Contributions to Conformable and non-Conformable Calculus

by

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To my mother

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- Fleitas, A., Méndez Bermúdez, J. A., Nápoles Valdés, J. E. & Sigarreta, J. M.(2019). On fractional Liénard-type systems. *Revista mexicana de física* 100 (1) 001-010.
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Abstract

In this work, we introduce a definition of a local fractional derivative and a fractional integral of order $\alpha > 0$ (where this parameter does not need to be integer), with which we overcome some deficiencies of known local derivatives, conformable or not. This definition allows to compute fractional derivatives of functions defined on any open set on the real line (and not just on the positive half-line). Moreover, we extend some classical results to the context of fractional derivatives. Also, applications of the fractional derivative through the direct and inverse problems are shown, as well as the feasibility of the fractional calculus in real and simulated problems.

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Chapter 1

Introduction

Throughout the centuries, Mathematics has played a crucial role in the development of human civilization, because among other reaches, it has allowed to describe and predict events in the real world, through mathematical representations. At this point, it is valid to emphasize the importance of Differential and Integral Calculus for the study of many of the laws of nature, either through problems of *direct* or *inverse* type.

In the present work we collect the different types of fractional derivatives known (global, conformable and non-conformable). In addition, we define a new derivative of fractional type, as well as the associated integral operator. In the same direction, different applications of the generalized fractional derivatives are shown, as well as their relations with respect to other types of derivatives.

The origins of fractional calculus date back to Newton and Leibniz in the seventeenth century (1695), and thus, this branch of mathematical analysis can be considered as old as classical calculus (see [62]). Fractional calculus extends derivation and integration to arbitrary non-integer orders, meaning by that, rational, real or even complex orders. The fact is that fractional calculus, at present, is successfully used to model a broad range of phenomena that occur in physics, economy and science.

In 1738, the first extension of the ordinary derivative appears in an article written by Euler in [18], where he notices that it makes sense to consider the derivative $\frac{d^n}{dx^n}$ of the power x^m for fractional p . Starting from the formulation for the derivative n -th derivative of $y = x^m$, he extends the result for the case of the fractional order

as follows:

$$\frac{d^p y}{dx^p} = \frac{\int_0^1 (-\ln t)^m dt}{\int_0^1 (-\ln t)^{m-p} dt} x^{m-p}, \quad m > p, \quad p \in \mathbb{R}^+. \quad (1.1)$$

Following the idea of Euler, in 1819 the article [49] by Lacroix, develops the derivative of the function $y = x^m$ by using Legendre's notation for the Gamma function as

$$\frac{d^p y}{dx^p} = \frac{\Gamma(m+1)}{\Gamma(m-p+1)} x^{m-p}. \quad (1.2)$$

In particular, when $y = x$ and $p = 1/2$, we have $\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$ and

$$\frac{d^{1/2} y}{dx^{1/2}} = \frac{2\sqrt{x}}{\sqrt{\pi}}. \quad (1.3)$$

This result coincides with what would be obtained by using the definition of Riemann-Liouville fractional derivative.

Later, in 1822, Fourier [25] suggests the use of the following integral representation for $f(x)$,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\alpha) d\alpha \int_{-\infty}^{\infty} \cos(p(x-\alpha)) dp, \quad (1.4)$$

to give the following definition of fractional derivative,

$$\frac{d^u}{dx^u} f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\alpha) d\alpha \int_{-\infty}^{+\infty} p^u \cos \left[p(x-\alpha) + u \frac{\pi}{2} \right] dp. \quad (1.5)$$

One year later, the first concrete application of fractional calculus is presented by Abel in [2, 3]. There, he used the derivative of order $1/2$ to solve the integral equation

$$\int_0^x (x-t)^{-1/2} f(t) dt = k, \quad (1.6)$$

arising from the formulation of the problem of *Tautocronia*.

Equation (1.6) is known as *Abel Integral Equation*, which except for the multiplicative factor $\frac{1}{\Gamma(1/2)}$, corresponds to Riemann-Liouville fractional integral of order $1/2$ of the function $f(x)$. Inspired by the work of Abel, in the papers [54, 53] in 1832, Liouville defines a fractional derivative based on the derivative of integer order

n of the exponential function

$$\frac{d^n}{dx^n} e^{ax} = a^n e^{ax}, \quad (1.7)$$

where $a \in \mathbb{R}$. Later, this result was extended for the derivatives of fractionary order p

$$\frac{d^p}{dx^p} e^{ax} = a^p e^{ax}. \quad (1.8)$$

Analogously, Liouville proposes that the derivative of arbitrary order of a function f , that can be expressed as the series of exponential functions

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}, \quad (1.9)$$

has the form

$$\frac{d^p f(x)}{dx^p} = \sum_{n=0}^{\infty} c_n a_n^p e^{a_n x}. \quad (1.10)$$

This formula is known as *Liouville's first definition of fractional derivative*. Liouville also adopted another method to treat the derivative of arbitrary order of functions of type x^{-a} , for real values of a with $a > 0$ and $x > 0$. In this approach, he considered the integral

$$I = \int_0^{\infty} u^{a-1} e^{-xu} du \quad (1.11)$$

and obtained

$$\frac{d^p}{dx^p} x^{-a} = \frac{(-1)^p \Gamma(a+p)}{\Gamma(a)} x^{-(a+p)}, \quad (1.12)$$

which is known as the *second definition of fractional derivative of Liouville*.

Both definitions were limited, because they were valid only for a restricted class of functions. Then, his attention shifted to the fractional integral. In 1832, in [54] he obtained the formula

$$\frac{d^{-p}}{dx^{-p}} f(x) = \frac{1}{(-1)^p \Gamma(p)} \int_0^{\infty} \tau^{p-1} f(x+\tau) d\tau, \quad \operatorname{Re}(p) > 0, \quad (1.13)$$

which, except for the term $(-1)^p$, is known today as the *definition of Liouville of the fractional integral of order α by the right*.

In 1867 and 1868, Anton Karl Grünwald in [32] and Aleksey Vasilievich Letnikov

in [51], respectively, propose a new definition of fractional derivative based on the classical definition of the iterated derivative,

$$D^n f(x) = \lim_{h \rightarrow 0} h^{-n} \sum_{k=0}^n (-1)^k \binom{n}{k} f(x - kh), \quad n \in \mathbb{N}, \quad (1.14)$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}. \quad (1.15)$$

Given $s \in \mathbb{R}$, we denote by $\lfloor s \rfloor$ and $\lceil s \rceil$ the lower and upper integer part of s , respectively.

Allowed them to define the following fractional derivative of order α

$${}^{GL}D_a^\alpha f(x) = \lim_{h \rightarrow 0} h^{-\alpha} \sum_{k=0}^{\lfloor \frac{x-a}{h} \rfloor} (-1)^k \binom{\alpha}{k} f(x - kh), \quad (1.16)$$

where $a < x$.

The current definition of *Riemann-Liouville fractional integral* is due to the work of N. Ya. Sonine in [72] and Laurent in [50], in the years 1870 and 1884, respectively. For this, both authors started from the *Cauchy formula for iterated integral* (1.17).

$$\begin{aligned} \frac{d^{-n} f(x)}{dx^{-n}} &= I_a^n f(x) \\ &= \int_a^x \int_a^{\tau_1} \cdots \int_a^{\tau_{n-1}} f(\tau) d\tau d\tau_{n-1} d\tau_{n-2} \cdots d\tau_2 d\tau_1 \\ &= \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt. \end{aligned} \quad (1.17)$$

Generalizing the factorial through the Gamma function, they defined

$$\frac{d^{-\alpha} f(x)}{dx^{-\alpha}} = I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad \operatorname{Re}(\alpha) > 0. \quad (1.18)$$

Next, the global fractional differential and integral operators currently used will be formally defined. For more information about some basic properties and relationships, see Podlubny [65], Samko [67], Kilbas [47] and Trujillo [11].

Definition 1 Let $\Omega = [a, b]$ ($-\infty < a < b < \infty$) be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals $I_{a+}^\alpha f$ and $I_{b-}^\alpha f$ of order $\alpha \in$

$\mathbb{C}(Re(\alpha) > 0)$ are defined by

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > a \quad (1.19)$$

and

$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x < b, \quad (1.20)$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the left and right sided fractional integrals (see [47]).

Definition 2 The Riemann-Liouville fractional derivatives ${}^{RL}D_{a+}^{\alpha} f$ and ${}^{RL}D_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by

$$\begin{aligned} {}^{RL}D_{a+}^{\alpha} f(x) &= D^n I_{a+}^{n-\alpha} f(x) \quad (n = \lceil \alpha \rceil; x > a) \\ &= \frac{1}{\Gamma(n-\alpha)} D^n \int_a^x \frac{f(t)}{(x-t)^{1-n+\alpha}} dt \end{aligned} \quad (1.21)$$

and

$$\begin{aligned} {}^{RL}D_{b-}^{\alpha} f(x) &= (-D)^n I_{b-}^{n-\alpha} f(x) \quad (n = \lceil \alpha \rceil; x < b) \\ &= \frac{1}{\Gamma(n-\alpha)} (-D)^n \int_x^b \frac{f(t)}{(x-t)^{1-n+\alpha}} dt, \end{aligned} \quad (1.22)$$

respectively (see [47]).

Definition 3 Given $\alpha, a > 0$ and $n = \lceil \alpha \rceil$, the Caputo fractional derivative ${}^CD_a^{\alpha} f$ of order α is defined by

$$\begin{aligned} {}^CD_a^{\alpha} f(x) &= I_{a+}^{n-\alpha} D^n f(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{D^n f(t)}{(a-t)^{1-n+\alpha}} dx \end{aligned} \quad (1.23)$$

(see [65]).

Definition 4 The left and right sided Grünwald-Letnikov fractional derivatives of order $\alpha > 0$, on a finite interval $[a, b]$, are defined by

$${}^{GL}D_{a+}^{\alpha} f(x) = \lim_{h \rightarrow +0} h^{-\alpha} \sum_{k=0}^{\lfloor \frac{x-a}{h} \rfloor} (-1)^k \binom{\alpha}{k} f(x - kh) \quad (1.24)$$

and

$${}^{GL}D_{b-}^{\alpha}f(x) = \lim_{h \rightarrow +0} h^{-\alpha} \sum_{k=0}^{\lfloor \frac{b-x}{h} \rfloor} (-1)^k \binom{\alpha}{k} f(x - kh) \quad (1.25)$$

respectively, where

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha + 1)}{\Gamma(k + 1)\Gamma(\alpha - k + 1)} \quad (1.26)$$

(see [47]).

One has to recognize that these fractional derivatives (Caputo, Riemann-Liouville and Grünwald-Letnikov) D^{α} show some drawbacks. Among them:

1. Most of the fractional derivatives (except Caputo-type), do not satisfy $D^{\alpha}(1) = 0$, if α is not a natural number.
2. Neither of the fractional derivatives satisfies the classical Product Rule $D^{\alpha}(fg) = gD^{\alpha}(f) + fD^{\alpha}(g)$.
3. Neither of the fractional derivatives satisfies the classical Quotient Rule $D^{\alpha}(\frac{f}{g}) = \frac{gD^{\alpha}(f) - fD^{\alpha}(g)}{g^2}$, with $g \neq 0$.
4. Neither of the fractional derivatives satisfies the Chain Rule $D^{\alpha}(f \circ g)(t) = D^{\alpha}(f(g))D^{\alpha}g(t)$.
5. Neither of the fractional derivatives satisfies the Indices Rule $D^{\alpha}D^{\beta}(f) = D^{\alpha+\beta}(f)$.
6. Fractional derivatives do not have a corresponding “calculus”.

Few yeras ago, the concept of conformable fractional derivative was introduced in [1]. One year later, Almeida *et. al.* in [7] defined a local fractional derivative, using for this purpose a certain incremental quotient. More recently, that result was generalized with the introduction of a non-conformable fractional derivative as a natural extension of the well-known definition of the derivative at a point [33]. This latter result is especially remarkable since it allows overcoming some inconsistencies of the fractional derivatives existing up to that moment. In this way, a new direction in fractional calculus was opened.

In 2014, Khalil *et. al.* publishes the article titled: *A new definition of fractional derivative* in [46], which is considered by many researchers, the work that starts the study of *conformable fractional derivatives*. This fractional derivative of order α is defined as follows.

A fractional derivative D^α is said conformable if $D^\alpha f(t) \rightarrow f'(t)$ when $\alpha \rightarrow 1$; i.e., when $\alpha \rightarrow 1$, $D^\alpha f(t)$ preserves the angle of the tangent line to the curve.

Definition 5 Given a function $f : [0, \infty) \rightarrow \mathbb{R}$, the conformable fractional derivative T_α of f of order α is defined by

$$T_\alpha f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad (1.27)$$

for all $t > 0$, $\alpha \in (0, 1)$. If f is α -differentiable in some $(0, a)$, $a > 0$ and $\lim_{t \rightarrow 0^+} f^\alpha(t)$ exists, then define

$$f^\alpha(0) = \lim_{t \rightarrow 0^+} f^\alpha(t).$$

Definition 6 Let $\alpha \in (n, n + 1]$, and f n -differentiable at $t > 0$. Then the conformable fractional derivative T_α of f of order α is defined as

$$T_\alpha f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f^{(\lceil \alpha \rceil - 1)}(t + \varepsilon t^{(\lceil \alpha \rceil - \alpha)}) - f^{(\lceil \alpha \rceil)}(t)}{\varepsilon}. \quad (1.28)$$

In [46] the following results are shown.

Theorem 7 If a function $f : [0, \infty) \rightarrow \mathbb{R}$ is α -differentiable at $t_0 > 0$, $\alpha \in (0, 1]$, then f is continuous at t_0 .

Theorem 8 Let $\alpha \in (0, 1]$ and f, g α -differentiable at a point in $(0, \infty)$. Then

1. $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for all $a, b \in \mathbb{R}$.
2. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
3. $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.
4. If, in addition, f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha} \frac{df}{dt}(t)$.
5. $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$.
6. $T_\alpha(\lambda) = 0$ for all functions $f(t) = \lambda$.

Also, [46] contains some results for this fractional derivative which are equivalent to some results of the classical calculus.

Theorem 9 (Rolle's Theorem type). Let $a > 0$ and $f : [a, b] \rightarrow \mathbb{R}$ be a given function satisfying:

1. f is continuous on $[a, b]$,

2. f is α -differentiable for some $\alpha \in (0, 1)$,

3. $f(a) = f(b)$.

Then, there exists $c \in (a, b)$ such that $f^{(\alpha)}(c) = 0$.

Theorem 10 (Mean Value Theorem type). Let $a > 0$ and $f : [a, b] \rightarrow \mathbb{R}$ be a given function that satisfying:

1. f is continuous on $[a, b]$,

2. f is α -differentiable for some $\alpha \in (0, 1)$.

Then, there exists $c \in (a, b)$ such that $f^{(\alpha)}(c) = \frac{f(b) - f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}$.

Later, Katugampola defines a new derivative in [44], as follows.

Definition 11 Let $f : [0, \infty) \rightarrow \mathbb{R}$ and $t > 0$. Then the fractional derivative $\mathcal{D}^\alpha$ of f of order $\alpha \in (0, 1)$ is defined by

$$\mathcal{D}^\alpha(f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(te^{\varepsilon t^{-\alpha}}) - f(t)}{\varepsilon}. \quad (1.29)$$

If f is α -differentiable in some interval $(0, a)$ and $\lim_{t \rightarrow 0^+} \mathcal{D}^\alpha(f)(t)$ exists, then we define

$$\mathcal{D}^\alpha(f)(0) = \lim_{t \rightarrow 0^+} \mathcal{D}^\alpha(f)(t).$$

This fractional derivative satisfies some properties which are similar to the ones of Khalil's operator T_α .

More recently, Karci in [42] proposes the following definition of fractional derivative.

Definition 12 Given $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function and $\alpha \in \mathbb{R}$, the fractional derivative $f^{(\alpha)}$ is defined as

$$f^{(\alpha)}(x) = \lim_{h \rightarrow 0} \frac{f^\alpha(x+h) - f^\alpha(x)}{(x+h)^\alpha - x^\alpha}. \quad (1.30)$$

A year later, Almeida *et al.* in [7] introduce a new type of fractional derivative with a kernel in the following way.

Definition 13 Let $k : [a, b] \rightarrow \mathbb{R}$ be a continuous nonnegative function such that $k(t) \neq 0$, whenever $t > a$. Given a function $f : [a, b] \rightarrow \mathbb{R}$ and $\alpha \in (0, 1)$, we say that f is α -differentiable at $t > a$, with respect to kernel k , if the limit

$$f^{(\alpha)}(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon k(t)^{1-\alpha}) - f(t)}{\varepsilon} \quad (1.31)$$

exists. The α -derivative at $t = a$ is defined by

$$f^{(\alpha)}(a) = \lim_{t \rightarrow a^+} f^{(\alpha)}(t), \quad (1.32)$$

if the limit exists.

The results of Khalil in [46] are generalized in [7].

Recently, in 2017, Vanterler *et al.* introduce in [73], a new type of fractional derivative based on the functions of Mittag-Leffler.

