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# Vibrations of Bernoulli-Euler beams using the two-phase nonlocal elasticity theory

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**Abstract**

In this work the problem of the in-plane free vibrations (axial and bending) of a Bernoulli-Euler nanobeam using the mixed local/nonlocal Eringen elasticity theory is studied. The natural frequencies of vibration have been analytically obtained solving two uncoupled integro-differential eigenvalue problems, which are properly transformed in differential eigenvalue problems. Different kinds of end supports have been considered, and the influence of both mixture parameter and length scale has been analysed. The results show the softening effect of the Eringen's nonlocality, which is more pronounced as the local phase fraction decreases.

A large number of papers devoted to the dynamics of Bernoulli-Euler beams considering the fully nonlocal Eringen elasticity theory has been previously published. However, as recently stated by Romano et al. (2017), the problem is ill-posed in general, and the existence of a solution is an exception, the rule being non-existence. Nevertheless, the presence of a local term in the constitutive equation, leading to the two-phase formulation, renders the problem well-posed. To the best knowledge of the authors, this is the first time an exact solution is presented for a dynamic problem involving structures with constitutive equations corresponding to nonlocal integral Eringen's elasticity.

*Keywords:* mixed local/nonlocal Eringen elasticity theory, dynamic, axial vibrations, bending vibrations, nanobeams

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## 1. Introduction

The explosive growth of the nanotechnology and of the applications in the field of nanostructures has soared the studies related to nonlocal elasticity theories, among other generalized continuum mechanics approaches. The reasons are: (i) the clasical elasticity is a scale-free theory which cannot adequately address the size effect commonly present in nanotecnology applications, and (ii) they are an attractive alternative to the huge computational cost of the Molecular Dynamic techniques.

The origin of nonlocal continuum mechanics theories can be found in papers by Kröner (1967), Krumhansl (1968) and Kunin (1968). Later, Eringen and coworkers (Eringen, 1972a,b; Eringen and Edelen, 1972; Eringen, 1983, 2002) simplified the above theories for linear homogeneous isotropic nonlocal elastic materials. Further, Eringen proposed a two-phase nonlocal model (Eringen, 1972a, 1987) which combine the classical local and the nonlocal constitutive theories. The basic feature of the Eringen theory of elasticity is that the stress at each point is related to the strain at all points in the domain. This influence decreases as the distance between the point of interest and the neighboring points increases.

The nonlocal approach enabled different authors (Eringen, 1977; Eringen et al., 1977; Zhou et al., 1999) to address problems related with stress singularities which arise in classical fracture mechanics formulations, showing that these disappear using the nonlocal treatment. Additionally, these theories could overcome the difficulties showed by the classical local approaches to correctly predict solutions for problems in which microstructural and size effects are significant. These effects are present in modern engineering applications such as nano-machines (Drexler, 1992; Han et al., 1997; Fennimore et al., 2003; Bourlon et al., 2004; Kim et al., 2015), micro- or nano-electromechanical (MEMS or NEMS) devices (Martin, 1996; Ekinci and Roukes, 2005; Arndt et al., 2011; Berman and Krim, 2013), or in biotechnology and biomedical areas (Bhushan, 2007; Saji et al., 2010).

The constitutive relation for the two-phase constitutive model originally proposed by Eringen (1972a, 1987) has been recovered by other authors (Altan, 1989; Polizzotto, 2001; Pisano and Fuschi, 2003; Zhu and Dai, 2012; Benvenuti and Simone, 2013; Khodabakhshia and Reddy, 2015; Eptaimeros et al., 2016; Wang et al., 2016; Zhu et al., 2017), and is given by:

$$\boldsymbol{\sigma}(\mathbf{x}) = \xi_1 \mathbf{C} : \boldsymbol{\varepsilon}(\mathbf{x}) + \xi_2 \int_V \alpha(\mathbf{x}, \mathbf{x}', \kappa)\, \mathbf{C} : \boldsymbol{\varepsilon}(\mathbf{x}')\, dV' \tag{1}$$

$\boldsymbol{\sigma}$ being the nonlocal Cauchy stress tensor, and $\boldsymbol{\varepsilon}$ the infinitesimal strain tensor at a reference point $\mathbf{x}$; $\alpha(\mathbf{x}, \mathbf{x}', \kappa)$ is the kernel that represents the nonlocal behaviour which is dependent on a internal characteristic length, $\kappa$, linked to some material properties (lattice parameter, size of the grain, granular distance), $\mathbf{C}$ is the fourth-order elasticity tensor, and $V$ is the solid domain.

The parameters $\xi_1$ and $\xi_2$ represent the volumen fraction of material complying with local and nonlocal integral elasticity, respectively. Thus, the relation $\xi_1 + \xi_2 = 1$ holds. The case $\xi_2 = 0$ corresponds to the pure local elasticity approach, while $\xi_1 = 0$ deals with the fully nonlocal integral elasticity formulation.

For this last case, $\xi_1 = 0$, the nonlocal constitutive theory, introduced by Eringen (1983), is recovered. Eringen showed that, for a specific class of kernel functions, the nonlocal integral constitutive equation can be transformed into a differential form. Peddieson et al. (2003) applied for the first time the differential Eringen nonlocal model to the Bernoulli-Euler beams, and from that, and due to its simplicity, this approach has been widely used to analyze the static, buckling, and dynamic behavior of nanostructures. It is not feasible to report here the whole of the related papers. Therefore we refer the recent reviews by Eltaher et al. (2016) and Rafii-Tabar et al. (2016) on the application of nonlocal continuum theories to nanostructures.

However, Benvenuti and Simone (2013) found incoherent results related to the static behavior of a bar subjected to axial loads. Moreover, several authors have pointed out the paradoxical results obtained from the Eringen differential model regarding a cantilever beam when compared to other boundary conditions, both for the static (Peddieson et al., 2003; Wang and Liew, 2007; Challamel and Wang, 2008; Wang et al., 2008; Challamel et al., 2014) and vibrational behaviour (Lu et al., 2006).

Although several attempts have been made to overcome these paradoxical results (Challamel and Wang, 2008; Fernández-Sáez et al., 2016), a clear picture of the problem has been pointed out by Romano et al. (2017) who shown that, in the majorities of the cases, the integral formulation of the fully nonlocal elasticity theory leads to problems that have to be considered as ill-posed. These problems have no solution in general. Only when certain constitutive boundary conditions are fulfilled (Romano et al., 2017; Polyanin and Manzhirov, 2008), the integral formulation is equivalent to the differential one, and the problem has a unique solution. Therefore, Fernández-Sáez et al. (2016); Tuna and Kirca (2016a,b); Eptaimeros et al. (2016) proposed improper solutions (numerical or analytical) for a problem which is in fact unsolvable in most of the practical cases, since the constitutive boundary conditions are not fulfilled. However, using the mixed constitutive model with $\xi_1 > 0$, the ill-posedness of the purely nonlocal problem is eliminated, and therefore true solutions can be achieved using this formulation (Romano et al., 2017).

Variational principles governing the integral form of the two-phase nonlocal approach were derived by Polizzotto (2001), and the model was applied latter to analyse several problems related to the static behaviour of nanostructures. Zhu and Dai (2012), Pisano and Fuschi (2003), and Benvenuti and Simone (2013) solved the problem of a bar subjected to static axial loads. Pisano et al. (2009) used this integro-differential nonlocal model to derive a finite element formulation for 2D problems. More recently, with the same constitutive model, the static bending of Bernoulli-Euler beams subjected to different boundary and load conditions has been studied, via a finite element approach (Khodabakhshia and Reddy, 2015), or with an analytical model (Wang et al., 2016). The buckling of Bernoulli-Euler beams was also addressed using this constitutive formulation (Zhu et al., 2017).

The vibrational behaviour of Euler-Bernoulli beams involving the two-phase Eringen nonlocality has been studied by Eptaimeros et al. (2016), using a $FEM$ approach to obtain the eigenfrequencies for different boundary conditions. They analyzed the effects of different nonlocal parameters in the dynamic response of the beams.

In this paper we formulate and analytically solve the problem of the free in-plane (axial and bending) vibrations of a beam using the mixed local/nonlocal Eringen elasticity theory. The movement equations have been obtained using the Hamilton's Principle, leading to two uncoupled integro-differential eigenvalue problems corresponding to axial and bending vibrations, respectively. For the case of axial vibrations the integro-differential eigenvalue problem was transformed to a fourth-order differential equation with four boundary conditions: two of them correspond to the classical ones (one for each end), while the other two come from the transformation process. The same procedure applied to the integro-differential eigenvalue problem related to the bending vibrations leads to a sixth-order differential equation with six boundary conditions: four of them correspond to the classical ones (two for each end), while the other two are related to the transformation process.

The natural frequencies have been obtained solving the corresponding eigenvalue problems in differential form. To that aim, a method based on the Krylov-Duncan procedure (Karnovsky and Lebed, 2010), originally developed for the case of classical beams, has been used. The method proved to be very efficient in all the examples considered. The influence of both mixture parameter and length scale has been analysed.

To the best knowledge of the authors, this is the first time an exact solution is presented for a dynamic problem involving structures with constitutive equations corresponding to nonlocal integral Eringen's elasticity.

## 2. Problem formulation

In the following section we present an application of the mixed local/nonlocal Eringen integral model to the study of the axial and bending behaviour of a Bernoulli-Euler beam.

