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Multiphase Mixed-Integer Optimal Control Approach to Aircraft Trajectory Optimization

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In this paper, an approach to aircraft trajectory optimization is presented in which integer and continuous variables are considered. Integer variables model decision-making processes, and continuous variables describe the state of the aircraft, which evolves according to differential-algebraic equations. The problem is formulated as a multiphase mixed-integer optimal control problem. It is transcribed into a mixed-integer nonlinear programming problem by applying a fifth degree Gauss–Lobatto direct collocation method and is then solved using a nonlinear-programming-based branch-and-bound algorithm. The approach is applied to the following en route flight planning problem: Given an aircraft point mass model, a wind forecast, an airspace structure, and the relevant flying information regions with their associated overflying costs, find the control inputs that steer the aircraft from the initial fix to the final fix, following a route of waypoints while minimizing the fuel consumption and overflying costs during the flight. The decision-making process arises in determining the optimal sequence of waypoints. The optimal times at which the waypoints are to be overflown are also to be determined. Numerical results are presented and discussed, showing the effectiveness of the approach.

I. Introduction

IN the future air traffic management system, the trajectory becomes the fundamental element of a new set of operating procedures collectively referred to as trajectory-based operations (TBO) [1]. TBO will provide the capabilities, decision-support tools, and automation to manage aircraft movement by trajectory. This shift to a trajectory-based air traffic control (ATC) will enable aircraft to fly negotiated flight paths in which ATC will issue restrictions to be met and the airline will decide the most economical way to meet them. In this way, the TBO concept will allow more flexible and efficient airspace usage, resulting in more efficient trajectories.

With such underlying motivation, this paper presents an approach to aircraft trajectory optimization in which both continuous and integer variables are included. Integer variables model decision-making processes and continuous variables describe the state of the aircraft, which evolves according to differential-algebraic equations.

Trajectory optimization has been studied for several decades. A survey on numerical methods for trajectory optimization is given in [2]. The trajectory optimization problem can be formulated as an optimal control problem ([3] Chaps. 2 and 3). Practical issues for solving optimal control problems are discussed in ([4] Chap. 4).

In the flight of an aircraft, several flight phases can be distinguished. Thus, different flight models can be used for different phases, resulting in a multistage or multiphase trajectory optimization problem [5], which can be typically formulated as multiphase optimal control problems. Many instances have been solved for practical applications in aerospace engineering. In [6,7], an unmanned air vehicle flight mission is solved considering the route as a given sequence of waypoints. In [5,8], the minimum fuel trajectory problem for commercial aircraft is discussed for a given sequence of phases constituting the flight plan. Multistage problems for space flight trajectory optimization have also been solved. They

include multistage launch [9], multistage ascent [10], or multistage orbit transfer [11].

However, due to an increasing sophistication of both manned and unmanned space missions, and due to the complex nature of the airspace structure in atmospheric flights, decision variables may be needed for better modeling of trajectory optimization problems [12]. Therefore, integer or binary variables are introduced to model decision-making processes in optimal control problems. Few works considering decision variables have been presented in the scope of aircraft trajectory optimization: In [12], two examples are solved, an asteroid mission and a refueling mission, using a pseudospectral knotting method to generate a mixed-variable programming problem; in [13,14], evolutionary algorithms including decision variables are derived for space mission planning. Therefore, more efforts in this direction are needed.

The main contribution of this paper is to present an approach based on mixed-integer nonlinear programming (MINLP) to aircraft trajectory optimization in which decision-making processes are modeled with application to commercial aircraft en route trajectory optimization.

The problem to be studied can be described as follows: Given an aircraft point mass dynamic model, a wind forecast, an airspace structure, and the flying information regions (FIRs) with their associated overflying charges, find the control inputs that steer the aircraft from the initial fix to the final fix following a route made of a sequence of waypoints, while minimizing the fuel consumption and overflying costs during the flight. The decision-making process arises in determining the ordered set of waypoints to be overflown. The times at which they are reached are also to be determined.

Such a problem can be formulated as a multiphase mixed-integer optimal control problem (MIOCP) [15,16]. The multiphase MIOCP

is transcribed into an MINLP problem, first converting the multiphase optimal control problem into a conventional optimal control problem by making the unknown switching times part of the state [17], then by applying a collocation method based on high-order Gauss–Lobatto quadrature rules [18] to convert the dynamic equations of the system into constraints, and finally introducing binary variables to model the decision-making process. Thus, the resulting optimization problem is an MINLP problem.

MINLP problems belong to a very difficult class of optimization problems. In theory, they are nondeterministic polynomial time hard to solve. In practice, they are among the most challenging problems in computational optimization. In particular, this is true when the feasible set is not convex and highly nonlinear, as is the case here. A commonly used algorithm to solve MINLP problems is the spatial branch-and-bound algorithm [19,20]. Several computer programs implementing this algorithm have been developed. Among the most advanced codes are the commercial solver BARON [21] and the open-source solver Couenne [22]. Unfortunately, these solvers rapidly become impractical when the feasible regions are highly nonconvex and the size of problem grows. Therefore, in the present paper, a heuristic approach is followed, which is based on a nonlinear solver able to find locally optimal solutions. In this approach, the discrete aspects of the problem are neglected and a branch-and-bound scheme aimed at finding integer values for the discrete variables is used. This algorithm is exact for problems in which both objective function and nonlinear region are convex, but it has shown its effectiveness in finding good solutions to problems with nonconvex features [23].

The general multiphase MIOCP is stated in Sec. II. The solution approach that has been followed is presented in Sec. III. In particular, in Sec. III.C, special focus has been put on detailing the MINLP algorithm used herein to find a solution. In Sec. IV, the application to en route trajectory optimization is presented. In Sec. V, results of the application of the approach to an instance of trajectory optimization problem are reported and discussed. Finally, Sec. VI contains the conclusions.

II. Problem Formulation

The multiphase motion of an aircraft can be modeled by a set of differential-algebraic dynamic subsystems:

$$\dot{x}^q(t) = f^q(x^q(t), u^q(t), z), \quad 0 = g^q(x^q(t), u^q(t), z)$$

where $q = 1, \dots, N$ is the index of phases, $t \in [t^l, t^F] \subset \mathbb{R}$ is the time, $x^q(t) \in \mathbb{R}^{n_{x^q}}$ is the state variable in phase q whose time derivative is $\dot{x}^q(t) \in \mathbb{R}^{n_{x^q}}$, $u^q(t) \in \mathbb{R}^{n_{u^q}}$ is the control function in phase q , which is assumed to be measurable, and $z \in \mathbb{R}^{n_z}$ represents a vector of parameters. Let

$$t^l = \tilde{t}^0 \leq \tilde{t}^1 \leq \dots \leq \tilde{t}^N = t^F$$

be the switching times between phases. Thus, in the time interval $[\tilde{t}^{q-1}, \tilde{t}^q]$ with $q = 1, \dots, N$, the system evolution is governed by the differential-algebraic subsystem q , being $\tilde{t}^0 = t^l$ and $\tilde{t}^N = t^F$. At time $\tilde{t}^q$ with $q = 1, \dots, N-1$, the differential-algebraic dynamic subsystem changes from q to $q+1$. The decision-making processes are modeled by means of one time-independent vector of binary variables $v^q \in \{0, 1\}^{n_{v^q}}$, $q = 1, \dots, N-1$, whose components are $v^{q,j}$, $q = 1, \dots, N-1$, $j = 1, \dots, n_{v^q}$.

