Approximate analytic temperature distribution and efficiency for annular fins of uniform thickness

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Abstract

The salient feature in the quasi one-dimensional differential equation for annular fins of uniform thickness is without question the presence of the variable coefficient $1/r$ multiplying the first order derivative, dT/dr. A good-natured manipulation of the variable coefficient $1/r$ is the principal objective of the present work. Specifically, the manipulation applies the mean value theorem for integration to $1/r$ in the proper fin domain extending from the inner radius r_1 to the outer radius r_2. It is demonstrated that approximate analytic temperature profiles and heat transfer rates of good quality are easily obtainable without resorting to the exact analytic temperature distribution and heat transfer rate embodying modified Bessel functions. For enhanced visualization, the computed temperature profiles, tip temperatures and fin efficiencies of approximate nature are graphed and tabulated for realistic combinations of the normalized radii ratio c and the thermo-geometric fin parameter ξ of interest in thermal engineering applications.

1. Introduction

Thermal designers face two fundamental questions when dealing with a bundle of annular fins of uniform thickness to be attached to a round tube or a solid rod at a temperature higher than the surrounding fluid. The two fundamental questions are: (1) What is the heat transfer rate from a single annular fin to the fluid and (2) what is the tip temperature of such a fin that complies with safety standards?

Under the classical one-dimensional formulation, the temperature descend along an annular fin with uniform thickness is governed by a differential equation of second order having a variable coefficient of intricate form, $1/r$, that accompanies the first order derivative dT/dr. By means of a temperature excess related to the dependent variable, the differential equation can be homogenized resulting in a modified Bessel equation of zero order. In the homogeneous differential equation, the variable coefficient $1/r$ is troublesome and may be viewed as a hyperbola segment inside the fin domain that extends from the inner radius r_1 to the outer radius r_2.

Previous efforts aimed at simplifying the modified Bessel equation for annular fins with uniform thickness by analytic means are nonexistent in the specialized literature. However, some marginal attempts linked to heat transfer estimates are worth mentioning. Harper and Brown [1] suggested that the hyperbolic function based efficiency for longitudinal fins of uniform thickness is suitable to approximate the efficiency of annular fins with uniform thickness provided that the radii ratio c is close to unity. This 'longitudinal fin approximation' was extended to intermediate c values by Schmidt [2] substituting the fin height by an equivalent height. This is perhaps the best known analytic simplification for the annular fin efficiency. Other recent works by Hong and Webb [3] and Perrotin and Clodic [4] added terms to the solution presented by Schmidt, but at the price of increasing its complexity and restricting the betterment of the results to a certain thermo-geometric region. It should be emphasized that the above-mentioned analytic ap-

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Nomenclature

Bi	transversal Biot number, ht/k	
c	normalized radii ratio, r_1/r_2	
EBi	enlarged Biot number, ξ^2	
E_η	relative error of the approximate η	
E_t	relative error of the approximate $\theta(1)$	
h	mean convection coefficient	$\mathrm{W\,m^{-2}\,K^{-1}}$
I_v	modified Bessel function of first kind and order v	
k	fin thermal conductivity	$\mathrm{W\,m^{-1}\,K^{-1}}$
K_v	modified Bessel function of second kind and order v	
L	fin length, $r_2 - r_1$	m
m	parameter related to the efficiency of the longitudinal fin in Table 3	
M_{f1}	mean value of the function $1/R$ in $[c, 1]$	
M_{f2}	$1/\overline{R}$	
M_{f3}	mixed mean, $\frac{1}{2}(M_{f1} + M_{f2})$	
n	parameter related to the efficiency of the longitudinal fin in Table 3	
Q	heat transfer rate	W
Q_i	ideal heat transfer rate	W
r	radial coordinate	m
r_1	inner radius	m
r_2	outer radius	m
R	normalized r, r/r_2	
$\bar{R}$	mean value of the function R in $[c, 1]$	
t	semi-thickness of the annular fin	m
T	temperature	K
T_b	base temperature	K
T_∞	fluid temperature	K

Greek letters

β	dimensional thermo-geometric fin parameter, $(h/kt)^{1/2}$	$\mathrm{m^{-1}}$
γ	dimensionless group, $\xi/(1-c)$	
η	fin efficiency or dimensionless Q, Q/Q_i	
θ	normalized T, $(T - T_\infty)/(T_b - T_\infty)$	
$\lambda_{1,2}$	roots of the auxiliary equation (14)	
ξ	dimensionless thermo-geometric fin parameter, $\beta \cdot L$	

Subscripts

b	base
i	ideal
t	tip
∞	surrounding fluid

proximations are unrelated to the temperature solution of the modified Bessel equation.

In order to generate uncomplicated and powerful temperature profiles and heat transfer rates for annular fins of uniform thickness, the present study pursues a new analytic methodology. It embraces the concept of the mean value theorem articulated with the simplicity inherent to the standard longitudinal fin of uniform thickness. Thereby, the idea is to replace the cumbersome variable coefficient $1/r$ in the fin equation governing an annular fin of uniform thickness with a constant coefficient thanks to the mean value theorem for integration. In the implementation of the mean value theorem, one viable possibility is to substitute $1/r$ by the mean value of $f(r) = 1/r$ in the proper fin domain $[r_1, r_2]$. Another attempt is to multiply the differential equation by the independent variable r, so that it contains two variable coefficients r, one accompanies the second order derivative dT^2/dr^2 in the first term and the other accompanies T in the third term. Thereby, the troublesome variable coefficient r is just a straight line segment inside the fin domain $[r_1, r_2]$, so that r is substituted by the mean radius.

