On polynomials associated with an Uvarov modification of a quartic potential Freud-like weight

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A B S T R A C T

In this contribution we consider sequences of monic polynomials orthogonal with respect to the standard Freud-like inner product involving a quartic potential

$$\langle p,q\rangle = \int_{\mathbb{R}} p(x)q(x)e^{-x^4+2tx^2}dx + Mp(0)q(0).$$

We analyze some properties of these polynomials, such as the ladder operators and the holonomic equation that they satisfy and, as an application, we give an electrostatic interpretation of their zero distribution in terms of a logarithmic potential interaction under the action of an external field. It is also shown that the coefficients of their three term recurrence relation satisfy a nonlinear difference string equation. Finally, an equation of motion for their zeros in terms of their dependence on t is given.

1. Introduction

Let us consider the so called Freud-like inner products

$$\langle p,q\rangle_t = \int_{\mathbb{R}} p(x)q(x)d\mu_t(x), \quad p,q \in \mathbb{P},$$

where $d\mu_t(x) = \omega_t(x)dx = e^{-V_t(x)}dx$ is a positive, nontrivial Borel measure supported in the whole real line $\mathbb{R}$. The properties of such sequences of polynomials are known only for certain values of the external potential $V_t(x)$. Thus, a well studied example in the literature corresponds to the case $V_t(x) = x^4 - 2tx^2$, which leads to the inner product

$$\langle p,q\rangle_t = \int_{\mathbb{R}} p(x)q(x)e^{-x^4+2tx^2}dx, \quad p,q \in \mathbb{P}. \tag{1}$$

Here $t \in \mathbb{R}_+$ is a parameter, which, in other contexts, can be interpreted as the "time" variable. We denote by $\{F_n^t\}_{n\geqslant 0}$ the corresponding sequence of monic orthogonal polynomials (SMOP, in short) which constitutes a family of semi-classical orthogonal polynomials (see [23,28]), because $V_t(x)$ is differentiable in $\mathbb{R}$ (the support of μ_t), and the weight function satisfies

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the distributional equation, known in the literature as Pearson equation (see [22,30])

$$[\omega_t(x)]' = (-4x^3 + 4tx)\omega_t(x).$$

where "$'$" denotes derivative with respect to the variable x.

We must point out that the class of such a family is $s = 2$, and that several SMOP related to (1) have been studied in the literature. For example, for $t = 0$ the asymptotic behavior of the corresponding SMOP has been deeply analyzed in [25]. In [5] a Riemann-Hilbert approach is done to find several semi-classical asymptotic results for these polynomials. As an application of the semi-classical asymptotics of the orthogonal polynomials, the authors prove the universality of the local distribution of eigenvalues in the matrix model with the double-well quartic interaction in the presence of two cuts. In [12] the authors study the varying quartic Freud-like weight $e^{-V(x)}$, with $V(x) = N(t\frac{x^4}{4} + \frac{x^2}{2})$ for $t < 0$, where the orthogonality takes place on certain contours of the complex plane. They use the Deift/Zhou steepest descent analysis of the Riemann-Hilbert problem associated with the corresponding polynomials, to present an alternative and a more direct proof of the Fokas, Its and Kitaev result, showing that there exists a critical value for t around which the asymptotics of the recurrence coefficients are described in terms of exactly specified solutions of the Painlevé I equation. In [19] the author explores the nonlinear difference equation satisfied by the coefficients in the three term recurrence relations for polynomials orthogonal with respect to exponential weights, and in [21] the author finds the relation of such Freud's equations with discrete Painlevé equations for certain potentials, as $V(x) = x^2$, $V(x) = \alpha x^4 + \beta x^2$ or $V(x) = x^6$, among others. The survey [18] presents analytic aspects of general orthogonal polynomials associated with several exponential weights on finite and infinite intervals. The works [4,9,10,25–27,31,32] deal with uniform and strong asymptotics for polynomials orthogonal with respect to certain exponential weights, and their connections with the random matrix theory. Magnus showed in [20] that the recurrence coefficients of semi-classical orthogonal polynomials are shown to be solutions of nonlinear differential equations with respect to a well-chosen parameter, according to principles established by D. Chudnovsky and G. Chudnovsky.

In the linear space $\mathbb{P}$ of polynomials with real coefficients, let us introduce the following inner product

$$\begin{aligned}\langle p, q\rangle &= \int_{\mathbb{R}} p(x)q(x)\omega_t(x)dx + Mp(0)q(0) \\ &= \langle p, q\rangle_t + Mp(0)q(0),\end{aligned} \tag{2}$$

with $M \in \mathbb{R}_+$. We denote by $\{Q_n^t\}_{n\geqslant 0}$ the SMOP with respect to the above inner product. $\{Q_n^t\}_{n\geqslant 0}$ is said to be the sequence of monic polynomials orthogonal with respect to the measure $d\nu(x) = d\mu_t(x) + M\delta_0(x)$, where $\delta_0(x)$ is the Dirac delta function at $x = 0$. These polynomials are known as *Freud-type* or *Freud-Krall* orthogonal polynomials. It is clear that $\{Q_n^t\}_{n\geqslant 0}$ is a standard sequence, since the operator of multiplication by x is symmetric with respect to such an inner product, i.e. $\langle xp, q\rangle = \langle p, xq\rangle$, for every $p, q \in \mathbb{P}$.

This inner product was first studied in [14] in the particular case $t = 0$. There, the authors introduce a Dirac mass point at $x = 0$ and provide the holonomic second order linear differential equation with varying polynomial coefficients. They also give an electrostatic interpretation for the distribution of zeros of the corresponding orthogonal polynomials.

Following this initial work, the aim of our contribution is twofold. First, we generalize the electrostatic interpretation of the zero distribution in terms of a logarithmic potential interaction under the action of an external field for values of t greater than zero. Second, we derive an equation of motion for the distribution of the zeros of the corresponding Freud-type orthogonal polynomials.

The remainder of this Section will be devoted to provide some basic structural properties of $\{F_n^t\}_{n\geqslant 0}$ to be used in the sequel.

Proposition 1. Let $\{F_n^t\}_{n\geqslant 0}$ denote the sequence of Freud monic polynomials orthogonal with respect to (1). The following statements hold.

1. Three term recurrence relation. For every $n \in \mathbb{N}$, the recurrence relation for $\{F_n^t\}_{n\geqslant 0}$, is given by (see [13])

$$xF_n^t(x) = F_{n+1}^t(x) + a_n^2(t)F_{n-1}^t(x) \tag{3}$$

   with $F_{-1}^t(x) = 0$ and $F_0^t(x) = 1$. An important feature of these polynomials is that the recurrence coefficients $a_n(t)$ in the above three term recurrence relation satisfy the following nonlinear difference equation

$$4a_n^2(t)\left(a_{n-1}^2(t) + a_n^2(t) + a_{n+1}^2(t) - t\right) = n, \quad n \geqslant 1, \tag{4}$$

   with $a_0^2(t) = 0$ and $a_1^2(t) = ||x||_t^2/||1||_t^2$. This is known in the literature as the *string equation* or *Freud equation* (see [2,13,15,17], among others). Moreover, notice that $\omega_t(x)$ is an even weight, i.e. $\{F_n^t\}_{n\geqslant 0}$ is a symmetric sequence of orthogonal polynomials.
2. We will denote by

$$||F_n^t||_t^2 = \left\langle F_n^t, F_n^t\right\rangle_t = \int_{\mathbb{R}} [F_n^t(x)]^2 e^{-x^4+2tx^2} dx.$$

the corresponding norm. Hence

$$\zeta_n(t) = a_1^2(t)\cdots a_n^2(t) > 0,$$

with $||F_n^t||_t^2 = \zeta_n(t)\,||1||_t^2$.

3. Evolution equation. The recurrence coefficients $a_n(t)$ in (3) satisfy

$$\dot{a}_k(t) = a_k(t)\left(a_{k+1}^2(t) - a_{k-1}^2(t)\right), \quad k \geqslant 1,$$

where "dot" denotes the derivative with respect to time variable $t > 0$ (see [1, (1.12)],[24, (2.4)]).

4. Structure relation ([15, Th. 3.2.1, and p.57]). For every $n \in \mathbb{N}$,

$$[F_n^t(x)]' = a(x,t;n)F_n^t(x) + b(x,t;n)F_{n-1}^t(x), \tag{5}$$

where

$$a(x,t;n) = -4a_n^2(t)\,x,$$
$$b(x,t;n) = 4a_n^2(t)\big(x^2 - t + a_n^2(t) + a_{n+1}^2(t)\big).$$

5. Holonomic equation ([15, Th. 3.2.3, and p.57]). The SMOP $\{F_n^t\}_{n\geqslant 0}$ satisfies the second order linear differential equation

$$[F_n^t(x)]'' + R_n^t(x)[F_n^t(x)]' + S_n^t(x)F_n^t(x) = 0, \tag{6}$$

where

$$R_n^t(x) = -4\big(x^3 - tx\big) - \frac{2x}{x^2 - t + a_n^2(t) + a_{n+1}^2(t)},$$

$$S_n^t(x) = 4a_n^2(t)\Bigg[4x^2\Bigg(a_{n-1}^2(t) + a_n^2(t) + a_{n+1}^2(t) - t - \frac{2}{x^2 - t + a_n^2(t) + a_{n+1}^2(t)}\Bigg) + \big(a_n^2(t) + a_{n+1}^2(t) - t\big)\big(a_{n-1}^2(t) + a_n^2(t) - t\big) + 1\Bigg].$$

The structure of the manuscript is as follows. Section 2 will be devoted to the Freud kernel polynomials. In Section 3, several connection formulas between Freud-type and Freud polynomials are given and we briefly analyze the coefficients in the three term recurrence relation for Freud-type polynomials and a string-type equation for these coefficients. In Section 4, we provide the lowering and raising operators associated with Freud-type polynomials, the corresponding holonomic equation, and the electrostatic interpretation of the zeros of Freud-type polynomials as positive unit charges interacting according to a logarithmic potential under the action of an external field. Notice that the second order linear differential equation is obtained as a composition of the ladder operators following the ideas in [6] for some absolutely continuous measures and [7] for measures with Dirac deltas. Nevertheless, we must point out that in our case we consider a measure with an extra time parameter t and a Dirac mass located at $x = 0$. The explicit expressions of the polynomial coefficients in the holonomic equation are given in our contribution. Its role in the study of the electrostatic interpretation is emphasized in this Section. In Section 5, we analyze the zero behavior of Freud and Freud-type polynomials in terms of the parameter t as well as we deduce some results concerning the monotonicity and speed of convergence of zeros of Freud-type polynomials in terms of the mass M. Finally, in Section 6, we explore some numerical results showing the location of certain zeros of Freud-type orthogonal polynomials.

