UNIVERSIDAD CARLOS III DE MADRID working papers

Working Paper 11-28
Statistics and Econometrics Series 21
July 2011

Departamento de Estadística
Universidad Carlos III de Madrid
Calle Madrid, 126
28903 Getafe (Spain)
Fax (34) 91 624-98-49

Free Completely Random Measures

Francesca Collet[1] and Fabrizio Leisen[2]

Abstract

Free probability is a noncommutative probability theory introduced by Voiculescu where the concept of independence of classical probability is replaced by the concept of freeness. An important connection between free and classical infinite divisibility was established by Bercovici and Pata (1999) in form of a bijection, mapping the class of classical infinitely divisible laws into the class of free infinitely divisible laws.

A particular class of infinitely divisible laws are the completely random measures introduced by Kingman (1967). In this paper, a free analogous of completely random measures is introduced and, a free Poisson process characterization is provided as well as a representation through a free cumulant transform. Furthermore, some examples are displayed.

Keywords: *Bayesian non parametrics, Bercovici-Pata bijection, free completely random measures, free infinite divisibility, free probability*

[1] Departamento de Ciencia e Ingenieria de Materiales e Ingenieria quimica, Universidad Carlos III de Madrid
, Avenida de la Universidad 30; 28911 - Leganes (Madrid), Spain
. e-mail: fcollet@ing.uc3m.es

[2] Departamento de Estadistica, Universidad Carlos III de Madrid
, Calle Madrid 126; 28903 - Getafe (Madrid), Spain
. e-mail: fabrizio.leisen@gmail.com

1 Introduction

From the seminal paper of Kingman (1967), the use of completely random measures (CRM) growth exponentially, due to their use in Bayesian statistics. The construction of priors through normalization of CRM is a standard procedure in Bayesian non parametrics, see James et al. (2006, 2009). For example, the most popular Bayesian non parametric prior, the Dirichlet process (refer to Ferguson (1973, 1974)), could be defined through a normalization of a completely random measure.

Free probability is a noncommutative probability theory where the classical concept of independence is replaced by the concept of freeness. This theory has been developed by Voiculescu (1986) and, objects and notions in classical probability have a free analogous. For example, it is possible to define a free convolution and, as a consequence, a notion of free infinite divisibility. In Bercovici and Pata (1999) an important connection in terms of a bijection was established between classical and free infinite divisibility. The infinite divisibility of CRM suggests an analogous object in the freeworld. In this paper, free completely random measures (free CRM) are introduced and studied. In particular, in Section 2 some preliminaries about free probability and free infinite divisibility are given. In Section 3, free CRM are defined and the existence is proved. Moreover, a free Poisson process characterization is provided. In Section 4, some examples of free CRM are displayed.

2 Preliminaries on Free Probability

In this section, some preliminaries about free probability are given. See also Barndorff-Nielsen and Thorbjørnsen (2006); Biane (2003); Nica and Speicher (2006); Speicher (2003); Voiculescu (1986) and Voiculescu et al. (1992).

2.1 Noncommutative Probability Space

Let $\mathcal{H}$ be a Hilbert space and consider the complete normed space $(B(\mathcal{H}), \|\cdot\|)$, where $\|\cdot\|$ is the operator norm and $B(\mathcal{H})$ denote the vector space of all bounded operators on $\mathcal{H}$. The composition of operators form a (noncommutative) multiplication on $B(\mathcal{H})$ and, together with the linear operations, turns $B(\mathcal{H})$ into an algebra. Moreover, if we also consider the adjoint operation as involutive, antilinear operation on $B(\mathcal{H})$, then we obtain a $*$-algebra.

For our purposes we are interested to a particular class of $*$-subalgebras of $B(\mathcal{H})$: the von Neumann algebras.

Definition 2.1 *A* von Neumann algebra $\mathcal{A}$ *acting on a Hilbert space* $\mathcal{H}$ *is a unital subalgebra of* $B(\mathcal{H})$*, which is closed under the adjoint operation, under the weak operator topology on* $B(\mathcal{H})$ *and topologically closed with respect to the operator norm.*

Definition 2.2 *A* state $\tau : \mathcal{A} \longrightarrow \mathbb{C}$ *on the von Neumann algebra* $\mathcal{A}$ *is a positive and linear functional such that* $\tau(\mathbf{1}_{\mathcal{A}}) = 1_{\mathbb{C}}$*. Moreover, if in addition*

- *for any* $L \in \mathcal{A}$*, denoted by* L^* *the adjoint of* L*, we have that* $\tau(L^*L) = 0$ *implies* $L = 0$*, then* τ *is called* faithful *state;*
- *for any* $L_1, L_2 \in \mathcal{A}$*, we have* $\tau(L_1L_2) = \tau(L_2L_1)$*, then* τ *is called* tracial *state;*
- *its restriction to the unit ball of* $\mathcal{A}$ *is continuous in the weak operator topology, then* τ *is called* normal *state.*

Definition 2.3 *A* W^**-probability space is a pair* $(\mathcal{A}, \tau)$ *where* $\mathcal{A}$ *is a von Neumann algebra and* τ *is a faithful, normal tracial state on* $\mathcal{A}$*. The elements of* $\mathcal{A}$ *are thought as noncommutative random variables.*

If $(\mathcal{A}, \tau)$ is a W^*-probability space acting on $\mathcal{H}$ and L is an unbounded operator on $\mathcal{H}$, L can not be an element of $\mathcal{A}$. The closest L can get to $\mathcal{A}$ is to be *affiliated* with $\mathcal{A}$, which means that L commutes with any unitary operator U, that commutes with all the elements of $\mathcal{A}$. If L is selfadjoint, L is affiliated with $\mathcal{A}$ if and only if $f(L) \in \mathcal{A}$ for any bounded Borel function $f : \mathbb{R} \longrightarrow \mathbb{R}$.

