanger effectiveness. Results obtained with different values of Δ_w show that κ_w^{opt} decreases with Δ_w , adopting values of order unity when Δ_w is also of order unity; e.g., $\kappa_w^{\text{opt}} = (2.6, 1.8)$ for $\Delta_w = (0.2, 0.3)$. As already mentioned in the discussion of Table 1, the existence of an optimum wall conductivity [11, 12, 25] is intimately linked to the non-monotonic behavior of the lowest order eigenvalues with κ_w . The exact results presented in this paper could be used to shed further light on this topic. However, the analysis goes beyond the scope of this paper and is left for future work.

The exact solution presented here not only provides overall results, such as the outlet bulk temperatures reported in Table 2 and in Figs. 3 and 4. It also gives detailed information, such as the spatial temperature fields, the bulk and interfacial temperature distributions, the longitudinal variation of the interfacial heat-transfer rates and wall-averaged axial heat flux, etc. Sample calculations are shown in Figs. 5 and 6, where the parameter values have been chosen so as to mimic the particular case studied by Maranzana et al. [12]. As can be

seen, the agreement in the temperature distribution predicted by the truncated exact solution and by finite-difference integrations is remarkable everywhere, with only minor differences being observed in the wall-averaged axial heat flux, which is slightly underestimated in the numerical solution. This fact can be traced back to the lack of spatial resolution of the numerical method in the thermal entry regions, where the interfacial heat transfer rates present a nearly-singular behavior that can not be solved accurately without a careful asymptotic analysis of these regions [48].

4.1. Nusselt numbers and overall heat-transfer coefficient

Using the concept of additivity of thermal resistances, the dimensionless overall heat-transfer coefficient can be written as

$$\frac{1}{\overline{U}} = \frac{4}{\text{Nu}_1} + \frac{1}{\kappa_w} + \frac{4}{k\text{Nu}_2}, \quad (51)$$

where $\text{Nu}_i = h_i(4a_i)/k_i$ represents the Nusselt number of fluid i , based on the hydraulic diameter, $4a_i$, and the heat-transfer coefficient, h_i . Introducing dimensionless variables, the Nusselt numbers appearing in (51) can be written as

$$\text{Nu}_1(\xi) = \frac{4\nu_1(\xi)}{\theta_{w1}(\xi) - \theta_{m1}(\xi)}, \quad (52)$$

$$\text{Nu}_2(\xi) = \frac{4\nu_2(\xi)}{k[\theta_{m2}(\xi) - \theta_{w2}(\xi)]}. \quad (53)$$

Figures 5(f) and 6(f) show the Nusselt number distributions $\text{Nu}_i(\xi)$ predicted by the truncated exact solution with $N = 10$ in two representative cases corresponding to long and moderately short heat exchangers, respectively. The figures also show the thermal entry length of both fluids, arbitrarily defined as the downstream distance required for the first eigenmode (i.e., $h_{\pm 1,i}$) to decay to 1% of its initial amplitude. With this definition the thermal entry length of streams 1 and 2 are given, respectively, by $e^{-|\lambda_1|\Delta\xi_{T1}} = 0.01$ and $e^{-|\lambda_{-1}|\Delta\xi_{T2}} = 0.01$, or, what is the same, by $\Delta\xi_{T1} \simeq 4.6|\lambda_1|^{-1}$ and $\Delta\xi_{T2} \simeq 4.6|\lambda_{-1}|^{-1}$.

As can be seen, in the long heat exchanger shown in Fig. 5 ($\xi_L = 5.89$) the thermal development regions extend to about 20% of the heat exchanger length. In those regions the Nusselt numbers take different values, with that of the incoming stream exhibiting a markedly singular behavior. In the central region, however, the Nusselt numbers decay to the constant value 8.235, corresponding to thermally developed flow with uniform heat flux between parallel plates [49] (as should be expected for a nearly-balanced heat exchanger, as the one considered here). By way of contrast, the heat exchanger shown in Fig. 6, which exhibits the same parameters m , k , κ_w , and Δ_w , and therefore, the same eigenvalues and eigenmodes, is significantly shorter ($\xi_L = 1.76$). In this case the thermal entry lengths overlap in the center of the heat exchanger, so that the Nusselt numbers do not quite reach their fully developed value.

In a nearly-balanced heat exchanger, the average heat flux conducted by the wall ν_w in the axial direction remains roughly constant along the fully developed region (see, e.g., Fig. 5(f)). It is in the thermal entry regions where, due to the imbalance between the interfacial heat transfer rates, $\nu_1 \neq \nu_2$, the wall

axial heat flux grows from virtually zero at the inlets to its uniform (fully developed) value for $\Delta\xi_{T1} < \xi < \xi_L - \Delta\xi_{T2}$. A clear picture therefore emerges in which heat is first transferred from fluid 2 to the wall in the thermal entry region of fluid 2, then conducted axially through the wall to the opposite end of the heat exchanger, and finally transferred back from the wall to fluid 1 in the thermal entry region of fluid 1.

This description may suffer small changes in unbalanced heat exchangers, where the average wall axial heat flux also experiences significant variations in the fully developed regime due to the exponential decay of the lowest order eigenmode ($\lambda_0 \neq 0$) throughout the heat exchanger length. Anyway, in both cases the main effect of axial wall conduction is to enhance the exchange of thermal energy between the thermal entry regions of fluids 1 and 2, up to the point that for large values of Δ_w the counterflow heat exchanger behaves effectively as a co-current flow system [7–9].

5. Conditions for ignoring axial wall conduction

This section discusses under which conditions it is justified to ignore the effects of axial wall conduction in the calculation of the eigenvalues and eigenfunctions of the problem, as well as of the heat exchanger effectiveness. With this aim, we shall first consider the leading order perturbations to eigencondition (41) in the limit of small axial conduction, $\Delta_w \ll 1$. If we introduce the notation $\beta_n \equiv \Delta_w \lambda_n / 2$, Eq. (41) can be written as

$$[f'_{n,1}(1; \lambda_n) - k f'_{n,2}(1; m\lambda_n)]^2 = [F \tan \beta_n + 4\kappa_w \beta_n] \left[\frac{F}{\tan \beta_n} - 4\kappa_w \beta_n \right] \quad (54)$$

where $F \equiv f'_{n,1}(1; \lambda_n) + k f'_{n,2}(1; m\lambda_n)$ is a quantity of order unity (or, more precisely, of order κ_w^{-1} , as implied by Eq. (56) below).