Consider a beam of length $L$, constant Young modulus $E$, and uniform cross section $A$ and inertia $I$. The variables $x$, $y$ and $z$ represent, respectively, the axial, out-of-plane and transverse coordinates. The variables $U_x$ and $U_y$, and $U_z$ correspond to the displacements in the coordinate directions. Using the kinematics of the Bernoulli-Euler beam, we have:

$$U_x(x,y,z,t) = u(x,t) - z\partial_x w(x,t); \quad U_y(x,y,z,t) = 0; \quad U_z(x,y,z,t) = w(x,t) \tag{2}$$

where $u$ and $w$ represent, respectively, the axial and transverse displacements of the cross-section's centroid. Symbol $t$ represents time. The strain $\varepsilon_x$ follows the expression

$$\varepsilon_x\left(x,t\right) = \partial_x u\left(x,t\right) - z\ \partial_{xx} w\left(x,t\right) \tag{3}$$

The nonlocal normal stress $\sigma_x\left(x,t\right)$ is given by the 1D mixed local/nonlocal Eringen integral constitutive equation

$$\sigma_x\left(x,t\right) = \xi_1 E\varepsilon\left(x,t\right) + \xi_2 E\int_0^L k\left(|x-\bar{x}|,\kappa\right)\ \varepsilon_x\left(\bar{x},t\right)\ d\bar{x} \tag{4}$$

with the Helmholtz kernel

$$k\left(|x-\bar{x}|,\kappa\right)=\frac{1}{2\kappa}e^{-\frac{|x-\bar{x}|}{\kappa}} \tag{5}$$

$\kappa = e_0 a$ being the non-local parameter, depending on both an internal length scale $a$ and a material constant $e_0$. Moreover, we consider that the relation $\xi_1+\xi_2=1$ holds.

Assuming the previous hypotheses, the axial force is given by

$$N\left(x,t\right)=\int_A \sigma_x\left(x,t\right)\ dA = EA\left[\xi_1\partial_x u\left(x,t\right)+\xi_2\int_0^L k\left(|x-\bar{x}|,\kappa\right)\partial_{\bar{x}}u\left(\bar{x},t\right)d\bar{x}\right] \tag{6}$$

and the bending moment is given by

$$M\left(x,t\right)=\int_A \sigma_x\left(x,t\right)z\ dA=-EI\left[\xi_1\ \partial_{xx}w\left(x,t\right)+\xi_2\int_0^L k\left(|x-\bar{x}|,\kappa\right)\partial_{\bar{x}\bar{x}}w\left(x,t\right)d\bar{x}\right] \tag{7}$$

The governing equation and the corresponding boundary conditions are derived applying the Hamilton's Principle. Let $\mathcal{L}$ be the Lagrangian of the system, defined as the difference between the kinetic energy, $\mathcal{K}$, and the total potential energy, $\Pi$

$$\mathcal{L}=\mathcal{K}-\Pi \tag{8}$$

Kinetic $\mathcal{K}$ and the potential $\Pi$ energies are given by:

$$\mathcal{K}=\frac{1}{2}\rho A\int_0^L\left[\left(\partial_t u\left(x,t\right)\right)^2+\left(\partial_t w\left(x,t\right)\right)^2\right]dx \tag{9}$$

and

$$\begin{aligned}\Pi=&\frac{1}{2}EA\left\{\xi_1\int_0^L\left(\partial_x u\left(x,t\right)\right)^2dx+\xi_2\int_0^L\left[\int_0^L k\left(|x-\bar{x}|,\kappa\right)\partial_{\bar{x}}u\left(\bar{x},t\right)d\bar{x}\right]\partial_x u\left(x\right)dx\right\}+\\&\frac{1}{2}EI\left\{\xi_1\int_0^L\left(\partial_{xx}w\left(x,t\right)\right)^2dx\ +\xi_2\int_0^L\left[\int_0^L k\left(|x-\bar{x}|,\kappa\right)\partial_{\bar{x}\bar{x}}w\left(\bar{x},t\right)d\bar{x}\right]\partial_{xx}w\left(x\right)dx\right\}-\\&\int_0^L q_x\left(x,t\right)u\left(x,t\right)dx-\int_0^L q_z\left(x,t\right)w\left(x,t\right)dx\end{aligned} \tag{10}$$

with $q_x\left(x,t\right)$ and $q_z\left(x,t\right)$ being the external loads in directions $x$ and $z$, respectively.

Equating the first variation of the action integral $\mathcal{A}=\int_{t_1}^{t_2}\mathcal{L}\ dt$ to zero, the Euler-Lagrange equations are determined

$$\xi_1 EA\partial_{xx}u\left(x,t\right)+\xi_2 EA\partial_x\left[\int_0^L k\left(|x-\bar{x}|,\kappa\right)\partial_{\bar{x}}u\left(\bar{x},t\right)d\bar{x}\right]+q_x\left(x,t\right)=\rho A\partial_{tt}u\left(x,t\right) \tag{11}$$

$$-\xi_1 EI\partial_{xxxx}w\left(x,t\right)-\xi_2 EI\partial_{xx}\int_0^L k\left(\left|x-\bar{x}\right|,\kappa\right)\partial_{\bar{x}\bar{x}}w\left(\bar{x},t\right)\;d\bar{x}+q_z\left(x,t\right)=\rho A\partial_{tt}w\left(x,t\right) \tag{12}$$

together with the following pairs of essential and natural boundary conditions at $x^*=0$ or $x^*=L$

$$u\left(x^*,t\right)=0;\quad \text{or}\quad EA\left[\xi_1\partial_x u\left(x^*,t\right)+\xi_2\int_0^L k\left(\left|x^*-\bar{x}\right|,\kappa\right)\partial_{\bar{x}}u\left(\bar{x},t\right)d\bar{x}\right]=0 \tag{13}$$

$$w\left(x^*,t\right)=0;\quad \text{or}\quad EI\left[\xi_1\partial_{xxx}w\left(x^*,t\right)+\xi_2\;\partial_x\int_0^L k\left(\left|x^*-\bar{x}\right|,\kappa\right)\partial_{\bar{x}\bar{x}}w\left(\bar{x},t\right)d\bar{x}\right]=0 \tag{14}$$

$$\partial_x w\left(x^*,t\right)=0;\quad \text{or}\quad EI\left[\xi_1\partial_{xx}w\left(x^*,t\right)+\xi_2\int_0^L k\left(\left|x^*-\bar{x}\right|,\kappa\right)\partial_{\bar{x}\bar{x}}w\left(\bar{x},t\right)d\bar{x}\right]=0 \tag{15}$$

A complete derivation of the previous equations can be found in Appendix A.

The initial conditions are stated as:

$$u\left(x,0\right)=u^0\left(x\right);\quad \partial_t u\left(x,0\right)=u_t^0\left(x\right) \tag{16}$$

$$w\left(x,0\right)=w^0\left(x\right);\quad \partial_t w\left(x,0\right)=w_t^0\left(x\right) \tag{17}$$

Assuming free vibrations ($q_x(x,t)=0$, $q_z(x,t)=0$) the governing equations, written in terms of axial force and bending moment, become

$$\partial_x N\left(x,t\right)=\rho A\;\partial_{tt}u\left(x,t\right) \tag{18}$$

$$\partial_{xx}M\left(x,t\right)=\rho A\;\partial_{tt}w\left(x,t\right) \tag{19}$$

and the boundary conditions at $x=0$ or $x=L$

$$u\left(x,t\right)=0;\quad \text{or}\quad N\left(x,t\right)=0 \tag{20}$$

$$w\left(x,t\right)=0;\quad \text{or}\quad \partial_x M\left(x,t\right)=0 \tag{21}$$

$$\partial_x w\left(x,t\right)=0;\quad \text{or}\quad M\left(x,t\right)=0 \tag{22}$$

According to the previous equations, the problems of axial and bending vibrations become uncoupled, as in the local elasticity theory. Next, both of them will be solved separately.

*2.1. Axial vibrations*

The axial vibration problem is defined by the governing Eq. (18), the boundary conditions (20), and the intial conditions (16).

Using the classical separation of variables method, the expression for the axial displacement is:

$$u\left(x,t\right) = L\; U\left(x\right) T_u\left(t\right) \tag{23}$$

Substituting (23) in (18), and using the following dimensionless variables and constants

$$\xi = \frac{x}{L}; \quad s = \frac{\bar{x}}{L}; \quad \tau_u = \omega_{u_0}\; t; \quad \omega_{u_0} = \frac{1}{L}\sqrt{\frac{E}{\rho}}; \quad h = \frac{\kappa}{L}; \quad \lambda = \left(\frac{\omega_u}{\omega_{u_0}}\right)^2 \tag{24}$$

the governing equation for the function $T_u(\tau_u)$ is written as

$$\ddot{T}_u(\tau_u) + \lambda_u T_u(\tau_u) = 0 \tag{25}$$

where $\dot{(\bullet)} \equiv \partial_{\tau_u}$. The solution of the previous equation can be expressed as:

$$T_u\left(\tau_u\right) = R_u \sin\sqrt{\lambda_u}\tau_u + S_u \cos\sqrt{\lambda_u}\tau_u \tag{26}$$

$R_u$ and $S_u$ being constants that can be obtained from the initial conditions, and $\lambda_u$, related with the frequency of the oscillation $\omega_u$ ($\lambda_u \sim \omega_u^2$), is obtained solving the eigenvalue problem stated as follows.

Substituting (23) in (6), we get $N\left(x,t\right) = \mathcal{N}\left(x\right) T_u\left(t\right)$, and the nondimensional axial force ($\bar{\mathcal{N}}\left(\xi\right) = \mathcal{N}\left(\xi\right) L/EA$) becomes

$$\bar{\mathcal{N}}\left(\xi\right) = \xi_1 U'\left(\xi\right) + \xi_2 \frac{1}{2h}\int_0^1 e^{-\frac{|\xi-s|}{h}} U'\left(s\right) ds \tag{27}$$

where $(\bullet)' \equiv \partial_\xi$, and the kernel given by Eq. (5) has been used. Then, the governing equation for the function $U(\xi)$ is written as an eigenvalue problem

$$\bar{\mathcal{N}}'\left(\xi\right) = -\lambda_u U(\xi) \tag{28}$$

or, written in terms of the displacement,

$$\left[\xi_1 U'\left(\xi\right) + \xi_2 \frac{1}{2h}\int_0^1 e^{-\frac{|\xi-s|}{h}} U'\left(s\right) ds\right]' = -\lambda_u U(\xi) \tag{29}$$

subject to one of the following sets of boundary conditions:

- Supported

$$U(0) = 0; \qquad U(1) = 0 \tag{30}$$

- Cantilever

$$U(0) = 0; \qquad \bar{\mathcal{N}}(1) = 0 \tag{31}$$

- Free

$$\bar{\mathcal{N}}(0) = 0; \quad \bar{\mathcal{N}}(1) = 0 \tag{32}$$

*2.2. Bending vibrations*

The bending vibration problem is defined by the governing Eq. (19), the boundary conditions (21) and (22), and the intial conditions (17).

