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A MULTIVARIATE GENERALIZED INDEPENDENT FACTOR GARCH MODEL WITH AN APPLICATION TO FINANCIAL STOCK RETURNS

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Keywords: ICA, Multivariate GARCH, Factor Models, Forecasting Volatility

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We propose a new multivariate factor GARCH model, the GICA-GARCH model, where the data are assumed to be generated by a set of independent components (ICs). This model applies independent component analysis (ICA) to search the conditionally heteroskedastic latent factors. We will use two ICA approaches to estimate the ICs. The first one estimates the components maximizing their non-gaussianity, and the second approach exploits the temporal structure of the data. After estimating the ICs, we fit an univariate GARCH model to the volatility of each IC. Thus, the GICA-GARCH reduces the complexity to estimate a multivariate GARCH model by transforming it into a small number of univariate volatility models. We report some simulation experiments to show the ability of ICA to discover leading factors in a multivariate vector of financial data. An empirical application to the Madrid stock market will be presented, where we compare the forecasting accuracy of the GICA-GARCH model versus the orthogonal GARCH one.

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1 Introduction

Since Engel (1982) introduced the ARCH model and Bollerslev (1986) generalized it, proposing the GARCH model, many researchers have been interested in modelling volatility in financial time series. In multivariate time series, financial volatilities tend to move together across markets, and in order to understand their comovements, a multivariate modelling approach is required. The first multivariate GARCH model (MGARCH) was proposed by Bollerslev, Engle and Wooldridge (1988) as an extension of the univariate GARCH model. Other different specifications for MGARCH have been proposed in the literature (see, for example, the survey of Bauwens *et al.*, 2006), but in these developments the number of parameters to estimate

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can be very large, and the restrictions to guarantee the positive definiteness of the conditional covariance matrix are difficult to implement. A possible solution to these problems is to use factor models, where the data set are explained by a small number of unobserved components. Comovements in stock returns reinforce the intuition that financial markets are driven by a few latent common sources.

Several factor models have been presented in the literature. The most popular one is the orthogonal GARCH model (O-GARCH) (Alexander, 2001) that estimates the unobserved factors using a small set of principal components (PCs), and fits an univariate GARCH model for each component. That is, the O-GARCH estimates a MGARCH model requiring only a small number of univariate GARCH estimations. Van der Weide (2002) introduces a vectorized version of the O-GARCH, the generalized orthogonal GARCH model (GO-GARCH), that does not reduce the dimension of the data and does not allow for idiosyncratic components. In order to solve these problems, Lanne and Saikkonen (2007) propose the generalized orthogonal factor GARCH model, that allows some diagonal elements of the conditional covariance matrix to be constant. All previous models use principal components analysis (PCA) to identify the set of underlying factors, which are unconditionally uncorrelated. However, to guarantee the diagonality of the conditional covariance matrix, an additional assumption is needed: the factors must be conditionally uncorrelated. Fan *et al.* (2008) show that this assumption could lead to serious errors in model fitting, and they propose to model multivariate volatilities using conditionally uncorrelated components (CUC-GARCH).

In this paper we propose a new alternative for modelling multivariate volatilities as linear combination of several univariate GARCH models. We introduce a multivariate generalized independent component analysis GARCH model (GICA-GARCH). Independent component analysis (ICA) can be seen as a factor model (Hyvärinen and Kano, 2003) where the unobserved components are non-gaussian, and mutually independent. Previous researchers, Back and Weigend (1997), Kiviluoto and Oja (1998), Cha and Chan (2000), and Mălăroiu *et al.* (2000) among others, have applied ICA to financial data. Furthermore, ICA can be considered as a generalization of PCA (Hyvärinen *et al.*, 2001), and seems to be, a priori, more suitable than PCA to explain the non-gaussian behaviour of financial data (Wu and Yu, 2005).

The key idea of the GICA-GARCH is to assume that financial data are generated by a linear combination of a small number of conditionally heteroskedastic independent components and idiosyncratic shocks. Then, GICA-GARCH can be seen as a generalization of the multivariate GARCH model proposed by Lanne and Saikkonen (2007), but assuming that the unobserved factors are estimated by ICA rather than by PCA. Therefore, the common components will be unconditionally independent and not only uncorrelated. Moreover, the GICA-GARCH is related to the O-GARCH because both models transform the problem of estimating a multivariate GARCH model into a small number of univariate GARCH models. Finally, note that the independence assumption of the GICA-GARCH model is stronger than the one corresponding to the CUC-GARCH model.

The paper is organized as follows. In Section 2 we present the ICA model, describe the three ICA algorithms that we apply to estimate the unobserved components, and explain a procedure to sort the ICA components in terms of their explained variability. Furthermore, the relationship between ICA and dynamic factor model (DFM) is analyzed. In Section 3, we introduce the GICA-GARCH model and its application to forecast the volatility of a vector of

stock returns from the volatility of a small number of components. Some simulation experiments to illustrate the ability of this model to estimate the unobserved components are presented in Section 4. Section 5 shows the results of the empirical application to a real-time dataset. Finally, Section 6 gives some concluding remarks.

2 The ICA model

In this section, we introduce the concept of ICA. First, we present the basic ICA model according to the formal definition given by Common (1994). Then, we briefly describe the three algorithms we use to estimate the ICA components. As the definition of ICA implies no ordering of the ICs, a procedure to weight and sort them is next explained. Finally, we formulate the ICA model as a particular DFM and analyze the relationship between both models.

2.1 Definition of ICA

ICA assumes that the observed multivariate vector is a linear combination of a set of unobserved components. Let $\mathbf{x}_t = (x_{1t}, x_{2t}, \ldots, x_{mt})'$ be the $m-$dimensional vector of stationary time series, with $E[\mathbf{x}_t] = \mathbf{0}$ and $E[\mathbf{x}_t\mathbf{x}_t'] = \Gamma_{\mathbf{x}}(0)$ positive definite. We assume that $\mathbf{x}_t$ is generated by a linear combination of r $(r \leq m)$ latent factors. That is,

$$\mathbf{x}_t = \mathbf{A}\mathbf{s}_t, \quad t = 1, 2, ..., T \tag{1}$$

where $\mathbf{A}$ is an unknown $m \times r$ full rank matrix, with elements a_{ij} that represent the effect of s_{jt} on x_{it}, for $i = 1, 2, ..., m$ and $j = 1, 2, ..., r$, and $\mathbf{s}_t = (s_{1t}, s_{2t}, \ldots, s_{rt})'$ is the vector of unobserved factors, which are called independent components (ICs). We assume that $E[\mathbf{s}_t] = \mathbf{0}$, $E[\mathbf{s}_t\mathbf{s}_t'] = \mathbf{I}_r$, and the components of $\mathbf{s}_t$ are statistically independent. Let $(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_T)$ be the observed multivariate time series. The problem is to estimate both $\mathbf{A}$ and $\mathbf{s}_t$ only from $(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_T)$. That is, ICA looks for an $r \times m$ matrix, $\mathbf{W}$, such that the components given by

$$\widehat{\mathbf{s}}_t = \mathbf{W}\mathbf{x}_t, \quad t = 1, 2, ..., T, \tag{2}$$

are as independent as possible. However, previous assumptions are not sufficient to estimate $\mathbf{A}$ and $\mathbf{s}_t$ uniquely, and it is required that no more than one IC is normally distributed. By (1), we have:

$$\begin{aligned} \mathbf{\Gamma}_{\mathbf{x}}(0) &= \mathbf{A}\mathbf{A}', \\ \mathbf{\Gamma}_{\mathbf{x}}(\tau) &= \mathbf{A}\mathbf{\Gamma}_{\mathbf{s}}(\tau)\mathbf{A}', \text{ for } \tau \geq 1. \end{aligned} \tag{3}$$

Note that, in spite of previous assumptions, ICA cannot determine either the sign or the order of the ICs. From now, through this paper, we focus on the most basic form of ICA, which considers that the number of observed variables is equal to the number of unobserved factors, i.e., $m = r$.

2.2 Procedures for estimating the ICs

Both ICA and PCA obtain the latent factors as linear combinations of the data. However their aims are slightly different. PCA tries to get uncorrelated factors and, for this purpose, it requires

the matrix $\mathbf{W}$ such that $\mathbf{WW}' = \mathbf{I}$, and the rows of $\mathbf{W}$ are the projection vectors that maximize the variance of the estimated unobserved factors, $\widehat{\mathbf{s}}_t$. On the other hand, the most often used methods for estimating the ICs impose the restriction that the rows of $\mathbf{W}$ are the directions that maximize the independence of $\widehat{\mathbf{s}}_t$.

Three main ICA algorithms have been proposed: JADE (Cardoso and Souloumiac, 1993) and FastICA (Hyvärinen, 1999; Hyvärinen and Oja, 1997) are based on the non-gaussianity of the ICs, while SOBI (Belouchrani et al., 1997) is based on the temporal independence of the data. Before the application of any of these algorithms, it is useful to standardize the data. Thus, we search for a linear transformation of $\mathbf{x}_t$, $\mathbf{z}_t = \mathbf{M}\mathbf{x}_t$, where $\mathbf{M}$ is an $m \times m$ matrix such that the $m-$dimensional vector $\mathbf{z}_t$ has identity covariance matrix. This multivariate standardization is carried out as follows. From

$$\mathbf{\Gamma}_{\mathbf{x}}(0) = E\{\mathbf{x}_t\mathbf{x}_t'\} = \mathbf{EDE}', \tag{4}$$

where $\mathbf{E}_{m\times m}$ is the orthogonal matrix of eigenvectors, and $\mathbf{D}_{m\times m}$ the diagonal matrix of eigenvalues, then $\mathbf{M} = \mathbf{D}^{-1/2}\mathbf{E}'$. The model (1) in terms of $\mathbf{z}_t$, is

$$\mathbf{z}_t = \mathbf{U}\mathbf{s}_t, \tag{5}$$

where $\mathbf{U} = \mathbf{MA}$ is, by (3) and (4), an $m \times m$ orthogonal matrix. Therefore, the multivariate standardization of the original data guarantees the orthogonality of the loading matrix.