2. Freud kernel polynomials

The n-th degree kernel polynomial

$$K_n(x,y;t) = \sum_{k=0}^{n} \frac{F_k^t(x)F_k^t(y)}{||F_k^t||_t^2} \tag{7}$$

associated with the polynomial sequence $\{F_n^t\}_{n\geqslant 0}$ will play a key role in order to prove some of the basic results of the manuscript. Notice that if $deg\, p \leq n$, then the n-th kernel polynomial satisfies the so-called "reproducing property"

$$\int_{\mathbb{R}} K_n(x,y;t)p(x)\omega_t(x)dx = p(y). \tag{8}$$

For $x \neq y$, according to the Christoffel-Darboux formula, for every $n \in \mathbb{N}$ we get

$$K_n(x,y;t) = \frac{1}{||F_n^t||_t^2}\frac{F_{n+1}^t(x)F_n^t(y) - F_{n+1}^t(y)F_n^t(x)}{x-y} \tag{9}$$

together with the confluent expression

$$K_n(x,x;t) = \sum_{k=0}^{n} \frac{[F_k^t(x)]^2}{||F_k^t||^2} = \frac{[F_{n+1}^t]'(x)F_n^t(x) - [F_n^t]'(x)F_{n+1}^t(x)}{||F_n^t||^2}.$$

From the above formula

$$K_n(0,0;t) = \frac{[F_{n+1}^t]'(0)F_n^t(0) - [F_n^t]'(0)F_{n+1}^t(0)}{||F_n^t||_t^2},$$

and also we conclude that $K_n(0, 0; t) > 0$, no matter the degree n or parity of the kernel, since it is a sum of strictly positive terms.

The weight function $\omega_t(x)$ is an even function, so the monic polynomial sequence $\{F_n^t(x)\}_{n\geqslant 0}$ is symmetric (see [8, Ch. I]) and the following expressions hold

$$\begin{cases} F_n^t(0) = [F_n^t]''(0) = 0, & \text{if } n \text{ is odd}, \\ [F_n^t]'(0) = [F_n^t]'''(0) = 0, & \text{if } n \text{ is even}. \end{cases} \tag{10}$$

From (7), (9) and (10) we have for $m \geq 1$

$$\begin{aligned} K_{2m-1}(x,0;t) &= K_{2m-2}(x,0;t), \\ K_{2m}(x,0;t) &= \frac{1}{||F_{2m}^t||_t^2} \frac{F_{2m}^t(0)F_{2m+1}^t(x)}{x} \end{aligned} \tag{11}$$

and, therefore, all $K_n(x, 0; t)$ has even degree for any n. It is obvious that

$$\begin{aligned} K_{2m-1}(0,0;t) &= K_{2m-2}(0,0;t), \\ K_{2m}(0,0;t) &= \frac{[F_{2m+1}^t]'(0)F_{2m}^t(0)}{||F_{2m}^t||_t^2}. \end{aligned} \tag{12}$$

From now on, $\{F_n^{t,[k]}\}_{n\geq 0}$ will denote the SMOP with respect to the inner product

$$\langle p, q\rangle_{t,[k]} = \int_{\mathbb{R}} p(x)q(x)x^k e^{-x^4+2tx^2}dx \tag{13}$$

which is a polynomial modification of the measure $d\mu_t(x) = e^{-x^4+2tx^2}dx$ called the *k-iterated Christoffel perturbation*. If $k = 1$ we have the *Christoffel canonical transformation of the measure* μ_t (see [33,34] among others). The SMOP $\{F_n^{t,[k]}\}_{n\geq 0}$ is known as the *monic k-iterated Freud kernel polynomials*. In this particular case, due to the parity of the weight function, when k is odd the perturbed measure is not quasi-definite and, as consequence, the corresponding SMOP does not exist. On the contrary, when k is even the perturbed measure is still positive definite, and therefore the corresponding k-iterated Freud kernel polynomials are well defined. We will denote by

$$||F_n^{t,[k]}||_{t,[k]}^2 = \langle F_n^{t,[k]}, F_n^{t,[k]}\rangle_{t,[k]} = \int_{\mathbb{R}} |F_n^{t,[k]}(x)|^2 x^k e^{-x^4+2tx^2}dx, \quad k \text{ even},$$

the corresponding norm.

Let $\{F_n^{t,[2]}(x)\}_{n\geq 0}$ denote the sequence of 2-iterated monic Freud kernel polynomials, or the SMOP associated with the measure $d\mu_t^{[2]} = x^2 d\mu_t$ which is the 2-*iterated Christoffel transformation of* μ_t.

The following two lemmas will be useful in the sequel.

Lemma 1. *The* 2*-iterated Freud kernel polynomials and the Freud orthogonal polynomials satisfy the connection formulas,*

$$x^2 F_{2m-1}^{t,[2]}(x) = F_{2m+1}^t(x) + \xi_{2m-1}^2 F_{2m-1}^t(x), \quad m \geq 1, \tag{14}$$

$$x^2 F_{2m}^{t,[2]}(x) = F_{2m+2}^t(x) + \xi_{2m}^2 F_{2m}^t(x), \quad m \geq 1, \tag{15}$$

where

$$\begin{aligned} \xi_{2m-1}^2 &= \frac{||F_{2m-1}^{t,[2]}||_{t,[2]}^2}{||F_{2m-1}^t||_t^2} = \frac{-[F_{2m+1}^t]'(0)}{[F_{2m-1}^t]'(0)}, \\ \xi_{2m}^2 &= \frac{||F_{2m}^{t,[2]}||_{t,[2]}^2}{||F_{2m}^t||_t^2} = \frac{-F_{2m+2}^t(0)}{F_{2m}^t(0)}. \end{aligned}$$

Furthermore,

$$x^2 F_{2m-1}^{t,[2]}(x) = xF_{2m}^t(x) + \left[\xi_{2m-1}^2 - a_{2m}^2(t)\right]F_{2m-1}^t(x), \tag{16}$$

$$xF_{2m}^{t,[2]}(x) = F_{2m+1}^t(x), \tag{17}$$

where

$$\left[\xi_{2m-1}^2 - a_{2m}^2(t)\right] > 0. \tag{18}$$

Proof. We can expand $x^2F_n^{t,[2]}(x)$ in terms of the SMOP $\{F_n^t\}_{n\geqslant 0}$ as

$$x^2F_n^{t,[2]}(x)=\sum_{k=0}^{n+2}b_{n,k}F_k^t(x).$$

For every $k<n$, we have $b_{n,k}=0$ by orthogonality. Also we have $b_{n,n+2}=1,$ since we deal with monic polynomials. The following coefficient

$$b_{n,n+1}=\frac{\langle x^2F_n^{t,[2]},F_{n+1}^t\rangle_t}{||F_{n+1}^t||_t^2}$$

vanishes, because the numerator is an integral in the whole $\mathbb{R}$ of an odd function, so

$$\langle x^2F_n^{t,[2]},F_{n+1}^t\rangle_t=\int_{-\infty}^{\infty}x^2F_n^{t,[2]}F_{n+1}^te^{-x^4+2tx^2}dx=0.$$

Next, we easily see that

$$b_{n,n}=\xi_n^2=\frac{\langle x^2F_n^{t,[2]},F_n^t\rangle_t}{||F_n^t||_t^2}=\frac{\langle F_n^{t,[2]},F_n^t\rangle_{t,[2]}}{||F_n^t||_t^2}=\frac{||F_n^{t,[2]}||_{t,[2]}^2}{||F_n^t||_t^2}>0.$$

Thus, we have

$$x^2F_n^{t,[2]}(x)=F_{n+2}^t(x)+\xi_n^2\,F_n^t(x),\tag{19}$$

Shifting the index in the above formula for n odd and n even, we obtain formulas (14) and (15) respectively. Taking $x=0$ in (15) yields

$$\xi_{2m}^2=\frac{||F_{2m}^{t,[2]}||_{t,[2]}^2}{||F_{2m}^t||_t^2}=\frac{-F_{2m+2}^t(0)}{F_{2m}^t(0)},$$

which is well defined since $x=0$ is not a zero of any Freud polynomial of even degree. In case of Freud polynomials of odd degree, the above expression is not defined. Letting $x\to 0$ in (14), we may apply L'Hôpital's rule obtaining

$$\xi_{2m-1}^2=\frac{||F_{2m-1}^{t,[2]}||_{t,[2]}^2}{||F_{2m-1}^t||_t^2}=\frac{-[F_{2m+1}^t]'(0)}{[F_{2m-1}^t]'(0)}.$$

From (3)

$$F_{2m+2}^t(x)=xF_{2m+1}^t(x)-a_{2m+1}^2(t)F_{2m}^t(x),\tag{20}$$

and taking $x=0$ we have

$$a_{2m+1}^2(t)=\frac{-F_{2m+2}^t(0)}{F_{2m}^t(0)},$$

which essentially equals to ξ_{2m}^2. Combining (15) with (20) we have

$$\begin{aligned}x^2F_{2m}^{t,[2]}(x)&=xF_{2m+1}^t(x)+\left[\xi_{2m}^2-a_{2m+1}^2(t)\right]F_{2m}^t(x)\\&=xF_{2m+1}^t(x).\end{aligned}$$

Dividing by x we obtain the connection formula (17).

Next, from (3)

$$F_{2m+1}^t(x)=xF_{2m}^t(x)-a_{2m}^2(t)F_{2m-1}^t(x),$$

and substituting this expression into (14) gives

$$x^2F_{2m-1}^{t,[2]}(x)=xF_{2m}^t(x)+\left[\xi_{2m-1}^2-a_{2m}^2(t)\right]F_{2m-1}^t(x)$$

which yields expression (16). The x derivative of (16) evaluated at $x=0,$ provides

$$[\xi_{2m-1}^2-a_{2m}^2(t)]=\frac{-F_{2m}^t(0)}{[F_{2m-1}^t]'(0)},\tag{21}$$

It is very well known (see [8, Ch. I, Section 8]) that, due to the symmetry of the weight function, there exist two n-th degree monic polynomials A_m and B_m such that

$$\begin{aligned}F_{2m}^t(x)&=A_m(x^2),\\F_{2m+1}^t(x)&=xB_m(x^2),\end{aligned}$$

being $B_m(x)$ the kernel polynomials, with parameter 0, of $A_n(x)$. Thus, we have that $sign\,F^t_{2m}(0) = sign\,A_m(0) = (-1)^m$ and $sign\,[F^t_{2m-1}]'(0) = sign\,B_{m-1}(0) = (-1)^{m-1}$, so therefore, concerning the sign of (21) we have

$$sign\frac{F^t_{2m}(0)}{[F^t_{2m-1}]'(0)} = sign\frac{A_m(0)}{B_m(0)} = \frac{(-1)^m}{(-1)^{m-1}} = -1,$$

which shows that

$$[\xi^2_{2m-1} - a^2_{2m}(t)] > 0.$$

This completes the proof. □

Lemma 2. *The SMOP $\{F^{t,[2]}_n\}_{n\geqslant 0}$ satisfies the following three term recurrence relation*

$$xF^{t,[2]}_n(x) = F^{t,[2]}_{n+1}(x) + \alpha^2_n F^{t,[2]}_{n-1}(x),$$

where

$$\alpha^2_n = \frac{||F^{t,[2]}_n||^2_{t,[2]}}{||F^{t,[2]}_{n-1}||^2_{t,[2]}} = \frac{\xi^2_n}{\xi^2_{n-1}} a^2_n(t).$$

$a^2_n(t)$ and ξ^2_n are given in (3) and Lemma 1, respectively.