Using again the separation of variables method for the transverse displacement

$$w\,(x,t) = L\ W\,(x)\, T_w\,(t) \tag{33}$$

Substituting (33) in (19), and defining the additional dimensionless parameters and variables

$$\tau_w = \omega_{w_0} t; \quad \omega_{w_0} = \frac{1}{L^2}\sqrt{\frac{EI}{\rho A}}; \quad \lambda_w = \left(\frac{\omega_w}{\omega_{w_0}}\right)^2 \tag{34}$$

the function $T_w(\tau_w)$ follows the differential equation

$$\ddot{T}_w(\tau_w) + \lambda_w T_w(\tau_w) = 0 \tag{35}$$

where now $(\dot{\bullet}) \equiv \partial_{\tau_w}$. The solution of Eq. (35) is of the form

$$T_w\,(\tau_w) = R_w \sin\sqrt{\lambda_w}\tau_w + S_w \cos\sqrt{\lambda_w}\tau_w \tag{36}$$

where the constants $R_w$ and $S_w$ can be obtained from the initial conditions, and $\lambda_w$, related with the oscillation frequency $\omega_w$ ($\lambda_w \sim \omega_w^2$), is determined by solving the eigenvalue problem described subsequently.

Substituting (33) in (7), we get $M\,(x,t) = \mathcal{M}\,(x)\, T_w\,(t)$, and the nondimensional bending moment ($\bar{\mathcal{M}}\,(\xi) = \mathcal{M}\,(\xi)\, L/EI$) becomes

$$\bar{\mathcal{M}}\,(\xi) = -\xi_1 W''\,(\xi) - \xi_2 \frac{1}{2h}\int_0^1 e^{-\frac{|\xi - s|}{h}} W''\,(s)\, ds \tag{37}$$

Therefore, the governing equation for the function $W(\xi)$ is presented as an eigenvalue problem

$$\bar{\mathcal{M}}''(\xi) = -\lambda_w W(\xi) \tag{38}$$

or, written in terms of the displacement,

$$\left[-\xi_1 W''(\xi) - \xi_2 \frac{1}{2h}\int_0^1 e^{-\frac{|\xi-s|}{h}} W''(s)\, ds\right]'' = -\lambda_w W(\xi) \tag{39}$$

subject to one of the following sets of boundary conditions:

- Supported

$$W(0) = 0; \quad \bar{\mathcal{M}}(0) = 0; \quad W(1) = 0; \quad \bar{\mathcal{M}}(1) = 0; \tag{40}$$

- Cantilever

$$W(0) = 0; \quad W'(0) = 0; \quad \bar{\mathcal{M}}(1) = 0; \quad \bar{\mathcal{M}}'(1) = 0; \tag{41}$$

- Free

$$\bar{\mathcal{M}}(0) = 0; \quad \bar{\mathcal{M}}'(0) = 0; \quad \bar{\mathcal{M}}(1) = 0; \quad \bar{\mathcal{M}}'(1) = 0; \tag{42}$$

## 3. Transformation of the integro-differential eigenvalue problems into differential ones

### *3.1. General methodology*

The integro-differential governing equation corresponding to the axial problem, Eq. (28) or (29), and to the bending problem, Eq. (38) or (39), can be transformed into equivalent differential ones. Integrating once Eq. (28) we get

$$\frac{\bar{\mathcal{N}}(\xi)}{\xi_1} = f_u(\xi) \tag{43}$$

with

$$f_u'(\xi) = -\frac{\lambda_u U(\xi)}{\xi_1} \tag{44}$$

Likewise, integrating twice Eq. (38), we obtain

$$\frac{\bar{\mathcal{M}}(\xi)}{\xi_1} = -f_w(\xi) \tag{45}$$

with

$$f_w''(\xi) = \frac{\lambda_w W(\xi)}{\xi_1} \tag{46}$$

Inserting the expressions for $\bar{\mathcal{N}}(\xi)$, Eq. (27), and $\bar{\mathcal{M}}(\xi)$, Eq. (37), in Eqs. (43) and (45) respectively, we get

$$U'(\xi) + \frac{\xi_2}{\xi_1}\frac{1}{2h}\int_0^1 e^{-\frac{|\xi-s|}{h}} U'(s)\,ds = f_u(\xi) \tag{47}$$

$$W''(\xi) + \frac{\xi_2}{\xi_1}\frac{1}{2h}\int_0^1 e^{-\frac{|\xi-s|}{h}} W''(s)\,ds = f_w(\xi) \tag{48}$$

Notice that both Eq. (47) and Eq. (48) are of the general form

$$y(\xi) + C\int_a^b e^{\mu|\xi-s|} y(s)\,ds = g(\xi) \tag{49}$$

As stated in Polyanin and Manzhirov (2008), the function $y(\xi)$ obeys the following second-order linear nonhomogeneous ordinary differential equation with constant coefficients

$$y''(\xi) + \mu(2C-\mu)\,y(\xi) = g''(\xi) - \mu^2 g(\xi) \tag{50}$$

provided that the so-called *constitutive boundary conditions* (Romano et al., 2017) are fulfilled (Polyanin and Manzhirov, 2008)

$$y'(a) + \mu y(a) = g'(a) + \mu g(a) \tag{51}$$

$$y'(b) - \mu y(b) = g'(b) - \mu g(b) \tag{52}$$

This permits the above mentioned transformation, that will be next developed for both axial and bending cases.

### *3.2. Differential eigenvalue problem for axial vibration*

In view of the above, using Eq. (50) and assuming the equivalences $y(\xi) = U'(\xi)$, $a = 0$, $b = 1$, $\mu = -1/h$, $C = \xi_2/(2\xi_1 h)$, and $g(\xi) = f_u(\xi)$, as well as the relation $\xi_1 + \xi_2 = 1$, the following expression of the axial force in terms of derivatives of the displacement can be obtained

$$\bar{\mathcal{N}}(\xi) = -h^2\xi_1 U'''(\xi) + \left(1 - h^2\lambda_u\right) U'(\xi) \tag{53}$$

Then, the integro-differential eigenvalue problem can be written in differential form

$$U^{IV}(\xi) - \frac{1}{\xi_1}\left(\frac{1}{h^2} - \lambda_u\right) U''(\xi) - \frac{\lambda_u}{\xi_1 h^2} U(\xi) = 0. \tag{54}$$

subject to the *constitutive* boundary conditions, Eqs. (51) and (52),

$$-h\ U'''(0) + U''(0) + \frac{1-\xi_1-h^2\lambda_u}{\xi_1 h}U'(0) + \frac{\lambda_u}{\xi_1}U(0) = 0 \tag{55}$$

$$h\ U'''(1) + U''(1) - \frac{1-\xi_1-h^2\lambda_u}{\xi_1 h}U'(1) + \frac{\lambda_u}{\xi_1}U(1) = 0 \tag{56}$$

and appropriate standard boundary conditions given by any of the Eqs. (30) to (32). Note that these can be written in terms of the displacement using Eq. (53).

### *3.3. Differential eigenvalue problem for bending vibration*

Using Eq. (50) with the equivalences $y(\xi) = W''(\xi)$, and $g(\xi) = f_w(\xi)$, the following relation between bending moment and transverse displacement can be obtained

$$\bar{\mathcal{M}}(\xi) = \xi_1 h^2 W^{IV}(\xi) - W''(\xi) - h^2\lambda_w\ W(\xi) \tag{57}$$

The integro-differential eigenvalue problem can then be written in differential form

$$W^{VI}(\xi) - \frac{1}{\xi_1 h^2}W^{IV}(\xi) - \frac{\lambda_w}{\xi_1}W''(\xi) + \frac{\lambda_w}{\xi_1 h^2}W(\xi) = 0 \tag{58}$$

subject to the *constitutive* boundary conditions, Eqs. (51) and (52)

$$h^2W^V(0) - hW^{IV}(0) - \frac{1-\xi_1}{\xi_1}W'''(0) + \frac{1-\xi_1}{\xi_1 h}W''(0) - \frac{h^2\lambda_w}{\xi_1}W'(0) + \frac{h\lambda_w}{\xi_1}W(0) = 0 \tag{59}$$

$$h^2W^V(1) + hW^{IV}(1) - \frac{1-\xi_1}{\xi_1}W'''(1) - \frac{1-\xi_1}{\xi_1 h}W''(1) - \frac{h^2\lambda_w}{\xi_1}W'(1) - \frac{h\lambda_w}{\xi_1}W(1) = 0 \tag{60}$$

as well as to the standard boundary conditions given by any of the Eqs. (40) to (42). Note that these can be written in terms of the displacement using Eq. (57).