A direct consequence of replacing the variable coefficient $1/r$ by the mean value of the function $f(r) = 1/r$, or the variable coefficient r by the mean value of the function $g(r) = r$ is that the two transformed differential equations hold constant coefficients and are no longer of Bessel type. It is envisioned that the two computational procedures may facilitate the determination of the approximate analytic temperature distribution and heat transfer rate for annular fins of uniform thickness in terms of the two controlling parameters, one the normalized radii ratio c and the other the dimensionless thermo-geometric parameter ξ. Undoubtedly, the main objective for undertaking this work is to bypass the evaluation of the pair I_v and K_v; the modified Bessel functions of first kind and second kind of order v. Even with contemporary numerical and symbolic algebra codes, like *Mathematica, Maple and Matlab*, these evaluations are elaborate and tedious.

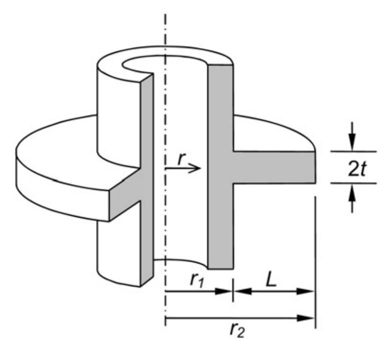

Fig. 1. Schematic of an annular fin of uniform thickness.

2. Formulation of the problem

An annular fin of constant thickness $2t$, inner radius r_1 and outer radius r_2 dissipating heat by convection to a surrounding fluid is shown in Fig. 1. In the modeling, the most common Murray–Gardner assumptions (Murray [5], Gardner [6] and Kraus [7]) are adopted: steadiness in heat flow; constant thermal conductivity k; uniform heat transfer coefficient h; unvarying fluid temperature T_∞; prescribed fin base temperature T_b; preponderance of radial temperature gradients over those in the transversal direction; negligible heat transfer at the outermost fin section (i.e., adiabatic fin tip); and null heat sources or sinks. In harmony with the classical formulation, the normalized temperature $\theta(R)$ along the radial direction obeys the following quasi one-dimensional fin equation in cylindrical coordinates (Incropera and DeWitt [8], Schneider [9]):

$$\frac{d^2\theta}{dR^2}+\frac{1}{R}\frac{d\theta}{dR}-\frac{\xi^2}{(1-c)^2}\theta=0 \quad \text{in } c\leqslant R\leqslant 1 \tag{1}$$

This second-order ordinary differential equation is classified as a modified Bessel equation of zero order. It possesses constant coefficients in the first and third terms and a variable coefficient $1/R$ of awkward form in the second term. Certainly, the presence of the variable coefficient complicates the analytic solution.

The boundary conditions for prescribed temperature at the base $R=c$ and zero heat loss at the tip $R=1$ are:

$$\theta(c)=1 \quad \text{and} \quad \frac{d\theta(1)}{dR}=0 \tag{2}$$

In the preceding trio of equations, θ is the normalized temperature $(T-T_\infty)/(T_b-T_\infty)$, R is the normalized radius r/r_2, c is the normalized radii ratio r_1/r_2, and ξ is the dimensionless thermo-geometric fin parameter $\beta L=(hr_2^2(1-c)^2/(kt))^{1/2}$; a quantity of relevance in the study of fin heat transfer [9]. Using the transversal Biot number $Bi=ht/k$ as an adequate reference parameter, ξ^2 is called the enlarged Biot number $EBi=(ht/k)((r_2-r_1)/t)^2$, which accounts for the fin length $L=r_2-r_1$ instead of the inner radius r_1 chosen by Ullmann and Kalman [10].

The indirect route for determining the heat transfer rate Q from straight fins of tapered profile and nonstraight fins of any profile to a neighboring fluid has been conveniently channeled through the dimensionless heat transfer or fin efficiency $\eta=Q/Q_i$. In this ratio proposed originally by Gardner [6], Q is the actual heat transfer rate and Q_i is an ideal heat transfer rate from an identical reference fin, but having infinite thermal conductivity $k\to\infty$. Thereby, the computation of η for the annular fin of constant thickness may be carried out in two ways: (1) differentiating $\theta(R)$ at the fin base $R=c$:

$$\eta_1=-\frac{1}{\xi^2}\left[\frac{2c(1-c)}{1+c}\right]\frac{d\theta(c)}{dR} \tag{3a}$$

or (2) integrating $\theta(R)$ over the dimensionless fin length from the base $R=c$ to the tip $R=1$:

$$\eta_2=\frac{2}{1-c^2}\int_c^1\theta(R)R\,dR \tag{3b}$$

3. Exact analytic procedure

The exact analytic solution of Eq. (1) satisfying Eq. (2) can be found in [8,9], among other heat transfer textbooks. It provides the dimensionless temperature distribution $\theta(R)$ involving modified Bessel functions:

$$\theta(R)=\frac{I_1(\gamma)K_0(\gamma R)+I_0(\gamma R)K_1(\gamma)}{I_1(\gamma)K_0(\gamma c)+I_0(\gamma c)K_1(\gamma)} \tag{4}$$

where for conciseness γ designates the dimensionless group $\xi/(1-c)$. Here, $I_v(\cdot)$ is the modified Bessel function of first kind and $K_v(\cdot)$ is the modified Bessel function of second kind, both of order v.

