Bending vibrations of rotating nonuniform nanocantilevers using the Eringen nonlocal elasticity theory

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A B S T R A C T

The natural frequencies of the flapwise bending vibrations of a nonuniform rotating nanocantilever has been calculated, considering the true spatial variation of the axial force due to the rotation. The area of the nanobeam cross section is assumed to change linearly. The problem has been formulated using the nonlocal Eringen elasticity theory and it was solved by a pseudo spectral collocation method based on Chebyshev polynomials. The effect of the nonlocal small scale, angular speed, nonuniformity of the sec tion and hub radius on the vibration behavior of the nanocantilever is discussed.

Keywords:
Nonlocal elasticity
Free vibrations
Rotation
Small scale effects

1. Introduction

Many micro or nanoelectromechanical systems (MEMS or NEMS) devices incorporate structural elements such as beams and plates in micro (or nano) length scale. Size effects are signif icant in the mechanical behavior of these structures in which dimensions are small and comparable to molecular distances. Since the atomic and molecular models require a great computational effort, simplified models are useful for analyzing the mechanical behavior of such devices.

The nanostructures that undergoing rotation are system with a promise future to be used in nanomachines [1,2] which include shaft of nanomotor [3,4] devices such as fullerene gears and carbon nanotube gears [5].

Classical continuum mechanics cannot predict the size effect, due to its scale free character. Despite some sporadic efforts in the 19th century and in the first half of the 20th century to capture the effects of microstructure using the continuum equations of elasticity with additional higher order derivatives, it was not until the 1960s that a major revival took place. From this time are the works of Toupin [6,7], Mindlin and Tiersten [3], Kröner [4], Green and Rivlin [8], Mindlin [9,10], Mindlin and Eshel [11].

More recently, Eringen derived a simple stress gradient theory from his earlier integral nonlocal theories [12].

In the early 1990s, Aifantis and coworkers suggested to extend the linear elastic constitutive relations with the Laplacian of the strain [1,13,14].

Askes and Gitman [15] show that both Eringen and Aifantes theories can be unified. An excellent overview on the historical development of theses theories, as well as its mining and imple mentation can be found in the paper by Askes and Aifantis [2].

Among the size dependent continuum theories, the theory of nonlocal continuum mechanics initiated by Eringen and coworkers [16,5,12] has been widely used to analyze many problems, such as wave propagation, dislocation, and crack singularities and, from the pioneer work of Peddieson et al. [17], for problems involving nanostructures. Thus, the nonlocal theory of elasticity has been used to address the behavior of beams [18 23], rods [24 29], plates [30], as well as carbon nanotubes (CNTS) [31 37].

Nowadays, a great effort is devoted to the vibration analysis of nanobeams and CNTS under rotation using the Eringen nonlocal elasticity theory [38 40].

Pradhan and Murmu [38], applied a nonlocal beam model to investigate the flap wise bending vibration characteristics of a uni form rotating nanocantilever. The nonlocal natural frequencies were obtained using the Differential Quadrature Method (DQM). They also discussed the effects of the nonlocal small scale, angular velocity and hub radius on vibration characteristics of the nanocantilever.

Murmu and Adhikari [39], investigated the same problem, but now considering an initially prestressed single walled carbon to analyze the effect on the initial preload in the vibration characteristics.

In both papers the nonlocal boundary conditions related to the free end of the nanobeam are not properly considered.

Narendar and Gopalakrishnan [40] analyzed the wave disper sion behavior of a uniform rotating nanotube modeled as an non local Euler Bernoulli beam. They consider the spatial variation of the centrifugal force in and average sense, replacing the variable ax ial effort by the maximum force (at the root of the nanocantilever).

In this paper we investigate the flap wise bending vibration characteristics of a nonuniform rotating nanocantilever considering

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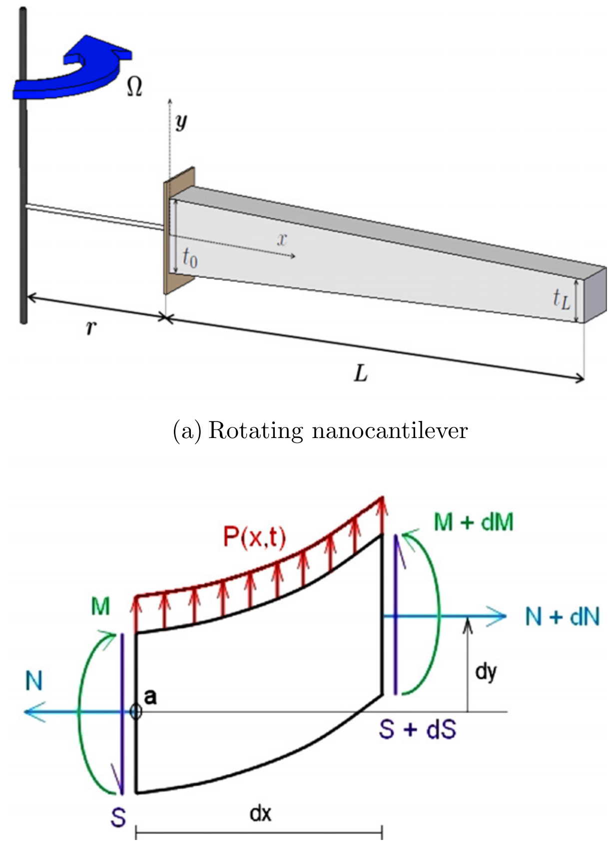
the true (quadratic) spatial variation of the axial force due to the rotation. The area of the nanobeam cross section of the nanocantilever is assumed to change linearly. The solution method of the corresponding nonlocal equations of motion are solved using a pseudo spectral collocation method based on Chebyshev polynomials. The effects of the nonlocal small scale, angular velocity, non uniformity of the section and hub radius on vibration characteristics of the nanocantilever are discussed.

The paper ir organized as follow: in Section 2 a brief resume of the constitutive equations of the Eringen nonlocal elasticity theory is given. In Section 3 the equation for the flapwise vibrations of the rotating nanocantilever is derived as well as the proper nonlocal boundary conditions. In Section 4 the pseudo spectral collocation method based on Chebyshev polynomials used to find the solution is briefly exposed. The numerical results and discussion of the effect of different variables in the nonlocal frequencies of the nanostructures appear in Section 5. Finally some conclusions are given in Section 6.



(a) Rotating nanocantilever

(b) Beam slice

Fig. 1. Schematic rotating nanocantilever system.

2. Nonlocal constitutive relations

The theory of nonlocal elasticity, developed by Eringen [16,41] and Eringen and Edelen [42] states that the nonlocal stress tensor components σ_{ij} at any point $\mathbf{x}$ in a body can be expressed as:

$$\sigma_{ij}(\mathbf{x}) \quad \int_{\Omega} \alpha(|\mathbf{x}' \quad \mathbf{x}|, \tau) t_{ij}(\mathbf{x}) d\Omega(\mathbf{x}') \tag{1}$$

where $t_{ij}(\mathbf{x})$ are the components of the classical local stress tensor at point $\mathbf{x}$, which are related to the components of the linear strain tensor ε_{kl} by the conventional constitutive relations for a Hookean material, i.e:

$$t_{ij} \quad C_{ijkl}\varepsilon_{kl} \tag{2}$$

The meaning of Eq. (1) is that the nonlocal stress at point $\mathbf{x}$ is the weighted average of the local stress of all points in the neighborhood of $\mathbf{x}$, the size of which is related to the nonlocal kernel $\alpha(|\mathbf{x}' \quad \mathbf{x}|, \tau)$. Here, $|\mathbf{x}' \quad \mathbf{x}|$ is the Euclidean distance and τ is a constant given by:

$$\tau \quad \frac{e_0 a}{l} \tag{3}$$

that represents the ratio between a characteristic internal length, a (such as the lattice spacing) and a characteristic external one, l (e.g. crack length, wavelength) trough an adjusting constant, e_0, dependent on each material.

According to [12], for a class of physically admissible kernel $\alpha(|\mathbf{x}' \quad \mathbf{x}|, \tau)$ it is possible to represent the integral constitutive relations given by Eq. (1) in an equivalent differential form as:

$$(1 \quad (e_0 a)^2 \nabla^2)\sigma_{kl} \quad t_{kl} \tag{4}$$

where ∇^2 is the Laplacian operator. Thus, the scale length $e_0 a$ takes into account the size effect on the response of nanostructures.

3. Problem formulation

Let us consider a beam of length L along the axial coordinate x, of constant thickness b and variable height $t(x)$. The beam is clamped at section O ($x = 0$) located at distance r from the axes around which rotates at constant angular velocity Ω as shown in Fig. 1(a). Let $v(x, t)$ be the transverse deflection along the coordinate y.