Proof. We can expand $xF^{t,[2]}_n(x)$ in terms of $\{F^{t,[2]}_n\}_{n\geqslant 0}$ and, for orthogonality reasons, the only terms remaining are $F^{t,[2]}_{n+1}(x)$ and $F^{t,[2]}_{n-1}(x)$. The coefficient of $F^{t,[2]}_{n+1}(x)$ is 1 because we deal with monic polynomials and the other one is

$$\alpha^2_n = \frac{||F^{t,[2]}_n||^2_{t,[2]}}{||F^{t,[2]}_{n-1}||^2_{t,[2]}} > 0.$$

Let notice that

$$\alpha^2_n = \frac{\langle F^{t,[2]}_n, F^t_n\rangle_{t,[2]}}{\langle F^{t,[2]}_{n-1}, F^t_{n-1}\rangle_{t,[2]}} = \frac{\langle x^2 F^{t,[2]}_n, F^t_n\rangle_t}{\langle x^2 F^{t,[2]}_{n-1}, F^t_{n-1}\rangle_t}.$$

Applying (19) we get

$$\begin{aligned}\alpha^2_n &= \frac{\langle F^t_{n+2}, F^t_n\rangle_t + \xi^2_n\langle F^t_n, F^t_n\rangle_t}{\langle F^t_{n+1}, F^t_{n-1}\rangle_t + \xi^2_{n-1}\langle F^t_{n-1}, F^t_{n-1}\rangle_t}\\ &= \frac{\xi^2_n}{\xi^2_{n-1}}\frac{||F^t_n||^2_t}{||F^t_{n-1}||^2_t}\\ &= \frac{\xi^2_n}{\xi^2_{n-1}} a^2_n(t).\end{aligned}$$

This completes the proof. □

Let $x_{n,k} = x_{n,k}(t)$, $x^{[2]}_{n,k} = x^{[2]}_{n,k}(t)$, $k = 1,\dots,n$, be the zeros of $F^t_n(x)$, $F^{t,[2]}_n(x)$, respectively, arranged in an increasing order. All of them are real and simple. By parity reasons, these zeros are symmetrically arranged with respect to the origin. That is, $-x^{[2]}_{n,1} = x^{[2]}_{n,n}$, $-x^{[2]}_{n,2} = x^{[2]}_{n,n-1}$ and so on.

We next prove that the zeros of $G_{2m}(x) = xF^{t,[2]}_{2m-1}(x)$ and $F^t_{2m}(x)$ interlace. Concerning the zeros $g_{2m,k} = g_{2m,k}(t)$, $k = 1,\dots,2m$, of $G_{2m}(x)$ are the same zeros of $F^{t,[2]}_{2m-1}(x)$ except one more zero at the origin, so $G_{2m}(x)$ has a double zero at $x = 0$. Thus, we have $g_{2m,l} = x^{[2]}_{2m-1,l}$, $l = 1,\dots m$, with $x^{[2]}_{2m-1,m} = g_{2m,m} = g_{2m,m+1} = 0$ and $g_{2m,r} = x^{[2]}_{2m-1,r-1}$, $r = m+1,\dots,2m$.

Both $G_{2m}(x)$ and $F^t_{2m}(x)$ are even polynomial functions, so their respective graphs are symmetric with respect to the point $x = 0$, which means that we only need to prove interlacing in the positive real semi-axis, being the situation in $\mathbb{R}_-$ the reflection with respect to the y-axis. Hence, we only need to prove that, for $x > 0$, between two consecutive zeros $(x_{2m,k}, x_{2m,k+1})$ there is only one zero of $F^{t,[2]}_{2m-1}(x)$. Consider (14) evaluated at the positive zeros of $F^t_{2m}(x)$ (i.e., $x_{2m,\,r} > 0$, with $r = m+1,\dots,2m$). We have

$$x_{2m,r}\,F^{t,[2]}_{2m-1}(x_{2m,r}) = \left[\xi^2_{2m-1} - a^2_{2m}(t)\right] F^t_{2m-1}(x_{2m,r}),$$

and taking into account (18), we get

$$sign[F^{t,[2]}_{2m-1}(x_{2m,r})] = sign[F^t_{2m-1}(x_{2m,r})],\ \ r = m+1,\dots,2m. \tag{22}$$

Thus, from (22), the relation between the zeros of $G_{2m}(x)$ and $F^{t,[2]}_{2m-1}(x)$, the symmetric reflection with respect to the y-axis, and the well known fact that the zeros of $F^t_{2m-1}(x)$ interlace with the zeros of $F^t_{2m}(x)$, we obtain the following interlacing property

Theorem 3. *The inequalities*

$$x_{2m,1} < g_{2m,1} < x_{2m,2} < \cdots < x_{2m,m} < g_{2m,m} = 0 = g_{2m,m+1} < \cdots < x_{2m,2m-1} < g_{2m,2m} < x_{2m,2m},$$

hold for every $m \in \mathbb{N}$.

3. Connection formulas

We can expand $Q_n^t(x)$ in terms of the SMOP $\{F_n^t\}_{n\geqslant 0}$, which is an orthogonal basis of $\mathbb{P}$, as follows

$$Q_n^t(x) = F_n^t(x) + \sum_{i=0}^{n-1} \lambda_{n,i} F_i^t(x),$$

where

$$\lambda_{n,i} = \frac{\left\langle F_i^t(x), Q_n^t(x)\right\rangle_t}{||F_i^t||_t^2}, \quad 0 \leqslant i \leqslant n-1.$$

From (2) and (7) the above equality becomes

$$Q_n^t(x) = F_n^t(x) - MQ_n^t(0)K_{n-1}(x, 0; t), \tag{23}$$

and evaluating the above expression at $x = 0$, we deduce

$$Q_n^t(0) = \frac{F_n^t(0)}{1 + MK_{n-1}(0, 0; t)}. \tag{24}$$

Hence, from (3) and (9), (23) reads as

$$xQ_n^t(x) = F_{n+1}^t(x) + A_n^t F_n^t(x) + B_n^t F_{n-1}^t(x),$$

where

$$A_n^t = -\frac{MF_n^t(0)F_{n-1}^t(0)}{\left\|F_{n-1}^t\right\|_t^2\left(1 + MK_{n-1}(0, 0; t)\right)},$$

$$B_n^t = a_n^2(t) + \frac{M\left(F_n^t(0)\right)^2}{\left\|F_{n-1}^t\right\|_t^2\left(1 + MK_{n-1}(0, 0; t)\right)} = a_n^2(t)\left(\frac{1 + MK_n(0, 0; t)}{1 + MK_{n-1}(0, 0; t)}\right).$$

Remark 1. Due to the fact that $\omega_t(x)$ is an even weight function, $x = 0$ is always a zero of $F_n^t(x)$ for n odd. Then, $F_n^t(0)F_{n-1}^t(0) = 0$ for every $n \geq 1$, and therefore $A_n^t = 0$ for all positive integer n.

From the above remark, we have

$$xQ_n^t(x) = F_{n+1}^t(x) + B_n^t F_{n-1}^t(x), \tag{25}$$

$$Q_{2n+1}^t(x) = F_{2n+1}^t(x).$$

Introducing the notation

$$b_n^t = \frac{1 + MK_n(0, 0; t)}{1 + MK_{n-1}(0, 0; t)} \tag{26}$$

we get

$$B_n^t = a_n^2(t)\, b_n^t.$$

This yields an expression for the ratio of the energy of polynomials $Q_n^t(x)$ and $F_n^t(x)$ with respect to the norms associated with their corresponding inner products.

Proposition 2. *Let* $\|\cdot\|^2$ *be the squared norm of Freud-type monic polynomials with respect to (2). Then*

$$\frac{||Q_n^t||^2}{||F_n^t||_t^2} = \frac{1 + MK_n(0, 0; t)}{1 + MK_{n-1}(0, 0; t)} = b_n^t, \quad n \geqslant 1.$$

Moreover, $b_n^t = 1$ *when* n *is odd and* $b_n^t > 1$ *when* n *is even, i.e., for every* $m \geq 0$,

$$||Q_{2m+1}^t||^2 = ||F_{2m+1}^t||_t^2.$$

Proof. Taking in account

$$\begin{aligned}\left\|Q_n^t\right\|^2 &= \left\langle Q_n^t(x), x^n\right\rangle \\ &= \left\langle Q_n^t(x), F_n^t(x)\right\rangle \\ &= \left\langle Q_n^t(x), F_n^t(x)\right\rangle_t + MQ_n^t(0)F_n^t(0)\end{aligned}$$

and using (24), we get

$$\begin{aligned}||Q_n^t||^2 &= ||F_n^t||_t^2 + \frac{M\left(F_n^t(0)\right)^2}{1+MK_{n-1}(0,0;t)}\\ &= ||F_n^t||_t^2\,\frac{\left(1+MK_{n-1}(0,0;t)\right)+\frac{M\left(F_n^t(0)\right)^2}{||F_n^t||_t^2}}{1+MK_{n-1}(0,0;t)}\\ &= ||F_n^t||_t^2\,\frac{1+MK_n(0,0;t)}{1+MK_{n-1}(0,0;t)}.\end{aligned}$$

Evaluating (11) at $x=0$ we have $b_n^t=1$ for n odd and $b_n^t>1$ for n even. This gives the result when combined with the above equation. □

As a consequence, we get

Theorem 4. *Let $\left\{Q_n^t\right\}_{n\geqslant 0}$ be the sequence of monic Freud-type polynomials orthogonal with respect to (2). Then*

$$\begin{aligned}Q_{2m+1}^t(x) &= F_{2m+1}^t(x), \quad m\geqslant 0,\\ Q_{2m}^t(x) &= F_{2m}^t(x) - \frac{MF_{2m}^t(0)}{1+MK_{2m-1}(0,0;t)}K_{2m-1}(x,0;t), \quad m\geqslant 1.\end{aligned}$$

Next, we provide an alternative way to represent the Freud-type polynomials of even degree $Q_{2m}^t(x)$ in terms of the polynomials $F_{2m}^t(x)$ and the 2-iterated monic Freud kernel polynomials $F_n^{t,[2]}(x)$. This representation will allow us to obtain the results about monotonicity and asymptotic behavior (presented below in this work) for the zeros of $Q_{2m}^t(x)$ in terms of the parameter M present in (2). We only need to consider Freud type polynomials of even degree, because they are the only ones affected by variations of M.

Theorem 5 (Connection formula)**.** *The sequence $\{\tilde{Q}_{2m}^t\}_{m\geqslant 0}$ can be represented as*

$$Q_{2m}^t(x) = F_{2m}^t(x) + MK_{2m-1}(0,0;t)G_{2m}(x), \quad m\geqslant 1, \tag{27}$$

with $G_{2m}(x)=xF_{2m-1}^{t,[2]}(x)$, $\tilde{Q}_{2m}^t(x)=\kappa_{2m}Q_n^t(x)$ and $\kappa_{2m}=1+MK_{2m-1}(0,0;t)>0$.

