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An insight into RBF-FD approximations augmented with polynomials

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Abstract

Radial basis function-generated finite differences (RBF-FD) based on the combination of polyharmonic splines (PHS) with high degree polynomials have recently emerged as a powerful and robust numerical approach for the local interpolation and derivative approximation of functions over scattered node layouts. Among the key features, (i) high orders of accuracy can be achieved without the need of selecting a shape parameter or the issues related to numerical ill-conditioning, and (ii) the harmful edge effects associated to the use of high order polynomials (better known as Runge's phenomenon) can be overcome by simply increasing the stencil size for a fixed polynomial degree. The present study complements our previous results, providing an analytical insight into RBF-FD approximations augmented with polynomials. It is based on a closed-form expression for the interpolant, which reveals the mechanisms underlying these features, including the role of polynomials and RBFs in the interpolant, the approximation error, and the behavior of the cardinal functions near boundaries. Numerical examples are included for illustration.

1 Introduction

RBF-FD methods appeared in the early 2000s as a local alternative to global RBF methods for numerically solving partial differential equations (PDEs) on irregular domains [15, 16, 17]. It is essentially a generalization of the classical finite difference (FD) method to scattered node layouts, leading to highly sparse differentiation matrices (better-conditioned than in the global approach), that can achieve high-order algebraic convergence. Since its appearance, it has been successfully applied in a wide range of problems for the numerical solution of PDEs (see [8] for a complete list of references). However, some drawbacks (such as the hassle of determining a shape parameter or the associated ill-conditioning issue) have conditioned its use.

Recently, it has been found [1, 3, 5, 6] that the combination of polyharmonic splines (PHS) with high degree polynomials offers a fresh alternative to the use of infinitely smooth RBFs in this local approach. Among the main features,

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1. Augmenting the PHS RBF interpolant with polynomials overcomes the stagnation errors (these are errors failing to decrease to zero under continuing node refinement, also known as saturation errors). The convergence order under refinement is determined by the highest augmented polynomial degree, and not the RBFs.
2. RBF-FD stencils based on PHS augmented with high-degree polynomials can overcome the Runge phenomenon without any special treatment (such as ghost nodes) by simply increasing the stencil size for a fixed polynomial degree.

Different areas of application have already experienced the benefits of this approach (such as the treatment of interfaces between different medias [10, 11], the computation of quadrature formulas over smooth surfaces with boundaries [13], or the tracer transport on the sphere using a semi-Lagrangian scheme [14]).

The present study, which is a follow-up to [6] and [3], provides an insight into these features from a different perspective. The key result here is a closed-form expression for the RBF interpolant augmented with polynomials. This leads to analytic expressions for the polynomial and RBF coefficients under refinement, as well as the approximation error. In agreement with [6], it is shown that all Taylor terms of a target function up to the augmented degree are treated exactly by the supplementary polynomials, determining the convergence order of the approximation, while the RBF part acts only on the remainder.

Moreover, it leads to exact expressions for the cardinal functions of the interpolant. As it is shown, RBF approximations augmented with polynomials can be written as pure RBF approximations plus a correction term, which accounts for the polynomial reproduction of the combined interpolant. This correction mixes RBFs and polynomials, and its magnitude decreases for increasingly large stencil sizes. We show that this is the key mechanism underlying the behavior of cubic PHS combined with polynomials based RBF-FD stencils found in [3]. These results are also applied in [2] to analyze the behavior of the interpolant near boundaries focusing more on heuristic perspectives and numerical demonstrations.

The structure of the paper is as follows. In Section 2 we obtain a closed-form expression for the RBF interpolant augmented with polynomials, which is used along the paper. Section 3 focuses on the behavior of the interpolant on a local stencil, explaining the convergence rates under refinement found in [6]. The exact expression of the cardinal functions is obtained in Section 4. This is used to analyze the non-oscillatory behavior of the interpolant at the edges observed in [3] when increasing the stencil size for a fixed polynomial degree. This is illustrated through 1-D and 2-D numerical tests. Finally, the main conclusions are summarized in Section 5.

2 RBF interpolation augmented with polynomials

2.1 Formulation

Consider the scattered data interpolation problem: given the data f_i at location $\underline{x}_i \in \mathbb{R}^d$, $i = 1, 2, \dots, n$, find a continuous function $s(\underline{x})$ (interpolant) satisfying $s(\underline{x}_i) = f_i$, $i = 1, \dots, n$. In RBF methods, the standard interpolant is equal to

$$s(\underline{x}) = \sum_{j=1}^n \lambda_j \phi(\|\underline{x} - \underline{x}_j\|) + \sum_{k=1}^s \beta_k p_k(\underline{x}), \quad (1)$$

Table 1: Some commonly used radial basis functions.

Piecewise smooth RBFs	
Polyharmonic spline (PHS)	$r^m, m = 1, 3, 5, \dots$ $r^m \log(r), m = 2, 4, 6, \dots$
Compact Support (“Wendland”)	$(1 - \varepsilon r)_+^m p(\varepsilon r), p$ certain polynomials, $m \in \mathbb{N}$
Infinitely smooth RBFs	
Gaussian (GA)	$e^{-(\varepsilon r)^2}$
Multiquadric (MQ)	$\sqrt{1 + (\varepsilon r)^2}$
Inverse Quadratic (IQ)	$1/(1 + (\varepsilon r)^2)$
Inverse Multiquadric (IMQ)	$1/\sqrt{1 + (\varepsilon r)^2}$
Bessel (BE) ($d = 1, 2, \dots$)	$J_{d/2-1}(\varepsilon r)/(\varepsilon r)^{d/2-1}$

with the matching conditions

$$\sum_{j=1}^n \lambda_j p_k(\underline{x}_j) = 0, \quad k = 1, \dots, s, \quad (2)$$

where $\phi(\|\underline{x} - \underline{x}_j\|)$ is an RBF centered on $\underline{x}_j$, $\|\cdot\|$ is the standard euclidean distance, and $\{p_k(\underline{x})\}_{k=1}^s$ is a basis for the multivariate polynomial space Π_l^d of total degree l in d dimensions with $s = \binom{l+d}{l}$. Table 1 lists a few of the many available RBF choices.