When an exact $\theta(R)$ is secured from Eq. (4), the two possible η-avenues in Eqs. (3a) and (3b) coalesce into the exact dimensionless heat transfer or fin efficiency,

$$\eta=\frac{1}{\xi}\frac{2c}{1+c}\frac{I_1(\gamma)K_1(\gamma c)-I_1(\gamma c)K_1(\gamma)}{I_1(\gamma)K_0(\gamma c)+I_0(\gamma c)K_1(\gamma)} \tag{5}$$

because the heat loss from the fin tip is zero[1]. In this regard, Arpaci [11] has stated that whenever $\theta(R)$ is approximate, the integration approach η_2 should be preferred over the differentiation approach η_1.

4. Approximate analytic procedures

Option 1: Let us isolate the disturbing variable coefficient $1/R$ in Eq. (1) and consider it as a function $f(R)=1/R$ outlining a hyperbola segment in the closed interval $c\leqslant R\leqslant 1$ in which R operates. Upon applying the mean value theorem for integration to this function, the end result is

$$M_{f1}=\frac{1}{1-c}\int_c^1\frac{1}{R}\,dR=\frac{\ln c}{c-1} \tag{6}$$

where M_{f1} designating the functional mean of $1/R$ depends solely on the radii ratio c. Therefore, replacing $1/R$ with M_{f1} in Eq. (1), the descriptive fin equation is transformed to

$$\frac{d^2\theta}{dR^2}+M_{f1}\frac{d\theta}{dR}-\frac{\xi^2}{(1-c)^2}\theta=0 \quad \text{in } c\leqslant R\leqslant 1 \tag{7}$$

Option 2: An alternative functional mean thorough the optic of R is plausible rewriting Eq. (1) as:

$$R\frac{d^2\theta}{dR^2}+\frac{d\theta}{dR}-\frac{\xi^2}{(1-c)^2}R\theta=0 \quad \text{in } c\leqslant R\leqslant 1 \tag{8}$$

Let us isolate the variable coefficient R in Eq. (8) and consider it as a function outlining a straight line segment in the closed interval $c\leqslant R\leqslant 1$. Owing to the mean value theorem for integration, the mean radius $\overline{R}$ leads to

$$\overline{R}=\frac{1}{1-c}\int_c^1 R\,dR=\frac{1+c}{2} \tag{9}$$

which depends on c only. Subsequently, upon defining $M_{f2}=\frac{1}{\overline{R}}=\frac{2}{1+c}$ to preserve uniformity, Eq. (8) is converted to:

$$\frac{d^2\theta}{dR^2}+M_{f2}\frac{d\theta}{dR}-\frac{\xi^2}{(1-c)^2}\theta=0 \quad \text{in } c\leqslant R\leqslant 1 \tag{10}$$

[1] Although the objective of this work is framed in the context of common heat dissipative fins, the calculation procedure applies equally well to the opposite situation dealing with heat absorbing fins.

Table 1
Dependence of the mean values M_f on the radii ratio c

c	M_{f1}	M_{f2}	M_{f3}
0.2	2.01	1.67	1.84
0.4	1.53	1.43	1.48
0.6	1.28	1.25	1.26
0.8	1.12	1.11	1.11
1.0	1.00	1.00	1.00

Option 3: The mixed mean of the two previous functional means M_{f1} and M_{f2} may be conceived as a compromise between *Option 1* and *Option 2*. That is,

$$M_{f3} = \frac{M_{f1} + M_{f2}}{2} = \frac{\ln(c)}{2(c-1)} + \frac{1}{1+c} \tag{11}$$

which is placed as the coefficient of the first order temperature derivative dT/dr, like in Eq. (7).

For the sake of generality, the fin equation (1) can be re-expressed as

$$\frac{d^2\theta}{dR^2} + M_f \frac{d\theta}{dR} - \frac{\xi^2}{(1-c)^2}\theta = 0 \quad \text{in } c \leqslant R \leqslant 1 \tag{12}$$

where the generic mean M_f equates to M_{f1}, M_{f2} or M_{f3}. Pausing here for a moment, it can be seen that Eq. (12) owns three constant coefficients, in contrast to the original equation (1) holding mixed coefficients; that is, two constant coefficients and a variable coefficient $1/R$.

The general solution of Eq. (12) is (Boyce and DiPrima [12]):

$$\theta(R) = C_1 e^{\lambda_1 R} + C_2 e^{\lambda_2 R} \tag{13}$$

where the roots of the auxiliary equation are

$$\lambda_1, \lambda_2 = \frac{-M_f \pm \sqrt{M_f^2 + \frac{4\xi^2}{(1-c)^2}}}{2} \tag{14}$$

The combination of Eqs. (13), (14) and (2) culminates in the particular solution:

$$\theta(R) = \frac{\lambda_2 e^{\lambda_1 (R-1)} - \lambda_1 e^{\lambda_2 (R-1)}}{\lambda_2 e^{\lambda_1 (c-1)} - \lambda_1 e^{\lambda_2 (c-1)}} \tag{15}$$

Unquestionably, this approximate analytic temperature distribution of compact form constitutes the centerpiece of the present work. Hence, it is reasonable to contrast the complex Bessel structure of the exact temperature distribution given by Eq. (4) against the simple exponential structure of the approximate temperature distribution given by Eq. (15).