## 4. Solution of the differential eigenvalue problem

### *4.1. Axial problem*

The general solution of the differential Eq. (54) is of the form

$$U(\xi) = \sum_{i=1}^{4} B_i e^{b_i \xi} \tag{61}$$

where $b_i$ are the roots of the characteristic polynomial

$$r^4 - \frac{1-h^2\lambda_u}{\xi_1 h^2} r^2 - \frac{\lambda_u}{\xi_1 h^2} = 0 \tag{62}$$

and $B_i$ are arbitrary constants that have to be determined by imposing two standard boundary conditions, selected from any of Eqs. (30) to (32), and two additional constitutive boundary conditions, Eqs. (55) and (56). The previous solution admits an alternative expression, following a procedure similar to that used to obtain the Krylov-Duncan functions (Karnovsky and Lebed, 2010). Thus, the displacement function $U(\xi)$ can be alternatively written as

$$U(\xi) = U_0 \; F_1(\xi) + \bar{\mathcal{N}}_0 \; F_2(\xi) + \kappa_0 \; F_3(\xi) + \kappa_1 \; F_4(\xi) \tag{63}$$

where $F_i(\xi)$ are functions to be determined, $U_0$ and $\bar{\mathcal{N}}_0$ the axial displacement and axial force at $\xi = 0$, respectively, and $\kappa_0$ and $\kappa_1$ the expressions that, equated to zero, define the constitutive boundary conditions, see Eqs. (55) and (56):

$$\kappa_0 = -h \; U_0''' + U_0'' + \frac{1-\xi_1-h^2\lambda_u}{\xi_1 h} U_0' + \frac{\lambda_u}{\xi_1} U_0 \tag{64}$$

$$\kappa_1 = h \; U_1''' + U_1'' - \frac{1-\xi_1-h^2\lambda_u}{\xi_1 h} U_1' + \frac{\lambda_u}{\xi_1} U_1 \tag{65}$$

with $U_0'$, $U_0''$, $U_0'''$, and $U_1'$, $U_1''$, $U_1'''$ being the first, second and third derivatives of $U(\xi)$ at $\xi = 0$ and $\xi = 1$, respectively. Using the general solution given by Eq. (61) to derive $U_0$, Eq. (53) to derive $\bar{\mathcal{N}}_0$, and Eqs. (64) and (65) to obtain $\kappa_0$ and $\kappa_1$, the linear system $\mathbf{H}_u \cdot \mathbf{B} = \mathbf{V}_u$ can be constructed and solved, where $\mathbf{B} = \{B_1, B_2, B_3, B_4\}^T$, $\mathbf{V}_u = \{U_0, \bar{\mathcal{N}}_0, \kappa_0, \kappa_1\}^T$, and $\mathbf{H}_u$ is a matrix of coefficients. Substituting back $B_i$ as a function of $U_0$, $\bar{\mathcal{N}}_0$, $\kappa_0$ and $\kappa_1$ in Eq. (61), and rearranging terms, the functions $F_i(\xi)$ can be derived.

Finally, taking into account that $\kappa_0$ and $\kappa_1$ are zero in order to satisfy the constitutive boundary conditions, the displacement solution reads

$$u(\xi) = U_0 \; F_1(\xi) + \bar{\mathcal{N}}_0 \; F_2(\xi) \tag{66}$$

and the characteristic equation can be obtained by imposing suitable standard boundary conditions to the solution:

- Supported: $U_0 = 0$, $U(1) = 0$.

$$F_2(1) = 0 \tag{67}$$

- Cantilever: $U_0 = 0$, $\bar{\mathcal{N}}(1) = 0$.

$$\left(1 - h^2\lambda_u\right) F_2'(1) - h^2\xi_1 F_2'''(1) = 0 \tag{68}$$

- Free: $\bar{\mathcal{N}}_0 = 0$, $\bar{\mathcal{N}}(1) = 0$.

$$\left(1 - h^2\lambda_u\right) F_1'(1) - h^2\xi_1 F_1'''(1) = 0 \tag{69}$$

Note that the functions $F_i$ depend on the eigenvalue $\lambda_u$ and on the nonlocal parameter $h$.

### *4.2. Bending problem*

The general solution of the differential Eq. (58) is of the form

$$W(\xi) = \sum_{i=1}^{6} D_i e^{d_i \xi} \tag{70}$$

where the coefficients $d_i$ are the roots of the characteristic polynomial

$$r^6 - \frac{1}{\xi_1 h^2} r^4 - \frac{\lambda_w}{\xi_1} r^2 + \frac{\lambda_w}{\xi_1 h^2} = 0 \tag{71}$$

and constants $D_i$ are determined by imposing four standard boundary conditions, selected from any of Eqs. (40) to (42), and two additional constitutive boundary conditions, Eqs. (59) and (60). Following the same procedure used for the solution of the axial vibration problem, the displacement function is written as

$$W(\xi) = W_0\ G_1(\xi) + \Phi_0\ G_2(\xi) + \bar{\mathcal{M}}_0\ G_3(\xi) + \bar{Q}_0\ G_4(\xi) + \kappa_0\ G_5(\xi) + \kappa_1\ G_6(\xi) \tag{72}$$

where $G_i(\xi)$ are functions to be determined, $W_0$ and $\bar{\mathcal{M}}_0$ are the transverse displacement and bending moment at $\xi = 0$, respectively, $\Phi_0 = W'(0)$, $\bar{Q}_0 = \bar{\mathcal{M}}'(0)$, and $\eta_0$ and $\eta_1$ the expressions that, equated to zero, define the constitutive boundary conditions, see Eqs. (59) and (60):

$$\eta_0 = h^2 W_0^{V} - h W_0^{IV} - \frac{1-\xi_1}{\xi_1} W_0''' + \frac{1-\xi_1}{\xi_1 h} W_0' - \frac{h^2\lambda_w}{\xi_1} W_0' + \frac{h\lambda_w}{\xi_1} W_0 \tag{73}$$

$$\eta_1 = h^2 W_1^V + h W_1^{IV}(1) - \frac{1-\xi_1}{\xi_1} W_1''' - \frac{1-\xi_1}{\xi_1 h} W_1'' - \frac{h^2\lambda_w}{\xi_1} W_1' - \frac{h\lambda_w}{\xi_1} W_1 \tag{74}$$

with $W_0'$, $W_0''$, $W_0'''$, $W_0^{IV}$, $W_0^V$, and $W_1'$, $W_1''$, $W_1'''$, $W_1^{IV}$, $W_1^V$ being, respectively, the first to fifth derivatives of $W(\xi)$ at the boundaries $\xi = 0$ and $\xi = 1$. Using the general solution given by Eq. (70) (and its derivative) to obtain $W_0$ and $\Phi_0$, Eq. (57) and its derivative to obtain $\bar{\mathcal{M}}_0$ and $\bar{Q}_0$, and Eqs. (73) and (74) to get $\eta_0$ and $\eta_1$, the linear system $\mathbf{H}_w \cdot \mathbf{D} = \mathbf{V}_w$ can be constructed and solved, where $\mathbf{D} = \{D_1, D_2, D_3, D_4, D_5, D_6\}^T$, $\mathbf{V}_w = \left\{W_0, \Phi_0, \bar{\mathcal{M}}_0, \bar{Q}_0, \eta_0, \eta_1\right\}^T$, and $\mathbf{H}_w$ is a matrix of coefficients. Substituting back $D_i$ as a function of $W_0$, $\Phi_0$, $\bar{\mathcal{M}}_0$, $\bar{Q}_0$, $\eta_0$, and $\eta_1$ in Eq. (70), and obtaining the coefficients for each of the elements in vector $\mathbf{V}_w$, the functions $G_i(\xi)$ are derived. Finally, the displacement solution reads

$$W(\xi) = W_0\ G_1(\xi) + \Phi_0\ G_2(\xi) + \bar{\mathcal{M}}_0\ G_3(\xi) + \bar{Q}_0\ G_4(\xi) \tag{75}$$

where $\eta_0 = 0$ and $\eta_1 = 0$ has been imposed, fulfilling the constitutive boundary conditions.

Finally, the characteristic equation can be obtained by imposing appropriate standard boundary conditions to the solution:

- Supported: $W_0 = 0$, $\bar{\mathcal{M}}_0 = 0$, $W(1) = 0$, $\bar{\mathcal{M}}(1) = 0$.

$$\begin{vmatrix} a_{11}^{Supp} & a_{12}^{Supp} \\ a_{21}^{Supp} & a_{22}^{Supp} \end{vmatrix} = 0 \tag{76}$$

- Cantilever: $W_0 = 0$, $\Phi_0 = 0$, $\bar{\mathcal{M}}(1) = 0$, $\bar{\mathcal{M}}'(1) = 0$.

$$\begin{vmatrix} a_{11}^{Cant} & a_{12}^{Cant} \\ a_{21}^{Cant} & a_{22}^{Cant} \end{vmatrix} = 0 \tag{77}$$

- Free: $\bar{\mathcal{M}}_0 = 0$, $\bar{Q}_0 = 0$, $\bar{\mathcal{M}}(1) = 0$, $\bar{\mathcal{M}}'(1) = 0$.

$$\begin{vmatrix} a_{11}^{Free} & a_{12}^{Free} \\ a_{21}^{Free} & a_{22}^{Free} \end{vmatrix} = 0 \tag{78}$$

The elements in the three previous determinants are given in Appendix B. Note that these elements are dependent on the functions $G_i$ which, in turn, depend on the eigenvalue $\lambda_u$ and on the nonlocal parameter $h$.

## 5. Results. Natural frequencies of vibration

In this section, the eigenfrequencies for axial and bending vibration of the two-phase nonlocal elastic beam are presented for three different boundary conditions in each case, *supported*, *cantilever*, and *free*, using the presented solution procedure. The influence of the material parameters in the vibrational behavior has been analysed by considering four values of the nonlocal parameter $h = \{0.010, 0.025, 0.050, 0.075\}$, and ranging the mixture parameter $\xi_1$ from 0.1 to 1.0. The eigenfrequencies have been obtained applying the capabilities of symbolic calculus provided by the Mathematica software (Wolfram Research, Inc., 2017).

### *5.1. Axial vibration*

Figs. 1 to 3 show, for each boundary condition, the first four eigenfrequencies $\omega_{u_n}$, $n = 1, \ldots, 4$, as a function of the mixture parameter $\xi_1$, and of the nonlocal parameter $h$. Frequencies presented in the Figures are normalized by their corresponding local counterparts given in Table 1.