In the limit $\beta_n \ll 1$, the right hand side of Eq. (54) can be expanded as

$$[F \tan \beta_n + 4\kappa_w \beta_n] \left[\frac{F}{\tan \beta_n} - 4\kappa_w \beta_n \right] = (4\kappa_w + F)F - 16\kappa_w \left[\kappa_w + \frac{F}{3} \right] \beta_n^2 + O(\beta_n^4). \quad (55)$$

Substituting (55) in (54), and after a little algebra, we have

$$f'_{n,1}(1; \lambda_n) + k f'_{n,2}(1; m\lambda_n) \left[1 + \frac{f'_{n,1}(1; \lambda_n)}{\kappa_w} \right] = 4 \left[\kappa_w + \frac{F}{3} \right] \beta_n^2 + O(F\beta_n^4), \quad (56)$$

which, in the formal limit $\beta_n \rightarrow 0$, reduces identically to the eigencondition

$$f'_{n,1}(1; \lambda_n) + k f'_{n,2}(1; m\lambda_n) \left[1 + \frac{f'_{n,1}(1; \lambda_n)}{\kappa_w} \right] = 0 \quad (57)$$

obtained by Quintero et al. [41, Eq. (30)] in the absence of axial wall conduction. In this limit, the wall-normal heat flux

is uniform across the wall, the wall eigenfunction reduces to a straight line, $g_{n,w}(y_w) = 1 + \alpha_n y_w$, and the fluid eigenfunctions exhibit a finite jump at the wall, $g_{n,2}(1) - g_{n,1}(1) = \alpha_n$, which grows for increasing values of the wall thermal resistance, κ_w^{-1} [41].

5.1. Determination of the eigenvalues

The above discussion shows that axial wall conduction starts to play a role as soon as the leading order term in the right-hand side of (56) becomes large enough to modify the roots of (57) in a significant way. If the dimensionless wall conductivity is of order unity or larger, $\kappa_w \gtrsim 1$, as typically occurs in applications with low-to-moderate wall thermal resistances, $\kappa_w^{-1} \lesssim 1$, the leading order term in the right-hand side of (56) is $4\kappa_w \beta_n^2 (= \kappa_w \Delta_w^2 \lambda_n^2)$. Accordingly, the condition for axial wall conduction to be negligible in the determination of the n -th eigenvalue can be simply stated as

$$\kappa_w \Delta_w^2 \lambda_n^2 \ll 1, \quad \Delta_w^2 \ll \frac{1}{\kappa_w \lambda_n^2}, \quad \text{or} \quad |\lambda_n| \ll \frac{1}{\sqrt{\kappa_w \Delta_w}}. \quad (58)$$

Note that, for given values of κ_w and Δ_w , the latter condition can only be satisfied by a finite set of eigenvalues, whose values could in principle be determined using (57) and ignoring the effect of axial wall conduction. The remaining (i.e., higher-order) eigenvalues, such that $|\lambda_n| \gtrsim 1/(\sqrt{\kappa_w \Delta_w})$, should however be calculated using the original eigencondition (41) with axial wall conduction effects fully taken into account.

The above discussion is not strictly applicable to the lowest order eigenvalue, λ_0 , which may be affected by axial wall conduction earlier than implied by condition (58). This is for instance the case in nearly-balanced heat exchangers, such that $\lambda_0 \sim 1 - (mk)^{-1} \ll 1$ [40, 41]. The analysis is facilitated in this case by introducing the asymptotic expansions

$$\begin{aligned} f'_{0,1}(1; \lambda_0) &= -\frac{\lambda_0}{2} - \frac{17\lambda_0^2}{140} + O(\lambda_0^3), \\ f'_{0,2}(1; m\lambda_0) &= \frac{m\lambda_0}{2} - \frac{17(m\lambda_0)^2}{140} + O[(m\lambda_0)^3], \end{aligned} \quad (59)$$

for the derivatives of the lowest order eigenfunction at the wall in terms of the small parameter λ_0 .

Using the above expansions in (56), writing $\beta_n = \Delta_w \lambda_0 / 2$, and expanding in series of λ_0 , the eigencondition reduces to

$$a\lambda_0 - b\lambda_0^2 = \kappa_w \Delta_w^2 \lambda_0^2 + O(\lambda_0^3), \quad (60)$$

where $a = (mk - 1)/2$ and $b = (17/140)(m^2k + 1) + (mk)/(4\kappa_w)$. Neglecting terms of order λ_0^3 and solving for λ_0 we get

$$\lambda_0 = 0 \quad \text{and} \quad \lambda_0 \simeq \frac{a}{b + \kappa_w \Delta_w^2}, \quad (61)$$

which, besides the trivial eigenvalue, always present in this problem, yields the following expression for the lowest order eigenvalue

$$\lambda_0 \simeq \frac{a/b}{1 + (\kappa_w \Delta_w^2 / a)(a/b)} = \frac{\bar{\lambda}_0^0}{\left[1 + \frac{2\kappa_w \Delta_w^2}{mk-1} \bar{\lambda}_0^0 \right]}, \quad (62)$$

where $\bar{\lambda}_0^0 = a/b$ denotes the approximate value of λ_0 obtained from (61) in the limit of infinitely thin walls, $\Delta_w \rightarrow 0$.

Although the last expression requires that λ_0 be small compared to unity (a condition that holds, as previously discussed, for nearly-balanced heat exchangers), our numerical results (not presented here for brevity) show that Eq. (62) provides also a good description of the variation of λ_0 with Δ_w in more general situations when we use λ_0^0 (the numerical value of λ_0 corresponding to $\Delta_w = 0$, determined from Eq. (57)) instead of $\bar{\lambda}_0^0$. Thus, in general, we may write

$$\lambda_0 \simeq \frac{\lambda_0^0}{\left[1 + \frac{2\kappa_w \Delta_w^2}{mk-1} \lambda_0^0\right]}, \quad (63)$$

an expression that may be used to obtain an accurate value for λ_0 to be used as starting guess in the numerical calculation from the full eigencondition (41).

The last expression shows that the condition for axial wall conduction to be negligible in the determination of λ_0 can be written as

$$\frac{2\kappa_w \Delta_w^2}{mk-1} \lambda_0^0 \ll 1, \quad \text{or} \quad \Delta_w^2 \ll \frac{mk-1}{2\kappa_w \lambda_0^0}. \quad (64)$$

This condition, derived for nearly-balanced heat exchangers, can be combined with condition (58) particularized for $\lambda_n = \lambda_0^0$ to write the following criterion of general validity

$$\Delta_w \ll \min \left\{ \sqrt{\frac{mk-1}{2\kappa_w \lambda_0^0}}, \frac{1}{\sqrt{\kappa_w \lambda_0^0}} \right\}, \quad (65)$$

where the first bracketed expression holds for $\lambda_0^0 \leq 2/(mk-1)$ and the second for $\lambda_0^0 \geq 2/(mk-1)$.

Table 3 shows the application of this general criterion to different parameter sets ranging from nearly-balanced ($m = 1.1$, $k = 1$) to highly unbalanced ($m = k = 2$) heat exchangers. It is interesting to note that the critical value of Δ_w given by Eq. (65) depends mainly on κ_w , whereas m and k (determining whether the heat exchanger is balanced or not) have only a minor impact.