On the other hand, inspection of the fin efficiency diagram for annular fins of uniform thickness in [8,9] reveals that the radii ratio c is commonly placed inside the interval $0.2 \leqslant c \leqslant 1$. In this regard, the emerging three M_f magnitudes are listed in Table 1 for selected radii ratios c. Correspondingly, the approximate fin efficiency relations through the tandem of Eqs. (3) are readily obtained from Eq. (15). Therefore, the fin efficiency reads:

(1) by differentiation of $\theta(R)$,

$$\eta_1 = \frac{2c}{1-c^2} \frac{e^{\lambda_1 (c-1)} - e^{\lambda_2 (c-1)}}{\lambda_2 e^{\lambda_1 (c-1)} - \lambda_1 e^{\lambda_2 (c-1)}} \tag{16a}$$

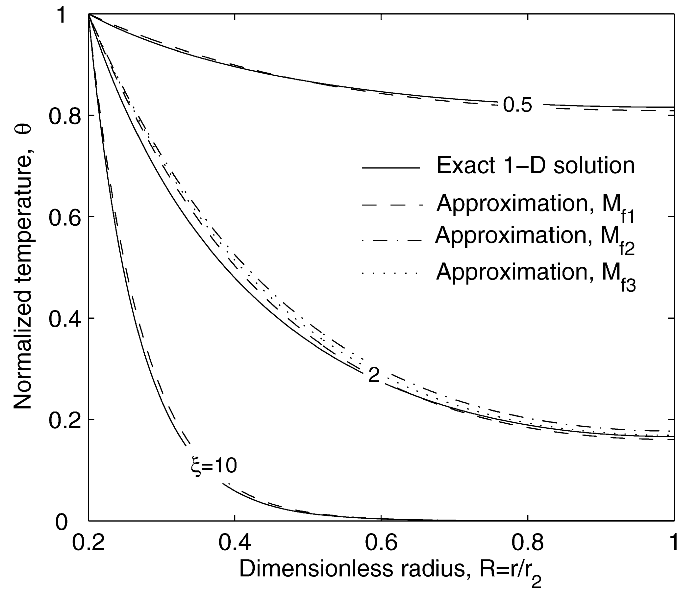

Fig. 2. Comparison between the approximate and exact temperature profiles for $c = 0.2$ and different dimensionless fin parameters ξ.

or (2) by integration of $\theta(R)$,

$$\eta_2 = \frac{2}{1-c^2}\left\{\lambda_2^3\left[\lambda_1 - 1 + (1 - c\lambda_1)e^{\lambda_1 (c-1)}\right] - \lambda_1^3\left[\lambda_2 - 1 + (1 - c\lambda_2)e^{\lambda_2 (c-1)}\right]\right\}\left\{\lambda_1^2\lambda_2^2\left[\lambda_2 e^{\lambda_1 (c-1)} - \lambda_1 e^{\lambda_2 (c-1)}\right]\right\}^{-1} \tag{16b}$$

In fact, it should be anticipated that the differentiation approach of the fin efficiency, Eq. (16a), may produce results that are different than those of the integral approach, Eq. (16b). This is so because the approximate temperature distribution, Eq. (15), does not satisfy exactly the descriptive fin equation, Eq. (1). As a result, the heat transported by conduction at the fin base and the heat dissipated by convection could have unequal values.

Lastly, important issues related to the safe-touch temperatures of solid objects were discussed by Arthur and Anderson [13]. In this framework, the outermost parts of an array of annular fins around $R = 1$ are prone to be accidentally touched by technical personnel in plant environments. Accordingly, the fin tip temperature $\theta(1)$ is considered a parameter of relevance for safety precautions. From Eq. (4), the exact dimensionless tip temperature is

$$\theta(1) = \frac{I_1(\gamma)K_0(\gamma) + I_0(\gamma)K_1(\gamma)}{I_1(\gamma)K_0(\gamma c) + I_0(\gamma c)K_1(\gamma)} \tag{17}$$

By virtue of Eq. (15), the approximate dimensionless tip temperature turns out to be

$$\theta(1) = \frac{\lambda_2 - \lambda_1}{\lambda_2 e^{\lambda_1 (c-1)} - \lambda_1 e^{\lambda_2 (c-1)}} \tag{18}$$

5. Presentation of approximate analytic results

Fig. 2 displays the exact dimensionless temperature profiles calculated with the modified Bessel functions, Eq. (4), and with the approximate analytic temperature distribution in Eq. (15)

Table 2
Comparison of the computed fin efficiencies

Procedure	c	ξ	Approximate (relative error E_η)						Exact
			M_{f1}	M_{f2}	M_{f3}	Schmidt [2]	Hong and Webb [3]	Perrotin and Clodic [4]	
derivative	0.2	3	−1.62e−1	−1.99e−1	−1.81e−1	2.40e−1	1.06e−1	−2.04e−2	0.1720
integral	0.2	3	3.50e−2	9.54e−2	6.49e−2				0.1720
derivative	0.2	1.5	−1.83e−1	−2.48e−1	−2.16e−1	4.16e−2	1.31e−2	−6.61e−3	0.4020
integral	0.2	1.5	−1.25e−3	6.12e−2	3.00e−2				0.4020
derivative	0.2	0.5	−1.77e−1	−2.91e−1	−2.36e−1	−1.27e−2	−1.57e−2	−6.36e−3	0.8470
integral	0.2	0.5	−4.37e−3	1.39e−2	5.07e−3				0.8470
derivative	0.8	3	−2.88e−3	−3.03e−3	−2.96e−3	4.55e−3	−4.75e−2	−3.05e−2	0.3068
integral	0.8	3	8.43e−4	9.77e−4	9.10e−4				0.3068
derivative	0.8	1.5	−3.69e−3	−3.97e−3	−3.83e−3	−7.87e−3	−2.08e−2	−1.13e−2	0.5760
integral	0.8	1.5	3.34e−4	4.57e−4	3.96e−4				0.5760
derivative	0.8	0.5	−4.08e−3	−4.51e−3	−4.30e−3	−3.00e−3	−4.45e−3	−2.58e−3	0.9160
integral	0.8	0.5	4.08e−5	6.99e−5	5.53e−5				0.9160