Table 1: Nondimensional eigenfrequencies ($n = 1, ..., 4$) corresponding to axial vibration for different boundary conditions. Fully local case ($\xi_1 = 1$).

| | $\omega_{u_1}^{local}$ | $\omega_{u_2}^{local}$ | $\omega_{u_3}^{local}$ | $\omega_{u_4}^{local}$ |
|---|---|---|---|---|
| Supported | $\pi$ | $2\pi$ | $3\pi$ | $4\pi$ |
| Cantilever | $\pi/2$ | $3\pi/2$ | $5\pi/2$ | $7\pi/2$ |
| Free | $\pi$ | $2\pi$ | $3\pi$ | $4\pi$ |

For values $\xi_1 < 1$ (thus $\xi_2 > 0$), the eigenfrequencies decrease as the nonlocal effect becomes more relevant, either by increasing the value of the nonlocal parameter $h$ or by decreasing the mixture parameter $\xi_1$. This softening effect of the Eringen's nonlocality has been previously reported in the literature for static behavior of rods subjected to axial loads (Benvenuti and Simone, 2013). It is worth to point out that, in all cases, the local eigenfrequencies are recovered for $\xi_1 = 1$.

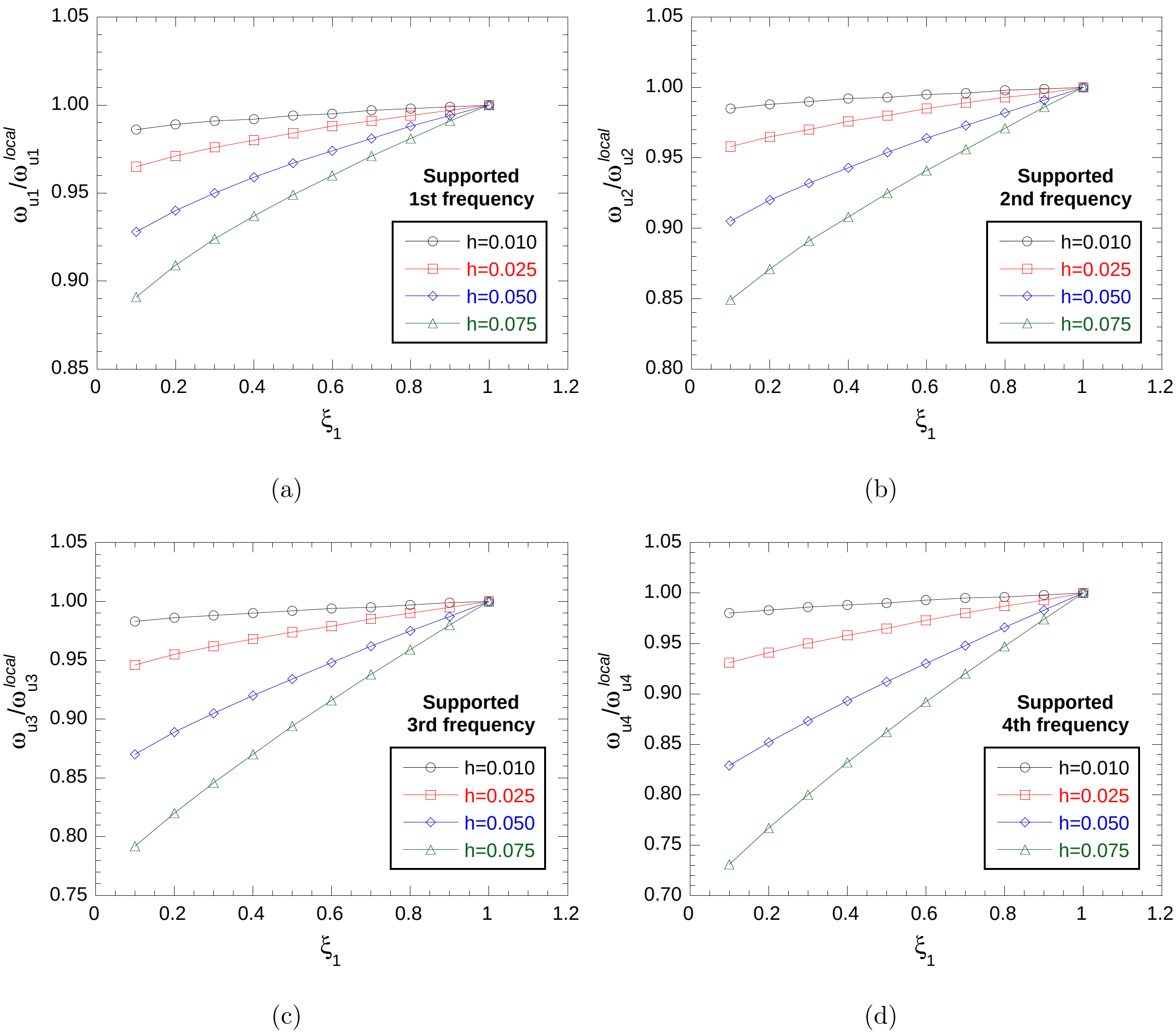



Figure 1: Axial vibration. First four natural frequencies of the *supported* beam as a function of the mixture parameter $\xi_1$, for four different values of the nonlocal parameter $h$. The frequency $\omega_{u_n}$ has been normalized by the frequency $\omega_{u_n}^{local}$ corresponding to the local case ($\xi_1 = 1$).

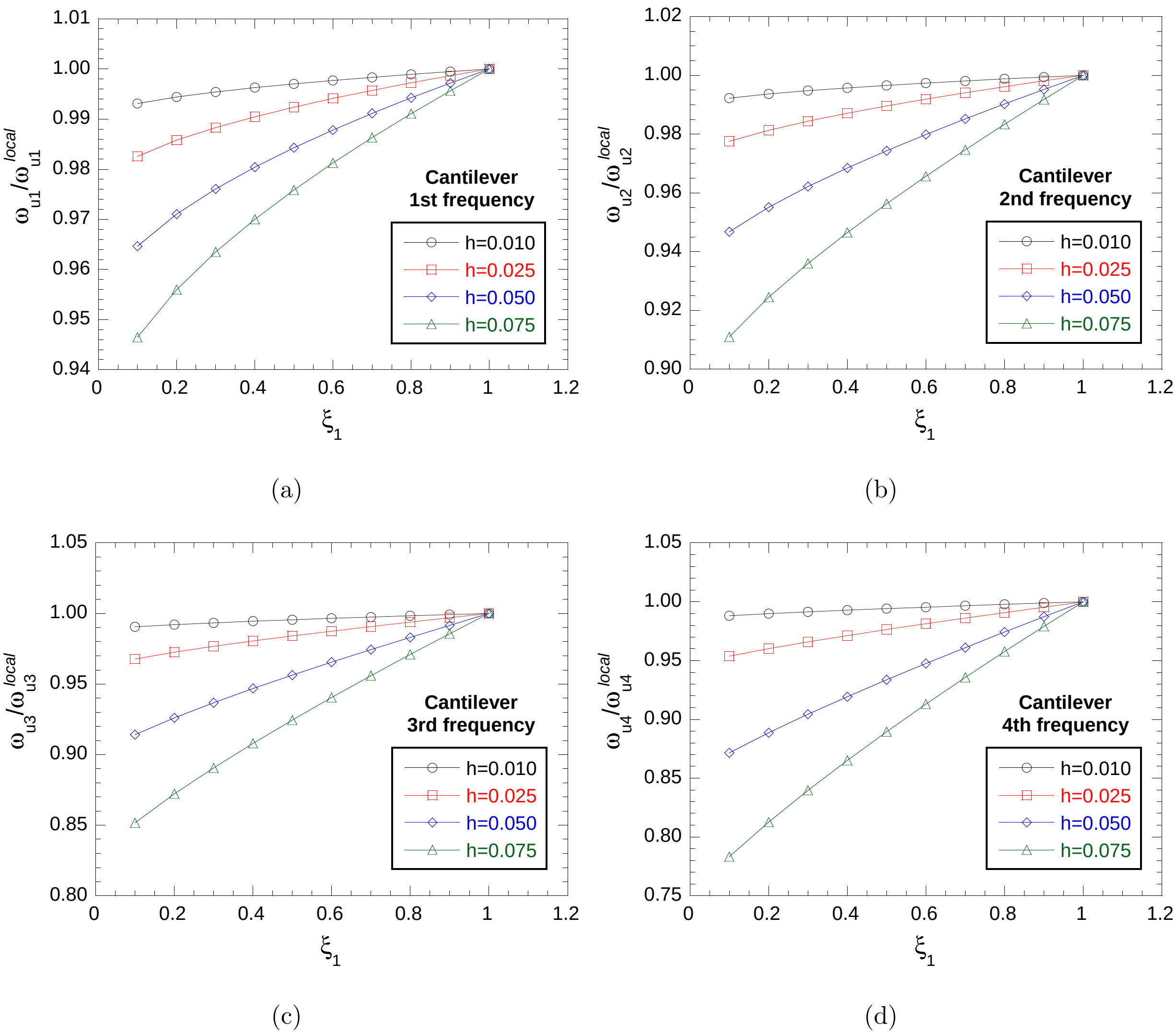
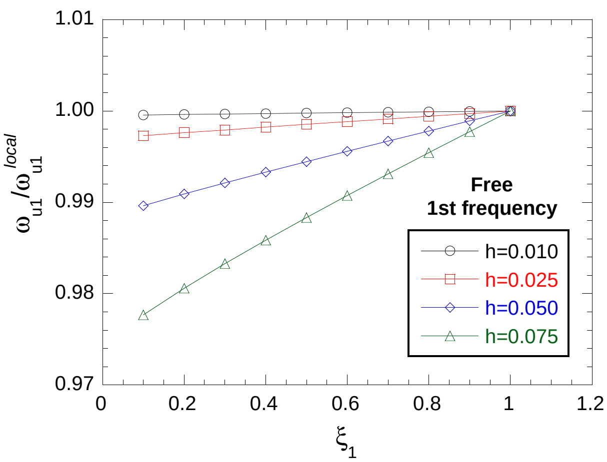
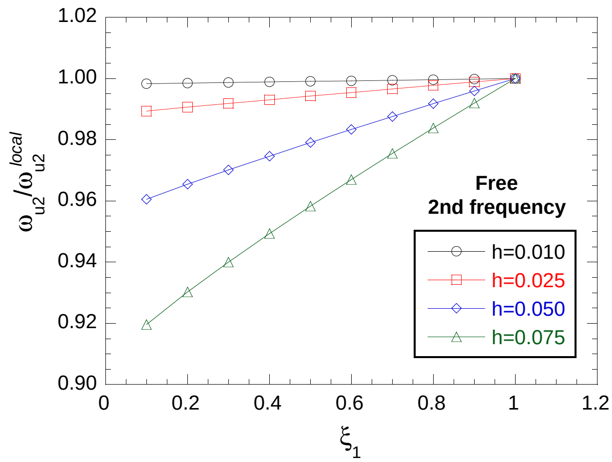