Proof. Let κ_{2m} be given by the positive quantity $\kappa_{2m}=1+MK_{2m-1}(0,0;t)$. From the expression for $Q_{2m}^t(x)$ in Theorem 4 we have

$$\kappa_{2m}Q_{2m}^t(x)=\kappa_{2m}F_{2m}^t(x)-MF_{2m}^t(0)K_{2m-1}(x,0;t), \quad m\geqslant 1.$$

Being $\kappa_{2m}>0$, we call $\tilde{Q}_{2m}^t(x)=\kappa_{2m}Q_{2m}^t(x)$ the polynomial with the same zeros that $Q_{2m}^t(x)$. Next, (11) and (12) yields

$$\begin{aligned}\tilde{Q}_{2m}^t(x) &= F_{2m}^t(x)+M\left[K_{2m-1}(0,0;t)F_{2m}^t(x)-F_{2m}^t(0)K_{2m-1}(x,0;t)\right]\\ &= F_{2m}^t(x)-M\frac{[F_{2m-1}^t]'(0)F_{2m}^t(0)}{||F_{2m-1}^t||_t^2}\left[F_{2m}^t(x)-\frac{F_{2m}^t(0)}{[F_{2m-1}^t]'(0)}\frac{F_{2m-1}^t(x)}{x}\right]\\ &= F_{2m}^t(x)+MK_{2m-1}(0,0;t)\left[F_{2m}^t(x)-\frac{F_{2m}^t(0)}{[F_{2m-1}^t]'(0)}\frac{F_{2m-1}^t(x)}{x}\right]\end{aligned} \tag{28}$$

On the other hand, replacing (21) into (16) and dividing by x yields

$$xF_{2m-1}^{t,[2]}(x)=F_{2m}^t(x)-\frac{F_{2m}^t(0)}{[F_{2m-1}^t]'(0)}\frac{F_{2m-1}^t(x)}{x}.$$

From the above expression, we can finally rewrite (28) as

$$Q_{2m}^t(x)=F_{2m}^t(x)+MK_{2m-1}(0,0;t)xF_{2m-1}^{t,[2]}(x).$$

This completes the proof. □

In the remaining of this section we will focus our attention on the coefficients of the three term recurrence relation satisfied by the Freud-type SMOP. Since $Q_n^t(x)$ are standard and symmetric, they satisfy the following fundamental recurrence relation.

Proposition 3. *The polynomials $Q_n^t(x)$ satisfy the three term recurrence relation*

$$xQ_n^t(x)=Q_{n+1}^t(x)+\gamma_n(t)Q_{n-1}^t(x), \tag{29}$$

where

$$\gamma_n(t)=\frac{b_n^t}{b_{n-1}^t}a_n^2(t). \tag{30}$$

Proof. We expand $xQ_n^t(x)$ in terms of the SMOP $\{Q_n^t\}_{n\geqslant 0}$ and, taking into account (2), the Lemma follows. □

Proposition 4 (String equation)**.** *The coefficients (30) of the above three term recurrence relation for* $\{Q_n^t\}_{n\geqslant 0}$ *satisfy the following nonlinear difference string equation*

$$4\gamma_n^2(t)\left(\frac{b_{n-2}^t}{b_n^t}\gamma_{n-1}^2(t)+\left(\frac{b_{n-1}^t}{b_n^t}\right)^2\gamma_n^2(t)+\frac{b_{n-1}^t}{b_{n+1}^t}\gamma_{n+1}^2(t)-\frac{t}{2}\right)=n.$$

Proof. It is enough to replace (30) in the string equation (4), and then the Proposition follows. □

4. Holonomic equation and electrostatic model

Next, we give details of the second order linear differential equation satisfied by $\{Q_n^t\}_{n\geqslant 0}$ when $t>0$. First, from (3) we can rewrite (25) as

$$xQ_n^t(x)=A_1(x,t;n)F_n^t(x)+B_1(t;n)F_{n-1}^t(x), \tag{31}$$

with

$$\begin{aligned} A_1(x,t;n)&=x,\\ B_1(t;n)&=a_n^2(t)\big(b_n^t-1\big). \end{aligned}$$

In order to obtain the ladder operators and the second order linear differential equation, we follow a different approach as in [15, Ch. 3]. Our technique is based on the connection formula (23), the three term recurrence relation (3) satisfied by the SMOP $\{F_n^t\}_{n\geqslant 0}$, and its corresponding structure relation (5).

We begin by proving several lemmas which are needed for the proof of Theorem 9.

Lemma 6. *For the SMOP* $\{Q_n^t\}_{n\geqslant 0}$ *and* $\{F_n^t\}_{n\geqslant 0}$ *we have*

$$x[Q_n^t(x)]'=C_1(x,t;n)F_n^t(x)+D_1(x,t;n)F_{n-1}^t(x), \tag{32}$$

where

$$\begin{aligned} C_1(x,t;n)&=-4a_n^2(t)\Big[b_n^t x^2+\big(b_n^t-1\big)\big(a_{n-1}^2(t)+a_n^2(t)-t\big)\Big],\\ D_1(x,t;n)&=4a_n^2(t)\,x\Big(a_{n+1}^2(t)+b_n^t\big[x^2-t+a_n^2(t)\big]\Big)-\frac{1}{x}a_n^2(t)\big(b_n^t-1\big). \end{aligned}$$

The coefficients $A_1(x,t;n)$, $B_1(t;n)$ are given in (31), b_n^t is given in (26), and $a(x,t;n)$, $b(x,t;n)$ come from the structure relation (5).

Proof. Shifting the index in (5) as $n\to n-1$, and using (3) we obtain

$$[F_{n-1}^t(x)]'=\tilde{a}(x,t;n)F_n^t(x)+\tilde{b}(x,t;n)F_{n-1}^t(x), \tag{33}$$

where

$$\begin{aligned} \tilde{a}(x,t;n)&=-4\big[x^2-t+a_{n-1}^2(t)+a_n^2(t)\big],\\ \tilde{b}(x,t;n)&=4x\big[x^2-t+a_n^2(t)\big]. \end{aligned}$$

Next, taking derivatives with respect to the variable x in both sides of (31), we get

$$x^2[Q^t{}_n(x)]'=x^2[F_n^t(x)]'+xB_1(t;n)[F_{n-1}^t(x)]'-B_1(t;n)F_{n-1}^t(x).$$

Substituting (5), (33) and the expression for $A_1(x,t;n)$, $B_1(t;n)$ into the above expression the Lemma follows. □

From Proposition 2, $b_n^t=1$ when n is odd, so the above results can be simplified as

$$x[Q_{2m+1}^t(x)]'=C_1(x,t;2m+1)F_{2m+1}^t(x)+D_1(x,t;2m+1)F_{2m}^t(x),$$

where

$$\begin{aligned} C_1(x,t;2m+1)&=-4a_{2m+1}^2(t)\,x^2,\\ D_1(x,t;2m+1)&=4a_{2m+1}^2(t)\,x\big(x^2-t+a_{2(n+1)}^2(t)+a_{2m+1}^2(t)\big), \end{aligned}$$

which corresponds to (5).

Lemma 7. *The sequences of monic polynomials* $\{Q_n^t\}_{n\geqslant 0}$ *and* $\{F_n^t\}_{n\geqslant 0}$ *are also related by*

$$xQ^t{}_{n-1}(x)=A_2(t;n)F_n^t(x)+B_2(x,t;n)F_{n-1}^t(x), \tag{34}$$

$$x[Q_{n-1}^t(x)]'=C_2(x,t;n)F_n^t(x)+D_2(x,t;n)F_{n-1}^t(x), \tag{35}$$

where

$$A_2(t;n) = 1 - b^t_{n-1},$$
$$B_2(x,t;n) = xb^t_{n-1},$$

and

$$C_2(x,t;n) = (b^t_{n-1} - 1)\left(4xa_n^2(t) + \frac{1}{x}\right) - 4xb^t_{n-1}\left[x^2 - t + a^2_{n-1}(t) + a^2_n(t)\right],$$
$$D_2(x,t;n) = 4a^2_n(t)\left(1 - b^t_{n-1}\right)\left[x^2 - t + a^2_n(t) + a^2_{n+1}(t)\right] + 4x^2 b^t_{n-1}\left[x^2 - t + a^2_n(t)\right].$$

Proof. The proof of (34) and (35) is a straightforward consequence of (3), (5), (31), and Lemma 6. □

By Proposition 2 it is obvious that (34) is exactly the first equation of Theorem 4 if n is even and (35) can be simplified as

$$x[Q^t_{2m-1}(x)]' = C_2(x,t;2m)F^t_{2m}(x) + D_2(x,t;2m)F^t_{2m-1}(x), \tag{36}$$

where

$$C_2(x,t;2m) = -4x\left(x^2 - t + a^2_{2m-1}(t) + a^2_{2m}(t)\right),$$
$$D_2(x,t;2m) = 4x^2\left(x^2 - t + a^2_{2m}(t)\right).$$

An equivalent formulation of (36) is (5) if we substitute $F^t_{2m}(x)$ in the above relation according to (3) when the index n is shifted by $2m-1$.

The following lemma shows the converse of (31)–(34) for the polynomials $F^t_n(x)$ and $F^t_{n-1}(x)$. Indeed, we express these two consecutive polynomials of the SMOP $\{F^t_n\}_{n\geqslant 0}$ in terms of only two consecutive Freud-type orthogonal polynomials of the SMOP $\{Q^t_n\}_{n\geqslant 0}$.

Lemma 8. *For* $t > 0$,

$$xb^t_{n-1}F^t_n(x) = xb^t_{n-1}Q^t_n(x) - a^2_n(t)\left(b^t_n - 1\right)Q^t_{n-1}(x), \tag{37}$$

$$xb^t_{n-1}F^t_{n-1}(x) = \left(b^t_{n-1} - 1\right)Q^t_n(x) + xQ^t_{n-1}(x). \tag{38}$$

Proof. Note that (31) and (34) can be interpreted as a system of two linear equations with two polynomial unknowns, namely $F^t_n(x)$ and $F^t_{n-1}(x)$. Hence from Cramer's rule, we have

$$\Delta(x,t;n) = x^2 b^t_{n-1} + a^2_n(t)\left(b^t_n - 1\right)\left(b^t_{n-1} - 1\right).$$

From Proposition 2 we know that $b^t_n = 1$ when n is odd and $b^t_n > 1$ when n is even. Therefore, the product $\left(b^t_n - 1\right)\left(b^t_{n-1} - 1\right)$ is always zero, and the Lemma easily follows. □

Theorem 9 (Ladder operators)**.** *Let* $\mathfrak{a}_n$ *and* $\mathfrak{a}^\dagger_n$ *be the differential operators*

$$\mathfrak{a}_n = \begin{vmatrix} A_2(t;n) & C_1(x,t;n) \\ B_2(x,t;n) & D_1(x,t;n) \end{vmatrix} + \Delta(x,t;n)\frac{d}{dx},$$
$$\mathfrak{a}^\dagger_n = \begin{vmatrix} A_1(x,t;n) & C_2(x,t;n) \\ B_1(t;n) & D_2(x,t;n) \end{vmatrix} - \Delta(x,t;n)\frac{d}{dx},$$

satisfying

$$\mathfrak{a}_n[Q^t_n(x)] = \begin{vmatrix} A_1(x,t;n) & C_1(x,t;n) \\ B_1(t;n) & D_1(x,t;n) \end{vmatrix} Q^t_{n-1}(x), \tag{39}$$
$$\mathfrak{a}^\dagger_n[Q^t_{n-1}(x)] = \begin{vmatrix} A_2(t;n) & C_2(x,t;n) \\ B_2(x,t;n) & D_2(x,t;n) \end{vmatrix} Q^t_n(x). \tag{40}$$

Let point out that all the above expressions are given only in terms of the coefficients in (31), (3), (32), (34), and (35) . Thus, $\mathfrak{a}_n$ *and* $\mathfrak{a}^\dagger_n$ *are, respectively, lowering and raising operators associated with the Freud-type SMOP* $\{Q^t_n\}_{n\geqslant 0}$.