2.2 Free Independence

The basic notion in free probability is free independence (or freeness) of noncommutative random variables. It was introduced by Voiculescu (1986) and it can be regarded as a parallel of the fundamental concept of independence in classical probability theory.

Definition 2.4 *Let $L_1, \ldots, L_r$ be self-adjoint operators affiliated with a W^*-probability space $(\mathcal{A}, \tau)$. We say that $L_1, \ldots, L_r$ are* freely independent *(or free) with respect to τ, if*

$$\tau\left\{[f_1(L_{i_1}) - \tau(f_1(L_{i_1}))]\,[f_2(L_{i_2}) - \tau(f_2(L_{i_2}))] \cdots \left[f_p(L_{i_p}) - \tau\left(f_p(L_{i_p})\right)\right]\right\} = 0$$

for any $p \in \mathbb{N}$, any bounded Borel functions $f_1, f_2 \ldots, f_p : \mathbb{R} \longrightarrow \mathbb{R}$ and any indeces $i_1, i_2, \ldots, i_p \in \{1, 2, \ldots, r\}$ satisfying the conditions $i_1 \neq i_2$, $i_2 \neq i_3$, $\ldots$, $i_{p-1} \neq i_p$.

The analogy between free and classical independence is that around free independence, several notions can be developed similar to those around independence: addition of free random variable, central limit theorem for free random variables, processes with free independent increments, stochastic calculus etc. In particular, the above definition of freeness provides an interesting free analogous of convolution theory.

2.3 Free Infinite Divisibility

By $\mathbb{C}^+$ (respectively $\mathbb{C}^-$) we denote the set of complex numbers with strictly positive (respectively strictly negative) imaginary part. Let μ a probability measure on $\mathbb{R}$ and consider its Cauchy transform $G_\mu : \mathbb{C}^+ \longrightarrow \mathbb{C}^-$ given by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z - t} \mu(dt)\,. \tag{1}$$

Then define the mapping $F_\mu : \mathbb{C}^+ \longrightarrow \mathbb{C}^+$ by

$$F_\mu(z) = \frac{1}{G_\mu(z)}\,, \tag{2}$$

and note that F_μ is analytic on $\mathbb{C}^+$. Moreover, it was proved in Bercovici and Voiculescu (1993) that there exist positive numbers α and K such that F_μ has an (analytic) right inverse F_μ^{-1} defined on the region

$$\Gamma_{\alpha,K} := \{z \in \mathbb{C} : |\mathrm{Re}\ (z)| < \alpha \mathrm{Im}\ (z), \mathrm{Im}\ (z) > K\}.$$

Following Barndorff-Nielsen and Thorbjørnsen (2006), the free cumulant transform $C_\mu^{\boxplus}$ of μ is defined by

$$C_\mu^{\boxplus}(z) = zF_\mu^{-1}\left(z^{-1}\right) - 1 \qquad \text{for } z^{-1} \in \Gamma_{\alpha,K} \tag{3}$$

and its key property is that, for any probability measures μ_1, μ_2 on $\mathbb{R}$, we have

$$C_{\mu_1 \boxplus \mu_2}(z) = C_{\mu_1}(z) + C_{\mu_2}(z)\,,$$

where $\mu_1 \boxplus \mu_2$ is the free convolution between μ_1 and μ_2. Moreover, in analogy to the classical case, the free Lévy-Khintchine characterization of the free cumulant transform establishes when a probability measure is free infinitely divisible; that is, when it is infinitely divisible with respect to the free convolution $\boxplus$.

Proposition 2.1 (Barndorff-Nielsen and Thorbjørnsen (2006); Bercovici and Voiculescu (1993)) *A probability measure μ on $\mathbb{R}$ is free infinitely divisible if and only if there exist a non-negative number a, a real number η and a Lévy measure ρ, satisfying $\rho(\{0\}) = 0$ and $\int_{\mathbb{R}} \min\{x^2, 1\} \rho(dx) < +\infty$, such that the free cumulant transform $C_\mu^{\boxplus}$ has the representation:*

$$C_\mu^{\boxplus}(z) = \eta z + az^2 + \int_{\mathbb{R}} \left(\frac{1}{1 - xz} - 1 - xz \, \mathbb{I}_{\{|x| \leq 1\}}(x) \right) \rho(dx) \quad \text{with } z \in \mathbb{C}^- . \tag{4}$$

In that case, the triplet (a, η, ρ) is uniquely determined and is called the free characteristic triplet for μ.

2.4 Connections between Free and Classical Infinite Divisibility

Let $\mathcal{ID}(*)$ and $\mathcal{ID}(\boxplus)$ denote the classes of the laws which are infinitely divisible with respect to the convolution $*$ and to the free convolution $\boxplus$ respectively. An important connection between free and classical infinite divisibility was established in Bercovici and Pata (1999), in form of a bijection Λ from $\mathcal{ID}(*)$ to $\mathcal{ID}(\boxplus)$.