The $\boldsymbol{\lambda}$ and $\boldsymbol{\beta}$ coefficients are determined by enforcing (1) to be exact on the collocation nodes, i.e. $s(\underline{x}_i) = f_i, i = 1, \dots, n$. This, together with (2), leads to the linear system of equations

$$\begin{bmatrix} A & P \\ P^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\beta} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix}, \quad (3)$$

where A is the square matrix with elements

$$A_{i,j} = \phi(\|\underline{x}_i - \underline{x}_j\|), \quad i, j = 1, \dots, n,$$

and P is the $n \times s$ matrix

$$P = \begin{bmatrix} p_1(\underline{x}_1) & p_2(\underline{x}_1) & \cdots & p_s(\underline{x}_1) \\ p_1(\underline{x}_2) & p_2(\underline{x}_2) & \cdots & p_s(\underline{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_1(\underline{x}_n) & p_2(\underline{x}_n) & \cdots & p_s(\underline{x}_n) \end{bmatrix}, \quad (4)$$

whose k -th column corresponds with the k -th element of the polynomial basis evaluated on the collocation nodes,

$$\mathbf{p}_k = [p_k(\underline{x}_1), p_k(\underline{x}_2), \dots, p_k(\underline{x}_n)]^T. \quad (5)$$

In the following, we will refer to this interpolant as *RBF+poly interpolant*. Depending on the RBF considered, the inclusion of polynomials up to a certain degree l may be necessary to guarantee a well-posed interpolation problem (for a rigorous discussion on unisolvency conditions see [4]).

2.2 Interpretation as a constrained optimization problem

RBF interpolation augmented with polynomials can be interpreted as a constrained optimization problem [18]. The linear system of equations (3) is the solution of the equality-constrained quadratic optimization problem

$$\min_{\boldsymbol{\lambda}} J(\boldsymbol{\lambda}) = \frac{1}{2} \boldsymbol{\lambda}^T A \boldsymbol{\lambda} - \boldsymbol{\lambda}^T \mathbf{f} \quad \text{subject to} \quad P^T \boldsymbol{\lambda} = \mathbf{0}. \quad (6)$$

To prove it, let Z denote the $n \times (n-s)$ matrix whose columns are a basis for the null space of P^T , i.e. all the elements $\mathbf{z} \neq 0 \in R^n$ satisfying $P^T \mathbf{z} = 0$. For positive definite RBFs, the A matrix is positive definite by definition. For strictly conditionally positive definite RBFs, such as PHS (which is emphasized in this work), the matrix $Z^T A Z$ is positive definite (see, for instance, [4, 12, 18]). Therefore, the problem is convex and has a unique solution. Constructing the Lagrangian

$$L(\boldsymbol{\lambda}, \boldsymbol{\beta}) = \frac{1}{2} \boldsymbol{\lambda}^T A \boldsymbol{\lambda} - \boldsymbol{\lambda}^T \mathbf{f} + \boldsymbol{\beta}^T P^T \boldsymbol{\lambda},$$

where $\boldsymbol{\beta}$ are the Lagrange multipliers, and enforcing $\nabla_{\boldsymbol{\lambda}, \boldsymbol{\beta}} L = \mathbf{0}$ yield the system of equations (3).

2.3 Exact expression for the RBF+poly interpolant

In this section we find the exact expression for the RBF+poly interpolant, which will be used in the rest of the paper to understand the role of polynomials in RBF-FD approximations.

Theorem 1. *Provided that A and P are full rank, the RBF+poly interpolant (1) is equal to*

$$s(\underline{x}) = \mathbf{f}^T [I - W P^T] \hat{\boldsymbol{\psi}}(\underline{x}) + \mathbf{f}^T W \mathbf{p}(\underline{x}), \quad (7)$$

where W is the $n \times s$ matrix

$$W = A^{-1} P (P^T A^{-1} P)^{-1}, \quad (8)$$

$\hat{\boldsymbol{\psi}}(\underline{x})$ is the RBF cardinal function without polynomial augmentation

$$\hat{\boldsymbol{\psi}}(\underline{x}) = A^{-1} \boldsymbol{\phi}(\underline{x}), \quad (9)$$

and the vectors

$$\boldsymbol{\phi}(\underline{x}) = [\phi(\|\underline{x} - \underline{x}_1\|), \phi(\|\underline{x} - \underline{x}_2\|), \dots, \phi(\|\underline{x} - \underline{x}_n\|)]^T$$

and

$$\mathbf{p}(\underline{x}) = [p_1(\underline{x}), p_2(\underline{x}), \dots, p_s(\underline{x})]^T,$$

contain a basis for the RBF space and polynomial space Π_l^d , respectively.

Proof. The exact solution for $\boldsymbol{\lambda}$ and $\boldsymbol{\beta}$ can be obtained using block decomposition in (3). For instance, multiplying the first row by $P^T A^{-1}$ yields

$$\begin{bmatrix} P^T & P^T A^{-1} P \\ P^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\beta} \end{bmatrix} = \begin{bmatrix} P^T A^{-1} \mathbf{f} \\ \mathbf{0} \end{bmatrix}.$$

Subtracting the two rows gives the values of the β coefficients,

$$\beta = (P^T A^{-1} P)^{-1} P^T A^{-1} \mathbf{f},$$

which substituted back in (3) yields the λ coefficients,

$$\lambda = \left[I - A^{-1} P (P^T A^{-1} P)^{-1} P^T \right] A^{-1} \mathbf{f},$$

where I is the $n \times n$ identity matrix. Therefore, the RBF interpolant (1) written as a function of the λ and β coefficients takes the form

$$s(\underline{x}) = \underbrace{\mathbf{f}^T \left[I - A^{-1} P (P^T A^{-1} P)^{-1} P^T \right] A^{-1}}_{\lambda^T} \phi(\underline{x}) + \underbrace{\mathbf{f}^T A^{-1} P (P^T A^{-1} P)^{-1}}_{\beta^T} \mathbf{p}(\underline{x}).$$

Equation (7) follows by simply defining $W = A^{-1} P (P^T A^{-1} P)^{-1}$. $\square$

This requires that both A and P are full rank matrices on the stencil nodes. Although it might seem restrictive, there are even more restrictive constraints in practical problems. For instance, when solving differential equations, not only the unisolvency of P , but some other features are controlled by the node distribution: the differentiation matrix (assembled with the RBF-FD weights) needs to fulfill a diagonal dominance criteria for the convergence of the iterative solvers for elliptic PDEs, or to verify a stability condition for the numerical time-step of parabolic/hyperbolic PDEs. In exchange, the nodes turn out to be smoothly scattered over the domain (instead of randomly placed) in such a way that the internodal distance is kept roughly constant within each stencil. On these node layouts, finding A and P to be singular is extremely unlikely.

