

# Coordination and growth: the Stag Hunt game on evolutionary networks

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**Abstract.** Recently, the study of evolutionary games on networks has attracted great interest, focused mainly on the problem of the emergence of cooperation. A well studied framework for this problem is the Prisoner's Dilemma game on fixed, evolving or growing networks. In this paper we present a complete picture of the behavior of another important social dilemma, the Stag Hunt game, under an evolutionary preferential attachment model, in which the network grows according to the dynamical states of the elements of the system. We observe the emergence of a scale-free and hierarchical organization of the strategies according to connectivity classes as a by-product of the diffusion of cooperation in the network. Depending on the parametrization of the game dynamics, we find a smooth transition from cooperation to defection and a polymorphic state with simultaneous presence of cooperator and defector hubs, which is very unusual in coordination games.

**Keywords:** applications to game theory and mathematical economics, interacting agent models, socio-economic networks

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## 1. Introduction

The problem of the emergence of cooperation in natural and social systems is related to important biological and socio-economical issues, and it has been the subject of intense research in recent years [1]–[3]. The mathematical approach to this problem, usually tackled under the general framework of evolutionary games on graphs, has attracted a lot of interest in the physics community recently [4]–[6]. It has been recognized that the well-mixed (mean-field) dynamics, in which every individual interacts with every other one, does not compare well in general with the results obtained by introducing some form of structure in the population. Here, structure is to be understood as a (possibly spatial, possibly complex) network that dictates who interacts with whom (typically, every individual with their nearest neighbors). On the other hand, interaction between individuals takes place through a game [7] that in general represents a social dilemma [8]. Social dilemmas are situations in which individual rationality leads to a poor result from the global viewpoint, and are thus particularly well suited to representing the problem of the emergence of altruism, given by the choice to cooperate (C), against short-term benefits of egoism achieved by defecting (D). Pioneering results on social dilemmas on structured populations have been obtained for both lattices and regular graphs (see, e.g., [9]–[15]) as well as for scale-free (SF) [16] networks (see, e.g., [17]–[19]) (see also [4, 5] for recent, comprehensive reviews). The results of all those studies were inconclusive in so far as it was found that the emergence of cooperation depended on the details of the network, the dynamics or update rule for the individuals and the specific game under consideration.

Among the different social dilemmas that can be considered, the great majority of the above mentioned works focus on the Prisoner’s Dilemma (PD), in which the emergence and the stability of cooperation are hampered by its most stringent conditions: to defect is a dominant strategy as it always yields larger payoff irrespective of what the other player does. Thus, cooperation is risky (it yields worse payoffs) and there would be temptation to

defect even if it were not [20]. However, there are many other types of dilemmas, prominent among which is the Stag Hunt game [21]. In this case, players want to coordinate, i.e., the best option is to do always as the opponent does; conflict arises because one of the two possibilities is more beneficial but more risky (when chosen by one player alone, it has a poorer payoff associated than coordinating in the other option). Correspondingly, in this game both mutual cooperation and mutual defection are evolutionary stable strategies [7] and thus global attractors of the game dynamics. This fact gives rise to an interesting bistable condition, so, depending on the initial frequencies of strategies, the system will end up in an all-C or in an all-D state. We note that from the dynamical viewpoint this is a much richer problem than the PD, where there is only one global attractor, namely all-D. This is more so when the problem is posed on a network, because then local initial conditions govern local convergence to one or the other attractor, leading to highly non-trivial dynamics [15] which is far from being understood.

From an applied perspective, and in spite of having been the subject of only a few studies, the Stag Hunt game captures the conflict between individual security and aggregate wellness, which naturally arises in many social patterns, and represents many situations better than the PD: indeed, this is the case when taking into account the effect of group selection, in which the social interactions work towards maximizing the group success [22], or when considering iterated games, in which the game is repeated and the players keep in mind the outcome of previous and future rounds [1]. The Stag Hunt game has been studied on networks to a much lesser extent than the Prisoner's Dilemma, particularly relevant examples being [15, 21], [23]–[27], the main conclusion being that, generally speaking, homogeneous degree networks (lattices or regular random graphs) promote the coordination in the preferred choice whereas heterogeneous graphs (e.g., SF ones) tend to reduce it (often slightly), albeit this depends on the update rule as is the case with the Prisoner's Dilemma. Finally, it is interesting to mention that several papers have addressed this question in an experimental framework, restricted of course to very small networks (less than ten individuals, see e.g. [28]–[30]). While these experiments shed some light on the micromotives governing individual behavior, it is clear that in order to understand larger (social) systems one needs to resort to theoretical studies like the present one.

In this paper, we address the issue of the emergence of cooperation in the Stag Hunt game on dynamic networks. The works quoted above have been performed on static networks, which are either endogenously or exogenously generated, but do not change in time. However, when networks coevolve with the game being played, high rates of cooperation have been reported, even for the Prisoner's Dilemma [31]–[39] (see [6] for a very recent review; see also [40, 41] for more economically oriented works). Our particular study is devoted to the case in which the network is not initially given, but rather it grows by successive addition of new individuals with new links. We note that this issue is relevant in the wider context of complex adaptive systems, as the debate ‘form follows function’ which arises in many situations of interest; what we are addressing here is indeed whether the mechanism of network growth can indeed give rise to structures that promote cooperation or have any other special property. As far as we know, the only attempt to address these issues in the context of evolutionary games on graphs is the evolutionary preferential attachment mechanism [42]–[44], which does not assume any *a priori* network: instead, new nodes are added to the network and link formation is ruled by the dynamical

states of its components, providing a feedback mechanism between evolutionary dynamics and topology that shapes the structure. The fitness of a node is neither determined by the instantaneous topology of the network nor imposed as an external constraint [45], but it is the result of the dynamical evolution of the system, being proportional to the payoffs obtained in the game. Nodes of the network are thus regarded as individuals involved in a social dilemma, and newcomers are more probably linked to nodes with high fitness, according to the preferential attachment rule.