2.2.1 Joint Approximate Diagonalization of Eigen-matrices: JADE

JADE (Cardoso and Souloumiac, 1993) estimates the ICs maximizing their non-gaussianity. After whitening the observed data, JADE looks for a matrix, $\mathbf{U}' = (\mathbf{MA})'$, such that the components given by

$$\widehat{\mathbf{s}}_t^J = \mathbf{U}'\mathbf{z}_t, \tag{6}$$

are maximally non-gaussian distributed. Under the non-gaussianity assumption, the information provided by the covariance matrix of the data, $\Gamma_{\mathbf{z}}(0) = E\{\mathbf{z}_t\mathbf{z}_t'\} = \mathbf{I}$, is not sufficient to compute (6), and higher-order information is needed. Cardoso and Souloumiac (1993) use the cumulants, which are the coefficients of the Taylor series expansion of the characteristic function. In practice, it is enough to take into account fourth-order cumulants, which are defined as:

$$\begin{aligned} cum_4(z_{it}, z_{jt}, z_{ht}, z_{lt}) &= E\{z_{it}z_{jt}z_{ht}z_{lt}\} - E\{z_{it}z_{jt}\}E\{z_{ht}z_{lt}\} - \\ &\quad -E\{z_{it}z_{ht}\}E\{z_{jt}z_{lt}\} - E\{z_{it}z_{lt}\}E\{z_{jt}z_{ht}\}, \end{aligned} \tag{7}$$

and the fourth-order cumulant tensor associated to $\mathbf{z}_t$ is a $m \times m$ matrix given by

$$[Q_{\mathbf{z}}(\mathbf{Q})]_{ij} = \sum_{k,l=1}^{r} cum_4(z_{it}, z_{jt}, z_{kt}, z_{lt})\, q_{kl},$$

where $\mathbf{Q} = (q_{kl})_{k,l=1}^m$ is an arbitrary $m\times m$ matrix, and $cum_4(z_i, z_j, z_k, z_l)$ is like in (7). It is easy to see that a set of random vectors is independent if all their cross-cumulants of order higher than two are equal to zero. Therefore, $\widehat{\mathbf{s}}_t^J$ will be maximally independent if its associated fourth order cumulant tensor, $Q_{\widehat{\mathbf{s}}_t^J}(\cdot)$, is maximally diagonal. Cardoso and Soloumiac (1993) show that given a set of $m \times m$ matrices, $\Im = \{\mathbf{Q}_1, \cdots, \mathbf{Q}_q\}$, there exists an orthogonal transformation,

$\mathbf{V}$, such that the matrices $\{\mathbf{V}'Q_{\mathbf{z}}(\mathbf{Q}_i)\mathbf{V}\}_{\mathbf{Q}_i\in\Im}$ are approximately diagonal. Then $\mathbf{V}=\mathbf{U}'$, and the latent factors are estimated as in (6). JADE uses an iterative process of Jacobi rotations to solve the joint diagonalization of several cumulants matrices. It is a very efficient algorithm in low dimensional problems, but when the dimension increases, it requires high computational cost.

2.2.2 Fast Fixed-Point Algorithm: FastICA

FastICA is a fixed-point algorithm for non-adaptative environments, which was proposed by Hyvärinen and Oja (1997). It estimates

$$\widehat{\mathbf{s}}_t^F = \mathbf{U}'\mathbf{z}_t \tag{8}$$

by maximizing their kurtosis. Thus, FastICA searches the directions of projection that maximize the absolute value of the kurtosis of the $\widehat{\mathbf{s}}_t^F$. As kurtosis is very sensitive to outliers, FastICA is not a robust algorithm. Hyvärinen (1999) proposes a more robust version of FastICA using an approximation of negentropy instead of kurtosis to measure the non-gaussianity of the ICs. Negentropy is the normalized version of entropy given by:

$$J\left(\widehat{\mathbf{s}}_t^F\right) = H\left(\widehat{\mathbf{s}}_t^g\right) - H\left(\widehat{\mathbf{s}}_t^F\right),$$

where $\widehat{\mathbf{s}}_t^g$ is a gaussian vector of the same correlation matrix as $\widehat{\mathbf{s}}_t^F$, and $H(\cdot)$ is the entropy of a random vector defined as $H\left(\widehat{\mathbf{s}}_t^F\right) = -E\left\{\log p_{\widehat{\mathbf{s}}_t^F}(\xi)\right\}$, where $p_{\widehat{\mathbf{s}}_t^F}(\cdot)$ is the density function of $\widehat{\mathbf{s}}_t^F$. Negentropy is a good index for non-gaussianity because it is always non-negative and it is zero *iff* the variable is gaussian distributed. Therefore, the ICs, given by (8), are estimated as the projections of the data in the directions such that the negentropy of $\widehat{\mathbf{s}}_t^F$ is maximum. The main advantage of FastICA is that it converges in a few number of iterations.

2.2.3 Second-Order Blind Identification: SOBI

Belouchrani *et al.* (1997) extended the previous work of Tong *et al.* (1990), and proposed the SOBI algorithm. SOBI requires that the ICs, given by

$$\widehat{\mathbf{s}}_t^S = \mathbf{U}'\mathbf{z}_t, \tag{9}$$

will be mutually uncorrelated for a set of time lags. That is, a set of time delayed covariance matrices of $\widehat{\mathbf{s}}_t^S$,

$$\Gamma_{\mathbf{s}}(\tau) = E\left\{\widehat{\mathbf{s}}_t^S\widehat{\mathbf{s}}_{t-\tau}^{S\prime}\right\}, \text{ for } \tau \geq 1, \tag{10}$$

should be diagonal. SOBI searches for an orthogonal transformation that jointly diagonalizes (10). This algorithm also applies whitening as a preprocessing procedure, and the covariance structure of the whitened data model (5) is given by:

$$\mathbf{\Gamma}_{\mathbf{z}}(\tau) = \mathbf{U}\mathbf{\Gamma}_{\mathbf{s}}(\tau)\mathbf{U}', \text{ for } \tau \geq 1, \tag{11}$$

where $\mathbf{U}$ is an orthogonal matrix. Therefore,

$$\mathbf{\Gamma}_{\mathbf{s}}(\tau) = \mathbf{U}'\mathbf{\Gamma}_{\mathbf{z}}(\tau)\mathbf{U}, \text{ for } \tau \geq 1. \tag{12}$$

Thus, SOBI searches for an orthogonal transformation that will be the joint diagonalizer of the set of time delayed covariance matrices, $\{\mathbf{\Gamma_s}(\tau_q)\}_{\tau_q \in \mathcal{J}}$. The optimization problem is to minimize

$$F(\mathbf{U}) = \sum_{\tau_q \in \mathcal{J}} \mathbf{off}\left(\mathbf{U}'\mathbf{\Gamma_z}(\tau)\,\mathbf{U}\right),$$

where $'\mathbf{off}'$ is a measure of the non-diagonality of a matrix, which is defined by the sum of the squares of their off-diagonal elements. SOBI solves this problem using Jacobi rotation techniques. Belouchrani et al. (1997) show that this problem has a unique solution: if there exists two different ICs that have different autocovariances for at least one time-lag, then the joint diagonalizer, $\mathbf{U}$, exists and it is unique. That is, if for all $1 \leq i \neq j \leq r$, there is any $q = 1, ..., K$ such that $\gamma_{\mathbf{s}_i}(\tau_q) \neq \gamma_{\mathbf{s}_j}(\tau_q)$, then the components of $\widehat{\mathbf{s}}_t^S$ can be separated, they are unique, and lagged uncorrelated. Note that SOBI cannot get the ICs if they have identical autocovariances for the lags considered.

2.2.4 Weighting the ICs

After estimating the components, we should decide which of them are more important to explain the underlying structure of the observed data. Note that the PCs are sorted in terms of variability, but the ICs are undeterminated with respect to the order. Following Back and Weigend (1997), we sort the ICs in terms of their explained variability. According to model (1), the ith observed variable is given by $x_{it} = \sum_{j=1}^{m} a_{ij} s_{jt}$, and its variance is

$$var(x_{it}) = \sum_{i=1}^{m} a_{ij}^2, \ \forall\ i = 1, ..., m. \tag{13}$$

For each x_{it}, with $i = 1, ..., m$, Back and Weigend (1997) define the weighted ICs in terms of the elements of the ith row of $\mathbf{A}$ as $\mathbf{s}_t^{w(i)} = diag(a_{i1}, a_{i2}, \ldots, a_{im})\,\mathbf{s}_t$. That is, for each x_{it}, the jth weighted IC is given by $s_{jt}^{w(i)} = a_{ij} s_{jt}$, for $j = 1, ..., m$, and its variance is

$$var\left(s_{jt}^{w(i)}\right) = a_{ij}^2, \ \forall\ i, j = 1, ..., m. \tag{14}$$

Therefore, from (13) and (14), the variance of x_{it} which is explained by $s_{jt}^{w(i)}$ is computed as:

$$\nu_j^i = \frac{a_{ij}^2}{\sum_{i=1}^{m} a_{ij}^2}. \tag{15}$$

The total variance of $\mathbf{x}_t$ explained by the jth IC is given by:

$$\vartheta_j = \frac{\sum_{i=1}^{m} \nu_j^i}{\sum_{j=1}^{m}\left(\sum_{i=1}^{m} \nu_j^i\right)}, \quad \forall j = 1, ..., m.$$

Thus, after getting $\{\vartheta_1, \vartheta_2, ..., \vartheta_m\}$, we can sort the ICs in terms of variability. The most important ICs will be those that explain the maximum variance of $\mathbf{x}_t$.