Proof. The proof of Theorem 9 follows from Lemmas 6–8. Replacing (37) and (38) in (32) and (35) one obtains the ladder equations

$$[Q^t_n(x)]' = \frac{C_1(x,t;n)B_2(x,t;n) - A_2(t;n)D_1(x,t;n)}{\Delta(x,t;n)}Q^t_n(x) + \frac{A_1(x,t;n)D_1(x,t;n) - B_1(t;n)C_1(x,t;n)}{\Delta(x,t;n)}Q^t_{n-1}(x)$$

Table 1
Coefficients of the polynomials $\mathcal{A}(x,t;n)$ and $\mathcal{B}(x,t;n)$ for every integer n.

Polynomial	Power	Coefficient
$\mathcal{A}(x,t;n)$	8	$4a_n^2(t)\, b_n^t\big[b_{n-1}^t\big]^2$
	6	$4a_n^2(t)\, b_{n-1}^t\big\{b_n^t(a_n^2(t)\big[2(1-b_n^t)+b_{n-1}^t(3b_n^t-2)\big]-tb_{n-1}^t)+a_{n+1}^2(t)b_{n-1}^t\big\}$
	4	$a_n^2(t)\big(b_n^t-1\big)\{4b_{n-1}^t a_n^2(t)(b_{n-1}^t[(b_n^t-1)a_{n-1}^2(t)+(1-3b_{n-1}^t)t]+2(b_{n-1}^t-1)a_{n+1}^2(t)$
		$..+2tb_n^t)+4a_n^4(t)[\big(3[b_n^t]^2-1\big)[b_{n-1}^t]^2+2b_n^tb_{n-1}^t(1-2b_n^t)+(b_n^t-1)b_n^t]-[b_{n-1}^t]^2\}$
	2	$2a_n^4(t)\big(b_{n-1}^t-1\big)\big(b_n^t-1\big)^2\{2a_n^2(t)[b_n^t(t-b_n^ta_n^2(t))+b_{n-1}^t(2(b_n^t-1)a_{n-1}^2(t)$
		$...+a_n^2(t)\big[b_n^t(b_n^t+2)-2\big]+t(2-3b_n^t))+a_{n+1}^2(t)\big(b_{n-1}^t-1\big)]-b_{n-1}^t\}$
	0	$-a_n^6(t)\big(b_{n-1}^t-1\big)^2\big(b_n^t-1\big)^3\{1-4a_n^2(t)\big[b_n^t-1\big][a_{n-1}^2(t)+a_n^2(t)-t]\}$
$\mathcal{B}(x,t;n)$	11	$-16a_n^2(t)\, b_n^t[b_{n-1}^t]^2$
	9	$-16a_n^2(t)\, b_{n-1}^t\big\{b_n^t\big(a_n^2(t)\big[b_{n-1}^t(3b_n^t-2)+2(1-b_n^t)\big]-2tb_{n-1}^t\big)+b_{n-1}^ta_{n+1}^2(t)\big\}$
	7	$-4a_n^2(t)\{[b_{n-1}^t]^2[1--4ta_{n+1}^2(t)+4t^2b_n^t+b_n^t]+4b_{n-1}^ta_n^2(t)(b_{n-1}^t[(b_n^t-1)^2a_{n-1}^2(t)$
		$..+t(6(1-b_n^t)b_n^t-1)]+2(b_{n-1}^t-1)(b_n^t-1)a_{n+1}^2(t)+4t(b_n^t-1)b_n^t)$
		$.+4(b_n^t-1)a_n^4(t)[(3[b_n^t]^2-1)[b_{n-1}^t]^2+2b_n^t(1--2b_n^t)b_{n-1}^t+(b_n^t-1)b_n^t]\}$
	5	$-4(b_n^t-1)a_n^2(t)\{2b_{n-1}^ta_n^2(t)(b_{n-1}^t(-2t(b_n^t-1)a_{n-1}^2(t)+(6t^2+1)b_n^t-2t^2+1)$
		$.-4ta_{n+1}^2(t)(b_{n-1}^t-1)-(4t^2+1)b_n^t-1)+4a_n^4(t)(2b_{n-1}^t(t(1-2b_n^t)^2$
		$.-a_{n-1}^2(t)(b_n^t-1)^2)-[b_{n-1}^t]^2(-2a_{n-1}^2(t)(b_n^t-1)^2+tb_n^t(6b_n^t-5)+t)$
		$.+a_{n+1}^2(t)(b_n^t-1)(b_{n-1}^t-1)^2-2t(b_n^t-1)b_n^t)+4a_n^6(t)(b_{n-1}^t-1)(b_n^t-1)$
		$.(b_{n-1}^t(b_n^t(b_n^t+2)-2)-[b_n^t]^2)+t[b_{n-1}^t]^2\}$
	3	$-2a_n^2(t)\big(b_n^t-1\big)\{2a_n^4(t)[b_n^t-1](-2b_{n-1}^t(-4ta_{n-1}^2(t)[b_n^t-1]$
		$.+8t^2b_n^t+b_n^t-4t^2+1)+[b_{n-1}^t]^2(-8ta_{n-1}^2(t)[b_n^t-1]+12t^2b_n^t+b_n^t-8t^2-1)$
		$.-4ta_{n+1}^2(t)[b_{n-1}^t-1]^2+4t^2b_n^t+b_n^t+1)-8a_n^6(t)[b_{n-1}^t-1][b_n^t-1]$
		$(-[b_{n-1}^t-1](a_{n-1}^2(t)[b_n^t-1]^2-2t[b_n^t]^2)-t[b_{n-1}^t-2b_n^t+1])+4a_n^2(t)\, b_{n-1}^t[b_n^t-1]$
		$.(b_{n-1}^t[2t-a_{n-1}^2(t)]-t)+8a_n^8(t)[b_{n-1}^t-1]^2[b_n^t-1]^3+[b_{n-1}^t]^2\}$
	1	$-4a_n^4(t)\big(b_{n-1}^t-1\big)\big(b_n^t-1\big)^2(1-4a_n^2(t)[b_n^t-1][a_{n-1}^2(t)+a_n^2(t)-t])$
		$(ta_n^2(t)[b_{n-1}^t-1][b_n^t-1]+b_{n-1}^t)$

and

$$
\begin{aligned}
[Q_{n-1}^t(x)]' &= \frac{C_2(x,t;n)B_2(x,t;n)-A_2(t;n)D_2(x,t;n)}{\Delta(x,t;n)}Q_n^t(x)\\
&\quad+\frac{A_1(x,t;n)D_2(x,t;n)-B_1(t;n)C_2(x,t;n)}{\Delta(x,t;n)}Q_{n-1}^t(x),
\end{aligned}
$$

which are equivalent to (39) and (40). This completes the proof of Theorem 9. □

Theorem 10 (Holonomic equation)**.** *The Freud-type polynomial $Q_n^t(x)$ satisfies the holonomic equation (second order linear differential equation)*

$$
\mathcal{A}(x,t;n)[Q_n^t(x)]''+\mathcal{B}(x,t;n)[Q_n^t(x)]'+\mathcal{C}(x,t;n)Q_n^t(x)=0, \tag{41}
$$

where

$$
\begin{aligned}
\mathcal{A}(x,t;n) &= \Psi_{1,1}(x,t;n)\Delta^2(x,t;n),\\
\mathcal{B}(x,t;n) &= \Delta(x,t;n)\big(W\{\Psi_{1,1}(x,t;n),\Delta(x,t;n)\}+\Psi_{1,1}(x,t;n)\big[\Psi_{2,1}(x,t;n)-\Psi_{1,2}(x,t;n)\big]\big),\\
\mathcal{C}(x,t;n) &= \Delta(x,t;n)W\{\Psi_{1,1}(x,t;n),\Psi_{2,1}(x,t;n)\}+\Psi_{1,1}(x,t;n)\begin{vmatrix}\Psi_{1,1}(x,t;n) & \Psi_{1,2}(x,t;n)\\ \Psi_{2,1}(x,t;n) & \Psi_{2,2}(x,t;n)\end{vmatrix},
\end{aligned}
$$

with

$$
\Psi_{i,j}(x,t;n)=\begin{vmatrix}A_i(x,t;n) & C_j(x,t;n)\\ B_i(x,t;n) & D_j(x,t;n)\end{vmatrix},\qquad i,j=1,2.
$$

Moreover, the polynomials $\mathcal{A}(x,t;n)$ and $\mathcal{C}(x,t;n)$ are even functions and the polynomial $\mathcal{B}(x,t;n)$ is an odd function, whose coefficients are showed in Table 1.

Proof. The proof of Theorem 10 comes directly from the ladder operators given in Theorem 9. The usual technique consists in applying the raising operator to both sides of the equation satisfied by the lowering operator, i.e.

$$
\mathfrak{a}_n^\dagger\big[Q_{n-1}^t(x)\big]=\mathfrak{a}_n^\dagger\left[\frac{1}{\Psi_{1,1}(x,t;n)}\mathfrak{a}_n[Q_n^t(x)]\right]=\Psi_{2,2}(x,t;n)Q_n^t(x).
$$

Thus,

$$
\frac{\Psi_{1,2}(x,t;n)}{\Psi_{1,1}(x,t;n)}\mathfrak{a}_n[Q_n^t(x)]-\Delta(x,t;n)\frac{d}{dx}\left(\frac{1}{\Psi_{1,1}(x,t;n)}\mathfrak{a}_n[Q_n^t(x)]\right)=\Psi_{2,2}(x,t;n)Q_n^t(x)
$$

Table 2
Coefficients of the polynomials $\mathcal{A}(x,t;2n)$ and $\mathcal{B}(x,t;2n)$ for every integer n.