Proposition 2.2 (Barndorff-Nielsen and Thorbjørnsen (2006); Bercovici and Voiculescu (1993)) *If μ is a measure in $\mathcal{ID}(*)$ with classical characteristic triplet (a, η, ρ), then $\Lambda(\mu)$ is a measure in $\mathcal{ID}(\boxplus)$ with free characteristic triplet (a, η, ρ).*

If $\mu \in \mathcal{ID}(\boxplus)$ and its Lévy measure ρ satisfies $\int_{\{|x| \leq 1\}} |x| \, \rho(dx) < +\infty$, then, for every $z \in \mathbb{C}^-$, we can rewrite (4) as

$$C_\mu^{\boxplus}(z) = \tilde{\eta} z + az^2 + \int_{\mathbb{R}} \left(\frac{1}{1 - xz} - 1 \right) \rho(dx) \, , \tag{5}$$

where $\tilde{\eta} = \eta - \int_{\{|x| \leq 1\}} x \rho(dx)$. This representation is called the *free drift type* cumulant transform of μ. By Bercovici-Pata bijection, if $\mu \in \mathcal{ID}(*)$ has classical drift type triplet $(a, \tilde{\eta}, \rho)$, then the free drift type triplet of $\Lambda(\mu)$ is also $(a, \tilde{\eta}, \rho)$.

We summarize some properties of the Bercovici-Pata bijection in the following proposition (see Barndorff-Nielsen and Thorbjørnsen (2006); Bercovici and Pata (1999); Bercovici and Voiculescu (1993)).

Proposition 2.3 *The mapping $\Lambda : \mathcal{ID}(*) \longrightarrow \mathcal{ID}(\boxplus)$ has the following properties:*

1. *if $\mu, \nu \in \mathcal{ID}(*)$, then $\Lambda(\mu * \nu) = \Lambda(\mu) \boxplus \Lambda(\nu)$.*

2. *if $\mu \in \mathcal{ID}(*)$ and $c > 0$, then $\Lambda(D_c\mu) = D_c\Lambda(\mu)$, where $D_c\mu$ means the spectral distribution of the operator cX with $\mu = \mathscr{L}(X)$.*

3. *if δ_c denotes the Dirac measure at $c \in \mathbb{R}$, then $\Lambda(\delta_c) = \delta_c$.*

4. *Λ is an homeomorphism with respect to the weak convergence; i.e., let μ be a measure in $\mathcal{ID}(*)$ and let $(\mu_n)_{n \geq 0}$ be a sequence of measures in $\mathcal{ID}(*)$, then $\mu_n \longrightarrow \mu$ if and only if $\Lambda(\mu_n) \longrightarrow \Lambda(\mu)$ in weak convergence.*

Remark 1 *Properties 2 and 3 mean that Λ is preserved under affine transforms: $\Lambda(D_c\mu * \delta_a) = D_c\Lambda(\mu) \boxplus \delta_a$ for any $c > 0$ and $a \in \mathbb{R}$.*

3 Free Completely Random Measure

Completely random measures were introduced by Kingman (1967). For an account about completely random measures see also Kingman (1993) and Çınlar (2011).

Definition 3.1 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and denote by $\mathbb{E}$ a Polish space endowed with its Borel σ-field $\mathcal{E}$. Moreover, let $\mathbb{M}_{\mathbb{E}}$ be the space of boundedly finite measures on $(\mathbb{E}, \mathcal{E})$ equipped with the topology of weak convergence and denote by $\mathcal{M}_{\mathbb{E}}$ the corresponding Borel σ-field. A completely random measure (CRM) is a measure $\Phi^* : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{M}_{\mathbb{E}}, \mathcal{M}_{\mathbb{E}})$ that satisfies*

1. *$\Phi^*(\emptyset) = 0$ almost surely;*
2. *for any collection of disjoint sets $E_1, E_2, \ldots, E_n$ in $\mathcal{E}$, the random variables $\Phi^*(E_1), \Phi^*(E_2), \ldots, \Phi^*(E_n)$ are mutually independent.*

Definition 3.2 *Let $(\mathcal{A}, \tau)$ be a W^*-probability space and let $\mathcal{A}_+$ denote the cone of positive, selfadjoint operators in $\mathcal{A}$. A* free completely random measure *(free CRM) on a Polish space $(\mathbb{E}, \mathcal{E})$, with values in $(\mathcal{A}, \tau)$, is a mapping $\Phi^{\boxplus} : \mathcal{E} \longrightarrow \mathcal{A}_+$ such that*

1. *for any set $E \in \mathcal{E}$, $\mathscr{L}\left\{\Phi^{\boxplus}(E)\right\} = \Lambda(\mathscr{L}\{\Phi^*(E)\})$, where Λ is the Bercovici-Pata bijection and Φ^* a (classical) completely random measure;*

and, for any collection of disjoint sets $E_1, E_2, \ldots, E_n$ in $\mathcal{E}$,

2. *the operators $\Phi^{\boxplus}(E_1), \Phi^{\boxplus}(E_2), \ldots, \Phi^{\boxplus}(E_n)$ are freely independent;*
3. $\Phi^{\boxplus}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \Phi^{\boxplus}(E_i)$.

