This document is published in:

AIAA Guidance, Navigation, and Control Conference. 2-5 August 2010, Toronto, Ontario Canada (2010). AIAA Meeting Papers.

DOI: <http://dx.doi.org/10.2514/6.2010-8453>

© 2010 American Institute of Aeronautics and Astronautics

Hybrid Optimal Control Approach to Commercial Aircrafts 3D Multiphase Trajectory Optimization

M. Soler ^{*}, A. Olivares [†] and E. Staffetti [‡]

Universidad Rey Juan Carlos, Móstoles, Madrid, 28933, Spain

^{*}Ph.D. Candidate, Department of Statistics and Operations Research, Móstoles, Madrid, 28933, Spain, Student AIAA Member.

[†]Associate Professor, Department of Statistics and Operations Research, Móstoles, Madrid, 28933, Spain.

[‡]Associate Professor, Department of Statistics and Operations Research, Móstoles, Madrid, 28933, Spain.

Given the sequence of phases and flight modes conforming the flight profile of a commercial aircraft, the initial and final states, a set of path constraints and real wind forecast data, we solve the multiphase problem of finding optimal control inputs, switching times between flight modes and the corresponding trajectory of the aircraft that minimizes fuel consumption. The aircraft in flight is modelled as a hybrid dynamical system, i.e., a system that has continuous and discrete dynamics, where the distinct discrete dynamics corresponds to different flight phases and switches between them occur either in response to control law or when the state of the system reaches prescribed regions of the state space. The three dimensional motion of the aircraft over a spherical earth is described by a point variable-mass dynamic model. The hybrid optimal control problem is converted into a conventional optimal control problem by a parameterization of the switching instants and solved using a collocation method. This approach provides an overall optimal solution for a complete flight including the optimal switching instants between phases. An application to a realistic 13-phase A-340-300 flight is solved and discussed.

Nomenclature

$a_{l,max(civ)}$	maximum longitudinal acceleration for civil flights
$a_{n,max(civ)}$	maximum normal acceleration for civil flights
C_D	coefficient of drag = $CD_0 + KC_L^2$
CD_0	parasite drag coefficient
C_L	coefficient of lift
CL_{max}	maximum lift coefficient
$C_{Tc,4}$	first thrust temperature coefficient
CV_{min}	minimum speed coefficient
D	drag force = $\frac{1}{2}\rho V^2 SCD$
G_t	temperature gradient on maximum altitude
G_W	mass gradient on maximum altitude
g	acceleration due to gravity
h	altitude
h_{M_0}	maximum operational height
h_{max}	maximum altitude at MTOW under ISA conditions for maximum mass
h_u	maximum altitude
K	induced drag coefficient
L	lift force = $\frac{1}{2}\rho V^2 SCL$
M	Mach number
m	mass
m_{max}	maximum mass (Maximum Take Off Weight)
m_{min}	minimum mass (Operating Empty Weight)
$\dot{m}_{min}$	minimum fuel flow
r	earth radius
T	thrust
T_{max}	maximum thrust
V	true airspeed
V_{CAS}	calibrated air speed
V_{M_0}	maximum operational calibrated airspeed
V_{stall}	stall speed

$V_{w\lambda}$	wind velocity West-East referred to the earth, fixed reference frame
$V_{w\theta}$	wind velocity South-North referred to the earth, fixed reference frame
α	angle of attack
ΔT_{ISA}	temperature deviation from ISA
η	thrust specific fuel flow
χ	heading angle
γ	flight path angle
λ	longitude
ϕ	bank angle
$\phi_{max,civ}$	maximum bank angle for civil flights
θ	latitude

I. Introduction

Air Traffic Management (ATM) is nowadays a very complex and highly regulated system that encompasses air traffic flow management, air traffic control and airspace management. A substantial change in the current ATM paradigm is needed because this system, responsible for sustainable, efficient and safe operation in civil aviation, is reaching the limits of its capabilities. Its capacity, efficiency, environmental impact and flexibility should be improved to accommodate airspace users requirements[§]. For instance, only in Europe the so-called route inefficiencies generated the annual cost of 2.6 B Euros ¶ in 2008. These inefficiencies are caused by many restrictions currently in use. The need to fit aircraft trajectories to ATM system requirements makes them difficult to be optimized and, therefore, generally suboptimal flight profiles are being flown. This results in higher operative costs and higher emissions due to non minimal fuel consumption. Air traffic is responsible for 2% of CO_2 global emissions. Thus, minimizing both fuel consumption and emissions are interesting and challenging research subjects. In this paper we focus on commercial aircraft minimum-fuel trajectory optimization towards a future, more flexible ATM concept of operation, meeting safety, efficiency, cost effectiveness and environmental sustainability requirements in flight profiles.

With this aim, this paper presents an approach to the problem of finding optimal aircraft trajectories where different flight phases and operational procedures can be combined so that a single optimal control problem is solved. The coupling of these discrete flight phases with the continuous aircraft dynamics results in a hybrid system. Some works that model aircraft flights as hybrid systems are^{1,2,3} and⁴. General hybrid systems models include both switches between discrete dynamics and jumps of state variables⁵. In particular, since the state is not subjected to discontinuous jumps, the flight of an aircraft has intrinsically the characteristics of a controlled switched dynamical system, i.e., a hybrid system in which discontinuous jumps of the state are not included in the model^{6,7}. Switches between flight modes can be autonomous or controlled. Autonomous switches take place when the continuous state hits prescribed regions of the state space, e.g., when the aircraft reaches a prescribed altitude, whereas controlled switches occur in response to a control input⁵.

Hybrid control systems involve both continuous and discrete state and control variables. In general, finding all components of the solution to hybrid optimal control problems is hard because the optimal sequence of discrete states is very difficult to determine. In our case, the phase sequence is given, but optimal switching times must be determined. Problems with known phase sequence have been frequently solved in aerospace engineering as multi-phase problems^{8,9,10,11,12}, most of them solved using pseudo-spectral methods,^{13,14,15,16,17} with different software,¹⁸ and¹⁹. However, none of these applications focused on commercial aircrafts. Furthermore, such multiphase problems need interior-point constraints and linkage constraints to enforce continuity across different phases. In this approach we convert the hybrid optimal control into an equivalent optimal control problem making the unknown switching times part of the state using a method similar to those presented in^{20,21}, so a conventional optimal control is stated and, thus, we do not need to enforce continuity with linkage constraints. The resulting optimal control problem is then solved numerically. There are many techniques for solving trajectory optimization, classified as either indirect or direct methods^{22,23}. Our formulation allows the use of direct transcription methods, such as^{24,25,26} and²⁷.