With the above notation, the λ and β coefficients can be written as

$$\lambda^T = \mathbf{f}^T (I - W P^T) A^{-1} \quad (10)$$

and

$$\beta^T = \mathbf{f}^T W. \quad (11)$$

Observe that W plays a key role in the approximation (both λ and β coefficients depend on $\mathbf{f}^T W$). The following result illustrates its meaning.

Theorem 2. *Let $p^{(\alpha)}(\underline{x}) = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ be the m -th element of total degree $\alpha = \alpha_1 + \cdots + \alpha_d$ from the augmented polynomial basis. Then, the m -th column of matrix $W = A^{-1} P (P^T A^{-1} P)^{-1}$ contains the differentiation weights $\mathbf{w}^{(m)}$ approximating the operator*

$$L^{(\alpha)} = \frac{1}{\alpha_1! \cdots \alpha_d!} \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}. \quad (12)$$

Proof. When applying the differential operator of total degree $\beta = \beta_1 + \cdots + \beta_d$,

$$L^{(\beta)} = \frac{1}{\beta_1! \cdots \beta_d!} \frac{\partial^\beta}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}},$$

to the multivariate polynomial term $p^{(\alpha)}(\underline{x} - \underline{x}_0)$, it is verified that

$$L^{(\beta)} p^{(\alpha)}(\underline{x} - \underline{x}_0) \Big|_{\underline{x}=\underline{x}_0} = \begin{cases} 1 & \text{if } \alpha_i = \beta_i, \quad i = 1, \dots, d \\ 0 & \text{otherwise} \end{cases}. \quad (13)$$

Alternatively, the product of P^T with W is equal to

$$P^T W = (P^T A^{-1} P) (P^T A^{-1} P)^{-1} = I.$$

This is, the m -th row of P^T yields the m -th row of the identity matrix,

$$\mathbf{p}_m^T W = (\mathbf{p}_m^T A^{-1} P) (P^T A^{-1} P)^{-1} = [0, \dots, 0, \underset{\substack{\uparrow \\ m\text{-th} \\ \text{column}}}{1}, 0, \dots, 0] = \boldsymbol{\delta}_m^T. \quad (14)$$

Observe that this is the discretized form of equation (13), where the m -th column of W contains the weights $\mathbf{w}^{(m)}$ approximating the action of (12) which exactly differentiate $\mathbf{p}_m$. $\square$

Some trivial properties of the RBF+poly interpolant, such as the polynomial reproduction property or the matching conditions (2), can be obtained analytically using these results. Since they will be used along the paper, we state them below.

Proposition 1. *When the data to be interpolated comes from one of the augmented polynomial terms, i.e. $f_i = p_k(\underline{x}_i)$, for any $k = 1, 2, \dots, s$, the $\boldsymbol{\lambda}$ and $\boldsymbol{\beta}$ coefficients are equal to*

$$\boldsymbol{\lambda} = \mathbf{0} \quad \text{and} \quad \boldsymbol{\beta} = \boldsymbol{\delta}_k.$$

Proof. Using identity (14), the $\boldsymbol{\lambda}$ coefficients (10) satisfy

$$\begin{aligned} \boldsymbol{\lambda}^T &= \mathbf{p}_k^T A^{-1} - \mathbf{p}_k^T W P^T A^{-1} \\ &= \mathbf{p}_k^T A^{-1} - \boldsymbol{\delta}_k^T P^T A^{-1} \\ &= \mathbf{p}_k^T A^{-1} - \mathbf{p}_k^T A^{-1} \\ &= \mathbf{0}^T, \end{aligned}$$

and the $\boldsymbol{\beta}$ coefficients (11),

$$\boldsymbol{\beta}^T = \mathbf{p}_k^T W = \boldsymbol{\delta}_k^T.$$

$\square$

Observe that the interpolant reproduces exactly the polynomial term

$$s(\underline{x}) = \boldsymbol{\lambda}^T \cdot \boldsymbol{\phi}(\underline{x}) + \boldsymbol{\beta}^T \cdot \mathbf{p}(\underline{x}) = \mathbf{0}^T \cdot \boldsymbol{\phi}(\underline{x}) + \boldsymbol{\delta}_k^T \cdot \mathbf{p}(\underline{x}) = p_k(\underline{x}),$$

as it is otherwise expected since $p_k(\underline{x})$ is in the range of P and so, in the range of the collocation matrix (3). This is the so-called *polynomial reproduction property* of RBF interpolation augmented with polynomials [4, 8].

Proposition 2. *The λ coefficients (10) satisfy the matching conditions (2) for all the elements of $\{p_k(\underline{x})\}_{k=1}^s$.*

Proof. Substituting (10) into the matching conditions (2), leads to

$$\begin{aligned}
\lambda^T \cdot p_k &= \underbrace{f^T \left[I - A^{-1} P (P^T A^{-1} P)^{-1} P^T \right] A^{-1}}_{\lambda^T} p_k = f^T A^{-1} [I - P W^T] \cdot p_k \\
&= f^T A^{-1} [p_k - P W^T p_k] \\
&= f^T A^{-1} [p_k - P \delta_k] \\
&= f^T A^{-1} [p_k - p_k] \\
&= 0,
\end{aligned}$$

which is verified for $k = 1, \dots, s$. Note that (14) has been used. $\square$

This result has implications on the nature of the interpolant at the edges. For instance, it was found in [7] that cubic RBF interpolant without polynomial augmentation on the interval $[-1, 1]$ has the end-conditions

$$\begin{aligned}
s''(-1) &= s'(+1) - 2s'(-1) - \frac{3}{2}(s(+1) + s(-1)), \\
s''(+1) &= 2s'(+1) - s'(-1) - \frac{3}{2}(s(+1) + s(-1)).
\end{aligned}$$

However, by augmenting the interpolant up to linear terms (i.e. enforcing the two constraints $\sum \lambda_i = 0$ and $\sum \lambda_i x_i = 0$) it changes to the natural cubic spline with end-conditions $s''(\pm 1) = 0$.

3 Convergence of local RBF+poly interpolant

In our previous work [6], we studied numerically the convergence under refinement of the λ and β coefficients, as well as the approximation error, concluding some important observations regarding the role of polynomials in RBF-FD approximations. In this section, we derive a simple analysis to explain these features.

Theorem 3. *Let f be a smooth multivariable function. For a local RBF+poly interpolant (1) of f ,*

1. *The polynomial coefficients, which are the Lagrange multipliers of the underlying minimization problem (6), are equal to*

$$\beta_k = L_k[f(x_0)] + O(h^{l+1}), \quad k = 1, \dots, s,$$

where L_k is the differential operator (12).