Polynomial	Power	Coefficient
$\mathcal{A}(x,t;2n)$	8	$4a_{2n}^2(t)\,b_{2n}^t$
	6	$4a_{2n}^2(t)\left(a_{2n+1}^2(t)+b_{2n}^t\left[a_{2n}^2(t)\,b_{2n}^t-t\right]\right)$
	4	$-a_{2n}^2(t)\left(1-4a_{2n}^2(t)\left[a_{2n}^2(t)+a_{2n-1}^2(t)-t\right]\left[b_{2n}^t-1\right]\right)\left(b_{2n}^t-1\right)$
	2	0
	0	0
$\mathcal{B}(x,t;2n)$	11	$-16a_{2n}^2(t)\,b_{2n}^t$
	9	$-16a_{2n}^2(t)\left(a_{2n+1}^2(t)+b_{2n}^t\left[a_{2n}^2(t)\,b_{2n}^t-2t\right]\right)$
	7	$-4a_{2n}^2(t)\left\{1-4ta_{2n+1}^2(t)+4a_{2n}^4(t)[b_{2n}^t-1]^2+b_{2n}^t+4t^2b_{2n}^t \right.$ $\left. +4a_{2n}^2(t)\left(a_{2n-1}^2(t)\left(b_{2n}^t-1\right)^2-t\left[1+2\left(b_{2n}^t-1\right)b_{2n}^t\right]\right)\right\}$
	5	$-4ta_{2n}^2(t)\left(1-4a_{2n}^2(t)\left[a_{2n}^2(t)+a_{2n-1}^2(t)-t\right]\left[b_{2n}^t-1\right]\right)\left(b_{2n}^t-1\right)$
	3	$-2a_{2n}^2(t)\left(1-4a_{2n}^2(t)\left[a_{2n}^2(t)+a_{2n-1}^2(t)-t\right]\left[b_{2n}^t-1\right]\right)\left(b_{2n}^t-1\right)$
	1	0

which becomes

$$\frac{\Psi_{1,2}(x,t;n)\Psi_{2,1}(x,t;n)}{\Psi_{1,1}(x,t;n)}[Q_n^t(x)]+\frac{\Psi_{1,2}(x,t;n)\Delta(x,t;n)}{\Psi_{1,1}(x,t;n)}[Q_n^t(x)]'$$
$$-\Delta(x,t;n)\frac{d}{dx}\left(\frac{\Psi_{2,1}(x,t;n)}{\Psi_{1,1}(x,t;n)}[Q_n^t(x)]+\frac{\Delta(x,t;n)}{\Psi_{1,1}(x,t;n)}[Q_n^t(x)]'\right)=\Psi_{2,2}(x,t;n)Q_n^t(x).$$

The intermediate computations above yield

$$\frac{d}{dx}\left(\frac{\Psi_{2,1}(x,t;n)}{\Psi_{1,1}(x,t;n)}[Q_n^t(x)]\right)=\begin{vmatrix}\Psi_{1,1}(x,t;n) & \Psi_{2,1}(x,t;n)\\ \Psi_{1,1}'(x,t;n) & \Psi_{2,1}'(x,t;n)\end{vmatrix}\frac{Q_n^t(x)}{\Psi_{1,1}^2(x,t;n)}+\frac{\Psi_{2,1}(x,t;n)}{\Psi_{1,1}(x,t;n)}[Q_n^t(x)]',$$
$$\frac{d}{dx}\left(\frac{\Delta(x,t;n)}{\Psi_{1,1}(x,t;n)}[Q_n^t(x)]'\right)=\begin{vmatrix}\Psi_{1,1}(x,t;n) & \Delta(x,t;n)\\ \Psi_{1,1}'(x,t;n) & \Delta'(x,t;n)\end{vmatrix}\frac{[Q_n^t(x)]'}{\Psi_{1,1}^2(x,t;n)}+\frac{\Delta(x,t;n)}{\Psi_{1,1}(x,t;n)}[Q_n^t(x)]'',$$

and

$$\frac{\Psi_{1,2}(x,t;n)\Psi_{2,1}(x,t;n)}{\Psi_{1,1}(x,t;n)}[Q_n^t(x)]+\frac{\Psi_{1,2}(x,t;n)\Delta(x,t;n)}{\Psi_{1,1}(x,t;n)}[Q_n^t(x)]'$$
$$-\frac{\Delta(x,t;n)}{\Psi_{1,1}^2(x,t;n)}\begin{vmatrix}\Psi_{1,1}(x,t;n) & \Psi_{2,1}(x,t;n)\\ \Psi_{1,1}'(x,t;n) & \Psi_{2,1}'(x,t;n)\end{vmatrix}[Q_n^t(x)]-\frac{\Delta(x,t;n)\Psi_{2,1}(x,t;n)}{\Psi_{1,1}(x,t;n)}[Q_n^t(x)]'$$
$$-\frac{\Delta(x,t;n)}{\Psi_{1,1}^2(x,t;n)}\begin{vmatrix}\Psi_{1,1}(x,t;n) & \Delta(x,t;n)\\ \Psi_{1,1}'(x,t;n) & \Delta'(x,t;n)\end{vmatrix}[Q_n^t(x)]'-\frac{\Delta^2(x,t;n)}{\Psi_{1,1}(x,t;n)}[Q_n^t(x)]''$$
$$=\Psi_{2,2}(x,t;n)Q_n^t(x).$$

Combining all the above expressions, and after some cumbersome computations, Theorem 10 follows. □

Corollary 1. *For every nonnegative integer n, when n is odd (41) is equivalent to $\mathcal{A}(x,t;n)$ times (6) and, in the other case the polynomial coefficients of (41) are contained in Table 2.*

Proof. The procedure is to observe that in the case where the degree is odd, (41) is reduced to the second order linear differential equation (6), which is satisfied by $F_{2m+1}^t(x)$ since

$$\mathcal{B}(x,t;2m+1)=\mathcal{A}(x,t;2m+1)\,R_{2m+1}^t(x)\quad\text{and}\quad\mathcal{C}(x,t;2m+1)=\mathcal{A}(x,t;2m+1)\,S_{2m+1}^t(x).$$

On the other hand, if the degree is even then we get $b_{2m-1}^t=1$. □

For a deeper discussion of ladder operators we refer the reader to [6,15]. We next provide the second order linear differential equation satisfied by the SMOP $\{Q_n^t\}_{n\geqslant 0}$ taking into account the measure μ_t is semi-classical. This is the main tool for the further electrostatic interpretation of zeros. Once we have the second order linear differential equation satisfied by the SMOP $\{Q_n^t\}_{n\geqslant 0}$ it is easy to obtain an electrostatic model for their zeros. We will study the asymptotic behavior of the position of the movable constant charges involved in the external field. As we have shown in Section 3 Freud-type orthogonal polynomials and Freud polynomials of odd degree coincide. Thus, in this Section we shall derive the electrostatic model for the zeros in the case when n is even (see [29]).

Let us evaluate (41) at the zeros $\{y_{2m,i}(t)\}_{i=1}^{2m}$ of the polynomial $Q_{2m}^t(x)$, yielding

$$\frac{[Q_{2m}^t\big(y_{2m,i}(t)\big)]''}{[Q_{2m}^t\big(y_{2m,i}(t)\big)]'} = -\frac{\mathcal{B}\big(y_{2m,i}(t),t;2m\big)}{\mathcal{A}\big(y_{2m,i}(t),t;2m\big)}$$
$$= \frac{16b_{2m}^t y_{2m,i}^3(t) + 8y_{2m,i}(t)\Big[\big(a_{2m}^2(t)\,b_{2m}^t - t\big)b_{2m}^t + a_{2m+1}^2(t)\Big]}{4b_{2m}^t y_{2m,i}^4(t) + 4y_{2m,i}^2(t)\Big[\big(a_{2m}^2(t)\,b_{2m}^t - t\big)b_{2m}^t + a_{2m+1}^2(t)\Big] + h_{2m}^t}$$
$$-\frac{2}{y_{2m,i}(t)} + 4\big(y_{2m,i}^3(t) - t\,y_{2m,i}(t)\big),$$

where

$$h_{2m}^t = \big(4a_{2m}^2(t)\big[a_{2m}^2(t) + a_{2m-1}^2(t) - t\big]\big[b_{2m}^t - 1\big] - 1\big)\big(b_{2m}^t - 1\big).$$

The above equation reads as the electrostatic equilibrium condition for $\{y_{2m,i}(t)\}_{i=1}^{2m}$. Having

$$u(x,t;2m) = 4b_{2m}^t x^4 + 4x^2\big([a_{2m}^2(t)\,b_{2m}^t - t]b_{2m}^t + a_{2m+1}^2(t)\big) + h_{2m}^t \tag{42}$$

the previous condition can be rewritten as

$$\sum_{j=1,\,j\neq i}^{2m}\frac{1}{y_{2m,j}(t) - y_{2m,i}(t)} + \frac{1}{2}\frac{[u]'\big(y_{2m,i}(t),t;2m\big)}{u\big(y_{2m,i}(t),t;2m\big)} - \frac{1}{y_{2m,i}(t)} + 2\big(y_{2m,i}^3(t) - ty_{2m,i}(t)\big) = 0,$$

which means that the set of zeros $\{y_{2m,i}(t)\}_{i=1}^{2m}$ are the critical points (zeros of the gradient) of the total energy. Hence, the electrostatic interpretation of the distribution of zeros means that we have an equilibrium position under the presence of an external potential

$$V^{ext}(x) = \frac{1}{2}\ln u(x,t;2m) - \frac{1}{2}\ln x^2 e^{-x^4+2tx^2},$$

where the first term represents a *short range potential* corresponding to a unit charge located at the real zeros of the quartic polynomial $u(x,\,t;\,2m)$, and the second one is a *long range potential* associated with the Freud weight function.

5. Zeros of Freud-type SMOP

Having in mind the techniques shown in [16], we will first study the motion of zeros of time depending of the polynomial $F_n^t(x)$ (Theorem 11) and, finally, in Subsection 5.2 we give the differential equation that the zeros of the polynomial $Q_n^t(x)$ satisfy.

5.1. Equations of motion for zeros of $\{F_n^t(x)\}_{n\geqslant 0}$

Notice that the three term recurrence relation (3) implies that

$$x^2\,F_n^t(x) = F_{n+2}^t(x) + \big[a_{n+1}^2(t) + a_n^2(t)\big]F_n^t(x) + a_n^2(t)\,a_{n-1}^2(t)\,F_{n-2}^t(x) \tag{43}$$

for all $n \geq 1$ and $F_{-1}^t(x) = 0$. The requirement on $\frac{\partial F_n^t}{\partial t}(x)$ is that

$$\frac{\partial F_n^t}{\partial t}(x) = \sum_{i=0}^{n-2}\flat_{n,i}\,F_i^t(x),$$

with $\flat_{n,i} = \frac{\langle\frac{\partial F_n^t}{\partial t},F_i^t\rangle_t}{\|F_i^t\|_t^2}$. From

$$0 = \int_{-\infty}^{\infty}F_n^t(x)\,F_i^t(x)\,\omega_t(x)dx, \quad 0\leqslant i\leqslant n-2,$$

differentiating the above expression with respect to t, we obtain

$$0 = \int_{-\infty}^{\infty}\left[F_i^t(x)\,\frac{\partial F_n^t}{\partial t}(x) + F_n^t(x)\,\frac{\partial F_i^t}{\partial t}(x)\right]\omega_t(x)dx + 2\int_{-\infty}^{\infty}x^2\,F_n^t(x)\,F_i^t(x)\,\omega_t(x)dx.$$

Hence, since $\frac{\partial F_n^t}{\partial t}(x)$ is a linear combination of the first n elements of the sequence $\{F_k^t\}_{k\geqslant 0}$, using (43) we get

$$\left\langle\frac{\partial F_n^t}{\partial t},F_i^t\right\rangle_t = -2a_n^2(t)\,a_{n-1}^2(t)\,\left\|F_{n-2}^t\right\|_t^2\,\delta_{n-2,i}, \quad 0\leqslant i\leqslant n-2,$$

and

$$\frac{\partial F_n^t}{\partial t}(x) = -2a_n^2(t)\, a_{n-1}^2(t)\, F_{n-2}^t(x) \tag{44}$$

as claimed for $t > 0$ and $n \geq 1$ with $F_{-1}^t(x) = 0$.