2.3 ICA and the Dynamic Factor Model

Suppose that, in the basic ICA model, there are r gaussian components, $\mathbf{s}_t^{(1)} = (s_{1t}, \ldots, s_{rt})'$ with $r < m$, representing the common dynamic of the time series, but the other $m - r$ components,

$\mathbf{s}_t^{(2)} = (s_{r+1t}, \ldots, s_{mt})'$, are Gaussian. Then we can split the matrix $\mathbf{A} = [\mathbf{A}_1\mathbf{A}_2]$ accordingly and write

$$\mathbf{x}_t = \mathbf{A}_1\mathbf{s}_t^{(1)} + \mathbf{A}_2\mathbf{s}_t^{(2)}. \tag{16}$$

Calling $\mathbf{n}_t = \mathbf{A}_2\mathbf{s}_t^{(2)}$ to the vector of Gaussian noise we have $\mathbf{x}_t = \mathbf{A}_1\mathbf{s}_t^{(1)} + \mathbf{n}_t$, which is similar to the DFM studied by Peña and Box (1987). However, there are two main differences between these models. First, in the factor model, the r common factors, $\mathbf{s}_t^{(1)}$, are assumed Gaussian and linear, whereas here they are non Gaussian. Second, in the standard factor model the covariance matrix of the noise is of full rank, whereas here it will have rank equal to $m - r$. This last constraint can be relaxed by assuming that the ICA model is contaminated with some Gaussian error model, as in $\mathbf{x}_t = \mathbf{A}\mathbf{s}_t + \mathbf{u}$, where $\mathbf{u}$ is Gaussian. Note that the latent factors of the DFM can be estimated by using PCA (see, for example, Stock and Watson, 2002).

3 The GICA-GARCH model

This section presents the GICA-GARCH model as a new multivariate volatility model. From now on, let $\mathbf{x}_t = (x_{1t}, x_{2t}, \ldots, x_{mt})'$ be the vector of m financial time series. First, we introduce the GICA-GARCH model, give its mathematical formulation, and describe the structure of the ICA components. Then, we explain how our model is used to forecast the volatility of a vector of financial data from the volatility of a set of ICs. Finally, we relate the GICA-GARCH model to other factor GARCH models. In particular, we show that the GICA-GARCH model can be seen as a particular dynamic factor GARCH model.

3.1 The model

Empirical evidence reveals that financial assets cannot be predicted at short horizons, but it is well known that we can forecast their conditional variance using a particular GARCH model (Engel, 1982; Bollerslev, 1986). Therefore, we focus our analysis on forecasting the volatility of the observed financial time series. Let us assume that $\mathbf{x}_t$ is a linear combination of a set of independent factors given by (1) . Financial time series are characterized by the presence of clusters of volatility, and this implies that the unobserved factors will follow conditionally heteroskedastic processes. We suppose that the vector of unobserved components, $\mathbf{s}_t$, follows an r-dimensional $ARMA(p,q)$ model with $GARCH\,(p',q')$ disturbances:

$$\mathbf{s}_t = \sum_{i=1}^{p} \mathbf{\Phi}_i \mathbf{s}_{t-i} + \sum_{l=0}^{q} \mathbf{\Theta}_l \mathbf{e}_{t-l}, \tag{17}$$

where $\mathbf{\Phi}_i = diag\left(\phi_i^{(1)}, ..., \phi_i^{(r)}\right)$ with $|\phi_i^{(j)}| < 1\ \forall j$, $\mathbf{\Theta}_l = diag\left(\theta_l^{(1)}, ..., \theta_l^{(r)}\right)$ with $\mathbf{\Theta}_0 = \mathbf{I}_r$ and $|\theta_l^{(j)}| < 1\ \forall j$, and $\mathbf{e}_t$ is an r-dimensional vector of conditionally heteroskedastic errors given by:

$$\mathbf{e}_t = \mathbf{H}_t^{1/2}\varepsilon_t, \tag{18}$$

where $\varepsilon_t \sim iid\,(\mathbf{0}, \mathbf{I}_r)$ and $\mathbf{H}_t^{1/2} = diag(\sqrt{h_{jt}})$ is an $r \times r$ positive definite diagonal matrix such that

$$h_{jt} = \alpha_0^{(j)} + \sum_{i=1}^{p'} \alpha_i^{(j)} e_{jt-i}^2 + \sum_{l=1}^{q'} \beta_l^{(j)} h_{jt-l},\ \text{for } j = 1, .., r, \tag{19}$$

where h_{jt} is a stationary process, independent of ε_{jt}, and it is the conditional variance of the jth IC: $h_{jt} = V\left(e_{jt}|\mathbf{I}_{t-1}\right) = V\left(s_{jt}|\mathbf{I}_{t-1}\right)$, where $\mathbf{I}_{t-1}$ is the past information available until time $t-1$. In order to ensure a positive $h_{jt} > 0$, $\forall j$, it is assumed that $\alpha_0^{(j)} > 0$, $\alpha_i^{(j)} \geq 0$, $\beta_i^{(j)} \geq 0$, and $\sum_{i=1}^{\max(p',q')} \left(\alpha_i^{(j)} + \beta_i^{(j)}\right) < 1$ (see Bollerslev, 1986).

From (1), we have that the conditional covariance matrix of $\mathbf{x}_t$ is:

$$\mathbf{\Omega}_t = V\left(\mathbf{x}_t|\mathbf{I}_{t-1}\right) = \mathbf{A}\mathbf{H_t}\mathbf{A}', \tag{20}$$

where $\mathbf{H}_t = diag(h_{1t}, ..., h_{rt})$ is the $r \times r$ conditional covariance matrix of $\mathbf{s}_t$ at time time t. That is, we can forecast the volatility of the stock returns from the predicted volatility of the components. Note that we have assumed that $\mathbf{H}_t$ is diagonal, but the ICs are only unconditionally independent. In order to guarantee the diagonality of $\mathbf{H}_t$, we should assume that the conditional correlations of the ICs are zero. This assumption allows us to achieve our purpose: estimating a multivariate GARCH model from a small number of GARCH univariate models, and therefore, reducing considerably the number of parameters to be estimated.

3.2 Identification of the factors

In practice, we separate the estimation of the factors from fitting their volatility models. First, we apply ICA to identify the underlying independent components. Any of the previous ICA algorithms standardizes the data as a preprocessing step, and solves the basic ICA model for the normalized data, which is given by equation (5). Thus, JADE, FastICA, and SOBI will estimate the loading matrix, that is orthogonal, and the m ICs, defined by equation (6), (8), and (9), respectively. After estimating the model, we weight the ICs according to the procedure explained in (2.2.4.): we sort the ICs in terms of their explained total variability and the first few are the most important ICs. Hence, we split the vector of ICs as $\mathbf{s}_t = [\mathbf{s}_t^{(1)}\mathbf{s}_t^{(2)}]$, where $\mathbf{s}_t^{(1)} = (s_{1t}, \ldots, s_{rt})'$ are the r ICs, with $r < m$, which we choose to represent the data, and $\mathbf{s}_t^{(2)} = (s_{r+1t}, \ldots, s_{mt})'$ are the $m-r$ ICs which we consider as noise. From now on, we focus on the r selected ICs and we fit an univariate $ARMA(p,q) - GARCH(p\prime, q\prime)$ for each one of them. We estimate the univariate volatility of each IC and generate the conditional covariance matrix of $\mathbf{s}_t^{(1)}$, $\mathbf{H}_t$. Finally, we get the conditional covariance matrix of the observed data from (20) and its ith diagonal term, $\gamma_{it}^2 = \sum_{j=1}^{r} h_{jt} a_{ij}^2$, is the conditional variance of x_{it}, for $i = 1, 2, ..., m$.

Note that the performance of the GICA-GARCH model depends on the method we apply to estimate the ICs. We will investigate the usefulness of the three algorithms presented in section 2. Since we have seen that they use different estimation principles (JADE and FastICA non-gaussianity, and SOBI dynamic uncorrelation) the performance of the algorithms is expected to depend on the features of the data.

Financial data usually tend to exhibit fat-tailed distributions. The kurtosis coefficient is a popular measure for the thickness of the tails and financial data have excess kurtosis. Furthermore, they have small lagged autocorrelation coefficients. Then, it seems that JADE and FastICA could work better for financial data than SOBI. On the other hand, when the data have significant correlation structure, SOBI may be the most appropiated algorithm to estimate the ICs.

3.3 The GICA-GARCH model as a dynamic factor GARCH model

The GICA-GARCH model can be seen as a particular multivariate factor GARCH model where the comovements of the data are driven by a small number of ICs. Suppose that in basic ICA model, the r components of $\mathbf{s}_t^{(1)}$, with $r < m$, are the key factors describing the common behavior of the financial markets, and the other $m - r$ components, $\mathbf{s}_t^{(2)}$, are considered as noise in the error term. That is, we can split the matrix $\mathbf{A} = [\mathbf{A}_1 \mathbf{A}_2]$ accordingly and write

$$\mathbf{x}_t = \mathbf{A}_1 \mathbf{s}_t^{(1)} + \mathbf{n}_t, \tag{21}$$

where $\mathbf{n}_t = \mathbf{A}_2 \mathbf{s}_t^{(2)}$ is the error term. We assume that common factors are conditionally heteroskedastic and evolve according to a univariate $ARMA(p,q) - GARCH(p\prime, q\prime)$, that is, $\mathbf{s}_t^{(1)} | \mathbf{I}_{t-1} \sim D(\mu_t, \mathbf{H}_t)$, where $\mathbf{H}_t$ is a diagonal $r \times r$ matrix containing the conditional variance of the common components. From this formulation, the GICA-GARCH model is related to the dynamic factor GARCH (Alessi *et al.*, 2006). However, there are two main differences between these two models. First, in the dynamic factor GARCH model, the heteroskedastic components $\mathbf{s}_t^{(1)}$ are assumed to be conditionally Gaussian and $\mu_t = \mathbf{0}$, whereas in the GICA-GARCH they can be non Gaussian, and they are allow to have a non zero conditional mean. Second, in the dynamic factor GARCH model, the conditional variance of the observed data depends on the common and idiosyncratic components, whereas in the GICA-GARCH the volatility of the data is estimated from the volatilities of the common components. Note also that in the dynamic factor GARCH model the covariance matrix of the noise is of full rank, whereas in the GICA-GARCH it will have rank equal to $m-r$, but this assumption can be relaxed easily by assuming an additional measurement noise so that the covariance matrix is a full rank matrix.