2. *The RBF coefficients are*

$$\lambda_i = 0 + O(h^{l+1}), \quad i = 1, \dots, n.$$

Proof. Consider an infinitely smooth multivariable function f , which admits a Taylor series expansion in the neighborhood of a point $\underline{x}_0$, i.e.

$$f(\underline{x}) = \sum_{k=1}^{\infty} L_k[f(\underline{x}_0)] p_k(\underline{x} - \underline{x}_0),$$

where the multivariate polynomial term $p_k(\underline{x} - \underline{x}_0)$ and the differential operator L_k are defined as in Theorem 2. Evaluation on the stencil nodes leads to the vector form

$$\mathbf{f} = \sum_{k=1}^{\infty} L_k[f(\underline{x}_0)] \mathbf{p}_k,$$

where $\mathbf{f}^T = [f(\underline{x}_1), f(\underline{x}_2), \dots, f(\underline{x}_n)]$ and $\mathbf{p}_k^T = [p_k(\underline{x}_1 - \underline{x}_0), p_k(\underline{x}_2 - \underline{x}_0), \dots, p_k(\underline{x}_n - \underline{x}_0)]$. It can be separated in two parts: one over the augmented polynomial terms and the other over the remainder terms,

$$\mathbf{f} = \underbrace{\sum_{k=1}^s L_k[f(\underline{x}_0)] \mathbf{p}_k}_{\text{Augmented polynomial terms}} + \underbrace{\sum_{k=s+1}^{\infty} L_k[f(\underline{x}_0)] \mathbf{p}_k}_{\text{Higher degree terms}}.$$

Substitution into the $\boldsymbol{\lambda}$ and $\boldsymbol{\beta}$ coefficients (10) and (11), followed by the application of Proposition 1 over the augmented polynomial terms, yields

$$\boldsymbol{\lambda}^T = \mathbf{0}^T + \sum_{k=s+1}^{\infty} L_k[f(\underline{x}_0)] [\mathbf{p}_k^T - \mathbf{p}_k^T W P^T] A^{-1}, \quad (15)$$

and

$$\boldsymbol{\beta}^T = [L_1[f(\underline{x}_0)], L_2[f(\underline{x}_0)], \dots, L_s[f(\underline{x}_0)]] + \sum_{k=s+1}^{\infty} L_k[f(\underline{x}_0)] \mathbf{p}_k^T W, \quad (16)$$

where the remaining terms of the Taylor series can be bound, behaving as $O(h^{l+1})$. $\square$

The approximation error of the local RBF+poly interpolant follows from this result. By simply substituting the $\boldsymbol{\lambda}$ and $\boldsymbol{\beta}$ coefficients above into (1), it is obtained the exact expression for the interpolant

$$s(\underline{x}) = \sum_{k=1}^s L_k[f(\underline{x}_0)] p_k(\underline{x} - \underline{x}_0) + \sum_{k=s+1}^{\infty} L_k[f(\underline{x}_0)] \left[\mathbf{p}_k^T \cdot \hat{\boldsymbol{\psi}}(\underline{x}) - \mathbf{p}_k^T W \left(P^T \hat{\boldsymbol{\psi}}(\underline{x}) - \mathbf{p}(\underline{x} - \underline{x}_0) \right) \right],$$

leading to the approximation error

$$s(\underline{x}) - f(\underline{x}) = \sum_{k=s+1}^{\infty} L_k[f(\underline{x}_0)] \left[\left(\mathbf{p}_k^T \cdot \hat{\boldsymbol{\psi}}(\underline{x}) - p_k(\underline{x} - \underline{x}_0) \right) - \mathbf{p}_k^T W \left(P^T \hat{\boldsymbol{\psi}}(\underline{x}) - \mathbf{p}(\underline{x} - \underline{x}_0) \right) \right], \quad (17)$$

which can be bounded over the stencil as

$$\|s(\underline{x}) - f(\underline{x})\|_{\infty} \leq Ch^{l+1} \max_{\underline{x} \in \Omega} |L_k[f(\underline{x})]|.$$

Observe that the convergence order is determined by the degree l of augmented polynomial terms in agreement with [6]. The RBFs act on the residual of the approximation through the terms

$$\tau_k(\underline{x}) = \mathbf{p}_k^T \cdot \hat{\psi}(\underline{x}) - p_k(\underline{x} - \underline{x}_0) = \sum_{i=1}^n p_k(\underline{x}_i - \underline{x}_0) \hat{\psi}_i(\underline{x}) - p_k(\underline{x} - \underline{x}_0), \quad k = 1, 2, \dots \quad (18)$$

This introduces an opportunity to improve the accuracy since $\mathbf{p}_k^T \cdot \hat{\psi}(\underline{x})$ reproduces $p_k(\underline{x} - \underline{x}_0)$ more accurate as n increases. These results are closely aligned with the arguments in [6] based on general assumptions. All Taylor terms of a target function f up through degree l are treated exactly by the supplementary polynomials, while the RBFs are left to deal on the remainder, implying an approximation error $O(h^{l+1})$.

3.1 Connection with numerical results

Consider the numerical experiment from Section 5 in [6] (also described in Section 5.15 in [8]). This is, the numerical approximation of function

$$f(x, y) = 1 + \sin(4x) + \cos(3x) + \sin(2y), \quad (19)$$

over a disc of radius R discretized with n minimum energy-like nodes using PHS augmented with polynomials. In this section, the numerical results obtained in [6] regarding the convergence under refinement of the $\boldsymbol{\lambda}$ and $\boldsymbol{\beta}$ coefficients, as well as the approximation error, are explained using the analysis above.

3.1.1 Convergence as a function of the internodal distance h

Figures 8 and 9 in [6] (Figure 8 is displayed as Figure 1 in the present manuscript) shows the convergence of the approximation error, and the $\boldsymbol{\lambda}$ and $\boldsymbol{\beta}$ coefficients, when function (19) is approximated using PHS and GA, respectively. In the following, we list each of the conclusions from [6], and provide a connection with our analysis.

1. The $\boldsymbol{\lambda}$ coefficients behave as $O(h^{l+1})$, where l is the maximum degree of augmented polynomial terms and h is the internodal distance (R in the figure).

This agrees with Theorem 3, Part 2.

2. The $\boldsymbol{\beta}$ coefficients of the augmented polynomials behave as $O(1)$ for constant terms, $O(h)$ for linear terms, $O(h^2)$ for quadratic terms, and so on.