The papers [42]–[44] considered the Prisoner’s Dilemma as the game giving rise to the network, finding that the resulting structures supported higher levels of cooperation than those reported for other networks. The aim of this paper is to give a complete picture of the behavior of the Stag Hunt game dynamics under the evolutionary preferential attachment model, also discussing what the role of the parameters involved is, the importance of the choice of the payoff matrix as well as the structure of the resulting networks. To this end, the paper is structured as follows. Section 2 introduces the model, including the evolutionary preferential attachment algorithm. Section 3 presents our main results, regarding the cooperation levels achieved and the structure of the network, as a function of the parameters of the attachment algorithm. Section 4 discusses those results and proposes explanations for them. Sections 5 and 6 are devoted to checking what the influence of the remaining parameters of the model is, including the game payoffs. Finally, section 7 summarizes our main conclusions and presents a few open questions.

## 2. The model

We begin by defining the game we are going to consider. Symmetrical  $2 \times 2$  games are often parametrized as follows: both players receive  $R = 1$  under mutual cooperation and  $P = 0$  under mutual defection, while a cooperator receives  $S \in [-1, 1]$  when confronted with a defector, who in turn receives  $T \in [0, 2]$ . In particular, the Stag Hunt game, the main object of our work, is characterized by the constraint  $1 > T > 0 > S$ . In order to simplify the problem, instead of using this two-parameter setup, we restrict ourselves to the following choice: taking the parameter  $r = T = -S \in [0, 1]$ , we have

$$\begin{array}{c} \text{C} \quad \text{D} \\ \text{C} \begin{pmatrix} 1 & -r \\ r & 0 \end{pmatrix} \\ \text{D} \end{array} \quad (1)$$

Thus we expect cooperation from the players for  $r \simeq 0$ , while defection will be more probable for  $r \simeq 1$ . The reason for this choice is that in the case of a well-mixed population and in many networks of interest, the whole parameter space for the Stag Hunt game divides into two regions, one with full cooperation and the other with full defection, with the boundary perpendicular to the line we have considered in the  $ST$ -plane (see, e.g., [5]).

We now introduce the dynamics. Following the previously discussed evolutionary preferential attachment mechanism [42, 44], the network starts at time  $t = 0$  with an initial core of  $m_0$  fully connected nodes. At equally spaced time intervals  $\tau_D$  every node  $i$  plays with all his  $k_i(t)$  neighbors, and receives a payoff  $f_i(t)$  according to matrix (1). At the end of the round each node  $i$  can imitate the strategy  $s_j$  of a random chosen  $j$

neighbor with a probability given by the replicator rule, in the form

$$P\{s_j^t \rightarrow s_i^{t+1}\} = \frac{f_j(t) - f_i(t)}{(1+r)\max[k_i, k_j]} \quad (2)$$

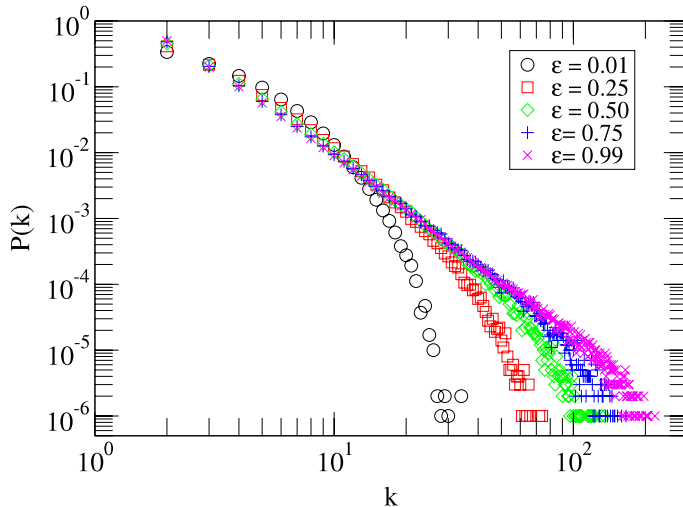
valid only if  $f_j(t) \geq f_i(t)$  (the probability to switch strategy is 0 in the opposite case). Then, at equally spaced time intervals  $\tau_T$ , a new element (cooperator or defector with equal probability) is added to the network, and the probability that a pre-existing node  $i$  receives one of the  $m$  links of the new one is linear in its payoff  $f_i$ , controlled by the selection parameter  $\varepsilon \in [0, 1]$  as

$$\Pi_i(t) = \frac{1 - \varepsilon + \varepsilon f_i(t)}{\sum_{j=1}^{N(t)} (1 - \varepsilon + \varepsilon f_j(t))} \quad (3)$$

where  $N(t)$  is the number of nodes of the network at time  $t$ . We prevent the possibility that a single element of the network can receive more than one link from an incoming node, and in the case of  $f_i(t) < 0$  for some  $i$ , we rescale all the payoffs by summing the minimum, in order to ensure  $\Pi_i \geq 0 \forall i$ , without changing the relative distance among the payoffs of each node in the system. Thus, within the framework of the preferential attachment mechanism, in the weak selection limit  $\varepsilon \rightarrow 0$  the growth is independent of the game dynamics, while in the strong selection limit  $\varepsilon \rightarrow 1$  high performing players with large payoff  $f_i$  are preferentially chosen and attract newcomers. The two associated time scales,  $\tau_T$  and  $\tau_D$ , control the ratio between the network's evolution and the game dynamics. If  $\tau_D/\tau_T > 1$  the network grows faster than the evolutionary dynamics, whereas in the other case the nodes play many times before adding a new element. Note that in the attachment procedure given by equation (3) we consider only the payoffs obtained in the last round played.