deduced in this work. The smallest radii ratio of magnitude $c = 0.2$ (i.e., $r_2 = 5r_1$) reported in the fin efficiency diagram in [8,9] has been chosen as a critical case in order to analyze the totality of the results. The three implementations of the mean value theorem, namely M_{f1}, M_{f2} and M_{f3} have been encapsulated in Eq. (15) for a representative value of the thermo-geometric fin parameter, $\xi = 2$, in the mid-curve of the family of curves. As Fig. 2 reveals, the mean value M_{f1} from Eq. (6) provides the most accurate results for small and medium R. The mean value M_{f2} from Eq. (9) furnishes temperature profiles that separate from the actual fin behavior in a larger extent. However, M_{f3} being the average between M_{f1} and M_{f2} in Eq. (11) supplies the best temperature profiles when R is close to the fin tip. This can be explained noting that the approximation obtained with the mean M_{f3} produces temperature profiles that are situated between the other two temperature approximations. Therefore, the underpredictive nature of the M_{f1} approximation at the fin tip is compensated by the overpredictive nature of M_{f2}. This same trend prevails for other values of the fin parameter ξ. Fig. 2 also includes curves for small $\xi = 0.5$ and large $\xi = 10$, but to preserve clarity the graph only contrasts the exact temperatures against the approximate temperatures using the mean M_{f1}. The approximate temperature profile does not degenerate for large ξ, whichever M_f is selected, because the analytic temperature distribution of Eq. (15) is physically consistent. In other words, it tends to zero when $\xi \to \infty$ and $R > c$.

The fin efficiency estimated with the approximate analytic temperature distribution united with the integral approach η_2 in Eq. (16b), furnishes very satisfactory results as can be confirmed in Fig. 3 and Table 2. In the table, the relative error in predicting the efficiency E_η is defined as:

$$E_\eta = \frac{\eta_{\text{approx.}} - \eta_{\text{exact}}}{\eta_{\text{exact}}} \tag{19}$$

It should be stressed that high accuracy is retained for a broad range of radii ratios c. Again, the best η_2 results correspond to the mean M_{f1} because the temperature profile obtained using this mean does not deviate significantly from the exact temperature profile along the radial direction of the fin. Concerning η_2 results for the mean M_{f1}, the largest relative error E_η that appears in Table 2 corresponds to 3.5% of the exact efficiency for the pair $c = 0.2$ and $\xi = 3$. In spite of its exactness circa $R = 1$, the mixed mean M_{f3} cannot surpass the accuracy inherent to M_{f1}, since close to the fin tip the temperature decreases and its contribution in the heat dissipation is weaker that in other radial positions.

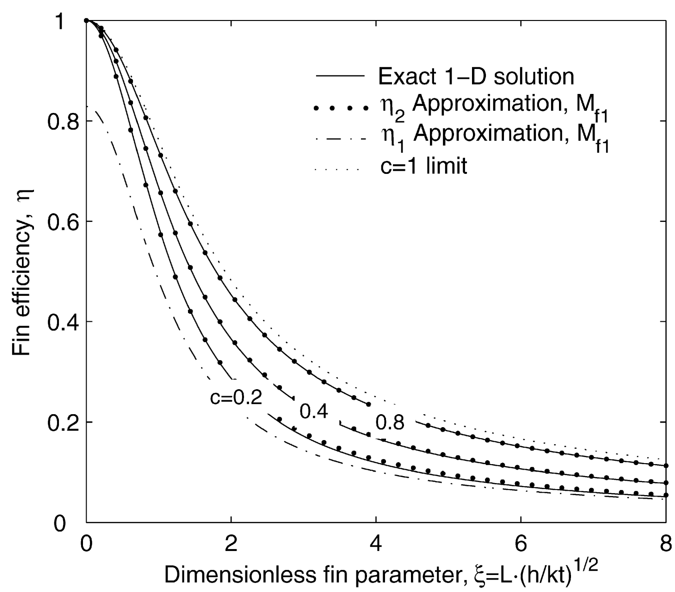

Fig. 3. Comparison between the approximate and exact fin efficiencies with the dimensionless fin parameter ξ for different radii ratios c.

In the items of Table 2 it is observable that the differences between the integral-based η_2 results for the three approximations are not significant. For this reason, the efficiency comparison shown in Fig. 3 has been restricted to the mean M_{f1} only. In contrast, inadmissible underpredictions appear when c is low and the derivative approach for the fin efficiency η_1 is calculated, Eq. (16a). This is illustrated in Fig. 3 for the smallest radii ratio under study, i.e., $c = 0.2$, where the discrepancies between η_2 and η_1 are most pronounced. The efficiency η_1 relies on the temperature derivative at a single point, i.e., the fin base $R = c$, whose value always underpredicts (in module) the real one. This is in accord with the tendency manifested in the curves in Fig. 2 and the negative relative error in Table 2. A fur-

Table 3
Parameters for the longitudinal fin based approximations of annular fin efficiency