The multiphase MIOCP can be stated as follows ([15] Chap. 1):

$$\begin{aligned} \min J(x^q, u^q, v^q, z) = & \sum_{q=1}^N \left[\int_{\tilde{t}^{q-1}}^{\tilde{t}^q} L^q(x^q(t), u^q(t), z) dt \right] \\ & + \sum_{q=1}^{N-1} E^q(x^q(\tilde{t}^q), v^q, z) \end{aligned} \quad (1a)$$

subject to

$$\dot{x}^q(t) = f^q(x^q(t), u^q(t), z), t \in [\tilde{t}^{q-1}, \tilde{t}^q], q = 1, \dots, N \quad (1b)$$

$$g^q(x^q(t), u^q(t), z) = 0, t \in [\tilde{t}^{q-1}, \tilde{t}^q], q = 1, \dots, N \quad (1c)$$

$$c^q(x^q(t), u^q(t), z) \leq 0, t \in [\tilde{t}^{q-1}, \tilde{t}^q], q = 1, \dots, N \quad (1d)$$

$$x^1(t^l) = x^l \quad (1e)$$

$$\psi(x^N(t^F)) = 0 \quad (1f)$$

$$r^{\text{ineq}}(x^1(\tilde{t}^1), v^1, x^2(\tilde{t}^2), v^2, \dots, x^{N-1}(\tilde{t}^{N-1}), v^{N-1}, z) \leq 0 \quad (1g)$$

$$r^{\text{eq}}(x^1(\tilde{t}^1), v^1, x^2(\tilde{t}^2), v^2, \dots, x^{N-1}(\tilde{t}^{N-1}), v^{N-1}, z) = 0 \quad (1h)$$

$$x^{q+1}(\tilde{t}^q) = \text{Tr}^q(x^q(\tilde{t}^q), u^q(\tilde{t}^q), v^q, z), q = 1, \dots, N-1 \quad (1i)$$

The terms of the objective functional (1a) are in Bolza form and contain a Lagrange term

$$\int_{\tilde{t}^{q-1}}^{\tilde{t}^q} L^q(x^q(t), u^q(t), z) dt$$

and a Mayer term $E^q(x^q(\tilde{t}^q), v^q, z)$. Equations (1b) and (1c) with $f^q \in \mathbb{R}^{n_{f^q}}$ and $g^q \in \mathbb{R}^{n_{g^q}}$ are the equations of the differential-algebraic system in phase q . Equation (1d) with $c^q \in \mathbb{R}^{n_{c^q}}$ are the inequality constraints in phase q . Equations (1e) and (1f) are the boundary conditions of the problem. Equation (1g) with $r^{\text{ineq}} \in \mathbb{R}^{n_{r^{\text{ineq}}}}$ and Eq. (1h) with $r^{\text{eq}} \in \mathbb{R}^{n_{r^{\text{eq}}}}$ are the inequality and equality interior point constraints, respectively.

Equation (1i) are the transition conditions between phases to ensure continuity, which are usually of the form $x^{q+1}(\tilde{t}^q) = x^q(\tilde{t}^q)$, for $q = 1, \dots, N-1$. These conditions are also referred to as linkage conditions. Functions $f^q \in \mathbb{R}^{n_{f^q}}$ are assumed to be piecewise Lipschitz and $\partial g^q / \partial x^q$ is assumed to be regular. L^q , E^q , c^q , r^{ineq} , and r^{eq} are assumed to be twice differentiable. The dimensions n_{x^q} , n_{u^q} , n_{v^q} , n_{c^q} , n_{f^q} , and n_{g^q} can be different for each phase.

The solution of this problem is composed of the functions $x^q(t)$, $u^q(t)$, $q = 1, \dots, N$, $t \in [t^l, t^F]$, the switching times $\tilde{t}^1, \dots, \tilde{t}^{N-1}$, the vector of binary variables v^q , $q = 1, \dots, N-1$, and the vector of parameters z . Note that $\tilde{t}^N = t^F$ is also a variable of the problem if the final time is unknown.

A more general form of multiphase mixed-integer optimal control problem that includes time-dependent binary control functions can be found in [15,16].

III. MIOCP Solution Approach

A. Treatment of the Switching Times

The multiphase MIOCP is tackled first making the unknown switching times part of the state vector and then introducing a new independent variable with respect to which the switching times are fixed. In this reformulated problem, there is a linear relation between the new variable and time, but the slope of this linear relation changes on each interval between two switches. These slopes, which are part of the solution to the multiphase MIOCP, are actually time scaling factors that determine the optimal switching times. The reformulated problem is a conventional MIOCP in which linkage conditions (1i) are not necessary. More details on this technique can be found in [17].

B. Fifth Degree Gauss–Lobatto Collocation Method

A collocation approach has been used to deal with the differential constraints Eq. (1b) and the Lagrange term in Eq. (1a) are modified,

and the switching times $\tilde{t}^q, q = 1, \dots, N-1$ are included in the state vector.

In particular, the fifth degree Gauss–Lobatto integration scheme has been used [18]. The motivation behind the use of this integration scheme is related to its order of accuracy, which is a measure of the effectiveness of a numerical method to approximate a solution. The fifth degree Gauss–Lobatto integration scheme has an order of accuracy of eight, whereas, for instance, the trapezoid and Simpson rules (the second and third degree Gauss–Lobatto integration schemes) have an order of accuracy of two and four, respectively. The greater the order of accuracy, the greater is the reduction in error if the step size is made smaller. Therefore, as the order of accuracy increases, a specified accuracy may be achieved with larger step sizes (i.e., with a smaller number of variables). Reducing the number of variables drastically improves the computation times for the resolution of the MINLP problem as will be explained later.

For phase $q, q = 1, \dots, N$, the time interval $[\tilde{t}^{q-1}, \tilde{t}^q]$ is subdivided into N_q subintervals $[\tilde{t}_{i-1}^q, \tilde{t}_i^q], i = 1, \dots, N_q$, where $\tilde{t}_0^q = \tilde{t}^{q-1}$ and

$$\tilde{t}_{N_q}^q = \tilde{t}^q$$

The state is approximated using a fifth degree interpolation polynomial, whereas a free control interpolation scheme has been used for the controls, that is, the discretized control variables taken into account represent discrete values for the controls at each discrete time at which the system equations are evaluated, $u_{i-1}^q, u_{i-1,a}^q, u_{i-1,C}^q, u_{i-1,b}^q, u_i^q$. Thus, for each subinterval $[\tilde{t}_{i-1}^q, \tilde{t}_i^q]$, the unknowns are

$$(x_{i-1}^q, x_{i-1,C}^q, x_i^q, u_{i-1}^q, u_{i-1,a}^q, u_{i-1,C}^q, u_{i-1,b}^q, u_i^q)$$

where

$$x_0^q = x_{N_q}^{q-1}, \quad x_{N_q}^q = x_0^{q+1}, \quad x_{N_q}^N = x^N(t^F), \quad \text{and} \quad x_0^1 = x^1(t^I) = x^I$$