We can now formulate this result

Theorem 11 (Equation of motion for zeros of $F_n^t(x)$)**.** *Let n be a positive integer and $t > 0$. If $x_{n,1}(t), \ldots, x_{n,n}(t)$ are the n zeros of $F_n^t(x)$, then*

$$\frac{\partial x_{n,k}(t)}{\partial t} = \frac{x_{n,k}(t)}{2\left(x_{n,k}^2(t) - t + a_n^2(t) + a_{n+1}^2(t)\right)}.$$

Proof. Given $n \geq 1$ and $t > 0$, we have

$$F_n^t\left(x_{n,k}(t)\right) = 0, \quad 1 \leqslant k \leqslant n. \tag{45}$$

If we differentiate the above equation with respect to time t, we get

$$\frac{\partial x_{n,k}(t)}{\partial t} \frac{\partial F_n^t}{\partial x}(x)\Big|_{x=x_{n,k}(t)} + \frac{\partial F_n^t}{\partial t}\left(x_{n,k}(t)\right) = 0. \tag{46}$$

On the other hand, from (5) and (45), we obtain

$$\frac{\partial F_n^t}{\partial x}(x)\Big|_{x=x_{n,k}(t)} = b\left(x_{n,k}(t), t; n\right) F_{n-1}^t\left(x_{n,k}(t)\right).$$

Hence, using (44), (46) and the above formula, it follows that

$$\frac{\partial x_{n,k}(t)}{\partial t} = \frac{a_{n-1}^2(t)\, F_{n-2}^t\left(x_{n,k}(t)\right)}{2\left(x_{n,k}^2(t) - t + a_n^2(t) + a_{n+1}^2(t)\right) F_{n-1}^t\left(x_{n,k}(t)\right)}. \tag{47}$$

Finally, we obtain the result since the three term recurrence relation (3) yields

$$F_{n-2}^t\left(x_{n,k}(t)\right) = \frac{x_{n,k}(t)}{a_{n-1}^2(t)} F_{n-1}^t\left(x_{n,k}(t)\right).$$

□

5.2. Equations of motion for zeros of Freud-type orthogonal polynomials

It is required that

$$\frac{\partial Q_n^t}{\partial t}(y) = \sum_{i=0}^{n-2} \widetilde{\flat}_{n,i}\, Q_i^t(y),$$

with $\widetilde{\flat}_{n,i} = \frac{\langle \frac{\partial Q_n^t}{\partial t}, Q_i^t \rangle}{\|Q_i^t\|^2}$. We first compute the coefficients $\widetilde{\flat}_{n,i}$, $0 \leqslant i \leqslant n-2$. From the orthogonality relations

$$0 = \int_{-\infty}^{\infty} Q_n^t(y)\, Q_i^t(y)\, \omega_t(y) dy + M Q_n^t(0) Q_i^t(0), \quad 0 \leqslant i \leqslant n-2.$$

Taking the partial derivative with respect to the variable t in the above equation we obtain, for $0 \leqslant i \leqslant n-1$,

$$\left\langle \frac{\partial Q_n^t}{\partial t}(y), Q_i^t(y) \right\rangle = -2\left\langle Q_n^t(y), y^2 Q_i^t(y)\right\rangle_t.$$

Applying (8) and (23) in the above equality we get

$$\widetilde{\flat}_{n,i} = 0, \quad 0 \leqslant i \leqslant n-3,$$

since

$$\left\langle Q_n^t(y), y^2 Q_i^t(y)\right\rangle_t = \left\langle F_n^t(y), y^2 Q_i^t(y)\right\rangle_t - M Q_n^t(0) \left\langle K_{n-1}(x, 0; t), y^2 Q_i^t(y)\right\rangle_t.$$

Then, we get

$$\frac{\partial Q_n^t}{\partial t}(y) = \widetilde{\flat}_{n,n-2}\, Q_{n-2}^t(y),$$

where

$$\widetilde{\mathfrak{b}}_{n,n-2} = -\frac{2}{b^t_{n-2}}\left(a^2_n(t)a^2_{n-1}(t) + \frac{M[F^t_n(0)]^2}{\left(1+MK_{n-1}(0,0;t)\right)\left\|F^t_{n-2}\right\|^2_t}\right).$$

Let us denote by $y_{n,1}(t), \ldots, y_{n,n}(t)$ the n zeros of polynomial $Q^t_n(y)$. From (29) it follows that

$$\frac{\partial Q^t_n}{\partial t}\left(y_{n,k}(t)\right) = \widetilde{\mathfrak{b}}_{n,n-2}\frac{y_{n,k}(t)\, b^t_{n-2}}{b^t_{n-1}\, a^2_{n-1}(t)} Q^t_{n-1}\left(y_{n,k}(t)\right). \tag{48}$$

We can now state the analogue of Theorem 11 for the zeros of Freud-type polynomials.

Theorem 12 (Equation of motion for zeros of $Q^t_n(y)$)**.** *Let n be a positive integer and $t > 0$. If $y_{n,1}(t), \ldots, y_{n,n}(t)$ are the n zeros of $Q^t_n(y)$, then*

$$\frac{\partial y_{n,k}}{\partial t}(t) = \frac{-y^2_{n,k}(t)\,\widetilde{\mathfrak{b}}_{n,n-2}\frac{b^t_{n-2}}{b^t_{n-1}\,a^2_{n-1}(t)} Q^t_{n-1}\left(y_{n,k}(t)\right)}{C_{k,1}(n,t)F^t_n\left(y_{n,k}(t)\right) + C_{k,2}(n,t)F^t_{n-1}\left(y_{n,k}(t)\right)},$$

where

$$C_{k,1}(n,t) = 1 - 4a^2_n(t)\left[b^t_n y^2_{n,k}(t) + \left(b^t_n - 1\right)\left(a^2_n(t) + a^2_{n-1}(t) - t\right)\right],$$
$$C_{k,2}(n,t) = 4a^2_n(t)\, y_{n,k}(t)\left[b^t_n\left(y^2_{n,k}(t) - t + a^2_n(t)\right) + a^2_{n+1}(t)\right].$$

Proof. Given $n \geq 1$ and $t > 0$, we have

$$Q^t_n\left(y_{n,k}(t)\right) = 0, \quad 1 \leqslant k \leqslant n. \tag{49}$$

If we differentiate the above equation with respect to t, we get

$$\frac{\partial y_{n,k}}{\partial t}(t)\,\frac{\partial Q^t_n}{\partial y}(y)\Big|_{y=y_{n,k}(t)} + \frac{\partial Q^t_n}{\partial t}\left(y_{n,k}(t)\right) = 0. \tag{50}$$

We only need to compute the y derivative of $Q^t_n(y)$ and to combine (48) with the above expression in order to get the result. Differentiating (31) and applying (3), (5), and (49), we have

$$y_{n,k}(t)\,\frac{\partial Q^t_n}{\partial y}(y)\Big|_{y=y_{n,k}(t)} = C_{k,1}(n,t)F^t_n\left(y_{n,k}(t)\right) + C_{k,2}(n,t)F^t_{n-1}\left(y_{n,k}(t)\right).$$

Hence, combining (48) and the above formula with (50) the Theorem follows. □

In the case when n is odd, the above Theorem provides (47), which establishes the formula of Theorem 11.

5.3. Behavior and monotonicity with M of the zeros of $Q^t_{2m}(x)$

Let assume that $y_{n,\,k}$, $k = 1, 2, \ldots, n$, are the zeros of $Q^t_n(x)$ arranged in an increasing order. From the analysis done before, it is clear that the zeros $y_{n,\,s}$ when n is odd are not affected by the mass M. Next, we analyze the behavior of zeros $y_{2m,s} = y_{2m,s}(M)$, $s = 1, \ldots, 2m$, as a function of the mass M and we obtain such a behavior when the positive real number M goes from zero to infinity. In order to do that, we use a technique developed in [3, Lemma 1] and [11, Lemmas 1 and 2] concerning the behavior and the asymptotics of the zeros of linear combinations of two n-th degree polynomials $h_n, g_n \in \mathbb{P}$ with interlacing zeros, such that $f(x) = h_n(x) + cg_n(x)$, $c \geq 0$. From now on, we will refer to this technique as the *Interlacing Lemma*. Here the linear combination of two polynomials of the same degree $2m$ is given by (27), and F^t_{2m}, G_{2m} play the role of $h_n(x)$, $g_n(x)$ respectively.

In order to apply this technique, we need to show that the hypotheses of the Interlacing Lemma are fulfilled. First, in Theorem 3 the interlacing of the zeros of F^t_{2m} and G_{2m} was proved. In our computations, we will only deal with the zero behavior in the positive real semi-axis, because the behavior in $\mathbb{R}_-$ follows by reflection through the y-axis by symmetry reasons as usual. Thus, from (27), the positivity of $K_{2m-1}(0,0;t)$, and Theorem 3 we are in the hypothesis of the Interlacing Lemma, and we immediately conclude the following results about monotonicity, asymptotics, and speed of convergence for the zeros of $Q^t_{2m}(x)$ in terms of the mass M.

Let us define the monic polynomials

$$G^l_m(x) = x(x-g_{2m,1})(x-g_{2m,2})\cdots(x-g_{2m,m-1})$$
$$= x(x-x^{[2]}_{2m-1,1})(x-x^{[2]}_{2m-1,2})\cdots(x-x^{[2]}_{2m-1,m-1})$$

and

$$G^r_m(x) = x(x-g_{2m,m+2})(x-g_{2m,m+3})\cdots(x-g_{2m,2m})$$
$$= x(x-x^{[2]}_{2m-1,m+1})(x-x^{[2]}_{2m-1,m+2})\cdots(x-x^{[2]}_{2m-1,2m-1}),$$

such that $G^l_{2m}(x) = G^l_m(x)G^r_m(x)$.

Table 3
Zeros of $Q_4^{0.5}(x)$ and $u(x, 0.5; 4)$ for some values of M.

	$t = 0.5$						
	$M = 0$	$M = 0.002$	$M = 0.05$	$M = 0.5$	$M = 5$	$M = 10$	$M = 50$
$Q_4^t(x)$							
	±1.1640	±1.1639	±1.1623	±1.1516	±1.1318	±1.1286	±1.1257
	±0.4839	±0.4836	±0.4755	±0.4154	±0.2256	±0.1689	±0.0794
$u(x, t; 4)$							
	$\pm 0.94861i$	$\pm 0.94869i$	$\pm 0.9505i$	$\pm 0.9618i$	$\pm 0.9827i$	$\pm 0.9863i$	$\pm 0.9898i$
	0	±0.0144	±0.0689	±0.1528	±0.1219	±0.0942	±0.0455

Table 4
Zeros of $Q_4^1(x)$ and $u(x, 1; 4)$ for some values of M.