A word is in order to comment about our choice of the dynamics. It is well known by now that the outcome of evolutionary game theory depends strongly on the dynamics, as stated above and shown in [5]. Therefore, it is important to clarify that the dynamics we are using is directly inspired in the imitation phenomena observed in most social networks. Indeed, from the experimental viewpoint, there are several reports that indicate that imitation is commonly used by humans [46, 47], and that it can be justified in psychological terms by looking at how confirmation and disconfirmation of beliefs are carried out [48]. Imitation has also been proposed as a relevant force to drive the evolution toward economic equilibrium [49]. Finally, it has been recently shown in a computer tournament that imitation is the tool of choice for learning rather than asocial information, even if the latter is cost-free [50]. We thus believe that, while admittedly our results will only be valid for the dynamics we are studying and related ones, their relevance in social terms cannot be overstated.

We monitor the properties of the network at various steps of the evolution, such as  $N = 100, 400, 1000$ , and at the end of the network's growth,  $N = 2000$  nodes. Then we let the agents play the game without adding new elements, until the network reaches an asymptotic equilibrium; we define that such an equilibrium has been reached if for ten consecutive time steps there are not changes of strategy in any node, and in any case we stop the game after playing a maximum number of  $10^5$  rounds.



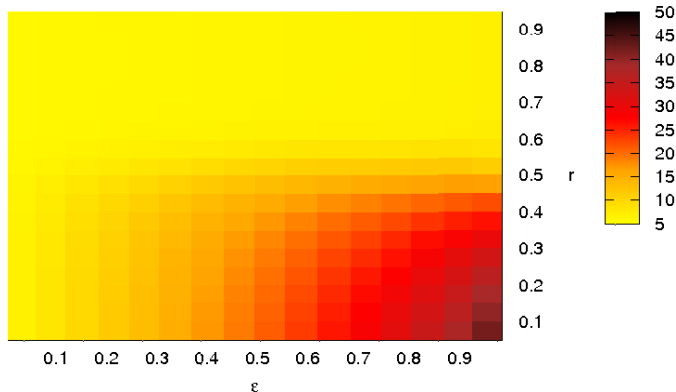
**Figure 1.** Degree distributions for several  $\varepsilon$ . A transition from homogeneous to SF networks is evident. The parameters are: temptation parameter,  $r = 0.3$ ; initial core,  $m_0 = 4$  nodes, 50% cooperators and 50% defectors; links per new node,  $m = 2$ ; and  $\tau_D = 10\tau_T$ , with  $\tau_T = 1$ , i.e., ten nodes are added for each game round we play. Results are the average of at least  $10^3$  realizations.

### 3. Results

Let us begin by looking at the phenomenology of the model in the  $\varepsilon$ - $r$  space while fixing the other parameters (whose influence will be considered in sections 5 and 6 below). We thus take an initial core of  $m_0 = 4$  nodes, made up of 50% cooperators and 50% defectors, each new element attaches to  $m = 2$  different pre-existing nodes, and we add a number of ten nodes for each game round we play (i.e.,  $\tau_D = 10\tau_T$ , with  $\tau_T = 1$ ). Each result we report is averaged over at least  $10^3$  realizations.

The evolutionary preferential attachment mechanism combined with the Stag Hunt game dynamics gives rise to a highly non-trivial interaction between the emergence of cooperation and the topology of the underlying network. As shown in figures 1 and 2, the degree heterogeneity of the resulting networks depends not only on the selection parameter  $\varepsilon$ , which is expected and consistent with previous findings [42], but also on the temptation parameter  $r$ , and thus on the final cooperation level. Figure 1 shows that in the low  $r$  region, two families of networks are obtained depending on the value of  $\varepsilon$ . In the limit  $\varepsilon \rightarrow 0$  we obtain homogeneous networks, characterized by a degree distribution with a tail that decays exponentially fast as the degree  $k$  increases. In the strong selection limit  $\varepsilon \rightarrow 1$ , on the contrary, scale-free networks with  $\gamma \simeq 3$  arise, when the temptation parameter  $r$  is sufficiently small to allow a high level of cooperation. The emergence of SF networks leads also to a hierarchical organization of cooperation in the local structure, the payoff received by the nodes and the strategies chosen being related to their degree, as we will show.