Definition 14 Let $f : [0, \infty) \rightarrow \mathbb{R}$. For $0 < \alpha < 1$ and $\beta > 0$, a truncated M -fractional type derivative of f of order α at $t > 0$, denoted by ${}_iD_M^{\alpha, \beta}$, is defined as

$${}_iD_M^{\alpha, \beta} f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t {}_iE_\beta(\varepsilon t^{-\alpha})) - f(t)}{\varepsilon}. \quad (1.33)$$

In 2018, Nápoles *et al.* define in [33], a new local derivative, in this case non-conformable, opening a new work area, because it allows obtaining nonclassical results when $\alpha \rightarrow 1$.

Definition 15 Given a function $f : [0, \infty) \rightarrow \mathbb{R}$. Then the N -derivative of f of order $\alpha \in (0, 1)$ at $t > 0$ is defined by

$$N_1^{(\alpha)} f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon e^{t^{-\alpha}}) - f(t)}{\varepsilon}. \quad (1.34)$$

If f is α -differentiable in some $(0, a)$ and $\lim_{t \rightarrow 0^+} N_1^{(\alpha)} f(t)$ exists, then $N_1^{(\alpha)} f(0) = \lim_{t \rightarrow 0^+} N_1^{(\alpha)} f(t)$.

As we have shown, the idea of fractional calculus is as old as traditional calculus (see [62]). Until recently, research on fractional calculus was confined to the field of mathematics. However, in the last two decades, many applications of fractional calculus appeared in several fields of engineering, applied sciences, physics, economy, etc (e.g. [28], [6] and [46]). Therefore, fractional calculus has become an important topic for researchers in several fields. This is not the result of chance; on the contrary, there is a strong mathematical foundation for it, as several general studies show ([9], [10], [27], [34], [6], [39], [40], [45], [47], [56] and [65]).

Chapter 2

On the generalized fractional derivative

2.1 Fractional derivative

In this section, a new definition of a local generalized fractional derivative of a function at a point t is introduced, and some similar results to those of classical calculus are obtained.

Given $s \in \mathbb{R}$, we denote by $\lceil s \rceil$ the *upper integer part* of s , i.e., the smallest integer greater than or equal to s .

Definition 16 *Given an interval $I \subseteq \mathbb{R}$, $f : I \rightarrow \mathbb{R}$, $\alpha \in \mathbb{R}^+$ and a positive continuous function $T(t, \alpha)$ on I , the derivative $G_T^\alpha f$ of f of order α at the point $t \in I$ is defined by*

$$G_T^\alpha f(t) = \lim_{h \rightarrow 0} \frac{1}{h^{\lceil \alpha \rceil}} \sum_{k=0}^{\lceil \alpha \rceil} (-1)^k \binom{\lceil \alpha \rceil}{k} f(t - khT(t, \alpha)). \quad (2.1)$$

If $a = \min\{t \in I\}$ (respectively, $b = \max\{t \in I\}$), then $G_T^\alpha f(a)$ (respectively, $G_T^\alpha f(b)$) is defined with $h \rightarrow 0^-$ (respectively, $h \rightarrow 0^+$) instead of $h \rightarrow 0$ in the limit.

Remark 17 *Notice that:*

(1) *If $T(t, \alpha) = 1$ when $\alpha \in \mathbb{N}$, then we obtain a conformable local fractional derivative of any order.*

(2) If $T(t, \alpha)$ depends on t when $\alpha \in \mathbb{N}$, then we get a non-conformable local fractional derivative of any order, something that has not been reported in the literature yet. If $\alpha \in (0, 1]$ and $T(t, \alpha) = e^{t^{-\alpha}}$, then the non-conformable fractional derivative defined in [33] is obtained.

If we choose the function $T(t, \alpha) = t^{\lceil \alpha \rceil - \alpha}$, then we obtain the following particular case of G_T^α , defined in [21]. Note that $T(t, \alpha) = t^{\lceil \alpha \rceil - \alpha} = 1$ for every $\alpha \in \mathbb{N}$.

Definition 18 Let I be an interval $I \subseteq (0, \infty)$, $f : I \rightarrow \mathbb{R}$ and $\alpha \in \mathbb{R}^+$. The conformable derivative $G^\alpha f$ of f of order α at the point $t \in I$ is defined by

$$G^\alpha f(t) = \lim_{h \rightarrow 0} \frac{1}{h^{\lceil \alpha \rceil}} \sum_{k=0}^{\lceil \alpha \rceil} (-1)^k \binom{\lceil \alpha \rceil}{k} f(t - kht^{\lceil \alpha \rceil - \alpha}). \quad (2.2)$$

We know from the classical calculus that if f is a function defined in a neighborhood of the point t , and there exists $D^n f(t)$, then

$$D^n f(t) = \lim_{h \rightarrow 0} \frac{1}{h^n} \sum_{k=0}^n (-1)^k \binom{n}{k} f(t - kh).$$

Therefore, if $\alpha = n \in \mathbb{N}$ and f is smooth enough, then Definition 18 coincides with the classical definition of the n -th derivative.

In [46] is defined a conformable derivative in the following way. Given $f : (0, \infty) \rightarrow \mathbb{R}$ and $\alpha \in (0, 1]$, the derivative of f of order α at the point t is defined by

$$T_\alpha f(t) = \lim_{h \rightarrow 0} \frac{f(t) - f(t - ht^{1-\alpha})}{h}. \quad (2.3)$$

It is clear then that T_α is a particular case of G_T^α when $\alpha \in (0, 1]$ and $T(t, \alpha) = t^{1-\alpha}$. See [1], [41] and [44] for more information on T_α .

In [46] is defined the following local fractional derivative.

Definition 19 Given $f : (0, \infty) \rightarrow \mathbb{R}$, the derivative $T_\alpha f$ of f of order $\alpha \in \mathbb{R}^+$ is defined by

$$T_\alpha f(t) = \lim_{h \rightarrow 0} \frac{f^{(\lceil \alpha \rceil - 1)}(t) - f^{(\lceil \alpha \rceil - 1)}(t - ht^{\lceil \alpha \rceil - \alpha})}{h}.$$

We can establish the following differences between our definition and the generalized derivative T_α :

(1) Definition 16 permits direct inferences for the fractional derivative of any positive order, whereas in Definition 19, the fractional derivative can be applied only to the last derivative.

(2) Definition 19 can be applied only to functions defined on a subset of $\mathbb{R}^+$, and the generality of Definition 16 allows to consider fractional derivatives of functions defined on $\mathbb{R}$.

We start by proving the following result.

Theorem 20 *Let I be an interval $I \subseteq \mathbb{R}$, $f : I \rightarrow \mathbb{R}$ and $\alpha \in \mathbb{R}^+$.*

(1) *If there exists $D^{\lceil \alpha \rceil} f$ at the point $t \in I$, then f is G_T^α -differentiable at t and $G_T^\alpha f(t) = T(t, \alpha)^{\lceil \alpha \rceil} D^{\lceil \alpha \rceil} f(t)$.*

(2) *If $\alpha \in (0, 1]$, then f is G_T^α -differentiable at $t \in I$ if and only if f is differentiable at t ; in this case, we have $G_T^\alpha f(t) = T(t, \alpha) f'(t)$.*

Proof. Assume first that there exists $D^{\lceil \alpha \rceil} f$ at the point t . If we take $q = hT(t, \alpha)$ in the definition of $G_T^\alpha f$, then we obtain

$$\begin{aligned} G_T^\alpha f(t) &= T(t, \alpha)^{\lceil \alpha \rceil} \lim_{q \rightarrow 0} \frac{1}{q^{\lceil \alpha \rceil}} \sum_{k=0}^{\lceil \alpha \rceil} (-1)^k \binom{\lceil \alpha \rceil}{k} f(t - kq) \\ &= T(t, \alpha)^{\lceil \alpha \rceil} D^{\lceil \alpha \rceil} f(t). \end{aligned}$$

Assume now that $\alpha \in (0, 1]$. Thus, $\lceil \alpha \rceil = 1$ and the previous argument gives

$$G_T^\alpha f(t) = T(t, \alpha) \lim_{q \rightarrow 0} \frac{f(t) - f(t - q)}{q} = T(t, \alpha) f'(t),$$

and there exists $G_T^\alpha f(t)$ if and only if there exists $f'(t)$. ■

Theorem 20 has the following consequences.

Proposition 21 *Let I be an interval $I \subseteq \mathbb{R}$, $f : I \rightarrow \mathbb{R}$ and $\alpha \in \mathbb{R}^+$. If there exists $D^{\lceil \alpha \rceil} f$ on I and $G_T^\alpha f = 0$ on I , then f is a polynomial of degree at most $\lceil \alpha \rceil - 1$ on I .*

Proposition 22 *Let f be a function with a local extreme at t_0 . If f is G_T^α -differentiable on t_0 for some $\alpha \in (0, 1]$, then $G_T^\alpha f(t_0) = 0$.*

Note that if $T(t, n) = 1$ when $n \in \mathbb{N}$ (in particular, if $T(t, \alpha) = t^{\lceil \alpha \rceil - \alpha}$) and there exists $D^n f(t)$, then Theorem 20 gives $G_T^\alpha f(t) = D^n f(t)$.

The following result contains some basic properties of the derivative G_T^α .

Theorem 23 *Let I be an interval $I \subseteq \mathbb{R}$, $f, g : I \rightarrow \mathbb{R}$ and $\alpha \in \mathbb{R}^+$. Assume that f, g are G_T^α -differentiable functions at $t \in I$. Then the following statements hold:*

(1) $af + bg$ is G_T^α -differentiable at t for every $a, b \in \mathbb{R}$, and $G_T^\alpha(af + bg)(t) = aG_T^\alpha f(t) + bG_T^\alpha g(t)$.

(2) If $\alpha \in (0, 1]$, then fg is G_T^α -differentiable at t and $G_T^\alpha(fg)(t) = f(t)G_T^\alpha g(t) + g(t)G_T^\alpha f(t)$.

(3) If $\alpha \in (0, 1]$ and $g(t) \neq 0$, then f/g is G_T^α -differentiable at t and $G_T^\alpha(\frac{f}{g})(t) = \frac{g(t)G_T^\alpha f(t) - f(t)G_T^\alpha g(t)}{g(t)^2}$.

(4) $G_T^\alpha(\lambda) = 0$, for every $\lambda \in \mathbb{R}$.

(5) $G_T^\alpha(t^p) = \frac{\Gamma(p+1)}{\Gamma(p-\lceil\alpha\rceil+1)} t^{p-\lceil\alpha\rceil} T(t, \alpha)^{\lceil\alpha\rceil}$, for every $p \in \mathbb{R} \setminus \mathbb{Z}^-$.

(6) $G_T^\alpha(t^{-n}) = (-1)^{\lceil\alpha\rceil} \frac{\Gamma(n+\lceil\alpha\rceil)}{\Gamma(n)} t^{-n-\lceil\alpha\rceil} T(t, \alpha)^{\lceil\alpha\rceil}$, for every $n \in \mathbb{Z}^+$.

Proof. Items (1) and (4) follow from the definition of G_T^α .

If $\alpha \in (0, 1]$, then Theorem 20 gives f, g are differentiable functions at t , and $G_T^\alpha h(t) = T(t, \alpha)h'(t)$ for any differentiable function h . These facts imply items (2) and (3).

Finally, since $f(t) = t^p$ is a C^∞ function on $(0, \infty)$, Theorem 20 gives $G_T^\alpha(t^p) = T(t, \alpha)^{\lceil\alpha\rceil} D^{\lceil\alpha\rceil}(t^p)$. ■

A similar argument to the one in the proof of Theorem 23 provides the chain rule for this fractional derivative.

Theorem 24 *Let $\alpha \in (0, 1]$, g a G_T^α -differentiable function at t and f a differentiable function at $g(t)$. Then $f \circ g$ is G_T^α -differentiable at t , and $G_T^\alpha(f \circ g)(t) = f'(g(t)) G_T^\alpha g(t)$.*

Following [46], if $f : [0, \infty) \rightarrow \mathbb{R}$, then we define $G^\alpha f(0) = \lim_{t \rightarrow 0^+} G^\alpha f(t)$ when this limit exists (in this case it is not possible to apply Definition 16, since $T(t, \alpha) = t^{\lceil\alpha\rceil-\alpha}$ and $T(0, \alpha) = 0$). Note that the statement in Theorem 20 does not hold in general for $G^\alpha f(0)$.

Proposition 25 *For each positive integer n and $\alpha \in (n-1, n)$ such that $n(n-\alpha) \notin \mathbb{Z}$, there are functions $f : [0, \infty) \rightarrow \mathbb{R}$ such that f is G^α -differentiable on $[0, \infty)$ and $D^{\lceil n(1-(n-\alpha)) \rceil} f(0)$ does not exist.*

Proof. Recall that $G^\alpha = G_T^\alpha$ with $T(t, \alpha) = t^{n-\alpha}$. Theorem 23 gives $G^\alpha(t^p) = \frac{\Gamma(p+1)}{\Gamma(p+1-n)} t^{p-n} (t^{n-\alpha})^n$, for every $p \in \mathbb{R} \setminus \mathbb{Z}^-$ and $t > 0$. If we choose $p = n(1 - (n - \alpha))$,

then we have

$$G^\alpha(t^p) = \frac{\Gamma(p+1)}{\Gamma(p+1-n)} t^{p-n} (t^{n-\alpha})^n = \frac{\Gamma(p+1)}{\Gamma(p+1-n)}$$

for every $t > 0$ and so, for every $t \geq 0$.

Since $p = n(1 - (n - \alpha)) \notin \mathbb{Z}$, we have that $(D^{\lceil p \rceil} t^p)(0) = (D^{\lceil n(1-(n-\alpha)) \rceil} t^p)(0)$ does not exist, and this finishes the proof. ■

If we take $\alpha = n - 1/2$ in Proposition 25, then we obtain the following result.

Corollary 26 *For each odd integer n greater than 1, there are functions $f : [0, \infty) \rightarrow \mathbb{R}$ such that f is $G^{n-1/2}$ -differentiable on $[0, \infty)$ and such that $D^{(n+1)/2} f(0)$ does not exist.*

As an application of fractional calculus, the following result provides a new proof of some known combinatorial identities.

Proposition 27 *For each positive integers $r < n$ we have*

$$\sum_{k=0}^n \binom{n}{k} (-1)^k k^r = 0, \quad \sum_{k=0}^n \binom{n}{k} (-1)^k k^n = (-1)^n n!$$

Proof. For each fixed n , let us choose $\alpha \in (n-1, n) \setminus \mathbb{Q}$; thus, $p = n(1 - (n - \alpha)) \notin \mathbb{Z}$. Let $g : [0, \infty) \rightarrow \mathbb{R}$ be the function $g(t) = t^p$. The argument in the proof of Proposition 25 gives

$$G^\alpha g(t) = \frac{\Gamma(p+1)}{\Gamma(p+1-n)} \neq 0$$

for every $t \geq 0$, since $p \notin \mathbb{Z}$. We have

$$\frac{\Gamma(p+1)}{\Gamma(p+1-n)} = G^\alpha g(1) = \lim_{h \rightarrow 0} h^{-n} \sum_{k=0}^n (-1)^k \binom{n}{k} (1 - kh)^p.$$

If

$$w(h) = \sum_{k=0}^n \binom{n}{k} (-1)^k (1 - kh)^p,$$

then

$$D^r w(h) = c_{p,r} \sum_{k=0}^n \binom{n}{k} (-1)^{k+r} (1 - kh)^{p-r} k^r,$$

with $r < n$ ($r \in \mathbb{N}$), and

$$c_{p,r} = p(p-1)(p-2) \cdots (p-(r-1)) = \frac{\Gamma(p+1)}{\Gamma(p+1-r)}.$$

Note that $c_{p,r} \neq 0$, since $p \notin \mathbb{Z}$. Hence, by applying Bernoulli-L'Hôpital's rule n times, we obtain

$$\begin{aligned} c_{p,n} &= G^\alpha g(1) = \lim_{h \rightarrow 0} \frac{c_{p,r} (-1)^r \sum_{k=0}^n \binom{n}{k} (-1)^k (1 - kh)^{p-r} k^r}{(n-r)! n(n-1) \cdots (n+1-r)} \\ &= \lim_{h \rightarrow 0} \frac{c_{p,n} (-1)^n \sum_{k=0}^n \binom{n}{k} (-1)^k (1 - kh)^{p-n} k^n}{n!} \\ &= c_{p,n} (-1)^n \frac{\sum_{k=0}^n \binom{n}{k} (-1)^k k^n}{n!}. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} (-1)^k k^r &= \lim_{h \rightarrow 0} \sum_{k=0}^n \binom{n}{k} (-1)^k (1 - kh)^{p-r} k^r = 0, \\ \sum_{k=0}^n \binom{n}{k} (-1)^k k^n &= (-1)^n n! \end{aligned}$$

■

Theorem 28 *Let I be an interval $I \subseteq \mathbb{R}$, $f : I \rightarrow \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}^+$ with $\lceil \alpha \rceil = \lceil \beta \rceil$. Then f is G_T^α -differentiable at $t \in I$ if and only if it is G_T^β -differentiable at t , and furthermore,*

$$G_T^\alpha f(t) = \left(\frac{T(t, \alpha)}{T(t, \beta)} \right)^{\lceil \alpha \rceil} G_T^\beta f(t).$$

In particular, f is G_T^α -differentiable if and only if it is $G_T^{\lceil \alpha \rceil}$ -differentiable.