Moreover, the GICA-GARCH model given by (21) can be seen as a conditionally heteroskedastic factor model (Diebold and Nerlove, 1989),

$$\begin{aligned} \mathbf{x}_t &= \mathbf{A}_1 \mathbf{s}_t^{(1)} + \mathbf{u}_t, \\ \begin{pmatrix} \mathbf{s}_t^{(1)} \\ \mathbf{u}_t \end{pmatrix} | \mathbf{I}_{t-1} &\sim D \left\{ \begin{pmatrix} \mu_t \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{H}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma} \end{pmatrix} \right\} \end{aligned}$$

where $\mathbf{u}_t$ is a $m \times 1$ vector of idiosyncratic noises, which are conditionally orthogonal to $\mathbf{s}_t^{(1)}$, and $\mathbf{\Gamma}$ is an $m \times m$ positive definite matrix that represents the constant conditional idiosyncratic variances. According to the conditionally heteroskedastic factor model, the conditional covariance matrix of the data is $\mathbf{\Omega}_t = \mathbf{A}_1 \mathbf{H}_t \mathbf{A}_1^{'} + \mathbf{\Gamma}$, and, in practice, it can be approximated as

$$\mathbf{\Omega}_t = \mathbf{A}_1 \mathbf{H}_t \mathbf{A}_1^{'}, \tag{22}$$

with an accuracy that depends on the number of common components, r, which are chosen. Note that the approximation given by (22) is equivalent to the way we estimate the conditional covariance matrix of $\mathbf{x}_t$ according to the GICA-GARCH model. The main difference between these two models is that, in the conditionally heteroskedastic factor model, the noise has a full rank conditional covariance matrix, whereas in the GICA-GARCH model it will have rank equal to $m - r$.

In summary, the GICA-GARCH model can be seen as a factor GARCH model with unconditionally independent factors, and offers a new approach for estimating multivariate GARCH models as linear combination of several univariate GARCH models. It is also an extension of

the (generalized) orthogonal GARCH model ((G)O-GARCH) (Alexander, 2001; van der Weide, 2002), that assumes only unconditionally uncorrelated factors. These models use a small number of factors compared to the number of observed financial time series, and transform the problem to estimate a multivariate GARCH model into a small number of univariate volatility models. Furthermore, the GICA-GARCH model is related to the work proposed by Fan *et al.* (2008) that models multivariate volatilities through conditionally uncorrelated components.

4 Simulation experiments

In this section, we present two simulation experiments to show the effectiveness of ICA versus PCA with non linear data. First, we consider a set of non-linear factors without conditional heteroskedasticity. Second, we consider factors which are conditionally heteroskedastic. In this second experiment we allow for both zero conditional mean and dynamic non-linear components. In order to analyze the performance of ICA and PCA we compute the correlation coefficient between each original component and its estimation. Moreover, we compute the Euclidean distance between the original and the estimated components as $d(s_j, \widehat{s}_j^{(\cdot)}) = \sqrt{\sum_{t=1}^{T}\left(s_{jt} - \widehat{s}_{jt}^{(\cdot)}\right)^2}$, for $j = 1, \ldots, r$, where $\widehat{s}_{jt}^{(\cdot)}$ is the jth estimated component by the corresponding method.

In the first simulation exercise, we generate 6 artificially time series of 528 observations which are defined in Table 1.

Table 1: Definition of the original factors

$s_{1t} = 2(\lfloor \frac{t}{10} \rfloor - \frac{t}{10}) + 0.2n_{1t}$
$s_{2t} = 10I_{\{t>(9+105\lfloor \frac{t}{105} \rfloor)\}} + 6I_{\{t\leq(9+105\lfloor \frac{t}{105} \rfloor)\}} + 0.01n_{2t}$
$s_{3t} = 70\sin(100\pi t^{-0.5}) + 2\sin(240\pi t^{-0.5}) + 0.4n_{3t}$
$s_{4t} = \log(n_{3t})$
$s_{5t} = -\frac{t^2}{2}\exp(3\sin(2t)) + 0.1n_{5t}$
$s_{6t} = s_{6,t-12} + 0.2n_{6t}$

NOTE: $n_{it} \sim N(0,1)$, $\forall i$, $\lfloor \cdot \rfloor$ is the nearest integer function, and $I_{\{\cdot\}}$ is an indicator function

All these components with zero mean and unit variance are given in Figure 1. We generate a 6×6 random matrix, $\mathbf{A}$, and we mix the original components according to (1). The resulting time series vector, $\mathbf{x}_t$, is plotted in Figure 2.

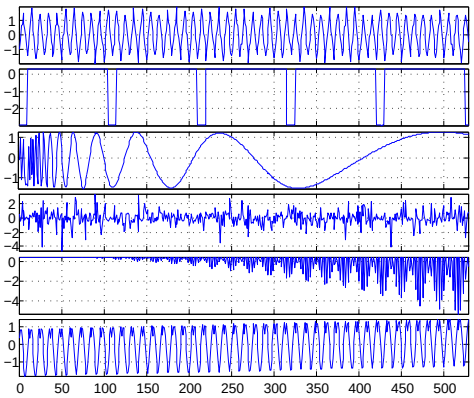


Figure 1: Original unobserved components

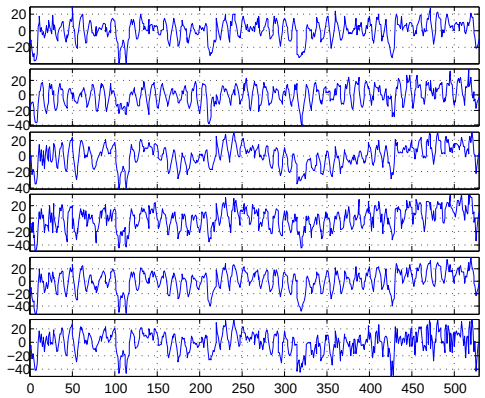
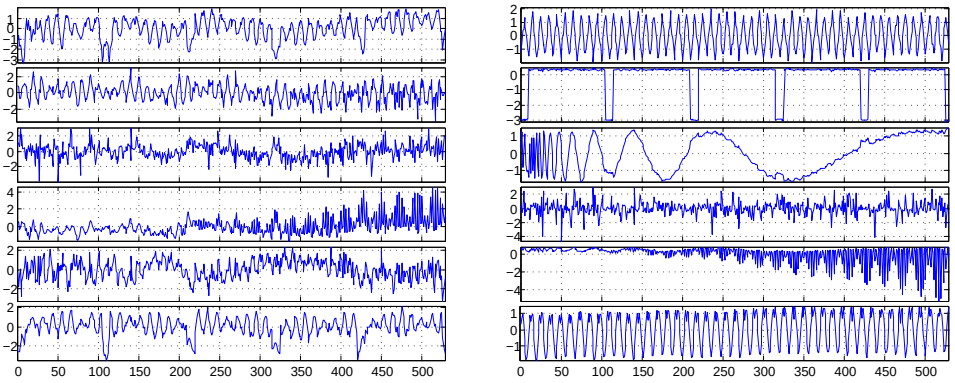
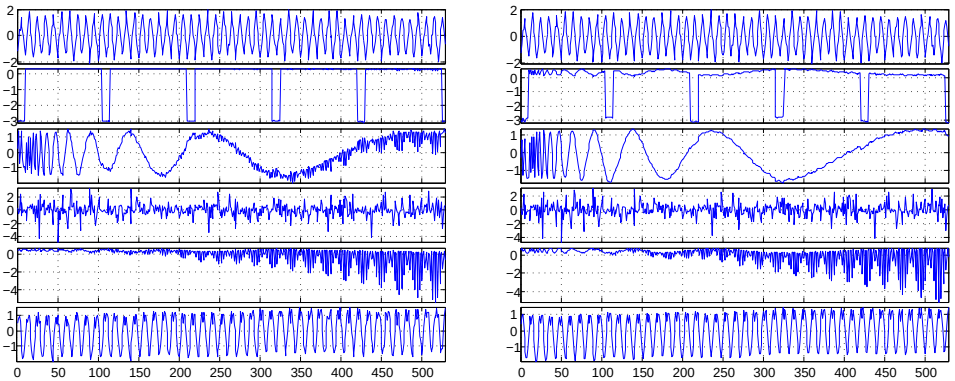


Figure 2: Observed data

We apply ICA and PCA to compute the estimates $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{s}}_t^{(\cdot)}$. Figures 3 shows $\widehat{\mathbf{s}}_t^{(\cdot)}$ obtained in the four estimation methods and in the same order as the original ones. We conclude that ICA seems to perform better than PCA to estimate the non-linear factors.