5.2. Determination of the overall heat exchanger effectiveness

Although axial wall conduction may not have any effect in the determination of the lowest order eigenvalue λ_0 , it may still affect overall heat exchanger performance through the lowest order eigenmode. This indirect effect comes from the expansion coefficients A and C_0 , which depend critically on the dimensionless heat exchanger length, ξ_L , a parameter that does not appear in the eigenvalue problem.

For estimative purposes, we shall consider the following approximate expression for the average dimensionless wall temperature

$$\bar{\theta}_w(\xi) = \frac{\theta_{w1}(\xi) + \theta_{w2}(\xi)}{2} \simeq \frac{1 + \frac{\lambda_0}{2} \frac{e^{-\lambda_0 \xi}}{g'_{0,1}(1)}}{1 - (mk)^{-1} e^{-\lambda_0 \xi_L}}, \quad (66)$$

where $g'_{0,1}(1) = f'_{0,1}(1)/[1 + f'_{0,1}(1)/(2\kappa_w)]$, derived from the work of Quintero et al. [41], where no axial wall conduction is

taken into account. This expression can be used to estimate the ratio of the overall axial heat transfer by conduction in the wall

$$\delta_w k_w \frac{d\bar{\theta}_w}{dX} = \delta_w k_w \frac{T_{2,\text{in}} - T_{1,\text{in}}}{\text{Pe}_1 a_1} \frac{d\bar{\theta}_w}{d\xi} \quad (67)$$

to the overall convective heat transfer in the flow channels $2a_1 V_1 \rho_1 c_1 (T_{2,\text{in}} - T_{1,\text{in}})$. In dimensionless form, this ratio is bounded by

$$\begin{aligned} \frac{\text{Axial wall conduction}}{\text{Convective heat flux}} &\simeq \kappa_w \Delta_w^2 \frac{d\bar{\theta}_w}{d\xi} \\ &\leq \frac{\kappa_w \Delta_w^2 \lambda_0^2 \left[1 + \frac{f'_{0,1}(1)}{2\kappa_w}\right]}{-2f'_{0,1}(1)[1 - (mk)^{-1} e^{-\lambda_0 \xi_L}]} \end{aligned} \quad (68)$$

where, without loss of generality, we have assumed that $\lambda_0 > 0$ (i.e., $mk > 1$), so that $e^{-\lambda_0 \xi} \leq 1$ throughout the exchanger.

It is now convenient to consider the limit of nearly-balanced heat exchangers ($\lambda_0 \ll 1$) in order to simplify the above expression and arrive to a simpler (and practical) criterion for neglecting axial wall conduction in the overall heat transfer process. To this end, we introduce the asymptotic expansions

$$\begin{aligned} f'_{0,1}(1) &= -\frac{\lambda_0}{2} + O(\lambda_0^2) \\ e^{-\lambda_0 \xi_L} &= 1 - \lambda_0 \xi_L + O(\lambda_0^2) \\ mk &= 1 + \left[\frac{17}{70} \frac{k+1}{k} + \frac{1}{2\kappa_w} \right] \lambda_0 + O(\lambda_0^2) \end{aligned} \quad (69)$$

in Eq. (68), which then takes the simplified form

$$\frac{\text{Axial wall conduction}}{\text{Convective heat flux}} \leq \frac{\kappa_w \Delta_w^2}{\xi_L + \left[\frac{17}{70} \frac{k+1}{k} + \frac{1}{2\kappa_w} \right]} \quad (70)$$

with the bracketed expression taking values from about 0.3 (for $1/m \simeq k = 0.1$, $\kappa_w = 1$) to 3 (for $1/m \simeq k = 10$, $\kappa_w \rightarrow \infty$). The above expression leads to the following criterion for axial wall conduction to be negligible

$$\frac{\kappa_w \Delta_w^2}{\xi_L + \left[\frac{17}{70} \frac{k+1}{k} + \frac{1}{2\kappa_w} \right]} \ll 1 \quad (71)$$

which generalizes the criterion previously established by Maranzana et al. [12] (stated in our nomenclature as $\kappa_w \Delta_w^2 / \xi_L < 0.01$ in Ref. [41]) by the inclusion of the mentioned bracketed expression. As a result, we anticipate our criterion (71) to have a wider generality than that of Maranzana et al. [12], since it accounts explicitly for all the dimensionless parameters involved in the problem, m , k , κ_w , Δ_w , and ξ_L (remember that $m \simeq 1/k$ in nearly-balanced heat exchangers), and includes a new correction term that modifies the value of ξ_L in the denominator by a factor of order unity.

To illustrate the application of this criterion to an actual device, let us consider the experimental work of Yang et al. [50] on a double-layered microchannel heat exchanger fitted with interchangeable foils to enable independent control of the wall thickness and thermal conductivity. It is worth noting that the

experimental setup chosen in [50], consisting of a gas-to-gas ($Pr_i \sim 1$) device with square cross section microchannels, does not match our liquid-to-liquid ($Pr_i \gg 1$) multi-layered parallel plate arrangement, which precludes quantitative comparisons with our analytical results. However, the qualitative importance of axial wall conduction can still be assessed using (71) if one considers the physical and operational parameters given in [50], i.e., $L \simeq 4 \times 10^{-2}$ m, $a_i \simeq 10^{-4}$ m, $k_i \simeq 0.0271$ W/(m K), $Pr_i \simeq 0.711$, $Pe_i = Pr_i Re_i \simeq 180 - 1070$, $\delta_w \simeq 5 \times 10^{-4}$ m, and $k_w \simeq 0.25$ W/(m K) (PEEK) and 400 W/(m K) (copper). Here the Peclet numbers are based on the side of the square cross-section of the channels, $2a_i = 2 \times 10^{-4}$ m = 200 μ m, and the gas thermal conductivities and Prandtl numbers are taken as those of air at 40°C. Using these values one may estimate the dimensionless parameters appearing in (71), giving $k = 1$, $\Delta_w = 0.028 - 0.005$, $\xi_L = 2.2 - 0.4$, and $\kappa_w = 2$ (PEEK) and 3000 (copper), where the higher (lower) values of Δ_w and ξ_L correspond to the lower (higher) Peclet numbers.

Table 4 gives the ratio of the axial heat transfer in the wall to the convective heat flux in the channels estimated from the left hand side of (71). It can be seen that for low conductive materials (e.g., PEEK, $\kappa_w = 2$) the small value of Δ_w , which appears squared in the numerator, makes the criterion for neglecting axial wall conduction to hold thoroughly for all Peclet numbers. By contrast, for high conductive materials (e.g., copper, $\kappa_w = 3000$) the small value of Δ_w is partially, or even fully, compensated by κ_w , resulting in a significant effect of axial wall conduction, particularly for the smallest Peclet numbers, for which Δ_w is not so small. The results, summarized in Table 4, are in qualitative agreement with those of Yang et al. [50] (see, e.g., Fig. 6 of their paper), who showed that the effect of axial wall conduction was completely negligible for PEEK, but not for copper. In the latter case the heat exchanger effectiveness suffered a significant degradation due to axial wall conduction, which increased from a small percent at the largest Peclet numbers to a factor of order unity for the lowest in agreement with the predictions shown in the last column of the table.