3.1 Existence of a free completely random measure

Next step consists in showing that a free completely random measure as in Definition 3.2 exists. Its general existence is proved adapting the proof of the existence of the free Poisson random measure found in Section 6.3 of Barndorff-Nielsen and Thorbjørnsen (2006). The proof is carried out in a series of lemmas (Lemmas 6.10-6.15 in Barndorff-Nielsen and Thorbjørnsen (2006)). For brevity, we will state and prove only the lemmas we generalized (Lemmas 6.10 and 6.11). Concerning the remaining lemmas (the demonstrations of Lemmas 6.12-6.15 are unchanged) we refer to Barndorff-Nielsen and Thorbjørnsen (2006).
We start introducing some notation. If $\mu_1, \ldots, \mu_n$ are probability measures on $\mathbb{R}$, we denote

$$\mathop{*}_{i=1}^{n} \mu_i = \mu_1 * \cdots * \mu_n \qquad \text{and} \qquad \mathop{\boxplus}_{i=1}^{n} \mu_i = \mu_1 \boxplus \cdots \boxplus \mu_n \,.$$

Consider the Polish space $(\mathbb{E}, \mathcal{E})$, we define the set

$$\mathcal{I} := \bigcup_{k \in \mathbb{N}} \mathcal{I}_k \,,$$

where

$$\mathcal{I}_k = \{(E_1, \ldots, E_k) : E_1, \ldots, E_k \in \mathcal{E} \setminus \{\emptyset\} \text{ and are disjoint}\}$$

and we identify $(E_1, \ldots, E_k)$ with $\left(E_{\pi(1)}, \ldots, E_{\pi(k)}\right)$ for any permutation π of the set of indices $\{1, \ldots, k\}$. Moreover, we introduce a partial order on $\mathcal{I}$. Let $(E_1, \ldots, E_k)$ and $(F_1, \ldots, F_h)$ two elements of $\mathcal{I}$, then

$$(E_1, \ldots, E_k) \leq (F_1, \ldots, F_h) \iff \text{each } E_i \text{ is a union of some of the } F_j\text{'s.}$$

Lemma 3.1 (Modification of Lemma 6.10 in Barndorff-Nielsen and Thorbjørnsen (2006)) *Given* $S = (E_1, \ldots, E_k) \in \mathcal{I}$*, there exists a* W^**-probability space* $(\mathcal{A}_S, \tau_S)$ *generated by freely independent positive operators* $\Phi_S^{\boxplus}(E_1), \ldots, \Phi_S^{\boxplus}(E_k) \in \mathcal{A}_S$*, satisfying*

$$\mathscr{L}\left\{\Phi_S^{\boxplus}(E_i)\right\} = \Lambda\left(\mathscr{L}\{\Phi^*(E_i)\}\right) \text{ for } i = 1, \ldots, k\,. \tag{6}$$

Proof: By Voiculescu's theory of (reduced) product of von Neumann algebras (see Voiculescu et al. (1992)), we can construct the space $(\mathcal{A}_S, \tau_S)$ as the free product of the W^*-probability spaces $\left(L^\infty(\mathbb{R}, \mu_i), E_{\mu_i}\right)_{i=1,\ldots,k}$, where $\mu_i = \mathscr{L}\left\{\Phi_S^{\boxplus}(E_i)\right\}$ and E_{μ_i} is the expectation with respect to μ_i. □

Lemma 3.2 (Modification of Lemma 6.11 in Barndorff-Nielsen and Thorbjørnsen (2006)) *Consider two elements* $S = (E_1, \ldots, E_k)$ *and* $T = (F_1, \ldots, F_h)$ *of* $\mathcal{I}$ *and suppose that* $S \leq T$*. Let* $(\mathcal{A}_S, \tau_S)$ *and* $(\mathcal{A}_T, \tau_T)$ *be* W^**-probability spaces as in Lemma 3.1. Then there exists an injective, unital, normal* $*$*-homomorphism* $\iota_{S,T} : \mathcal{A}_S \longrightarrow \mathcal{A}_T$*, such that* $\tau_S = \tau_T \circ \iota_{S,T}$*.*

Proof: We adapt the notation from Lemma 3.1. For any fixed index $i \in \{1, \ldots, k\}$, we have $E_i = F_{j(i,1)} \cup \cdots \cup F_{j(i,h_i)}$ for suitable (distinct) indices $j(i,1), \ldots, j(i,h_i) \in \{1, \ldots, h\}$. Then

$$\begin{aligned}
\mathscr{L}\left\{\sum_{\ell=1}^{h_i} \Phi_T^{\boxplus}\left(F_{j(i,\ell)}\right)\right\} &= \boxplus_{\ell=1}^{h_i} \mathscr{L}\left\{\Phi_T^{\boxplus}\left(F_{j(i,\ell)}\right)\right\} \\
&= \boxplus_{\ell=1}^{h_i} \Lambda\left(\mathscr{L}\left\{\Phi^*\left(F_{j(i,\ell)}\right)\right\}\right) \qquad \text{by property (6)} \\
&= \Lambda\left(\mathop{*}_{\ell=1}^{h_i} \mathscr{L}\left\{\Phi^*\left(F_{j(i,\ell)}\right)\right\}\right) \qquad \text{by 1 in Proposition 2.3} \\
&= \Lambda\left(\mathscr{L}\left\{\sum_{\ell=1}^{h_i} \Phi^*\left(F_{j(i,\ell)}\right)\right\}\right) \\
&= \Lambda\left(\mathscr{L}\left\{\Phi^*\left(\bigcup_{\ell=1}^{h_i} F_{j(i,\ell)}\right)\right\}\right) \qquad \text{by measure property} \\
&= \Lambda\left(\mathscr{L}\left\{\Phi^*\left(E_i\right)\right\}\right) \\
&= \mathscr{L}\left\{\Phi_S^{\boxplus}(E_i)\right\} \qquad \text{by property (6).}
\end{aligned}$$