In order to present a global picture of the degree of heterogeneity of the resulting networks in the  $\varepsilon$ - $r$  space, we plot in figure 2 the variance  $\langle k^2 \rangle$  of the degree distribution



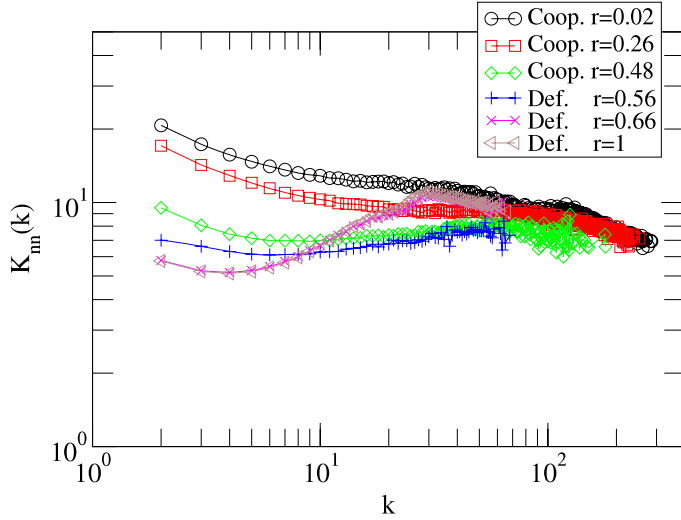
**Figure 2.** The variance  $\langle k^2 \rangle$  of the degree distributions, from (4), represented with a color code in the  $\varepsilon$ - $r$  space as indicated in the plot. The parameters are as in figure 1.

$P(k)$ , computed as

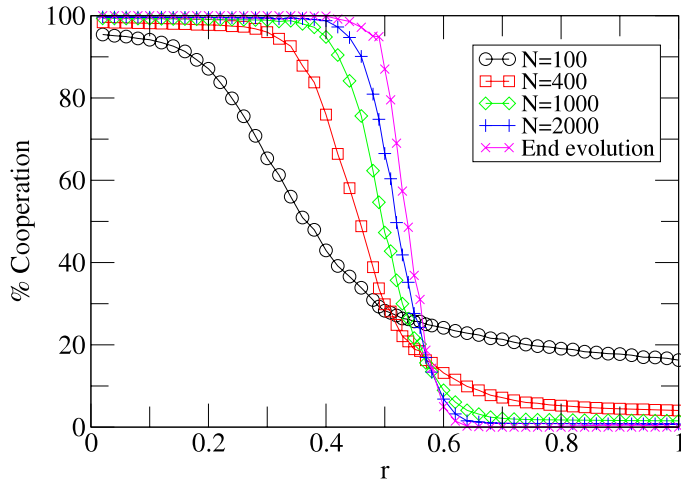
$$\langle k^2 \rangle = \sum_{k \in P(k)} (k - \bar{k})^2 p_k. \quad (4)$$

This magnitude provides us with a heuristic picture of the transition from homogeneous to SF networks, because the variance is expected to be higher for heterogeneous networks than for homogeneous ones, although it has to be understood as an indication because it is affected by strong finite-size effects. We see that the plot in figure 2 confirms the ideas suggested by figure 1, in the sense that networks with high variance (probably SF networks) are restricted to the high  $\varepsilon$ , low  $r$  region.

In order to provide more information about the network structure, it is convenient to look at other quantities, such as the degree correlation  $K_{nn}(k)$ , defined as the mean degree  $K$  of the neighbors of a node of degree  $k$ . This quantity is related to the degree of heterogeneity of the  $P(k)$  of the resulting topologies, and it presents two distinct behaviors in the  $\varepsilon$ - $r$  space. Indeed, in figure 3 we show that, when the network is homogeneous, the degree correlation has an assortative character to a certain extent, which means that lowly connected nodes are linked to each other, and the same takes place for highly connected elements. On the contrary, when SF networks arise, for  $\varepsilon \rightarrow 1$  and small  $r$ ,  $K_{nn}(k)$  presents more of a disassortative behavior, with cooperators hubs more often linked to lowly connected nodes and vice versa. Another relevant magnitude to monitor is the clustering coefficient, i.e., the number of triangles actually realized on the network over the possible ones. This quantity (figure not shown) remains far away from typical values of real networks, and its behavior in the  $\varepsilon$ - $r$  space is similar to that of the degree of heterogeneity. The maximum value,  $\langle CC \rangle_{\max} = 3\%$ , is reached in the region of SF networks, for  $\varepsilon \rightarrow 1$  and  $r \rightarrow 0$ , while in the rest of the plot it falls down to a constant value close to that of ER random graphs. This is clearly related to the fact that, when forming SF networks, indeed, new elements attach with high probability to the nodes of the initial core (which will be the hubs of the network), which are linked to each other, closing the triangle and increasing the clustering coefficient.



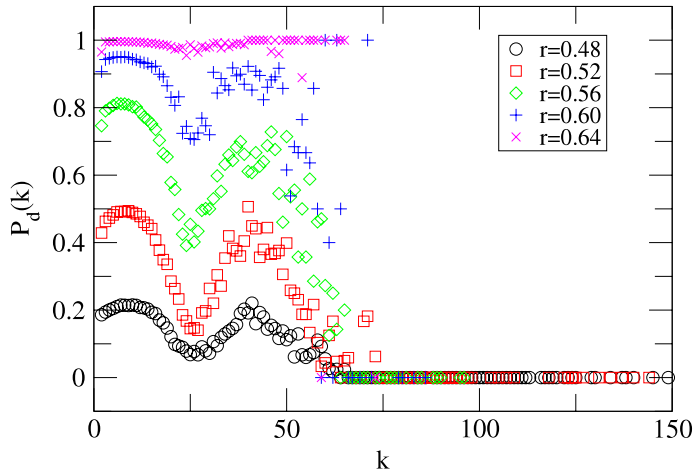
**Figure 3.** Degree-degree correlations obtained for  $\varepsilon = 0.99$  and several values of  $r$ . A change from some disassortative to some assortative character increasing  $r$  can be observed. The parameters are as in figure 1.