The fundamental prior research work on commercial aircraft trajectory optimization within the current ATM concept was presented in²⁸, where the flight of an aircraft was modeled as a collection of phases and procedures. A sequence of minimum-fuel consumption problems where continuity of the state variables was imposed in order to link phases was solved, permitting discontinuities in control variables and flight path angle. Together with the presence of these discontinuities, combining phases in this way is computationally inefficient and tedious. Moreover, concatenating optimal phase-by-phase solutions does not lead to an overall optimal trajectory. Recently, in²⁹, a method to compute overall optimal trajectories has been presented. This work was carried out in consistence with the current ATM concept, and a gradient algorithm was used to solve the problem. This algorithm does not allow switching times

[§] SESAR ATM Master Plan: <http://www.eurocontrol.int/sesar>

[¶] Data source: Eurocontrol Performance Review Commission 2008 Report

to be calculated. Additionally, neither a control law nor the evolution of the states within time history were reported.

Many works not based on the current ATM concept focused on minimizing fuel consumption only during aircraft cruise, e.g.,^{30, 31} and³². In³² the interplay between aircraft mass and speed at constant altitude was studied to minimize fuel consumption, and important fuel savings were obtained when compared to steady cruise. However, these results do not consider the fact that cruise is just one of the different phases of a complete flight, and therefore, they are not realistic because the optimal solution could significantly differ if we take into account the influence of previous and following phases.

Modeling the actual ATM paradigm, as in²⁹, enforces to specify two operative procedures per phase, for instance, to climb with constant V_{CAS} and constant throttle setting, or to perform a steady cruise, i.e., with constant Mach and constant altitude. On the contrary, we only set one operative procedure per phase, e.g., to climb with constant V_{CAS} , or to cruise at constant altitude as in³². This assumption increases the complexity of aircraft dynamic equations, but also gives more room for planning more efficient flight profiles.

Such hybrid optimal control approach provides an overall optimal solution for a realistic 13-phase complete flight pursuing a more efficient ATM concept. Hybrid theory and formulation includes optimal switching times between phases as part of the solution. Such an approach was first presented in³³ with an application to a vertical 7-phase profile. In³⁴, vertical free flights, vertical typical flights and vertical optimized procedure flights were compared and analyzed, obtaining fuel savings of 10.9% for short range profiles and 4.6% for medium range profiles when comparing typical flights with optimized flights. In this paper, an application to trajectory optimization of a three dimensional point mass model over a spherical earth of an Airbus A-340-300 is reported. Real wind data are considered in the model. The results show the accuracy and effectiveness of our approach.

This paper is organized as follows. In Section II., a brief overview of hybrid systems is given. In Section III., the formulation of the hybrid optimal control problem is presented. The hybrid optimal control problem is then reformulated into an equivalent optimal control problem in Section IV.. In Section V., we define the dynamic model of the aircraft, and in Section VI. an instance of a minimum-fuel consumption control problem is stated. The results are showed in Section VII.. Finally, in Section VIII. some conclusion are drawn.

II. Hybrid Systems

As previously stated, hybrid systems are dynamical systems that involve both continuous and discrete state variables. The evolution of the continuous state is described by a set of differential equations, whereas the evolution of the discrete state is usually modeled as a finite state machine. Thus, a hybrid dynamical system can be considered as a set of interacting dynamical systems, each evolving in different continuous state spaces, in which switching among dynamical systems occur when the state variable hits specified subsets of the composing state spaces. This phenomenon is called autonomous switching in⁵. It has to be noticed that the state space of each dynamic systems may be different. Another phenomenon considered in⁵ is due to the presence of autonomous jumps which take place when the continuous state changes discontinuously as certain regions of the state space are reached.

Hybrid control systems are control systems that involve both continuous and discrete state variables and continuous and discrete control variables. A control system for a hybrid dynamical system is composed by a collection of controllers for the continuous states of each constituent dynamical system. Besides issuing continuous controls, they are also able to switch the dynamic equation of the continuous state variable as well as to reset the state when it satisfies certain conditions. So, the phenomena that must be included in mathematical models of controlled hybrid dynamical systems are controlled switches and controlled jumps⁵. Controlled switches are changes in the dynamic equation that occur in response to control commands. Controlled jumps are changes in the state in response to a certain control input. In general, both controlled switches and controlled jumps have an associated cost. Hybrid dynamical systems whose state does not exhibit state jumps at intervention times are called switched dynamical systems.

The first work on optimal control of hybrid systems is³⁵. Other papers on this topic are^{36, 37, 38} and³⁹. In⁷ a general model of a switched system including both controlled and autonomous switches was first introduced. Then, two different classes of optimal control problems were formulated. Some relevant theoretical results were reviewed and a two stage optimization method was presented.

In⁶ and²⁰, a two-stage approach for solving optimal control problems of switched systems was presented, where the number of switches and the sequence of active subsystems was given. To solve these problems, it was necessary to seek both optimal switching instants and optimal piecewise continuous inputs. In⁶ the method was based on direct differentiations of value functions and nonlinear optimization techniques were used. In²⁰, in order to determine the optimal switching instants, the derivatives of the optimal cost with respect to the switching instants needed to be computed. A method that first transcribes an optimal control problem into an equivalent problem parameterized by the switching instants was presented, where the values of the derivatives were obtained based on the solution of a two point boundary value differential algebraic equation. To solve motion planning problems for robotics systems, a reformulation similar to the one presented in²⁰ was proposed in²¹.

III. Hybrid Optimal Control Problem Formulation

The flight of an aircraft intrinsically has the characteristics of a controlled switched dynamical system. Indeed, several flight modes can be distinguished for climb, cruise and descent, each with an associated dynamic model and a set of path constraints. The switch between flight modes can be autonomous or controlled. For instance, an autonomous switch between two phases may occur when the aircraft reaches a prescribed altitude. On the contrary, a controlled switch takes place in response to control inputs established by the solution to the optimal control problem. Trajectory planning for an aircraft can be solved as an optimal control of a switched dynamical system with autonomous and controlled switches.