The resulting problem is an MINLP problem of the form

$$\begin{aligned} \min \sum_{q=1}^N \sum_{i=1}^{N_q} \hat{L}_i^q(x_{i-1}^q, x_{i-1,C}^q, x_i^q, u_{i-1}^q, u_{i-1,a}^q, u_{i-1,C}^q, u_{i-1,b}^q, u_i^q, z) \\ + \sum_{q=1}^{N-1} E^q(x_{N_q}^q, u_{N_q}^q, v^q, z) \end{aligned} \quad (2a)$$

subject to

$$\phi_{5,a}(x_{i-1}^q, x_{i-1,C}^q, x_i^q, u_{i-1}^q, u_{i-1,a}^q, u_{i-1,C}^q, u_i^q, z) = 0, q = 1, \dots, N, \\ i = 1, \dots, N_q \quad (2b)$$

$$\phi_{5,b}(x_{i-1}^q, x_{i-1,C}^q, x_i^q, u_{i-1}^q, u_{i-1,a}^q, u_{i-1,C}^q, u_{i-1,b}^q, u_i^q, z) = 0, q = 1, \dots, N, \\ i = 1, \dots, N_q \quad (2c)$$

$$g(x_{i-1}^q, x_{i-1,C}^q, x_i^q, u_{i-1}^q, u_{i-1,a}^q, u_{i-1,C}^q, u_{i-1,b}^q, u_i^q, z) = 1, q = 1, \dots, N, \\ i = 1, \dots, N_q \quad (2d)$$

$$c(x_{i-1}^q, x_{i-1,C}^q, x_i^q, u_{i-1}^q, u_{i-1,a}^q, u_{i-1,C}^q, u_{i-1,b}^q, u_i^q, z) \leq 0, q = 1, \dots, N, \\ i = 1, \dots, N_q \quad (2e)$$

$$x_0^1 = x^I \quad (2f)$$

$$\psi(x_{N_q}^N) = x^F \quad (2g)$$

$$r^{\text{ineq}}(x_{N_1}^1, v^1, x_{N_2}^2, v^2, \dots, x_{N_{q-1}}^{N-1}, v^{N-1}, z) \leq 0 \quad (2h)$$

$$r^{\text{eq}}(x_{N_1}^1, v^1, x_{N_2}^2, v^2, \dots, x_{N_{q-1}}^{N-1}, v^{N-1}, z) = 0 \quad (2i)$$

$$v^q \in \{0, 1\}^{n_{v^q}}, q = 1, \dots, N-1 \quad (2j)$$

where $\hat{L}_i^q$ is obtained applying the Gauss–Lobatto fifth degree integration rule to the Lagrange term in Eq. (1a), and $E^q(x_{N_q}^q, u_{N_q}^q, v^q, z)$ corresponds to the discretized Mayer term in Eq. (1a). The two defect equations $\phi_{5,a}$ and $\phi_{5,b}$ result from the discretization of the differential constraints (1b). Constraints (2d–2i) are the discretized versions of constraints (1c–1h), respectively. Notice that x^I corresponds to the initial state and x^F corresponds to the final state. The unknowns of this problem are

$$(x_{i-1}^q, x_{i-1,C}^q, x_i^q, u_{i-1}^q, u_{i-1,a}^q, u_{i-1,C}^q, u_{i-1,b}^q, u_i^q)$$

for $q = 1, \dots, N, i = 1, \dots, N_q$, together with the vector of binary variables $v^q \in \{0, 1\}^{n_{v^q}}, q = 1, \dots, N-1$.

C. MINLP Resolution

Fixing all variables $v^{q,j}, q = 1, \dots, N-1, j = 1, \dots, n_{v^q}$, is equivalent to fixing the sequence of alternatives and, if this is done, the multiphase MIOCP becomes a conventional multiphase optimal control problem. A simple algorithmic approach could therefore be to enumerate all possible values for $v^{q,j}$, solve the associated multiphase optimal control problems, and pick the best solution. Unfortunately, a rapid calculation of the number of problems to solve if one follows this approach shows that it is impractical for more than a handful of possible values for $v^{q,j}$. A common approach to try to address bigger problems is to do an implicit enumeration via a branch-and-bound algorithm [24,25]. Branch-and-bound is a standard algorithm for integer programming (see, for example, [26] and references therein). A brief sketch of it in the context of this paper is given next, with an emphasis on the particularities that arise in the context of multiphase MIOCP. For a more complete exposition, the reader is invited to refer to the preceding references.

Branch-and-bound is a divide-and-conquer method. The problem is divided by partitioning the set of feasible solutions into smaller and smaller subsets. The conquering is done by computing bounds on the value of the best feasible solution in each subset and discarding subsets based on this bound. Branch-and-bound is an exact algorithm when the bound used in the fathoming phase is a valid lower bound. However, the problem of interest here is particular in that systematically since obtaining a good lower bound on the value of the multiphase MIOCP is a daunting task.

Indeed, to compute a valid lower bound, one has to build a convex approximation of the optimization problem in which a convex function is minimized over a convex feasible region. Here, there are two sources of nonconvexities: the binary variables of the problem and the nonlinear equations used to describe the feasible region. Dealing with binary variables is standard and can simply be done by replacing the set $\{0, 1\}$ with the interval $[0, 1]$ (i.e., relaxing them). Dealing with nonlinear and highly nonconvex equations is much more involved. Although systematic methods exist to compute convex approximations for such nonlinear equations (for example, within the solvers Baron [20,21] and Couenne [22]), their efficiency is limited. In the experiment herein presented, they are not able to yield any good lower bound.

Therefore, the approach does not rely on a true lower bound but rather uses approximate solutions. In that case, the procedure is heuristic (i.e., does not return the exact optimal solution). The quality of the final solution depends on the quality of the approximation. To the best of our knowledge, there is no theoretical guarantee on the quality of the approximation. Its practical efficiency is the subject of the computational section of this paper.

Approximations are computed by using the relaxed integer optimal control problem (RIOCP), where the constraints $v^{q,j} \in \{0, 1\}$ are relaxed to $v^{q,j} \in [0, 1]$, for $q = 1, \dots, N-1$, and $j = 1, \dots, n_{v^q}$. A locally optimal solution to the RIOCP can be computed with a nonlinear programming algorithm, for instance, the interior point algorithm implemented by Interior Point OPTimizer (IPOPT) [27].

Algorithm 1 NLP BB

0. Initialize.

$\mathcal{T} \leftarrow \{(\emptyset, \emptyset)\}$. $\beta_{\mathcal{U}} = \infty$. $v^* \leftarrow \text{NONE}$.

1. Terminate?

Is $\mathcal{T} = \emptyset$? If so, stop and return the sequence described by v^* .

2. Select.

Choose and delete a problem $\mathcal{N}^l = (\mathcal{L}^l, \mathcal{U}^l)$ from $\mathcal{T}$.

3. Evaluate.

Solve the RIOCP $(\mathcal{L}^l, \mathcal{U}^l)$. If no solution can be found, go to step 1, else let $\beta_{\mathcal{L}^l, \mathcal{U}^l}^R$ be its objective function value and $\hat{v}$ be the values for the relaxed binary variables.

4. Prune.

If $\beta_{\mathcal{L}^l, \mathcal{U}^l}^R \geq \beta_{\mathcal{U}}$, go to step 1. If $\hat{v} \in \{0, 1\}^{N-1} \times \{0, 1\}^{n_{\text{var}}}$, go to step 5, else let $\beta_{\mathcal{U}} \leftarrow \beta_{\mathcal{L}^l, \mathcal{U}^l}^R$, $v^* \leftarrow \hat{v}$, and delete from $\mathcal{T}$ all problems with $\beta_{\mathcal{L}}^R \geq \beta_{\mathcal{U}}$. Go to step 1.