**Figure 4.** Average cooperation level for  $\varepsilon = 0.99$  as a function of  $r$  at various steps of the network evolution. The shape of the transition becomes abrupt when the network reaches the equilibrium. The parameters are as in figure 1.

Having discussed the structure of the generated networks, we now turn to the cooperative behavior of the agents. We find that the final average cooperation level reached does not depend on  $\varepsilon$ , which means that cooperation is not specially favored within the framework of evolutionary preferential attachment, unlike the case of the Prisoner's Dilemma [42]. In figure 4 we focus on the strong selection limit  $\varepsilon = 0.99$ , and we show that the shape of the transition from cooperation to defection changes during various steps of network's evolution, producing an overall enhancement of the average cooperation level. While at the beginning the transition is smooth, as the network grows it becomes more abrupt, until, at the end of the game, the range of  $r$  polarizes into two distinct regions,





**Figure 5.** Probability that a node with connectivity  $k$  plays as a defector for different values of  $r$  and  $\varepsilon = 0.99$  at the end of the growth of the network. Hubs or highly connected nodes are always occupied by cooperators. The parameters are as in figure 1.

one with an all-C configuration, for small  $r$ , and the other with all-D, for large  $r$ . If we check the relaxation time needed to get the asymptotic equilibrium, measured in number of rounds, we discover a peak just on the critical point of the transition from cooperation to defection, due to the unstable character of the resulting network, which takes a long time to polarize into an all-C or all-D configuration. We believe that the fact that in our model cooperation is not promoted with respect to the case of static networks is not surprising: indeed, previous findings [15, 17] show that the enhancement of cooperation due to a heterogeneous underlying structure is quite small for Stag Hunt games (unlike other social dilemmas), and in all cases it is restricted to the region for  $T \rightarrow 1$  and  $S \rightarrow 0$ , while we are considering the representation  $S = -T$ . It is interesting to note, however, that the transition from cooperation to defection takes place at a value of  $r$  slightly larger than that found for standard SF networks. On the other hand, cooperation in Stag Hunt games is increased by the presence of clustering, but as in our case it remains quite low, it does not lead to a noticeable improvement of the cooperation level.

In spite of the fact that cooperation as a whole is not much enhanced, the resulting networks show a hierarchical organization of cooperation in the local structure of the network, which is a key feature in order to understand how the evolutionary preferential attachment mechanism works. In figure 5 we show that when there is simultaneous presence of cooperators and defectors in the network, for  $r$  close to the critical point, the hubs, or highly connected nodes, are always occupied by cooperators, receiving a very large payoff, while defectors are localized in the intermediate and low degree classes, with small payoffs. The same hierarchical organization of cooperation, with the hubs always occupied by cooperators, has been also found for the PD on static SF networks [18]. On the other hand, using the evolutionary preferential attachment mechanism based on the PD game, this statement is no longer true and hubs may be occupied by defectors too [44]. Therefore, it is clear that the organization of the agents on the network depends on the game and is not only a consequence of the attachment rule for the links.

## 4. Discussion

Having presented our main simulation results, we now can understand how the evolutionary preferential attachment model acts coupled with the game dynamics, and why SF networks arise only in the presence of cooperation for  $\varepsilon \rightarrow 1$ . Imagine the evolution of a cooperator  $i$  of degree  $k$  in the initial phase of the network growth. If the node  $i$  reaches a ‘critical’ degree  $k_c$ , its payoff could be high enough (in particular if  $i$  is connected to a majority of cooperators) to attract new nodes. If they are cooperators, they increase  $i$ ’s payoff, while if they are defectors, they cannot change the strategy of node  $i$  and in the end they will imitate  $i$ ’s strategy, reinforcing even more the group, until the cooperator ( $i$ ) becomes a hub. Conversely, if  $k$  is less than  $k_c$ ,  $i$  cannot defend itself from the incoming defectors’ attack, which invades first the neighborhood of  $i$  and sooner or later itself too. The group becomes composed of only defectors, all the elements receive a zero payoff and they are not likely to attract new nodes, and hence the group eventually stops its growth at an intermediate degree.

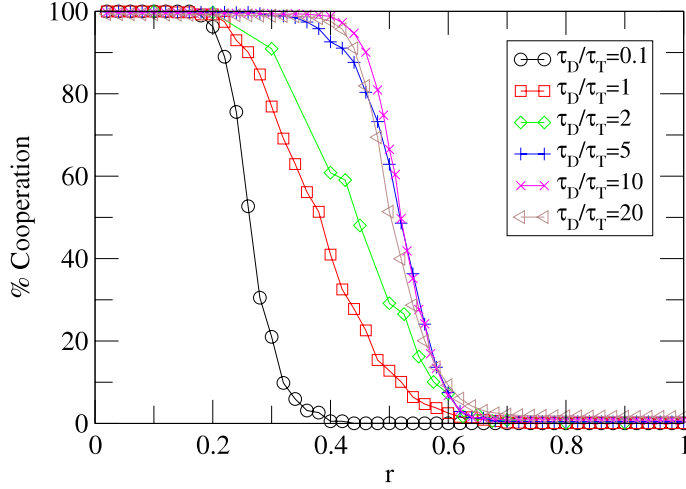
If  $r$  is small, there are many cooperators in the first phase of the network’s evolution, and some of them can easily reach the critical degree  $k_c$ , becoming hubs and giving rise to an SF network. On the contrary, large  $r$  prevents this mechanism and homogeneous networks are produced. Cooperators are not able to achieve  $k_c$ , because they become defectors long before, and the presence of only defectors implements a generalized zero payoff, preventing the preferential attachment mechanism from working, and new nodes simply attach randomly.