A switched dynamical system is composed of a set of dynamic systems,

$$\dot{x} = f_k[x(t), u(t), t], k \in \{1, 2, \dots, N_D\}, \quad (1)$$

where x represent the n -dimensional state vector, and the set $\{1, 2, \dots, N_D\}$ represents the different dynamical systems, and a switching sequence in $[t_I, t_F]$, defined as the timed sequence of $N + 1$ active dynamical systems,

$$\sigma = [(t_I, k_I), (t_1, k_1), \dots, (t_N, k_N)], \quad (2)$$

where $0 \leq N < \infty$, $t_I \leq t_1 \leq \dots \leq t_N \leq t_F$ and $k_j \in \{1, 2, \dots, N_D\}$.

In order to control a switched dynamical system, both a m -dimensional control input, $u(t)$, and a switching sequence, σ , have to be specified. We suppose that the set of admissible control inputs is a set of piecewise continuous functions in $t \in [t_I, t_F]$. In this sequence, the pair (t_j, k_j) indicates that at time t_j the dynamic equation of the switched system changes from k_{j-1} to k_j , being $t_0 = t_I$ and $t_{N+1} = t_F$. As a consequence, in the time interval $[t_j, t_{j+1})$ the system evolution is governed by the dynamic equation k_j . In the interval $[t_N, t_F]$ the active dynamic system is k_N .

The pairs (t_j, k_j) can be classified in two categories: those corresponding to autonomous switches; and those corresponding to controlled switches. For instance, an autonomous switch may occur when the aircraft reaches a prescribed altitude. On the contrary, a controlled switch takes place in response to control inputs established by the solution to the optimal control problem. In this work, we assume that the sequence of phases is given by a flight profile, i.e., the untimed sequence of active systems $\zeta = (k_I, k_1, \dots, k_N)$ is known.

The hybrid optimal control problem can be stated as follows: Consider the switched dynamic system (1) whose state and control variables are subjected to a set of equality and inequality constraints

$$g_k[x(t), u(t), t] = 0, h_k[x(t), u(t), t] \leq 0, k \in \{1, 2, \dots, N_D\}. \quad (3)$$

Given an initial state, $x(t_I)$, a final state, $x(t_F)$, a time interval, $[t_I, t_F]$, and a prescribed untimed sequence of active dynamic systems, ζ , find a piecewise continuous input, $u(t)$, the switching instants, $(t_1, \dots, t_N)$, and the corresponding piecewise smooth trajectory, $x(t)$, between $x(t_I)$ and $x(t_F)$ that fulfill (1), (3), and minimize

$$J = \phi[x(t_F)] + \int_{t_I}^{t_F} L[x(t), u(t), t] dt. \quad (4)$$

The final time, t_F , may be fixed or left free. We assume that f_k, g_k, h_k and ϕ are smooth enough functions.

IV. Problem Reformulation

Using a method similar to²⁰ and²¹, this hybrid optimal control problem is converted into an optimal control problem making the unknown switching times part of the state and introducing a new independent variable to which the switching times are fixed. In this reformulated problem, there is a linear relation between the new variable and time, but the slope of this linear relation changes on each interval between two switches. These slopes, which are part of the solution to the optimal control problem, are actually time scaling factors that determine the optimal switching times. The resulting optimal control problem is then solved numerically.

As previously stated, the number of switches, N , and the sequence of discrete states, ζ , in which the system evolves through, are known. Without loss of generality, we can assume that $t_I = t_0 = 0$ and $t_F = t_{N+1} = 1$. The first step is to introduce the new state variables, $x_{n+1}, \dots, x_{n+N}$, corresponding to the switching times, t_i

$$\begin{aligned} x_{n+i} &= t_i, \\ \dot{x}_{n+i} &= 0. \end{aligned}$$

We then introduce the new independent variable, τ . The relation between τ and t changes on each interval $[t_i, t_{i+1}]$. We establish piecewise linear correspondence between time, t , and the new independent variable, τ , so that for every

Figure 1. Relation between scaled time, τ , and real (unscaled) time t .

chosen fixed point, $\tau_i, i = 1, \dots, N+1$, t equals t_i . Any monotonically increasing sequence of N numbers on interval $[0, 1]$ could be used. We set $\tau_i = i/(N+1)$ (see Figure 1). As a result we obtain the following expression

$$t = \begin{cases} (N+1)x_{n+1}\tau, & 0 \leq \tau \leq \frac{1}{N+1} \\ \dots & \\ (N+1)(x_{n+i+1} - x_{n+i})\tau + (i+1)x_{n+i} - ix_{n+i+1}, & \frac{i}{N+1} < \tau \leq \frac{i+1}{N+1} \\ \dots & \\ (N+1)(1 - x_{n+N})\tau + (N+1)x_{n+N} - N, & \frac{N}{N+1} < \tau \leq 1 \end{cases}$$

By introducing the new independent variable, τ , the evolution equation on the interval $[t_i, t_{i+1}]$ given by Equation (1) becomes

$$x' = (N+1)(x_{n+i+1} - x_{n+i})\hat{f}_i(x, u, \tau), \quad (5)$$

where $(\cdot)'$ denotes the derivative of $(\cdot)$ with respect to the new independent variable, τ , and

$$\hat{f}_i(x, u, \tau) = f_i(x, u, t(\tau)).$$

let $\hat{x}$ be the extended state vector

$$\hat{x} = [x_1, \dots, x_n, x_{n+1}, \dots, x_{n+N}]^T,$$

Then, define on each interval $\frac{i}{N+1} < \tau \leq \frac{i+1}{N+1}$

$$\hat{L}(\hat{x}, u, \tau) = (N+1)(x_{n+i+1} - x_{n+i})L(x, u, t(\tau)).$$

We can rewrite the functional (4) as

$$J = \psi(\hat{x}(1)) + \int_0^{\frac{1}{N+1}} \hat{L}(\hat{x}, u, \tau) d\tau + \dots + \int_{\frac{N}{N+1}}^1 \hat{L}(\hat{x}, u, \tau) d\tau = \psi(\hat{x}(1)) + \int_0^1 \hat{L}(\hat{x}, u, \tau) d\tau, \quad (6)$$

and the task is to minimize J in the extended state space, subject to the parametrized system given in (5), and to the corresponding path constraints in (3).