5. Divide.

Create two new nodes $\mathcal{N}^{|\mathcal{T}|}$, and $\mathcal{N}^{|\mathcal{T}|+1}$. Choose $\hat{q}$ and $\hat{j}$ such that $\hat{v}^{\hat{q}, \hat{j}} \in \{0, 1\}$. Let $\beta^{|\mathcal{T}|} \leftarrow \beta^{|\mathcal{T}|+1} \leftarrow \beta_{\mathcal{L}, \mathcal{U}}^R$ and add the problem $\mathcal{N}^{|\mathcal{T}|} = (\mathcal{L} \cup v^{\hat{q}, \hat{j}}, \mathcal{U}^l)$ and $\mathcal{N}^{|\mathcal{T}|+1} = (\mathcal{L}^l, \mathcal{U}^l \cup v^{\hat{q}, \hat{j}})$. Go to step 1.

The branch-and-bound framework is then used to find a solution that also satisfies the integrity requirements $v^{q,j} \in \{0, 1\}$, $q = 1, \dots, N-1$, and $j = 1, \dots, n_{\text{var}}$. This variant of branch-and-bound is usually called nonlinear programming based branch and bound (NLP-BB). For more details, see, e.g., [28] and references therein).

The first step of the branch-and-bound algorithm is to solve the RIOCP. If the solution obtained by solving the RIOCP is integer feasible (all variables $v^{q,j}$ take value zero or one), it specifies a sequence of points and the algorithm stops. If no solution to the RIOCP is found, the algorithm stops. If an upper bound $\beta_{\mathcal{U}}$ on the value of the optimal solution is known and the value of the solution of the RIOCP is above $\beta_{\mathcal{U}}$, the algorithm also stops (fixing the infeasibility of the solution should increase the objective value of the solution). Otherwise, the algorithm divides the feasible region in two by fixing one of the variables $v^{q,j}$ such that $\hat{v}^{q,j} \in \{0, 1\}$ to zero and one successively.

Applying the preceding steps recursively, yields to a tree $\mathcal{T}$ of partial assignment for the binary variables. At each node of this tree, a subset $\mathcal{L}$ of the variables v are fixed to zero and a subset $\mathcal{U}$ are fixed to one, and a local optimum of the restriction of the RIOCP where the variables in $\mathcal{L}$ and $\mathcal{U}$ are fixed is to be sought. This restricted relaxed optimal control program is referred to as RIOCP $(\mathcal{L}, \mathcal{U})$. The value of the upper bound $\beta_{\mathcal{U}}$ is initially $+\infty$ and is updated whenever a new integer-feasible solution is found such that the cost is improved.

The pseudocode of the NLP BB is given in Algorithm 1. Several solvers implement this algorithm, for example, MINLP BB [29] and SBB [30]. In this paper, the solver BONMIN has been used [31]. BONMIN is an open-source MINLP solver implementing several different algorithms for solving mixed-integer nonlinear optimization problems. Source code and binaries of BONMIN are available from COIN-OR.[†] BONMIN is called through the AMPL modeling language.

Two critical steps for the practical efficiency of Algorithm 1, which have not been explicated, are the selection of the next subproblem to evaluate (step 2), and the choice of the variable to divide the feasible region (step 5). For these two steps, standard rules implemented in BONMIN are used. The subproblem selected in step 2 is always the one with lowest $\beta^{q,j}$ (best-bound rule). Whereas, for choice of the variable $\hat{v}^{q,j}$, a default strategy in BONMIN is used, which is a combination of strong branching and pseudocosts [28].

IV. Commercial Aircraft Trajectory Optimization

In this section, the commercial aircraft trajectory optimization problem is stated as a multiphase MIOCP. The statement of the problem involves the specification of the models that have been used to describe the horizontal flight dynamics of the aircraft and the wind acting on it, together with the assumptions that have been made on the

airspace structure and the scheme used to compute the en route overflying charges.

A. Horizontal Flight Dynamics

The horizontal motion over a spherical earth including wind effects is considered. A common assumption in aircraft trajectory optimization is to consider a two-degree-of-freedom dynamic model, which describes the point variable-mass motion of the aircraft. A standard atmosphere is assumed, so that the density of the air is only a function of altitude. Because the altitude is assumed to be constant, the density of the air is also constant. The airplane is a conventional jet airplane and BADA 3.6 [32] is used as the aircraft performance model.

The differential-algebraic equations governing the translational horizontal motion of the airplane are the following:

$$\begin{aligned}\dot{\lambda}(t) &= \frac{V(t) \cos \chi(t) + W_x(\lambda(t), \theta(t))}{R_e \cos \theta(t)}, \\ \dot{\theta}(t) &= \frac{V(t) \sin \chi(t) + W_y(\lambda(t), \theta(t))}{R_e}, \\ \dot{V}(t) &= \frac{T(t) - D(V(t))}{m(t)}, \dot{\chi}(t) = \frac{L(V(t))}{m(t)V(t)} \sin \mu(t), \\ L(V(t)) \cos \mu(t) &= m(t)g, \dot{m}(t) = -\eta(V(t))T(t)\end{aligned}\quad (3)$$

In the set of Eq. (3), the state vector is $x(t) = (\lambda(t), \theta(t), V(t), \chi(t), m(t))$, where λ is the longitude, θ the latitude, V the true air speed, χ the heading angle, and m the mass of the aircraft. W_x and W_y are functions of longitude and latitude at a given altitude and correspond to the east and north wind components, respectively. Lift $L = C_L S \hat{q}$ and drag $D = C_D S \hat{q}$ are the components of the aerodynamic force, where C_D is the coefficient of drag, C_L is the coefficient of lift, S is the reference wing surface area, and $\hat{q} = 1/2 \rho V^2$ is the dynamic pressure, with ρ being the density of the air. A parabolic drag polar is also assumed (i.e., $C_D = C_{D_0} + C_{D_i}$), where C_{D_0} is the parasite coefficient of drag and C_{D_i} is the induced coefficient of drag. R_e is the radius of Earth, and g is the force due to gravity, which is assumed to be constant; η corresponds to specific fuel consumption. In general, the engine thrust T and bank angle μ are the control variables of the aircraft, that is, $u(t) = (T(t), \mu(t))$. The thrust is commanded by the engine throttle and the bank angle is commanded combining rudder and ailerons trims. Note that the differential-algebraic system is nonlinear and nonconvex.

The path constraints of the problem are those that define the aircraft's flight envelope and restrictions in the control actions. They can be found in the BADA database manual [32]:

$$\begin{aligned}C_{V_{\min}} V_s(m(t)) &\leq V(t) \leq V_{M_0}, & M(V(t)) &\leq M_{M_0}, \\ m_{\min} &\leq m(t) \leq m_{\max}, & 0 &\leq C_L(t) \leq C_{L_{\max}}, \\ T_{\min} &\leq T(t) \leq T_{\max}, & \mu(t) &\leq \mu_{\max, \text{civ}}\end{aligned}\quad (4)$$

where C_v is the minimum speed coefficient, V_s is the stall speed, which is a function of the mass for a given altitude, and V_{M_0} is the maximum operating calibrated airspeed. M is the Mach number, which is a function of the velocity at a given altitude, and M_{M_0} is the maximum operating Mach number; $m_{\min}$ and $m_{\max}$ correspond to the operating empty mass and the takeoff mass, respectively; and $C_{L_{\max}}$ is the maximum coefficient of lift. $T_{\min}$ and $T_{\max}$ correspond, respectively, to the minimum and maximum available thrust at a given altitude, and $\mu_{\max, \text{civ}}$ corresponds to the maximum bank angle due to structural limitations. Note that this set of inequality constraints describes a nonlinear, nonconvex set.