In the case  $\varepsilon \rightarrow 0$ , the preferential attachment is switched off, so homogeneous networks with zero cooperation level are produced. The game dynamics prevails, preventing the emergence of cooperation even in the first phase of the evolution. On the other hand, for  $\varepsilon \rightarrow 1$  and for hostile dynamics conditions for cooperation ( $r \rightarrow 1$ ), when just a few nodes are present at the beginning of the growth, the incoming cooperators attach to the highest connected nodes and preserve a small payoff, thanks to the preferential attachment. In this phase ( $N \simeq 100$ ) the network is ‘unstable’, because small clusters of cooperators can survive together with a majority of defectors. Then at a certain instant of the evolution, the game dynamics prevails and the existing cooperators are invaded by defectors, while the incoming nodes are not enough compared to the network’s size to preserve a considerable payoff. Thus new elements begin to attach randomly and the network loses the possibility of becoming SF.

## 5. Influence of the parameters

In section 4, we have presented our general understanding of the model behavior. It is now time to focus on how the variation of some of its parameters, such as the initial conditions or the time scales, affects the picture obtained with the standard parameters discussed above. This is an important issue in so far as, in the case of static networks, as well as for the evolutionary approach on well-mixed populations, the Stag Hunt game has been found to be very sensitive to the initial conditions, more than other social dilemmas, basically due to its bistable character.

*Initial conditions.* In our model the initial conditions are given by the size of the initial core  $m_0$  and its composition, but they strongly influence the average level of cooperation



**Figure 6.** Average cooperation level as a function of  $r$  for several values of the time scale  $\tau_D/\tau_T$ . The enhancement of cooperation for high time scales is evident. The other parameters are the same as in figure 1.

reached by the system, thus being consistent with previous results. Besides, in the model we present here, the topological structure of the resulting networks is greatly affected by these initial conditions as well. Indeed, increasing the size of the core or its composition in cooperators leads to a remarkable enhancement of cooperation. For instance, if we start with an initial core of  $m_0 = 10$  and all cooperator nodes, we get a final state of all-C even for  $r \simeq 1$ . Although the initial core is small compared to the final size of the network, the evolutionary mechanism of the preferential attachment emphasizes the role of the first elements, which determine the entire evolution of the network. Conversely, letting  $\varepsilon \rightarrow 0$ , the effect of the initial core is considerably smaller.

*Time scales.* The time scale  $\tau_D/\tau_T$  plays a fundamental role in determining the action of the evolutionary preferential attachment mechanism on the network's structure and on the emergence of cooperation. We have verified that, by modifying the ratio  $\tau_D/\tau_T = 0.1, 1, 2, 5, 10, 20$ , we change the interaction between the game dynamics and the evolutionary topology. If  $\tau_D/\tau_T > 1$ , then the evolution of the network is faster than the selection pressure given by the game dynamics, while the choice  $\tau_D/\tau_T < 1$  is not very realistic, because selection, understood as strategies evolution, should be slower than the network growth. In figure 6 we show that a small time scale  $\tau_D/\tau_T \lesssim 1$  gives rise to a reasonably low average level of cooperation, but if we increase that time ratio, the level of cooperation also increases for all values of  $r$ . Of course, there is a value  $\tau_D/\tau_T \simeq 10, 20$ , for which the cooperation in the system cannot be improved any further. The preferential attachment mechanism, clearly, makes sense only if a relatively large number of nodes are attached at each round, i.e.,  $\tau_D/\tau_T \gg 1$ . In this way, at each game round the stochastic fluctuations due to the incoming elements make it possible that a node, with a payoff slightly higher than the others, attracts a good number of new elements. Thus it can increase its payoff, attracting more and more nodes in a cascade process. On the other hand, if we attach only a few nodes each round or we play many times before adding a node,  $\tau_T/\tau_D \lesssim 1$ , this possibility does not exist: the game dynamics, given by the payoff

matrix, is the only element which determines the network's evolution. The system evolves quickly either to an all-C or to an all-D configuration, depending on whether  $r$  is small enough to make the cooperators' payoffs higher than the defectors' ones, or vice versa.

We have looked for confirmation of this behavior by considering the transition from cooperation to defection during the network growth at various steps. Unlike the case  $\tau_D/\tau_T = 10, 20$ , if  $\tau_D/\tau_T \lesssim 1$  the shape of this curve does not change during the evolution, and the cooperation level reached at the starting point is the same as that at the end of the game, for any  $r$ . Cooperation and defection cannot coexist, and the strategies of the elements of the network do not evolve, remaining frozen at the initial state: either all-C or all-D. In this case, the composition of the initial core is crucial for the final cooperation level reached, because the payoffs of the core determine all the evolution. Correspondingly, the transition point from cooperation to defection at small time scales, indeed, is close to the value of  $r$ , which ensures the same probability for either a cooperator or a defector of the core to be chosen.