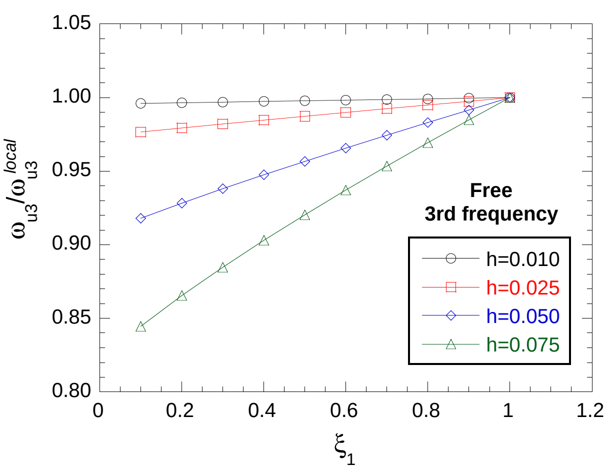

Figure 2: Axial vibration. First four natural frequencies of the *cantilever* beam as a function of the mixture parameter $\xi_1$, for four different values of the nonlocal parameter $h$. The frequency $\omega_{u_n}$ has been normalized by the frequency $\omega_{u_n}^{local}$ corresponding to the local case ($\xi_1 = 1$).



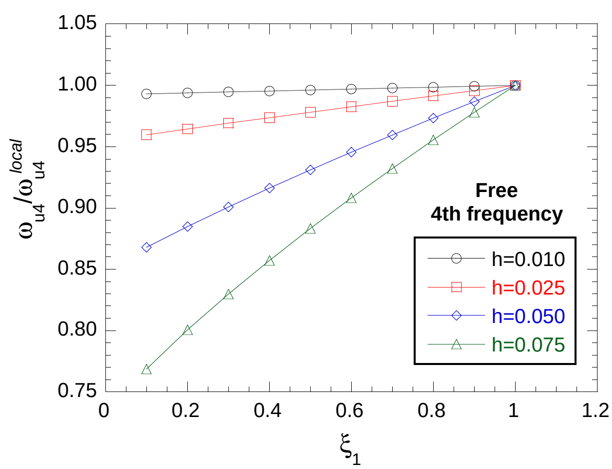

(a)




(b)




(c)




(d)

Figure 3: Axial vibration. First four natural frequencies of the *free* beam as a function of the mixture parameter $\xi_1$, for four different values of the nonlocal parameter $h$. The frequency $\omega_{u_n}$ has been normalized by the frequency $\omega_{u_n}^{local}$ corresponding to the local case ($\xi_1 = 1$).

### 5.2. Bending vibration

Figs. 4 to 6 show, for each boundary condition, the first four eigenfrequencies $\omega_{w_n}$, $n = 1, \ldots, 4$, as a function of $h$ and $\xi_1$. Frequencies presented in these Figures are normalized by their corresponding local counterparts (Table 2).

Table 2: Nondimensional eigenfrequencies ($n = 1, ..., 4$) corresponding to bending vibration for different boundary conditions. Fully local case ($\xi_1 = 1$).

| | $\omega_{w_1}^{local}$ | $\omega_{w_2}^{local}$ | $\omega_{w_3}^{local}$ | $\omega_{w_4}^{local}$ |
|---|---|---|---|---|
| Supported | $\pi^2$ | $4\pi^2$ | $9\pi^2$ | $16\pi^2$ |
| Cantilever | 3.516 | 22.034 | 61.697 | 120.902 |
| Free | 22.373 | 61.673 | 120.903 | 199.859 |

As in the previous analysis devoted to axial vibrations, in all considered configurations, the local eigenfrequencies are recovered for $\xi_1 = 1$, when the contribution of the nonlocal phase is neglected. Moreover, the results show the same trends independently of the boundary condition; thus the paradoxical behavior of the cantilever beam that can be found when using the (ill-posed) fully nonlocal Eringen model is not addressed with the mixed formulation. A similar remark has been stated by Eptaimeros et al. (2016).

The eigenfrequencies decrease as the nonlocal effect becomes more relevant, either by increasing the value of the nonlocal parameter $h$ or by decreasing the mixture parameter $\xi_1$. Moreover, the influence of nonlocality becomes more relevant for higher modes. This softening effect of the Eringen's nonlocality has been reported for static bending (Khodabakhshia and Reddy, 2015; Wang et al., 2016), as well as for bending vibrations (Eptaimeros et al., 2016).

It is interesting to notice that, for the *free* case, it is possible to find a solution for the fully nonlocal case ($\xi_1 = 0, \xi_2 = 1$). The reason is that, for this particular situation, the fulfilment of the classical boundary conditions guarantees the accomplishment of the constitutive boundary conditions associated to the transformation process from the integro-differential eugenvalue problem to the differential one (see Appendix C for details). Thus, for the free case, the four first eigenfrequencies corresponding to the fully nonlocal problem for different values of the nonlocal parameter $h$ have been calculated and are quoted in Fig. 6. It can be seen that the obtained frequencies are the limit of the corresponding mixed problem for $\xi_1 \to 0$.

## 6. Summary and conclusions

In this paper we presented an exact solution for the free in-plane vibrations (axial and bending) of a Bernoulli-Euler beam using the mixed local/nonlocal constitutive equations related to the Eringen elasticity theory. The two original uncoupled integro-differential eigenvalue problems, governing the axial and bending vibrations, have been transformed into the equivalent differential ones. Then, the governing equations in differential form become of fourth-order (axial) or sixth-order (bending), and two constitutive boundary conditions are added to the two (axial) or four (bending) standard ones. Moreover, expressions for the axial force and bending moment in terms of the derivatives of displacement have been derived.

The solution of the differential eigenvalue problems has been solved following a procedure similar to that used to obtain the Krylov-Duncan functions. This permitted to reduce the order of the determinant from which the characteristic equation is derived.

The method has been applied to study the in-plane free vibration of a Bernoulli-Euler beam with different classical boundary conditions: supported, cantilever and free.

The corresponding eigenfrequencies for different values of the nonlocal parameter $h$, and of the local phase fraction $\xi_1$ have been calculated. Thus, the softening effect of the Eringen's nonlocality has been highlighted, which is more pronounced as the mixture parameter $\xi_1$ decreases. It is worth highlighting that existence of a solution for the fully nonlocal problem with $\xi_1 = 0$ is an exception (for instance, bending with free boundary conditions), the rule being non-existence, as shown by Romano et al. (2017). However, the presence of the local term in the constitutive equation introduces a regularization parameter $\xi_1 > 1$ which renders the problem well-posed.

To the authors knowledge, this is the first time an exact solution has been derived for a dynamic problem involving the nonlocal integral Eringen constitutive equation. This opens a pathway to the analysis of the vibrational behavior for other typologies of nonlocal