B. Wind Data

To take into account the influence of wind, forecasts provided by the National Oceanic and Atmospheric Administration Forecasts System Laboratory via gridded binary (GRIB) files are considered. GRIB files provide wind forecasts as tabular data that give the three

[†]Data available online at <http://www.coin-or.org> [retrieved 25 June 2012].

Fig. 1 Waypoints and navigation aids AIRAC June 2012.

components (vertical, north, and east) of the wind vector at each node of the grid. Wind forecast tabular data are fitted into analytic functions by means of nonlinear regression analysis ([33] Chap. 15).

C. Airspace Structure

The airspace is structured to ensure the safe development of aircraft operations. To this end, a network of routes has been established and equipped with navigation aids, so that the aircraft can navigate following them. The routes of this network are referred to as air traffic services (ATS) routes, and they are composed of waypoints and airways. Waypoints may be simple named points in the space or they may be associated with existing navigational aids, intersections, or fixes. Airways are imaginary corridors connecting waypoints.

The ATS routes are published in the basic manual for aeronautical information called the Aeronautical Information Publication, which is usually updated once a month coinciding with the Aeronautical Information Regulation and Control (AIRAC) cycle. Ocean tracks might change twice a day to take advantage of any favorable wind. In free-flight areas, the path is defined by the user, and thus finding the optimal path considering the affect of wind is a crucial issue in these zones.

Figure 1 shows the waypoints and navigation aids corresponding the AIRAC cycle published in June 2012. Airways have been omitted for the sake of clarity. A flight plan must be defined specifying a certain number of waypoints that the aircraft is going to fly. The huge number of waypoints in this figure reflects the inherent complexity of defining an efficient flight plan in a structured airspace.

In the model assumed in this paper, airways are not considered and it is supposed that the aircraft can fly an arbitrary route among waypoints. However, considering a complete directed graph structure over the set of waypoints is redundant because aircraft must fly

through closer waypoints before reaching farther waypoints. A graph structure that is capable of reflecting this simple observation is the multipartite graph structure.

Thus, the airspace structure is modeled as a complete multipartite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, whose vertex set $\mathcal{V}$ is partitioned into pairwise disjoint independent subsets, which are called partite sets. In this model, nodes represent waypoints and arcs represent possible transitions between them. In a complete multipartite graph, vertices are adjacent if and only if they belong to different (adjacent) partite sets. The complete multipartite graph considered is composed of a sequence of $N + 1$ partite sets, $\mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_N$, where $\mathcal{V}_0$ and $\mathcal{V}_N$ contain one node each, the initial and final waypoints p^I and p^F , respectively, and $\mathcal{V}_q$, $q = 1, \dots, N - 1$, contains n_{v^q} nodes. Let $P = \{p^{1,1}, \dots, p^{N-1, n_{v^{N-1}}}\}$ be the collection of waypoints of the partite sets $\mathcal{V}_1, \dots, \mathcal{V}_{N-1}$ (see Fig. 2).

D. En Route Overflying Charges

ATS routes go through flight regions, which are portions of the airspace in which a single national aviation authority provides navigation, surveillance, and control services. These regions are referred to as FIRs/upper information regions (UIRs). Figure 3 shows the FIR/UIR structure of the North Atlantic airspace. In general, national aviation authorities apply overflying fees for the services they provide. Very different charging schemes are applied, including purely traveled distance-based charges, aircraft weight, and traveled distance charges, flat rate charges (FR), or communication rate charges (CR) [34].

The charging methodologies in the relevant regions for the flight to be analyzed in the experiment, namely, the United States, Canada, and Europe, including the North Atlantic oceanic regions, are briefly presented.

Fig. 2 Multipartite graph structure.

Fig. 3 Relevant FIR/UIR regions for flights between North America and Europe.

In Europe, the standard European Organisation for the Safety of Air Navigation (EUROCONTROL) charge formula for en route services in the EUROCONTROL members' countries is

$$r_i^{\text{Eur}} = \text{UR}_i \frac{\text{GCD}_i}{100} \sqrt{\frac{\text{MTOW}}{50}}$$

where UR_i is the service unit rate in FIR_i (referring to member country i), GCD_i is the great circle distance in kilometers traveled in FIR_i , and MTOW is the maximum takeoff weight in metric tonnes of the aircraft. The unit rates of en route charges are established by each EUROCONTROL member state and updated every month [35]. In the United States, the Federal Aviation Administration only charges overflight fees to operators who fly in the United States controlled airspace, but neither take off nor land in the United States. In the continental airspace, the en route charges r^{USACon} are \$38.44 per 100 n mile (measured in GCD). In the oceanic airspace, the fee r^{USAOc} is \$17.22 per 100 n mile (in GCD).** NAV CANADA applies different fees for its oceanic and continental airspaces. Canadian oceanic charges in Gander Oceanic FIR are based on a flat rate that can be decomposed into navigation fee FR^{GanOc} of \$93.24 Canadian and a communication fee CR^{GanOc} of \$22.04 Canadian [36]. Canadian continental airspace charges are based on aircraft weight and traveled distance as follows:

$$r^{\text{CanCon}} = \text{UR} \cdot \text{GCD} \cdot \sqrt{\text{MTOW}}$$

where the UR is \$0.03445, the traveled GCD is in kilometers, and MTOW is in tonnes. Charges for services provided in the Shanwick Oceanic FIR comprise a flat communication rate CR^{ShOc} of €45 (charged by Ireland) and a flat navigation fee FR^{ShOc} of £65.70 (charged by the United Kingdom) [37].

E. Problem Statement

Suppose the aircraft is constrained to pass through one edge within every partite set of the graph in Fig. 2, that is, the aircraft is constrained to fly over the initial waypoint, $N-1$ waypoints, each belonging to a different partite set $\mathcal{V}_1, \dots, \mathcal{V}_{N-1}$, and the final waypoint. Thus, N phases can be identified during the motion of the aircraft.