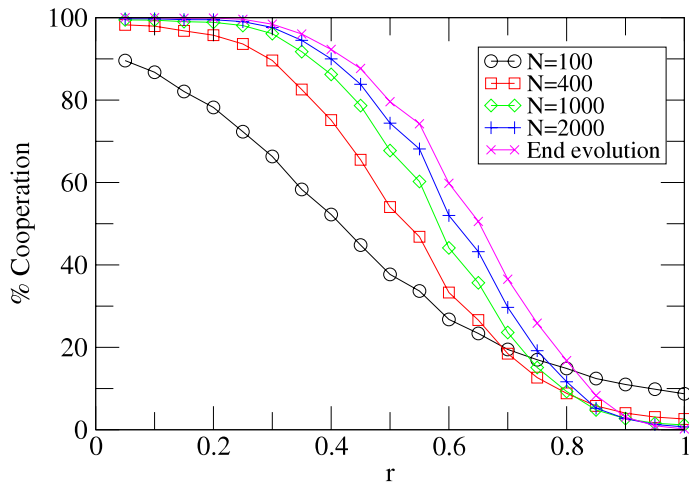
*Mean degree and final size.* In contrast to the two other aspects we have mentioned above, the mean degree  $\langle k \rangle$  and the final size  $N$  of the resulting network affect only slightly the outcome of the model. If we increase the mean degree  $\langle k \rangle = 2m$ , determined by the number  $m$  of nodes which every new element attaches to, we make the network more interconnected, in the sense that each node interacts with many others, and as a consequence the local structure is less relevant and we go toward a mean-field approximation. Our results indicate that increasing  $m = 2, 3, 4, 5$  leads to a slight but progressive decrease of the average level of cooperation, while on increasing the final size of the network  $N = 1000, 2000, 5000, 10\,000$  the transition from cooperation to defection becomes more abrupt. In view of this, it is reasonable to put forward the hypothesis that for large  $\langle k \rangle$  and in the limit  $N \rightarrow \infty$ , the shape of the transition approaches asymptotically that of a complete graph (well-mixed population).

## 6. Role of the game dynamics

Aside from the parameters considered in section 5, there is another important question to address, namely how the parametrization of the game dynamics affects model behavior. This is what we would like to discuss now in a section of its own, because of its dramatic effect on the results. In principle, one would expect that choosing different forms of the payoff matrix, while maintaining the Stag Hunt game constraints, would not affect the average level of cooperation reached and the main topological properties of the resulting networks with respect to those found with the choice (1). In order to verify this hypothesis, we have carried out simulations of our model with a different parametrization of the payoff matrix, without changing the relative difference between the payoffs of the players and fixing the other parameters to  $\varepsilon = 0.99$ ,  $\tau_D/\tau_T = 10$ ,  $m_0 = 4$ ,  $m = 2$  and  $N = 2000$  as before. As it turns out, the most interesting results arise when each term of the payoff matrix is positive, and therefore we present here a representative situation, given by

$$\begin{pmatrix} 2 & 1 - r \\ 1 + r & 1 \end{pmatrix}. \quad (5)$$

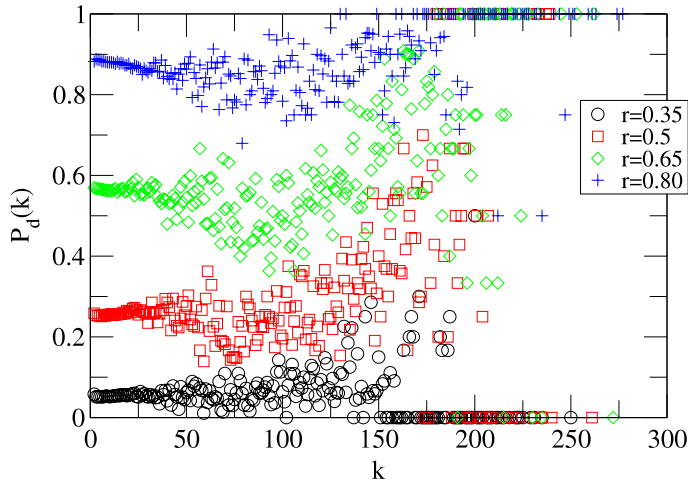
With the above choice, the emergence of cooperation and the topological structure of the resulting networks are completely different from the case of the form (1). This is



**Figure 7.** Average cooperation level as a function of  $r$  at various steps of the network’s evolution with the form (5) of the payoff matrix. Even when the dynamics stops, the transition from cooperation to defection remains smooth. The parameters are  $\varepsilon = 0.99$ ,  $\tau_D/\tau_T = 10$ ,  $m_0 = 4$ ,  $m = 2$  and  $N = 2000$ .

clearly demonstrated in figure 7, where we show that there is a progressive increment of the average level of cooperation during the network growth and that, even more importantly, the shape of the transition from cooperation to defection is very smooth at every step of the evolution, even after the achievement of the asymptotic equilibrium. Thus, although the Stag Hunt game dynamics predicts a bistable character with two evolutionary equilibria, given by an all-C or an all-D configuration, in this case there is a wide range of game payoffs in which the network ends up in a mixed-strategy or in a polymorphic state, without converging to either of the two Nash equilibria of the basic game. This result is confirmed by the behavior of the relaxation time needed to get the equilibrium, which does not have a peak but remains very high for the values of  $r$  which correspond to this smooth crossover.

To our knowledge, smooth transitions from cooperation to defection in the Stag Hunt game have been observed only in models built using real social networks with peculiar structure or on artificial models designed to mimic real ones [25], while generally the final configuration in static networks is given by only cooperators or only defectors. The reason for this mixed state has recently been traced back to the appearance of topological traps or bottlenecks in the network that prevent the propagation of one of the strategies to the rest of the network [27]. This is in agreement with our finding here that the topological structure of the resulting networks is now independent of the cooperation level. Indeed, using this form of the payoff matrix, scale-free networks with a disassortative character of the degree correlation arise for any value of  $r$ . The clustering coefficient is constant with  $r$  and quite low  $\langle CC \rangle = 2\%$ . In addition, figure 8 shows another striking feature emerging from the action of the game dynamics: plotting the strategies as a function of the nodes’ degree for some values of  $r$ , we realize that the hubs are occupied by both cooperators and defectors which can even coexist, unlike the other case presented above, but in good agreement with some previous results [44].