In addition, $\Phi_T^{\boxplus}(E_1), \ldots, \Phi_T^{\boxplus}(E_k)$ are freely independent selfadjoint operators and, similarly, the operators $\sum_{\ell=1}^{h_i} \Phi_T^{\boxplus}\left(F_{j(i,\ell)}\right)$, for $i = 1, \ldots, k$, are freely independent and selfadjoint. Combining these observations with Remark 1.8 in Voiculescu (1990), it follows that there exists an injective, unital, normal $*$-homomorphism $\iota_{S,T} : \mathcal{A}_S \longrightarrow \mathcal{A}_T$, such that

$$\iota_{S,T}\left(\Phi_S^{\boxplus}(E_i)\right) = \sum_{\ell=1}^{h_i} \Phi_T^{\boxplus}\left(F_{j(i,\ell)}\right) \text{ for } i = 1, \ldots k$$

and $\tau_S = \tau_T \circ \iota_{S,T}$. □

Thus, it remains proved the following.

Theorem 3.1 *Let $(\mathbb{E}, \mathcal{E})$ be a Polish space. Then there exist a W^*-probability space $(\mathcal{A}, \tau)$ and a free completely random measure $\Phi^{\boxplus}$ on $(\mathbb{E}, \mathcal{E})$ with values in $(\mathcal{A}, \tau)$.*

3.2 Free Poisson representation of a free CRM

First we recall the definition of free Poisson random measure (taken from Barndorff-Nielsen and Thorbjørnsen (2002, 2006)) and then we use it to provide a characterization of free CRMs'.

Definition 3.3 *Let $(\mathbb{E}, \mathcal{E}, \nu)$ be a σ-finite measure space and define*

$$\mathcal{E}_0 = \{E \in \mathcal{E} : \nu(E) < +\infty\} .$$

Furthermore let $(\mathcal{A}, \tau)$ be a W^*-probability space and let $\mathcal{A}^+$ denote the cone of positive operators in $\mathcal{A}$. A free Poisson random measure *on $(\mathbb{E}, \mathcal{E}, \nu)$, with values in $(\mathcal{A}, \tau)$, is a mapping $M : \mathcal{E}_0 \longrightarrow \mathcal{A}_+$ such that*

1. *for any $E \in \mathcal{E}_0$, $\mathscr{L}\{M(E)\} = \Lambda(\mathscr{L}\{N(E)\})$, where Λ is the Bercovici-Pata bijection and N a classical Poisson random measure (i.e., $\mathscr{L}\{N(E)\} = \mathrm{Po}(\nu(E))$);*

and, for any collection of disjoint sets $E_1, E_2, \ldots, E_n$ in $\mathcal{E}_0$,

2. *the operators $M(E_1), M(E_2), \ldots, M(E_n)$ are freely independent;*
3. $M\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} M(E_i)$.

Notice that by Definition 3.3 the free Poisson random measure is clearly a free CRM.

Proposition 3.1 *Let M be a free Poisson random measure on a σ-finite measure space $(\mathbb{E} \times \mathbb{R}^+, \mathcal{E} \times \mathcal{B}(\mathbb{R}^+), \nu_{\mathcal{E}} \otimes \nu_{\mathcal{B}})$ with values in the W^*-probability space $(\mathcal{A}, \tau)$. Furthermore, assume that $\int_{\{|x| \le 1\}} |x| \nu_{\mathcal{B}}(dx) < +\infty$. Then,*

1. *For every $E \in \mathcal{E}$, the operator*
$$H(E) = \int_{E \times \mathbb{R}^+} x M(dt, dx) \tag{7}$$
is a free completely random measure.

2. *Given a free completely random measure $\Phi^{\boxplus}$ on $(\mathbb{E} \times \mathbb{R}^+, \mathcal{E} \times \mathcal{B}(\mathbb{R}^+))$, for every $E \in \mathcal{E}$, $\Phi^{\boxplus}(E)$ has a representation of the form*
$$\Phi^{\boxplus}(E) \stackrel{d}{=} \int_{E \times \mathbb{R}^+} x M(dt, dx) , \tag{8}$$
where the symbol "$\stackrel{d}{=}$" means equality in distribution.

Proof: First we observe that the technical assumption $\int_{\{|x| \le 1\}} |x| \nu_{\mathcal{B}}(dx) < +\infty$ ensures the stochastic integral in the right-hand side of (7) and (8) is well-defined (refer to Proposition 6.25(ii) in Barndorff-Nielsen and Thorbjørnsen (2006)). We now proceed proving propositions 1 and 2.

1. We have to verify that the operator $H(\cdot)$ satisfies properties 1-3 of Definition 3.2. Statement 3 follows directly by the properties of free Poisson random measure (see 3 in Definition 3.3) and the linearity of integral operators. Self-adjointness and free independence are guaranteed by Proposition 6.22 in Barndorff-Nielsen and Thorbjørnsen (2006). It remains to show that, for any $E \in \mathcal{E}$, it holds true $\mathscr{L}\{H(E)\} = \Lambda(\mathscr{L}\{\Phi^*(E)\})$ with Φ^* (classical) completely random measure.