	$t = 1$						
	$M = 0$	$M = 0.002$	$M = 0.05$	$M = 0.5$	$M = 5$	$M = 10$	$M = 50$
$Q_4^t(x)$							
	±1.3002	±1.3001	±1.2988	±1.2891	±1.2659	±1.2615	±1.2570
	±0.6156	±0.6153	±0.6084	±0.5533	±0.3335	±0.2551	±0.1227
$u(x, t; 4)$							
	$\pm 0.7653i$	$\pm 0.7654i$	$\pm 0.7687i$	$\pm 0.7894i$	$\pm 0.8354i$	$\pm 0.8454i$	$\pm 0.8563i$
	0	±0.0171	±0.0827	±0.1962	±0.1800	±0.1424	±0.0703

Theorem 13. *In the negative real semiaxis, the following interlacing property holds*

$$x_{2m,1} < y_{2m,1} < g_{2m,1} < x_{2m,2} < y_{2m,2} < \cdots < g_{2m,m-1} < x_{2m,m} < y_{2m,m} < g_{2m,m} = 0.$$

Moreover, each $y_{n,l} = y_{n,l}(M)$ is an increasing function of M and, for each $l = 1, \ldots, m$,

$$\lim_{M\to\infty} y_{2m,l}(M) = g_{2m,l}\,,$$

as well as

$$\lim_{M\to\infty} M[g_{2m,l} - y_{2m,l}] = \frac{F_{2m}^t(g_{2m,l})}{K_{2m-1}(0, 0; t)[G_m^l]'(g_{2m,l})}.$$

Applying symmetry properties through y-axis, in the positive real semiaxis the following interlacing property holds

$$0 = g_{2m,m+1} < y_{2m,m+1} < x_{2m,m+1} < g_{2m,m+2} < \cdots < x_{2m,2m-1} < g_{2m,2m} < y_{2m,2m} < x_{2m,2m}.$$

Moreover, each $y_{n,r} = y_{n,r}(M)$ is a decreasing function of M and, for each $r = m + 1, \ldots, 2m$,

$$\lim_{M\to\infty} y_{2m,r} = g_{2m,r}\,,$$

as well as

$$\lim_{M\to\infty} M[y_{2m,r} - g_{2m,r}] = \frac{-F_{2m}^t(g_{2m,r})}{K_{2m-1}(0, 0; t)[G_m^r]'(g_{2m,r})}.$$

Notice that the mass point at $x = 0$ attracts two zeros of $Q_{2m}^t(x)$, i.e. when $M \to \infty$, it captures $y_{2m,\,m}$ and $y_{2m,m+1}$ at the same time.

6. Numerical experiments

We next provide some numerical experiments using Mathematica ® software, dealing with the zeros of Freud-type polynomials $\{Q_n^t\}_{n\geqslant 0}$. More specifically, we will show the position of the two symmetric and closest-to-the-origin zeros of some even polynomials of the sub-sequence $\{Q_{2k}^t\}_{k\geqslant 0}$. We choose $Q_4^t(x)$ for the following first experiment varying t. We show the location of its zeros for several values of t and M, and we also show the position of the source-charges of the short range potential

$$\upsilon_{short}(x) = \frac{1}{2}\ln u(x, t; 2m), \quad m = 1, 2, 3, \ldots,$$

which are the zeros of the polynomial $u(x, t; 4)$ defined in (42). In Tables 3–6 we provide numerical evidence of the position of its zeros when t is equal to 1/2, 1, 3/2, and 2 respectively, for several choices of M. Notice that the polynomial $u(x, t; 4)$ has exactly degree four, and its zeros are always two real and two simple conjugate complex numbers. We also remark that we recover the results in [14, Cor. 3.5] when $t = 0$.

Table 5
Zeros of $Q_4^{1.5}(x)$ and $u(x, 1.5; 4)$ for some values of M.

	$t = 1.5$						
	$M = 0$	$M = 0.002$	$M = 0.05$	$M = 0.5$	$M = 5$	$M = 10$	$M = 50$
$Q_4^t(x)$							
	± 1.4485	± 1.4484	± 1.4474	± 1.4395	± 1.4120	± 1.4047	± 1.3964
	± 0.8059	± 0.8057	± 0.8010	± 0.7603	± 0.5363	± 0.4290	± 0.2176
$u(x, t; 4)$							
	$\pm 0.5175i$	$\pm 0.5179i$	$\pm 0.5263i$	$\pm 0.5714i$	$\pm 0.663i$	$\pm 0.6899i$	$\pm 0.7282i$
	0	± 0.0233	± 0.1125	± 0.2770	± 0.3021	± 0.2477	± 0.1261

Table 6
Zeros of $Q_4^2(x)$ and $u(x, 2; 4)$ for some values of M.

	$t = 2$						
	$M = 0$	$M = 0.002$	$M = 0.05$	$M = 0.5$	$M = 5$	$M = 10$	$M = 50$
$Q_4^t(x)$							
	± 1.60437	± 1.6043	± 1.6038	± 1.5989	± 1.5717	± 1.5594	± 1.5406
	± 1.0429	± 1.0428	± 1.0408	± 1.0220	± 0.8736	± 0.7644	± 0.4490
$u(x, t; 4)$							
	0	$\pm 0.0605i$	$\pm 0.1906i$	$\pm 0.3316i$	$\pm 0.4564i$	$\pm 0.4885i$	$\pm 0.5748i$
	± 0.1487	± 0.1613	± 0.2539	± 0.4256	± 0.5625	± 0.5117	± 0.2832

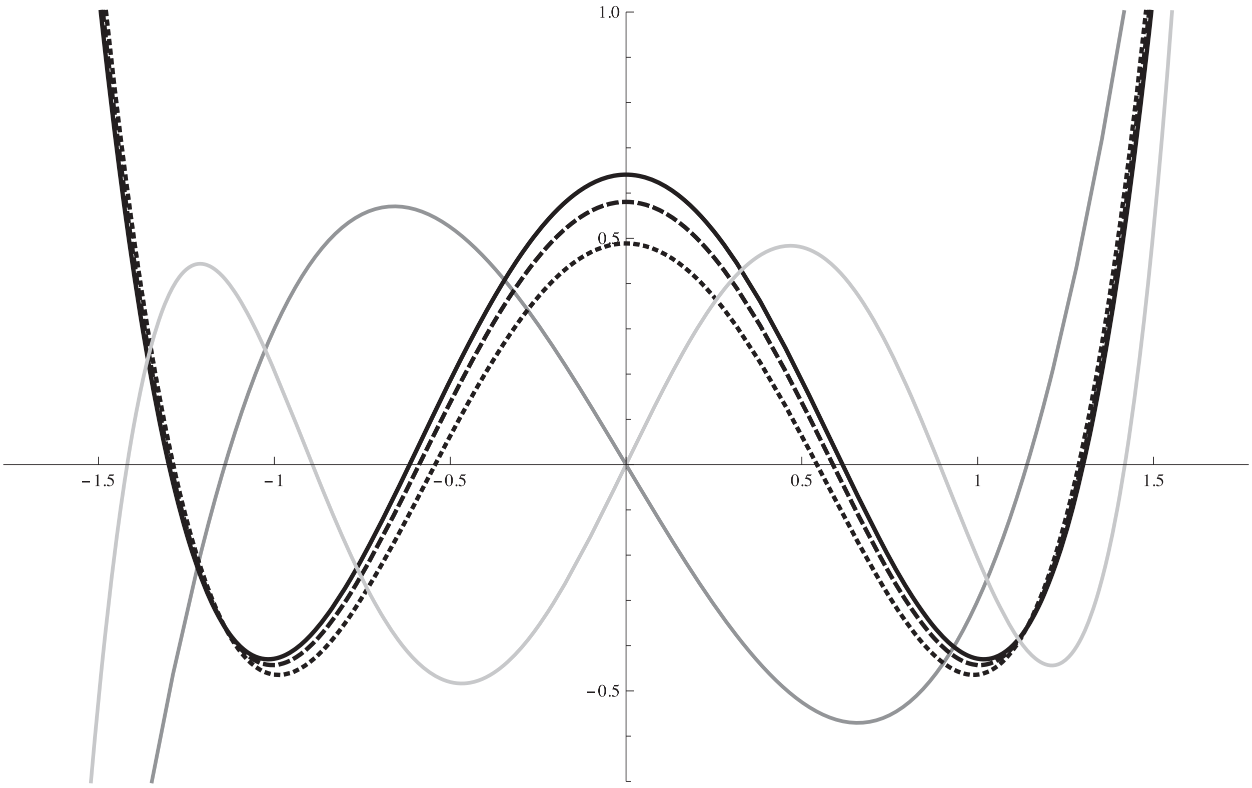

Fig. 1. The graphs of $Q_3^1(x)$ and $Q_5^1(x)$ (gray) and $Q_4^1(x)$ for some values of M (black lines).

Fig. 1 illustrates the change in the even Freud-type polynomials when M varies as described in Theorem 13. We enclose the graphs of $Q_4^1(x)$ for three different values of M. The black continuous, dashed, and dotted lines correspond to $M = 0$, $M = 0.2$, and $M = 0.6$, respectively. We also include, with different tones of gray color, the graphs of $Q_3^1(x)$ and $Q_5^1(x)$, showing that the odd degree polynomials are not affected by the variation of the mass M.

Table 7
Zeros of $Q_6^1(x)$ and $u(x,\ 1;\ 6)$ for some values of M.

	$t = 1$						
	$M = 0$	$M = 0.002$	$M = 0.05$	$M = 0.5$	$M = 5$	$M = 10$	$M = 50$
$Q_6^t(x)$							
	±1.51614	±1.51612	±1.5153	±1.5103	±1.5018	±1.5005	±1.4993
	±1.0730	±1.0729	±1.0711	±1.0600	±1.0403	±1.0374	±1.0346
	±0.4530	±0.4526	±0.4445	±0.3846	±0.2044	±0.1524	±0.0714
$u(x,\ t;\ 6)$							
	$\pm 0.9164i$	$\pm 0.9165i$	$\pm 0.9185i$	$\pm 0.9300i$	$\pm 0.9501i$	$\pm 0.9533i$	$\pm 0.9564i$
	±0	±0.0139	±0.0665	±0.1444	±0.1109	±0.0852	±0.0409

Table 8
Zeros of $Q_{10}^1(x)$ and $u(x,\ 1;\ 10)$ for some values of M.

	$t = 1$						
	$M = 0$	$M = 0.002$	$M = 0.05$	$M = 0.5$	$M = 5$	$M = 10$	$M = 50$
$Q_{10}^t(x)$							
	±1.79469	±1.79467	±1.7942	±1.7921	±1.7896	±1.7893	±1.7890
	±1.49286	±1.49284	±1.4922	±1.4888	±1.4849	±1.4845	±1.4841
	±1.17419	±1.17414	±1.1730	±1.1674	±1.1608	±1.1600	±1.1593
	±0.7931	±0.7930	±0.7907	±0.7789	±0.7647	±0.7630	±0.7616
	±0.2950	±0.2947	±0.2858	±0.2291	±0.1067	±0.0780	±0.0359
$u(x,\ t;\ 10)$							
	$\pm 1.1107i$	$\pm 1.1108i$	$\pm 1.1117i$	$\pm 1.1166i$	$\pm 1.1223i$	$\pm 1.1230i$	$\pm 1.1236i$
	±0	±0.0110	±0.0509	±0.0950	±0.0589	±0.0440	±0.0206

Finally, the last two Tables 7 and 8 show the position of the zeros of Freud-type polynomials of $Q_6^1(x)$ and $Q_{10}^1(x)$ and the zeros of the corresponding ghost polynomials. Notice that the zeros of the ghost polynomials continue to be two real and two complex conjugate numbers.

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