Each of the binary variables $v^{q,j}$, $q = 1, \dots, N-1$, $j = 1, \dots, n_{v,q}$, is associated with a waypoint $p^{q,j}$ of the set, and $v^{q,j} = 1$ means that the aircraft flies over waypoint j of partite set $\mathcal{V}_q$ at time $\tilde{t}^q$. Let

$$\bar{X} = \{\bar{x}^{1,1}, \dots, \bar{x}^{N-1, n_{v,N-1}}\}$$

with $\bar{x}^{q,j} = (\lambda^{q,j}, \theta^{q,j})$ be the set of waypoint locations. Let $\bar{x}^I, \bar{x}^F$ be the positions of the initial and final waypoints. The constraints on the waypoints can be expressed as follows:

**Data available online at http://www.faa.gov/air_traffic/international_aviation/overflight_fees [retrieved 25 June 2012].

$$\bar{x}(\tilde{t}^q) = \sum_{j=1}^{n_{v,q}} v^{q,j} \bar{x}^{q,j}, q = 1, \dots, N-1 \quad (5)$$

Additional constraints are

$$\sum_{j=1}^{n_{v,q}} v^{q,j} = 1, q = 1, \dots, N-1 \quad (6)$$

Condition (5) means that, if $v^{q,j} = 1$, it will be $\bar{x}^{q,j}(\tilde{t}^q) = \bar{x}^{q,j}$, that is, the aircraft will overfly waypoint in location $\bar{x}^{q,j}$ at time $\tilde{t}^q$, $q = 1, \dots, N-1$. Condition (6) means that the aircraft must overfly only a single waypoint of partite set $\mathcal{V}_q$ at time $\tilde{t}^q$, $q = 1, \dots, N-1$. Notice that upper and lower bounds on the overflying times could also be specified, which could be interesting, for instance, to issue sector capacity constraints in air traffic flow management (ATFM).

In the Lagrange term of the objective functional (1a), $L^q(x^q(t), u^q(t), z) = \dot{m}^q(V(t), T(t))$ is the fuel flow of the aircraft during phase q , whereas the Mayer term in objective functional (1a) is the cost due to overflying charges, which is computed taking into account the actual traveled distances in the relevant FIRs/UIRs when flying between couples of waypoints belonging to adjacent partite sets.

Notice that the objective functional (1a) could be seen as a multiobjective functional, balancing continuous costs corresponding to fuel consumption and costs corresponding to overflying charges. The natural choice followed here is to balance these two terms using their monetary cost. In Sec. V.E, the impact of the costs due to overflying charges will be discussed by varying the weight of these costs in the objective function.

V. Case Study

In this section, the results of the application of the method described in the preceding sections to a realistic instance of the aircraft optimization problem will be described. More specifically, the trajectory optimization problem of an A330-301 aircraft performing the en route part of a flight New York–Rome between the waypoint Yahoo [$p^I = (-69.74^\circ, 41.69^\circ)$], as the initial fix, and the waypoint Amtel [$p^F = (11.60^\circ, 43.21^\circ)$], as the final fix, is presented. The altitude of the route has been considered constant at 38,000 ft, that is, at flight level 380. The initial conditions of the problem were $V(t^I) = 235$ m/s, $\gamma(t^I) = 0$ deg, $\chi(t^I) = 0$ deg, $m(t^I) = 174,000$ kg.

The en route part of the flight has been divided into $N = 9$ phases, also referred to as legs, with an initial and final waypoint, and eight intermediate waypoints. Thus, eight partite sets with $n_{v,q} = 5$ waypoints in each set have been considered.

The selection of the number of intervals has been done comparing solutions to the MINLP computed with increasing number of intervals in each phase until a negligible change in the objective function was observed. Based on this criterion, the fifth degree

Table 1 Coordinates and designators of the waypoints

Coordinates	Designator	Coordinates	Designator
$p^{1,1} = (41.11^\circ, -67.00^\circ)$	DOVEY	$p^{1,2} = (40.11^\circ, -67.00^\circ)$	JABOC
$p^{1,3} = (41.78^\circ, -67.00^\circ)$	VITOL	$p^{1,4} = (42.63^\circ, -67.00^\circ)$	KANNI
$p^{1,5} = (43.56^\circ, -67.00^\circ)$	TUSKY	$p^{2,1} = (44.93^\circ, -51.00^\circ)$	VODOR
$p^{2,2} = (45.83^\circ, -51.00^\circ)$	URTAQ	$p^{2,3} = (46.87^\circ, -51.00^\circ)$	RONPO
$p^{2,4} = (47.82^\circ, -51.00^\circ)$	NOVEP	$p^{2,5} = (48.77^\circ, -51.00^\circ)$	LOGSU
$p^{3,1} = (46.00^\circ, -8.00^\circ)$	RIVAK	$p^{3,2} = (47.00^\circ, -8.00^\circ)$	LAPEX
$p^{3,3} = (48.00^\circ, -8.00^\circ)$	REGHI	$p^{3,4} = (49.50^\circ, -8.00^\circ)$	RATKA
$p^{3,5} = (45.00^\circ, -8.00^\circ)$	BEGAS	$p^{4,1} = (45.93^\circ, -5.22^\circ)$	ERWAN
$p^{4,2} = (44.50^\circ, -4.94^\circ)$	ATLEN	$p^{4,3} = (44.61^\circ, -5.40^\circ)$	KOLEK
$p^{4,4} = (46.32^\circ, -3.69^\circ)$	NOVAN	$p^{4,5} = (45.08^\circ, -3.86^\circ)$	PEPET
$p^{5,1} = (43.69^\circ, -1.41^\circ)$	SIGOS	$p^{5,2} = (45.01^\circ, -0.78^\circ)$	BMC11
$p^{5,3} = (46.05^\circ, -2.25^\circ)$	GODEM	$p^{5,4} = (45.73^\circ, -1.06^\circ)$	MAREN
$p^{5,5} = (44.55^\circ, -1.12^\circ)$	CAZAUX NDB	$p^{6,1} = (44.78^\circ, 1.47^\circ)$	RATRA
$p^{6,2} = (43.54^\circ, 1.36^\circ)$	LFBF 12 GS	$p^{6,3} = (45.33^\circ, 1.23^\circ)$	MAKOX
$p^{6,4} = (44.12^\circ, 2.16^\circ)$	DEPES	$p^{6,5} = (44.95^\circ, 2.36^\circ)$	AURILLAC NDB
$p^{7,1} = (43.38^\circ, 4.84^\circ)$	RHONE	$p^{7,2} = (44.37^\circ, 5.26^\circ)$	XIRBI
$p^{7,3} = (45.66^\circ, 4.89^\circ)$	RUSIT	$p^{7,4} = (45.10^\circ, 5.16^\circ)$	ROMAM
$p^{7,5} = (46.50^\circ, 4.95^\circ)$	ALURA	$p^{8,1} = (42.89^\circ, 8.67^\circ)$	RAPUR
$p^{8,2} = (44.59^\circ, 8.66^\circ)$	TESTO	$p^{8,3} = (44.04^\circ, 8.03^\circ)$	LBN32
$p^{8,4} = (45.15^\circ, 7.99^\circ)$	SIRLO	$p^{8,5} = (43.45^\circ, 7.59^\circ)$	GONTO

Gauss–Lobatto collocation method has been applied with a discretization using a total of 72 intervals in which, for the first three phases, the number of intervals is $N_1 = N_2 = N_3 = 12$ and, for the other phases, $N_4 = N_5 = N_6 = N_7 = N_8 = N_9 = 6$.

A. Waypoints

The waypoints and navigational aids of the AIRAC cycle published in June 2012 have been considered (see Fig. 1) and a set of 8×5 waypoints have been selected from them. For those phases entering or exiting oceanic regions, the waypoints have been selected manually coincident with the FIR/UIR bounds. This was the case of the first, second, and third partite sets. On the contrary, for those phases overflying the intra-European area, the waypoints have been selected randomly according to the following algorithm. First, the trajectory optimization problem has been solved without waypoint constraints, obtaining the free-flight trajectory. Then, the subpath from the intersection point between the free-flight path and the French FIR/UIR to the final waypoint was considered. Because six phases had to be defined in this subpath, five equidistant points along it have been selected. Finally, a random selection of the waypoints has been done using bivariate Gaussian probability density functions centered at the selected points, in which the directions of the principal axes of the ellipses that correspond to equidensity contours are parallel to the tangent and normal directions to the subpath at each point. The resulting set of waypoints is given in Table 1.