Proof. Since $\lceil \alpha \rceil = \lceil \beta \rceil$, if we take $q = hT(t, \alpha)$ in the definition of $G_T^\alpha f$, then we obtain

$$\begin{aligned} G_T^\alpha f(t) &= T(t, \alpha)^{\lceil \alpha \rceil} \lim_{q \rightarrow 0} \frac{1}{q^{\lceil \alpha \rceil}} \sum_{k=0}^{\lceil \alpha \rceil} (-1)^k \binom{\lceil \alpha \rceil}{k} f(t - kq) \\ &= \left(\frac{T(t, \alpha)}{T(t, \beta)} \right)^{\lceil \alpha \rceil} T(t, \beta)^{\lceil \alpha \rceil} \lim_{q \rightarrow 0} \frac{1}{q^{\lceil \beta \rceil}} \sum_{k=0}^{\lceil \beta \rceil} (-1)^k \binom{\lceil \beta \rceil}{k} f(t - kq) \\ &= \left(\frac{T(t, \alpha)}{T(t, \beta)} \right)^{\lceil \alpha \rceil} G_T^\beta f(t), \end{aligned}$$

and there exists $G_T^\alpha f(t)$ if and only if there exists $G_T^\beta f(t)$. ■

Proposition 29 *Given any $t_0 \in \mathbb{R}$, there exists a function $g : \mathbb{R} \rightarrow \mathbb{R}$ which is G_T^α -differentiable at t_0 for every $\alpha > 1$, and which is not G_T^α -differentiable at t_0 for $\alpha \in (0, 1]$.*

Proof. Let us define a function $f : \mathbb{R} \rightarrow \mathbb{R}$ as follows: $f(t) = 0$ if $t \in \mathbb{Q}$ and $f(t) = t$ otherwise.

Since f is not differentiable at 0, Theorem 20 gives that f is not G_T^α -differentiable at 0 for $\alpha \in (0, 1]$.

Since $-h \in \mathbb{Q}$ if and only if $-kh \in \mathbb{Q}$ for every $k \in \mathbb{Z}^+$, we have for every $h \in \mathbb{R}$,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} f(-kh) = 0$$

for every $n \in \mathbb{N}$ with $n \geq 2$. Thus, we have

$$\sum_{k=0}^n (-1)^k \binom{n}{k} f(-khT(0, \alpha)) = 0$$

for every $h \in \mathbb{R}$ and $n \in \mathbb{N}$ with $n \geq 2$. Hence, f is G_T^n -differentiable at 0 for every $n \in \mathbb{N}$ with $n \geq 2$ and so, Theorem 28 gives that f is G_T^α -differentiable at 0 for every $\alpha > 1$.

Finally, given any $t_0 \in \mathbb{R}$, the function $g(t) = f(t - t_0)$ is G_T^α -differentiable at t_0 for every $\alpha > 1$, and it is not G_T^α -differentiable at t_0 for $\alpha \in (0, 1]$. ■

Proposition 30 *Given any $t_0 \in \mathbb{R}$, there exists a G_T^α -differentiable function $g : \mathbb{R} \rightarrow \mathbb{R}$ at t_0 for every $\alpha \in (1, 2]$, and which is not continuous at t_0 .*

Proof. Consider the sequence of prime numbers $\{p_k\}_{k=1}^\infty$ greater than 2.

Let us define a function $f : \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$\begin{aligned} f\left(-\frac{1}{2^n p_k}\right) &= \frac{p_k}{2^n}, & \text{for } k, n \geq 1, \\ f(t) &= 0, & \text{otherwise.} \end{aligned}$$

Since

$$\lim_{k \rightarrow \infty} f\left(-\frac{1}{p_k}\right) = \lim_{k \rightarrow \infty} p_k = \infty,$$

f is not continuous at 0.

If $q \in (0, 1/2)$ with $2q \neq 1/(2^n p_k)$ for every $k, n \geq 1$, then

$$f(0) - 2f(-q) + f(-2q) = 0.$$

If $q \in (0, 1/2)$ with $2q = 1/(2^n p_k)$ for some $k, n \geq 1$, then

$$f(0) - 2f\left(-\frac{1}{2^{n+1} p_k}\right) + f\left(-\frac{1}{2^n p_k}\right) = -2\frac{p_k}{2^{n+1}} + \frac{p_k}{2^n} = 0.$$

Hence, for each fixed $\alpha \in (1, 2]$, we have for enough small $h > 0$

$$f(0) - 2f(-hT(0, \alpha)) + f(-2hT(0, \alpha)) = 0,$$

and so, $G_T^\alpha f(0) = 0$ for every $\alpha \in (1, 2]$.

Therefore, given any $t_0 > 0$, the function $g(t) = f(t - t_0)$ is G_T^α -differentiable at t_0 for every $\alpha \in (1, 2]$, and it is not continuous at t_0 . ■

The following result shows an advantage of Definition 1 over some previous definitions of fractional derivatives: a general rule in order to compute iterated derivatives.

Theorem 31 *Let I be an interval $I \subseteq (0, \infty)$, $f : I \rightarrow \mathbb{R}$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in (0, 1]$. If there exists $D^n f(t)$ for some $t \in I$, then*

$$G^{\alpha_n} \dots G^{\alpha_2} G^{\alpha_1} f(t) = t^{-\alpha_1 - \alpha_2 - \dots - \alpha_n} \sum_{k=1}^n b_{n,k} t^k f^{(k)}(t),$$

with $b_{n,n} = 1$, $b_{n,0} = 0$, for $n \geq 1$,

$$\begin{aligned} b_{n,k} &= (k - \alpha_1 - \dots - \alpha_{n-1}) b_{n-1,k} + b_{n-1,k-1}, \quad \text{for } k < n, \\ b_{n,1} &= \prod_{j=1}^{n-1} (1 - \alpha_1 - \dots - \alpha_{j-1}), \quad \text{for } n > 1. \end{aligned}$$

Proof. Since $T(t, \alpha) = t^{1-\alpha}$ for $\alpha \in (0, 1]$, Theorem 20 gives the result if $n = 1$.

Assume that the result holds for $n - 1$ and let us prove that it holds for n . Theorem 20 gives

$$\begin{aligned} G^{\alpha_n} \dots G^{\alpha_2} G^{\alpha_1} f(t) &= G^{\alpha_n} \left(\sum_{k=1}^{n-1} b_{n-1,k} t^{k-\alpha_1-\dots-\alpha_{n-1}} f^{(k)}(t) \right) \\ &= \sum_{k=1}^{n-1} b_{n-1,k} (k - \alpha_1 - \dots - \alpha_{n-1}) t^{k-\alpha_1-\dots-\alpha_{n-1}-1} t^{1-\alpha_n} f^{(k)}(t) \\ &\quad + \sum_{k=1}^{n-1} b_{n-1,k} t^{k-\alpha_1-\dots-\alpha_{n-1}} t^{1-\alpha_n} f^{(k+1)}(t) \\ &= \sum_{k=1}^{n-1} b_{n-1,k} (k - \alpha_1 - \dots - \alpha_{n-1}) t^{k-\alpha_1-\dots-\alpha_n} f^{(k)}(t) \\ &\quad + \sum_{k=2}^n b_{n-1,k-1} t^{k-\alpha_1-\dots-\alpha_n} f^{(k)}(t), \end{aligned}$$

and this gives $b_{n,k} = (k - \alpha_1 - \dots - \alpha_{n-1}) b_{n-1,k} + b_{n-1,k-1}$ for $k < n$ (since $b_{n,0} = 0$)

and $b_{n,n} = b_{n-1,n-1} = 1$. Also,

$$\begin{aligned} b_{n,1} &= (1 - \alpha_1 - \cdots - \alpha_{n-1}) b_{n-1,1} \\ &= (1 - \alpha_1 - \cdots - \alpha_{n-1}) \prod_{j=1}^{n-2} (1 - \alpha_1 - \cdots - \alpha_{j-1}) \\ &= \prod_{j=1}^{n-1} (1 - \alpha_1 - \cdots - \alpha_{j-1}) \end{aligned}$$

for $n > 1$, and this finishes the proof. ■

Theorem 20 gives that the derivatives of any order are also conformable and provides a simple way to obtain $G^\alpha f$. Theorem 31 shows a rule in order to compute iterated derivatives although, unfortunately, the index law for iterated derivatives is not fulfilled.

Theorem 32 [*Rolle's Theorem*] Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function such that $f(a) = f(b)$ and f is G_T^α -differentiable on (a, b) for some $\alpha \in (0, 1]$. Then there exists $c \in (a, b)$ such that $G_T^\alpha f(c) = 0$.

Proof. Since f is continuous on $[a, b]$, and $f(a) = f(b)$, there is at least a point of local extreme $c \in (a, b)$. Since f is G_T^α -differentiable at c with $\alpha \in (0, 1]$, we have

$$\begin{aligned} G_T^\alpha f(c) &= G_T^\alpha f(c^+) = \lim_{h \rightarrow 0^+} \frac{f(c) - f(c - hT(c, \alpha))}{h} \\ &= G_T^\alpha f(c^-) = \lim_{h \rightarrow 0^-} \frac{f(c) - f(c - hT(c, \alpha))}{h}. \end{aligned}$$

Since $T(c, \alpha) > 0$ and c is a point of local extreme of f , we conclude that $G_T^\alpha f(c^+)$ and $G_T^\alpha f(c^-)$ have opposite signs, and so, Proposition 22 gives $G_T^\alpha f(c) = 0$. ■

Theorem 33 [*Mean Value Theorem*] Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function which is G_T^α -differentiable on (a, b) for some $\alpha \in (0, 1]$. Then there exists $c \in (a, b)$ such that

$$\frac{c^{1-\alpha}}{\alpha T(c, \alpha)} G_T^\alpha f(c) = \frac{f(b) - f(a)}{b^\alpha - a^\alpha}.$$

Proof. Let us consider the function

$$g(t) = f(t) - f(a) - \frac{f(b) - f(a)}{b^\alpha - a^\alpha} (t^\alpha - a^\alpha).$$

This function g satisfies the hypotheses of Rolle's Theorem (Theorem 32) and, there-

fore, there exists $c \in (a, b) \subset (0, \infty)$ such that $G_T^\alpha g(c) = 0$. Then we have

$$\begin{aligned} G_T^\alpha f(t) &= G_T^\alpha \left(g(t) + f(a) + \frac{f(b) - f(a)}{b^\alpha - a^\alpha} (t^\alpha - a^\alpha) \right) \\ &= G_T^\alpha g(t) + \frac{f(b) - f(a)}{b^\alpha - a^\alpha} G_T^\alpha (t^\alpha), \\ G_T^\alpha f(c) &= \frac{f(b) - f(a)}{b^\alpha - a^\alpha} G_T^\alpha (t^\alpha)(c). \end{aligned}$$

Since $\alpha \in (0, 1]$, Theorem 20 gives $G_T^\alpha (t^\alpha) = T(t, \alpha) \alpha t^{\alpha-1}$, and we conclude

$$G_T^\alpha f(c) = T(c, \alpha) \alpha c^{\alpha-1} \frac{f(b) - f(a)}{b^\alpha - a^\alpha}.$$

■

Theorem 33 allows to obtain the following results.

Theorem 34 *Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function which is G_T^α -differentiable on (a, b) for some $\alpha \in (0, 1]$. If $G_T^\alpha f(t) = 0$ for all $t \in (a, b)$, then f is constant on $[a, b]$.*

Proof. Let us consider $s, t \in [a, b]$ with $s \neq t$. Without loss of generality we can assume that $s < t$. By Theorem 33, there exists $c \in (s, t) \subseteq (a, b) \subset (0, \infty)$ such that

$$\frac{c^{1-\alpha}}{\alpha T(c, \alpha)} G_T^\alpha f(c) = \frac{f(t) - f(s)}{t^\alpha - s^\alpha}.$$

Since $G_T^\alpha f(c) = 0$, we conclude that $f(s) = f(t)$, and so, f is constant on $[a, b]$. ■

Theorem 35 *Let f be a continuous function on $[a, b]$ which is G_T^α -differentiable on (a, b) for some $\alpha \in (0, 1]$. Then $G_T^\alpha f(t) \geq 0$ (respectively, ≤ 0) if and only if f is a non-decreasing (respectively, non-increasing) function. If $G_T^\alpha f(t) > 0$ (respectively, $G_T^\alpha f(t) < 0$), then f is a strictly increasing (respectively, decreasing) function.*

Theorem 36 *Let f, g be continuous functions on $[a, b]$ which are G_T^α -differentiable on (a, b) for some $\alpha \in (0, 1]$. If $G_T^\alpha f(t) \geq G_T^\alpha g(t)$ for all $t \in (a, b)$, then the following statements hold:*

- (1) *If $f(a) = g(a)$, then $f(t) \geq g(t)$ for all $t \in [a, b]$.*
- (2) *If $f(b) = g(b)$, then $f(t) \leq g(t)$ for all $t \in [a, b]$.*

Proof. Consider the function $h(t) = f(t) - g(t)$. Then h is continuous on $[a, b]$ and G_T^α -differentiable on (a, b) . Theorem 23 gives that $G_T^\alpha h(t) = G_T^\alpha f(t) - G_T^\alpha g(t) \geq 0$

for all $t \in (a, b)$, and so, h is a non-decreasing function by Theorem 35. Hence, $h(a) \leq h(t) \leq h(b)$ for every $t \in [a, b]$.

If $h(a) = f(a) - g(a) = 0$, then $f(t) - g(t) = h(t) \geq h(a) = 0$ for every $t \in [a, b]$.

If $h(b) = f(b) - g(b) = 0$, then $f(t) - g(t) = h(t) \leq h(b) = 0$ for every $t \in [a, b]$.

■

We can use again Rolle's Theorem (Theorem 32) to prove the following result, which generalizes the Mean Value Theorem (Theorem 33).

Theorem 37 *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous functions which are G_T^α -differentiable on (a, b) for some $\alpha \in (0, 1]$. Then there exists $c \in (a, b)$ such that*

$$(g(b) - g(a)) G_T^\alpha f(c) = (f(b) - f(a)) G_T^\alpha g(c).$$

Furthermore, if $g(a) \neq g(b)$ and $G_T^\alpha g(c) \neq 0$, then

$$\frac{G_T^\alpha f(c)}{G_T^\alpha g(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. Let us define

$$F(t) = (g(b) - g(a))(f(t) - f(a)) - (f(b) - f(a))(g(t) - g(a)).$$

Since F satisfies the assumptions of Rolle's Theorem (Theorem 32), there exists $c \in (a, b)$ such that $G_T^\alpha F(c) = 0$. Hence,

$$G_T^\alpha F(c) = (g(b) - g(a)) G_T^\alpha f(c) - (f(b) - f(a)) G_T^\alpha g(c) = 0,$$

and this gives the result. ■

This result has a direct consequence which is a fractional version of Bernoulli-L'Hôpital's rule.

Corollary 38 *Let f, g be functions such that $\lim_{t \rightarrow a} f(t) = \lim_{t \rightarrow a} g(t) = 0$. If there exists $\lim_{t \rightarrow a} G_T^\alpha f(t)/G_T^\alpha g(t)$ for some $\alpha \in (0, 1]$, then*

$$\lim_{t \rightarrow a} \frac{f(t)}{g(t)} = \lim_{t \rightarrow a} \frac{G_T^\alpha f(t)}{G_T^\alpha g(t)}.$$

Let us state now some results on fractional derivatives of order $\alpha > 1$.

Theorem 39 *Let I be an interval $I \subseteq \mathbb{R}$, $f : I \rightarrow \mathbb{R}$ and $\alpha \in (1, 2]$. If f has a turning point in $t \in I$ and it is G_T^α -differentiable at t , then $G_T^\alpha f(t) = 0$.*

Proof. We can assume that there exists $\varepsilon > 0$ such that f is concave in $(t, t + \varepsilon)$ and convex in $(t - \varepsilon, t)$, since otherwise we can consider $-f$ instead of f . Since f is convex in $(t - \varepsilon, t)$, if $0 < h < \varepsilon/(2T(1, \alpha))$, then

$$\begin{aligned} f(t - hT(1, \alpha)) &\leq \frac{1}{2} f(t - 2hT(1, \alpha)) + \frac{1}{2} f(t), \\ \frac{f(t) - 2f(t - hT(1, \alpha)) + f(t - 2hT(1, \alpha))}{h^2} &\geq 0, \\ G_T^\alpha f(t) &\geq 0. \end{aligned}$$

Since f is concave in $(t, t + \varepsilon)$, a similar argument gives $G_T^\alpha f(t) \leq 0$, and we conclude $G_T^\alpha f(t) = 0$. ■

By using the argument in the proof of Theorem 39, we obtain the following result.

Theorem 40 *Let I be an interval $I \subseteq \mathbb{R}$, $f : I \rightarrow \mathbb{R}$ a convex (respectively, concave) function and $\alpha \in (1, 2]$. If f is G_T^α -differentiable at $t \in I$, then $G_T^\alpha f(t) \geq 0$ (respectively, $G_T^\alpha f(t) \leq 0$).*

Recall that Alexandrov theorem gives that every convex function has in almost every t a second-order Taylor expansion at t with a local error smaller than any quadratic [5].

Now we present our fractional version of Flett's theorem.