The equation of motion in the vertical direction for a beam slice of length dx (see Fig. 1(b)) can be written as:

$$\frac{\partial S}{\partial x} + p \quad \rho A(x) \frac{\partial^2 v}{\partial t^2} \tag{5}$$

where ρ is the density of the material, $A(x)$ the cross sectional area of the beam, p the distributed transverse force along axis x, and S the shear force on the cross section defined as $S \quad \int_A \sigma_{xy} dA$. If the rotational inertia is neglected, the momentum balance can be expressed as:

$$S + \frac{\partial M}{\partial x} \quad N \frac{\partial v}{\partial x} \quad 0 \tag{6}$$

M and N being, respectively, the resultant bending moment and axial load, given by $M \quad \int_A \sigma_{xx} y dA$ and $N \quad \int_A \sigma_{xx} dA$, where σ_{xx} is the normal stress in the x direction.

The one dimensional specialization of general nonlocal constitutive equations, Eq. (4), gives:

$$\sigma_{xx} \quad (ae_0)^2 \frac{\partial^2 \sigma}{\partial x^2} \quad E\varepsilon_{xx} \tag{7}$$

It is possible to integrate the one dimensional nonlocal constitutive equation, Eq. (7), multiplied by y, along the cross section of the beam:

$$\int_A \sigma_{xx} y dA \quad (ae_0)^2 \int_A \frac{\partial^2 \sigma_{xx}}{\partial x^2} y dA \quad E \int_A \varepsilon_{xx} y dA \tag{8}$$

and using the basic hypothesis of the Euler Bernoulli beam theory:

$$\varepsilon_{xx} \quad y \frac{\partial^2 v}{\partial x^2} \tag{9}$$

the following differential relation between the bending moment, M, and the vertical displacement, v, is found:

$$M(x) \quad EI \frac{\partial^2 v}{\partial x^2} + (ae_0)^2 \frac{\partial^2 M}{\partial x^2} \tag{10}$$

where I is the moment of inertia ($I \quad \int_A y^2 dA$). From Eqs. (5), (6), and (10), the nonlocal expression of both bending moment and shear force can be determined as a function of the displacement v and the external force per unit of length p:

$$M(x) \quad EI\frac{\partial^2 v}{\partial x^2} + (ae_0)^2\left(p + \frac{\partial}{\partial x}\left(N\frac{\partial v}{\partial x}\right) \quad \rho A\frac{\partial^2 v}{\partial t^2}\right) \tag{11}$$

$$S(x) \quad \frac{\partial}{\partial x}\left(EI\frac{\partial^2 v}{\partial x^2}\right) + N\frac{\partial v}{\partial x}$$
$$(ae_0)^2\left(\frac{\partial p}{\partial x} + \frac{\partial^2}{\partial x^2}\left(N\frac{\partial v}{\partial x}\right) \quad \frac{\partial}{\partial x}\left(\rho A\frac{\partial^2 v}{\partial t^2}\right)\right) \tag{12}$$

From now on, we consider a linear variation of the edge of the beam, defined as:

$$t(x) \quad t_0\left(1 + \beta\frac{x}{L}\right) \tag{13}$$

where the parameter β, is a constant of the form:

$$\beta \quad \frac{t_L \quad t_0}{t_0} \tag{14}$$

being t_0 and t_L the height of the beam at the clamped section O and free end L, respectively. This variation results in a nonuniform nano cantilever whose cross section is given by:

$$A(x) \quad A_0\left(1 + \beta\frac{x}{L}\right) \tag{15}$$

which is related to a moment of inertia:

$$I(x) \quad I_0\left(1 + \beta\frac{x}{L}\right)^3 \tag{16}$$

where A_0 and I_0 are the cross section and moment of inertia of the beam, respectively, at the clamped section O.

In the same way as the cross section variation with the spatial coordinate x is considered, graded mechanical properties of the material, e.g. Young's modulus $E(x)$, could be taken into account. This procedure would allow to study heterogeneous materials.

Differentiating the Eq. (12) with respect to x, considering E and ρ as constants, setting the external load at zero ($p = 0$), and substituting into Eq. (5), we can formulate the differential equation that governs the flapwise bending vibrations of the nonlocal Euler Bernoulli rotating beam with nonuniform section:

$$\rho A\frac{\partial^2 v}{\partial t^2} + E\frac{\partial^2}{\partial x^2}\left(I\frac{\partial^2 v}{\partial x^2}\right) \quad \frac{\partial}{\partial x}\left(N\frac{\partial v}{\partial x}\right)$$
$$+ (ae_0)^2\left[\frac{\partial^3}{\partial x^3}\left(N\frac{\partial v}{\partial x}\right) \quad \rho\frac{\partial^2}{\partial x^2}\left(A\frac{\partial^2 v}{\partial t^2}\right)\right] \quad 0 \tag{17}$$

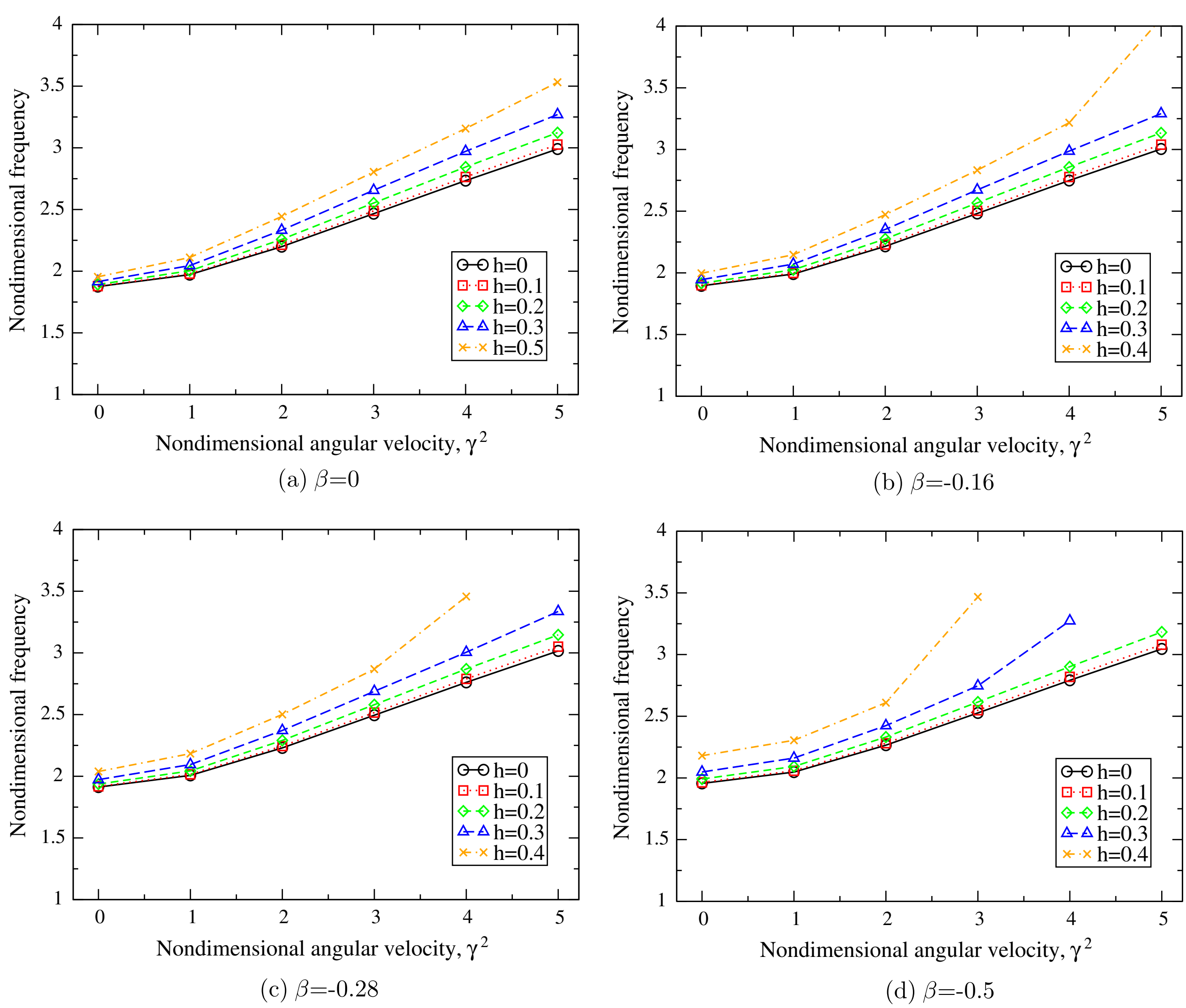


Fig. 2. Variation of nondimensional fundamental frequency with nondimensional angular velocity of a rotating nanocantilever for different nonlocal parameters, β = cte. δ = 1.