6. Conclusions

A theoretical study of multilayered laminar counterflow parallel-plate heat exchangers has been presented. The analysis, valid for constant property fluids, neglects viscous dissipation and upstream conduction in the fluids, but includes both axial and transverse conduction in the wall. Five dimensionless parameters emerge in the mathematical formulation of the problem: the ratio of the heat capacity flow rates, $m = (a_2 Pe_2)/(a_1 Pe_1)$, the normalized fluid-to-fluid and wall-to-fluid conductivity ratios, $k = (a_1 k_2)/(a_2 k_1)$ and $\kappa_w = (k_w a_1)/(\delta_w k_1)$, the dimensionless wall thickness, $\Delta_w = \delta_w/(a_1 Pe_1)$, and the dimensionless heat exchanger length, $\xi_L = L/(a_1 Pe_1)$.

The mathematical model has been solved in terms of eigenfunction expansions involving infinite sets of both positive and negative eigenvalues [20]. Closed-form analytical expressions for the eigenfunctions have been presented, which were used to evaluate the eigenvalues λ_n numerically through

an analytical eigencondition. Highly precise numerical values for λ_n , $n = 0, \pm 1, \dots, \pm 4$, with plots of the corresponding eigenfunctions, have been presented for selected values of m , k , κ_w , and Δ_w .

The results of the exact solution, truncated to $N = 5$ and 10 positive and negative eigenmodes, have been validated with the numerical solution of the problem obtained by standard finite-difference methods, showing negligibly small errors. The exact solution has also been validated against previous results obtained by Mori et al. [25] and Maranzana et al. [12] in similar configurations. The truncated exact solution confirms that i) the optimal wall conductivity κ_w^{opt} depends on the dimensionless wall thickness Δ_w , ii) the Nusselt numbers of both streams can not be assumed constant for short heat exchangers, and are different from each other in the thermal development region, and iii) for a fixed wall thermal resistance the effect of axial wall conduction is always to reduce the effectiveness of laminar counterflow heat exchangers.

The analysis finishes by establishing the conditions under which axial wall conduction can be neglected. After obtaining an upper bound for the ratio between axial wall conduction and the convective heat flux, a general criterion is proposed by imposing that this ratio must be small compared to unity. The criterion, which reduces to that of Maranzana et al. [12] in the particular case of nearly-balanced heat exchangers, was tested against recent experimental results reported in the literature, showing good qualitative agreement in terms of the role played by axial wall conduction.

Future work should address the influence of the different dimensionless parameters on the heat exchanger performance, aiming in particular to clarify the concept of optimal wall conductivity. The analytical results presented here pave the way for a thorough analysis of the eigenvalue spectrum, including the dependence of λ_n with the different dimensionless parameters, notably κ_w and Δ_w , which would be useful for this purpose.

Acknowledgements

This work was supported by Projects ENE2011-24574 and ENE2015-68703-C2-1-R of the Spanish *Ministerio de Economía y Competitividad*.

Appendix A. Orthogonality condition

As seen in the main text, the orthogonality condition (44) based on the inner product (45) plays a critical role in the determination of the solution of the problem. To come to this condition, we start writing Eq. (25) for the eigenfunctions $h_{n,1}$ and $h_{q,1}$, with $n \neq q$; the first equation is then multiplied by $h_{q,1}$, the second by $h_{n,1}$, and the resulting equations are subtracted to get

$$\begin{aligned}
 & -(\lambda_n - \lambda_q) \frac{3}{4} (1 - y_1^2) h_{n,1}(y_1) h_{q,1}(y_1) \\
 & = h_{n,1}''(y_1) h_{q,1}(y_1) - h_{q,1}''(y_1) h_{n,1}(y_1). \quad (\text{A.1})
 \end{aligned}$$

Integrating between $y_1 = 0$ and 1, applying integration by parts to the terms in the right hand side, and using the boundary condition (28) at $y_1 = 0$, results in

$$\int_0^1 \frac{3}{4} (1 - y_1^2) h_{n,1}(y_1) h_{q,1}(y_1) dy_1 = \frac{h_{n,1}(1) h'_{q,1}(1) - h_{q,1}(1) h'_{n,1}(1)}{\lambda_n - \lambda_q}. \quad (\text{A.2})$$

Operating similarly with Eq. (26) and the boundary condition (28) gives

$$-mk \int_0^1 \frac{3}{4} (1 - y_2^2) h_{n,2}(y_2) h_{q,2}(y_2) dy_2 = k \frac{h_{n,2}(1) h'_{q,2}(1) - h_{q,2}(1) h'_{n,2}(1)}{\lambda_n - \lambda_q}, \quad (\text{A.3})$$

and using Eq. (27) with the boundary conditions (29) and (30) leads to

$$\kappa_w (\lambda_n + \lambda_q) (\Delta_w)^2 \int_{-1/2}^{1/2} h_{n,w}(y_w) h_{q,w}(y_w) dy_w = - \frac{h_{n,1}(1) h'_{q,1}(1) - h_{q,1}(1) h'_{n,1}(1)}{\lambda_n - \lambda_q} - k \frac{h_{n,2}(1) h'_{q,2}(1) - h_{q,2}(1) h'_{n,2}(1)}{\lambda_n - \lambda_q}. \quad (\text{A.4})$$

Adding Eqs. (A.2), (A.3) and (A.4) we finally obtain the orthogonality condition stated in (44) and (45).

Following the same procedure it is easy to check that the eigenfunction $h_{n,1}(y_1)$ is also orthogonal with respect to unity. Note that the invariance of the problem (1)–(13) under arbitrary shifts in θ makes zero to be always an eigenvalue, aside from the λ_n , $n = 0, \pm 1, \pm 2, \dots$. The associated constant eigenfunction, which due to the normalization condition (31) turns out to be equal to unity, appears explicitly as the expansion coefficient A in (24). Thus, to check the orthogonality of $h_{n,1}(y_1)$ with respect to unity we integrate both sides of Eq. (25) between $y_1 = 0$ and 1 and use the boundary condition (28)

$$\int_0^1 \frac{3}{4} (1 - y_1^2) A h_{n,1}(y_1) dy_1 = -A \frac{h'_{n,1}(1)}{\lambda_n}. \quad (\text{A.5})$$

Then we integrate Eq. (26) with the boundary condition (28)

$$-mk \int_0^1 \frac{3}{4} (1 - y_2^2) A h_{n,2}(y_2) dy_2 = -A \frac{k h'_{n,2}(1)}{\lambda_n}, \quad (\text{A.6})$$

and Eq. (27) with the boundary conditions (29) and (30)

$$\kappa_w \lambda_n (\Delta_w)^2 \int_{-1/2}^{1/2} A h_{n,w}(y_w) dy_w = A \frac{h'_{n,1}(1)}{\lambda_n} + A \frac{k h'_{n,2}(1)}{\lambda_n}. \quad (\text{A.7})$$

Adding Eqs. (A.5), (A.6) and (A.7) results in this case in the orthogonality condition (46).