Theorem 3, Part 1, states that the $\boldsymbol{\beta}$ coefficients are the coefficients of the Taylor series expansion of function (19) with an error $O(h^{l+1})$. In this experiment, the coordinates are scaled as $x' = Rx$ and $y' = Ry$, where R is a scaling factor controlling the smoothness of the function. As a result,

Figure 1: Same as Fig. 8 in [6]. Behavior of the RBFs and polynomial coefficients under refinement, where the latter have been separated according to polynomial degree. For the RBFs r^3 , r^5 , r^7 , and r^9 , the condition numbers of the collocation matrix are 3×10^4 , 3×10^4 , 3×10^4 , and 3×10^4 , respectively. In every case, the RBF coefficients converge to zero at the same rate as the interpolation error.

Figure 2: Left: 56 minimum energy-like nodes discretizing the unit disc. The smaller dots around the origin within a disc of radius 0.2 are the evaluation points. This node distribution is scaled by a parameter R , controlling the smoothness of the function. Right: error under refinement of the β coefficients at the evaluation points using PHS r^5 augmented with 4th degree polynomials. They have been separated according to its polynomial degree. Observe that in every case, the RBF coefficients, the error of the β coefficients, and the interpolation error converge to zero as $O(R^5)$.

the β coefficients up to quadratic terms are equal to

$$\begin{aligned}
\beta_1 &= f(x_0, y_0) = 1 + \sin(4Rx_0) + \cos(3Rx_0) + \sin(2Ry_0), \\
\beta_2 &= \frac{\partial f}{\partial x}(x_0, y_0) = 4R \cos(4Rx_0) - 3R \sin(3Rx_0) \\
\beta_3 &= \frac{\partial f}{\partial y}(x_0, y_0) = 2R \cos(2Ry_0) \\
\beta_4 &= \frac{1}{2!} \frac{\partial^2 f}{\partial x^2}(x_0, y_0) = -9/2 R^2 \cos(3Rx_0) - 8R^2 \sin(4Rx_0) \\
\beta_5 &= \frac{1}{2!} \frac{\partial^2 f}{\partial y^2}(x_0, y_0) = -2R^2 \sin(2Ry_0) \\
\beta_6 &= \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = 0
\end{aligned} \tag{20}$$

behaving as $\|\beta\| = O(1)$ for constant terms (where $\beta = \beta_1$); $\|\beta\| = O(R)$ for linear terms (where $\beta = \{\beta_j\}_{j=2}^3$); $\|\beta\| = O(R^2)$ for quadratic terms (where $\beta = \{\beta_j\}_{j=4}^6$); and so on; in agreement with [6].

To verify numerically that the residual is actually $O(h^{l+1})$, we have repeated the same experiment and computed the difference between the β coefficients (20) and the corresponding numerical values under refinement when using PHS r^5 augmented with 4th degree polynomials. The disc of radius R (which in this case has the role of the internodal distance h) is discretized using 56 minimum energy-like nodes, and the error of the β coefficients is evaluated at different locations within a disc of radius 0.2. Figure 2 displays the collocation and evaluation points (left) and the error of the β coefficients as a function of R (right), which have been separated according to polynomial degree. In every case, the error of the β coefficients (but also the RBF coefficients and the interpolation error) converges to zero at the expected rate $O(R^5)$.

Figure 3: Numerical error (solid) and analytically predicted error (17) (dashed) under refinement, when function (19) is approximated on the disc of radius R shown in Figure 2 at $(x_0, y_0) = (0.1, 0.15)$, using PHS r^7 augmented with different polynomial terms, from constants up to 8th degree polynomials.

Figure 14: Convergence comparison for approximation of the test function in (12). In all cases, polynomials up to degree two are included, and in the RBF-FD cases PHS $\phi(r) = r^5$ are used. Note that increasing the stencil size has no effect on the polynomial least squares error (the three least-square error curves are indistinguishable in the figure.) Including RBFs with the polynomials does not change the rate of convergence, but greatly improves the error for each value of R as the stencil is increased, consistent with the low polynomial degree curves in Figure 12.

Figure 4: Same as Fig. 14 in [6]: convergence comparison for approximation of the test function in (19). In all cases, polynomials up to degree two are included, and PHS $\phi(r) = r^5$ are used.

3. The convergence order of the RBF+poly approximation is $O(h^{l+1})$, being determined by the maximum degree l of polynomial terms included.

One of the implications of Theorem 3 is the formula for the approximation error (17), whose convergence is $O(h^{l+1})$. To verify it numerically, function (19) is approximated under refinement at one arbitrary evaluation point (for instance, $(x_0, y_0) = (0.1, 0.15)$) over the stencil from Figure 2. Figure 3 displays the numerical errors (solid) obtained when using PHS r^7 augmented with increasingly high polynomial degrees, from constants up to 8th degree polynomials. The corresponding analytical errors for each degree term (dashed) are also displayed in the figure. Observe that the analytical errors match the numerical errors accurately up to machine precision as R is refined.

3.1.2 Convergence as a function of the stencil size n

Figure 14 in [6] (displayed here as Figure 4) shows the error when using PHS r^5 augmented with 2nd degree polynomials to approximate a function with increasingly large stencil sizes n and compares

Figure 5: Follow-up to Figure 4 using the collocation nodes displayed in Figure 2. (a) Convergence under refinement for $n = 10, 28, 55$: numerical error (solid), analytical error (dashed), fixed resolution $R = 0.05$ (dotted). (b) Convergence as a function of the stencil size n for $R = 0.05$: numerical error (solid); $\|\tau\|_2$ (dash-dotted), where $\tau = \{\tau_k\}_{k=1}^{10}$.

its performance with polynomial least-squares approximations. In this section, we will focus on understanding the mechanism that controls the improvement in the accuracy as a function of the stencil size n , specially for low degree polynomials.

The convergence as a function of the stencil size n can be explained using the approximation error (17). Observe it depends on how pure RBFs approximate polynomials (18). RBF approximations do not reproduce polynomials exactly unless they are augmented with polynomials appropriately. However, it becomes more accurate as the stencil size (or, equivalently, the number of RBF centers) increases [4, 8].

The same numerical experiment as in Figure 14 from [6] (i.e. approximation of function (19) using PHS r^5 combined with 2nd degree polynomials over a disc of radius R centered at the origin) is repeated here on the stencil displayed in Figure 2. The ℓ_2 -norm of both, the numerical errors and the analytical predicted errors (17), are computed under refinement. In this case, it is obtained the results displayed in Figure 5 (a) for $n = 10, 28, 55$. Notice the analytical errors (dashed) reproduce accurately the numerical errors (solid) for all the stencil sizes n as R decreases.