(a) $\widehat{\mathbf{s}}_t^P$: Estimated factors obtained by PCA (b) $\widehat{\mathbf{s}}_t^F$: Estimated factors obtained by FastICA



(c) $\widehat{\mathbf{s}}_t^J$: Estimated factors obtained by JADE (d) $\widehat{\mathbf{s}}_t^S$:Estimated factors obtained by SOBI

Figure 3: Estimated factors using different procedures

In order to validate this intuition, Table 2 presents the correlation between the estimated and generated components and we found that the correlations coefficients are higher for the ICs. Furthermore, the Euclidean distance between the original components and the ICs is smaller

than the one corresponding to PCs (see Table 3). Then, we conclude that ICA performs better than PCA to separate non-linear factors without heteroskedasticity. All ICA algorithms have a similar performance.

Table 2: Correlation coefficient between the original and the estimated components

	FAST	JADE	SOBI	PCA
1st	0.996	0.998	0.997	0.733
2nd	0.999	0.999	0.989	0.687
3rd	0.994	0.965	0.998	0.576
4th	0.991	0.993	0.991	0.834
5th	0.989	0.995	0.986	0.862
6th	0.994	0.988	0.998	0.653
Average	0.994	0.990	0.994	0.724

Table 3: Euclidean distance between the original and the estimated components

	FAST	JADE	SOBI	PCA
1st	1.991	1.519	1.664	16.782
2nd	0.886	0.628	3.416	18.160
3rd	2.510	6.090	1.345	21.131
4th	3.095	2.640	3.125	13.230
5th	3.454	2.404	3.896	12.072
6th	2.523	3.531	1.589	19.130
Average	2.410	2.802	2.506	16.751

In the first part of the second experiment, we generate components which have constant conditional mean but they are conditionally heteroskedastic. We generate six components of 1000 observations: three of them are gaussian random noises, and the other three are conditionally heteroskedastic processes defined in Table 4.

Table 4: Definition of the original factors

$$\breve{s}_{1t} = \sqrt{h_{1t}}\varepsilon_{1t};\ \text{where } h_{1t} = 0.2 + 0.7\breve{s}^2_{1t-1},$$
$$\breve{s}_{2t} = \sqrt{h_{2t}}\varepsilon_{2t};\ \text{where } h_{2t} = 0.021 + 0.073\breve{s}^2_{2t-1} + 0.906h_{2t-1},$$
$$\breve{s}_{3t} = \sqrt{h_{3t}}\varepsilon_{3t};\ \text{where } h_{3t} = 1.692 + 0.245\breve{s}^2_{3t-1} + 0.337h_{3t-1} + 0.310h_{3t-2},$$

NOTE: ε_{jt} is a random noise with zero-mean and unit variance, and it is independent of h_{jt}, $\forall\ j = 1, 2, 3$

We standardize the components to satisfy the requirements of ICA, and we plot them in Figure 4. Next, we generate a 6×6 random matrix, $\mathbf{A}$, and we compute $\mathbf{x}_t$ (see Figure 5) according to (1) .

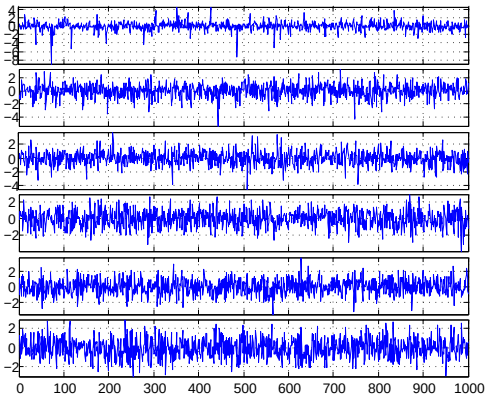


Figure 4: Original unobserved components

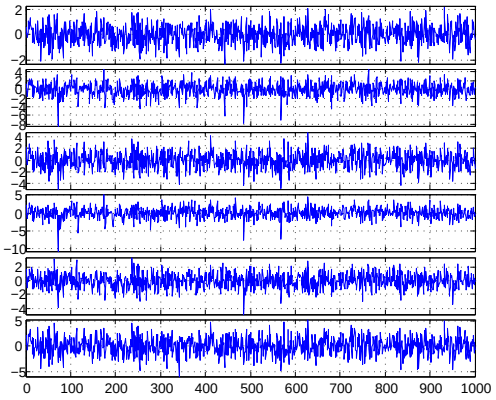


Figure 5: Observed data

We apply ICA and PCA to estimate the components, and compute the correlation coefficient and the Euclidean distances between $\breve{s}_{jt}$ and $\widehat{\breve{s}}^{(\cdot)}_{jt}$, for $j = 1, \ldots, 6$. Looking at the results, which

are shown in Tables 5 and 6 respectively, we conclude that the fitting of the ICs to the original components is better than the one corresponding to PCA. However, in this case SOBI performs worse than FastICA and JADE. This is to be expected as heteroskedastic processes have excess kurtosis and small autocorrelation coefficients.

Table 5: Correlation coefficient between the original and the estimated components

	FAST	JADE	SOBI	PCA
1st	0.985	0.995	0.907	0.778
2nd	0.993	0.992	0.747	0.750
3rd	0.934	0.928	0.838	0.708
4th	0.821	0.942	0.753	0.707
5th	0.869	0.909	0.637	0.530
6th	0.715	0.847	0.624	0.623
Average	0.886	0.934	0.751	0.683

Table 6: Euclidean distance between the original and the estimated components

	FAST	JADE	SOBI	PCA
1st	5.521	5.505	13.637	21.056
2nd	3.777	4.080	22.476	22.358
3rd	11.484	12.004	18.015	24.175
4th	18.920	10.736	22.210	24.181
5th	16.214	13.500	26.950	30.630
6th	23.872	17.472	27.410	27.450
Average	13.298	10.549	21.783	24.975

In the second part of this experiment we make a mixture of the previous $\mathbf{s}_t$ and $\breve{\mathbf{s}}_t$ and have components that are both non-linear and conditionally heteroskedastic. We consider six time series of 1000 observations. Three of them are zero-mean and unit variance random noises, and the others are generated as:

$$\widetilde{s}_{1t} = s_{2t} + \breve{s}_{1t} \qquad \widetilde{s}_{2t} = s_{3t} + \breve{s}_{3t} \qquad \widetilde{s}_{3t} = s_{6t} + \breve{s}_{2t}$$

where s_{jt}, for $j = 2, 3, 6$, is defined in Table 1, and $\breve{s}_{it}$, for $i = 1, 2, 3$, is defined in Table 2. The standardized components and the observed data set given by (1) , where $\mathbf{A}$ is randomly generated, are shown in Figures 6 and 7.

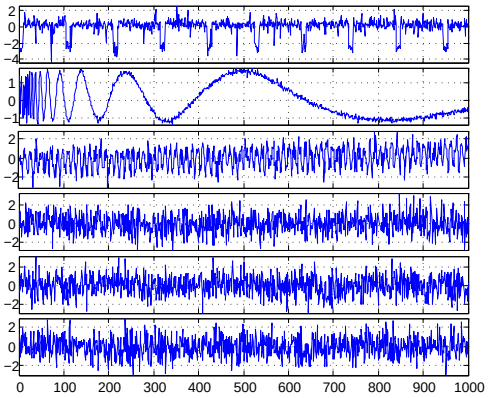


Figure 6: Original unobserved components

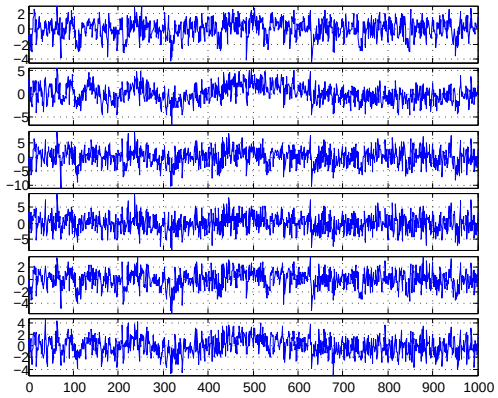
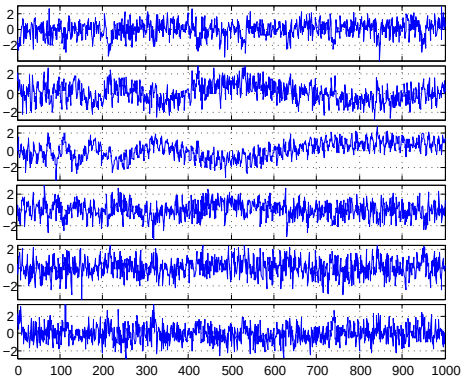


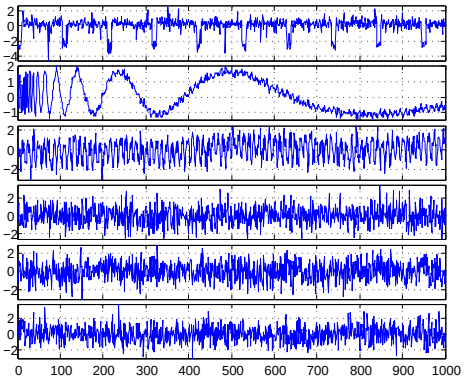
Figure 7: Observed data

Figure 8 shows the estimated components by ICA and PCA, and again ICA performs better than PCA, specially for the non-noisy components.

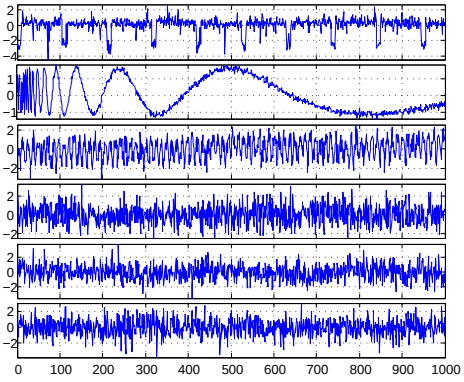
Since the components are now heteroskedastic and also have a non-linear conditional mean, SOBI performs as well as, or even better than, FastICA and JADE. The correlation coefficients (Table 7) and the Euclidean distances (Table 8) between the original and the estimated components, confirm our visual results. In fact, according to the average results, SOBI has the best performance for non-linear and conditionally heteroskedastic components.