Appendix B. Determination of the expansion coefficients

Appendix B.1. Adiabatic case

The coefficients A and C_n appearing in (24) can be obtained following the procedure introduced by Nunge and Gill [20]. With this aim, we take the inner product of the temperature profile $\theta_i(\xi, y_i)$ with the eigenfunction $h_{n,i}$, at both ends of the heat exchanger length

$$\langle \theta_i(0, y_i), h_{n,i} \rangle = \langle A, h_{n,i} \rangle + \sum_{q=-\infty}^{+\infty} C_q \langle h_{q,i}, h_{n,i} \rangle \quad (\text{B.1})$$

$$\langle \theta_i(\xi_L, y_i), h_{n,i} \rangle = \langle A, h_{n,i} \rangle + \sum_{q=-\infty}^{+\infty} C_q e^{-\lambda_q \xi_L} \langle h_{q,i}, h_{n,i} \rangle. \quad (\text{B.2})$$

According to the orthogonality conditions (45) and (46), the inner products $\langle A, h_{n,i} \rangle$ and $\langle h_{q,i}, h_{n,i} \rangle$, with $q \neq n$, are both equal to zero. Accordingly, from (B.1) and (B.2) we have

$$C_n = \frac{\langle \theta_i(0, y_i), h_{n,i} \rangle}{\langle h_{n,i}, h_{n,i} \rangle} = \frac{e^{\lambda_n \xi_L} \langle \theta_i(\xi_L, y_i), h_{n,i} \rangle}{\langle h_{n,i}, h_{n,i} \rangle}. \quad (\text{B.3})$$

Equating the numerators of both expressions for C_n and evaluating the inner products taking into account the inlet and outlet conditions

$$\theta_1(0, y_1) = A + \sum_{q=-\infty}^{+\infty} C_q h_{q,1}(y_1) = 0, \quad (\text{B.4})$$

$$\theta_2(0, y_2) = A + \sum_{q=-\infty}^{+\infty} C_q h_{q,2}(y_2) = \theta_{2,\text{out}}(y_2), \quad (\text{B.5})$$

$$\left. \frac{\partial \theta_w}{\partial \xi} \right|_{\xi=0} = - \sum_{q=-\infty}^{+\infty} C_q \lambda_q h_{q,w}(y_w) = 0, \quad (\text{B.6})$$

$$\theta_1(\xi_L, y_1) = A + \sum_{q=-\infty}^{+\infty} C_q e^{-\lambda_q \xi_L} h_{q,1}(y_1) = \theta_{1,\text{out}}(y_1), \quad (\text{B.7})$$

$$\theta_2(\xi_L, y_2) = A + \sum_{q=-\infty}^{+\infty} C_q e^{-\lambda_q \xi_L} h_{q,2}(y_2) = 1, \quad (\text{B.8})$$

$$\left. \frac{\partial \theta_w}{\partial \xi} \right|_{\xi=\xi_L} = - \sum_{q=-\infty}^{+\infty} C_q \lambda_q e^{-\lambda_q \xi_L} h_{q,w}(y_w) = 0, \quad (\text{B.9})$$

we finally get

$$\begin{aligned} & -mk \int_0^1 w(y_2) \theta_{2,\text{out}}(y_2) h_{n,2}(y_2) dy_2 \\ & + \lambda_n \kappa_w \Delta_w^2 \int_{-1/2}^{1/2} \theta_w(0, y_w) h_{n,w}(y_w) dy_w \\ & = e^{\lambda_n \xi_L} \left[\int_0^1 w(y_1) \theta_{1,\text{out}}(y_1) h_{n,1}(y_1) dy_1 - mk \int_0^1 w(y_2) h_{n,2}(y_2) dy_2 \right. \\ & \quad \left. + \lambda_n \kappa_w \Delta_w^2 \int_{-1/2}^{1/2} \theta_w(\xi_L, y_w) h_{n,w}(y_w) dy_w \right]. \quad (\text{B.10}) \end{aligned}$$

Expanding $\theta_{1,\text{out}}(y_1)$ and $\theta_{2,\text{out}}(y_2)$ using Eqs. (B.7) and (B.5), $\theta_w(0, y_w)$ and $\theta_w(\xi_L, y_w)$ using Eq. (24b), and applying standard

simplifying techniques based on (25)–(31), it is possible to write analytical expressions for all the integrals that appear in the process, eventually leading to the set of equations given in Eq. (48). Substituting $h_{n,i}(y_i)$ by 1 and λ_n by 0 and operating similarly we obtain Eq. (47).

Appendix B.2. Isothermic case

Substituting Eqs. (B.6) and (B.9) by

$$\theta_w(0, y_w) = A + \sum_{q=-\infty}^{+\infty} C_q h_{q,w}(y_w) = 0 \quad (\text{B.6}')$$

and

$$\theta_w(\xi_L, y_w) = A + \sum_{q=-\infty}^{+\infty} C_q e^{-\lambda_q \xi_L} h_{q,w}(y_w) = 1 \quad (\text{B.9}')$$

and operating as in the adiabatic case, we may write

$$\begin{aligned} & -mk \int_0^1 w(y_2) \theta_{2,\text{out}}(y_2) h_{n,2}(y_2) dy_2 \\ & = e^{\lambda_n \xi_L} \left[\int_0^1 w(y_1) \theta_{1,\text{out}}(y_1) h_{n,1}(y_1) dy_1 - mk \int_0^1 w(y_2) h_{n,2}(y_2) dy_2 + \right. \\ & \quad \left. \lambda_n \kappa_w \Delta_w^2 \int_{-1/2}^{1/2} h_{n,w}(y_w) dy_w \right]. \quad (\text{B.10}') \end{aligned}$$

Expanding $\theta_{1,\text{out}}(y_1)$ and $\theta_{2,\text{out}}(y_2)$ using (B.7) and (B.5), and applying standard simplifying techniques based on (25)–(31), we are led to the set of equations given in Eq. (48'). Substituting $h_{n,i}(y_i)$ by 1 and λ_n by 0 and operating similarly we finally get Eq. (47').