B. Wind Data

The wind forecast of 3 July 2012 has been considered. Let $W_x(\lambda, \theta)$ and $W_y(\lambda, \theta)$ be the analytic functions that result from the regression analysis to represent the east and north components, respectively. These functions are valid within a domain covering the North Atlantic and some parts of Europe (i.e., for $\lambda \in [-70^\circ, 12^\circ]$ and $\theta \in [40^\circ, 55^\circ]$) and can be included in the set of Eq. (3). The goodness of fit, measured in terms of R-squared parameter, yielded 0.78 for W_x and 0.71 for W_y . Figure 4 shows both forecast tabular data (solid dots) and analytic functions (surfaces) for W_x and W_y at 200 hpa ($h = 11,769$ m).

Figure 5 shows the great circle distance (minimum distance) path and the free-flight path that has been computed including wind effects. Wind vectors (arrows) represent the direction and speed of the wind in which longer arrows represent faster winds. It is easy to see that there is a region in which the phenomena of stronger eastward winds can be identified. This phenomenon is referred to as the jet stream, which is characterized by fast flowing, narrow air currents found in the atmosphere. The main jet streams are located near the tropopause at different latitudes.

C. En Route Overflying Charges

The overflying cost for a flight from a U.S. airport to Europe through Canadian continental airspace, Gander Oceanic, and Shanwick Oceanic FIRs can be expressed as

Fig. 4 North and east component of the wind speed at 200 hPa ($h = 11,769$ m) and the corresponding analytic functions that result from the regression analysis.

Fig. 5 Great circle distance path (dashed line) and approximated optimal free-flight path (solid line).

$$r^{\text{CanCon}} + FR^{\text{GanOc}} + CR^{\text{GanOc}} + FR^{\text{ShanOc}} + CR^{\text{ShanOc}} + \sum_i r_i^{\text{Eur}}$$

where r_i makes reference to the i th relevant European FIR/UIR. The components of this cost have been defined in Sec. IV.D. Notice that, because the flight departs from John F. Kennedy International Airport, the United States does not apply any navigation fee. The unit rates employed in the European regions are those corresponding to European regions' adjusted unit rates applicable to April 2012 flights.^{††} For continental Canada and oceanic regions, the rates are those given in Sec. IV.D. All rates have been converted to the Euro (€). It has been assumed that 1 kg of fuel costs €1.

D. Results

The computed sequence of waypoints, denoted by the active set of binary variables $v^{q,j}$, is given in Table 2. The corresponding route is: YAHOO, DOVEY, VODOR, RIVAK, PEPET, BMC11, RATRA, XIRBI, LBN32, AMTEL.

The approximated optimal path has been depicted in Fig. 6, in which the dots represent the computed discrete samples. The switching and final times of the approximated optimal solution are given in Table 3 together with the accumulated consumed fuel at the end of each leg and the overflying costs for each leg. The approximated optimal evolution of both state and control variables within the time domain are represented in Fig. 7, in which the dots represent the computed discrete samples and the vertical lines correspond to the switching times.

E. Discussion of the Results

The MINLP model used to solve this problem had 1730 variables (40 of them being integer variables) and 1672 constraints. The MINLP solver took 4726 iterations and 30 nodes, with a maximum depth of six on tree, to find a solution whose objective value was €32,778.48.

The velocity of the aircraft of the approximated optimal solution represented in Fig. 7a shows a nonsteady behavior in which it slightly decreases as fuel is burnt. Nonsteady flight performances are interesting in defining more efficient flight profiles within a future TBO concept of operations. For instance, varying velocity profiles has been shown to save fuel [38,39], but could also serve to absorb ground delays on air or for defining a more efficient ATFM [40].

The heading angle of the aircraft of the approximated optimal solution represented in Fig. 7b is also nonsteady. Instead of flying segments with a constant course, the aircraft flies with a slightly modified course by acting on ailerons and rudder so that the bank angle is in general nonzero. These maneuvers can be also observed in the evolution of the bank angle reported in Fig. 7f.

The trajectory planning problem can be solved on the ground, as part of the strategic planning, or during the tactical planning to redefine the trajectory. In both cases, a reduced computational time is a key factor to deliver flight plans or to agree to modifications of planned trajectories with ATC. Therefore, as explained in Sec. III.B, a special effort has been spent on reducing the computational time and two different strategies have been combined to achieve this goal.

On the one hand, a fifth degree Gauss–Lobatto integration rule has been used with 72 intervals. The computation time was 598.69 s on a Mac OS X 2.56 GHz laptop computer with 4 GB RAM. To give a quantitative measure of the computational time reduction that can be achieved with this integration rule, it is worth mentioning that the resolution of the MINLP problem discretized using a Hermite–Simpson collocation method with 290 intervals took 2987 s on the same computer, and the same approximated optimal solution was obtained. These results are congruent with the analysis given in [18].

On the other hand, an efficient heuristic has been implemented. As exposed in Sec. III.C, this heuristic approach has two main steps. First, the integer values are relaxed to the continuous domain $[0, 1]$, and the resulting NLP subproblems are solved to local optimality. Then, using a branch-and-bound framework, a solution that satisfies the integer requirements is sought. In the branching process, the branching rules can be seen as heuristics aimed at reducing the size of the search tree (i.e., some regions in which no good integer-feasible solution is expected are discarded, but there is no theoretical guarantee that supports the choice).

As pointed out before, there exists exact MINLP solvers for problems such as the one presented in this paper, although they are typically limited to problems of medium difficulty. To show that exact solvers are not adequate for the problem at hand, the state-of-the-art solver Couenne [22] was tested. It was not able to compute any feasible trajectory. It is worth stressing again that the difficulty in solving exactly the aircraft trajectory optimization problem lies not only in its size but also in its highly nonlinear and nonconvex nature. To the best of our knowledge, there exists no practical method to compute the optimal solution of this problem. Therefore, assessments on the quality of the solution can only be heuristic. To this purpose, three different tests have been carried out and will be described next.

First, the problem presented in this paper has $5^8 = 390,625$ feasible solutions for the binary variables. Because it is impractical to solve all the corresponding NLP subproblems and compare the obtained values of the objective function with the solution found by BONMIN, a sample of feasible values for the binary variables have been selected. This sample has been generated based on slight modifications of BONMIN's computed sequence of waypoints, in which one of the BONMIN's computed waypoints is permuted by another waypoint within its partite set, whereas the remaining waypoints are unchanged. This sampling results in 32 integer-feasible NLP subproblems, which have been solved using IPOPT and have been compared to BONMIN's computed solution. Results show that BONMIN's computed solution is always the best.