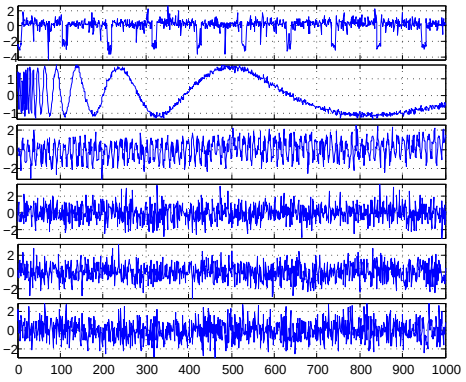
(a) $\widehat{\mathbf{s}}_t^P$: Estimated factors obtained by PCA



(b) $\widehat{\mathbf{s}}_t^F$: Estimated factors obtained by FastICA



(c) $\widehat{\mathbf{s}}_t^J$: Estimated factors obtained by JADE



(d) $\widehat{\mathbf{s}}_t^S$: Estimated factors obtained by SOBI

Figure 8: Estimated unobserved components using different procedures

Table 7: Correlation coefficient between the original and the estimated components

	FAST	JADE	SOBI	PCA
1st	0.988	0.996	0.998	0.602
2nd	0.991	0.998	0.999	0.536
3rd	0.966	0.983	0.990	0.650
4th	0.931	0.633	0.871	0.447
5th	0.827	0.731	0.965	0.626
6th	0.746	0.755	0.827	0.737
Average	0.908	0.849	0.942	0.600

Table 8: Euclidean distance between the original and the estimated components

	FAST	JADE	SOBI	PCA
1st	4.839	2.812	1.771	28.197
2nd	4.238	2.151	0.995	30.457
3rd	8.222	5.788	4.530	26.446
4th	11.779	27.098	16.035	33.227
5th	18.586	23.210	8.313	27.343
6th	22.543	22.122	18.611	22.937
Average	11.701	13.863	8.376	28.101

From these simulations, we conclude that, under non-gaussian data, ICA recovers the components better than PCA. The performance of the three ICA algorithms is as expected: SOBI is better than JADE or FastICA if the components have dynamics in the mean, both when the components are heteroskedastic and when they are not. Furthermore, when the components are conditionally heteroskedastic with constant conditional mean JADE and FastICA perform well, whereas SOBI is similar to PCA. This is not a surprising result because the SOBI components are the rotation of the PCs that diagonalize a set of time delayed covariance matrices, and the

conditionally heteroskedastic components are not time dependent.

5 Empirical application

In this section we apply our procedure to a real dataset of stock returns. First, we describe the data used; second, we explain the procedure to estimate the components; and third, we present the results of using ICs and PCs to forecast the volatility of the stock returns.

We use daily closing prices from the Madrid stock market. The data are from the 19 assets which were always included in the IBEX 35 from 2000 to 2004[1]. The IBEX 35 index is the main stock market index of the Bolsa de Madrid. Its composition is revised twice a year and it comprises the 35 companies with the largest trading volume of the Madrid stock exchange. We apply some preprocessing steps to the data. First of all, we transform daily closing prices to daily stock returns to achieve stationarity. The daily stock returns of the ith company are computed as:

$$r_{it} = \log\left(p_{it+1}\right) - log\left(p_{it}\right), \qquad i = 1, ..., 19, \tag{23}$$

where p_{it} is the daily closing price of the ith asset at time t. Then, we have a 19×1250 multivariate vector of stock returns, which is denoted by $\mathbf{r}_t$, whose columns are the value of these 19 stocks in the 1250 trading days in the period 2000-2004. There are some extreme observations that correspond to outliers, which are due to well know changes, such as stock splits or other legal changes. Finally, after the outliers have been removed, we also remove the mean from the stocks returns, $\mathbf{x}_t = \mathbf{r}_t - E[\mathbf{r}_t]$, and these 19 preprocessed daily stock returns time series are shown in Figure 9.

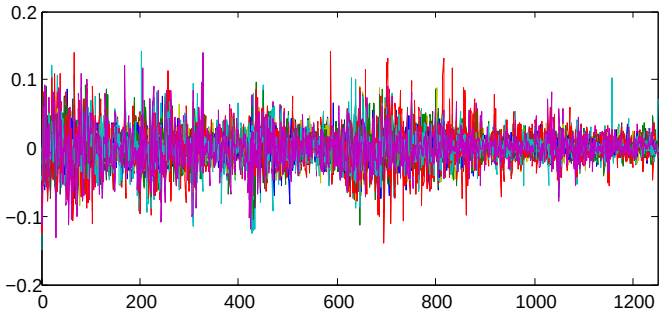


Figure 9: Series of daily stock returns from 2000 to 2004 (without outliers)

We compute the kurtosis coefficients of $\mathbf{x}_t$ and the results are displayed in Table 9. The distribution of daily stock returns is leptokurtic, that is, they are far away from gaussianity.

[1]The 19 stocks are listed in the Appendix.

Table 9: Kurtosis coefficients of the transformed daily stock returns

ACS	**ACX**	**ALT**	**AMS**	**ANA**	**BBVA**	**BKT**	**ELE**
5.3619	4.8558	5.9571	5.3303	5.8949	4.7493	5.9641	5.4053
FCC	**FER**	**IBE**	**IDR**	**NHH**	**POP**	**REP**	**SAN**
5.0625	4.6196	5.3139	4.8257	4.776	4.9358	4.9307	5.0546
SGC	**TEF**	**TPI**					
4.8686	4.0998	5.5625					

The 19 unobserved factors are estimated using both PCA and ICA and they are sorted in terms of the explained total variance. The results are displayed in Table 10.

Table 10: Sorted components in terms of their explained variability

PCA	$\%_{PCA}$	FAST	$\%_{Fast}$	JADE	$\%_{JADE}$	SOBI	$\%_{SOBI}$
$\widehat{s}^P_{1t}$	35.30	$\widehat{s}^F_{1t}$	17.72	$\widehat{s}^J_{1t}$	11.75	$\widehat{s}^S_{1t}$	11.13
$\widehat{s}^P_{2t}$	7.00	$\widehat{s}^F_{2t}$	10.22	$\widehat{s}^J_{2t}$	7.29	$\widehat{s}^S_{2t}$	9.65
$\widehat{s}^P_{3t}$	5.91	$\widehat{s}^F_{3t}$	6.40	$\widehat{s}^J_{3t}$	6.48	$\widehat{s}^S_{3t}$	9.16
$\widehat{s}^P_{4t}$	4.78	$\widehat{s}^F_{4t}$	5.92	$\widehat{s}^J_{4t}$	6.36	$\widehat{s}^S_{4t}$	8.15
$\widehat{s}^P_{5t}$	4.73	$\widehat{s}^F_{5t}$	5.76	$\widehat{s}^J_{5t}$	6.17	$\widehat{s}^S_{5t}$	7.52
$\widehat{s}^P_{6t}$	4.35	$\widehat{s}^F_{6t}$	4.89	$\widehat{s}^J_{6t}$	5.70	$\widehat{s}^S_{6t}$	5.42
$\widehat{s}^P_{7t}$	4.24	$\widehat{s}^F_{7t}$	4.65	$\widehat{s}^J_{7t}$	5.61	$\widehat{s}^S_{7t}$	5.23
$\widehat{s}^P_{8t}$	4.04	$\widehat{s}^F_{8t}$	4.62	$\widehat{s}^J_{8t}$	5.52	$\widehat{s}^S_{8t}$	4.37
$\widehat{s}^P_{9t}$	3.62	$\widehat{s}^F_{9t}$	4.43	$\widehat{s}^J_{9t}$	5.20	$\widehat{s}^S_{9t}$	4.11
$\widehat{s}^P_{10t}$	3.60	$\widehat{s}^F_{10t}$	4.15	$\widehat{s}^J_{10t}$	5.16	$\widehat{s}^S_{10t}$	4.00
$\widehat{s}^P_{11t}$	3.31	$\widehat{s}^F_{11t}$	3.86	$\widehat{s}^J_{11t}$	5.12	$\widehat{s}^S_{11t}$	3.83
$\widehat{s}^P_{12t}$	3.13	$\widehat{s}^F_{12t}$	3.85	$\widehat{s}^J_{12t}$	4.74	$\widehat{s}^S_{12t}$	3.73
$\widehat{s}^P_{13t}$	3.03	$\widehat{s}^F_{13t}$	3.69	$\widehat{s}^J_{13t}$	4.01	$\widehat{s}^S_{13t}$	3.57
$\widehat{s}^P_{14t}$	2.86	$\widehat{s}^F_{14t}$	3.67	$\widehat{s}^J_{14t}$	3.85	$\widehat{s}^S_{14t}$	3.57
$\widehat{s}^P_{15t}$	2.66	$\widehat{s}^F_{15t}$	3.56	$\widehat{s}^J_{15t}$	3.84	$\widehat{s}^S_{15t}$	3.57
$\widehat{s}^P_{16t}$	2.56	$\widehat{s}^F_{16t}$	3.47	$\widehat{s}^J_{16t}$	3.76	$\widehat{s}^S_{16t}$	3.42
$\widehat{s}^P_{17t}$	2.21	$\widehat{s}^F_{17t}$	3.26	$\widehat{s}^J_{17t}$	3.51	$\widehat{s}^S_{17t}$	3.26
$\widehat{s}^P_{18t}$	1.71	$\widehat{s}^F_{18t}$	2.97	$\widehat{s}^J_{18t}$	3.41	$\widehat{s}^S_{18t}$	3.22
$\widehat{s}^P_{19t}$	0.93	$\widehat{s}^F_{19t}$	2.89	$\widehat{s}^J_{19t}$	2.51	$\widehat{s}^S_{19t}$	3.10
	1.00		1.00		1.00		1.00

We use Figure 10, that shows the explained variability by the components estimated by the four algorithms, to decide the optimal number of components for each method. The results are given in Table 11, that also includes the absolute explained variability by the r selected components.

