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A NOTE ON REPRESENTATION OF PREFERENCES

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Abstract

We consider a class of relations which includes irreflexive preference relations and interdependent preferences. For this class, we obtain necessary and sufficient conditions for representation of the relation by two numerical functions in the sense of $a \prec x$ if and only if $u(a) < v(x)$.

Key words
Preference, continuous representation, pseudotransitivity, biorders, countable bounded preferences.

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1 Introduction

This work concerns the existence of a numerical representation for a class of relations which includes irreflexive preference relations and interdependent preferences.

As it is well known, if X is a connected and separable topological space, continuous preference orderings on X always have utility representations. (Eilenberg (1941)). The assumption of connectedness is not necessary in the setting of metric spaces; this fact is a consequence of a result of Debreu (1954) which establishes that if X is perfectly separable, every continuous preference ordering is representable by an utility function. However, as it was noted by Monteiro (1987), the above results may not be useful in infinite dimensional spaces because we lack, in general, the separability of the space; he proved that a continuous preference relation on a path connected space has a continuous utility representation if and only if it is countably bounded.

If the preference is given by an asymmetric binary relation, it is not possible to have an utility representation but it has been suggested by some authors that a preference relation on a set X could be represented with the help of two real valued functions u and v, where $v(x)$ and $u(x)$ can be interpreted as the lower and upper bounds of the utility perceived of the objet x (see Fishburn (1973) and Doignon *et al.* (1984)).

The representation by two numerical functions generalizes the classical utility function theory, because it allows the relation "$\sim$" (absence of strict preference) not to be transitive, which seems more in accordance with economic choices, since $x \sim y$ may correspond not only to true indifference between x and y, but also to an inability to choose between them. This type of relations are pseudotransitives; that is, if $x \prec y \preceq y' \prec z$ then $x \prec z$. Pseudotransitivity implies the transitivity of the strict preference $\prec$ but does not imply the transitivity of the indifference. Bridges (1983) is the first interested in the existence of a continuous representation in the case of a binary preference relation defined on a closed convex set of $\boldsymbol{R}^n$ and Chateauneuf (1987) gives, in terms of strong separability, necessary and sufficient conditions for the existence of representation of a pseudotransitive preference relation by two continuous functions in a connected topological space.

In the recent literature about walrasian equilibrium (see Florenzano (1990), Yannelis (1991)) it is frequent to consider interdependent preferences (the preference of an agent depend of the choice of the rest of the agents) or preferences with externalities. For example, if we consider a set of n agents and X_i represents the consumption set of the agent i, the preferences of the agent i are given by a correspondence P_i defined on the cartesian product of the consumption sets $A = \prod_{i=1}^{n} X_i$ into X_i. If $a = (a_1, \cdots, a_n) \in A$, $P_i(a)$ is interpreted as the set of consumptions $x \in X_i$ which the agent i prefers (strictly) to a_i when the consumption of the agent $k \neq i$ is a_k. It is usual to consider a condition of irreflexivity expressed by $a_i \notin P_i(a)$ (or even more, a_i is not in the convex hull of $P_i(a)$). Formally, this class of relation is a subset $\prec$ of $A \times X_i$, that is $\prec = \{(a, x); x \in P_i(a)\} = \{(a, x); a \prec x\}$. We have an analogous situation if a factor is the consumption set of an agent and the rest of the factors represents the set of possible externalities. Although these preferences don't have an utility representation, in some cases they can be represented by two numerical functions.

To formalize these ideas, if A and X are topological spaces, we say that the relation "$\prec$"$\subset A \times X$ has a numerical representation if there exist two functions $u : A \longrightarrow \mathbb{R}$, $v : X \longrightarrow \mathbb{R}$ such that $a \prec x$ if and only if $u(a) < v(x)$. This kind of representation is useful, for instance, to characterize the optimal allocations; the allocation $a \in A$ is optimal if and only if the "more prefered" set $P(a) = \{x \in X; a \prec x\}$ is empty and it occurs when $u(a)$ is an upper bound of v.

In this work we consider the class of relations, between two topological spaces, which verify a property of irreflexivity which generalizes the ordinary irreflexivity for a binary relation on a set; this class includes the class of interdependent preferences or preferences with externalities. For this class, we give necessary conditions for the existence of a continuous numerical representation and we prove that these conditions are sufficient in the setting of path connected topological spaces.

2 Definitions and notations

A relation between two sets A and X is a subset P of $A \times X$. When $(a, x) \in P$, we write $a \prec x$. The notation $x \preceq a$ signifies that $(a, x) \notin P$. We say that the relation is representable by two functions $u : A \longrightarrow \mathbb{R}$, $v : X \longrightarrow \mathbb{R}$ if

$a \prec x$ is equivalent to $u(a) < v(x)$. The relation is said a biorder if for all $a, b \in A$, $x, y \in X$, the property

$$[a \prec x, b \prec y] \Longrightarrow a \prec y \text{ or } b \prec x \text{ holds}$$

That property can be also expressed by

$$b \prec y \preceq a \prec x \Longrightarrow b \prec x$$

If a relation is representable by two functions is, obviously, a biorder. A relation between A and X induces relations $\preceq_1$ on A and $\preceq_2$ on X on the natural way

$$a \preceq_1 b \text{ if and only if } z \preceq a \text{ implies } z \preceq b; a, b \in A, z \in X$$

$$x \preceq_2 y \text{ if and only if } y \preceq c \text{ implies } x \preceq c; x, y \in X, c \in A$$

The relations $\preceq_1, \preceq_2$ are preorders on A and X respectively. They are complete preorders if and only if $\prec$ is a biorder. In this case, the strict relations associated to $\preceq_1, \preceq_2$ are defined by

$$a \prec_1 b \text{ if there exists } x \in X \text{ such that } a \prec x \preceq b; a, b \in A$$

$$x \prec_2 y \text{ if there exists } a \in A \text{ such that } x \preceq a \prec y; x, y \in X$$

If A and X are topological spaces, the relation $\prec$ is continuous if the sets $(a_0, \rightarrow) = \{x \in X; a_0 \prec x\}$ and $(\leftarrow, x_0) = \{a \in A; a \prec x_0\}$ are open in X and A, respectively, for all $a_0 \in A, x_0 \in X$

A preorder $\preceq$ on a set Y is countably bounded if there is some countable subset B of Y such that for all $y \in Y$ there exist $b_1, b_2 \in B$ with $b_1 \preceq y \preceq b_2$.

3 Results

In this work we consider the class of relations "$\prec$"$\subset A \times X$ verifying the following property:

For all $a \in A$, there exists $x_a \in X$ such that $x_a \preceq a$ and for all $x \in X$, there exists $a_x \in A$ such that $x \preceq a_x$. [1]

This class of relations includes the class of binary irreflexive relations on a set and the class of interdependent preference relations.

Proposition 1 *If there exists a representation u, v of $\prec$, then*

1) u is a pseudoutility for $\prec_1$, that is $a \prec_1 b \Longrightarrow u(a) < u(b)$ for all $a, b \in A$

2) v is a pseudoutility for $\prec_2$, that is $x \prec_2 y \Longrightarrow v(x) < v(y)$ for all $x, y \in X$.

3) the complete preorders $\preceq_1$, $\preceq_2$ are countably bounded.

Proof. It is clear, from our definition of $\preceq, \preceq_1, \preceq_2$, that the parts (1) and (2), are true. Let us prove the part (3). To show that $\preceq_1$ is countably bounded, we can suppose that u is a bounded function; if we denote $I = \inf_{a \in A} u(a)$ and $S = \sup_{a \in A} u(a)$, we have two possibilities:

a) if $S = \max_{a \in A} u(a)$, there exists $a_M \in A$ such that $S = u(a_M)$ and then $u(a) \leq S = u(a_M)$; from the part (1) we have $a \preceq_1 a_M$, that is, $\preceq_1$ is upper countably bounded.

b) if u has not maximum, for each $n \in I\!\!N$ we can choose $s_n \in A$ such that $u(s_n) \geq S - \dfrac{1}{n}$. If $a \in A$, there exists $n_0 \in I\!\!N$ such that $u(a) \leq S - \dfrac{1}{n_0} \leq u(s_{n_0})$; then, $a \preceq_1 s_{n_0}$ and $\preceq_1$ is upper countably bounded by the set $\{s_n\}_{n \in I\!\!N}$.

In the analogous way, we can also have either $a_m \in A$ with $I = u(a_m)$ or a countably set $\{I_n\}$ to prove that $\preceq_1$ is lower countably bounded. Then, $\preceq_1$ is countably bounded.

Proposition 2 *If A and X are connected spaces and the relation $\prec$ verifying [1] is representable by two continuous functions u, v, then $\prec_1, \prec_2$ and $\prec$ are continuous.*

Proof. We prove that $\prec_1$ is continuous. If $b \in (a, \rightarrow)_1$, there exists $x \in X$ such that $a \prec x \preceq b$. For condition [1], there is $x_a \in X$ such that $x_a \preceq a$ and then $v(x_a) \leq u(a) < v(x) \leq u(b)$. Since v is continuous and X is connected, it follows that there exists $z \in X$ such that

$$v(x_a) \leq u(a) < v(z) < v(x) \leq u(b)$$

From continuity of u, one deduces the existence of a neighborhood U of b such that $u(c) > v(z)$ if $c \in U$. Hence $a \prec z \preceq c$ or equivalently $a \prec_1 c$ if $c \in U$.

We have shown that $(a,\rightarrow)_1$ is open. In the analogous way we can prove that $(\leftarrow,a)_1$ is open and then it is proved that $\prec_1$ is continuous. The proof of continuity of $\prec_2$ and $\prec$ is similar.

Our aim is to prove that the necessary conditions above established are also sufficient in the setting of path connected topological spaces. The idea is to prove that there exists a connected and separable subset of A where the relation is representable and to show after that this representation can be extended. If $A' \subset A$ and $X' \subset X$, the restricted relation $\prec' = \prec \cap A' \times X'$ induces $\prec_1'$ and $\prec_2'$. Note that in general $\prec_1'$ and $\prec_2'$ do not coincide with $\prec_1 \cap A' \times A'$ and $\prec_2 \cap X' \times X'$, respectively, as we see in the following example.

Example 3 *Let* $A = X = [1,10]$ *and* $a \prec x$ *if and anly if* $a^2 < x$. *Let* $A' = X' = [1,3]$*: we have* $2 \prec_1 3$ *because there exists* $x \in A$ *such that* $2 \prec x \preceq 3$ *(any* x *such that* $4 < x \leq 9$*) but we don't have* $2 \prec_1' 3$.

Proposition 4 *Let "*$\prec$*"*$\subset A \times X$ *a biorder verifying the condition [1] such that* $\prec$, $\preceq_1$, $\preceq_2$ *are continuous. If* $A' \subset A$ *and* $X' \subset X$ *are connected and* A' *bounds* $\preceq_1$ *and* X' *bounds* $\preceq_2$*, then,* $\prec_1 = \prec_1'$, $\prec_2 = \prec_2'$

Proof. Let us remark that the relation $\prec'$ verifies also the condition [1]: since $\prec$ verifies the condition [1] and X' bounds $\preceq_2$, for each $a \in A'$ there exist $x_a \in X$, $x_a' \in X'$ such that

$$x_a' \preceq_2 x_a \preceq a \Longrightarrow x_a' \preceq a$$

It is similar to prove that for each $x \in X'$, there is $a_x' \in A'$ such that $x \preceq a_x'$.

Let us show now that $a \prec_1 b \Longleftrightarrow a \prec_1' b$ for all $a, b \in A'$. If $a \prec_1 b$; there exists $x \in X$ such that $a \prec x \preceq b$: since X' bounds $\preceq_2$, $x \preceq_2 x'$ for some $x' \in X'$ and then $a \prec x \preceq_2 x' \Longrightarrow a \prec x'$

From condition [1] we have $x_a' \in X'$ such that $x_a' \preceq a \prec x \Longrightarrow x_a' \prec_2 x$. Moreover if $z \in X'$ is such that $z \notin (a,\rightarrow)$, $z \preceq a \prec x \Longrightarrow z \prec_2 x$

Then, $X' = [(a,\rightarrow) \cap X'] \cup [(\leftarrow,x)_2 \cap X']$; that is, X' is the union of two non empty open sets. Since X' is connected, there is $y' \in X'$ such that $y' \in (a,\rightarrow) \cap (\leftarrow,x)_2 \Longrightarrow a \prec y' \prec_2 x \preceq b \Longrightarrow a \prec y' \preceq b \Longrightarrow a \prec_1' b$.

It follows that $\prec_1 = \prec_1'$ and in the similar way it is showed that $\prec_2 = \prec_2'$.

Note that if $X' = X$ (respectively $A' = A$) then $\prec_1$ is always identical to $\prec_1'$ (respectively $\prec_2 = \prec_2'$).

Theorem 5 *Let A be a path connected space, X a connected space and "$\prec$"$\subset A \times X$ verifying the condition [1]. There exists a continuous representation u, v of $\prec$ if and only if*

a) $\prec$ is a biorder.

b) the complete preorders $\preceq_1, \preceq_2$ are countably bounded.

c) the relations $\prec$, $\prec_1$, $\prec_2$ are continuous.

Moreover, when such a representation exists, there exists one such that u and v are utility functions for, repectively, the complete preorders $\preceq_1$ and $\preceq_2$.

Proof. The necessary conditions have already been proved. Let us now turn to the sufficiency part.

The space A is path connected; then, by using a result of Monteiro (1987), there exists a connected and separable subset of A, A', which bounds $\preceq_1$ (Let $a \in X$; if $\{x_n\}_{n\in \mathbb{N}}$ is a countable set which bounds $\preceq_1$, for each $n \geq 1$, let f_n be a path connecting a to x_n. We define $Y = \bigcup_{n\in \mathbb{N}} f_n[0,1]$; the set Y bounds $\preceq_1$ and it is path connected, then it is connected; moreover, it is separable because if A is a countable dense subset of $[0,1]$, the set $\bigcup_{n\in \mathbb{N}} f_n(A)$ is dense in Y.)

The relation $\preceq_1$ is continuous and $\preceq_1 = \preceq_1'$; by applying the classical result of Eilenberg (1941), there exists $u' : A' \longrightarrow [0,1]$ such that $a \preceq_1' b \iff u'(a) \leq u'(b)$ for all $a, b \in A'$, that is u' is a continuous utility representation for $\preceq_1'$.

The function u' can be extended to A: for each $a \in A$, the sets $\{a' \in A'; a' \preceq_1 a\}$ and $\{a' \in A'; a \preceq_1 a'\}$ are non empty (A' bounds $\preceq_1$) and closed sets on A'. Since A' is connected, there is $\bar{a} \in A'$ such that $\bar{a} \sim_1 a$. Let us define $u(a) = u'(\bar{a})$. We have, for all $a, b \in A$,

$$a \preceq_1 b \iff \bar{a} \preceq_1' \bar{b} \iff u'(\bar{a}) \leq u'(\bar{b}) \iff u(a) \leq u(b)$$

that is, u is an utility representation for $\preceq_1$. It is not difficult to prove that for all $\alpha \in \mathbb{R}$, the sets $\{a \in A; u(a) \geq \alpha\}$ and $\{a \in A; u(a) \leq \alpha\}$ are closed; then u is continuous.

Now, we define $v : X \longrightarrow [0,1]$ by

$$v(x) = \begin{cases} 0 & \textit{if there is not } a \in A \textit{ such that } a \prec x \\ \sup\{u(a), a \in A, a \prec x\} & \textit{in other case} \end{cases}$$

The pair u, v is a representation of $\prec$ on $A \times X$: let $a \in A, x \in X$ such that $a \prec x$; from condition [1], $x \preceq a_x$ from some $a_x \in A$ and then $a \prec_1 a_x$. Moreover, if $c \in A$ is such that $c \notin (\leftarrow, x), a \prec x \preceq c \Longrightarrow a \prec_1 c$. Then, $A = (\leftarrow, x) \cap (a, \rightarrow)_1$; that is, A is the union of two non empty open sets; since A is connected, there is $b \in A$ such that $a \prec_1 b \prec x \Longrightarrow u(a) < u(b) \leq v(x) \Longrightarrow u(a) < v(x)$.

Reciprocally, if $x \preceq a$, there are two possibilities: if there is not $b \in A$ such that $b \prec x \Longrightarrow v(x) = 0 \Longrightarrow v(x) \leq u(a)$; if there is $b \in A$ such that $b \prec x$, we have $b \prec x \preceq a \Longrightarrow b \prec_1 a \Longrightarrow u(b) < u(a)$ for all $b \prec x \Longrightarrow v(x) \leq u(a)$.

The function v is a continuous utility representation for $\preceq_2$: let $x \preceq_2 y$; there are two possibilities: if there is not $a \in A$ such that $a \prec x \Longrightarrow v(x) = 0 \Longrightarrow v(x) \leq v(y)$; if there exists $a \in A$ such that $a \prec x \Longrightarrow a \prec x \preceq_2 y \Longrightarrow a \prec y \Longrightarrow v(x) \leq v(y)$. Reciprocally, if $y \prec_2 x$, there exists $a \in A$ such that $y \preceq a \prec x \Longrightarrow v(y) \leq u(a) < v(x) \Longrightarrow v(y) < v(x)$. Moreover, v is continuous because the sets $\{x \in X; v(x) > \alpha\}$ and $\{x \in X; v(x) < \alpha\}$ are open for all $\alpha \in \mathbb{R}$.

4 Final remarks

We remark that our theorem generalizes the Chateauneuf's result in the sense that we allow relations between two different spaces A and X and even in the case where $A = X$, we don't require the relation be irreflexive. By other side, our characterization is in terms of the property of being countably bounded instead of the property of strong separability used by Chateauneuf. We remark that strong separability implies that the associate preorders are countably bounded and in general they are not equivalents. However, as a consequence of the result, we see that in the hypothesis of the theorem both properties are equivalents. In infinite dimensional spaces, the strong separability can be difficult to test and sometimes the property of being countably bounded can be easy to test; this is the case when the spaces are compact or σ-compact. As a consequence, we see that if A is a σ-compact and path connected space, every biorder in $A \times X$ with continuous associate preorders is representable by two continuous functions. As a particular case, every complete and continuous preorder in A is representable by a continuous utility function.

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