which can be rewritten, taking into account the Eqs. (15) and (16), as:

$$\rho A_0\left(1+\beta\left(\frac{x}{L}\right)\right)\frac{\partial^2 v}{\partial t^2}+EI_0\frac{\partial^2}{\partial x^2}\left(\left(1+\beta\left(\frac{x}{L}\right)\right)^3\frac{\partial^2 v}{\partial x^2}\right)\quad\frac{\partial}{\partial x}\left(N\frac{\partial v}{\partial x}\right)$$
$$+(ae_0)^2\left[\frac{\partial^3}{\partial x^3}\left(N\frac{\partial v}{\partial x}\right)\quad\rho A_0\frac{\partial^2}{\partial x^2}\left(\left(1+\beta\left(\frac{x}{L}\right)\right)\frac{\partial^2 v}{\partial t^2}\right)\right]\quad 0 \tag{18}$$

The axial force $N(x)$ due to the rotation, is given by:

$$N(x)\quad\int_x^L\rho A(\eta)\Omega^2(r+\eta)d\eta \tag{19}$$

and using Eq. (15):

$$N(x)\quad\frac{1}{6L}\Big[A_0\rho\Omega^2(L\quad x)\big[3L(L+2r+x)$$
$$+\beta(2L^2+3Lr+2Lx+3xr+2x^2)\big]\Big] \tag{20}$$

The Eq. (18) can be solved by using the classical separation of variables method as:

$$v(x,t)\quad\overline{V}(x)\,e^{i\omega t} \tag{21}$$

where ω is the natural frequency of vibrations.

Substituting Eq. (21) in Eq. (18) and using the new dimension less variables and constants given by:

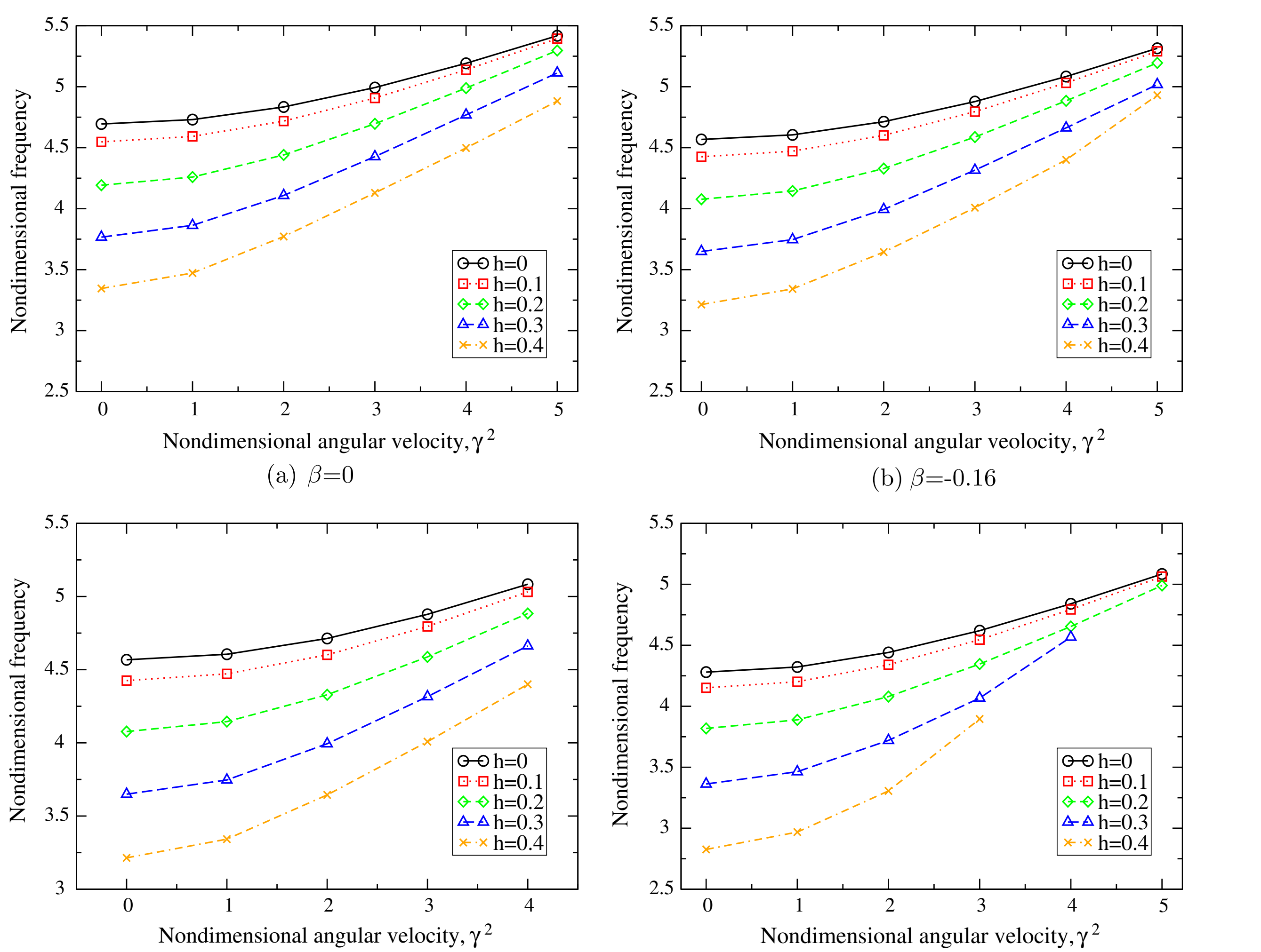
$$\xi\quad x/L;\quad V\quad\overline{V}/L;\quad h\quad\frac{ae_0}{L};\quad\delta\quad r/L;$$
$$\lambda^4\quad\frac{\rho AL^4}{EI}\omega^2;\quad\gamma^4\quad\frac{\rho AL^4}{EI}\Omega^2;\quad F\quad\frac{NL^2}{EI} \tag{22}$$

we get the spatial equation as:

$$\Big[6\beta^2(1+\beta\xi)V''(\xi)+6\beta(1+\beta\xi)^2V'''(\xi)+(1+\beta\xi)^3V^{IV}\Big]$$
$$+h^2\Big[F'''V'(\xi)+3F''V''(\xi)+3F'V'''(\xi)+FV^{IV}(\xi)$$
$$+\lambda^4\big[2\beta V'(\xi)+(1+\beta\xi)V''(\xi)\big]\Big]\quad F'V'(\xi)\quad FV''(\xi)$$
$$\lambda^4(1+\beta\xi)V(\xi)\quad 0 \tag{23}$$

where $(\ )'$ represents the derivative with respect to ξ. Note that the dimensionless rotational velocity γ^2 is implicitly included in the new dimensionless variable F. The above differential equation must be solved with the following boundary conditions corresponding to the rotating nanocantilever:

$$V(0)\quad 0 \tag{24}$$



(a) β=0
(b) β=-0.16
(c) β=-0.28
(d) β=-0.5

Fig. 3. Variation of nondimensional second mode frequency with nondimensional angular velocity of a rotating nanocantilever for different nonlocal parameters, β = cte. δ = 1.

$$V'(0) \quad 0 \tag{25}$$

$$(1+\beta)^3 V''(1) + h^2\left[\lambda^4(1+\beta)V(1)\right. \\ \left. \gamma^4(1+\beta)(1+\delta)V'(1)\right] \quad 0 \tag{26}$$

$$3\beta(1+\beta)^2 V''(1) + (1+\beta)^3 V'''(1) \\ h^2\{\gamma^4\left[(1+\beta(2+\delta))V'(1) + 2(1+\beta)(1+\delta)V''(1)\right] \\ + \lambda^4\left[\beta V(1) + (1+\beta)V'(1)\right]\} \quad 0 \tag{27}$$

The Eqs. (26) and (27) state, respectively, that the bending moment and shear force are zero at the free end of the nanocantilever. Note that these conditions are not equivalent to cancel the second and third derivatives of the displacements.

It is important to highlight that the equations and boundary conditions of classical local elasticity with Euler Bernoulli beam theory, are recovered when the parameter h is null.

Similar problems has been addressed by Pradhan and Murmu for the case of rotating nanocantilevers with constant cross section [38], and Murmu and Adhikari for prestressed nanotubes undergoing rotation also with constant cross section [39], but in these papers not proper boundary conditions are used.

4. Pseudo-spectral collocation solution based on Chebyshev polynomials

In order to obtain the natural frequencies of the problem defined by Eqs. (23) (27), a pseudo spectral collocation method based on Chebyshev polynomials will be used.