Consider a completely random measure Φ^* as in Definition 3.1 and recall (refer to Kingman (1993)) that, for any set $E \in \mathcal{E}$, it admits the representation

$$\Phi^*(E) \overset{\text{a.s.}}{=} \int_{E\times\mathbb{R}^+} xN(dt, dx)\,, \tag{9}$$

where the symbol "$\overset{\text{a.s.}}{=}$" means the equality holds almost surely and N is a classical Poisson random measure on $\mathbb{E}\times\mathbb{R}^+$. In particular, we choose N to be the Poisson random measure corresponding to M via property 1 in Definition 3.3. Then, by Barndorff-Nielsen and Thorbjørnsen (2006, Corollary 6.20), it follows

$$\mathscr{L}\left\{\int_{E\times\mathbb{R}^+} xM(dt, dx)\right\} = \Lambda\left(\mathscr{L}\left\{\int_{E\times\mathbb{R}^+} xN(dt, dx)\right\}\right) \tag{10}$$

and we obtain

$$\mathscr{L}\{H(E)\} = \mathscr{L}\left\{\int_{E\times\mathbb{R}^+} xM(dt, dx)\right\} = \Lambda\left(\mathscr{L}\left\{\int_{E\times\mathbb{R}^+} xN(dt, dx)\right\}\right) = \Lambda(\mathscr{L}\{\Phi^*(E)\})\,.$$

2. Let M and N be free and classical Poisson random measures on $(\mathbb{E}\times\mathbb{R}^+, \mathcal{E}\times\mathcal{B}(\mathbb{R}^+))$, such that the condition 1 of Definition 3.3 is satisfied. Furthermore, let Φ^* be the completely random measure associated with N via (9). Thus, for any $E \in \mathcal{E}$, we get

$$\begin{aligned}
\mathscr{L}\left\{\Phi^{\boxplus}(E)\right\} &= \Lambda\left(\mathscr{L}\left\{\Phi^*(E)\right\}\right) && \text{by property 1 in Definition 3.2}\\
&= \Lambda\left(\mathscr{L}\left\{\int_{E\times\mathbb{R}^+} xN(dt, dx)\right\}\right) && \text{by the representation (9)}\\
&= \mathscr{L}\left\{\int_{E\times\mathbb{R}^+} xM(dt, dx)\right\} && \text{by (10)}
\end{aligned}$$

and we can conclude.

□

4 Examples

This section is devoted to show some examples of free completely random measures such as the free Lévy process that appeared in the literature and the free stable process with $\alpha = \frac{1}{2}$.

4.1 Free Lévy Process and Free Brownian Motion

The following definition of free Lévy process is taken from Barndorff-Nielsen and Thorbjørnsen (2002).

Definition 4.1 *A* free Lévy process (in law), *affiliated with a W^*-probability space $(\mathcal{A}, \tau)$, is a family $(Z_t)_{t\geq 0}$ of self-adjoint operators affiliated with $\mathcal{A}$, which satisfies the following conditions:*

1. *whenever $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \cdots < t_n$, the increments $Z_{t_0}, Z_{t_1} - Z_{t_0}, Z_{t_2} - Z_{t_1}, \ldots, Z_{t_n} - Z_{t_{n-1}}$ are freely independent operators;*

2. *$Z_0 = 0$;*

3. *for any $s, t \in [0, +\infty[$, the distribution of $Z_{s+t} - Z_s$ is independent of s;*

4. *for any $s \in [0, +\infty[$, Z_{s+t} converges to Z_s in probability, as $t \to 0$; in other words, the distribution $\mathscr{L}\{Z_{s+t} - Z_s\}$ converges weakly to δ_0 (the Dirac measure at 0), as $t \to 0$.*

Proposition 4.1 *A free Lévy process $(Z_t)_{t\geq 0}$, affiliated with a W^*-probability space $(\mathcal{A}, \tau)$, is a free completely random measure on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$.*

Proof: We have to prove properties 1-3 of Definition 3.2. Statements 2 and 3 are a direct consequence of the definition of free Lévy process. In fact, thanks to the stationarity of the increments (properties 3 and 4 in Definition 4.1) we obtain directly 3 in Definition 3.2 and, using in addition the free independence of the increments (property 1 in Definition 4.1), we get the free independence. It remains to show that, for any $t \in \mathbb{R}^+$, the law of Z_t is the image through the Bercovici-Pata bijection Λ of the law of a completely random measure on the set $[0, t]$.
By Proposition 1.2 in Barndorff-Nielsen and Thorbjørnsen (2002) we know that, for any $t \in \mathbb{R}^+$, $\mathscr{L}\{Z_t\} = \Lambda(\mathscr{L}\{X_t\})$, where $(X_t)_{t\geq 0}$ is a (classical) Lévy process. Since $(X_t)_{t\geq 0}$ is a completely random measure (refer to Chapter VI, Section 4 in Çınlar (2011)), the conclusion follows. □

Corollary 4.1 *A free Brownian motion $(W_t)_{t\geq 0}$, affiliated with a W^*-probability space $(\mathcal{A}, \tau)$, is a free completely random measure on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$.*

Proof: A free Brownian motion is a free Lévy process (see Example 2.13 in Barndorff-Nielsen and Thorbjørnsen (2005)). Thus, by Proposition 4.1, we can conclude that it is also a free completely random measure. □

4.2 Free Stable Process

Free stable laws are a great matter of interest in free probability. For example, recently Demni (2011) characterized the class of positive free stable distributions. In this section, the free analogous of the stable process with $\alpha = \frac{1}{2}$ is displayed.