Table 11: Number of unobserved components and percentage of total explained variability

	PCA	FAST	JADE	SOBI
r	1	2	2	5
% variability	35.30	27.95	19.04	45.62

We are interested in investigating which assets are more important to define each component. From (2) , $\{\widehat{s}_{it}\}_{i=1}^{19}$ can be written as a linear combination of the stock returns, $\widehat{s}_{it} = \sum_{j=1}^{19} w_{ij} x_{jt}$, where w_{ij} represents the effect of the jth stock returns on the ith component, and the largest weights correspond to the most important assets. The ICs and the PCs have different interpretation. For example, if we focus on the first component, we have that, on one hand, the first PC

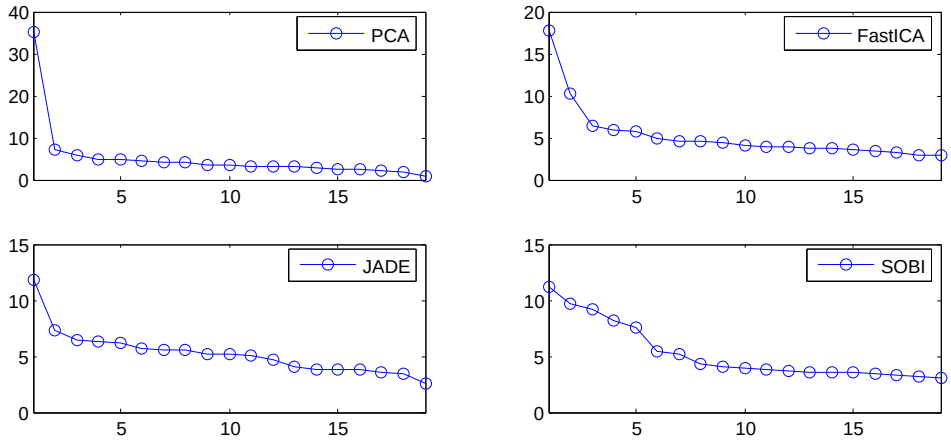


Figure 10: Explained total variability by the components.

can be seen as a weighted mean of the 19 daily stock returns time series, it is an index of the market. Indeed, if we plot the variation of variability of the first PC and the IBEX 35 index, considering groups of ten observations, it is clear that the first PC reflects the main movements of the index IBEX 35 (see Figure 11). Then, if we forecast the volatility of $\mathbf{x}_t$ from the volatility of the first PC, the 19 stock returns will tend to move together.

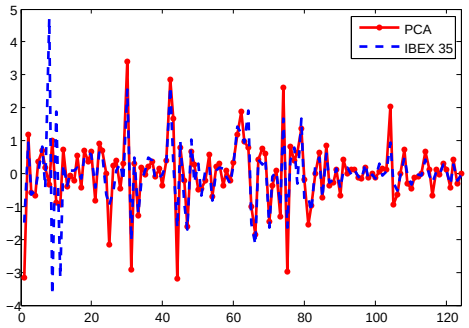


Figure 11: Variation of variability of $\widehat{s}_{1t}^{P}$ and the IBEX 35 index

On the other hand, the first ICs cannot be seen as indexes of the market. They are mainly associated with electricity, building industries, and banking[2], and separate the stock returns in terms of the individual explained variability, $\{\nu_1^i\}_{i=1}^{19}$ (see (15)). As an example, we analyze the first FastICA, $\widehat{s}_{1t}^{F}$. In Figure 12, that shows the variation of variability of $\widehat{s}_{1t}^{F}$ and the largest weighted assets on $\widehat{s}_{1t}^{F}$, we see that all assets present a cluster of high variability from observation 600 to 750. The assets which are positively weighted only have this period of higher variability, but the negative ones are also volatile at the beginning of the sample.

The forecasting performance of our model is checked as follows:

1. We estimate $\mathbf{A}$ and the unobserved components, by ICA and PCA, using the whole sample. Then, the components are sorted and r is fixed.

[2]The sectorial economic classification is detailed in the Appendix.

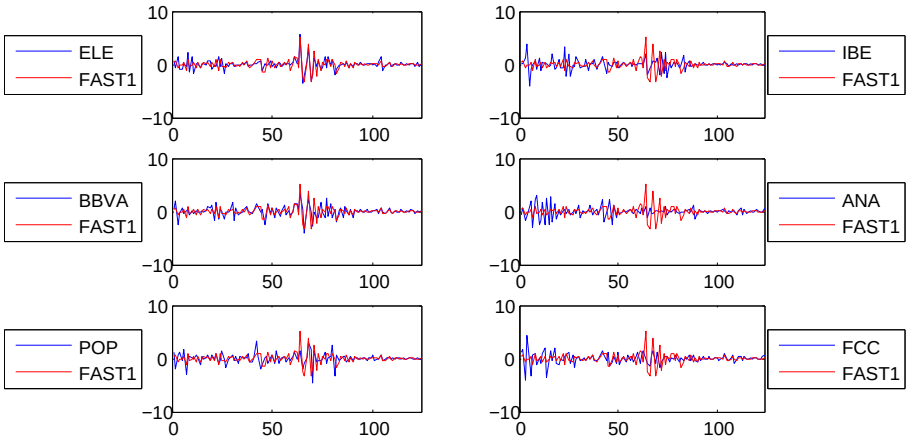


Figure 12: Variation of variability of $\widehat{s}_{1t}^{F}$ and the stock returns with the largest weights: a) on the left, the positive ones; b) on the right, the negative ones.

2. Using the whole sample, we fit an $ARMA(p,q)$ with $GARCH(p',q')$ disturbances for each component $\widehat{s}_{jt}$, with $j = 1, ..., r$.

3. We estimate the parameters of the $ARMA(p,q)$-$GARCH(p',q')$ model with a sample of 1000 observations. Then, we generate the one-step-ahead forecast for the volatility of each $\widehat{s}_{jt}$,

$$\widehat{h}_{j,1001|1000} = V\left[\widehat{s}_{j1001}|\mathbf{I}_{1000}\right], \quad j = 1, ..., r. \tag{24}$$

Thus, by rolling prediction for $t = 1001, ..., 1250$, we have:

$$\widehat{\mathbf{H}}_{t|1000} = diag(\widehat{h}_{1,t|1000}, ..., \widehat{h}_{r,t|1000}), \quad t = 1001, ..., 1250, \tag{25}$$

which is the conditional covariance matrix of $\widehat{\mathbf{s}}_t = (\widehat{s}_{1t}, ..., \widehat{s}_{rt})'$ at time t.

4. According to (20), we compute the conditional variance of $\mathbf{x}_t$ at time t, $\mathbf{\Omega}_t$, and the conditional variance of $\mathbf{x}_i$ at time t is given by the ith diagonal term of $\mathbf{\Omega}_t$,

$$\widehat{\gamma}_{i,t|1000}^2 = \sum_{j=1}^{r} \widehat{h}_{j,t|1000} a_{ij}^2, \ i = 1, 2, ..., 19, \ t = 1001, ..., 1250. \tag{26}$$

Then, we forecast the conditional variance of the stock returns from the predicted volatility of the unobserved components.

5. To evaluate the forecasting performance of GICA-GARCH and O-GARCH models, we need to compare the predicted volatility with respect to the real one. However, volatility cannot be observed. Following Franses and van Dijk (1996), we measure the 'true volatility' of the ith stock return at time t by:

$$\upsilon_{it} = (x_{it} - \overline{x_i})^2, \quad i = 1, ..., 19, \ t = 1001, ..., 1250, \tag{27}$$

where $\overline{x_i}$ is the average return of $\mathbf{x}_i$ over the last 1000 observations. Then, the one-step-ahead forecast error is given by:

$$\epsilon_{it} = \upsilon_{it} - \widehat{\gamma}_{i,t|1000}^2, \ i = 1, 2, ..., 19, \ t = 1001, ..., 1250. \tag{28}$$

6. To evaluate the accuracy of the model we divide the prediction error (28) by some alternative benchmark. This benchmark is obtained using another standard method of forecasting. Let us assume that the benchmark method predicts the volatility of the stock returns by their marginal variance. Then, we define the relative ratio as:

$$RE_{it} = \frac{\epsilon_{it}}{\epsilon^*_{it}}, \quad i = 1, 2, ..., 19, \ t = 1001, ..., 1250, \tag{29}$$

where ϵ^*_{it} is the forecast error of the ith stock return obtained by the benchmark method. That is,

$$\epsilon^*_{it} = v_{it} - \widehat{\sigma}^2_i, \ i = 1, 2, ..., 19, \ t = 1001, ..., 1250. \tag{30}$$

where $\widehat{\sigma}^2_i$ is the marginal variance of the ith stock return at time t. To minimize the impact of outliers when we analyze the volatility forecasting performance of GICA-GARCH and O-GARCH models, we use the Median Relative Absolute Error (MdRAE) criteria[3]:

$$MdRAE(RE_{it}) = median(|RE_{it}|)$$

Instead of using relative errors, we can use the relative measures by computing the ratio of the corresponding measure for the ICA method to respect the PCA one:

$$RelMdRAE = \frac{MdRAE_{ICA}}{MdRAE_{PCA}} \tag{31}$$

In order to analyze the effect of increasing the number of components, we vary r from 1 to 5 to evaluate the forecasting performance of the model. The results are displayed in Table 12. For each forecast model, this table shows the mean average of the $RelMdRAE$ measured over the 19 stock returns.