Theorem 41 *Let $f : [a, b] \rightarrow \mathbb{R}$ be a G_T^α -differentiable function on $[a, b]$ for some $\alpha \in (0, 1]$ such that $f'(a) = f'(b)$. Then there exists $c \in (a, b)$ such that*

$$\frac{G_T^\alpha f(c)}{T(c, \alpha)} = \frac{f(c) - f(a)}{c - a}.$$

Proof. Without loss of generality we can assume that $f'(a) = f'(b) = 0$, for if this is not the case, we could work with $f(t) - tf'(a)$. Let us consider the function

$$H(t) = \begin{cases} \frac{f(t) - f(a)}{t - a}, & t \neq a, \\ 0, & t = a. \end{cases}$$

Theorem 20 gives that H is a continuous function on $(a, b]$; since $f'(a) = 0$, H is continuous on $[a, b]$. Theorem 23 gives that H is G_T^α -differentiable on $(a, b]$. So we have, for $a < t \leq b$,

$$G_T^\alpha H(t) = G_T^\alpha \left[\frac{f(t) - f(a)}{t - a} \right] = -\frac{(f(t) - f(a))T(t, \alpha)}{(t - a)^2} + \frac{G_T^\alpha f(t)}{t - a}. \quad (2.4)$$

If $H(b) = 0$, then we have $H(a) = H(b)$ and the result is an immediate consequence of (2.4) and Rolle's Theorem (Theorem 32).

Suppose then $H(b) \neq 0$. We can assume $H(b) > 0$ (the case $H(b) < 0$ is similar). Since $G_T^\alpha f(b) = 0$, (2.4) gives

$$G_T^\alpha H(b) = -\frac{H(b)T(b, \alpha)}{b-a} < 0.$$

By Theorem 35, there exists $c_1 \in (a, b)$ such that $H(c_1) > H(b)$. Since H is a continuous function in $[a, c_1]$ and $0 = H(a) < H(b) < H(c_1)$, by the Intermediate Values Theorem there is $c_2 \in (a, c_1)$ such that $H(c_2) = H(b)$. From the Rolle's Theorem on $[c_2, b]$ we have

$$G_T^\alpha H(c) = -\frac{(f(c) - f(a))T(c, \alpha)}{(c-a)^2} + \frac{G_T^\alpha f(c)}{c-a} = 0.$$

■

Following a similar procedure we can prove the result below.

Theorem 42 *Let $f, g : [a, b] \rightarrow \mathbb{R}$ be G_T^α -differentiable functions on $[a, b]$ for some $\alpha \in (0, 1]$, with $G_T^\alpha g(a) \neq 0$, $g(t) \neq g(a)$ for all $t \in (a, b]$, $G_T^\alpha g(b)(g(b) - g(a)) > 0$ and $\frac{f'(a)}{g'(a)} = \frac{f'(b)}{g'(b)}$. Then there exists $c \in (a, b)$ such that*

$$(g(c) - g(a))G_T^\alpha f(c) = (f(c) - f(a))G_T^\alpha g(c).$$

Theorem 43 *Let $a \geq 0$ and $f : [a, b] \rightarrow \mathbb{R}$ a continuous function such that $f(a) = f(b)$ and f is G_T^α -differentiable on (a, b) for some $\alpha \in (0, 1]$. Then there exists $c \in (a, b)$ such that $G_T^\alpha f(c) = 0$.*

Proposition 44 *Let f be a continuous function on $[a, b]$ and G^α -differentiable in (a, b) for some $\alpha \in (0, 1]$, and $\lambda, \theta \in \mathbb{R}$. Then there exists $c \in (a, b)$ such that*

$$G^\alpha f(c) = \frac{c^{1-\alpha}[(\theta - \lambda)f(c) + \lambda f(b) - \theta f(a)]}{\lambda(c-a) + \theta(b-c)}. \quad (2.5)$$

Proof. Let us consider the auxiliary function

$$F(t) = \lambda(t-a)[f(t) - f(b)] - \theta(t-b)[f(t) - f(a)].$$

The function F is continuous on $[a, b]$ and G^α -differentiable in (a, b) . Also, we have

$F(a) = F(b) = 0$. By Theorem 32, there exists $c \in (a, b)$ such that

$$G^\alpha F(c) = 0.$$

Since

$$\begin{aligned} G^\alpha F(t) &= \lambda t^{1-\alpha}[f(t) - f(b)] + \lambda(t-a)G^\alpha f(t) - \theta t^{1-\alpha}[f(t) - f(a)] \\ &\quad - \theta(t-b)G^\alpha f(t), \end{aligned}$$

we have

$$\begin{aligned} G^\alpha F(c) &= \lambda c^{1-\alpha}[f(c) - f(b)] + \lambda(c-a)G^\alpha f(c) - \theta c^{1-\alpha}[f(c) - f(a)] \\ &\quad - \theta(c-b)G^\alpha f(c) = 0, \\ G^\alpha f(c) &= \frac{-\lambda c^{1-\alpha}[f(c) - f(b)] + \theta c^{1-\alpha}[f(c) - f(a)]}{\lambda(c-a) + \theta(b-c)}, \end{aligned}$$

which provides the desired result. ■

Using the previously developed ideas we can demonstrate the following classical result.

Proposition 45 *Let $a \geq 0$ and $f, g : [a, b] \rightarrow \mathbb{R}$ continuous functions which are G^α -differentiable on (a, b) for some $\alpha \in (0, 1]$. If $G^\alpha f(t) = G^\alpha g(t)$ for all $t \in (a, b)$, then there exists a constant C such that $f(t) = g(t) + C$ for every $t \in [a, b]$.*

Although the index law for iterated derivatives is not fulfilled, the following result provides a general rule in order to compute iterated derivatives.

2.2 Existence and uniqueness of solutions

Next we present our first result. The main part of Theorem 46 is the equivalence of the existence of $G^{\alpha_n} \dots G^{\alpha_2} G^{\alpha_1} f(t)$ and $f^{(n)}(t)$. Although the recurrence relation in this result appears in [23], we include a proof for the sake of completeness.

Theorem 46 *Let I be an interval $I \subseteq (0, \infty)$, $f : I \rightarrow \mathbb{R}$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in (0, 1]$. There exists $G^{\alpha_n} \dots G^{\alpha_2} G^{\alpha_1} f(t)$ for some $t \in I$ if and only if there exists $f^{(n)}(t)$. Furthermore,*

$$G^{\alpha_n} \dots G^{\alpha_2} G^{\alpha_1} f(t) = t^{-\alpha_1 - \alpha_2 - \dots - \alpha_n} \sum_{k=1}^n b_{n,k} t^k f^{(k)}(t),$$

with $b_{n,n} = 1$ for $n \geq 1$,

$$b_{n,k} = (k - \alpha_1 - \cdots - \alpha_{n-1}) b_{n-1,k} + b_{n-1,k-1}, \quad \text{for } 1 < k < n,$$

$$b_{n,1} = \prod_{j=1}^{n-1} (1 - \alpha_1 - \cdots - \alpha_j), \quad \text{for } n > 1.$$

Proof. Since $T(t, \alpha) = t^{1-\alpha}$ for $\alpha \in (0, 1]$, Theorem 20 gives the result if $n = 1$.

Assume that the result holds for $n - 1$ and let us show that it holds for n .

Assume first that there exists

$$G^{\alpha_n} \cdots G^{\alpha_1} f(t) = G^{\alpha_n} \left(G^{\alpha_{n-1}} \cdots G^{\alpha_1} f(t) \right)$$

for some $t \in I$. Thus, there exists $G^{\alpha_{n-1}} \cdots G^{\alpha_1} f(t)$, and the induction hypothesis gives that there exists $f^{(n-1)}(t)$, and

$$G^{\alpha_{n-1}} \cdots G^{\alpha_1} f(t) = \sum_{k=1}^{n-1} b_{n-1,k} t^{k-\alpha_1-\cdots-\alpha_{n-1}} f^{(k)}(t),$$

with $b_{n-1,n-1} = 1$ and

$$b_{n-1,1} = \prod_{j=1}^{n-2} (1 - \alpha_1 - \cdots - \alpha_j).$$

Theorem 20 gives that there exists

$$\left(\sum_{k=1}^{n-1} b_{n-1,k} t^{k-\alpha_1-\cdots-\alpha_{n-1}} f^{(k)}(t) \right)'$$

and

$$\begin{aligned} G^{\alpha_n} \cdots G^{\alpha_1} f(t) &= G^{\alpha_n} \left(\sum_{k=1}^{n-1} b_{n-1,k} t^{k-\alpha_1-\cdots-\alpha_{n-1}} f^{(k)}(t) \right) \\ &= t^{1-\alpha_n} \left(\sum_{k=1}^{n-1} b_{n-1,k} t^{k-\alpha_1-\cdots-\alpha_{n-1}} f^{(k)}(t) \right)'. \end{aligned}$$

Since t^β is a positive C^∞ function on $I \subseteq (0, \infty)$ for every $\beta \in \mathbb{R}$, and there exists

$f^{(n-1)}(t)$, we have that there exists

$$\begin{aligned}
& t^{\alpha_n-1} G^{\alpha_n} \dots G^{\alpha_1} f(t) - \left(\sum_{k=1}^{n-2} b_{n-1,k} t^{k-\alpha_1-\dots-\alpha_{n-1}} f^{(k)}(t) \right)' \\
&= \left(\sum_{k=1}^{n-1} b_{n-1,k} t^{k-\alpha_1-\dots-\alpha_{n-1}} f^{(k)}(t) \right)' - \left(\sum_{k=1}^{n-2} b_{n-1,k} t^{k-\alpha_1-\dots-\alpha_{n-1}} f^{(k)}(t) \right)' \\
&= \left(b_{n-1,n-1} t^{n-1-\alpha_1-\dots-\alpha_{n-1}} f^{(n-1)}(t) \right)' \\
&= \left(t^{n-1-\alpha_1-\dots-\alpha_{n-1}} f^{(n-1)}(t) \right)'.
\end{aligned}$$

Since $t^{n-1-\alpha_1-\dots-\alpha_{n-1}}$ is a positive C^∞ function on $I \subseteq (0, \infty)$, we conclude that there exists $f^{(n)}(t)$.

Furthermore,

$$\begin{aligned}
G^{\alpha_n} \dots G^{\alpha_1} f(t) &= G^{\alpha_n} \left(\sum_{k=1}^{n-1} b_{n-1,k} t^{k-\alpha_1-\dots-\alpha_{n-1}} f^{(k)}(t) \right) \\
&= t^{1-\alpha_n} \left(\sum_{k=1}^{n-1} b_{n-1,k} t^{k-\alpha_1-\dots-\alpha_{n-1}} f^{(k)}(t) \right)' \\
&= \sum_{k=1}^{n-1} b_{n-1,k} (k - \alpha_1 - \dots - \alpha_{n-1}) t^{k-\alpha_1-\dots-\alpha_{n-1}-1} t^{1-\alpha_n} f^{(k)}(t) \\
&\quad + \sum_{k=1}^{n-1} b_{n-1,k} t^{k-\alpha_1-\dots-\alpha_{n-1}} t^{1-\alpha_n} f^{(k+1)}(t) \\
&= \sum_{k=1}^{n-1} b_{n-1,k} (k - \alpha_1 - \dots - \alpha_{n-1}) t^{k-\alpha_1-\dots-\alpha_n} f^{(k)}(t) \\
&\quad + \sum_{k=2}^n b_{n-1,k-1} t^{k-\alpha_1-\dots-\alpha_n} f^{(k)}(t),
\end{aligned}$$

and this gives $b_{n,k} = (k - \alpha_1 - \dots - \alpha_{n-1}) b_{n-1,k} + b_{n-1,k-1}$ for $1 < k < n$, $b_{n,n} = b_{n-1,n-1} = 1$ and

$$\begin{aligned}
b_{n,1} &= (1 - \alpha_1 - \dots - \alpha_{n-1}) b_{n-1,1} \\
&= (1 - \alpha_1 - \dots - \alpha_{n-1}) \prod_{j=1}^{n-2} (1 - \alpha_1 - \dots - \alpha_j) \\
&= \prod_{j=1}^{n-1} (1 - \alpha_1 - \dots - \alpha_j)
\end{aligned}$$

for $n > 1$, and this finishes the proof of the equations.

Assume now that there exists $f^{(n)}(t)$. Thus, there exists $f^{(n-1)}(t)$ and, by the

induction hypothesis, there exists $G^{\alpha_{n-1}} \dots G^{\alpha_1} f(t)$, and

$$G^{\alpha_{n-1}} \dots G^{\alpha_1} f(t) = \sum_{k=1}^{n-1} b_{n-1,k} t^{k-\alpha_1-\dots-\alpha_{n-1}} f^{(k)}(t).$$

Since there exists $f^{(n)}(t)$, we have that there exists

$$\left(\sum_{k=1}^{n-1} b_{n-1,k} t^{k-\alpha_1-\dots-\alpha_{n-1}} f^{(k)}(t) \right)' = \left(G^{\alpha_{n-1}} \dots G^{\alpha_1} f(t) \right)'$$

and, by Theorem 20, there exists

$$G^{\alpha_n} \dots G^{\alpha_1} f(t) = G^{\alpha_n} \left(G^{\alpha_{n-1}} \dots G^{\alpha_1} f(t) \right) = t^{1-\alpha_n} \left(G^{\alpha_{n-1}} \dots G^{\alpha_1} f(t) \right)'.$$

■

Let I be an interval in $\mathbb{R}$ and Ω an open set in $\mathbb{R}^n$. The function $H : I \times \Omega \rightarrow \mathbb{R}^N$ is *uniformly Lipschitz* with respect to the second variable x if there is a constant L with $|H(t, x) - H(t, y)| \leq L|x - y|$ for every $t \in I$ and $x, y \in \Omega$. The function H is in (C, Lip) on $I \times \Omega$ if it is a continuous function on $I \times \Omega$ and it is uniformly Lipschitz continuous with respect to the second variable on this set.

Theorem 47 *Let $\alpha_1, \alpha_2, \dots, \alpha_n \in (0, 1]$, $x = (x_1, \dots, x_n)$, I an interval in $\mathbb{R}$, Ω an open set in $\mathbb{R}^n$, $t_0 \in I$ and $x_0 \in \Omega$. Let $F = (F_1, \dots, F_n) : I \times \Omega \rightarrow \mathbb{R}^n$ be in (C, Lip) on some open neighborhood of the point (t_0, x_0) , and consider the initial value problem*

$$G_T^{\alpha_j} x_j = F_j(t, x), \quad 1 \leq j \leq n, \quad x(t_0) = x_0. \quad (2.6)$$

Then there exists $h > 0$ such that (2.6) has a unique solution on $[t_0 - h, t_0 + h] \cap I$.

Furthermore, if $\Omega = \mathbb{R}^n$ and F is in (C, Lip) on $J \times \mathbb{R}^n$ for each compact interval $J \subseteq I$, then (2.6) has a unique solution on I .

Proof. Since $\alpha_1, \alpha_2, \dots, \alpha_n \in (0, 1]$, Theorem 20 gives that there is $G_T^{\alpha_j} x_j(t)$ if and only if there exists $x_j'(t)$ and, furthermore,

$$F_j(t, x) = G_T^{\alpha_j} x_j(t) = T(t, \alpha_j) x_j'(t),$$

for $1 \leq j \leq n$. Hence, (2.6) is equivalent to

$$x_j' = \frac{1}{T(t, \alpha_j)} F_j(t, x), \quad 1 \leq j \leq n, \quad x(t_0) = x_0. \quad (2.7)$$

By hypothesis, there are an interval $I_0 \subset I$ and an open set $\Omega_0 \subseteq \Omega$ such that $t_0 \in I_0$, $x_0 \in \Omega_0$ and F is in (C, Lip) on $I_0 \times \Omega_0$ with constant L . Since $T(t, \alpha_j)$ ($1 \leq j \leq n$) are positive continuous functions, there is a positive λ with $T(t, \alpha_j) \geq \lambda$ for every $t \in I_0$ and $1 \leq j \leq n$. If we define

$$H(t, x) = \left(\frac{1}{T(t, \alpha_1)} F_1(t, x), \dots, \frac{1}{T(t, \alpha_n)} F_n(t, x) \right),$$

then

$$\begin{aligned} |H(t, x) - H(t, y)|^2 &= \sum_{j=1}^n \frac{1}{T(t, \alpha_j)^2} (F_j(t, x) - F_j(t, y))^2 \\ &\leq \frac{1}{\lambda^2} |F(t, x) - F(t, y)|^2 \leq \frac{1}{\lambda^2} L^2 |x - y|^2 \end{aligned}$$

for every $t \in I_0$, $x, y \in \Omega_0$ and $1 \leq j \leq n$, and so, H is in (C, Lip) on $I_0 \times \Omega_0$ with constant L/λ . Thus, local Picard's Theorem guarantees that there exists $h > 0$ such that (2.7) (and so, (2.6)) has a unique solution on $[t_0 - h, t_0 + h] \cap I$.

Finally, assume that $\Omega = \mathbb{R}^n$ and F is in (C, Lip) on $J \times \mathbb{R}^n$ for each compact interval $J \subseteq I$. Choose compact intervals $\{J_k\}_{k=1}^\infty$ with $t_0 \in J_1$ and $\cup_{k=1}^\infty J_k = I$. For each k , global Picard's Theorem guarantees that (2.7) (and so, (2.6)) has a unique solution on J_k . Hence, (2.6) has a unique solution on I . ■

The argument in the proof of Theorem 47, by using Peano's Theorem instead of local Picard's Theorem, gives the following result.