It has to be noticed that points τ_i at which the system described by the function $\hat{f}$ switches between the discrete states are known. In the optimal solution, $\hat{x}^*$, the last N components will be the optimal switching times $t_i, i =$

$1, \dots, N$. The new equivalent problem is a conventional optimal control problem, that is, without varying switching instants. This new problem can be regarded as an optimal control problem parametrized by the switching instants x_{n+i} . This reformulation allows stating the optimal control problem either in Mayer form, in Lagrange form as a multiobjective problem, or in Bolza form, by combining both. Moreover, such a reformulation includes the switching costs $\delta[x(t_k), i_{k-1}, i_k]$.

V. Aircraft Equations of Motion

In the scope of aircraft trajectory optimization it is commonly considered a 3 Degree Of Freedom (DOF) dynamic model that describes the three dimensional variable-mass motion of the aircraft over a spherical earth model. A standard atmosphere is defined with $\Delta T_{ISA} = 0$. C_L is, in general, a function of the angle of attack and the Mach number, i.e., $C_L = C_L(\alpha, M)$. As a convention, the lift coefficient is used as a variable rather than the angle of attack. We assume a parabolic drag polar, i.e., $C_D = C_{D_0} + KC_L^2$. BADA 3.6^{||} is used as aircraft performance models. In order to faithfully represent the aircraft performance, this paper includes a realistic wind model in the equations of motion based on the NOAA^{**} forecasts through GRIB data.

These hypotheses lead to the following set of Ordinary Differential Equations (ODE) system for aircraft performance:

$$\begin{aligned} m(\dot{V} + \dot{V}_{w\lambda} \cos \gamma \cos \chi + \dot{V}_{w\theta} \cos \gamma \sin \chi) &= T - D - mg \sin \gamma, \\ m(V\dot{\gamma} + \dot{V}_{w\lambda} \sin \gamma \cos \chi + \dot{V}_{w\theta} \sin \gamma \sin \chi) &= L \cos \phi - mg \cos \gamma, \\ m(V\dot{\chi} \cos \gamma - \dot{V}_{w\lambda} \sin \chi + \dot{V}_{w\theta} \cos \chi) &= L \sin \phi, \\ \dot{\lambda} &= \frac{V \cos \gamma \cos \chi + V_{w\lambda}}{r \cos \theta}, \\ \dot{\theta} &= \frac{V \cos \gamma \sin \chi + V_{w\theta}}{r}, \\ \dot{h} &= V \sin \gamma, \\ \dot{m} &= -\eta T. \end{aligned} \tag{7}$$

This paper presents an aircraft dynamic model consisting of 7 equations and 10 variables ($T, C_L, \phi, V, \lambda, \theta, h, \gamma, \chi, m$), where T, C_L and ϕ are control inputs and $V, \lambda, \theta, h, \gamma, \chi, m$ state variables.

Several flight phases, aerodynamic configurations and flight procedures can be distinguished in commercial flights. The hybrid dynamic model is the ideal framework to bring together all these elements and to solve a single optimal control problem. Although many phases are generally defined in a commercial flight profile, without loss of generality, thirteen main flight phases could be distinguished: Take Off, Initial Climbing, Free Climb (associated to Standard Instrument Departure (SID) procedure), Accelerating (associated to velocity restrictions), Climb low (constant V_{cas}), Climb High (constant Mach), Cruise, Descent High (constant Mach), Descent low (constant V_{cas}), Decelerating (associated to velocity restrictions), Free Descent (associated to and Standard Terminal Arrival Route (STAR) procedure), Approach, and Landing. The aerodynamics of every phase may differ and depends on flap configuration. Moreover, different flight procedures can be identified for each phase. Therefore, in order to model a typical commercial aircraft flight, in this paper we adopt a 13-phase hybrid model with different flight procedures and aerodynamic configurations.

VI. Case study

Given a commercial flight profile as a sequence of phases, the initial and final conditions, a set of path constraints and real wind forecast data, our goal is to find the minimum fuel consumption trajectory of the aircraft. The optimal switching times between phases and the total flight time are also to be determined.

The 13-phase flight profile used for the numerical simulation with its corresponding aerodynamic configurations and the operative procedures are given in Table 1. An Airbus A-340-300 BADA model is used for simulation. The different A-340-300 aerodynamic configurations and the value of the aerodynamic parameters are listed in Table 2. Finally, the boundary conditions of the state variables have been selected according to a realistic flight of an A-340-300 flying from JFK^{††} to LEMD^{‡‡}. These boundary conditions are expressed in Table 3.

The path constraints, (8), of the problem are those that conform aircraft's flight envelope and can also be consulted in BADA database manual⁴⁰. CL_{max_i} and V_{Stall_i} , $i = 1, \dots, 13$, vary depending on the aerodynamic configuration. The rest of constraints are the same for all phases:

^{||} <http://www.eurocontrol.int>

^{**} <http://www.noaa.gov/>

^{††} John F. Kennedy Airport

^{‡‡} Madrid Barajas Airport

Table 1. Flight Profile

Phase	Name	Configuration	End Trigger	Procedure	Op. Constraint
1	Take Off	TO	$V = 1.3V_{stall_{IC}}$	$\chi = \chi_i, T = T_{max}$	$V_{CAS} < 250kt$
2	Initial Climb	IC	$V = 1.3V_{stall_{CR}}$	$\chi = \chi_i, T = T_{max}$	$V_{CAS} < 250kt$
3	Res. Free Climb	CR	$h = 10000ft$	$T = T_{max}$	$V_{CAS} < 250kt$
4	Climb Accel	CR	-	$h = 10000ft$	
5	Climb CAS	CR	-	$V_{CAS} = c_1, \chi = c_2, T = T_{max}$	
6	Climb Mach	CR	-	$Mach = c_3, \chi = c_2, T = T_{max}$	
7	Cruise	CR	-	$h = c_4$	
8	Descent Mach	CR	-	$Mach = c_5, \chi = c_6$	
9	Descent CAS	CR	$h = 10000ft$	$V_{CAS} = c_7, \chi = c_6$	
10	Descent Decel	CR	-	$h = 10000ft$	
11	Res. Free Descent	CR	$h = 6000ft$	T_{min}	$V_{CAS} < 250kt$
12	Approach	AP	$h = 2000ft.$	$\gamma = c_8, \chi = c_9$	$V_{CAS} < 250kt$
13	Landing	LD	Final cond.	$\gamma = c_8, \chi = c_9$	$V_{CAS} < 250kt$

* Being $c_1, \dots, c_9$ constant values to be determined by the optimal solution of the problem.