Chebyshev polynomials are recursive orthogonal polynomials defined as:

$$T_m(z) \quad \cos(m \cos^{\ 1}(z)) \quad \text{for } k \quad 0, 1, 2, \ldots \tag{28}$$

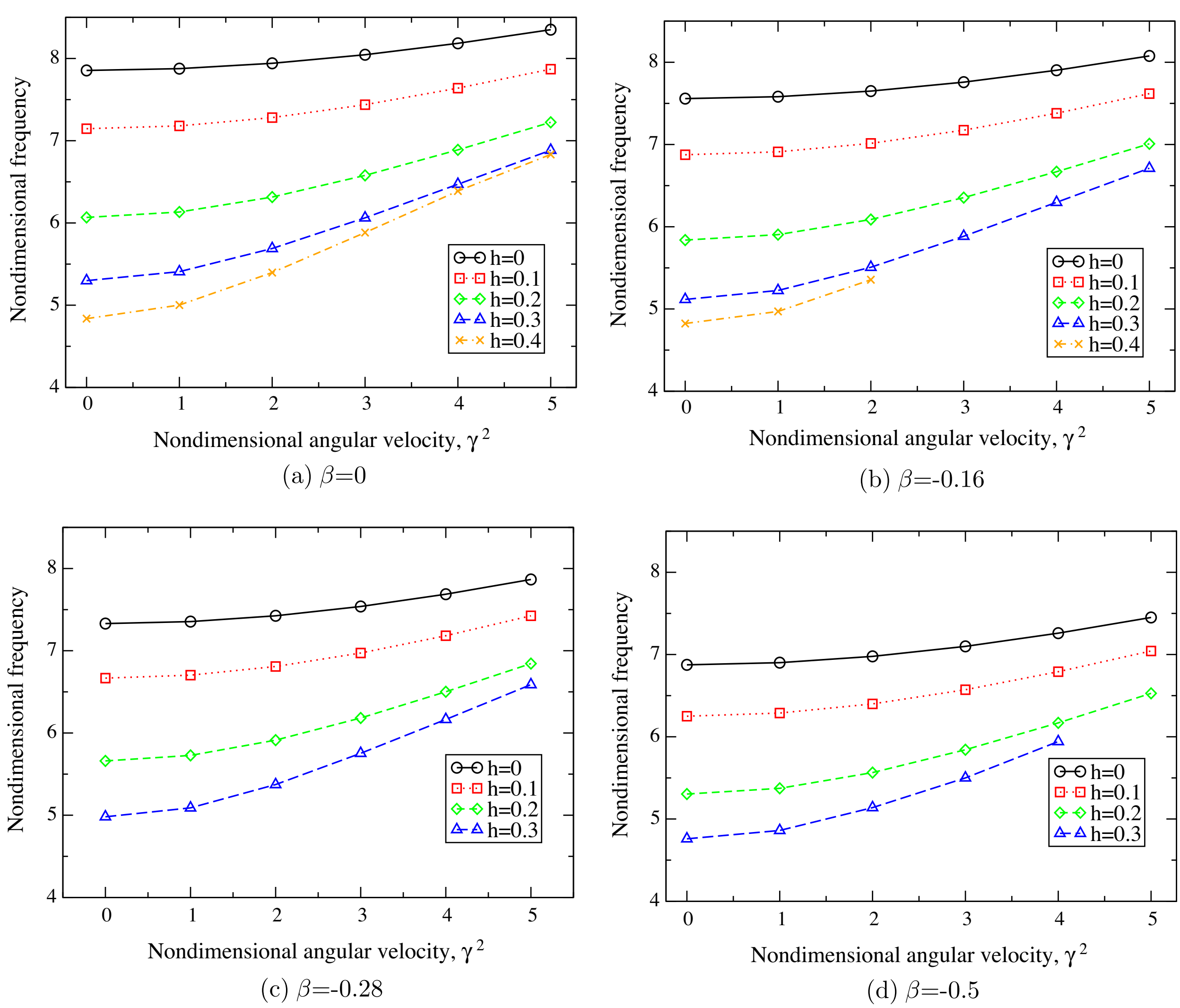
where m is an integer. Due to their recursive nature and fast convergence characteristics, Chebyshev polynomials have been used in the literature for the solution of boundary value problems [43 46]. This polynomials are a stable representation only on the interval $(\ \ 1, 1)$, but mostly functions of interest lie on the interval (l_1, l_2). Because of this, a change of variable can be defined as [47]:

$$z(\zeta) \quad \frac{l_2 \quad l_1}{2}\zeta + \frac{l_2 + l_1}{2}, \qquad \zeta(z) \quad \frac{2}{l_2 \quad l_1} z \quad \frac{l_2 + l_1}{l_2 \quad l_1} \tag{29}$$

where $\zeta \in (\ \ 1, 1)$ and $z \in (l_1, l_2)$.

Thus, the scaled Chebyshev polynomials can be defined as:

$$\mathcal{T}_m(z) \quad T_m(\zeta(z)) \tag{30}$$



(a) β=0 (b) β=-0.16 (c) β=-0.28 (d) β=-0.5

Fig. 4. Variation of nondimensional third mode frequency with nondimensional angular velocity of a rotating nanocantilever for different nonlocal parameters, β = cte. δ = 1.

This polynomials form a complete set in the interval (l_1, l_2), so that any square integrable function $y(z)$ that lies on the interval, can be expressed using the series expansion:

$$y(z) \quad \sum_{m \quad 0}^{\infty} a_m \mathcal{T}_m(z) \tag{31}$$

where a_m are the coefficients of the series.

If the function $y(z)$ is infinitely differentiable and well behaved on the interval (l_1, l_2), only a small number of terms, N, in the above series will be sufficient to represent the function accurately, so, for numerical solutions, truncated series expansions are used. Thus, a function $y(z)$ can be approximated as:

$$y(z) \approx y_N(z) \quad \sum_{i \quad 0}^{N} a_i \mathcal{T}_{N \quad i}(z) \tag{32}$$

The procedure from now on will be approximate the functions $V(\xi)$ by its series expansion $V_N(\xi)$ according to Eq. (32), and replace them in the Eqs. (23) (27) that define the problem. Thus, the functions $V_N(\xi)$ satisfy both differential equation, in a certain collocation points to be detailed later, and boundary conditions.

In these equations, successive derivatives of the functions $V(\xi)$ appear, so it is necessary to differentiate the functions:

$$V_N(\xi) \quad \sum_{i \quad 0}^{N} a_i \mathcal{T}_{N \quad i}(\xi) \tag{33}$$

as follows:

$$\frac{d^n V_N(\xi)}{d\xi^n} \quad \sum_{i \quad 0}^{N} a_i \frac{d^n \mathcal{T}_{N \quad i}(\xi)}{d\xi^n} \quad \sum_{i \quad 0}^{N} a_i \frac{d^n T_{N \quad i}(\zeta(\xi))}{d\xi^n} \tag{34}$$

In this case, from Eq. (29), $\zeta(\xi) = 2\xi \quad 1$, since $\xi \in (l_1, l_2) = (0, 1)$. Therefore:

$$\frac{d^n T_{N \quad i}(\zeta(\xi))}{d\xi^n} \quad \frac{d^n T_{N \quad i}(\zeta(\xi))}{d\zeta^n} \left(\frac{d\zeta(\xi)}{d\xi}\right)^n \quad 2^n \frac{d^n T_{N \quad i}(\zeta)}{d\zeta^n} \tag{35}$$

Finally, substituting last expression in Eq. (34), we obtain the n th derivative of the functions $V_N(\xi)$ as:

$$\frac{d^n V_N(\xi)}{d\xi^n} \quad 2^n \sum_{i \quad 0}^{N} a_i \frac{d^n T_{N \quad i}(\zeta)}{d\zeta^n} \tag{36}$$

The four first derivatives, $d^n T_{N \quad i}(\zeta)/d\zeta^n$, of Chebyshev polynomials, which are shown in Eqs. (23) (27), are defined, from Eq. (28), as:

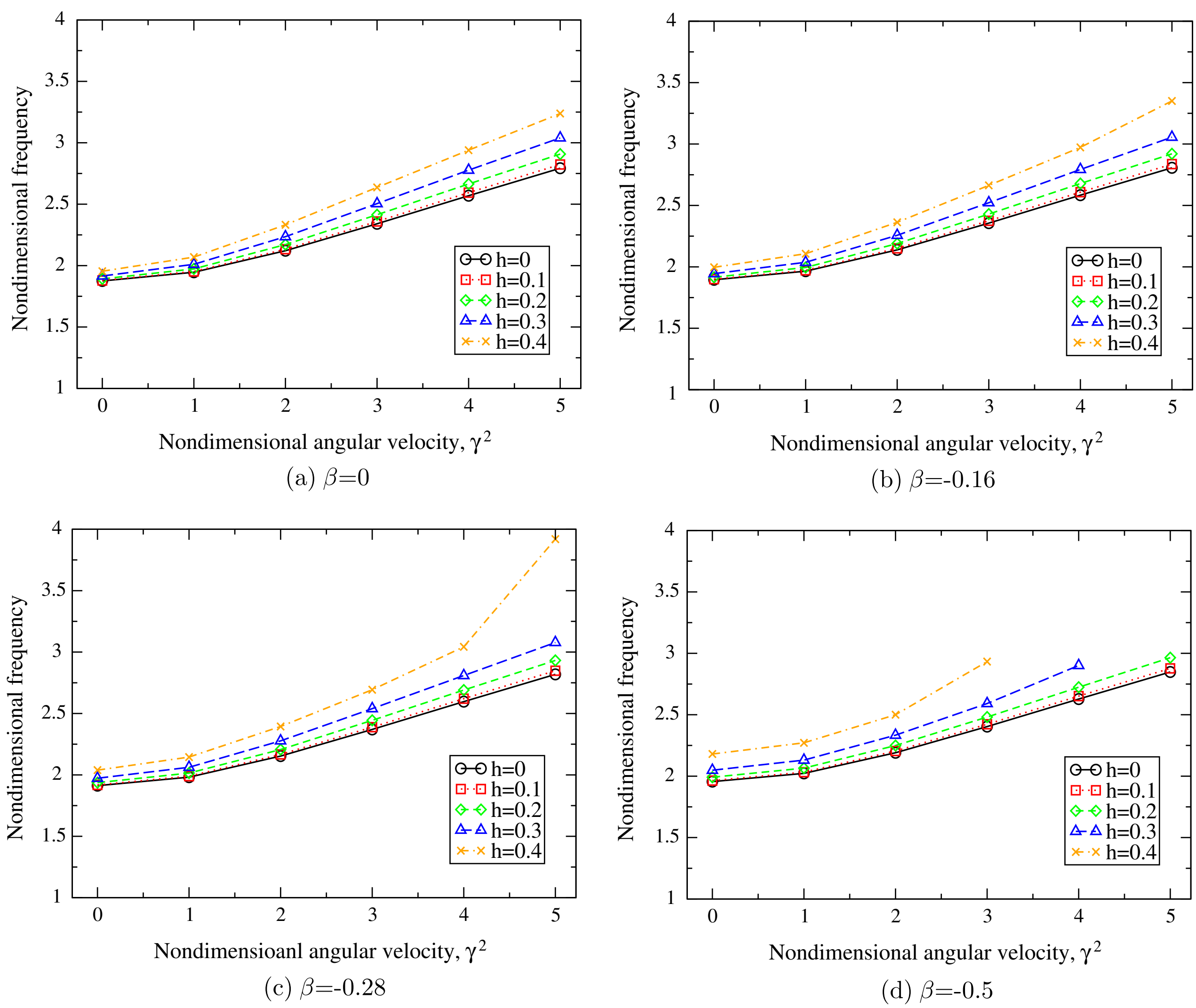


Fig. 5. Variation of nondimensional fundamental frequency with nondimensional angular velocity of a rotating nanocantilever for different nonlocal parameters, β = cte. δ = 0.5.

$$\frac{dT_m(\zeta)}{d\zeta} \quad \frac{m}{\sqrt{1 \quad \zeta^2}} \sin(m \cos^{\quad 1}(\zeta)) \tag{37}$$

$$\frac{d^2T_m(\zeta)}{d\zeta^2} \quad \frac{1}{1 \quad \zeta^2}\left[\zeta\frac{dT_m(\zeta)}{d\zeta} \quad m^2T_m(\zeta)\right] \tag{38}$$

$$\frac{d^3T_m(\zeta)}{d\zeta^3} \quad \frac{1}{(1 \quad \zeta^2)^2}\left\{[1+2\zeta^2+m^2(\zeta^2 \quad 1)]\frac{dT_m(\zeta)}{d\zeta} \quad 3\zeta m^2T_m(\zeta)\right\} \tag{39}$$

$$\frac{d^4T_m(\zeta)}{d\zeta^4} \quad \frac{1}{(1 \quad \zeta^2)^3}\left\{3\zeta[3+2\zeta^2+2m^2(\zeta^2 \quad 1)]\frac{dT_m(\zeta)}{d\zeta} \quad m^2[4+11\zeta^2+m^2(\zeta^2 \quad 1)]T_m(\zeta)\right\} \tag{40}$$

As stated above, we impose that the functions $V_N(\xi)$ satisfy the differential equation given by Eq. (23) in a certain collocation points. These collocation points or nodes will be the so called Gauss Chebyshev points, which are the roots of the Chebyshev polynomials of the first kind defined in Eq. (28), and are defined as:

$$\zeta_i \quad \cos\left(\frac{(2i \quad 1)\pi}{2M}\right) \quad i \quad 1,\ldots,M \tag{41}$$

in the variable $\zeta \in (\quad 1,1)$, and:

$$\xi_i \quad \frac{1}{2}\left[1+\cos\left(\frac{(2i \quad 1)\pi}{2M}\right)\right] \quad i \quad 1,\ldots,M \tag{42}$$

in the variable $\xi \in (0,1)$. The number of collocation points that we use is $M = (N+1 \quad k)$, where N is the number of terms in the series given by Eq. (33), and k is the number of boundary conditions.

Thus, with the equations obtained from the compliance of the Eq. (23) at the $(N+1 \quad k)$ Gauss Chebyshev points, and the k resulting equations from applying the boundary conditions, we ob tain a system of $(N+1)$ equations with $(N+1)$ unknowns, which are the series coefficients a_i:

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,N+1} \\ \vdots & \ddots & \vdots \\ A_{N+1,1} & \cdots & A_{N+1,N+1} \end{pmatrix} \begin{Bmatrix} a_0 \\ \vdots \\ a_N \end{Bmatrix} \quad \begin{Bmatrix} 0 \\ \vdots \\ 0 \end{Bmatrix} \tag{43}$$

This system of equations can be written in a compact form as:

$$[\mathbf{A}]\{\mathbf{a}\} \quad \{\mathbf{0}\} \tag{44}$$

being, as can be seen, a homogeneous system.

Hence, the terms of the matrix $[\mathbf{A}]$ are defined, from Eqs. (23) (27), as:

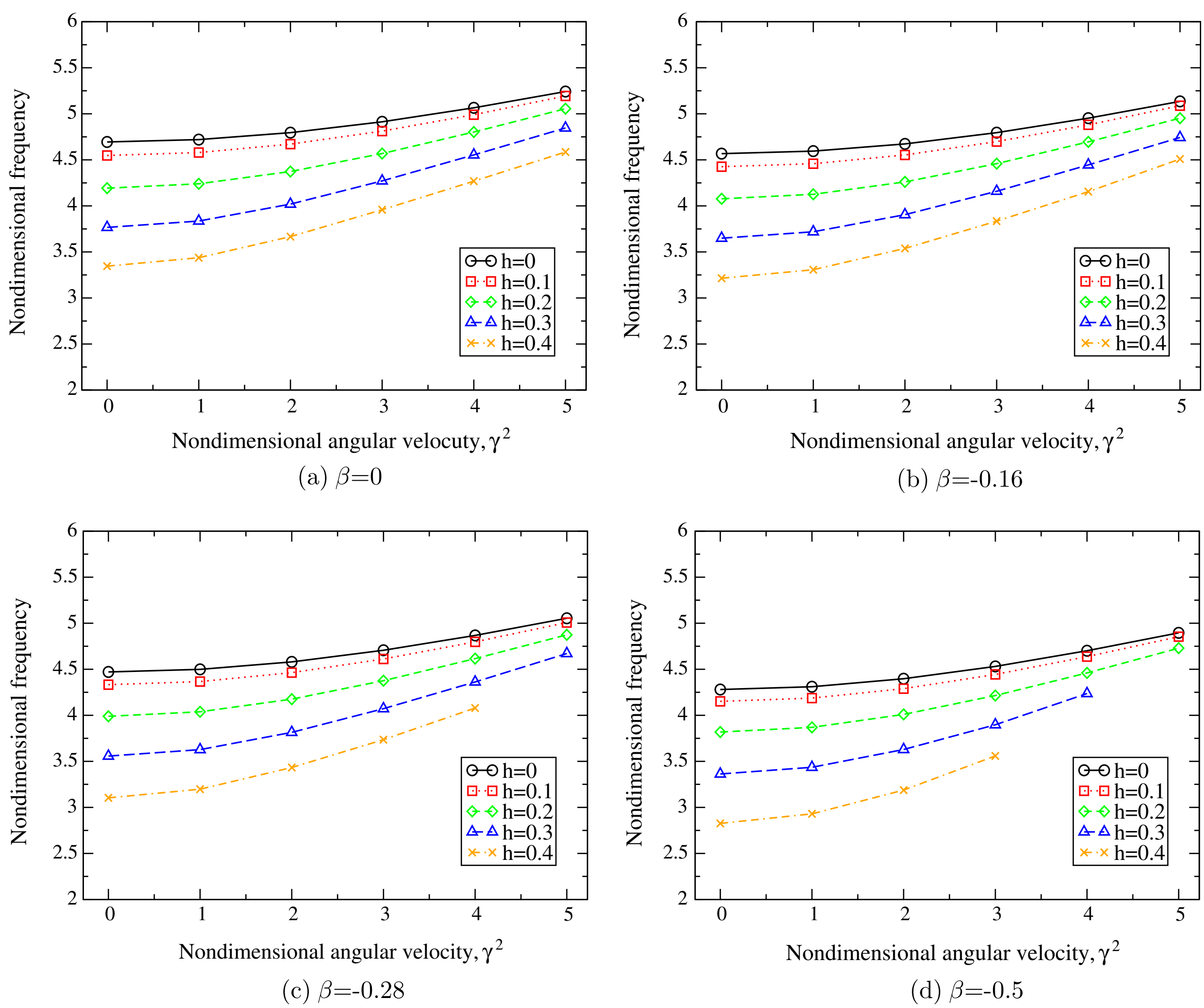


Fig. 6. Variation of nondimensional second mode frequency with nondimensional angular velocity of a rotating nanocantilever for different nonlocal parameters, β = cte. δ = 0.5.

$$A_{j,k} \quad 6\beta^2(1+\beta\xi_j)\mathcal{T}''_{N+1\ \ k}(\xi_j) + 6\beta(1+\beta\xi_j)^2\mathcal{T}'''_{N+1\ \ k}(\xi_j)$$
$$+ (1+\beta\xi_j)^3\mathcal{T}^{IV}_{N+1\ \ k}(\xi_j) + \gamma^4(\delta+\xi_j)(1+\beta\xi_j)\mathcal{T}'_{N+1\ \ k}(\xi_j)$$
$$F(\xi_j)\mathcal{T}''_{N+1\ \ k}(\xi_j) + h^2\left[F(\xi_j)\mathcal{T}^{IV}_{N+1\ \ k}(\xi_j)\right.$$
$$3\gamma^4(\delta+\xi_j)(1+\beta\xi_j)\mathcal{T}'''_{N+1\ \ k}(\xi_j)$$
$$\left.3\gamma^4(1+\beta(\delta+2\xi_j))\mathcal{T}''_{N+1\ \ k}(\xi_j) \quad 2\beta\gamma^4\mathcal{T}'_{N+1\ \ k}(\xi_j)\right]$$
$$+\lambda^4\left\{h^2\left[2\beta\mathcal{T}'_{N+1\ \ k}(\xi_j) + (1+\beta\xi_j)\mathcal{T}''_{N+1\ \ k}(\xi_j)\right]\right.$$
$$\left.(1+\beta\xi_j)\mathcal{T}_{N+1\ \ k}(\xi_j)\right\} \tag{45}$$

for $j = 1, \ldots, N\ \ 3$ and $k = 1, \ldots, N+1$. And:

$$A_{N\ \ 2,k} \quad \mathcal{T}_{N+1\ \ k}(0) \tag{46}$$

$$A_{N\ \ 1,k} \quad \mathcal{T}'_{N+1\ \ k}(0) \tag{47}$$

$$A_{N,k} \quad (1+\beta)^3\mathcal{T}''_{N+1\ \ k}(1)$$
$$+h^2\left[\lambda^4(1+\beta)\mathcal{T}_{N+1\ \ k}(1) \quad \gamma^4(1+\delta)(1+\beta)\mathcal{T}'_{N+1\ \ k}(1)\right] \tag{48}$$

$$A_{N+1,k} \quad 3\beta(1+\beta)^2\mathcal{T}''_{N+1\ \ k}(1) + (1+\beta)^3\mathcal{T}'''_{N+1\ \ k}(1)$$
$$h^2\left\{\gamma^4\left[\mathcal{T}'_{N+1\ \ k}(1)(1+\beta(2+\delta))\right.\right.$$
$$\left.+2(1+\delta)(1+\beta)\mathcal{T}''_{N+1\ \ k}(1)\right] \quad \lambda^4\left[\beta\mathcal{T}_{N+1\ \ k}(1)\right.$$
$$\left.\left.+(1+\beta)\mathcal{T}'_{N+1\ \ k}(1)\right]\right\} \tag{49}$$

for $k = 1, \ldots, N+1$.

To obtain the natural frequencies of vibration of the nonuniform rotating nanocantilever, it is not necessary to calculate the coefficients a_i of the expansion series, and therefore, it is not necessary to solve the system given in Eq. (43).

In order to get the values of the dimensionless vibration frequencies λ, we impose the system of equations given by Eq. (44) as singular, a necessary condition for obtain other solutions than the trivial. Thus, the determinant of matrix $[\mathbf{A}]$ must be zero, leading to a trascendental equation for λ, which solutions, λ_j, are the dimensionless vibration frequencies of the problem.

5. Results and discussion

In this article the small scale effects on the vibration response of rotating nanocantilever beams are shown with respect to the

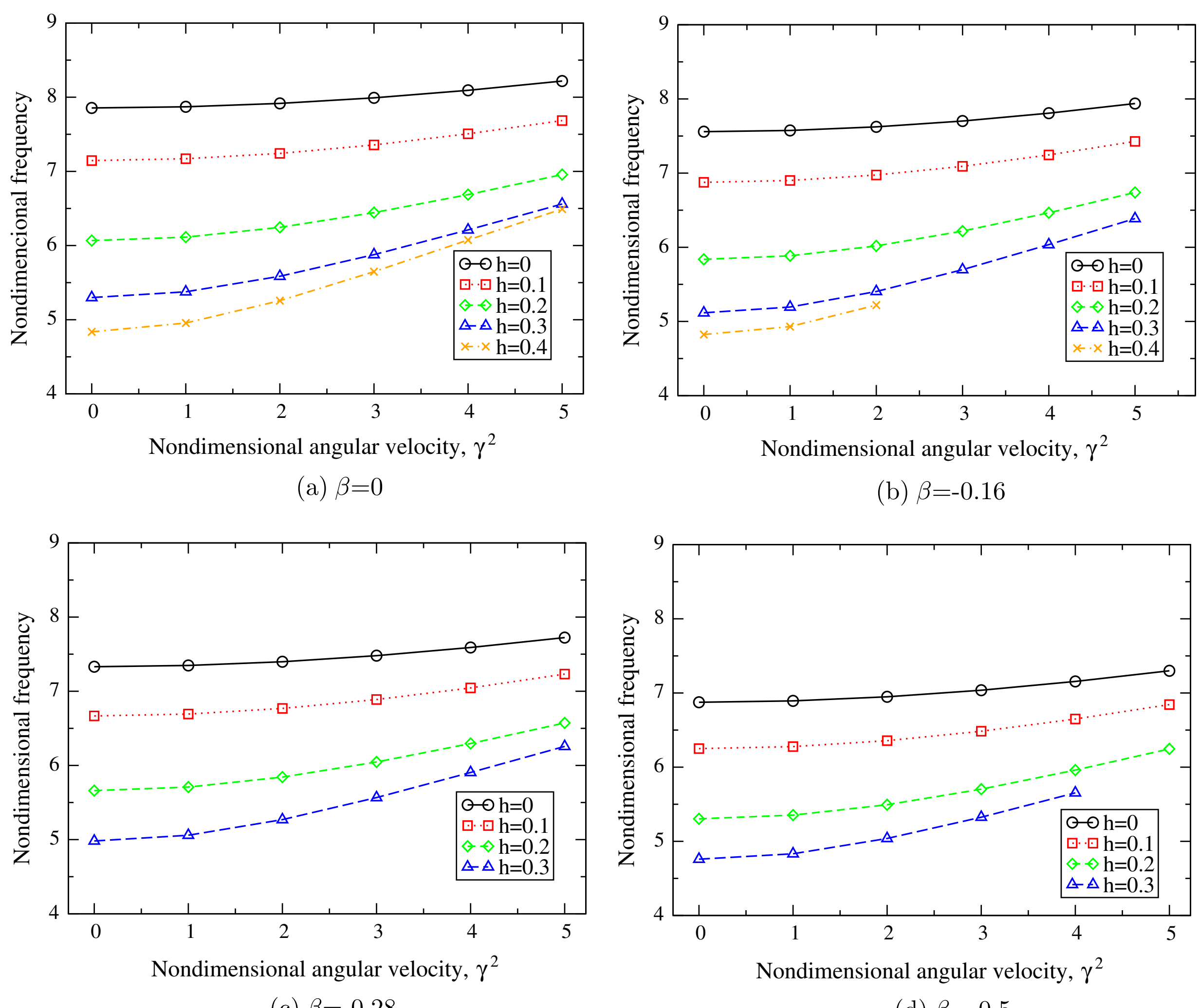


Fig. 7. Variation of nondimensional third mode frequency with nondimensional angular velocity of a rotating nanocantilever for different nonlocal parameters, β = cte. δ = 0.5.

angular velocity of rotating and to the cross section variation for different hub radius.