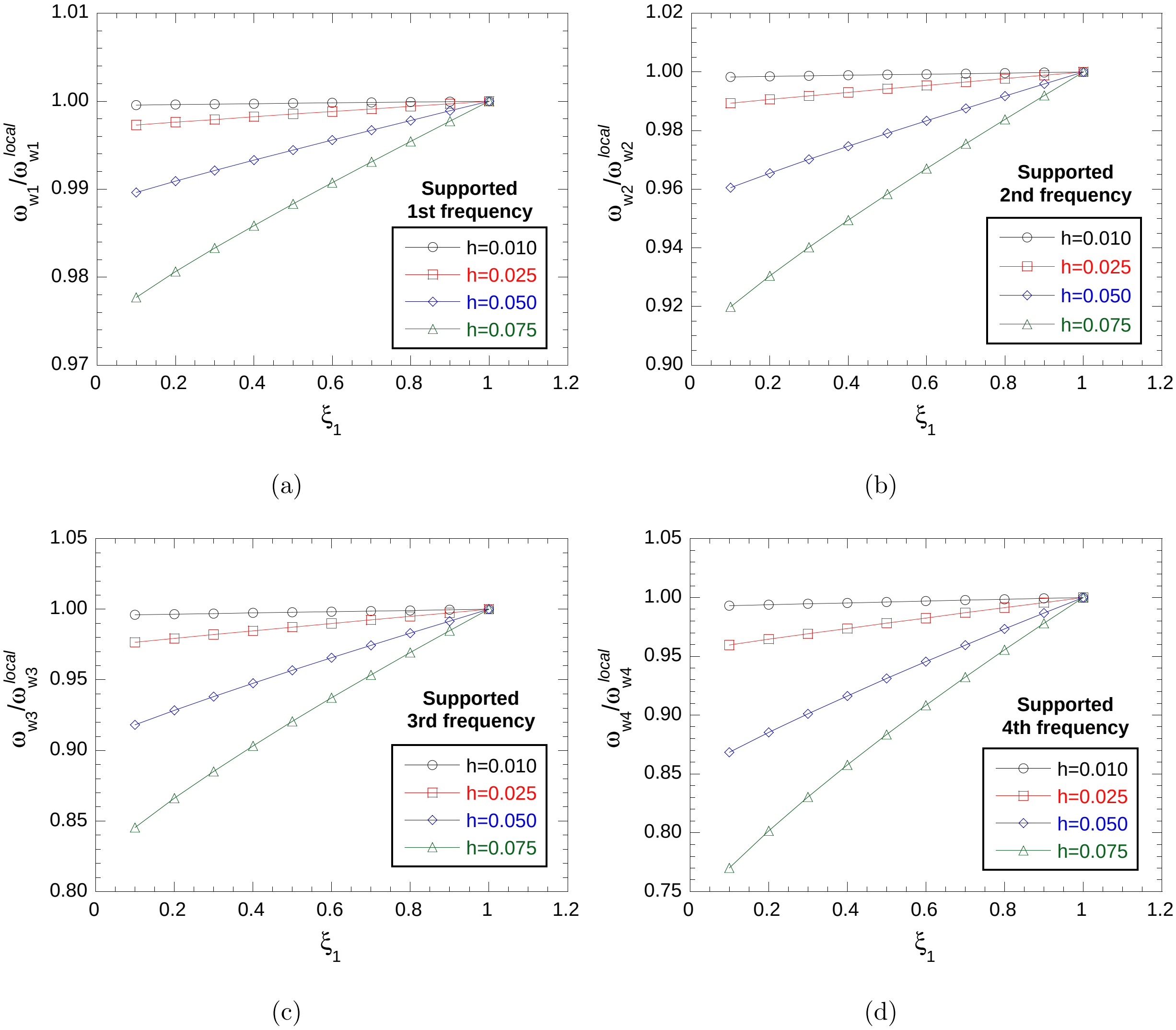
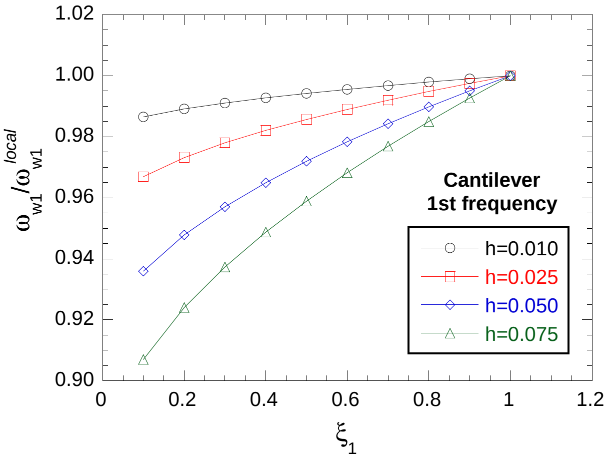
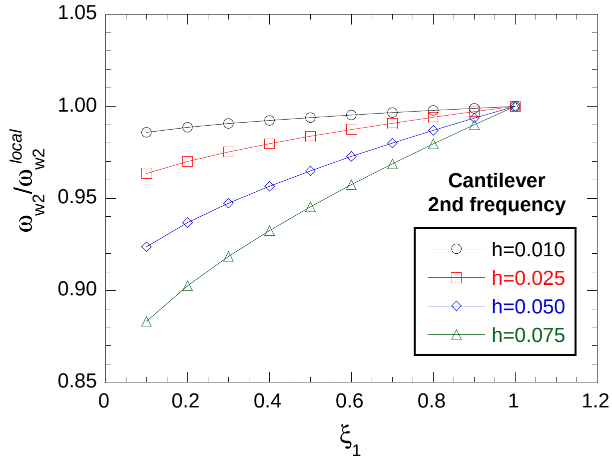


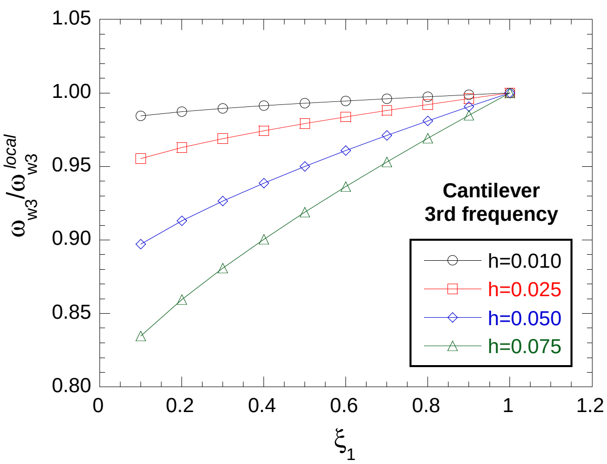

Figure 4: Bending vibration. First four natural frequencies of the *supported* beam, as a function of the mixture parameter $\xi_1$, for four different values of the nonlocal parameter $h$. The frequency $\omega_{w_n}$ has been normalized by the frequency $\omega_{w_n}^{local}$ corresponding to the local case ($\xi_1 = 1$).



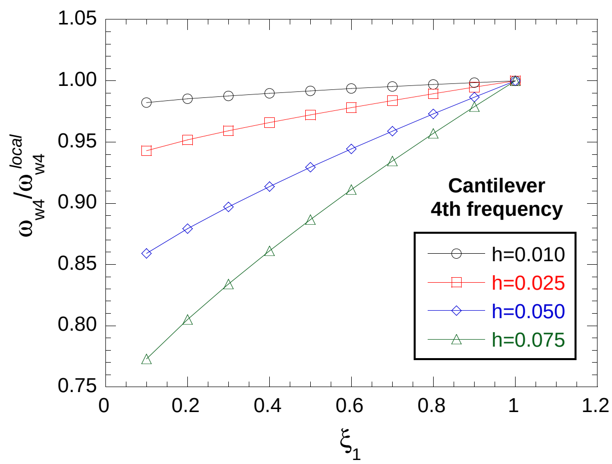

(a)




(b)




(c)




(d)

Figure 5: Bending vibration. First four natural frequencies of the *cantilever* beam as a function of the mixture parameter $\xi_1$, for four different values of the nonlocal parameter $h$. The frequency $\omega_{w_n}$ has been normalized by the frequency $\omega_{w_n}^{local}$ corresponding to the local case ($\xi_1 = 1$).

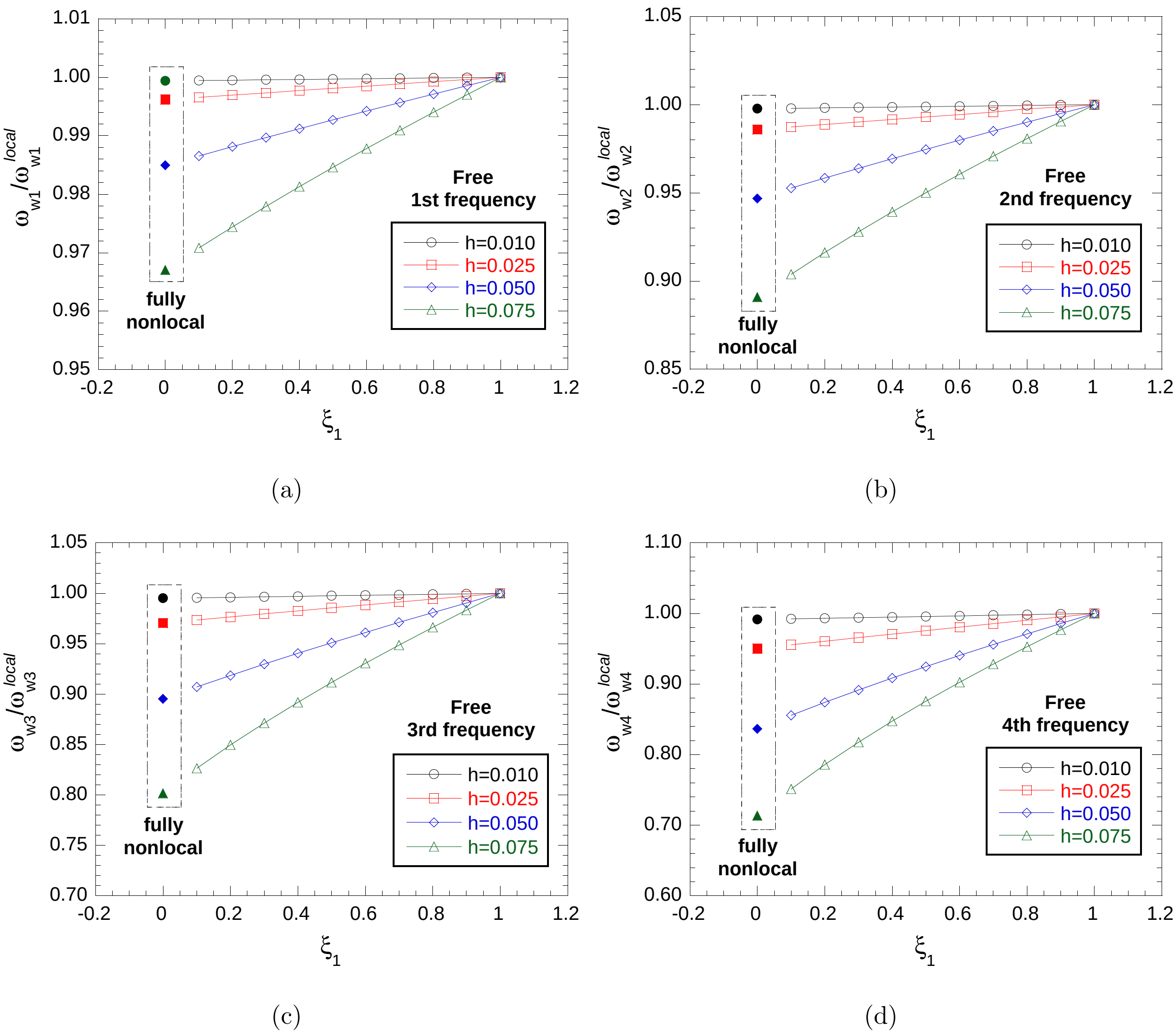



Figure 6: Bending vibration. First four natural frequencies of the *free* beam as a function of the mixture parameter $\xi_1$, for four different values of the nonlocal parameter $h$. The frequency $\omega_{w_n}$ has been normalized by the frequency $\omega_{w_n}^{local}$ corresponding to the local case ($\xi_1 = 1$).

solids.