Table 2. A-340-300 Aerodynamic Configurations

Configuration	Flap	$C_{L_{max}}$	CD_0	K
TO	1+F	3.02	0.043267	0.035018
IC	1	2.57	0.032091	0.037329
CR	Clean	1.74	0.0185	0.036
AP	2	3.25	0.053983	0.035524
LD	Full	3.45	0.076637	0.033017

Table 3. Boundary conditions

Boundary	Time	Altitude [ft]	λ [deg]	θ [deg]	γ [deg]	χ [deg]	V	Mass [Ton]
Initial	0	13	-73.77W	40.63N	4.5	90	$1.2V_{stall_{TO}}$	210
Final	Free	2000	-3.56W	40.47N	Free	Free	Free	Free

$$\begin{aligned}
0 &\leq h \leq \min[h_{M0}, h_u], \\
C_{V_{min}} V_{stall_i} &\leq V \leq V_{M0}, \\
M &\leq M_{M0}, \\
m_{min} &\leq m \leq m_{max}, \\
0 &\leq CL \leq CL_{max_i}, \\
T &\leq T_{max}, \\
\phi &\leq \phi_{max,civ} \\
\dot{V} &\leq a_{l,max(civ)} \\
\dot{\gamma} &\leq \frac{a_{n,max(civ)}}{V} \\
\dot{m}_{min} &\leq \dot{m},
\end{aligned} \tag{8}$$

where $h_u = h_{max} + G_t(\Delta T_{ISA} - C_{Tc,4}) + G_W(m_{max} - m)$, and $C_{V_{min}} = 1.2$. For the landing phase, γ has also been constrained between the typical values of an aircraft final descent path.

$$-6^\circ \leq \gamma_{Landing} \leq -3^\circ. \tag{9}$$

As mentioned above, wind conditions have been considered in aircraft dynamics due to its considerable effects in fuel consumption. Seeking a realistic model, herein we represent the wind function as a multiple regression of GRIB data provided by NOAA wind forecasts. In particular, we adjust these GRIB tabular data to analytical functions, so that both West-East component, $V_{w\lambda} = f(\lambda, \theta, h)$, and South-Noth component, $V_{w\theta} = f(\lambda, \theta, h)$, are functions of λ , θ and h . As the component perpendicular to earth, V_{wh} , is negligible, we have set it to zero. Such functions can be included in aircraft dynamics so the wind influence is taking into account by the dynamical system.

In this particular case, we have used the wind forecast of November the 4th, 2009. Once filtered to the area subject of study, the North-Atlantic, we converted those tabular data into analytical functions by means of multiple regression. A 4th degree polynomial is fitted to the data with the appropriate study of the residual and the regression coefficients statistical significance. The Goodness of fit, measured in terms of R-Squared parameter, yielded 0.782563 for $V_{w\lambda}$ and 0.651598 for $V_{w\theta}$. Real data and analytical functions are presented in Figure 2 for a cross section at 250 [Hpa] -h=10395 [m]-.

Figure 2. Wind data at 250 [Hpa] -h=10395 [m]-

The reformulated optimal control problem is stated as follows

$$\min J = \psi(\hat{x}(1)) + \int_0^{\frac{1}{N+1}} \hat{L}(\hat{x}, u, \tau) d\tau + \dots + \int_{\frac{N}{N+1}}^1 \hat{L}(\hat{x}, u, \tau) d\tau \quad (10)$$

Subject to

$$\dot{x}' = (N+1)(x_{n+i+1} - x_{n+i})\hat{f}_i(x, u, \tau), \text{ dynamic constraint}, \quad (11)$$

$$x'_{n+1} = \dots = x'_{n+N} = 0, \text{ switching dynamic constraint}, \quad (12)$$

$$\phi(t_I) = A, \text{ initial-boundary condition}, \quad (13)$$

$$\phi(t_F) = B, \text{ final-boundary condition}, \quad (14)$$

$$\phi_{l_i} \leq \phi_i[x, u] \leq \phi_{u_i}, \text{ path constraints}, \quad (15)$$

where $i = 1, \dots, 13$ corresponds to the sequence of phases of the flight given in Table 1 and $N = 12$ is the number of switches. Equation (11) corresponds to the aircraft ODE system given in (7), parametrized and particularized to the dynamic mode of each phase. Equation (12) represent the dynamics for the switching variables. Both together, (12) and (11), conform the ODE system for the extended vector $\hat{x}$. (13) and (14) are the initial and final conditions listed in Table 3. Finally, (15) correspond to the set of constraints of each phase given in (8) and (9).

The minimum-fuel consumption problem can be formulated in two different ways. The aircraft final mass can be maximized using Mayer term of the objective function (10). On the contrary, instantaneous consumption can be minimized using the Lagrange term of (10). Using Mayer term, (10) would become

$$\max J = m(1),$$

whereas employing Lagrange term, (10) would be expressed as

$$\min J = \int_0^{\frac{1}{N+1}} \hat{L}(\hat{x}, u, \tau) d\tau + \dots + \int_{\frac{N}{N+1}}^1 \hat{L}(\hat{x}, u, \tau) d\tau,$$

where $\hat{L}(\hat{x}, u, \tau)$ is the fuel flow, m' , of the aircraft. By using Lagrange term, although we focus on minimum-fuel consumption problems, our reformulation could be straightforwardly extended to solve multi-criterion optimization problems in which in each phase a different cost functional may be considered.

Referring the reader to Table 1, it can be observed that the profile is defined with some trigger conditions, the so called capture conditions for switching. When this occur, the switchings are claimed as autonomous, e.g., they occur when the aircraft reaches the respective altitudes or velocities. On the contrary, switches between phases with non-capture conditions are claimed as controlled switches since they are given by the control law within the optimal solution. As a consequence, the transition Mach, the cruising altitude or the constant calibrated speed of descent are not pre-fixed, but are set by the optimal solution, leading the system to the overall minimum fuel consumption. Such a proposed procedure profile is nothing more than the overall-best-fit adjustment of the free flight to the imposed procedures. Such en-route imposed procedures are derived from a better ATC and ATM performance instead of a pure en-route free flight.