Approach	m	n
Schmidt	0	$1+0.35\cdot\ln\left(\frac{1}{c}\right)$
Hong and Webb	0.1	$1+0.35\cdot\ln\left(\frac{1}{c}\right)$
Perrotin and Clodic	0.1	$1+\ln\left(\frac{1}{c}\right)\left[0.3+\left(\frac{0.26}{c^{0.3}}-0.3\right)\left(\frac{\xi}{2.5}\right)^{1.5-\frac{1}{12c}}\right]$

ther explanation of this underprediction can be conceived taking into consideration that M_f also underestimates the value of $1/R$ at small R. This translates into lowering the second and first derivative of the approximate solution in the simplified equation (12). In contrast, M_f overestimates $1/R$ when R is close to unity. As a result, the differences between the exact and the approximate temperature profiles at small R are partially rectified from the mid radius to the fin tip. This enlightens why the integral-based fin efficiency, η_2, relying on the whole temperature profile, from the base to the fin tip (and not only on the gradient at the fin base) delivers improved results over the differential-based approach η_1.

As the numbers listed in Table 2 demonstrate, the differences between the efficiency results based on the integral and derivative approach diminish for large c. In fact, in the limiting case dictated by $c = 1$, both the approximate and the exact predictions coincide, thus collapsing to:

$$\eta_{c=1} = \frac{\tanh(\xi)}{\xi} \tag{20}$$

This expression can be easily deduced from the approximate Eqs. (16) taking into account that the roots of the simplified Bessel equation confirm that $\lambda_{1,2}(1-c) \to \pm\xi$ when $c \to 1$. It should be noted that Eq. (20) supplies also the fin efficiency for a longitudinal fin of uniform thickness [9] and the same ξ, which is a logical similitude owing to the null curvature in the annular fin when c tends to unity and L is maintained constant.

Existing expressions of the annular fin efficiency based on the longitudinal fin approximation [2–4] have been included in Table 2 for comparison purposes. Table 3 summarizes these expressions, which can be conveniently channeled through the following relation:

$$\eta = \frac{\tanh(\xi n)}{\xi n}\cos(\xi n m) \tag{21}$$

The longitudinal fin based approximations of the efficiency relation, Eq. (21), are plotted in Fig. 4 along with the integral-based efficiency η_2 from Eq. (16b) for an intermediate radii ratio $c = 0.4$.

Although the efficiency approximations by Schmidt [2], Hong and Webb [3], and Perrotin and Clodic [4] may grant high accuracy for certain combinations of ξ and c, their fin efficiency predictions deteriorate out of the range in which these approximation were adjusted. This assertion can be observed in Table 2 in which the Schmidt [2] as well as Hong and Webb [3] approximations yield accurate results for large values of the

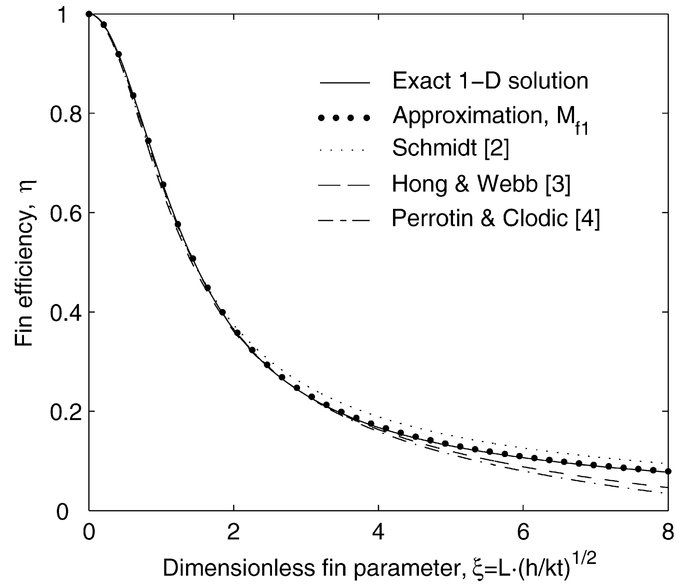

Fig. 4. Comparison between the different approximations for the fin efficiency varying with ξ for a fixed $c = 0.4$.

radii ratio. In contrast, the efficiency calculated with the Perrotin and Clodic approximation [4] delivers better results when c is small. If M_{f1} and the integral-based efficiency η_2 are selected, the mean value theorem approximation performs always better than the classical efficiency estimation due to Schmidt [2] and the Hong and Webb approximation [3]. The Perrotin and Clodic approximation [4] slightly surpasses in precision the mean value results M_{f1} of η_2 for smaller c and medium ξ. For instance, given $c = 0.2$ and $\xi = 3$, the mean M_{f1} when articulated with η_2 supplies a 3.5% relative error while the Perrotin and Clodic [4] is conducive to 2.04%. Those are remarkably precise values when compared to the large 24% relative error of the Schmidt approximation. However, for large radii ratio, e.g., $c = 0.8$, in combination with a high thermo-geometric fin parameter, $\xi = 3$, the relative error of the Perrotin and Clodic approximation [4] increases over the error associated with the Schmidt approach, being the η_2 estimation with the mean M_{f1} the most accurate method.