Theorem 48 *Let $\alpha_1, \alpha_2, \dots, \alpha_n \in (0, 1]$, $x = (x_1, \dots, x_n)$, I an interval in $\mathbb{R}$, Ω an open set in $\mathbb{R}^n$, $t_0 \in I$ and $x_0 \in \Omega$. Let $F = (F_1, \dots, F_n) : I \times \Omega \rightarrow \mathbb{R}^n$ be a continuous function on some open neighborhood of the point (t_0, x_0) , and consider*

$$G_T^{\alpha_j} x_j = F_j(t, x), \quad 1 \leq j \leq n, \quad x(t_0) = x_0. \quad (2.8)$$

Then there exists $h > 0$ such that (2.8) has a solution on $[t_0 - h, t_0 + h] \cap I$.

Let us define

$$G_T^{\alpha_k, \dots, \alpha_1} f(t) = G_T^{\alpha_k} \dots G_T^{\alpha_1} f(t).$$

Theorem 47 has the following consequence.

Theorem 49 *Let $\alpha_1, \alpha_2, \dots, \alpha_n \in (0, 1]$, I an interval in $\mathbb{R}$, Ω an open set in $\mathbb{R}^n$, $t_0 \in I$ and $y_0 = (y_{0,0}, y_{0,1}, \dots, y_{0,n-1}) \in \Omega$. Let $f : I \times \Omega \rightarrow \mathbb{R}$ be in (C, Lip) on some open neighborhood of the point (t_0, y_0) , and consider*

$$\begin{aligned} G_T^{\alpha_n, \dots, \alpha_1} y &= f(t, y, G_T^{\alpha_1} y, G_T^{\alpha_2, \alpha_1} y, \dots, G_T^{\alpha_{n-1}, \dots, \alpha_1} y), \\ y(t_0) &= y_{0,0}, \quad G_T^{\alpha_j, \dots, \alpha_1} y(t_0) = y_{0,j}, \quad 1 \leq j \leq n-1. \end{aligned} \quad (2.9)$$

Then there exists $h > 0$ such that (2.9) has a unique solution on $[t_0 - h, t_0 + h] \cap I$.

Furthermore, if $\Omega = \mathbb{R}^n$ and f is in (C, Lip) on $J \times \mathbb{R}^n$ for each compact interval $J \subseteq I$, then (2.9) has a unique solution on I .

Proof. Consider the new variables

$$x_1(t) = y(t), \quad x_j(t) = G_T^{\alpha_{j-1}, \dots, \alpha_1} y(t), \quad 2 \leq j \leq n.$$

Thus, (2.9) is equivalent to

$$\begin{aligned} G_T^{\alpha_1} x_1(t) &= x_2(t), & x_1(t_0) &= y_{0,0}, \\ G_T^{\alpha_2} x_2(t) &= x_3(t), & x_2(t_0) &= y_{0,1}, \\ &\dots & \dots & \\ G_T^{\alpha_{n-1}} x_{n-1}(t) &= x_n(t), & x_{n-1}(t_0) &= y_{0,n-2}, \\ G_T^{\alpha_n} x_n(t) &= f(t, x_1(t), x_2(t), \dots, x_n(t)), & x_n(t_0) &= y_{0,n-1}. \end{aligned}$$

Let us consider $x = (x_1, \dots, x_n)$ and $H = (H_1, \dots, H_n) : I \times \Omega \rightarrow \mathbb{R}^n$ given by $H_1(t, x) = x_2$, $H_2(t, x) = x_3$, $\dots$, $H_{n-1}(t, x) = x_n$, $H_n(t, x) = f(t, x)$. Since

$$\left| H(t, x) - H(t, x') \right|^2 = \sum_{j=2}^n (x_j - x'_j)^2 + \left(f(t, x) - f(t, x') \right)^2,$$

H is in (C, Lip) on some open neighborhood of the point (t_0, y_0) . Furthermore, if $\Omega = \mathbb{R}^n$ and f is in (C, Lip) on $J \times \mathbb{R}^n$ for each compact interval $J \subseteq I$, then H is in (C, Lip) on $J \times \mathbb{R}^n$ for each compact interval $J \subseteq I$. Hence, Theorem 47 gives the result. ■

Theorem 50 Let $p_0, p_1, \dots, p_{n-1}$ be continuous functions on I , and $\alpha_1, \alpha_2, \dots, \alpha_n \in (0, 1]$. Then the general solution of the linear FDE

$$G_T^{\alpha_n, \dots, \alpha_1} y + p_{n-1}(t) G_T^{\alpha_{n-1}, \dots, \alpha_1} y + \dots + p_1(t) G_T^{\alpha_1} y + p_0(t) y = 0 \quad (2.10)$$

on I is a linear space with dimension n .

Proof. For each $t_0 \in I$, and $y_0 = (y_{0,0}, y_{0,1}, \dots, y_{0,n-1}) \in \mathbb{R}^n$, let us consider the initial value problem

$$\begin{aligned} G_T^{\alpha_n, \dots, \alpha_1} y + p_{n-1}(t) G_T^{\alpha_{n-1}, \dots, \alpha_1} y + \dots + p_1(t) G_T^{\alpha_1} y + p_0(t) y &= 0, \\ y(t_0) &= y_{0,0}, \quad G_T^{\alpha_j, \dots, \alpha_1} y(t_0) = y_{0,j}, \quad 1 \leq j \leq n-1. \end{aligned} \quad (2.11)$$

Let us define $f(t, x) = -p_{n-1}(t)x_n - \dots - p_1(t)x_2 - p_0(t)x_1$, and consider any compact interval $J \subseteq I$. We can choose N with $|p_j(t)| \leq N$ for every $t \in J$ and $0 \leq j \leq n-1$.

Thus,

$$\left| f(t, x) - f(t, x') \right| \leq \sum_{j=1}^n \left| p_{j-1}(t)(x_j - x'_j) \right| \leq N \sum_{j=1}^n |x_j - x'_j| \leq N n^{1/2} |x - x'|$$

and f is in (C, Lip) on $J \times \mathbb{R}^n$. Therefore, Theorem 49 gives that (2.11) has a unique solution on I .

For each $0 \leq i \leq n-1$, let z_i be the unique solution of (2.11) on I with initial values $y_{0,j} = \delta_{i,j}$ for $0 \leq j \leq n-1$. Since (2.10) is a linear equation, the unique solution y of (2.11) with general initial values $y_0 = (y_{0,0}, y_{0,1}, \dots, y_{0,n-1}) \in \mathbb{R}^n$ is $y = y_{0,0}z_0 + y_{0,1}z_1 + \dots + y_{0,n-1}z_{n-1}$. Therefore, the general solution of the fractional linear equation (2.10) on I is a linear space with dimension n . ■

Theorem 51 *Let $p_0, p_1, \dots, p_{n-1}, q$ be continuous functions on the interval $I \subseteq \mathbb{R}$, and $\alpha_1, \alpha_2, \dots, \alpha_n \in (0, 1]$. Then the general solution of the linear FDE*

$$G_T^{\alpha_n, \dots, \alpha_1} y + p_{n-1}(t) G_T^{\alpha_{n-1}, \dots, \alpha_1} y + \dots + p_1(t) G_T^{\alpha_1} y + p_0(t) y = q(t) \quad (2.12)$$

on I is an affine space with dimension n , and so, it is the sum of a particular solution and the general solution of the homogeneous equation (2.10).

Proof. The argument in the proof of Theorem 50, with $f(t, x) = q(t) - p_{n-1}(t)x_n - \dots - p_1(t)x_2 - p_0(t)x_1$, gives that (2.12) has a solution z on I . If we write a solution of (2.12) as $y = z + x$, then x is a solution of (2.10). Hence, the general solution of (2.12) is the sum of z and the general solution of (2.10), and Theorem 50 gives the result. ■

Theorem 52 *Let I be an interval $I \subseteq (0, \infty)$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in (0, 1]$. The general solution of the equation $G^{\alpha_n} \dots G^{\alpha_2} G^{\alpha_1} f = 0$ on I is*

$$f(t) = c_1 + \sum_{k=1}^{n-1} c_k t^{\alpha_1 + \alpha_2 + \dots + \alpha_k} \quad (2.13)$$

with $c_1, c_2, \dots, c_n \in \mathbb{R}$.

Proof. Theorem 50 gives that the general solution of the equation $G^{\alpha_n} \dots G^{\alpha_2} G^{\alpha_1} f = 0$ on I is a linear space with dimension n . So, it suffices to check that the n linearly independent functions

$$1, t^{\alpha_1}, t^{\alpha_1 + \alpha_2}, \dots, t^{\alpha_1 + \alpha_2 + \dots + \alpha_{n-1}}$$

are solutions of the equation $G^{\alpha_n} \dots G^{\alpha_1} f = 0$.

Note that Theorem 20 gives

$$G^{\alpha_j} t^p = p t^{p-1} T(t, \alpha_j) = p t^{p-1} t^{1-\alpha_j} = p t^{p-\alpha_j}$$

for every $t > 0$, $p \in \mathbb{R}$ and $1 \leq j \leq n$. Therefore,

$$G^{\alpha_j, \dots, \alpha_1} t^p = p(p - \alpha_1)(p - \alpha_1 - \alpha_2) \cdots (p - \alpha_1 - \cdots - \alpha_{j-1}) t^{p-\alpha_1-\cdots-\alpha_j}$$

and so,

$$\begin{aligned} G^{\alpha_1} 1 &= 0, \\ G^{\alpha_2, \alpha_1} t^{\alpha_1} &= G^{\alpha_2} G^{\alpha_1} t^{\alpha_1} = G^{\alpha_2} \alpha_1 = 0, \\ G^{\alpha_3, \alpha_2, \alpha_1} t^{\alpha_1 + \alpha_2} &= G^{\alpha_3} G^{\alpha_2, \alpha_1} t^{\alpha_1 + \alpha_2} = G^{\alpha_3} (\alpha_1 + \alpha_2) \alpha_2 = 0, \\ &\dots \\ G^{\alpha_n, \dots, \alpha_1} t^{\alpha_1 + \cdots + \alpha_{n-1}} &= G^{\alpha_n} G^{\alpha_{n-1}, \dots, \alpha_1} t^{\alpha_1 + \cdots + \alpha_{n-1}} \\ &= G^{\alpha_n} (\alpha_1 + \cdots + \alpha_{n-1}) (\alpha_2 + \cdots + \alpha_{n-1}) \cdots \alpha_{n-1} = 0, \end{aligned}$$

and this finishes the proof. ■

2.3 Integral operator

Let I be an interval, $a, t \in I$ and $\alpha \in \mathbb{R}$. Following [46], the integral operator $J_{T,a}^\alpha$ is defined for every locally integrable function f on I as

$$J_{T,a}^\alpha(f)(t) = \int_a^t \frac{f(s)}{T(s, \alpha)} ds.$$

Proposition 53 *Let I be an interval, $a \in I$, $0 < \alpha \leq 1$ and f a differentiable function on I with f' locally integrable on I . Then, we have for all $t \in I$*

$$J_{T,a}^\alpha(G_T^\alpha(f))(t) = f(t) - f(a).$$

Proof. Since f' is a locally integrable function on I , Theorem 20 gives

$$J_{T,a}^\alpha(G_T^\alpha(f))(t) = \int_a^t \frac{G_T^\alpha(f)(s)}{T(s, \alpha)} ds = \int_a^t f'(s) ds = f(t) - f(a),$$

which is the desired result. ■

Proposition 54 *Let I be an interval, $a \in I$ and $\alpha \in (0, 1]$.*

$$G_T^\alpha(J_{T,a}^\alpha(f))(t) = f(t),$$

for every continuous function f on I and $a, t \in I$.

Proof. Let f be a continuous function on I . Proposition 53 gives for $a, t \in I$

$$\begin{aligned} \left(J_{T,a}^\alpha(f)(t) \right)' &= \left(\int_a^t \frac{f(s)}{T(s, \alpha)} ds \right)' = \frac{f(t)}{T(t, \alpha)}, \\ G_T^\alpha J_{T,a}^\alpha(f)(t) &= T(t, \alpha) \left(J_{T,a}^\alpha(f)(t) \right)' = T(t, \alpha) \frac{f(t)}{T(t, \alpha)} = f(t). \end{aligned}$$

■

In [46] it is defined the integral operator $J_{T,a}^\alpha$ for the choice of T given by $T(t, \alpha) = t^{1-\alpha}$, and [46, Theorem 3.1] shows

$$G^\alpha J_{t^{1-\alpha},a}^\alpha(f)(t) = f(t),$$

for every continuous function f on I , $a, t \in I$ and $\alpha \in (0, 1]$. Hence, Proposition 54 extends to any T this important equality.

Theorem 55 below contains some elementary facts on the integral operator $J_{T,a}^\alpha$.

Theorem 55 *Let I be an interval, $a, b \in I$ and $\alpha \in \mathbb{R}$. Let f, g be locally integrable functions on I , and $k_1, k_2 \in \mathbb{R}$. We have*

- (1) $J_{T,a}^\alpha(k_1f + k_2g)(t) = k_1J_{T,a}^\alpha f(t) + k_2J_{T,a}^\alpha g(t)$,
- (2) if $f \geq g$, then $J_{T,a}^\alpha f(t) \geq J_{T,a}^\alpha g(t)$ for every $t \in I$ with $t \geq a$,
- (3) $|J_{T,a}^\alpha f(t)| \leq J_{T,a}^\alpha |f|(t)$ for every $t \in I$ with $t \geq a$,
- (4) $\int_a^b \frac{f(s)}{T(s, \alpha)} ds = J_{T,a}^\alpha f(t) - J_{T,b}^\alpha f(t) = J_{T,a}^\alpha f(t)(b)$ for every $t \in I$.

Given $a, b \in I$ ($b > a$), let us denote by $F_{a,b}$ the usual inner product in $L^2[a, b]$

$$F_{a,b}(f, g) = \int_a^b f(t)g(t) dt.$$

Proposition 56 *Let I be an interval, $a, b \in I$ with $a < b$ and $\alpha \in \mathbb{R}$. The adjoint of $J_{T,a}^\alpha$ in $L^2[a, b]$ with respect to the inner product $F_{a,b}$ is the operator*

$$A_{T,a,b}^\alpha(f)(t) = \frac{1}{T(t, \alpha)} \int_t^b f(s) ds.$$

Proof. We obtain, by using integration by parts,

$$\begin{aligned}
 F_{a,b}(J_{T,a}^\alpha(f), g) &= \int_a^b g(t) J_{T,a}^\alpha(f)(t) dt \\
 &= \left[-J_{T,a}^\alpha(f)(t) \int_t^b g(s) ds \right]_{t=a}^{t=b} + \int_a^b \frac{f(t)}{T(t, \alpha)} \int_t^b g(s) ds dt \\
 &= \int_a^b \frac{f(t)}{T(t, \alpha)} \int_t^b g(s) ds dt = \int_a^b f(t) A_{T,a,b}^\alpha(g)(t) dt \\
 &= F_{a,b}(f, A_{T,a,b}^\alpha(g)).
 \end{aligned}$$

■

Proposition 57 *Let I be an interval, $a, b \in I$ with $a < b$ and $\alpha \in \mathbb{R}$. Then $J_{T,a}^\alpha$ is a Hilbert-Schmidt integral operator on $L^2[a, b]$, and so, a continuous and compact operator.*

Proof. Let us denote by χ_A the *characteristic function* of the set A (i.e., the function such that $\chi_A(t) = 1$ when $t \in A$ and $\chi_A(t) = 0$ otherwise). Then

$$J_{T,a}^\alpha(f)(t) = \int_a^t \frac{f(s)}{T(s, \alpha)} ds = \int_a^b k(t, s) f(s) ds,$$

with

$$k(t, s) = \chi_{[a,t]}(s) \frac{1}{T(s, \alpha)}.$$

We have

$$\int_a^b \int_a^b |k(t, s)|^2 ds dt \leq \int_a^b \int_a^b \frac{1}{T(s, \alpha)^2} ds dt = \int_a^b \frac{b-a}{T(s, \alpha)^2} ds < \infty,$$

since $T(s, \alpha)$ is a positive continuous function on I . Thus, $J_{T,a}^\alpha$ is a Hilbert-Schmidt integral operator on $L^2[a, b]$, and so, a compact operator. ■

Chapter 3

Applications

3.1 Fractional logistic growth models

We start this section by studying the fractional equation associated with the logistic model.

The logistic model allows to describe many phenomena of the real world such as: population growth, spread of diseases, diffusion in social networks, etc. This model constitutes a refinement of the classical exponential model growth. The parameters of this logistic model are: K charge capacity, a rate of change and y_0 initial population (see [75]).

Proposition 58 *Let $\alpha \in (0, 1]$, $a, K > 0$ and $t_0 \in I \subseteq \mathbb{R}$. Then the general solution of the fractional logistic equation*

$$G_T^\alpha y(t) = a y(t) (K - y(t)), \quad (3.1)$$

is given by

$$y(t) = \frac{KC}{(K - C) e^{-aK J_{T, t_0}^\alpha (1)(t)} + C}, \quad (3.2)$$

with $C \in \mathbb{R}$.