Alternatively, by fixing the resolution to $R = 0.05$ (dotted line in Figure 5 (a)) and computing the error as a function of the stencil size n , it is obtained the solid curve displayed in Figure 5 (b). Observe in (17) that the approximation error depends to leading order on how RBFs approximate polynomials, in this case up to 3rd degree. If τ_k represents the residual (18) for the k -th polynomial term, Figure 5 (b) displays $\|\tau\|_2$ (dashed curve) for $\tau = \{\tau_k\}_{k=1}^{10}$. Notice that both converge as a function of n at the same rate.

In order to get a deeper insight into RBF approximations - and since it will be otherwise useful in the following section - consider the experiment from Figure 10 in [6]. This is, the RBF approximation of polynomial terms on the interval $[-1, 1]$. In this experiment, we use PHS r^3 to approximate constant terms for the stencil sizes $n = 8, 10, 16, 32, 64$. With the notation (18), this is written as

$$\tau_1(x) = \sum_{i=1}^n p_1(x_i) \hat{\psi}_i(x) - p_1(x).$$

Figure 6: Numerical error when using PHS r^3 to approximate polynomials on the interval $[-1, 1]$. (a) Absolute value for constants. In our notation, $\tau_1(x) = \sum_{i=1}^n p_1(x_i) \hat{\psi}_i(x) - p_1(x)$, where $p_1(x) = 1$ and $n = 8, 16, 32, 64$. The errors decrease towards the interval center following the theoretical rates predicted in [6] (dashed). (b) $||\tau_k(x)||_\infty$ as a function of the stencil size n when approximating polynomials, from constants up to 7th degree.

Figure 6 (a) displays the corresponding absolute values of $\tau_1(x)$ in $[-1, 1]$. In all cases, the maximum error is reached at the edges, decreasing towards the interior. As mentioned in [6], this is due to the non-intuitive end-conditions of the RBF interpolant, which produce a non-constant interpolant with severe errors near the edge of the domain. These large edge errors oscillate and decay as $O(0.2679^{|k|})$ for r^3 (dashed), when k steps (nodes) are taken from the stencil edge towards the interior. Observe also that the errors at the edges decrease as a function of n (as otherwise expected since RBF interpolants reproduce polynomials more accurately for increasingly large stencil sizes). Figure 6 (b) shows the corresponding ℓ_∞ -norm of the residual $\tau_k(x)$ against n when augmenting the interpolant with polynomials (from constants up to 7th degree). The largest amplitude decreases as $O(1/n^2)$, which will be important when studying the edge oscillations of the cardinal functions in Section 4.

4 Cardinal functions and RBF-FD weights

In [3] it is found that the combination of PHS RBFs with supplementary polynomials leads to accurate approximations without edge oscillations for increasing values of the stencil size n . This observation, although useful to overcome the harmful effects of near-one-sidedness at the edges, is counter intuitive since larger stencils might be expected to suffer more than smaller ones. The purpose of this section is to provide an insight into this feature, which will be useful when studying the behavior of PHS+poly interpolants near domain boundaries in [2].

4.1 Cardinal functions

Consider the interpolation problem described in Section 2.1. An interpolant $s(\underline{x})$ can be expanded as a linear combination of the cardinal functions $\psi_i(\underline{x})$ as

$$s(\underline{x}) = \sum_{i=1}^n f(\underline{x}_i) \psi_i(\underline{x}),$$

where the cardinal conditions $\psi_i(\underline{x}_j) = \delta_{ij}$, $i, j = 1, \dots, n$, are satisfied on the collocation nodes. This observation leads to the following result:

Corollary 1. *The cardinal functions for the RBF+poly interpolant (1) are*

$$\psi(\underline{x}) = \hat{\psi}(\underline{x}) - W \left[P^T \hat{\psi}(\underline{x}) - \mathbf{p}(\underline{x}) \right]. \quad (21)$$

Proof. The result follows from (7) by simple inspection. $\square$

As discussed in the previous section, the RBF cardinal functions $\hat{\psi}(\underline{x})$ do not reproduce polynomials exactly, but approximate them with an error $\tau_k(\underline{x})$,

$$\mathbf{p}_k^T \cdot \hat{\psi}(\underline{x}) = \sum_{i=1}^n p_k(\underline{x}_i) \hat{\psi}_i(\underline{x}) = p_k(\underline{x}) + \tau_k(\underline{x}), \quad k = 1, 2, \dots$$

where $\mathbf{p}_k^T = [p_k(\underline{x}_1), p_k(\underline{x}_2), \dots, p_k(\underline{x}_n)]$ is the k -th row of matrix P^T . Writing the term $P^T \hat{\psi}(\underline{x}) - \mathbf{p}(\underline{x})$ as the residual

$$\boldsymbol{\tau}(\underline{x}) = [\tau_1(\underline{x}), \dots, \tau_s(\underline{x})]^T = \left[P^T \hat{\psi}(\underline{x}) - \mathbf{p}(\underline{x}) \right],$$

the RBF+poly cardinal functions (21) take the form

$$\psi(\underline{x}) = \hat{\psi}(\underline{x}) - W \boldsymbol{\tau}(\underline{x}). \quad (22)$$

If the data comes from a polynomial $p_k(\underline{x})$ up to degree l , the RBF+poly interpolant is equal to

$$\mathbf{p}_k^T \cdot \psi(\underline{x}) = \mathbf{p}_k^T \cdot \hat{\psi}(\underline{x}) - \mathbf{p}_k^T W \boldsymbol{\tau}(\underline{x}).$$

The term $-W \boldsymbol{\tau}(\underline{x})$, which contains both RBFs and polynomial terms and is zero at all data points, behaves as a ‘‘correction’’ term. It accounts for the error of $\hat{\psi}(\underline{x})$ when reproducing polynomials,

$$-\mathbf{p}_k^T W \boldsymbol{\tau}(\underline{x}) = -\boldsymbol{\delta}_k^T \cdot \boldsymbol{\tau}(\underline{x}) = -\tau_k(\underline{x}),$$

in such a way that $\psi(\underline{x})$ becomes exact,

$$\mathbf{p}_k^T \cdot \psi(\underline{x}) = \mathbf{p}_k^T \cdot \hat{\psi}(\underline{x}) - \mathbf{p}_k^T W \boldsymbol{\tau}(\underline{x}) = p_k(\underline{x}).$$

From the perspective of the minimization problem (6), we are *optimizing* RBF cardinal functions $\hat{\psi}(\underline{x})$, by ensuring that the polynomial reproduction property is satisfied through the addition of a correction term ‘ $-W \boldsymbol{\tau}(\underline{x})$ ’ obtained via Lagrange multipliers.