References

- [1] P. Rosa, T. G. Karayiannis, M. W. Collins, Single-phase heat transfer in microchannels: the importance of scaling effects, *Appl. Therm. Eng.* 29(17) (2009) 3447–3468.
- [2] A. C. Fernandez-Pello, Micropower generation using combustion: issues and approaches, *P. Combust. Inst.* 29(1) (2002) 883–899.
- [3] P. D. Ronney, Analysis of non-adiabatic heat-recirculating combustors, *Combust. Flame* 135(4) (2003) 421–439.
- [4] H. A. Mohammed, G. Bhaskaran, N. H. Shuaib, R. Saidur, Heat transfer and fluid flow characteristics in microchannels heat exchanger using nanofluids: a review, *Renew. Sust. Energ. Rev.* 15(3) (2011) 1502–1512.
- [5] M. Sánchez-Sanz, Premixed flame extinction in narrow channels with and without heat recirculation, *Combust. Flame* 159(10) (2012) 3158–3167.
- [6] R. B. Peterson, Numerical modeling of conduction effects in microscale counterflow heat exchangers, *Microscale Therm. Eng.* 3(1) (1999) 17–30.
- [7] J. E. Hesselgreaves, *Compact Heat Exchangers: Selection, Design and Operation*, Gulf Professional Publishing, 2001.
- [8] J. C. Pacio, C. A. Dorao, A review on heat exchanger thermal hydraulic models for cryogenic applications, *Cryogenics* 51(7) (2011) 366–379.
- [9] K. Chowdhury, S. Sarangi, Effect of finite thermal conductivity of the separating wall on the performance of counterflow heat exchangers, *Cryogenics* 23(4) (1983) 212–216.
- [10] W. Bier, W. Keller, G. Linder, D. Seidel, K. Schubert, H. Martin, Gas to gas heat transfer in micro heat exchangers, *Chem. Eng. Process.* 32(1) (1993) 33–43.
- [11] T. Stief, O.U. Langer, K. Schubert, Numerical investigations of optimal heat conductivity in micro heat exchangers, *Chem. Eng. Technol.* 22(4) (1999) 297–303.
- [12] G. Maranzana, I. Perry, D. Maillet, Mini-and micro-channels: influence of axial conduction in the walls, *Int. J. Heat Mass Tran.* 47(17) (2004) 3993–4004.
- [13] E. D. Marquardt, R. Radebaugh, Compact High Effectiveness Parallel Plate Heat Exchangers, In *Cryocoolers 12* (R. G. Ross, Jr., Ed.), Kluwer Academic/Plenum Publishers, 2003, pp. 507–516.
- [14] G. F. Nellis, A heat exchanger model that includes axial conduction, parasitic heat loads, and property variations, *Cryogenics*, 43 (2003) 523–538.
- [15] P. G. Kroeger, Performance Deterioration in High Effectiveness Heat Exchangers due to Axial Heat Conduction Effects, *Advances in Cryogenic Engineering*, Plenum Press, New York, pp. 363–372, 1967.
- [16] R. K. Shah and A. L. London, *Laminar Flow Forced Convection in Ducts: A Source Book for Compact Heat Exchanger Analytical Data*, Academic Press, New York, pp. 196–207, 1978.
- [17] W. M. Kays, A. L. London, *Compact Heat Exchangers*, 1984.
- [18] L. Graetz, Ueber die Wärmeleitungsfähigkeit von Flüssigkeiten, *Ann. Phys.* 254 (1882) 79–94.
- [19] R. P. Stein, The Graetz problem in cocurrent-flow, double-pipe, heat exchangers, *ANL Report 6889* (1964).
- [20] R. J. Nunge, W. N. Gill, Analysis of heat or mass transfer in some countercurrent flows, *Int. J. Heat Mass Tran.* 8 (1965) 873–886.
- [21] R. J. Nunge, W. N. Gill, An analytical study of laminar counterflow double-pipe heat exchangers, *AIChE J.* 12 (1966) 279–289.
- [22] R. Nunge, E. W. Porta, W. N. Gill, Axial conduction in the fluid streams of multistream heat exchangers, *Chem. Eng. Progr. Symp. Series* 63 (1967) 80–91.
- [23] S. N. Singh, Heat transfer by laminar flow in a cylindrical tube, *Appl. Sci. Res.* 7(5) (1958) 325–340.
- [24] E. J. Davis and W. N. Gill, The effects of axial conduction in the wall on heat transfer with laminar flow, *Int. J. Heat Mass Tran.* 13 (1970) 459–470.
- [25] S. Mori, M. Kataya, A. Tanimoto, Performance of counterflow, parallel plate heat exchangers under laminar flow conditions, *Heat Transfer Eng.* 2 (1980) 28–38.
- [26] G. S. Barozzi, G. Pagliarini, Experimental investigation of coupled conduction and laminar convection in a circular tube, *Int. J. Heat Mass Tran.* 27(12) (1984) 2321–2329.
- [27] G. S. Barozzi, G. Pagliarini, A method to solve conjugate heat transfer problems: the case of fully developed laminar flow in a pipe, *J. Heat Transf.* 107(1) (1985) 77–83.
- [28] G. Pagliarini, G. S. Barozzi, Thermal coupling in laminar flow double-pipe heat exchangers, *J. Heat Transf.* 113 (1991) 526–534.
- [29] S. Piva, Axial wall conduction preheating effects in high Péclet number laminar forced convection, *Int. J. Heat Mass Tran.* 39(16) (1996) 3511–3517.
- [30] E. Papoutsakis, D. Ramkrishna, H. C. Lim, The extended Graetz problem with Dirichlet wall boundary conditions, *Appl. Sci. Res.* 36(1) (1980) 13–34.
- [31] E. Papoutsakis, D. Ramkrishna, H. C. Lim, The extended Graetz problem with prescribed wall flux, *AIChE J.* 26(5) (1980) 779–787.
- [32] E. Papoutsakis, D. Ramkrishna, Conjugated Graetz problems. I. General formalism and a class of solid-fluid problems, *Chem. Eng. Sci.* 36 (1981) 1381–1390.
- [33] E. Papoutsakis, D. Ramkrishna, Conjugated Graetz problems. II. Fluid-fluid problems, *Chem. Eng. Sci.* 36 (1981) 1393–1399.
- [34] B. Weigand, An exact analytical solution for the extended turbulent Graetz problem with Dirichlet wall boundary conditions for pipe and channel flows, *Int. J. Heat Mass Tran.* 39(8) (1996) 1625–1637.
- [35] B. Weigand, M. Kanzamar, H. Beer, The extended Graetz problem with piecewise constant wall heat flux for pipe and channel flows, *Int. J. Heat Mass Tran.* 44(20) (2001) 3941–3952.
- [36] B. Weigand, D. Lauffer, The extended Graetz problem with piecewise constant wall temperature for pipe and channel flows, *Int. J. Heat Mass Tran.* 47(24) (2004) 5303–5312.
- [37] B. Weigand, G. Gassner, The effect of wall conduction for the extended Graetz problem for laminar and turbulent channel flows, *Int. J. Heat Mass Tran.* 50(5) (2007) 1097–1105.
- [38] C. Pierre, F. Plouraboué, Numerical analysis of a new mixed formulation for eigenvalue convection-diffusion problems, *SIAM J. Appl. Math.* 70(3) (2009) 658–676.