We start emphasizing the important role played by the Bercovici-Pata bijection Λ in understanding the characterization of free analogous of specific (classical) completely random measure. Consider a completely random measure Φ^* on the Polish space $(\mathbb{E}, \mathcal{E})$. Fixed a set $E \in \mathcal{E}$, $\mathscr{L}\{\Phi^*(E)\} \in \mathcal{ID}(*)$ and moreover the Laplace transform for $\Phi^*(E)$ is known (see Kingman (1993); Çınlar (2011)) and thus the characteristic triplet (a, η, ρ) of the Lévy-Khintcine representation is easy to compute. The Bercovici-Pata bijection maps $\mathscr{L}\{\Phi^*(E)\}$ into $\Lambda(\mathscr{L}\{\Phi^*(E)\})$. As a consequence, $\Lambda(\mathscr{L}\{\Phi^*(E)\}) \in \mathcal{ID}(\boxplus)$ and it is described by its free cumulant transform that can be obtained via (4), using the same triplet (a, η, ρ) of the corresponding classical object.

Proposition 4.2 *Let $(\mathbb{E}, \mathcal{E})$ a Polish space equipped with the Borel σ-field and let λ a finite measure on it. Given $\alpha \in (0, 1)$, consider the law μ of a (classical) α-stable process characterized by the Lévy intensity*

$$\rho(dt, dx) = \lambda(dt) \cdot \frac{\alpha}{\Gamma(1-\alpha)} x^{-1-\alpha} \, \mathbb{I}_{\{x>0\}}(x) \, dx \,,$$

where Γ is the Gamma function. Then, the free cumulant transform of the law of the free α-stable process is given by

$$C^{\boxplus}_{\Lambda(\mu),E}(z) = \lambda(E) \cdot (-1)^{\alpha-1} \, \alpha \, \Gamma(\alpha) \, z^{\alpha} \qquad \textit{with } E \in \mathcal{E} \textit{ and } z \in \mathbb{C}^-. \tag{11}$$

Proof: Since $\int_0^1 x^{-\alpha} dx < +\infty$, we work with the drift type representation (5) where $\tilde{\eta} = a = 0$. Consider a set $E \in \mathcal{E}$ and a complex number $z \in \mathbb{C}^-$. Recalling that μ and $\Lambda(\mu)$ have the same classical and free characteristic triplet, from (5) we obtain the free cumulant transform of $\Lambda(\mu)$ as

$$\begin{aligned}
C^{\boxplus}_{\Lambda(\mu),E}(z) &= \int_{E\times\mathbb{R}} \left(\frac{1}{1-xz}-1\right)\rho(dt,dx) \\
&= \lambda(E)\cdot\frac{\alpha}{\Gamma(1-\alpha)}\int_{\mathbb{R}^+}\left(\frac{1}{1-xz}-1\right)x^{-1-\alpha}\,dx \\
&= \lambda(E)\cdot\frac{\alpha z}{\Gamma(1-\alpha)}\int_{\mathbb{R}^+}\frac{x^{-\alpha}}{1-xz}\,dx \\
&= \lambda(E)\cdot\frac{B(1-\alpha,\alpha)}{\Gamma(1-\alpha)}(-1)^{\alpha-1}\,\alpha\,z^{\alpha} \qquad \text{where } B \text{ is the Beta function} \\
&= \lambda(E)\cdot(-1)^{\alpha-1}\,\alpha\,\Gamma(\alpha)\,z^{\alpha}.
\end{aligned}$$

□

In particular, if we set $\alpha=\frac{1}{2}$ and we consider the $\frac{1}{2}$-stable process is possible to determine explicitly the law of the corresponding (via Bercovici-Pata bijection) free $\frac{1}{2}$-stable process.

Proposition 4.3 *Let $(\mathbb{E},\mathcal{E})$ a Polish space equipped with the Borel σ-field and let λ a finite measure on it. Consider the law μ of a (classical) $\frac{1}{2}$-stable process characterized by the Lévy intensity*

$$\rho(dt,dx)=\lambda(dt)\cdot\frac{1}{2\sqrt{\pi}}x^{-3/2}\,\mathbb{I}_{\{x>0\}}(x)\,dx\,.$$

Then, for any fixed $E\in\mathcal{E}$, the distribution $\Lambda(\mu)$ of the free $\frac{1}{2}$-stable process is a generalized Beta distribution of the second kind $GB_2\left(\frac{3}{2},\frac{1}{2},1,\frac{\pi}{16}(\lambda(E))^2\right)$.

Proof: Fixed a set $E\in\mathcal{E}$, the strategy consists in deriving the Cauchy transform $G_{\Lambda(\mu),E}$ of $\Lambda(\mu)$ (see formula (1) for the definition) and then getting the distribution itself by Stieltjes inversion formula

$$d\Lambda(\mu)(x)=-\frac{1}{\pi}\lim_{y\to 0}\Im\left[G_{\Lambda(\mu),E}(x+iy)\right]\,, \tag{12}$$

where $\Im$ stands for the operation of taking the imaginary part of a complex number. The latter limit is meant in the weak topology on the space of the probability measures on $\mathbb{R}$.
By setting $\alpha=\frac{1}{2}$ in (11), we get the free cumulant transform of $\Lambda(\mu)$

$$C^{\boxplus}_{\Lambda(\mu),E}(z)=-\frac{i\sqrt{\pi}}{2}\lambda(E)\,z^{1/2}\,.$$