A discretization grid has been defined for τ with $n = 650$ ($n_1 = n_2 = \dots = n_{13} = 50$). In order to solve the optimal control problem formulated in (10)-(15) a Simpson collocation method,²⁴ and²⁶, is used. The resulting non linear programming problem is solved using large scale non linear optimization problems solver IPOPT⁴¹

VII. Results

The reformulated optimal control problem stated in Section VI. has been solved in Lagrange form. In this section the results obtained are reported and discussed.

The values $x_{n+1}, \dots, x_{n+N}$ of the extended state vector, $\hat{x}$, which correspond to the switching instants $t_1, \dots, t_{12}$, are listed in Table 4. Those values are variables in the reformulated problem, remaining constant along each phase in terms of τ , and only corresponding with the real time, t , at the end of each phase.

Table 4. Results

Phase	Name	Switches, t_i , [s]	Consumption [Kg]	Value
1	Take Off	11.4	62.7	$\chi = \chi_i^\dagger$
2	Initial Climb	51.7	287.2	$\chi = \chi_i$
3	Res. Free Climb	282.9	1472	Free
4	Climb Accel	338.2	1725.5	$h = 10000ft$
5	Climb CAS	851.8	3656.4	$V_{CAS} = 169.03$ [m/s], $\chi = 28.73$ [deg]
6	Climb Mach	1816.3	5864.1	$Mach = 0.784$, $\chi = 28.73$ [deg]
7	Cruise	20109.6	35271.6	$h = 11880.1$
8	Descent Mach	20197.2	35312.3	$Mach = 0.705$, $\chi = -31.01$ [deg]
9	Descent CAS	21723.2	35948.6	$V_{CAS} = 116.87$ [m/s], $\chi = -31.01$ [deg]
10	Descent Decel	21723.2	35948.6	$h = 10000ft$
11	Res. Free Descent	21890.1	36036.5	Free
12	Approach	22102.3	361431	$\gamma = -3.2$, $\chi = -32.27$ [deg]
13	Landing	22246.4	36218.2	$\gamma = -3.2$, $\chi = -32.27$ [deg]

[†] Given by the initial condition.

Table 4 shows switching times between phases for every phase, the accumulated consumption at the end of every phase and the constant values that describe aircraft performance in the different flight procedures. For instance, in the seventh row of Table 4, the first two elements, 7 and Cruise, correspond to the index number and the name of the phase, respectively. The third element, 20109.6 [s], is the switching time between Phase 7 and Phase 8, i.e., the time in which the aircraft reaches the top of descent. The fourth element, 35271 [kg], corresponds to the accumulate consumption at the end of cruise phase. The last element, $h = 11880$ [m/s], represents the constant value of performance according to the flight profile given in Table 1. A similar interpretation holds for the other phases except for the last one, where the switching time, 22246.4 [s], is indeed the total flight time, and the consumption, 36218.2[kg], represents the total fuel consumption.

Flight optimal path, optimal control law, which leads the system to minimum-fuel consumption, and the evolution of the state variables are presented in Figures 3-7. Switching instants are also depicted in the figures. For the sake of clarity, both control inputs and state variables are represented with respect to scaled time, τ , and in terms of real time, t . Scaled time, τ , ranges from 0 to 1. Real time, t , ranges from initial time to final flight time. The instants in which the system shifts between phases are given by switching times and, therefore, the homogenous grid with respect to τ becomes a non-homogenous grid in terms of t .

Figure 3 shows the three dimensional flight path. It is important to point out the effects of wind in aircraft trajectory. The optimal solution seeks the most favorable wind conditions towards minimum consumption. As Figure 3.b

illustrates, the great circle, which would be the minimum distance path, is modified by the effects of wind.

Figure 3. Flight optimal path.

Figure 4. Altitude and maximum altitude, h_u

In Figures 4-7 the thirteen phases are clearly distinguishable in both timescales. Figures 4, 5 and 6 show the state variables. In general, except for the case of γ , all state variables vary smoothly. γ exhibit high-frequency dynamics in in some phases, but within reasonable values for aircraft performance. The behavior of control inputs -Figure 7- shows expected values, even though they also exhibit some oscillations.

Regarding aircraft performances, result exhibit as well high concordance. For a more extended, detailed information on vertical aircraft optimal performance, the reader is referred to³⁴. Since the 3D ascent and descent, except for SID and STAR procedures near airports (phases 3 and 11) and accelerating and decelerating (phases 4 and 10), can be model as vertical prof les, all issues stated in³⁴ for vertical free f ight ascent and descent optimal performance remains valid: The optimum vertical free f ight starts seeking to achieve maximum altitude in minimum time, since f ying at low altitudes with maximum thrust setting is very fuel consuming. As the aircraft gets higher, it progressively softens its rate of climb to make a smooth transition to the subsequent pseudo-cruise phase, in which the aircraft asymptoti-

Figure 5. Longitude, Latitude, heading angle and fight path angle.

Figure 6. True airspeed, Mach number and calibrated airspeed.

Figure 7. CL, thrust, bank angle and consumption

cally approaches its operating ceiling. The optimization target during cruise is to maximize the specific range, which is the distance traveled per unit of fuel consumed. As the aircraft mass decreases due to fuel burn, the optimum profile shows an increasing trend in altitude following the also-increasing operating ceiling, while speed is conveniently adjusted, typically in a slightly decreasing trend. Such a performance is known as continuous cruise climb. Cruise phase ends when the optimum descent path is intercepted. The optimum vertical descent path is the result of descending at maximum gradient speed, which allows the aircraft to fly the greatest distance possible at idle thrust, thus minimizing fuel consumption. This optimum speed is the minimum drag speed, also known as base speed. This speed decreases as air density increases, so the aircraft lowers its speed as it descends. The base speed also changes as the aircraft deploys high-lift devices, so the aircraft speed is adjusted by regulating the flight path angle when a change in aerodynamic configuration is to be performed. At the very end of the flight, when the only aim is to land at any flyable speed, the flight path angle is increased to cover the greatest distance possible before touching ground, consuming the speed excess above minimum speed.