The difference between RBF+poly cardinal functions and pure RBF cardinal functions is, therefore, proportional to the correction term

$$\psi(\underline{x}) - \hat{\psi}(\underline{x}) = -W \boldsymbol{\tau}(\underline{x}) = -\sum_{k=1}^s \mathbf{w}^{(k)} \tau_k(\underline{x}),$$

where $\mathbf{w}^{(k)}$ are the differentiation weights contained in the columns of W . This can be bound as

$$\|\psi(\underline{x}) - \hat{\psi}(\underline{x})\|_\infty \leq \left[\sum_{k=1}^s \|\mathbf{w}^{(k)}\|_\infty \right] \max_{\underline{x} \in \Omega} \|\boldsymbol{\tau}(\underline{x})\|_\infty, \quad (23)$$

which depends on both, the differentiation weights $\mathbf{w}^{(k)}$ and the polynomial approximation error of pure RBFs $\boldsymbol{\tau}(\underline{x})$. This observation will be useful in Section 4.3 to explain the behavior of cubic PHS+poly approximations as a function of the stencil size (for a fixed polynomial degree l) observed in [3]. It will be also the mechanism underlying the behavior of PHS+poly interpolants near domain boundaries analyzed in [2].

4.2 RBF-FD weights

The RBF-FD weights $\mathbf{w}$ approximating a linear differential operator L at $\underline{x}_0$ are obtained as derivatives of the cardinal functions $\psi(\underline{x})$,

$$\mathbf{w} = L\psi(\underline{x}_0) = L\hat{\psi}(\underline{x}_0) - WL\tau(\underline{x}_0). \quad (24)$$

These are the combination of both, pure RBF weights $L\hat{\psi}(\underline{x}_0)$ and a correction term $-WL\tau(\underline{x}_0)$, which ensures the specified order of accuracy.

When the stencil size n is equal to the number of polynomial terms $s = \binom{l+d}{l}$, the P matrix is square. In this case,

$$W = A^{-1}P(P^T A^{-1}P)^{-1} = (P^T)^{-1},$$

and the RBF-FD weights (24) become the classical polynomial weights over scattered nodes

$$\mathbf{w} = L\psi(\underline{x}_0) = L\hat{\psi}(\underline{x}_0) - (P^T)^{-1} \left[P^T L\hat{\psi}(\underline{x}_0) - L\mathbf{p}(\underline{x}_0) \right] = (P^T)^{-1} L\mathbf{p}(\underline{x}_0).$$

This is well-posed as long as P is unisolvent (notice this is equivalent to standard FD if the nodes are grid based), but suffers from Runge's phenomenon when using high-degree polynomials.

Increasing the stencil size leads to approximations based on both RBFs and polynomials. The correction term required to enforce the specified order of accuracy in (22) decreases as a function of n . As a result, $\psi(\underline{x})$ converges to $\hat{\psi}(\underline{x})$ as given by (23). If $\hat{\psi}(\underline{x})$ behaves well, this is inherited by $\psi(\underline{x})$. We will show numerically in the next section that this is the mechanism controlling the recovery of the pattern of weights for the approximation of the second derivative observed in [3], where the weights of $L\psi(\underline{x}_0)$ recover the structure $[1, -2, 1]$ when using cubic PHS combined with 7th degree polynomials.

4.3 Connection with numerical results

Consider the 1-D numerical experiment from Section 4.2 in [3], where the cardinal functions of cubic PHS combined with 7th degree polynomials over the stencil $x = -1, 0, 1, 2, 3, \dots, n-2$ are displayed for $n = 8, 10$ and 16 , together with the corresponding RBF-FD weights approximating the second order derivative at $x_0 = 0$. The main conclusion of this experiment is that the harmful edge effects related to the use of high degree polynomials can be bypassed by simply increasing the stencil size for a fixed polynomial degree, recovering both stability and accuracy. The goal of this section is to explain these features using the analysis above.

The left column in Figure 7 displays the same cardinal functions as Figure 5 in [3] when using cubic PHS combined with 7th degree polynomials for $n = 8, 10, 16$ (rows). The RBF-FD weights approximating the second derivate at $x_0 = 0$, which are computed as derivatives of the cardinal functions (i.e. $w_i = \psi_i''(0)$), are also displayed in the figure. They are split according to equation (22) into pure PHS cardinal function $\hat{\psi}(x)$ (middle column) and correction term $-W\tau(x)$ (right column).

For $n = 8$, the cubic PHS (middle column) displays nice locality properties. However, for such a small stencil size, the error when approximating polynomials is very large - specially at the edges - as observed in Figure 6. When combined with polynomials, it requires a proportionally large

Figure 7: PHS+poly cardinal functions (left column), together with the corresponding RBF-FD weights approximating the second order derivative at $x_0 = 0$ (as displayed in Figure 5 of [3]), when using PHS r^3 augmented with 7th degree polynomials over the 1-D stencil $x = -1, 0, 1, 2, 3, \dots, n-2$, for $n = 8, 10, 16$ (rows). According to equation (22), they are split into the contribution of pure PHS cardinal functions $\hat{\psi}(x)$ (middle column) and the correction term $-W\tau(x)$ (right column). Note that the correction term ensures the specified order of accuracy in PHS+poly approximations.

Figure 8: $\|\psi(x_0) - \hat{\psi}(x_0)\|_\infty$ as a function of n at half a step in from each of the two boundaries ($x_0 = -0.5$ and $x_0 = n - 2.5$), and also from the stencil center ($x_0 = (n - 2)/2 + 0.5$).

correction term with large edge oscillations to fulfill the polynomial reproduction property (right column). The resulting RBF+poly cardinal functions (left column) inherit these oscillations, and the RBF-FD weights are large and oscillatory well into the stencil. These RBF-FD weights would potentially lead to non-diagonally dominant differentiation matrices.