- [39] J. Fehrenbach, F. De Gournay, C. Pierre, F. Plouraboué, The Generalized Graetz problem in finite domains, *SIAM J. Appl. Math.* 72(1) (2012) 99–123.
- [40] M. Vera, A. Liñan, Laminar counterflow parallel-plate heat exchangers: Exact and approximate solutions, *Int. J. Heat Mass Tran.* 53 (2010) 4885–4898.
- [41] A. E. Quintero, M. Vera, B. Rivero-de-Aguilar, Wall conduction effects in laminar counterflow parallel-plate heat exchangers, *Int. J. Heat Mass Tran.* 70 (2014) 939–953.
- [42] L. T. Yeh, Review of heat transfer technologies in electronic equipment, *J. Electron. Packaging* 117(4) (1995) 333–339.
- [43] X. Wei, Y. Joshi, M. K. Patterson, Experimental and numerical study of a stacked microchannel heat sink for liquid cooling of microelectronic devices, *J. Heat Transf.* 129(10) (2007) 1432–1444.
- [44] P. J. Schmid and D. S. Henningson, *Stability and Transition in Shear Flows*, Applied Math. Sciences, 142, Springer-Verlag, New York, 2001.
- [45] E. Whittaker, G. Watson, *Modern Analysis*, Camb. Univ. Pres., p. 337, 1952.
- [46] M. Abramowitz, I. Stegun, Eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, chapter 13, National Bureau of Standards, Applied Mathematics Series 55, 1964.
- [47] H. A. Lauwerier, The use of confluent hypergeometric functions in mathematical physics and the solution of an eigenvalue problem, *Appl. Sci. Res.* 2(1) (1951) 184–204.
- [48] M. Vera, A. Liñan, Exact solution for the conjugate fluid-fluid problem in the thermal entrance region of laminar counterflow heat exchangers, *Int. J. Heat Mass Tran.* 54 (2011) 490–499.
- [49] G.F. Hewitt, G.L. Shires, T.R. Bott, *Process heat transfer*, Vol. 113, Boca Raton, FL: CRC press, 1994.
- [50] Y. Yang, G. L. Morini, J. J. Brandner, Experimental analysis of the influence of wall axial conduction on gas-to-gas micro heat exchanger effectiveness, *Int. J. Heat Mass Tran.* 69 (2014) 17–25.

Table 1: Numerical values of λ_n , $n = 0, \pm 1, \dots, \pm 4$ corresponding to $m = 2$, $k = 1$ and selected values of κ_w and Δ_w .

κ_w	λ_0	λ_{-1}	λ_{+1}	λ_{-2}	λ_{+2}	λ_{-3}	λ_{+3}	λ_{-4}	λ_{+4}
$\Delta_w = 0$									
∞	1.03864104	-15.83966947	34.17607579	-52.70020334	109.6852995	-110.8839544	227.8753347	-190.4062448	388.7399459
50	1.01735984	-15.74607415	33.85007324	-52.44501396	108.8388790	-110.4128774	226.3384949	-189.6722514	386.3674986
10	0.94033136	-15.41629609	32.73633277	-51.58912376	106.1027789	-108.8970423	221.5991974	-187.3948149	379.3494039
2	0.68243803	-14.40508053	29.64839735	-49.33933718	99.67341953	-105.3509632	211.7820699	-182.5466567	366.2190597
$\Delta_w = 10^{-3}$									
50	1.01724499	-15.73076368	33.76071330	-52.24520124	107.6912049	-109.3905971	219.9339826	-186.1311210	359.1713035
10	0.94031169	-15.41346365	32.71939734	-51.55234636	105.8861106	-108.7142581	220.4983489	-186.8014178	375.5879479
2	0.68243593	-14.40463483	29.64547631	-49.33326992	99.63244345	-105.3202617	211.5690954	-182.4467384	365.5070094
$\Delta_w = 10^{-2}$									
50	1.00599724	-14.16308052	24.20804123	-33.40735117	49.65519462	-65.59487100	125.4432442	-125.2301657	247.1868897
10	0.93836784	-15.12902602	30.91249237	-47.16903739	76.35988941	-83.22692926	132.0986819	-130.5274518	243.5677185
2	0.68222879	-14.36035679	29.35235820	-48.70132119	94.81890789	-101.5139767	174.0357240	-163.7331405	249.0099923
$\Delta_w = 10^{-1}$									
50	0.48262275	-3.116241554	4.586299475	-21.29619853	31.49990153	-31.79750629	42.96831405	-61.23901234	63.07603798
10	0.77759708	-5.59492862	7.03758869	-20.93403564	31.79364896	-33.13273100	43.42225337	-59.88308186	64.01301066
2	0.66214372	-10.10198524	13.03303895	-21.09074314	32.55815993	-37.52500713	46.02708004	-57.95897220	67.92908802

Table 2: Outlet bulk temperature of fluid 1 as obtained from the truncated exact solution, Eq. (50), with $N = 10$ (22-term solution, 22TS) and $N = 5$ (12-term solution, 12TS), and from the finite-difference solution of the problem (NUM), corresponding to $m = 2$, $k = 1$ and selected values of ξ_L , κ_w , and Δ_w . The table also shows the relative error of the different solution approaches compared to the truncated exact solution with $N = 10$, taken here as **reference solution**.

I. Adiabatic end-wall boundary conditions									
ξ_L	1/2				2				Method
κ_w	2		50		2		50		
Δ_w	$\theta_{m1,out}$	Error	$\theta_{m1,out}$	Error	$\theta_{m1,out}$	Error	$\theta_{m1,out}$	Error	
10^{-3}	0.4614	—	0.5915	—	0.8561	—	0.9327	—	
	0.4614	0.000%	0.5915	0.007%	0.8561	0.000%	0.9327	0.002%	22TS exact ^a
	0.4628	0.305%	0.5916	0.022%	0.8572	0.122%	0.9328	0.006%	12TS exact ^b
									NUM ^c
10^{-2}	0.4611	—	0.5853	—	0.8560	—	0.9307	—	22TS exact ^a
	0.4611	0.000%	0.5852	0.001%	0.8560	0.000%	0.9307	0.000%	12TS exact ^b
	0.4625	0.304%	0.5852	0.017%	0.8571	0.122%	0.9307	0.002%	NUM ^c
II. Specified end-wall temperature conditions									
ξ_L	1/2				2				Method
κ_w	2		50		2		50		
Δ_w	$\theta_{m1,out}$	Error	$\theta_{m1,out}$	Error	$\theta_{m1,out}$	Error	$\theta_{m1,out}$	Error	
10^{-3}	0.4537	—	0.5944	—	0.8605	—	0.9333	—	
	0.4595	1.285%	0.5944	0.005%	0.8562	0.492%	0.9333	0.002%	22TS exact ^{a'}
	0.4642	2.324%	0.5946	0.031%	0.8576	0.330%	0.9333	0.007%	12TS exact ^{b'}
									NUM ^c
10^{-2}	0.4679	—	0.6048	—	0.8597	—	0.9355	—	22TS exact ^{a'}
	0.4689	0.195%	0.6013	0.571%	0.8585	0.132%	0.9355	0.000%	12TS exact ^{b'}
	0.4719	0.859%	0.6051	0.052%	0.8601	0.046%	0.9356	0.009%	NUM ^c

^a (50) with A and C_n , $n = 0, \pm 1, \dots, \pm N$ given by the $2(N + 1)$ -term truncated system (47)–(48) with $N = 10$.