The proposed procedure flight tries always to follow the patterns of free flight optimal performance. Consequently, it seems that the algorithm aims to minimize climb phase duration since it is very fuel consuming. But at the same time it needs to prevent negatively affecting the optimality of the subsequent flight phases. It performs take off and initial climbing with maximum throttle setting within the heading angle set by runways layout. Then it performs a turn while keeping climbing to head the aircraft to the desired course. After the acceleration phase at constant altitude, the algorithm selects the V_{cas}/Mach vertical ascent profile. The selected 328[kt]/0.78 climb speeds are slightly different to the typical speeds profile (300Kt/0.8; see BADA database), because it provides an increased rate of climb so that the aircraft can intercept a higher altitude to perform the cruise. It is in the cruising phase where the effects of wind play a fundamental role, the aircraft varies its minimum distance path seeking the best wind. It is also noticeable that including a couple of phases as step climbs during the cruise would let the aircraft fly nearer the ceiling constraint, which is indeed the optimal altitude. Eventually the aircraft intercepts the descending path, performing it at 0.7/226, considerable different than the typical 0.8/250[kt]. This is optimal thanks to the minimum drag due to CAS being almost constant along the descent, and the short duration of Mach phase. Finally, the constant flight path angle of initial and final approach, -3.2 [deg], is slightly more inclined than the typical one, -3 [deg].

It's remarkable that, though 13 phases are specified, the algorithm neglects or minimizes duration of those spurious ones. In the case of phase 10, Descent Deceleration, is dismissed by the optimal solution by setting the corresponding step to zero, i.e., setting equal switching times. Consequently, the algorithm leads us to a Continuous Descent Approach (CDA). In the case of phase 8, Descent Mach, the duration is minimized. As pointed out above, the optimal descent velocity is the base velocity, far from a constant Mach descent profile, and therefore a V_{cas} descent profile could be expected. However, the top of descent altitude is too high to start descending with the appropriate calibrated airspeed, so that the constant Mach phase is used to descent and decelerate until the optimal descent CAS velocity is intercepted.

According to these results, and taking into account the fuel savings achieved in³⁴, 10.9% for short range profiles and 4.6% for medium range profiles, is likely to claim that important fuel savings could be achieved for 3D flights when compared with current typical 3D flights.

The resulting sparse nonlinear programming problem, solved using IPOPT, had 5744 variables, 4944 equality constraints and 4150 inequality constraints. The total computational time on a 2.56 [GHz] laptop with 4 [GB] RAM was 4026.9 [s], showing a rather high computational efficiency of the method.

VIII. Conclusions

In this paper we addressed the problem of finding minimum-fuel consumption 3D realistic trajectories for commercial aircrafts using an approach based on hybrid optimal control theory, showing that gate to gate trajectory optimization could be address with such an approach. The results obtained using this approach show that overall optimization of a realistic, more efficient 13-phase flight profile can be carried out in the presence of real wind forecasts at a low computational cost, yielding reasonable values for both states and controls.

Hybrid optimal control lays the foundations of the optimization of aircraft trajectories as multi-objective optimal control problems by combining phases subject to highly restricted flight procedures, more flexible specific in-route flight procedures and even phases of free flight over non restricted areas. A first conclusion is that the algorithm showed robustness when dealing with spurious phases and was capable to neglect them, leading to CDA optimal descent performance. A further conclusion is that another concept of operation in Air Traffic Management is possible, planning more efficient flight profiles towards cost effectiveness, environmental sustainability and, at the same time, meeting safety requirements.

For further research, a more realistic model must be taken into account, specifically considering compressibility at high Mach numbers. Waypoints and overflight costs, emissions costs, or wind stochastic models could be some extensions of this work. Regarding conflict resolution, in the time being it seems more likely to obtain one-by-one gate to gate optimal paths and then address "ad hoc" conflict resolution at every sector.

Acknowledgements

This work is partially supported by the Spanish Government through the Ministerio de Ciencia e Innovación, the Comunidad de Madrid, and the project i-Math Ingenio Mathematica. This work has been carried out within the framework of the Atlantida project, partially funded by the Spanish Centro para el Desarrollo Tecnológico e Industrial, in which the Universidad Rey Juan Carlos is collaborating with GVM Aerospace and Defence.