Figs. 2 7 show the variation of the three first modes nondimensional frequencies, λ^2, with nondimensional angular velocity parameter, γ^2, for different values of the nonlocal parameter, h, and different values of β, which represents the cross section variation.

For the nonlocal model, h can be 0, 0.1, 0.2, 0.3 and 0.4, in a similar range employed by Pradham and Murmu [38] and Lu et al. [48]. For higher values of the nonlocal parameter, real values of nondimensional frequencies could not be obtained, as were reported by Lu et al. [48]. In the particular case of $h = 0$, model solution is agree with local elasticity theory.

The rotating nondimensional angular velocity, γ^2, is assumed to be in the range of 0 4, as had been employed by Pradham and Murmu [38]. Here is considered the nondimensional hub radius δ to be 1 (Figs. 2 4), and 0.5 (Figs. 5 7), and the cross section of the nanocantilever change linerly, being $\beta = 0$, 0.16, 0.28 and 0.5.

Fig. 2 shows the nondimensional fundamental frequency versus the nondimensional angular velocity γ^2 for different values of β and $\delta = 1$. It is observed that the nondimensional fundamental frequency increases with γ^2 for both the local and nonlocal elastic models, being higher as the nonlocal parameter h does. These observations are in line with those of Pradham and Murmu [38], Lu et al. [48] and Zhang et al. [49].

The difference between the frequencies by local and nonlocal models increases with the value of angular velocity, being close to each other at low rotating velocities. Difference are more relevant as the cross section changes faster. The combined effect of high values of β and h, may cause impossibility to obtain real values of nondimensional frequencies.

Figs. 3 and 4 show, respectively, the variation of the second and third nondimensional frequency with the nondimensional angular velocity. For second and third modes, the frequencies obtained by nonlocal models are smaller to that of local models, being smaller as h increases. In both modes, opposite to the first mode, the nonlocal effects on the frequencies diminish as the rotating velocity is higher, and the effect of cross section variation is almost neglected.

Figs. 5 7 show the variation of nondimensional first, second and third frequency with nondimensional angular velocity for $\delta = 0.5$. Similar conclusions can be obtained for this value of hub radius δ.

Figs. 8 10 show, respectively, the variation of the nondimensional frequencies corresponding to the three first modes respect to the cross section variation parameter, β, for different values of the nonlocal parameter and nondimensional angular velocity parameter.

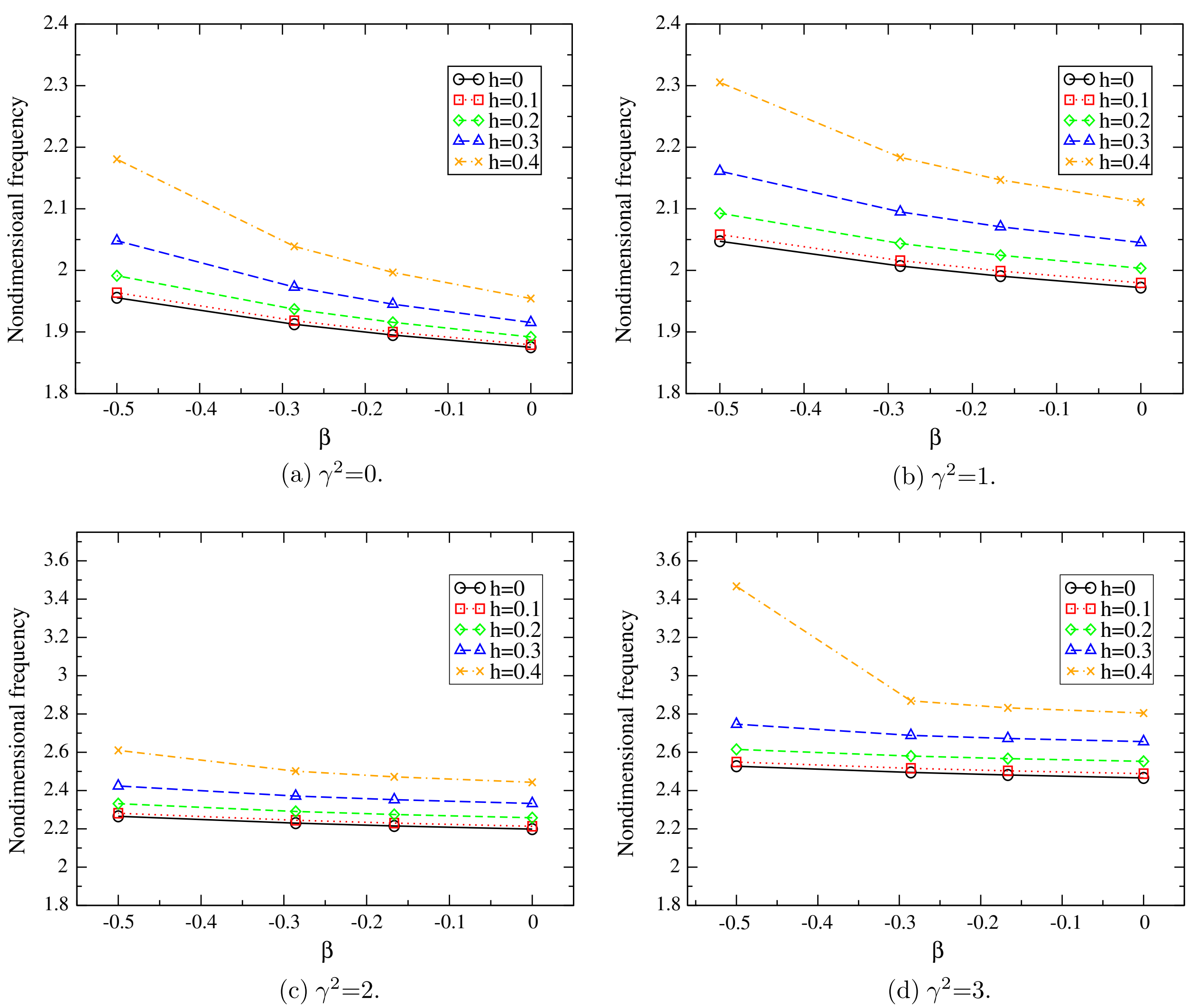
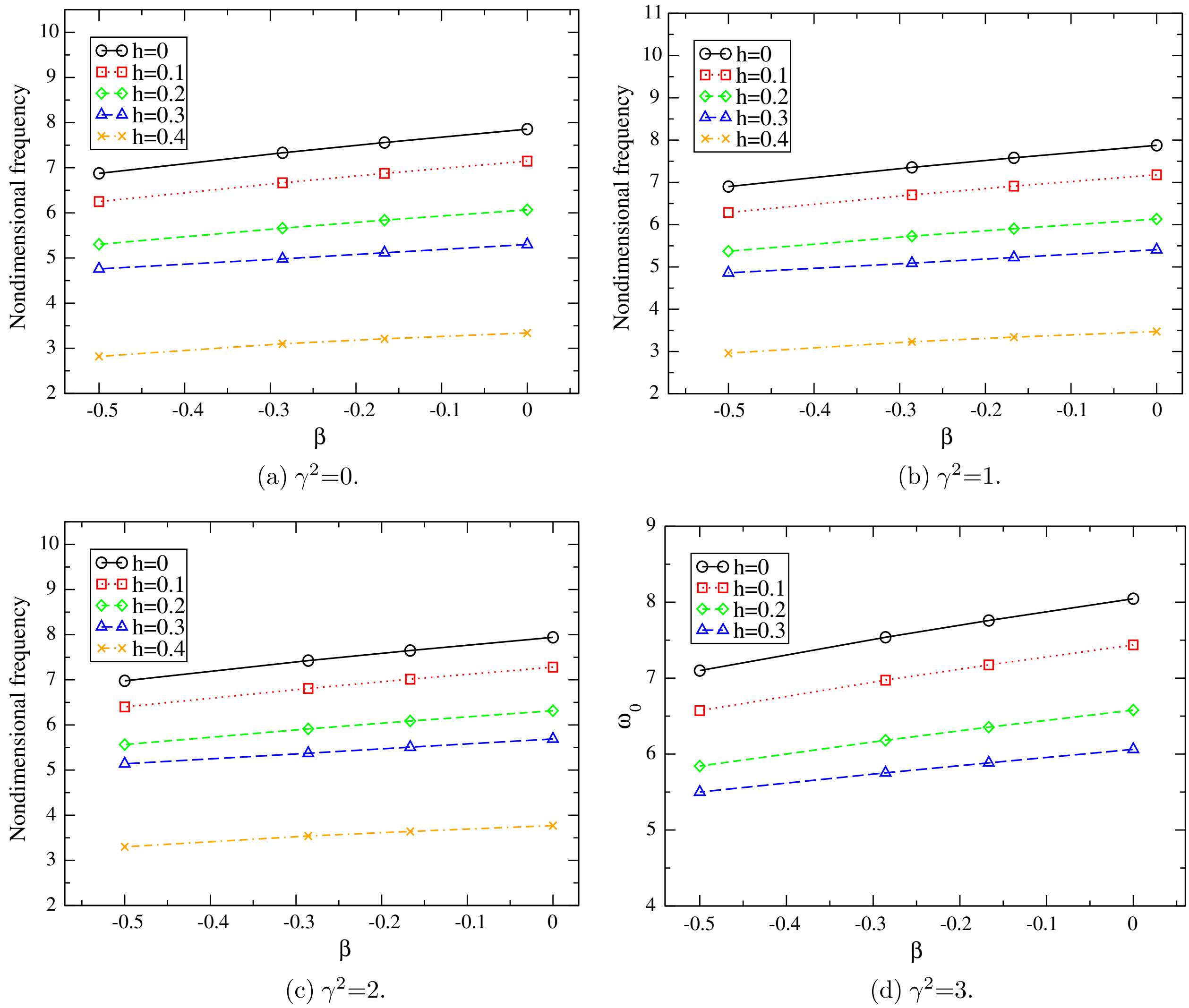