Proof. If y is given by (3.2), then Theorems 23 and 24 and Proposition 54 give

$$\begin{aligned}
 G_T^\alpha y(t) &= \frac{-KC G_T^\alpha \left((K-C) e^{-aK J_{T,t_0}^\alpha(1)} + C \right) (t)}{\left((K-C) e^{-aK J_{T,t_0}^\alpha(1)(t)} + C \right)^2} \\
 &= \frac{aK^2 C (K-C) e^{-aK J_{T,t_0}^\alpha(1)(t)}}{\left((K-C) e^{-aK J_{T,t_0}^\alpha(1)(t)} + C \right)^2} \\
 &= a y(t) \frac{K(K-C) e^{-aK J_{T,t_0}^\alpha(1)(t)}}{(K-C) e^{-aK J_{T,t_0}^\alpha(1)(t)} + C} \\
 &= a y(t) \left(K - \frac{KC}{(K-C) e^{-aK J_{T,t_0}^\alpha(1)(t)} + C} \right) \\
 &= a y(t) (K - y(t)),
 \end{aligned}$$

and y is a solution of (3.1).

Let us consider now the initial value problem (3.1) with $y(t_0) = y_0$. If we choose $C = y_0$, then

$$y(t) = \frac{K y_0}{(K - y_0) e^{-aK J_{T,t_0}^\alpha(1)(t)} + y_0} \quad (3.3)$$

is a solution of (3.1) satisfying $y(t_0) = y_0$, as desired. Since the initial value problem (3.1) with $y(t_0) = y_0$ has a unique solution, the proof is finished. ■

A proof of the following useful result appears in [24].

Lemma 59 *Assume that $y(t)$ is a positive differentiable function on $[t_0, \infty)$ and $a(t)$ is a continuous function on $[t_0, \infty)$.*

(1) *If $G_T^\alpha y \leq a(t)y$ for every $t \in [t_0, \infty)$, then*

$$y(t) \leq y(t_0) e^{J_{T,t_0}^\alpha(a)(t)}.$$

(2) *If $G_T^\alpha y \geq a(t)y$ for every $t \in [t_0, \infty)$, then*

$$y(t) \geq y(t_0) e^{J_{T,t_0}^\alpha(a)(t)}.$$

If the solutions starting near an equilibrium point y_e stay close to y_e for all $t \geq t_0$, then we say that y_e is (Lyapunov) *stable*. Furthermore, y_e is *asymptotically stable* if it is stable and every solution starting near y_e converges to y_e .

Next, we prove a result on stability of equilibrium points of fractional differential equations.

Theorem 60 *Let I be an interval with $[t_0, \infty) \subseteq I$, $P_0 \in \mathbb{R}$, $M_0 > 0$ and $\alpha \in (0, 1]$. Let p be a continuous function with $p(t) \geq P_0$ for every $t \in [t_0, \infty)$, and g a continuous function on $[t_0, \infty) \times \mathbb{R}$ and locally Lipschitz in the second variable, such that*

$$\left| \frac{g(t, z)}{z} \right| \leq M_0 \quad \text{for every } t \geq t_0, z \in \mathbb{R} \setminus \{0\}. \quad (3.4)$$

If $\lim_{t \rightarrow \infty} J_{T, t_0}^\alpha(1)(t) < \infty$, then $y(t) \equiv 0$ is an stable solution of the fractional differential equation

$$G_T^\alpha y + p(t)y = g(t, y). \quad (3.5)$$

Proof. By (3.4), we have

$$|g(t, z)| \leq M_0 |z|, \quad \text{for all } t \geq t_0, z \in \mathbb{R} \setminus \{0\}. \quad (3.6)$$

Since g is a continuous function, $g(t, 0) = 0$ for every $t \in [t_0, \infty)$, and $y(t) \equiv 0$ is a solution of (3.5). The hypotheses guarantee that the initial value problem (3.5) with $y(t_0) = y_0$ has a unique solution. Hence, two solutions of (3.5) do not intersect. Choose any $y_0 \neq 0$. Thus, we have either $y(t) > 0$ for $t \geq t_0$ or $y(t) < 0$ for $t \geq t_0$.

Assume first $y(t) > 0$. (3.6) gives $-p(t)y + g(t, y) \leq -P_0 y + M_0 y$ for $t \geq t_0$. Therefore, we have $G_T^\alpha y \leq (M_0 - P_0)y$ and Lemma 59 gives $0 < y(t) \leq y_0 e^{(M_0 - P_0)J_{T, t_0}^\alpha(1)(t)}$. Since $T(t, \alpha)(t) > 0$, there exists $\lim_{t \rightarrow \infty} J_{T, t_0}^\alpha(1)(t)$. Since this limit is finite by hypothesis, $J_{T, t_0}^\alpha(1)(t)$ is a bounded function and there exists M_1 such that $e^{(M_0 - P_0)J_{T, t_0}^\alpha(1)(t)} \leq M_1$ for all $t \geq t_0$. Thus, $0 < y(t) \leq y_0 M_1$ for every $t \geq t_0$.

If $y(t) < 0$, then a similar argument gives $0 > y(t) \geq y_0 M_2$ for every $t \geq t_0$.

Hence, given any $\varepsilon > 0$, if $|y_0| < \varepsilon / \min\{M_1, M_2\}$, then $|y(t)| < \varepsilon$ for every $t \geq t_0$, and so, the trivial solution $y(x) \equiv 0$ is stable. ■

Theorem 61 *Let I be an interval with $[t_0, \infty) \subseteq I$, $K > 0$ and $\alpha \in (0, 1]$. Let $a(t)$ be a positive continuous function on $[t_0, \infty)$. If $\lim_{t \rightarrow \infty} J_{T, t_0}^\alpha(a)(t) = \infty$, then $y(t) \equiv 0$ is not an stable solution and $y(t) \equiv K$ is an asymptotically stable solution of the fractional differential equation*

$$G_T^\alpha y = a(t)y(K - y). \quad (3.7)$$

Proof. The hypotheses guarantee that the initial value problem (3.7) with $y(t_0) = y_0$ has a unique solution. Hence, two solutions of (3.7) do not intersect, and so, if $y_0 \in \mathbb{R} \setminus \{0, K\}$, then we have either $y(t) < 0$ for all $t \geq t_0$, $0 < y(t) < K$ for all $t \geq t_0$ or $y(t) > K$ for all $t \geq t_0$. Choose any y_0 with $0 < y_0 < K$. Since $0 < y_0 < K$, we have $G_T^\alpha y = a(t)y(K - y) > 0$, and so, $y(t)$ is an increasing function and there

exists $\lim_{t \rightarrow \infty} y(t) =: K_0 \leq K$. Therefore, $0 < y_0 < K_0$ for every $t \geq t_0$.

Seeking for a contradiction assume that $K_0 < K$. We have $G_T^\alpha y \geq a(t)(K - K_0)y > 0$ for every $t > t_0$, and Lemma 59 gives $y(t) \geq y_0 e^{(K-K_0)J_{T,t_0}^\alpha(a)(t)}$ for all $t \geq t_0$. Since $\lim_{t \rightarrow \infty} J_{T,t_0}^\alpha(a)(t) = \infty$, we conclude that $\lim_{t \rightarrow \infty} y(t) = \infty$, a contradiction. Thus, $\lim_{t \rightarrow \infty} y(t) = K$.

If $y(t) > K$, then a similar argument gives $\lim_{t \rightarrow \infty} y(t) = K$.

Hence, $y(t) \equiv 0$ is not an stable solution and $y(t) \equiv K$ is an asymptotically stable solution of (3.7). ■

Next, we study a fractional logistic growth model via the conformable fractional derivative G_T^α , with kernel $T(t, \alpha) = e^{(\alpha-1)t}$:

$$\begin{aligned} G_T^\alpha y(t) &= a y(t)(K - y(t)), & K, a > 0, & \alpha \in (0, 1], \\ y(t_0) &= y_0, \end{aligned} \quad (3.8)$$

where $y(t)$ is the size of the population in time t , y_0 is the size of the initial population, a is the exchange rate and K is the average loading capacity.

An observation equation can be associated with the model (3.8):

$$O_i = g(y(t_i)) + \varepsilon_i, \quad i = 1, 2, 3, \dots, n, \quad (3.9)$$

where O_i correspond to the i -th observed value under uncertainty from a solution of (3.8) at the discrete time $t_i \in [0, T]$, $i = 1, 2, 3, \dots, n$, g is the observation function, and ε_i are measurement errors, which are considered as independent and identically distributed random variables from a normal distribution with mean zero and constant variance σ^2 (denoted by $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$).

From the definition of G_T^α in (16), with kernel $T(t, \alpha) = e^{(\alpha-1)t}$, we have

$$G_T^\alpha f(t) = \lim_{h \rightarrow 0} \frac{f(t) - f(t - h e^{(\alpha-1)t})}{h}. \quad (3.10)$$

Note that $T(t, \alpha) = e^{(\alpha-1)t} \rightarrow 1$ as $\alpha \rightarrow 1$, and so, G_T^α is a conformable fractional derivative.

Taking into account (3.3), the solution of the model (3.8) is given by

$$y(t) = \frac{K y_0}{y_0 + (K - y_0) e^{\frac{\alpha K}{1-\alpha} (e^{(1-\alpha)t_0} - e^{(1-\alpha)t})}}. \quad (3.11)$$

If we take the limit $\alpha \rightarrow 1$ in (3.11), we get the solution of (3.8) for the ordinary

case

$$y(t) = \frac{Ky_0}{y_0 + (K - y_0)e^{aK(t_0-t)}}. \quad (3.12)$$

In [8] the same fractional logistic model is studied by using Caputo fractional derivative, through its numerical relation with Grünwald-Letnikov fractional derivative. A similar numerical study can be developed by using Riemann-Liouville fractional derivative, see [65] for more information about this topic.

In [8, 69], the following numerical approximation is used for Grünwald-Letnikov fractional derivative:

$$\begin{aligned} {}^{GL}D_a^\alpha y(t_m) &\approx h^{-\alpha} \sum_{k=0}^m (-1)^k \binom{\alpha}{k} y(t_m - kh), \\ t_m &= mh, \quad m = 0, 1, 2, \dots \end{aligned} \quad (3.13)$$

$$y(t_m) = h^\alpha a y(t_{m-1})(K - y(t_{m-1})) + \sum_{k=1}^m (-1)^{k+1} \binom{\alpha}{k} y(t_{m-1} - kh). \quad (3.14)$$

Figure 3.1 shows the curves associated to a direct type problem for a logistic growth model, with the following values of the parameters: $a = 1$, $K = 1$, $y_0 = 0.1$, $\alpha = 0.85$, for the approaches: Ordinary Derivative (*black*), Fractional Conformable Derivative with $T(t, \alpha) = e^{(\alpha-1)t}$ (*red*), Khalil (*blue*) and Caputo Fractional Derivative (*green*).

Figure 3.1

3.1.1 Estimation of the derivative order for fractional logistic models

The Bayesian theory has been efficiently applied to solve inverse problems for fractional models, as [8] and [19] show. In this context, the parameters are taken as random variables which are assigned prior distributions that match the information known about their values before collecting any observational data. Next, we estimate the derivative order for our Fractional Gaussian Model following a Bayesian viewpoint. According to Bayesian statistical inversion theory, and provided that all information available has been assimilated into the model, the solution to an inverse problem is given by the subsequent distribution of the quantity of interest. For the model defined in (3.8) and (3.9) the parameter of interest is $\phi = (a, \alpha, \sigma^2)$. Next, some prior distributions are proposed: $a \sim \mathcal{G}(\gamma_a, \mu_a)$ and $\tau = 1/\sigma^2 \sim \mathcal{G}(\gamma_\tau, \mu_\tau)$; where $\mathcal{G}(\gamma, \mu)$ denotes the Gamma distribution with shape parameter γ and rate parameter μ . $\mathcal{U}$ is the continuous Uniform distribution on the interval $(0, 1]$; $\alpha \sim \mathcal{U}(0, 1]$, $\tau = 1/\sigma^2 \sim \mathcal{G}(0.001, 0.001)$ and $a \sim \mathcal{G}(0.001, 0.001)$.

The parameters introduced in the prior distributions are called hyperparameters. As mentioned before, the prior distributions have been defined in order to reflect the information already known about the possible values of the parameters

of interest. In the model (3.8), a is positive and $\alpha \in (0, 1]$. Taking for granted prior independence of the parameters, the joint prior distribution can be written as: $p(\phi|hyperparameters) = p(a|\gamma_a, \mu_a)p(\alpha)p(\tau|\gamma_\tau, \mu_\tau)$, where $p(a|\gamma_a, \mu_a)$, $p(\alpha)$ and $p(\tau|\gamma_\tau, \mu_\tau)$ are previously defined.

Denote by $O' = (O_1, O_2, \dots, O_n)$ the observed data at times $(t_1, t_2, \dots, t_n)$, which are independent and identically distributed from the model defined by (3.8) and (3.9). The likelihood function is given by

$$L(O|\phi) = \prod_{i=1}^n f_O(O_i) = \frac{1}{(\sigma\sqrt{2\pi})^n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^n (O_i - g(y(t_i)))^2 \right\}, \quad (3.15)$$

where $y(t_i), i = 1, \dots, n$ is a solution of (3.8).

Then, applying Bayes theorem, the posterior distribution of the parameters of interest is given by

$$p(\phi|O) = \frac{L(O|\phi)p(\phi)}{\int_{\Theta} L(O|\theta)p(\theta)d\theta}, \quad (3.16)$$

where Θ denotes the parameter space of ϕ . It is known that

$$p(\phi|O) \propto L(O|\phi)p(\phi). \quad (3.17)$$

Adopting a loss quadratic function, the Bayesian point estimation is the posterior mean of $\hat{\phi}_B$, which is given by $\hat{\phi}_B = E(\phi|O)$. Markov Chain Monte Carlo simulations (MCMC) are employed to work with (3.17). One of the most popular MCMC techniques is the Metropolis-Hastings [55, 15]. WinBUGS and JAGS are some computer programs that implement MCMC algorithms. For more information on this section see [8].

Now we show two applications to real problems of the logistic model by different approaches: Khalil (*blue*), Conformable Derivative with $T(t, \alpha) = e^{(\alpha-1)t}$ (*red*), Caputo (*green*) and Ordinary (*black*). For this study we used data on the Grass Growth [38], and the Efficiency of Solar Panels [8, 70]. Figure 3.2 shows the adjustments corresponding to the observations (black points) associated with grass growth, according to the different fractional approaches.

Figure 3.2: Data and estimates of Grass Growth.

The ℓ^1 -errors in the adjustments according to the different approaches are the following: Khalil et al. (0.3015582), Conformable Derivative with $T(t, \alpha) = e^{(\alpha-1)t}$ (**0.1449984**), Caputo (0.2759660) and Ordinary (0.2250631).

Figures 3.3 (Grass Growth) shows the trace and estimated posterior distributions of the parameters of interest for G_T^α with $T(t, \alpha) = e^{(\alpha-1)t}$.

Figure 3.3: Trace and estimated posterior densities of a , α and σ .

Figure 3.4 shows the adjustments corresponding to the observations (black points) associated with the efficiency of solar panels, according to the different fractional approaches: Khalil (*blue*), G_T^α with $T(t, \alpha) = e^{(\alpha-1)t}$ (*red*), Caputo (*green*) and Ordinary (*black*).

On the other hand, the ℓ^1 -errors in the adjustments according to the different approaches are the following: Khalil et al.(0.4893058), G_T^α with $T(t, \alpha) = e^{(\alpha-1)t}$ (**0.3861131**), Caputo (0.4159481) and Ordinary (0.4369251).

Figure 3.4: Data and estimates of Efficiency of Solar Panels

Figure 3.5 (Efficiency of Solar Panels) shows the trace and estimated posterior distributions of the parameters of interest for G_T^α with $T(t, \alpha) = e^{(\alpha-1)t}$.

Figure 3.5: Trace and estimated posterior densities of a , α and σ .

3.1.2 Remarks

In this work we used a generalized conformable fractional derivative (G_T^α with $T(t, \alpha) = e^{(\alpha-1)t}$) in order to study a logistic growth model. Taking into account an experimental dataset, we solve an inverse problem to estimate the order of the involved fractional derivative.

The estimates of the parameters a , α and τ related to Grass Growth data, according to the different approaches are:

Fractional derivative	a	α	τ
Ordinary	0.07304052	-	0.06570552
G_T^α wiht $T(t, \alpha) = e^{\alpha-1}$	0.05402704	0.98486882	0.06196433
Khalil et. al.	0.09228812	0.91608834	0.07442681
Caputo	0.10076710	0.92960698	0.06484054

On the other hand, the ℓ^1 -errors in the adjustments were Khalil (0.3015582), Conformable Derivative with $T(t, \alpha) = e^{(\alpha-1)t}$ (**0.1449984**), Caputo (0.2759660) and Ordinary (0.2250631).

The estimates of the parameters a , α and τ related to Efficiency of Solar Panels data, according to the different approaches are:

Fractional derivative	a	α	τ
Ordinary	0.39490535	-	0.06019291
G_T^α wiht $T(t, \alpha) = e^{\alpha-1}$	0.30696471	0.92073882	0.05617161
Khalil et. al.	0.41021761	0.95234783	0.06416603
Caputo	0.46705471	0.94285900	0.05768138

In this case, the ℓ^1 -errors in the adjustments were Khalil et al. (0.4893058), G_T^α with $T(t, \alpha) = e^{(\alpha-1)t}$ (**0.3861131**), Caputo (0.4159481) and Ordinary (0.4369251).