It follows from (3) that

$$F^{-1}_{\Lambda(\mu),E}\left(z^{-1}\right)=-\frac{i\sqrt{\pi}}{2}\lambda(E)\,z^{-1/2}+z^{-1}$$

and this gives

$$F_{\Lambda(\mu),E}(z)=-\frac{\pi}{8}(\lambda(E))^2+z\pm\frac{i\sqrt{\pi}}{2}\lambda(E)\sqrt{z-\frac{\pi}{16}(\lambda(E))^2}\,.$$

Since we want a density as result, we have to chose the "+"-sign in the above solution. By making the reciprocal (2), we compute the Cauchy transform of $\Lambda(\mu)$

$$G_{\Lambda(\mu),E}(z)=\frac{1}{z^2}\left[-\frac{\pi}{8}(\lambda(E))^2+z-\frac{i\sqrt{\pi}}{2}\lambda(E)\sqrt{z-\frac{\pi}{16}(\lambda(E))^2}\right]$$

and thus, by applying (12), we obtain the requested distribution

$$d\Lambda(\mu)(x) = -\frac{1}{\pi}\lim_{y\to 0}\Im\left\{\frac{1}{(x+iy)^2}\left[-\frac{\pi}{8}(\lambda(E))^2 + x + iy - \frac{i\sqrt{\pi}}{2}\lambda(E)\sqrt{x+iy-\frac{\pi}{16}(\lambda(E))^2}\right]\right\}$$

$$= -\frac{1}{\pi}\Im\left\{\frac{1}{x^2}\left[-\frac{\pi}{8}(\lambda(E))^2 + x - \frac{i\sqrt{\pi}}{2}\lambda(E)\sqrt{x-\frac{\pi}{16}(\lambda(E))^2}\right]\right\}$$

$$= \frac{1}{2\sqrt{\pi}}\frac{\lambda(E)}{x^2}\sqrt{x-\frac{\pi}{16}(\lambda(E))^2} \quad \text{with } x > \frac{\pi}{16}(\lambda(E))^2 . \tag{13}$$

It is easy to see that the Laplace transform for (13) is given by

$$[\mathcal{L}\Lambda(\mu)](r) = \frac{2}{\pi}\int_{\mathbb{R}^+}\exp\left\{-\frac{\pi(\lambda(E))^2}{16}(1+x)r\right\} x^{1/2}(1+x)^{-2}\, dx \qquad \text{with } r \in \mathbb{R}^+$$

from which the conclusion follows. □

Acknowledgements

The authors are very grateful to Antonio Lijoi for the inspiration, the patience and for the stimulating discussions.

References

O. E. Barndorff-Nielsen and S. Thorbjørnsen. Lévy processes in free probability. *Proc. Natl. Acad. Sci. USA*, 99:16576–16580 (electronic), 2002.

O. E. Barndorff-Nielsen and S. Thorbjørnsen. The Lévy-Itô decomposition in free probability. *Probab. Theory Related Fields*, 131:197–228, 2005.

O. E. Barndorff-Nielsen and S. Thorbjørnsen. *Quantum independent increment processes. II*, chapter Classical and free infinite divisibility and Lévy processes. Lect. Notes Math. 1866, Springer, Berlin, 2006.

H. Bercovici and V. Pata. Stable laws and domains of attraction in free probability theory. *Ann. of Math*, 149:1023–1060, 1999.

H. Bercovici and D. V. Voiculescu. Free convolution of measures with unbounded support. *Indiana Univ. Math. J.*, 42:733–773, 1993.

P. Biane. Free probability for probabilists. In *Quantum probability communications, Vol. XI (Grenoble, 1998)*, QP-PQ, XI, pages 55–71. World Sci. Publ., River Edge, NJ, 2003.

E. Çınlar. *Probability and stochastics*, volume 261 of *Graduate Texts in Mathematics*. Springer, New York, 2011.

N. Demni. Kanter random variable and positive free stable distributions. *Electron. Commun. Probab.*, 16: 137–149, 2011.

T. S. Ferguson. A Bayesian analysis of some nonparametric problems. *Ann. Statist.*, 1:209–230, 1973.

T. S. Ferguson. Prior distributions on spaces of probability measures. *Ann. Statist.*, 2:615–629, 1974.

L. F. James, A. Lijoi, and I. Prünster. Conjugacy as a distinctive feature of the Dirichlet process. *Scand. J. Statist.*, 33(1):105–120, 2006.

L. F. James, A. Lijoi, and I. Prünster. Posterior analysis for normalized random measures with independent increments. *Scand. J. Stat.*, 36(1):76–97, 2009.

J. F. C. Kingman. Completely random measures. *Pacific J. Math.*, 21:59–78, 1967.

J. F. C. Kingman. *Poisson processes*, volume 3 of *Oxford Studies in Probability*. The Clarendon Press Oxford University Press, New York, 1993. Oxford Science Publications.

A. Nica and R. Speicher. *Lectures on the combinatorics of free probability*, volume 335 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006.

R. Speicher. Free calculus. In *Quantum probability communications, Vol. XII (Grenoble, 1998)*, QP-PQ, XII, pages 209–235. World Sci. Publ., River Edge, NJ, 2003.

D. V. Voiculescu. Addition of certain noncommuting random variables. *J. Funct. Anal.*, 66(3):323–346, 1986.

D. V. Voiculescu. Circular and semicircular systems and free product factors. In *Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989)*, volume 92 of *Progr. Math.*, pages 45–60. Birkhäuser Boston, Boston, MA, 1990.

D. V. Voiculescu, K. J. Dykema, and A. Nica. *Free random variables*, volume 1 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 1992. A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups.