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ON NON REPRESENTABLE PREFERENCES

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Abstract

In this note, we prove that for every non-separable metric space there is a continuous preference ordering which is non representable by an utility function.

Key words
Preference Ordening; Utility Function; Non Separable Metric Space.

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1. Introduction

This work is concerned with the numerical representation of all continuous preference orderings on a topological space. As it is well known, if X is a connected and separable topological space, then continuous preference orderings on X always have utility representations (see Eilenberg (1941) and Debreu (1954)). The assumption of connectedness is not necessary in the setting of metric spaces: if X is perfectly separable, every continuous preference ordering is representable by an utility function (Debreu (1954)).

However, we show here that separability is also a necessary condition for the representability of all continuous preference orderings on a metric space. That is, if X is a non separable metric space, there exists a continuous preference ordering which does not admit an utility representation. This is relevant since consumption sets in infinite dimensional commodity spaces are not separable, in general.

2. Definitions

A preference ordering on the set X is, to be precise, a binary relation on X, say $\preceq$, which is reflexive, transitive and complete.

An utility representation for the preference ordering $\preceq$ on X is a function $u : X \to R$ such that $x \preceq y$ if and only if $u(x) \leq u(y)$.

Let X be a topological space. We say that X is separable if it contains a countable subset whose closure is X. We say that X is perfectly separable (or that X satisfies the second countability axiom) if there is a countable class of open subsets such that every open subset in X is the union of sets of that class. Every perfectly separable topological space is separable. Every separable metric space is perfectly separable. A topological space X is connected if there is no partition of X into two disjoint, non-empty closed sets. We say that X is path connected if for all x, y in X there is a continuous function $f : [0,1] \to X$ with $f(0) = x$ and $f(1) = y$. Note that every path connected space is connected and every convex set in a linear topological space is path connected.

A preference ordering $\preceq$ on a topological space X is continuous if the sets $\{x \in X \,;\, x \preceq x'\}$, $\{x \in X \,;\, x' \preceq x\}$ are closed for all $x' \in X$. A subset $B \subset X$ bounds $\preceq$ if for every $x \in X$ there are a, b in B with $a \preceq x \preceq b$. A preference ordering $\preceq$ is countably bounded if there exists a countable set $B \subset X$ that bounds $\preceq$. Any preference ordering which has an utility representation is countably bounded.

3. The existence theorem

THEOREM: Let X be a non separable metric space. Then there is a continuous preference ordering on X which cannot be represented by an utility function.

To prove the theorem we shall make use of an auxiliar space L called the long line (see Monteiro (1987), example 5, p. 151). Let Ω_1 be the least non-countable ordinal. We denote by Ω the set of all ordinals α such that $\alpha < \Omega_1$. That is to say that Ω is the set of all countable ordinals. Note that Ω is a well ordered set, non-countable and such that for all $\alpha \in \Omega$, $\{\beta \in \Omega \ ; \ \beta \leq \alpha\}$ is countable.

Between each $\alpha \in \Omega$ and its follower $\alpha + 1$ put one copy of the real interval $(0,1)$. The space L that we get, ordered in the obviuous way, is called the long line. We consider on L the order topology. The details on the topological space L can be seen in Steen and Seebach (1970, pp.71,72).

LEMMA: For each $a \in L$, $a \neq 0$ the order interval $[0,a] = \{x \in L \ ; \ 0 \leq x \leq a\}$ is a compact set homeomorphic to the real interval $[0,1]$.

Proof. It is clear that it suffices to prove the result when $a = \alpha \in \Omega$. As $\{\beta \in \Omega \ ; \ \beta \leq \alpha\}$ is a well ordered countable set, there is an order preserving $f : \{0, 1, \ldots, \alpha\} \to [0,1]$ such that $f(0) = 0$ and $f(\alpha) = 1$. We define $\tilde{f} : [0,\alpha] \to [0,1]$ by

$$\tilde{f}(b) = f(b) \quad \text{if } b \in \Omega \text{ and}$$

$$\tilde{f}(b) = f(\beta) + t(f(\beta+1) - f(\beta)) \quad \text{if } \ b = \beta + t, \ \beta \in \Omega, \ t \in (0,1).$$

It is clear that $\tilde{f}$ is an isomorphism of the order structures.

PROOF OF THE THEOREM: Let X be a non separable metric space. Non separable metric spaces are caracterized by the following property:

There are $\varepsilon > 0$ and an uncountable set $D \subset X$ such that

$$\text{for all } x, y \in D, \ x \neq y \text{ implies } d(x,y) \geq 3\varepsilon. \tag{1}$$

Otherwise, for each $\epsilon = \frac{1}{n}$, $n \in N$, there exists a countable set D_n verifying (1) such that $X = \bigcup_{a \in D_n} B(a, \frac{1}{n})$, where $B(a, \frac{1}{n}) = \{x \in X, \ d(x,a) < \frac{1}{n}\}$. Then the set $D = \bigcup D_n$ will be countable and dense.

As D is uncountable, for each $\alpha \in \Omega$ we can choose an $x_\alpha \in D$ in such a way that $\alpha \neq \beta$ implies $x_\alpha \neq x_\beta$. By the lemma, for each $\alpha \in \Omega$ there exist $\varphi_\alpha : [0, \varepsilon] \to L$, which is an isomorphism between the order structures of $[0, \varepsilon] \subset R$ and $[\varphi_\alpha(0), \varphi_\alpha(\varepsilon)] = [0, \alpha] \subset L$

Let $U : X \to L$ be defined by

$$U(x) = \begin{cases} 0 & \text{if } x \notin \bigcup_{\alpha\in\Omega} B(x_\alpha, \varepsilon) \\ \varphi_\alpha(\varepsilon - d(x_\alpha, x)) & \text{if } x \in B(x_\alpha, \varepsilon) \end{cases} .$$

It is clear that U is continuous in $B(x_\alpha, \varepsilon)$ because φ_α and d are continuous. If $x \in X$ is such that $d(x_\alpha, x) = \varepsilon$, we have $U(x) = 0$, and $\varphi_\alpha(0) = 0$, then U is continuous in x. As the intersection of two different balls is empty and U is constant in the exterior of $\bigcup_{\alpha\in\Omega} B(x_\alpha, \varepsilon)$, we have that U is continuous in X.

For x, y in X, we define $x \preceq y$ if and only if $U(x) \leq U(y)$. It is clear that $\preceq$ is a continuous preference ordering on X, but has no utility representation because is not countably bounded. To see it, note that given a countable set $B \subset X$ there exists $\alpha_B \in \Omega$ such that $\sup_{b\in B} U(b) < \alpha$ and then there is not a countable set $B \subset X$ that bounds $\preceq$.

4. Final remark

We remark that separability is not a necessary condition for the representability of all preference orderings on a general topological space X. Monteiro (1987) proves that a continuous preference ordering on a path connected topological space X is representable if and only if it is numerably bounded. A continuous preference ordering on a compact topological space has one best and one worst point. Then any continuous preference ordering on a compact or σ-compact (an union of a countable family of compact sets) path connected topological space is representable by utility functions. Note that any compact or σ-compact metric space is separable but compact topological spaces in general need not to be separable.

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