Second, the sensitivity of the MINLP algorithm to costs due to overflying charges has been analyzed. For this purpose, a parameter $\delta \in [0, 1]$ that multiplies the Mayer term in the objective functional

Table 2 Switching sequence

$v^{q,j}$	$v^{q,1}$	$v^{q,2}$	$v^{q,3}$	$v^{q,4}$	$v^{q,5}$
$v^{1,j}$	1	0	0	0	0
$v^{2,j}$	1	0	0	0	0
$v^{3,j}$	1	0	0	0	0
$v^{4,j}$	0	0	0	0	1
$v^{5,j}$	0	1	0	0	0
$v^{6,j}$	1	0	0	0	0
$v^{7,j}$	0	1	0	0	0
$v^{8,j}$	0	0	1	0	0

^{††}Santa Maria: €9.79; United Kingdom France: €83.23; Spain (continent): €71.84; France: €64.63; Italy: €78.69; Portugal (Lisbon UIR): €33.06.

Fig. 6 Approximated optimal path: The dots correspond to the computed samples. The triangles correspond to the waypoints of set P .

Fig. 7 State and control variables of the approximated optimal solution.

(2a) has been considered. In this way, $\delta = 1$ corresponds to the problem in which overflying charges are considered, whereas $\delta = 0$ corresponds to the problem considering only fuel consumption cost. Table 4 shows the performances of the algorithm for different values

Table 3 Switching times, fuel consumption, and overflying costs

Switching times, s	Accumulate consumption, kg	Overflying costs, €
$\bar{t}^1 = 951.0$	1,192.8	0
$\bar{t}^2 = 6,064.3$	7,995.1	0
$\bar{t}^3 = 18,407.7$	23,150.3	218.2
$\bar{t}^4 = 19,720.1$	24,687.4	447.7
$\bar{t}^5 = 20,671.4$	25,794.2	326.0
$\bar{t}^6 = 21,388.5$	26,626.2	245.9
$\bar{t}^7 = 22,613.5$	28,042	413.0
$\bar{t}^8 = 23,533.4$	29,103.6	310.5
$\bar{t}^9 = 24,829$	30,417.1	399.4

of the parameter δ . It can be observed that the number of explored nodes decreases as the weight assigned to overflying costs increases, resulting in faster computation, and that the selected route is very sensitive to changes of the values of δ . The seven values of δ reported in Table 4 give rise to four different routes A, B, C, and D, listed in Table 5.

Another test has been conducted to establish if the algorithm provides the most efficient route for different values of the parameter δ . In Table 6, the values of the objective function corresponding to different routes and different values of δ are reported. It can be seen that the algorithm always selected the most efficient route for each value of δ as shown in bold in Table 6.

Third, the sensitivity of the algorithm to the initial guess of both continuous and discrete variables has been analyzed. To generate the initial guess of the continuous variables of each NLP subproblem, using information on the physical system is important because a bad initial guess might lead to nonconvergence of the NLP problem. In general, common flight performances are sufficient to create a

Table 4 Sensitivity of the algorithm to changes of the overflying costs

δ	Iterations	Nodes	$t_{\text{comp}}, \text{ s}$	Objective function, €	Route	Consumption, kg	Overflying costs, €
0	21,723	401	1,835.67	30,014.8	D	30,014.8	— —
0.05	16,553	258	1,505.15	30,390.38	C	30,015.7	374.68
0.1	9,147	58	953.09	30,647.45	B	30,410.1	237.35
0.25	7,602	49	788.80	31,002.87	B	30,410.1	592.77
0.5	6,095	36	633.11	31,595.23	B	30,410.1	1,185.1
0.75	5,873	37	607.07	32,187.6	B	30,410.1	1,777.5
1	4,726	30	598.69	32,778.48	A	30,417.1	2,361.38

Table 5 Routes A, B, C, and D

Route	Waypoints									
A	YAHOO	DOVEY	VODOR	RIVAK	PEPET	BMC11	RATRA	XIRBI	LBN32	AMTEL
B	YAHOO	DOVEY	VODOR	RIVAK	ERWAN	BMC11	RATRA	XIRBI	LBN32	AMTEL
C	YAHOO	KANNI	RONPO	LAPEX	KOLEK	MAREN	AURILLAC NDB	XIRBI	LBN32	AMTEL
D	YAHOO	KANNI	NOVEP	LAPEX	KOLEK	MAREN	AURILLAC NDB	XIRBI	LBN32	AMTEL

Table 6 Values of the objective function € for different routes

δ	Route A	Route B	Route C	Route D
0	30,417.6	30,410.5	30,016.1	30,014.8
0.05	30,535.6	30,528.9	30,390.38	30,690.86
0.75	32,188.26	32,187.6	35,629.9	36,047.5
1	32,778.48	32,779.96	37,501.21	37,960.65

general initial guess that leads to relatively rapid and robust convergence. Such an initial guess for the NLP subproblems contributes to reduce the overall computational time of the MINLP algorithm because a large amount of NLP subproblems must be solved in the branch-and-bound algorithm. Moreover, having a good initial guess in terms of convergence is important in the branch-and-bound algorithm because a failure in solving any of the NLP subproblems might lead to rejection of a branch of the search tree that contains the optimal solution. It is worth pointing out that every single NLP subproblem computed in the different tests, whose results are reported in Table 4, resulted in a feasible solution and that the interior point strategy implemented in IPOPT to deal with infeasible initial guesses has contributed to this result.

In regard to the initial guesses for the binary variables, as said before, the first step of the branch-and-bound algorithm is to solve the relaxed MINLP problem, which is obtained by replacing the integrality constraint $v^q \in \{0, 1\}^{n_v^q}$ by $v^q \in [0, 1]^{n_v^q}$, $q = 1, \dots, N - 1$. Therefore, the initial guesses for the variables v^q , $q = 1, \dots, N - 1$ are actually vectors of real numbers in the interval $[0, 1]$. Relaxing the integrity of the variables v^q corresponds to a request that, at the switching times, the aircraft overflies points that are convex combinations of the waypoints $p^{q,j}$, $q = 1, \dots, N - 1$, $j = 1, \dots, n_{v^q}$. In the case in which overfly costs are not considered, the best option is to set the initial values so that its convex combination results in the best achievable solution to the relaxed problem. Using these initial guesses, the results presented in Table 4 for $\delta = 0$ have been obtained. Using different initial guesses for the variables v^q , $q = 1, \dots, N - 1$, the algorithm converged to the same solution but with higher computation time.

In the case in which overflying costs are considered (i.e., $\delta \neq 0$), it is not easy to calculate the best achievable solution to the relaxed problem, and thus applying the aforementioned strategy to compute the initial guesses for the variables v^q , $q = 1, \dots, N - 1$ is difficult. Therefore, the previously described procedure to compute initial guesses for $\delta = 0$ has also been used for $\delta = 1$ and the algorithm converged with an objective function value of €32,779.13. To check the quality of this strategy, 25 different initial guesses have been generated and tested. The algorithm converged 19 times with a slightly lower value of the objective function €32,778.48, three times with €32,779.13, and three times with the value €37,497.92 of the

objective function. This shows that, if overflying costs are considered, the algorithm is more sensitive to the initial guesses for the binary variables.

VI. Conclusions

In this paper, an effective approach to the solution of the aircraft trajectory optimization problem using a multiphase mixed-integer optimal control technique has been presented. The effectiveness of this approach has been proven by solving a realistic flight planning problem for a commercial aircraft, in which the used mixed-integer nonlinear programming solver gives a heuristic solution in time frames compatible not only with strategic but also with tactical planning. Moreover, the heuristic solution has been shown to be accurate. It can be concluded that the combination of integer and continuous variables has a strong potentiality in flight planning.

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