Table 12: Mean average of the $RelMdRAE$ measured over the 19 stock returns

Number of Components (r)	FAST	JADE	SOBI	PCA
1	0.767	0.774	**0.741**	1
2	0.824	0.800	**0.760**	1
3	0.743	0.711	**0.686**	1
4	0.708	**0.667**	0.678	1
5	0.735	0.708	**0.690**	1

From Table 12, we conclude that the forecasting performance of GICA-GARCH model is better than the corresponding to O-GARCH one, independently of the ICA algorithms we use and of the number of factors we consider. SOBI gives the best forecasting overall results except when $r = 4$. However, note that for $r = 4$ the differences between JADE and SOBI are not significant. Results are specially surprising when $r = 1$. Remember that PCA considers only one component as the optimal number of factors, and its explained variability is higher than the corresponding to the first ICs. One would expect that, when $r = 1$, PCA would show the best forecasting performance, but this is not so: any of the ICA algorithms performs better than PCA. Note that because we are dealing with relative ratios, we cannot say anything about the performance of each algorithm when the number of factors increases.

[3]See Hyndman and Koehler (2006) for a complete revision of measures of forecast accuracy.

Evaluating the forecasting performance of the model using the Relative Geometric Mean Relative Absolute Error ($RelGMRAE$) gives similar results which are available from the authors upon request.

6 Concluding remarks

We have proposed a new framework for modelling multivariate volatilities. We have introduced the GICA-GARCH model that can be seen as an extension of the orthogonal factor GARCH models. The GICA-GARCH model assumes that the comovements of a vector of financial data are driven a few independent components which are estimated by ICA. The GICA-GARCH model allows to estimate a multivariate GARCH model using a small number of independent and conditionally heteroskedastic factors, which evolve according to univariate GARCH models. Then, our model gives a parsimonious representation of the data, and reduces the number of parameters to be estimated. Interestingly, the GICA-GARCH model can also be seen as a particular dynamic factor GARCH model. Note, however, that while in the dynamic factor GARCH model the covariance matrix of the noise is of full rank, the GICA-GARCH will have rank equal to $m - r$. Nevertheless, this assumption can be relaxed easily by assuming an additional measurement noise so that the covariance matrix is of full rank.

In this paper, we have used three ICA algorithms for estimating the ICs. Their performance have been tested on some simulation experiments, and we conclude that in all cases ICA methods performs better than PCA to estimate non-linear and/or heteroskedastic components. However, the results among the different ICA algorithms are mixed. For non-linear factors (conditionally heteroskedastic or not), all work well but when the factors are conditionally heteroskedastic and do not have dynamics in the mean, SOBI performs worse than both JADE and FastICA.

The GICA-GARCH seems to work better for forecasting the volatility of the financial stock returns than the O-GARCH model.

In the future, we intend to design an alternative procedure to sort the ICs and to choose the optimal number of factors. Also, comparing the forecasting performance of our model with other multivariate GARCH and extending the GICA-GARCH model to other applications, may be challenges for the future.

References

Alessi, L., Barigozzi, M., and Capasso, M. (2006). *Dynamic factor GARCH: Multivariate volatility forecast for a large number of series.* LEM Papers Series 2006/25. Laboratory of Economics and Management (LEM). Sant'Anna School of Advanced Studies. Pisa, Italy.

Alexander, C. (2001). *Market Models: A guide to financial data analysis.* Chichester, U.K.: Wiley.

Back, A. D. and Weigend, A. S. (1997). A first application of independent component analysis to extracting structure from stock returns. *International Journal of Neural Systems* 8, 473-484.

Bauwens, L., Laurent, S., and Rombouts, J. V. K. (2006). Multivariate GARCH models: a survey. *Journal of Applied Econometrics* 21, 79-109.

Belouchrani, A., Abed Meraim, K., Cardoso, J.-F., and Moulines, E. (1997). A blind source separation tecnique based on second order statistics. *IEEE Transactions on Signal Processing* 45, 434-444.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroscedasticity. *Journal of Econometrics* 31, 307-327.

Bollerslev, T., Engle, R. F., and Wooldridge, J. W. (1988). A capital asset pricing model with time-varying covariances. *Journal of Political Economy* 96, 116-131.

Cardoso, J.-F. and Souloumiac, A. (1993). Blind beamforming for non-gaussian signals. *IEE-Proceedings-F* 140, 362-370.

Cha, S.-M. and Chan, L.-W. (2000). Applying independent component analysis to factor model in finance. *IDEAL*, 538-544.

Comon, P. (1994). Independent component analysis - a new concept?. *Signal Processing* 36, 287-314.

Diebold, F. X. and Nerlove, M. (1989). The dynamics of exchange rate volatility: a multivariate latent factor ARCH model. *Journal of Applied Econometrics* 4, 1–21.

Engel, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica* 50, 987-1007.

Fan, J., Wang, M., and Yao, Q. (2008). Modelling multivariate volatilities via conditionally uncorrelated components. *Journal of The Royal Statistical Society: Series B (Statistical Methodology)* 70(4), 679-702.

Franses, Ph. H. and van Dijk, D. (1996). Forecasting stock market volatility using nonlinear GARCH models. *Journal of Forecasting* 15, 229-235.

Hyndman, R. J. and Koehler, A. B. (2006). Another look at measures of forecast accuracy. *International Journal of Forecasting* 22, 679-688.

Hyvärinen, A. (1999). Fast and robust fixed-point algorithms for independent component analysis. *IEEE. Transaction on Neural Networks* 10, 626-634.

Hyvärinen, A. and Kano, K. (2003). Independent component analysis for non-normal factor analysis. In *H. Yanai et al. (Ed.) New Developments in Psychometrics*, pp. 649-656. Tokyo: Springer Verlag.

Hyvärinen, A., Karhunen, J., and Oja, E. (2001). *Independent component analysis.* New York: Wiley.

Hyvärinen, A. and Oja, E. (1997). A fast fixed-point algorithm for independent component analysis. *Neural Computation* 9, 1483-1492.

Kiviluoto, K., and Oja, E. (1998). Independent component analysis for parallel financial time series. In *Proc. Int. Conf. on Neural Information Processing*, vol. 2, pp. 895-898. Tokyo.

Lanne, M. and Saikkonen, P. (2007). A multivariate generalized orthogonal factor GARCH model. *Journal of Business and Economic Statistics* 25, 61-75.

Malaroiu, S., Kiviluoto, K., and Oja, E. (2000). Time series prediction with independent

component analysis. *Technical Report*. Helsinki University of Technology. Helsinki.

Peña, D. and Box, G. (1987). Identify a simplifying structure in time series. *Journal of The American Statistical Association* 82, 836-843.

Stock, J. H. and Watson, M. W. (2002). Macroeconomic forecasting using diffusion indexes. *Journal of Business and Economic Statistics* 20, 147-162.

Tong, L., Liu, R.-W., Soon, V. C., Huang, Y.-F., and Liu, R. (1990). AMUSE: A new blind identification algorithm. *Proceedings of IEEE ISCAS*.

van der Weide, R. (2002). GO-GARCH: a multivariate generalized orthogonal GARCH model. *Journal of Applied Econometrics* 17, 549-564.

Wu, E. H. C. and Yu, P. L. H. (2005). Volatility modelling of multivariate financial time series by using ICA-GARCH models. *IDEAL*, 571-579.

Appendix

Components of the IBEX 35 from 2000 to 2004 classified by sectors

Consumption		
Other goods of consumption	ALT	Altadis
Consumption services		
Leisure time / Tourism / Hotel industry	AMS	Amadeus
	NHH	NH Hoteles
Mass media / Publicity	SGC	Sogecable
	TPI	Telefónica Publicidad e Información
Financial Services / Estate Agencies		
Banking	BBVA	Banco Bilbao Vizcaya Argentaria
	BKT	Bankinter
	POP	Banco Popular
	SAN	Banco Santander Central Hispano(*)
Oil and Energy		
Oil	REP	Repsol
Electricity and Gas	ELE	Endesa
	IBE	Iberdrola
Materials / Industry / Building		
Minerals / Metals	ACX	Acerinos
Building	ACS	Grupo ACS
	ANA	Acciona
	FCC	Fomento de Construcciones y Contratas S.A.
	FER	Grupo Ferrovial
Technology / Telecommunications		
Telecommunications and others	TEF	Telefónica
Electronic and Software	TPI	Indra

(*) From 01/01/2000 to 31/10/2001, its name was SCH.

Univariate GARCH models: Specification chosen for the components in the four estimation methods.

PCA	FastICA
$s_{1t}^P \sim GARCH(1,1)$	$s_{1t}^F \sim GARCH(1,1)$
$s_{2t}^P \sim AR(1)+GARCH(1,1)$	$s_{2t}^F \sim AR(2)+GARCH(1,1)$
$s_{3t}^P \sim AR(1)+GARCH(1,1)$	$s_{3t}^F \sim GARCH(1,1)$
$s_{4t}^P \sim AR(2)+ARCH(1)$	$s_{4t}^F \sim GARCH(2,2)$
$s_{5t}^P \sim GARCH(1,1)$	$s_{5t}^F \sim AR(1)+GARCH(1,1)$
JADE	SOBI
$s_{1t}^J \sim GARCH(1,1)$	$s_{1t}^S \sim MA(1)+GARCH(1,1)$
$s_{2t}^J \sim ARCH(1,1)$	$s_{2t}^S \sim ARMA(1,1)+GARCH(1,1)$
$s_{3t}^J \sim ARMA(1,1)+GARCH(1,1)$	$s_{3t}^S \sim GARCH(1,1)$
$s_{4t}^J \sim GARCH(1,1)$	$s_{4t}^S \sim MA(1)+GARCH(1,1)$
$s_{5t}^J \sim GARCH(1,1)$	$s_{5t}^S \sim AR(1)+GARCH(1,1)$