From the above results it is inferred that the conformable approach proposed by us minimizes the ℓ^1 -error in the adjustments associated to both logistic models.

3.2 Local fractional Drude model

The Drude model or Lorentz-Drude model was developed around 1900 by Paul Drude to explain the transport properties of electrons in materials (especially in metals). The Drude model provides a basis for mechanics classical for the conductivity of metals, is based on the application of the kinetic theory to electrons in a solid [17].

The classic Drude model of motion for a charged particle of mass m_e and charge e affected by an external electrical field can be written as

$$m_e \frac{dv(t)}{dt} + \frac{m_e}{\tau} v(t) = -eE, \quad (3.18)$$

where $v(t)$ is the velocity, E is the electric field, t is time and τ is a damping term known as relaxation time. In this work we consider a conformable fractional

generalization of the Drude model, of the form

$$G_T^\alpha v(t) + \frac{1}{\tau} v(t) = -\frac{eE}{m_e}, \quad 0 < \alpha \leq 1. \quad (3.19)$$

Proposition 62 *Let p and q be continuous functions on an interval $I \subseteq \mathbb{R}$, $a \in I$ and $\alpha \in (0, 1]$. Then the solution of the linear fractional equation*

$$G_T^\alpha y + p(t)y = q(t), \quad (3.20)$$

on I is given by

$$y(t) = e^{-J_{T,a}^\alpha(p)(t)} \left(J_{T,a}^\alpha \left(q e^{J_{T,a}^\alpha(p)} \right) (t) + C \right). \quad (3.21)$$

Proof. If y is given by Eq. (3.21), then Theorems 23 and 24 and Proposition 54 gives

$$\begin{aligned} G_T^\alpha y &= G_T^\alpha \left(-J_{T,a}^\alpha(p)(t) e^{-J_{T,a}^\alpha(p)(t)} \left(J_{T,a}^\alpha \left(q e^{J_{T,a}^\alpha(p)} \right) (t) + C \right) \right. \\ &\quad \left. + e^{-J_{T,a}^\alpha(p)(t)} G_T^\alpha \left(J_{T,a}^\alpha \left(q e^{J_{T,a}^\alpha(p)} \right) \right) (t) \right) \\ &= -p(t) e^{-J_{T,a}^\alpha(p)(t)} \left(J_{T,a}^\alpha \left(q e^{J_{T,a}^\alpha(p)} \right) (t) + C \right) + e^{-J_{T,a}^\alpha(p)(t)} q(t) e^{J_{T,a}^\alpha(p)(t)} \\ &= -p(t)y + q(t), \end{aligned}$$

and y is a solution of (3.20). Therefore, it suffices to check that for each $t_0 \in I$ and $y_0 \in \mathbb{R}$ there exists $C = C(t_0, y_0)$ such that y is the solution of (3.20) with $y(t_0) = y_0$. If we choose

$$C = y_0 e^{J_{T,a}^\alpha(p)(t_0)} - J_{T,a}^\alpha \left(q e^{J_{T,a}^\alpha(p)} \right) (t_0), \quad (3.22)$$

then

$$y(t) = e^{-J_{T,a}^\alpha(p)(t)} \left(J_{T,a}^\alpha \left(q e^{J_{T,a}^\alpha(p)} \right) (t) + y_0 e^{J_{T,a}^\alpha(p)(t_0)} - J_{T,a}^\alpha \left(q e^{J_{T,a}^\alpha(p)} \right) (t_0) \right), \quad (3.23)$$

satisfies

$$y(t_0) = e^{-J_{T,a}^\alpha(p)(t_0)} \left(J_{T,a}^\alpha \left(q e^{J_{T,a}^\alpha(p)} \right) (t_0) + y_0 e^{J_{T,a}^\alpha(p)(t_0)} - J_{T,a}^\alpha \left(q e^{J_{T,a}^\alpha(p)} \right) (t_0) \right) = y_0, \quad (3.24)$$

and the proof is finished. ■

As it is well pointed out in, the fractional model has some faults (see [17]), the fractional operator has a fractional dimension $(time)^{-\alpha}$. However, this dimension mismatch may be mathematically corrected, for this we consider that the time relaxation t^α , the electron charge e^α and the mass of electrons m_e^α are nonlocal pa-

rameters. We will use this procedure to model our problem. So, from Eqs. (3.21) in (3.19) and considering $v(0) = v_0$ ($v_0 > 0$), we have

$$v(t) = e^{-J_{T,0}^\alpha(r)(t)} \left[J_{T,0}^\alpha \left(AE(t) e^{J_{T,0}^\alpha(r)} \right) (t) + v_0 \right], \quad (3.25)$$

where $r = \frac{1}{t^\alpha}$ is the fractional relaxation time and $A = -\left(\frac{e}{m_e}\right)^\alpha$.

For a conductor that complies Ohm's law with a number N of electrons per unit volume moving at the same velocity, the current density $J = -Ne v(t)$ can be written as

$$J = -Ne v(t) = -Ne \left\{ e^{-J_{T,0}^\alpha(r)(t)} \left[J_{T,0}^\alpha \left(AE(t) e^{J_{T,0}^\alpha(r)} \right) (t) + v_0 \right] \right\}, \quad (3.26)$$

where N is the number of particles for volume unity, e is the charge of electron and $v(t)$ is the velocity obtained in Eq. (3.25).

From the relation

$$J = \sigma(t)E, \quad (3.27)$$

where $\sigma(t)$ is the electrical conductivity and E is the electrical field. The electrical conductivity, in the fractional case is given by

$$\sigma(t) = \frac{Ne}{E} \left\{ e^{-J_{T,0}^\alpha(r)(t)} \left[J_{T,0}^\alpha \left(AE(t) e^{J_{T,0}^\alpha(r)} \right) (t) + v_0 \right] \right\}. \quad (3.28)$$

Eq. (3.28) represents the generalized expression for the electrical conductivity. If $\alpha = 1$, from the above equation, we obtain

$$\sigma(t) = \frac{Ne^2\tau}{m_e} \left(1 - \exp \left(-t/\tau \right) \right). \quad (3.29)$$

If $t \rightarrow \infty$, then we obtain the classic case

$$\sigma = \frac{Ne^2\tau}{m_e}. \quad (3.30)$$

For the electrical resistivity $\rho(t) = \frac{1}{\sigma(t)}$, we have

$$\rho(t) = \frac{1}{\sigma(t)} = \frac{1}{\frac{Ne}{E} \left\{ e^{-J_{T,0}^\alpha(r)(t)} \left[J_{T,0}^\alpha \left(AE(t) e^{J_{T,0}^\alpha(r)} \right) (t) + v_0 \right] \right\}}. \quad (3.31)$$

For a more complete study of the Drude model, we will consider some particular

cases of interest, in the kernel of definition of the derivative G . Thus, we can consider (among others) the following cases:

3.2.1 Local conformable model

a) We consider in Eq. (3.25), $T(t, \alpha) = e^{(1-\alpha)t}$.

Numerical solutions. Consider $e = -1.602 \times 10^{-19}C$, $m_e = 9.109 \times 10^{-31}kg$, $E = 100V/m$, $\tau = 1 \times 10^{-14}$, $N = 1 \times 10^{28}e/m^3$ and $v_0 = 0$. The numerical results given in Figs. 3.6a-3.6f show numerical simulations for the velocity, the current density and the electrical resistivity given by Eqs. (3.25), (3.26) and (3.31), respectively, for different values of α arbitrarily chosen.

Figure 3.6: In (a)-(b) Velocity of the particle measured in m/s for different values of α arbitrarily chosen for $E = 100V/m$ and $E = 100 \sin(t)V/m$, respectively. In (c)-(d) Current density measured in A/m^2 for different values of α arbitrarily chosen for $E = 100V/m$ and $E = 100 \sin(t)V/m$, respectively. In (e)-(f) Electrical resistivity measured in $\Omega \cdot m$ for different values of α arbitrarily chosen for $E = 100V/m$ and $E = 100 \sin(t)V/m$, respectively.

b) In this case we take in Eq. (3.25), $T(t, \alpha) = t^{(1-\alpha)}$, the derivative T^α given in [46].

Numerical solutions. Consider $e = -1.602 \times 10^{-19}C$, $m_e = 9.109 \times 10^{-31}kg$, $E = 100V/m$, $\tau = 1 \times 10^{-14}$, $N = 1 \times 10^{28}e/m^3$ and $v_0 = 0$. The numerical results given in Figs. 3.7a-3.7f show numerical simulations for the velocity, the current density and the electrical resistivity given by Eqs. (3.25), (3.26) and (3.31), respectively,

for different values of α arbitrarily chosen.

Figure 3.7: In (a)-(b) Velocity of the particle measured in m/s for different values of α arbitrarily chosen for $E = 100V/m$ and $E = 100 \sin(t)V/m$, respectively. In (c)-(d) Current density measured in A/m^2 for different values of α arbitrarily chosen for $E = 100V/m$ and $E = 100 \sin(t)V/m$, respectively. In (e)-(f) Electrical resistivity measured in $\Omega \cdot m$ for different values of α arbitrarily chosen for $E = 100V/m$ and $E = 100 \sin(t)V/m$, respectively.

3.2.2 Local non-conformable model.

a) We consider $T(t, \alpha) = e^{t^{-\alpha}}$, the N-derivative given in [33].

Numerical solutions. Consider $e = -1.602 \times 10^{-19}C$, $m_e = 9.109 \times 10^{-31}kg$,

$E = 100V/m$, $\tau = 1 \times 10^{-14}$, $N = 1 \times 10^{28}e/m^3$ and $v_0 = 0$. The numerical results given in Figs. 3.8a-3.8f show numerical simulations for the velocity, the current density and the electrical resistivity given by Eqs. (3.25), (3.26) and (3.31), respectively, for different values of α arbitrarily chosen.

Figure 3.8: In (a)-(b) Velocity of the particle measured in m/s for different values of α arbitrarily chosen for $E = 100V/m$ and $E = 100 \sin(t)V/m$, respectively. In (c)-(d) Current density measured in A/m^2 for different values of α arbitrarily chosen for $E = 100V/m$ and $E = 100 \sin(t)V/m$, respectively. In (e)-(f) Electrical resistivity measured in $\Omega \cdot m$ for different values of α arbitrarily chosen for $E = 100V/m$ and $E = 100 \sin(t)V/m$, respectively.

b) We consider $T(t, \alpha) = t^{-\alpha}$.

Numerical solutions. Consider $e = -1.602 \times 10^{-19}C$, $m_e = 9.109 \times 10^{-31}kg$, $E = 100V/m$, $\tau = 1 \times 10^{-14}$, $N = 1 \times 10^{28}e/m^3$ and $v_0 = 0$. The numerical results given in Figs. 3.9a-3.9f show numerical simulations for the velocity, the current density and the electrical resistivity given by Eqs. (3.25), (3.26) and (3.31), respectively,

for different values of α arbitrarily chosen.

Figure 3.9: In (a)-(b) Velocity of the particle measured in m/s for different values of α arbitrarily chosen for $E = 100V/m$ and $E = 100 \sin(t)V/m$, respectively. In (c)-(d) Current density measured in A/m^2 for different values of α arbitrarily chosen for $E = 100V/m$ and $E = 100 \sin(t)V/m$, respectively. In (e)-(f) Electrical resistivity measured in $\Omega \cdot m$ for different values of α arbitrarily chosen for $E = 100V/m$ and $E = 100 \sin(t)V/m$, respectively.

3.2.3 Remarks

Fractional calculus is successfully used to model a broad range of problems, since the evolution of many physical processes can be more precisely described with fractional derivatives. These differential and integral operators are the frequently used to con-

struct mathematical models in several scientific areas. Above the concept of local conformable and non-conformable fractional calculus, in this work, we suggested general conditions for the existence and the uniqueness of the exact solutions for differential equations. Also, we introduce an integral operator which is the inverse of this fractional derivative, and we study its properties, which allow to solve some fractional differential equations. In this work, considering the generalized fractional derivative, we have extended the classical Drude model. This model provides a good explanation for the electrical conductivity in metals, optical properties, the electrical resistivity and the thermal conductivity in metals. To test the new concepts, we consider some particular cases of interest in the kernel of definition of the derivative G , these cases involves local conformable and local non-conformable models. For both cases, we presented numerical simulation for different values of fractional order α for the velocity of the particle, the current density and the electrical resistivity. We obtained new behaviors which proves that the new operator can capture more complexities than existing classical, conformable or fractional differential and integral operators. It is worth noting that at limit case, that is., when $\alpha = 1$, the classical cases were recovered.

New investigations and applications in various fields of science and engineering can be explored with the use of this novel derivative; for example, the behavior in frequency of the polarizability, optical properties and conductivity will be presented in future research work being processed by authors of the present work.

3.3 On fractional Liénard–type systems

The classical Liénard equation [52]

$$x'' + f(x)x' + g(x) = 0$$

and the equivalent system

$$\begin{aligned} x' &= y - F(x) \\ y' &= -g(x) \end{aligned} \tag{3.32}$$

with $F(x) = \int_0^x f(s)ds$, have been extensively investigated since they appear in many problems of applied and basic research, such as in theory of feedback in electronic circuits and motion of mass-spring systems, see e.g. [12, 16, 63]. Also, several reviews of the literature that focus on Eq. (3.32) have been made. In fact, the book by Sansone and Conti [68] contains an excellent summary up to 1960; it was updated up to 1962 in [66] and further extended in the lists of references of [16, 13, 14]. Among the relevant papers that investigate the Liénard system, that were recently

published, we can mention Refs. [43, 57, 60]. Additionally, for a description of the problems associated to the oscillatory nature of Eq. (3.32) the book by Minorsky [57] is highly recommended.

In [60] the system (3.32) was studied in an attempt to unify, in a general result, the methods known for particular cases on an important qualitative aspect: the boundedness of solutions. The Liénard system (3.32) is often taken as the typical example of nonlinear self-excited vibration problem and since its appearance (cf. [48]) it has become a paradigm of nonlinear analysis. It is clear then that the study of (3.32) is of paramount importance in physics and other sciences.

However, the case of Liénard-type systems having non-integer derivatives has received scarce attention, although the nature of many process and phenomena makes them suitable for modeling with fractional differential equations. Over the years many mathematicians have built up a large body of mathematical knowledge on fractional calculus, but despite fractional calculus is a natural generalization of classical calculus, and although its mathematical history is equally long, it has, until recently, played a negligible role in applications. One reason for this could be that until recently, until a few years ago, the basic facts about fractional calculus were not readily accessible even in the mathematical literature (see [64, 65], for example).

On the other hand, taking into account the existence of different fractional derivatives, both global (Gronwald-Letnikov, etc.) and local (conformable and non-conformable) versus classical derivatives, one may ask: How significant is the use of fractional models in the study of processes and phenomenon? How advantageous is the use of these non-integer order derivatives? Is any information lost when using fractional derivatives instead of classical derivatives? In fact, fractional calculus is now applied successfully to address several problems of physics, see some examples in Refs. [26, 30, 29, 76, 31].

Therefore, the purpose of this work is to investigate the qualitative behavior of systems of type (3.32) under non-usual assumptions; that is, the functions evolved in (3.32) are the same but the derivatives considered are of different types (i.e., integer order and fractional order such as global, Gronwald-Letnikov and local, conformable and non-conformable) and elucidate the questions posed above. To that end we apply fractional derivatives to six known examples of Liénard-type systems, each one showing relevant dynamical properties: In example I the origin $(0, 0)$ is an asymptotically stable equilibrium position, in example II the system has a stable limit cycle, system III has no continuable solutions, example IV has unbounded solutions, example V requires additional conditions for the existence of a stable limit cycle, as compared to example II, and example VI is a particular case of a linear system with constant coefficients. Thus, starting with numerical experiments we conclude with some theoretical-methodological remarks.

3.3.1 Fractional approach to Liénard–type systems

Fractional calculus is a generalization of integration and differentiation to non-integer-order fundamental operations ${}_aD_t^\alpha$, where a and t are the bounds of the operation and $\alpha \in \mathbb{R}$. The continuous integrodifferential operator is defined as

$${}_aD_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha}, & \text{if } \alpha > 0 \\ 1, & \text{if } \alpha = 0 \\ \int_a^t (d\tau)^\alpha, & \text{if } \alpha < 0 \end{cases}. \quad (3.33)$$

The three most frequently used definitions for the general fractional differintegral are: the Grünwald-Letnikov (GL) definition, the Riemann-Liouville and the Caputo definition (see [65]). Other definitions are associated to well-known mathematicians as for instance Weyl, Fourier, Cauchy, Abel, etc.

We recall that the Grünwald-Letnikov definition is given as (with $a = 0$)

$${}^{GL}D_t^\alpha f(t) = \lim_{h \rightarrow 0} h^{-\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(t - kh)$$

with

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha + 1)}{\Gamma(k + 1)\Gamma(\alpha - k + 1)},$$

or the analogous representation

$$D^\alpha f(t) = \lim_{h \rightarrow 0} h^{-\alpha} \sum_{k=0}^{\left[\frac{t}{h}\right]} (-1)^k \binom{\alpha}{k} f(t - kh),$$

where $[\cdot]$ denotes the integer part. In the case of discrete times

$$\{t_0 = 0, t_1, t_2, t_3 \dots, t_N\}$$

and

$$t_n = t_0 + nh,$$

we have

$${}^{GL}D^\alpha f(t_n) \approx h^{-\alpha} \sum_{k=0}^n (-1)^k \binom{\alpha}{k} f(t_{n-k}). \quad (3.34)$$

To overcome some difficulties of the classical fractional derivative, Khalil et al. [46], came up with an interesting idea that generalizes the familiar limit definition of the derivative which allows to successfully introduce a conformable fractional

derivative (see Definition 19 below).