The predicted behavior at the fin tip temperature, Eq. (18), has been already explained when commenting the θ profiles. Curves for the tip temperature are presented in Fig. 5 while some examples of the relative error E_t for the three approximation approaches are included in Table 4. Similar to Eq. (19), E_t has been calculated with the following expression:

$$E_t = \frac{\theta(1)_{\text{approx.}} - \theta(1)_{\text{exact}}}{\theta(1)_{\text{exact}}} \tag{22}$$

According to the numbers that appear in Table 4, good results are expected for $\theta(1)$ when using any of the three approximate mean avenues, provided that the radii ratio c is not extremely small. As previously indicated, the mean M_{f3} produces the best tip temperatures. This is particularly obvious inspecting Fig. 5 and Table 4 concurrently, where the unreal value $c = 0.02$ (i.e., $r_2 = 50r_1$) has been included intentionally. It is surprising to see that for this very low radii ratio the accuracy

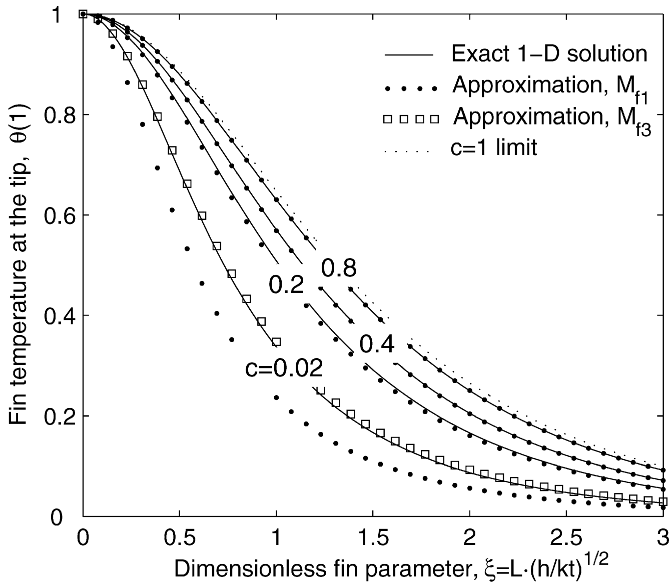

Fig. 5. Comparison between the approximate and exact tip temperatures with the dimensionless fin parameter ξ for different radii ratios c.

Table 4
Comparison of the computed fin tip temperatures

c	ξ	Approximate (relative error E_t)			Exact
		M_{f1}	M_{f2}	M_{f3}	
0.02	3	−3.53e−1	7.57e−1	9.57e−2	0.0268
0.02	1.5	−3.44e−1	5.31e−1	4.62e−2	0.1669
0.02	0.5	−1.7e−1	1.45e−1	1.21e−2	0.6870
0.2	3	−3.65e−2	8.16e−2	2.14e−2	0.0559
0.2	1.5	−3.04e−2	5.41e−2	1.17e−2	0.2918
0.2	0.5	−8.35e−3	1.30e−2	2.61e−3	0.8159
0.8	3	−1.02e−4	2.18e−4	5.81e−5	0.0921
0.8	1.5	−7.18e−5	1.24e−4	2.59e−5	0.4061
0.8	0.5	−1.50e−5	2.37e−5	4.36e−6	0.8790

of the approximate mixed mean M_{f3} continues to be less than a 10% error. Whenever $c \geqslant 0.2$, the results that emanate from the three mean approximations in Table 4 tend to be more similar, especially for M_{f1} and M_{f3}. For instance, fixing $c = 0.2$, the largest inaccuracies in the estimation of $\theta(1)$ are encountered in Table 4 when $\xi = 3$; being in the worst case less than 8.2% with M_{f2} and in the best 2.14% with M_{f3}. For the sake of clarity, Fig. 5 only includes the approximation connected to the mean M_{f1} when $c \geqslant 0.2$. Notice that the efficiency approximations by Schmidt [2], Hong and Webb [3], and Perrotin and Clodic [4] cannot provide the fin tip temperature.

If c tends to unity while maintaining L constant, the approximate equation (18) for the tip temperature simplifies to

$$\theta_{c=1}(1) = \frac{1}{\cosh(\xi)} \tag{23}$$

This outcome is also included in Fig. 5 and clearly coincides with the tip temperature of a basic longitudinal fin of uniform thickness. Differences between the approximate and exact 1-D tip temperatures are literally null if $c = 1$.

In general, the approximate results for both $\theta(1)$ and η deteriorate when the radii ratio c decreases because the differences between the constant mean values M_f and the variable coefficient $1/R$ of the descriptive fin equation turn out to be more pronounced. The influence of the thermo-geometric fin parameter ξ is subtle and obscure regarding the relative accuracy of the approximate fin efficiency. An increment in ξ usually elevates E_η. As far as the tip fin temperature is concerned, the relative error E_t increases with ξ, but this is not the case for the absolute error in the tip temperature prediction, which experiences a reduction.

Finally, Tables 5 and 6 collectively gather the minimum radii ratio c that can be handled with the mean value procedure if a pre-set relative error cannot be surpassed in the estimation of the fin efficiency and the tip temperature. Obviously, in harmony with the previous explanations in this subsection, the minimum c increases with the imposed accuracy. For a given relative error in the fin efficiency, the mismatches between the three M_f approximations are not significant, being the mean M_{f1} the best choice consistently, as reflected in Table 5. Owing to the nonlinear dependence on c in Eq. (15), this trend is broken when both the thermo-geometric fin parameter is small and the pre-set accuracy is not demanding. For example, if a stiff 1% relative error in estimating η_2 is required, all the ap-

Table 5
Minimum c for a pre-set accuracy in approximating the integration-based fin efficiency η_2