**Figure 8.** Probability that a node with degree  $k$  plays as a defector for different values of  $r$  at the end of the network’s growth, with the parametrization (5). Hubs are occupied by both cooperators and defectors. The parameters are as in figure 7.

Understanding the behavior of the model is not difficult if we note that the payoff of highly connected nodes is enhanced more than for the rest of the nodes by using this form of the payoff matrix. Adding a positive quantity to each term of the form (1), we are in fact giving an advantage both to the growth of the hubs and to their influence in the opinion changing. On the one hand, highly connected nodes collect more payoff than the others and then they attract with more probability the new incoming elements, thanks to the preferential attachment mechanism. On the other hand, the update rule becomes

$$P'\{s_j^t \rightarrow s_i^{t+1}\} \propto f_j'(t) - f_i'(t) = f_j(t) - f_i(t) + \Delta k_{ij}, \quad (6)$$

making it more probable that lowly connected nodes will imitate the hubs. Therefore, in this case, a defector with an intermediate degree can grow, becoming a hub, because he collects a good payoff also if he is linked to other defectors, contrary to what happens with the parametrization (1). For this reason, hubs of cooperators and defectors can simultaneously grow, leading to a polymorphic state in which a part of the network plays as cooperator and the other as defector, and neither one of the two populations can invade the other. The disassortative structure of the degree correlation confirms this hypothesis, because the defector and cooperator hubs are not directly linked to each other, as found also in [44] and in agreement with the existence of topological traps [27]. We thus find a picture in which a set of nodes with low degree oscillate between cooperation and defection, also found in [51], while the hubs do not change their strategies, giving rise to high relaxation times. Therefore, within this form of the payoff matrix, the evolutionary preferential attachment mechanism ‘works’ for any value of  $r$ : it produces SF networks which support a high level of cooperation and, even more interestingly, a polymorphic state in which hubs of cooperators and defectors coexist. We have checked that in this case SF networks are better than homogeneous ones at supporting cooperation: the average level of cooperation reached, even with the presence of defector hubs, is higher than in the case of static networks.

## 7. Conclusions

In summary, we have introduced and studied a model which incorporates a competition between structural and dynamical patterns, because the rules governing the formation of the network are linked to the dynamics of its components. While this competition was first discussed in the framework of the Prisoner's Dilemma, for this work we have considered a less demanding social dilemma such as the Stag Hunt game, in which agents basically aim to coordinate their choices. Our main conclusion is that, even if some of the results found for the Prisoner's Dilemma are recovered, there are many new features arising here that make clear that both the attachment dynamics and the game through which agents interact are determinant to the fate of the resulting network. This in turn implies that the non-trivial interaction between the emergence of cooperation and the topology of the underlying network depends on the specific social situation one is considering.

A very important result arising from the specific choice of the Stag Hunt game is that the model behavior and the hierarchical organization of cooperation strongly depend on the parametrization of the game dynamics. With some choices of the payoff matrix, indeed, the emergence of SF networks, with hubs always occupied by cooperators, depends on the temptation parameter  $r$ , and thus it is the direct product of the diffusion of cooperation in the network. On the other hand, choosing a different parametrization without changing the difference between the players' payoff, the topological structure of the resulting networks is independent of the game dynamics, and we obtain a polymorphic state (uncommon for the Stag Hunt game) with the simultaneous presence of cooperator and defector hubs. Even more interestingly, this configuration gives rise to a smooth transition from cooperation to defection, also very unusual given the Stag Hunt bistable game dynamics, while the average cooperation level is considerably increased with respect to the static networks case. For a given payoff matrix, we want to stress that the model behavior strongly depends on the setting of some of the parameters involved, such as the initial condition, due to the evolutionary nature of the preferential attachment mechanism, and the time scale at which the dynamical process takes place on the structure, and therefore their choices to model specific problems or systems should be carefully justified. This conclusion is in agreement with the realization that the outcome of evolutionary games depends strongly on details such as the payoff structure of the game, the characteristics of the update rule or simply the parameters involved [5, 15], and confirms that even if the evolutionary preferential attachment rule tends to generate SF networks in many cases, it is not an easy task to predict the cooperation level they will give rise to.

From a more general viewpoint, we have seen that the evolutionary preferential attachment model is a powerful and flexible instrument in order to explore the effect of different game dynamics on the emergence and diffusion of cooperation in heterogeneous networks, and thus it is a good candidate to explain why scale-free networks are among the best ones to support cooperation. On more theoretical grounds, the interaction between evolutionary games and complex network theory is a new and hot topic in the scientific research scenario. Along the lines presented here, an open question to address in future research is to explore the behavior of the Stag Hunt game in the  $S$ - $T$  plane, which can give us a complete picture of how the outcome of the model is determined by the game dynamics. It is interesting, moreover, to search for some form of continuity between

different games in the cooperative behavior, looking at the boundary regions with other social dilemmas. Finally, we leave for future research the question of whether equation (3) can be applied to other sorts of dynamics by appropriately defining the dynamical variable  $f_i(t)$  and adjusting the growth rules. In this respect, we note that modifications of the algorithm including some link rewiring could be of interest in connection with recent experimental results [52]. Work along these lines is in progress.

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