The non-conformable derivative (see [33]) is defined as:

Given a function $f : (0, \infty) \rightarrow \mathbb{R}$ and $\alpha \in (0, 1]$, the N_1^α -derivative of f at the point t is defined by

$$N_1^\alpha f(t) = \lim_{h \rightarrow 0} \frac{f(t + he^{t^{-\alpha}}) - f(t)}{h} .$$

If f is N_1^α -differentiable and there exists $\lim_{t \rightarrow 0^+} N_1^\alpha f(t)$, then we define

$$N_1^\alpha f(0) = \lim_{t \rightarrow 0^+} N_1^\alpha f(t) .$$

So for discrete times we have

$$N_1^\alpha f(t_n) \approx e^{t_n^{-\alpha}} \frac{f(t_n) - f(t_{n-1})}{h} .$$

Also, note that

$$N_1^\alpha f(t) = e^{t^{-\alpha}} f'(t) . \quad (3.35)$$

Some Liénard-type systems

For the numerical experiments we present below we use the standard Euler method that for the classical Liénard system of Eq. (3.32),

$$\begin{aligned} x' &\approx \frac{x_n - x_{n-1}}{h} = y_{n-1} - F(x_{n-1}) , \\ y' &\approx \frac{y_n - y_{n-1}}{h} = -g(x_{n-1}) , \end{aligned}$$

provides the discrete orbits

$$\begin{aligned} x_n &= x_{n-1} + h [y_{n-1} - F(x_{n-1})] , \\ y_n &= y_{n-1} - hg(x_{n-1}) . \end{aligned}$$

Moreover, we consider here the general system

$$\begin{aligned} x^{(\alpha)} &= y - F(x) , \\ y^{(\beta)} &= -g(x) , \end{aligned} \quad (3.36)$$

where $x^{(\alpha)}$ and $y^{(\beta)}$ are fractional operators and $0 < \alpha, \beta \leq 1$. Therefore, in the conformable case [from (3.36) and Theorem 20] we have

$$\begin{aligned} t^{1-\alpha} x' &= y - F(x) , \\ t^{1-\beta} y' &= -g(x) , \end{aligned}$$

with solutions

$$\begin{aligned} x_n &= x_{n-1} + h(t_{n-1})^{\alpha-1} [y_{n-1} - F(x_{n-1})] , \\ y_n &= y_{n-1} - h(t_{n-1})^{\beta-1} g(x_{n-1}) ; \end{aligned}$$

while in the non-conformable case [from (3.35) and (3.36)]

$$\begin{aligned} e^{t^{-\alpha}} x' &= y - F(x) , \\ e^{t^{-\beta}} y' &= -g(x) . \end{aligned}$$

the solutions are written as

$$\begin{aligned} x_n &= x_{n-1} + h e^{-(t_{n-1})^{-\alpha}} [y_{n-1} - F(x_{n-1})] , \\ y_n &= y_{n-1} - h e^{-(t_{n-1})^{-\beta}} g(x_{n-1}) . \end{aligned}$$

Note that the standard Euler method has a computational cost of n and an error of $O(h^2)$. But for the Grünwald-Letnikov case we use an *iterative* Euler method with computational cost of n^2 and error of $O(h)$: from (3.34) and (3.36) we have

$$\begin{aligned} h^{-\alpha} \sum_{k=0}^n (-1)^k \binom{\alpha}{k} x_{n-k} &= y_{n-1} - F(x_{n-1}) , \\ h^{-\beta} \sum_{k=0}^n (-1)^k \binom{\beta}{k} y_{n-k} &= -g_{n-1} , \end{aligned}$$

with solutions

$$\begin{aligned} x_n &= \sum_{k=1}^n (-1)^{k+1} \binom{\alpha}{k} x_{n-k} + h^{-\alpha} [y_{n-1} - F(x_{n-1})] , \\ y_n &= \sum_{k=1}^n (-1)^{k+1} \binom{\alpha}{k} y_{n-k} - h^{-\beta} g_{n-1} . \end{aligned}$$

Below we use a time step of size $h = 10^{-6}$. We verified that we obtain qualitatively the same solutions when using smaller values of h .

Now we consider six relevant cases of Liénard-type systems subject to different conditions and within the framework of different notions of derivatives (integer order and fractional order, global and local). For them, we will take as a base some works that have presented diverse qualitative results on these systems, under the general following conditions:

(a) $xg(x) > 0$ for $x \neq 0$,

(b) $\int_0^{\pm\infty} g(r)dr = +\infty$,

(c) $f(x) > 0$ for all $x \in \mathbb{R}$ and consequently $xF(x) > 0$ for $x \neq 0$.

Case I. The classical Liénard system. Consider the system (3.32) under assumptions (a) and (c). In this case the equilibrium point $(0, 0)$ is asymptotically stable (see for example [58] and [59] for additional results). Here we consider as an example the system

$$\begin{aligned} x' &= y - (x^3 + x) , \\ y' &= -x . \end{aligned} \tag{3.37}$$

In Figure 3.10 we present numerical solutions of the Liénard system of Eqs. (3.37) for (a) classical, (b) Grünwald-Letnikov, (c) conformable, and (d) non-conformable derivatives. The classical derivative solution is also included in panels (b-d) as a reference (black thick curve). Red thin curves in panels (b-d) are solutions for the orders $\alpha = 0.975, 0.925$, and 0.9 such that their proximity to the black curve is greater the closer to 1 is the order of its derivative; as expected. The blue dot indicates the initial condition. The rest of the figures, Figure 3.11 to 3.13 corresponding to Cases II to VI, respectively, display the same structure than Figure 3.10 for comparison purposes.

Notice that if we consider the system

$$\begin{aligned} x' &= y + \left(\frac{x^3}{3} + 2x \right) , \\ y' &= -2x^3 , \end{aligned} \tag{3.38}$$

instead of (3.37), we get $x'' - (x^2 + 2)x' + 2x^3 = 0$, which does not satisfies either (a) or (c) and has the unbound solution $x = e^{2t}$.

Case II. The study of the existence of limit cycles is one of the fundamental topics of the Qualitative Theory, since Poincaré treated it at the end of the XIX century. In [61] the author studies the existence of limit cycles and its algebraic character, being a fundamental example the system

$$\begin{aligned} x' &= y - \left(\frac{x^3}{3} - x \right) , \\ y' &= -x \left[1 + \frac{x^2(x^2 - 4)}{16} \right] . \end{aligned} \tag{3.39}$$

The numerical solutions of this system are shown in Figure 3.11.

Case III. Hricisakova in [35] studied the system (see also [36]):

$$\begin{aligned}x' &= y - \frac{3x^2}{2} , \\ y' &= -x^3 .\end{aligned}\tag{3.40}$$

This system under the assumptions $xf(x) > 0$, $xg(x) > 0$ for all $x \neq 0$, and f and g continuous on $\mathbb{R}$, may have a negative solution which is not continuable to the right as the following example shows: $x'' + 3xx' + x^3 = 0$, has the solution $x(t) = t^{-1}$. In general $x(t) = (x - \alpha)^{-1}$ with $t < \alpha$ is a noncontinuable solution of the above equation and $x(t) = (x - \alpha)^{-1}$, $t > \alpha$, is also a noncontinuable solution of this equation.

Figure 3.10: Numerical solutions of the Liénard system of Eqs. (3.37) for (a) classical, (b) Grünwald-Letnikov, (c) conformable, and (d) non-conformable derivatives. Red thin curves in panels (b-d) are solutions for the orders $\alpha = 0.975$, 0.925 , and 0.9 . The classical derivative solution is also included in panels (b-d) as a reference (black thick curve). The blue dot indicates the initial condition.

Figure 3.11: Numerical solutions of the Liénard system of Eqs. (3.39) for (a) classical, (b) Grünwald-Letnikov, (c) conformable, and (d) non-conformable derivatives. Red thin curves in panels (b-d) are solutions for the orders $\alpha = 0.975, 0.925$, and 0.9 . The classical derivative solution is also included in panels (b-d) as a reference (black thick curve). The blue dot indicates the initial condition.

The numerical solutions of system (3.40) are shown in Figure 3.12.

Case IV. Yu and Zhang in [71] studied the system (example 1, page 54):

$$\begin{aligned} x' &= y + \sqrt{2}x^2, \\ y' &= -x^3, \end{aligned} \tag{3.41}$$

which satisfies condition (c) and has the solution $y = -\sqrt{2}x^2/2$.

Then, in Figure 3.13 we present numerical solutions of (3.41).

Case V. Kooij and Jianhua considered in [48] the system:

$$\begin{aligned} x' &= \frac{1}{5}y - x(x-1)(x+1.1), \\ y' &= -x^3, \end{aligned} \tag{3.42}$$

which has a stable limit cycle at the origin (see example 4.1, page 272 of [37]). They showed that if the condition on the monotonicity of $F(x)$ is violated then an additional condition to guarantee the uniqueness of the limit cycle is needed.

The numerical solutions of (3.42) are presented in Figure 3.14.

Figure 3.12: Numerical solutions of the Liénard system of Eqs. (3.40) for (a) classical, (b) Grünwald-Letnikov, (c) conformable, and (d) non-conformable derivatives. Red thin curves in panels (b-d) are solutions for the orders $\alpha = 0.975, 0.925$, and 0.9 . The classical derivative solution is also included in panels (b-d) as a reference (black thick curve). The blue dot indicates the initial condition.

Figure 3.13: Numerical solutions of the Liénard system of Eqs. (3.41) for (a) classical, (b) Grünwald-Letnikov, (c) conformable, and (d) non-conformable derivatives. Red thin curves in panels (b-d) are solutions for the orders $\alpha = 0.975, 0.925$, and 0.9 . The classical derivative solution is also included in panels (b-d) as a reference (black thick curve). The blue dot indicates the initial condition.

Figure 3.14: Numerical solutions of the Liénard system of Eqs. (3.42) for (a) classical, (b) Grünwald-Letnikov, (c) conformable, and (d) non-conformable derivatives. Red thin curves in panels (b-d) are solutions for the orders $\alpha = 0.975, 0.925$, and 0.9 . The classical derivative solution is also included in panels (b-d) as a reference (black thick curve). The blue dot indicates the initial condition.

Figure 3.15: Numerical solutions of the Liénard system of Eqs. (3.43) for (a) classical, (b) Grünwald-Letnikov, (c) conformable, and (d) non-conformable derivatives. Red thin curves in panels (b-d) are solutions for the orders $\alpha = 0.975, 0.925$, and 0.9 . The classical derivative solution is also included in panels (b-d) as a reference (black thick curve). The blue dot indicates the initial condition.

Case VI. The linear system

$$\begin{aligned} x' &= y - x, \\ y' &= -x, \end{aligned} \tag{3.43}$$

can be considered as a limiting case for the Liénard system of Eq. (3.32). Its roots $(-1 \pm \sqrt{3}i)/2$ indicate that this system has an asymptotically stable focus at the origin, of a global nature.

Finally, Figure 3.15 shows numerical solutions of system (3.43).

3.3.2 Theoretical-methodological results

In addition to the standard Liénard system of Eq. (3.32) we consider here the more general system of Eq. (3.36):

$$\begin{aligned} x^{(\alpha)} &= y - F(x), \\ y^{(\beta)} &= -g(x), \end{aligned}$$

where the notation $x^{(\alpha)}$ and $y^{(\beta)}$ labels a fractional operator; for example the Gronwald-Letnikov fractional derivative (with $0 < \alpha, \beta \leq 1$). From (3.36) we have,

$$\frac{dy}{dx} = \frac{y^{(1-\beta)}}{x^{(1-\alpha)}} \frac{-g(x)}{y - F(x)}. \quad (3.44)$$

Consider the orbit $T(x, y)$ of (3.32) and the curve $C(x, y)$ which is the image of T by the application of Φ . We can show that C is an orbit of (3.36) as follows.

In the region $x > 0$, if a point (x, y) belongs to the curve C we have from (3.36) that

$$\frac{dy^\beta}{dx^\alpha} = \frac{y^{(\beta-1)}}{x^{(\alpha-1)}} \frac{dy}{dx}.$$

Thus the curve C in the region $x > 0$ is an orbit of (3.36) obtained by multiplying the orbit of (3.32) by a "factor" which contains a fractional integral of complementary order to the derivative considered; i.e., the Jacobian of the isomorphism. If the orbit T crosses the y -axis in some point $(0, y_1)$ then the orbit C crosses the y -axis in $(0, \Phi(0, y_1))$, with $y_1 > 0$ or $y_1 < 0$.

Let $C'(x, y)$ be an orbit of (3.36) passing through the point $(0, \Phi(0, y_1))$ and $T'(x, y)$ be the inverse image of C' by the application of Φ . From unicity of solutions of (3.32), T and T' coincide and C and C' therefore also coincide by injectivity of Φ (the Jacobian $y^{(\beta-1)}/x^{(\alpha-1)} \neq 0$). Thus Φ maps orbits of (3.32) into those of (3.36).

A similar result is obtained in the region $x < 0$. Therefore notice that if $xF(x) > 0$ and $xg(x) > 0$, for $x \neq 0$, then there exists an isomorphism Φ of the phase plane of the system (3.32) on the phase plane of the system (3.36), which is a one to one correspondence between all orbits of (3.32) and those of (3.36).

3.3.3 Remarks

Identifying the parameters that lead to stable or unstable behaviors in dynamical systems is of paramount importance; since stability represents the ability to return to the state of equilibrium after changes or temporary perturbations, according to external and internal factors. Also, the study of the behavior of orbits around critical points for the identification of possible stationary states in dynamical systems is of main relevance: When a small perturbation is made to a given system and there is a departure from the critical point, if the system is in a state of stable equilibrium it will approach again the critical point; on the contrary, it is in a state of unstable equilibrium.

Here, we made a stability analysis of Liénard-type systems where the perturbation is the application of non-integer differential operators. It is clear that the

Jacobian of the fractional differential operator can “distort” the geometric behavior of the orbits around the equilibrium points, thus we identified in which cases this distortion influences importantly the dynamics of the system (as compared to the dynamics under the integer differential operator).

It is well known that obtaining analytical solutions for given dynamical systems, mainly in the nonlinear case, is practically impossible. Thus, a numerical approach together with a qualitative analysis becomes a suitable strategy; hence the relevance of the methodology used in this work. Here we have studied the behavior of Liénard-type systems using different derivative operators: integer and fractional. The fractional derivatives we used were Grünwald-Letnikov (GL), conformable (G), and non-conformable (N), while the ordinary derivative was also shown as a reference. The main disadvantage of GL derivative is the non-compliance of several classical properties of the ordinary derivative, which makes it unable to “replicate” the case studies of the integer order. The local fractional G and N derivatives, do not have that disadvantage thus have some points in favor. For example, the G derivative is conformable which ensures an “equivalence” with the results of the integer order at all finite times. In the case of the N derivative, although it is non-conformable, at infinity behaves like the ordinary derivative, which is of paramount importance in the Qualitative Theory. Moreover, we have demonstrated theoretically that there is a “certain geometrical similarity” between the trajectories of Liénard-type systems with different differential operators; also, with numerical examples we showed that some cases suffer more distortions than others.

In addition from the figures presented in Sec. 2.2, we can make the following methodological observations:

- (1) If the equilibrium position is globally asymptotically stable (or completely unstable), the orbits of the system “resist” the deformations and still offer a good approximation when α and β are close to 1.
- (2) If the equilibrium position is not structurally “good”, the distortion manifests itself easily; i.e., the factor $y^{(1-\beta)}/x^{(1-\alpha)}$ strongly influences the configuration of the orbits.

Finally, can we contribute to answer the questions posed at the beginning of the work? Well, we realized that the study of the equilibrium points of the ordinary system provides hints of the behavior of the system under fractional differential operators; i.e., the study of a fractional model should be accompanied by the qualitative study of the corresponding ordinary system. Indeed the previous analysis of the corresponding ordinary system may considerably help to understand and improve processes or phenomenon studied in the frame of fractional derivatives.

Chapter 4

Conclusions

The present work introduces a new fractional derivative G_T^α and its associated integral operator $J_{T,a}^\alpha$. Furthermore, we develop a theoretical study on these operators, and they are compared with other existing ones. We show the feasibility of using these operators in practical problems, by solving problems of direct and inverse type with real and simulated data.

Open problems

1. What would be a natural extension of our derivative to the Euclidean space R^n ?
2. What types of kernel would allow to obtain the indices rule?
3. Compare the proposed derivative and the classical one in other real problems.

Contributions of this work

The results in this work appear in [21, 24, 20, 22, 4]; these works have been published or submitted to prestigious international mathematical journals. These results were presented in the following international and national conferences.

- Congreso Sur-Sureste de Matemáticas, Diciembre 2018, Acapulco, México.
- 19th International Conference Computacional and Mathematical Methods in Science and Engineering, June-July 2019, Cadiz, Spain.

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