References

- ¹J. Lygeros, S. Sastry, and C. Tomlin. The art of hybrid system. Compendium of Lecture Notes for the Hybrid Systems Class, 2002.
- ²I.M. Ross and C.N. D'Souza. Hybrid optimal control framework for mission planning. *Journal of Guidance Control and Dynamics*, 28(4):686, 2005.
- ³W. Glover and J. Lygeros. A stochastic hybrid model for air traffic control simulation. *Lecture Notes in Computer Science*, 2993:372–386, 2004.
- ⁴I. Lympieropoulos, A. Lecchini, W. Glover, J. Maciejowski, and J. Lygeros. A stochastic hybrid model for air traffic management processes. *University of Cambridge, Cambridge, CB2 1PZ, UK, Tech. Rep. CUED/F-INFENG/TR, 572*, 2007.
- ⁵M. S. Branicky, V. S. Borkar, and S. K. Mitter. A unified framework for hybrid control: Model and optimal control theory. *IEEE Transactions on Automatic Control*, 43(1):31–45, 1998.
- ⁶X. Xu and P. J. Antsaklis. Optimal control of switched systems via nonlinear optimization based on direct differentiations of value functions. *International Journal of Control*, 75(16/17):1406–1426, 2002.
- ⁷X. Xu and P. J. Antsaklis. Results and perspectives on computational methods for optimal control of switched systems. In *Hybrid Systems: Computation and Control*, pages 540–555. Springer, 2003.
- ⁸W. Roh and Y. Kim. Trajectory optimization for a multi-stage launch vehicle using time finite element and direct collocation methods. *Engineering Optimization*, 34(1):15–32, 2002.
- ⁹I.M. Ross, C. D'Souza, F. Fahroo, and JB Ross. A fast approach to multi-stage launch vehicle trajectory optimization. In *AIAA Guidance, Navigation, and Control Conference and Exhibit*, pages 11–14, 2003.
- ¹⁰G.T. Huntington and A.V. Rao. Optimal Configuration of Spacecraft Formations via a Gauss Pseudospectral Method. *Advances in the Astronautical Sciences*, 120:33–50, 2005.
- ¹¹T. R. Jorris and R. G. Cobb. Multiple method 2-d trajectory optimization satisfying waypoints and no-fly zone constraints. *Journal of Guidance, Control, and Dynamics*, 31(3):May–June, 2008.
- ¹²T. R. Jorris and R. G. Cobb. Three-dimensional trajectory optimization satisfying waypoints and no-fly constraints. *Journal of Guidance, Control, and Dynamics*, 32(2), March–April 2009.
- ¹³I.M. Ross and F. Fahroo. A perspective on methods for trajectory optimization. In *Proceedings of the AIAA/AAS Astrodynamics Specialist Conference, Monterey, CA*, pages 5–8, 2002.
- ¹⁴I.M. Ross and F. Fahroo. A direct method for solving nonsmooth optimal control problems. In *Proceedings of the 2002 World Congress of The International Federation on Automatic Control*, 2002.
- ¹⁵D. Benson. *A Gauss pseudospectral transcription for optimal control*. PhD thesis, Ph. D. Dissertation, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, 2004.
- ¹⁶I.M. Ross and F. Fahroo. Pseudospectral knotting methods for solving optimal control problems. *Journal of Guidance, Control, and Dynamics*, 27(3):397–405, 2004.
- ¹⁷T. Thorvaldsen, S.R. Hall, and D. Benson. A Gauss pseudospectral transcription for optimal control. 2005.
- ¹⁸A.V. Rao, D.A. Benson, C. Darby, M.A. Patterson, C. Francolin, I. Sanders, and G.T. Huntington. Algorithm xxx: GPOPS, A MATLAB Software for Solving Multiple-Phase Optimal Control Problems Using the Gauss Pseudospectral Method.
- ¹⁹P.E. Rutquist and M.M. Edvall. Propt-matlab optimal control software. *Tomlab Optimization Inc*, 260, 2009.
- ²⁰X. Xu and P. J. Antsaklis. Optimal control of switched systems based on parameterization of the switching instants. *IEEE Transactions on Automatic Control*, 49(1):2–16, 2004.
- ²¹M. Žefran. *Continuous Methods for Motion Planning*. PhD thesis, University of Pennsylvania, Computer and Information Science, 1996.
- ²²J. T. Betts. Survey of numerical methods for trajectory optimization. *Journal of Guidance control and dynamics*, 21(2):193–207, 1998.
- ²³O. von Stryk and R. Bulirsch. Direct and indirect methods for trajectory optimization. *Annals of Operations Research*, 37(1):357–373, 1992.
- ²⁴C. R. Hargraves and S. W. Paris. Direct trajectory optimization using nonlinear programming and collocation. *Journal of Guidance, Control, and Dynamics*, 10(4):338–342, 1987.
- ²⁵O. von Stryk. Numerical solution of optimal control problems by direct collocation. In Günter. Hoffmann, Karl-Heinz; Leugering, editor, *International Series of Numerical Mathematics*, pages 129–143. Springer, 1993.
- ²⁶A. L. Herman and B. A. Conway. Direct optimization using collocation based on high-order gauss-lobatto quadrature rules. *Journal of Guidance, Control, and Dynamics*, 19(3):592–599, May–June 1996.
- ²⁷P. Williams. Hermite-Legendre-Gauss-Lobatto Direct Transcription Methods in Trajectory Optimization. *Advances in the Astronautical Sciences*, 120, 2005.
- ²⁸J. T. Betts and E. J. Cramer. Application of direct transcription to commercial aircraft trajectory optimization. *Journal of Guidance, Control, and Dynamics*, 18(1):151–159, January–February 1995.
- ²⁹D. Wu and Y. J. Zhao. Performances and sensitivities of optimal trajectory generation for air traffic control automation. In *AIAA, editor, Proceedings of 2009 AIAA Guidance, Navigation and Control Conference*. AIAA, 2009.
- ³⁰G. Sachs and T. Christodoulou. Reducing fuel consumption of subsonic aircraft by optimal cyclic cruise. *Journal of Aircraft*, 24(9):616–622, 1987.
- ³¹P. K. A. Menon. Study of aircraft cruise. *Journal of Guidance, Control and Dynamics*, 15(5):631–639, 1987.
- ³²D. M. Pargett and M. D. Ardema. Flight path optimization at constant altitude. *Journal of Guidance Control and Dynamics*, 30(4):1197, 2007.
- ³³M. Soler, A. Olivares, and E. Staffetti. Hybrid optimal control approach to commercial aircraft trajectory optimization. *Journal of Guidance Control and Dynamics*, 33(3):985–991, May–June 2010.

- ³⁴M. Soler; D. Zapata; A. Olivares; E. Staffetti and J.Cegarra. Comparative analysis of commercial aircraft trajectory performance. In *Proceedings of the 2nd international conference on Engineering and Optimization*, volume To appear, 2010.
- ³⁵H. Witsenhausen. A class of hybrid-state continuous-time dynamic systems. *IEEE Transactions on Automatic Control*, 11(2):161–167, 1966.
- ³⁶M. Garavello and B. Piccoli. Hybrid necessary principles. *SIAM Journal on Mathematical Analysis*, 43:1867–1887, 2005.
- ³⁷H. J. Sussmann. A nonsmooth hybrid maximum principle. In D. Aeyels, F. Lamnabhi-Lagarigue, and A. van der Schaft, editors, *Stability and Stabilization of Nonlinear Systems*, pages 325–354. Springer, 1999.
- ³⁸J. Yong. Systems governed by ordinary differential equations with continuous, switching and impulse controls. *Applied Mathematics and Optimization*, 20(1):223–235, 1989.
- ³⁹S. Hedlund and A. Rantzer. Convex dynamic programming for hybrid systems. *IEEE Transactions on Automatic Control*, 47(9):1536–1540, 2002.
- ⁴⁰A. Nuic. *User Manual for the base of Aircraft Data (BADA) Revision 3.6*. Eurocontrol Experimental Center, 2005.
- ⁴¹Y. Kawajir, C. Laird, and A. Wachter. Introduction to IPOPT: A tutorial for downloading, installing, and using IPOPT, 2006.