Pure RBF approximations require a lower correction to reproduce polynomials exactly for increasingly large stencil sizes. This is observed numerically in Figure 7, where the entries of the correction term $-W\tau(x)$ decrease as a function of n . As a result, $\psi(x)$ and $\hat{\psi}(x)$ becomes similar, and the pattern of RBF-FD weights $\psi''(0)$ inherits the structure of $\hat{\psi}''(0)$. In this case, we have used r^3 , which is entirely free of edge oscillations (as pointed out in [7, 9]). Hence, the weights recover the pattern $[1, -2, 1]$ centered at the evaluation point, which would potentially lead to differentiation matrices well suited for numerical solvers in the context of solving elliptic PDEs (preserving, for instance, the diagonal dominance).

The convergence rate of $\|\psi(x_0) - \hat{\psi}(x_0)\|_\infty$ as a function of n is computed numerically at half a step in from each of the two boundaries (i.e. $x_0 = -0.5$ and $x_0 = n - 2.5$), and also from the stencil center ($x_0 = (n - 2)/2 + 0.5$). The results are displayed in Figure 8. Observe that $\|\psi(x_0) - \hat{\psi}(x_0)\|_\infty$ is equivalent at both $x_0 = 0.5$ and $x_0 = n - 2.5$ (as otherwise expected because of the symmetry of the cardinal functions), converging at a rate $O(1/n)$. Observe also that the convergence rate is much faster near the stencil center at $x_0 = (n - 2)/2 + 0.5$, in agreement with the results discussed in the previous section (see Figure 6).

When deriving equation (22), no assumptions were made regarding the space dimension. Therefore, this behavior is expected to hold in higher dimensions. To illustrate it, consider the numerical experiment from [2], where PHS+poly approximations are analyzed over different 2-D stencils. Figure 9 displays the stencil used here for $n = 31$. In this case, we have computed the cardinal functions using cubic PHS augmented with 4th degree polynomials, as well as the corresponding RBF-FD weights approximating the Laplacian at $(0, 0)$.

Figure 10 shows one of the PHS+poly cardinal functions $\psi_i(x, y)$ (left column) for $n = 16, 21, 31$ (rows). These are split into pure PHS cardinal functions $\hat{\psi}_i(x, y)$ (middle column) and correction term $-(W\tau(x, y))_i$ (right column). Observe that for $n = \binom{4+2}{4} + 1 = 16$, the correction term is large and oscillatory, and both $\psi_i(x, y)$ and $\hat{\psi}_i(x, y)$ differ significantly. However, as the stencil size

Figure 9: 2-D scattered node distribution, displaying the largest stencil used here for $n = 31$.

increases, $-(W \tau(x, y))_i$ becomes nearly zero, and $\psi_i(x, y)$ converges to $\hat{\psi}_i(x, y)$.

In Figure 11 it is depicted the RBF-FD weights that approximate the Laplacian at $(0, 0)$ for $n = 16, 21, 31$ (rows), separated as well into the contribution of pure cubic PHS weights $\Delta \hat{\psi}(0, 0)$ (middle column) and correction term $-W \Delta \tau(0, 0)$ (right column). Note that the weights in the right column ensures the specified order of accuracy when added to the ones in the middle column. As in the 1-D case, the pattern of RBF-FD weights $\Delta \psi(0, 0)$ converges to $\Delta \hat{\psi}(0, 0)$ as $-W \Delta \tau(0, 0)$ decreases for increasingly large stencil sizes.

In this case, the convergence rate of $\|\psi(x_0) - \hat{\psi}(x_0)\|_\infty$ as a function of n is also computed at an arbitrary point near the stencil center (for instance, $\underline{x}_0 = (-0.15, -0.15)$), together with the maximum difference over the whole stencil. The results are plotted in Figure 12. As in the 1-D case, the lowest convergence rate is $O(1/n)$, achieved near the edges. Around the stencil center, the convergence is much faster.

5 Conclusions

In this study, we have presented an insight into the mechanisms underlying the features previously observed in [6] and [3]. The analysis is based on a closed-form expression for the interpolant, which allows to analyze the convergence of PHS interpolants combined with high-degree polynomials and the nature of the cardinal functions.

In particular, we have proven analytically that when a smooth function f is locally approximated by an RBF+poly interpolant, the polynomial coefficients are actually the coefficients of the Taylor series expansions of f up to the augmented degree l , with an error $O(h^{l+1})$, while the coefficients of the RBFs are 0 up to $O(h^{l+1})$. Therefore, the target function is approximated by the polynomials and the RBFs act only on the remainder, leading to a convergence rate $O(h^{l+1})$. This agrees closely with our previous results in [6] based on general arguments.

We have also obtained an exact expression for the RBF+poly cardinal functions, which shows that the interpolants can be decomposed into pure RBF interpolants plus a correction term. Such a correction depends on the polynomial approximation error of the RBFs, and ensures the order of accuracy of the approximation. It is found to decrease in size as a function of the stencil size n (actually, at a numerically observed rate $O(1/n)$). The RBF+poly approximations become then dominated by the RBF part, explaining why when using cubic PHS (which behaves well also at the boundaries) the Runge phenomenon is removed and high order of accuracy is maintained (in the sense of being exact for polynomials up to specified degree) as found in [3] and studied further in [2].

Figure 10: A 2-D RBF+poly cardinal function $\psi_i(x, y)$ (left column) computed over the node distribution from Figure 9, using cubic PHS augmented with 4th degree polynomials and stencil sizes $n = 16, 21, 31$ (rows). As in the previous 1-D case, they are split into the contribution of pure PHS cardinal functions $\hat{\psi}_i(x, y)$ (middle column) and the corresponding correction term $-(W \tau(x, y))_i$ (right column). Note that the correction term ensures, when added to the pure PHS cardinal function, the specified order of accuracy of the RBF+poly cardinal function.

Figure 11: RBF-FD weights (left column) approximating the Laplacian at $(0,0)$ - computed as derivatives of the cardinal functions over the 2-D node distribution from Figure 9 - using cubic PHS augmented with 4th degree polynomials for stencil sizes $n = 16, 21, 31$ (rows). Blue corresponds to negative differentiation weights and yellow to positive. They are split as well into the contribution of pure PHS weights $\Delta\hat{\psi}(0,0)$ (middle column) and the correction term $-W \Delta\tau(0,0)$ (right column). Note again that the weights in the right column ensures, when added to the ones in the middle column, the specified order of accuracy of the RBF-FD weights.

Figure 12: $\|\psi(\underline{x}_0) - \hat{\psi}(\underline{x}_0)\|_\infty$ as a function of the stencil size n at $\underline{x}_0 = (-0.15, -0.15)$ (circle), and maximum value over stencil (triangle).

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