Fig. 8. Variation of nondimensional fundamental frequency with cross-section variation parameter, β, in a rotating nanocantilever for different nonlocal parameters, γ^2 = cte. $\delta = 1$.



(a) γ^2=0. (b) γ^2=1. (c) γ^2=2. (d) γ^2=3.

Fig. 9. Variation of nondimensional second frequency with cross-section variation in a rotating nanocantilever for different nonlocal parameters, γ^2 = cte. δ = 1.

In Fig. 8 is observed that the nondimensional fundamental frequency decreases with the cross section variation for both the local and nonlocal elastic models, being higher as the nonlocal parameter h does.

As the angular velocity increases, the effect induced by this parameter begins to be more important compared to the cross section variation. Globally, the frequencies in local and nonlocal models increases with the velocity, but the influence of the cross section variation diminishes. Figs. 9 and 10 show, respectively, the variation of the second and third nondimensional frequency with the nondimensional angular velocity.

For second and third modes, the frequencies obtained by nonlocal models are smaller to that of local models, being smaller as h increases. Here, frequencies increases with the cross section variation. The effect of nonlocal parameter seems to be not very sensitive to β and γ^2. As in the fundamental case, frequencies increase slightly with angular velocity.

6. Conclusions

In this work, The effects of the nonlocal small scale, angular velocity, nonuniformity of the section and hub radius on the three first flapwise vibration frequencies have been considered. By means of a pseudo spectral collocation method, based on Chebyshev polynomials, the nonlocal equations of motion for the beam are solved.

It is observed that the nondimensional frequencies increase with the rotating angular velocity for both local and nonlocal elastic models. For the same geometry and angular velocity, the nondimensional fundamental frequency obtained increases with the nonlocal parameter, h (note that h = 0 case correspond to local elastic model). This trend is opposite for the second and third mode of vibration, where the nonlocal frequencies are lower than local ones.

The fundamental frequency is not very influenced by the small scale parameter at low velocities, but difference between local and nonlocal theory increases with the rotation velocity of the beam. In higher modes, the frequencies obtained tends to be closer as the velocity increases, being the effect of h more significant at low velocities.

If the cross section decrease linearly from the fixed to the free end of the beam, the mass gravity center of the beam tends to move towards the fixed end of the beam and the fundamental frequency decreases. In this case, the effect of the nonlocal parameter is more important as the rotation and cross section variation increase. In higher modes, the effect of the cross section variation in the effective mass is opposite, and the effect of the nonlocal

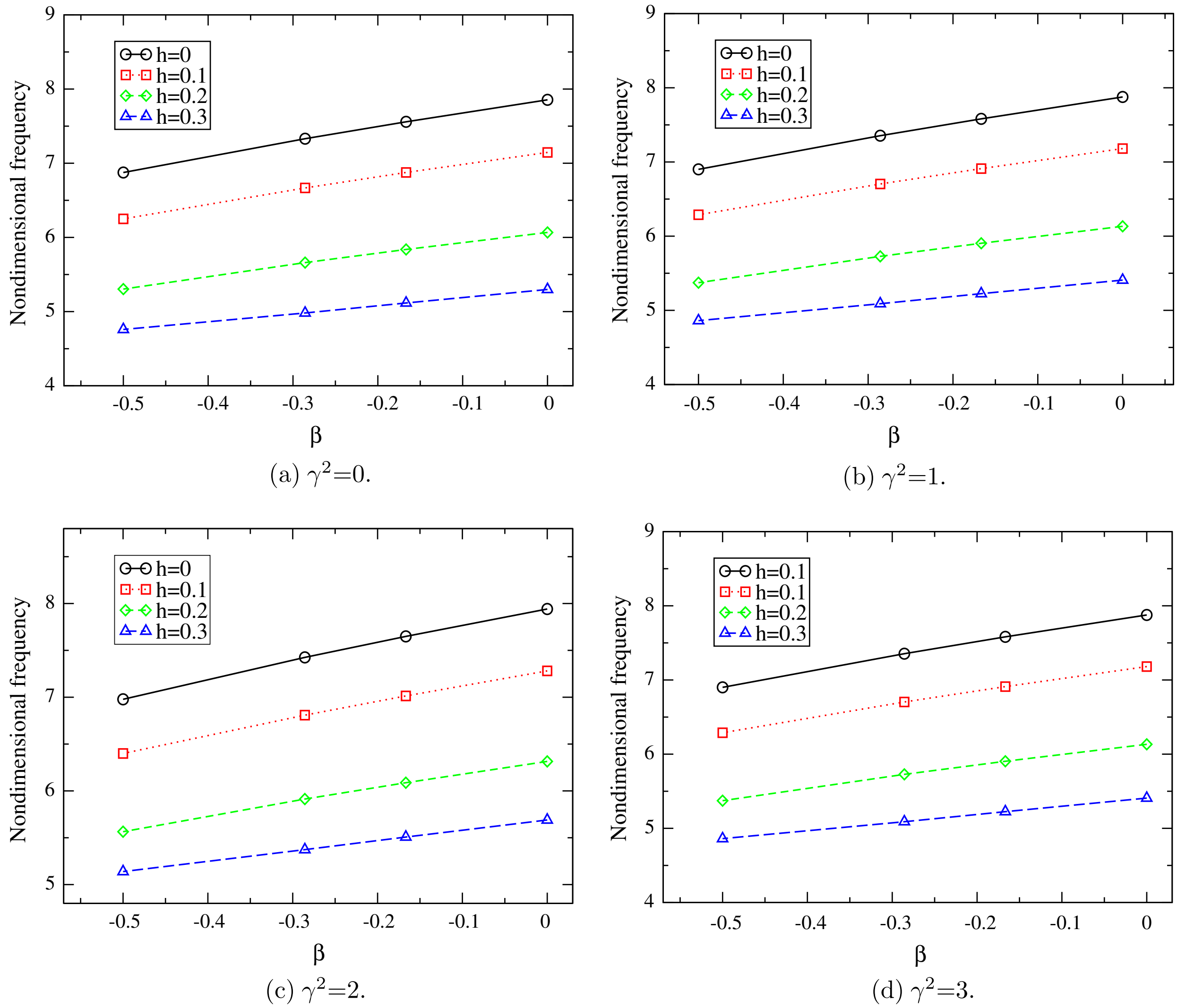


Fig. 10. Variation of nondimensional third frequency with cross-section variation in a rotating nanocantilever for different nonlocal parameters, γ^2 = cte. δ = 1.

parameter is similar for the different values of β and rotation velocities. Similar results were obtained for different hub radius.

In this work, nonuniform cross section has been considered but similar procedure can be used in case of heterogenous materials with graded mechanical properties.

A new work is in progress extending the present analysis to the case of rotating nanocantilevers using the Timoshenko beam theory.

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