^{a'} (50) with A and C_n , $n = 0, \pm 1, \dots, \pm N$ given by the $2(N + 1)$ -term truncated system (47')–(48') with $N = 10$.

^b (50) with A and C_n , $n = 0, \pm 1, \dots, \pm N$ given by the $2(N + 1)$ -term truncated system (47)–(48) with $N = 5$.

^{b'} (50) with A and C_n , $n = 0, \pm 1, \dots, \pm N$ given by the $2(N + 1)$ -term truncated system (47')–(48') with $N = 5$.

^c Finite-difference solution obtained with a grid of $n_\xi \times n_{y_i}$ (n_w) = 200×200 (50) points in the fluid (wall) domain.

Table 3: Sample application of criterion (65) establishing the conditions for axial wall conduction to be negligible in the determination of λ_0 . Note that the critical value of Δ_w depends mainly on κ_w , with m and k having only a minor impact.

m	k	κ_w	λ_0^0	$\geq \frac{2}{mk-1}$	$\Delta_w \ll \min \left\{ \sqrt{\frac{mk-1}{2\kappa_w\lambda_0^0}}, \frac{1}{\sqrt{\kappa_w\lambda_0^0}} \right\}$		
1.1	1	50	0.183	< 20	$\Delta_w \ll$	0.074	0.771
1.2	1.2	50	0.671	< 4.55	$\Delta_w \ll$	0.081	0.211
2	1	50	1.017	< 2	$\Delta_w \ll$	0.099	0.139
2	2	50	1.993	> 0.67	$\Delta_w \ll$	0.123	0.071
1.1	1	10	0.170	< 20	$\Delta_w \ll$	0.172	1.863
1.2	1.2	10	0.617	< 4.55	$\Delta_w \ll$	0.189	0.513
2	1	10	0.940	< 2	$\Delta_w \ll$	0.231	0.336
2	2	10	1.803	> 0.67	$\Delta_w \ll$	0.288	0.175
1.1	1	2	0.124	< 20	$\Delta_w \ll$	0.450	5.722
1.2	1.2	2	0.439	< 4.55	$\Delta_w \ll$	0.500	1.609
2	1	2	0.682	< 2	$\Delta_w \ll$	0.605	1.036
2	2	2	1.219	> 0.67	$\Delta_w \ll$	0.784	0.580

Table 4: Sample application of criterion (71) establishing the conditions for axial wall conduction to be negligible in the determination of heat exchanger performance. The dimensionless parameters are obtained from the physical and operational parameters reported by Yang et al. [50].

$\frac{\kappa_w \Delta_w^2}{\xi_L + \left[\frac{17}{70} \frac{k+1}{k} + \frac{1}{2\kappa_w} \right]}$			κ_w	
Pe	Δ_w	ξ_L	PEEK	copper
			2	3000
180	0.028	2.2	$5.3 \times 10^{-4} \ll 1$	$0.80 \sim 1$
1070	0.005	0.4	$5.6 \times 10^{-5} \ll 1$	$0.084 \lesssim 1$

Figure 1: Unitary cell of the idealized counterflow parallel-plate heat exchanger showing the physical dimensions, the coordinate system, the velocity field, and the physical properties of the fluids and the heat conducting wall. The domain under study is shaded gray.

Figure 2: Eigenfunctions $h_{n,i}(y_i)$, $n = 0, \pm 1, \dots, \pm 4$, for $m = 2$, $k = 1$, and selected values of κ_w (rows), corresponding to $\Delta_w = 10^{-3}$ (left column) and $\Delta_w = 10^{-2}$ (right column). Note the different vertical scale used in the left and right subplots. The insets in the upper and middle left panels show closer views of the wall eigenfunctions $h_{n,w}(y_w)$.

Figure 3: Variation of the outlet bulk temperature of fluid 1 with the dimensionless heat exchanger length, ξ_L , corresponding to adiabatic end-wall boundary conditions with $k = 1$, $\Delta_w/\xi_L = 0.01$, and different values of m and κ_w . Numerical results provided by the truncated exact solution, Eq. (50), with $N = 5$ (lines), by Mori et al. [25] (O), and by the finite-difference solution of the problem ($\Diamond$).

Figure 4: Variation of the outlet bulk temperature of fluid 1 with the dimensionless wall conductivity, κ_w , corresponding to adiabatic end-wall boundary conditions with $m = 0.97$, $k = 1.03$, $\xi_L = 5.89$, and different values of Δ_w . Numerical results provided by the truncated exact solution, Eq. (50), with $N = 10$ (lines), and by Maranzana et al. [12] for $\Delta_w = 0.3$ (O).

Figure 5: Numerical results provided by the truncated exact solution with $N = 10$ (a, b: filled contours; c–f: lines as specified in the legends) and by the finite-difference solution of the problem (a, b: thin dashed lines; c–f: $\diamond$) for a heat exchanger with $m = 0.97$, $k = 1.03$, $\kappa_w = 20$, $\Delta_w = 0.30$, and $\xi_L = 5.89$, almost identical to that studied by Maranzana et al. [12]. Spatial temperature distributions (a) in the fluids, $\theta_i(\xi, y_i)$, and (b) in the wall, $\theta_w(\xi, y_w)$, along with the longitudinal distributions of (c) the bulk temperatures, $\theta_{mi}(\xi)$, (d) the interfacial wall-fluid temperatures, $\theta_{wi}(\xi)$, (e) the interfacial heat-transfer rates, $\nu_i(\xi)$, and the average wall axial heat flux, $\nu_w(\xi)$, and (f) the Nusselt numbers, $Nu_i(\xi)$. The temperature isocontours shown in panels (a) and (b) are plotted at intervals $\Delta\theta = 0.05$.

Figure 6: Results for a heat exchanger corresponding to $m = 0.97$, $k = 1.03$, $\kappa_w = 20$, $\Delta_w = 0.30$ and $\xi_L = 1.76$. For further details see caption of Figure 5.