Accuracy	ξ	Minimum c					
$\lvert E_\eta \rvert$		M_{f1}	M_{f2}	M_{f3}	Schmidt [2]	Hong and Webb [3]	Perrotin and Clodic [4]
0.001	0.5	0.302	0.524	0.438	0.931	–	–
0.01	0.5	0.143	0.237	0.117	0.380	0.493	0.154
0.1	0.5	2.97e−2	3.66e−2	6.52e−4	1.45e−6	1.90e−5	6.71e−2
0.001	1.5	0.666	0.729	0.707	0.978	(0.243–0.252)*	–
0.01	1.5	0.144	0.443	0.371	0.709	(0.210–0.300)*	(0.182–0.665)*
0.1	1.5	4.17e−2	0.144	5.36e−2	0.106	6.62e−2	8.40e−2
0.001	3	0.784	0.798	0.792	0.888	(0.396–0.403)*	–
0.01	3	0.455	0.529	0.500	0.726	(0.369–0.436)*	(0.240–0.527)*
0.1	3	1.43e−3	0.194	0.141	0.366	0.207	0.102
0.001	10	0.792	0.797	0.795	1	–	–
0.01	10	0.499	0.519	0.510	0.917	–	–
0.1	10	0.147	0.179	0.164	0.590	–	–

* The limited interval of c values for which the cases comply with the pre-set accuracy is indicated in brackets.

Table 6
Minimum c for a pre-set accuracy in approximating the fin tip temperature

Accuracy $\|E_t\|$	ξ	Minimum c M_{f1}	M_{f2}	M_{f3}
0.001	0.5	0.428	0.484	0.306
0.01	0.5	0.183	0.227	6.41e−2
0.1	0.5	3.58e−2	3.58e−2	1.05e−3
0.001	1.5	0.589	0.645	0.491
0.01	1.5	0.327	0.398	0.219
0.1	1.5	8.95e−2	0.136	1.79e−3
0.001	3	0.620	0.692	0.569
0.01	3	0.356	0.456	0.296
0.1	3	9.89e−2	0.177	1.01e−3
0.001	10	0.574	0.729	0.653
0.01	10	0.296	0.505	0.394
0.1	10	6.41e−2	0.219	0.108

proximations based on the mean value theorem are valid for the combinations: $c \geqslant 0.237$ if $\xi = 0.5$ and $c \geqslant 0.529$ if $\xi = 3$. Moreover, the smallest radii ratio c required for the Schmidt [2], Hong and Webb [3], and Perrotin and Clodic [4] approximations is always higher than the c required for the mean value theorem M_{f1}, except in the case encompassing both $\xi = 0.5$ and a not demanding accuracy of $E_\eta = 0.1$.

Observe that the Perrotin and Clodic approximation [4] is unable to provide results with $|E_\eta| \leqslant 0.001$ for any combination of ξ and c presented in Table 5. This aspect has been indicated in Table 5 with a dash line. Regarding the Hong and Webb approximation [3], it can reach accuracies with $|E_\eta| \leqslant 0.01$ but usually in a restricted interval, if any. For example, if $\xi = 1.5$ the relative error is below 1% only when $0.21 \leqslant c \leqslant 0.3$. In contrast, using the mean M_{f1} the approximation of the fin efficiency does not exceed this error provided $c \geqslant 0.144$. Notice also that the Hong and Webb [3] and Perrotin and Clodic [4] approximations are incapable of reaching any of the pre-set accuracies in the items of Table 5 if $\xi = 10$.

Turning the attention to the fin tip temperature, the mixed mean M_{f3} reduces the minimum allowable c, sometimes by an order of magnitude, when compared to the other two mean approximations. Assigning a relative error limit of 1% for $\xi = 0.5$, the tip temperature estimate using M_{f1} is adequate if $c \geqslant 0.183$, but this limit diminishes markedly to a trivial $c \geqslant 0.0641$ (a factor of three) when M_{f3} is applied. However, despite the stunning M_{f3} results, it should be pointed out that practical engineering applications rarely involve very small radii ratios of c less than 0.2. It is for this reason that the M_{f1} avenue can be considered excellent in the determination of the tip fin temperature.

6. Conclusions

In calculating the temperature variation in annular fins of uniform thickness, the use of the mean value theorem for simplifying the modified Bessel differential equation gives way to approximate temperature solutions endowed with an unsurpassed combination of accuracy and easiness. Certainly, it can be inferred that the mean value theorem constitutes an interesting computational avenue for practical thermal engineering applications. It has been demonstrated that the mean M_{f1} in the descriptive differential equation with a variable coefficient $1/R$ delivers the most accurate results for fin efficiency prediction. On the other hand, in the case of the fin tip temperature the precise estimates are usually obtained by the mixed mean M_{f3}. Regardless of the mean value approach employed, the fin efficiency conveyed through the integral-based η furnishes more accurate results than the alternate derivative-based η. This last statement harmonizes with the recommendations made in [11]. Differences between the analytic temperature approximation developed in the present work and the exact analytic temperature distribution relying on Bessel functions are probably below the level of inaccuracy introduced by the Murray–Gardner assumptions on both exact and approximate temperatures.

In contrast to the common expressions for the annular fin efficiency estimates based on the longitudinal fin equivalence, the mean value theorem approach provides simultaneous predictions of the fin efficiency and temperature distribution, including the fin tip temperature. As an additional validation, the approximate analytic solution retain consistency relative to the exact solution in the limiting case $c = 1$. The computational methodology described in this work may find application in a broad class of fins of variable cross section, such as longitudinal and annular fins of triangular, parabolic or hyperbolic profile.

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