## Acknowledgements

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## Appendix A. Application of Hamilton's Principle

According to Hamilton's Principle, the first variation of the functional $\int_{t_1}^{t_2} \mathcal{L} dt$ is zero, $\mathcal{L}$ being the Lagrangian of the system. Then

$$\delta \int_{t_1}^{t_2} (\mathcal{K} - \Pi) \, dt = 0 \tag{A.1}$$

where the kinetic energy $\mathcal{K}$ and the total potential energy $\Pi$ are defined by Eqs. (9) and (10), respectively.

The term associated to the kinetic energy becomes, after intergating once by parts and using $\delta u\,(x, t_1) = \delta u\,(x, t_2) = 0$, $\delta w\,(x, t_1) = \delta w\,(x, t_2) = 0$:

$$\delta \int_{t_1}^{t_2} \mathcal{K} \; dt = -\int_{t_1}^{t_2} \left[ \rho A \int_0^L \; (\partial_{tt} u \; \delta u + \partial_{tt} w \; \delta w) \, dx \right] dt \tag{A.2}$$

where functional dependences have been obviated for clarity.

The term associated with the total potential energy becomes:

$$\begin{aligned} \delta \int_{t_1}^{t_2} \Pi \; dt = \\ \int_{t_1}^{t_2} \Bigg\{ \xi_1 EA \int_0^L \partial_x u \; \partial_x \delta u \; dx \; + \xi_2 EA \int_0^L \left[ \int_0^L k\left(|x - \bar{x}|, \kappa\right) \partial_{\bar{x}} u \; d\bar{x} \right] \partial_x \delta u \; dx \; + \\ \xi_1 EI \int_0^L \partial_{xx} w \; \partial_{xx} \delta w \; dx \; + \xi_2 EI \int_0^L \left[ \int_0^L k\left(|x - \bar{x}|, \kappa\right) \partial_{\bar{x}\bar{x}} w \; d\bar{x} \right] \partial_{xx} \delta w \; dx \; - \\ \int_0^L q_x \; \delta u \; dx - \int_0^L q_z \; \delta w \; dx \Bigg\} \end{aligned} \tag{A.3}$$

and, after integration by parts,

$$
\begin{aligned}
&\delta \int_{t_1}^{t_2} \Pi \; dt = \\
&\int_{t_1}^{t_2} \Bigg\{ \xi_1 EA \left[ \partial_x u \; \delta u \mid_0^L - \int_0^L \partial_{xx} u \; \delta u \; dx \right] + \\
&\xi_2 EA \left[ \left( \int_0^L k\left(|x-\bar{x}|, \kappa\right) \partial_{\bar{x}} u \; d\bar{x} \right) \delta u \Big|_0^L - \int_0^L \partial_x \left[ \int_0^L k\left(|x-\bar{x}|, \kappa\right) \partial_{\bar{x}} u \; d\bar{x} \right] \delta u \; dx \right] + \\
&\xi_1 EI \left[ \partial_{xx} w \; \delta \partial_x w|_0^L - \partial_{xxx} w \; \delta w|_0^L + \int_0^L \partial_{xxxx} w \; \delta w \; dx \right] + \\
&\xi_2 EI \Bigg[ \left[ \left( \int_0^L k\left(|x-\bar{x}|, \kappa\right) \partial_{\bar{x}\bar{x}} w \; d\bar{x} \right) \delta \partial_x w \right]_0^L - \left[ \left( \partial_x \int_0^L k\left(|x-\bar{x}|, \kappa\right) \partial_{\bar{x}\bar{x}} w \; d\bar{x} \right) \delta w \right]_0^L + \\
&\int_0^L \partial_{xx} \left( \int_0^L k\left(|x-\bar{x}|, \kappa\right) \partial_{\bar{x}\bar{x}} w \; d\bar{x} \right) \delta w \; dx \Bigg] - \\
&\int_0^L q_x \; \delta u \; dx - \int_0^L q_z \; \delta w \; dx \Bigg\}
\end{aligned}
\tag{A.4}
$$

The Euler-Lagrange Eqs. (11) and (12) are then obtained from Eq. (A.1) by setting the coefficients of $\delta u$ and of $\delta w$ in $(0, L)$ to zero for all $t$ in $(t_1, t_2)$. If the terms evaluated at $x = 0$ and $x = L$ in Eq. (A.1) are likewise canceled, the essential and natural boundary conditions given by Eqs. (13) to (15) are derived.

## Appendix B. Elements of the determinants defining the characteristic equations for bending

- Supported: After the boundary conditions $W_0 = 0$ and $\bar{\mathcal{M}}_0 = 0$, Eq. (75) becomes

$$
W(\xi) = \Phi_0 \; G_2(\xi) + \bar{Q}_0 \; G_4(\xi) \tag{B.1}
$$

and Eq. (57) becomes

$$
\begin{aligned}
\bar{\mathcal{M}}(\xi) = &\left[ -h^2 \lambda_w G_2(\xi) - G_2''(\xi) + \xi_1 h^2 G_2^{IV}(\xi) \right] \Phi_0 + \\
&\left[ -h^2 \lambda_w G_4(\xi) - G_4''(\xi) + \xi_1 h^2 G_4^{IV}(\xi) \right] \bar{Q}_0
\end{aligned}
\tag{B.2}
$$

Imposing the boundary conditions $W(1) = 0$ and $\bar{\mathcal{M}}(1) = 0$, the following system of equations is obtained

$$\left.\begin{aligned} a_{11}^{Supp}\Phi_0 + a_{12}^{Supp}\bar{Q}_0 = 0 \\ a_{21}^{Supp}\Phi_0 + a_{22}^{Supp}\bar{Q}_0 = 0 \end{aligned}\right\} \tag{B.3}$$

with

$$a_{11}^{Supp} = G_2\left(1\right) \tag{B.4}$$

$$a_{12}^{Supp} = G_4\left(1\right) \tag{B.5}$$

$$a_{21}^{Supp} = -h^2\lambda_w G_2\left(1\right) - G_2''\left(1\right) + \xi_1 h^2 G_2^{IV}\left(1\right) \tag{B.6}$$

$$a_{22}^{Supp} = -h^2\lambda_w G_4\left(1\right) - G_4''\left(1\right) + \xi_1 h^2 G_4^{IV}\left(1\right) \tag{B.7}$$

thus providing the characteristic Eq. (76).

- Cantilever: Proceeding in a similar way, we get the elements of the characteristic Eq. (77):

$$a_{11}^{Cant)} = -h^2\lambda_w G_3\left(1\right) - G_3''\left(1\right) + \xi_1 h^2 G_3^{IV}\left(1\right) \tag{B.8}$$

$$a_{12}^{Cant} = -h^2\lambda_w G_4\left(1\right) - G_4''\left(1\right) + \xi_1 h^2 G_4^{IV}\left(1\right) \tag{B.9}$$

$$a_{21}^{Cant} = -h^2\lambda_w G_3'\left(1\right) - G_3'''\left(1\right) + \xi_1 h^2 G_3^{V}\left(1\right) \tag{B.10}$$

$$a_{22}^{Cant} = -h^2\lambda_w G_4'\left(1\right) - G_4'''\left(1\right) + \xi_1 h^2 G_4^{V}\left(1\right) \tag{B.11}$$

- Free: Proceeding in a similar way, we get the elements of the characteristic Eq. (78);

$$a_{11}^{Free} = -h^2\lambda_w G_1\left(1\right) - G_1''\left(1\right) + \xi_1 h^2 G_1^{IV}\left(1\right) \tag{B.12}$$

$$a_{12}^{Free} = -h^2\lambda_w G_2\left(1\right) - G_2''\left(1\right) + \xi_1 h^2 G_2^{IV}\left(1\right) \tag{B.13}$$

$$a_{21}^{Free} = -h^2\lambda_w G_1'(1) - G_1'''(1) + \xi_1 h^2 G_1^V(1) \tag{B.14}$$

$$a_{22}^{Free} = -h^2\lambda_w G_2'(1) - G_2'''(1) + \xi_1 h^2 G_2^V(1) \tag{B.15}$$

## Appendix C. Fully nonlocal case for free bending vibration

For the fully nonlocal case ($\xi_1 = 0$), the governing Eq. (58) becomes

$$W^{IV}(\xi) + h^2\lambda_w W''(\xi) - \lambda_w W(\xi) = 0 \tag{C.1}$$

and for free bending vibration, the standard boundary conditions given by Eqs. (42) become

$$\bar{\mathcal{M}}(0) = -W''(0) - h^2\lambda_w W(0) = 0 \tag{C.2}$$

$$\bar{\mathcal{M}}'(0) = -W'''(0) - h^2\lambda_w W'(0) = 0 \tag{C.3}$$

$$\bar{\mathcal{M}}(1) = -W''(1) - h^2\lambda_w W(1) = 0 \tag{C.4}$$

$$\bar{\mathcal{M}}'(1) = -W'''(1) - h^2\lambda_w W'(1) = 0 \tag{C.5}$$

Fullfilment of the previous boundary conditions automatically satisfies the constitutive boundary conditions (73) and (74)

$$-hW'''(0) + W''(0) - h^3\lambda_w W'(0) + h^2\lambda_w W(0) = 0 \tag{C.6}$$

$$-hW'''(1) - W''(1) - h^3\lambda_w W'(1) - h^2\lambda_w W(1) = 0 \tag{C.7}$$

Note that, for the other studied boundary conditions, the constitutive boundary conditions are not fulfilled.

Finally, following a solution procedure similar to that described in section 4, the characteristic equation and eigenfrequencies can be derived.

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