Ph.D. Thesis

Strategic Profit Sharing Between Firms¹

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Abstract

The purpose of our thesis "*Strategic Profit Sharing Between Firms*" is to study the effects of the unilateral and unconditional profit sharing strategy on firms' behavior. The basic model is a two-stage game. In the first stage, firms decide simultaneously what part of their profits to give away to their rival and then, in a second stage, the equilibrium price and quantities are determined. Notice that the action of giving away profits is decided unilaterally and unconditionally in a non-cooperative framework. After the decision to give profits is taken, the giving firm is committed to it. Our thesis contribution is to explore issues on collusion if the cross-ownership is the product of this decision.

The thesis is divided into two chapters. The first chapter "*Strategic Profit Sharing Between Firms*" starts by presenting the general model that shows the direct (negative) and indirect (may be positive) effects of this strategy, and, then, concentrate on the effects of its inclusion in some oligopolistic models (Cournot, Bertrand, Hotelling). We show that giving away profits is a rewarding strategy for firms in some (but not all) of the above oligopolistic competition models. Our analysis of the standard models shows that firms may have incentives to share profits in equilibrium in the Bertrand and Hotelling models (at least for some values of the parameters that define the model) and not so much in the Cournot ones (requiring an unrealistic value for the relevant parameter).

For the linear demand cases of study, and for the most standard assumptions on both Cournot and Bertrand models, a necessary (but not sufficient) condition to find a profit-sharing equilibrium is that strategies are complementary for substitutes goods. This, in turn, implies that profit sharing is never an equilibrium in the standard Cournot models. The complementarity of the strategies is necessary for the equilibrium to internalize the effect on the other firm's profits.

Chapter I finishes by relaxing the profit sharing key assumptions of our model to understand the extent to which they are necessary conditions. We first explore the possibility that only one firm has the possibility to share profits. We show that profit sharing between firms is a winning and sustainable strategy only if both firms are sharing when competing in prices. We then explore how the profit sharing strategy may be concealed behind other arrangements. In particular, we see that the implication of the firms in a

joint venture may be such a strategy. This is important, as the standard profit sharing strategy may be easy to detect.

The second chapter "*Strategic Profit Sharing Leads to Collusion in Bertrand Oligopolies*" focuses on the Bertrand model with homogenous goods and adds a first stage of profit sharing. We show that firms may be able to support prices between the marginal cost and the monopoly price, thus obtaining positive profits in almost all of the equilibria. This remarkable result, that resembles a Folk theorem, is robust to the number of firms and to cost asymmetries. Furthermore, for a given equilibrium price, any share of the market can also be supported in a subgame perfect equilibrium. For the duopoly case we completely characterize the set of pure strategy equilibria. However, in the extension to more than two firms, and due to the increasing complication in the multiplicity of equilibria, we only show the existence of the equilibria in the subgames that is sufficient to support the desired equilibrium price.

Chapter 1

Strategic Profit Sharing Between Firms¹

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1 Introduction

The study of oligopolies is a recurrent topic that attracts the attention of economists. Models of oligopolistic competition are studied both because of their increasing number of applications and because of the theoretical insights they provide. The applications include the fields of industrial organization, macroeconomics, public economics and international trade. The theoretical works deal with issues such as the analysis of different choices of strategic variables, the effects of dynamic considerations and, more recently and related to the present work, the role of cross-ownership (see, for example, FritzRoy and Kraft, 1986, Farrell and Shapiro, 1990, Malueg 1992, Reitman 1994, Caminal and Vives, 1996, Jackson and Wilkie 2005 and Gilo *et al.* 2006)

Along this last line of research, the present Chapter focuses on the new strategy of firms that consists on unilaterally and voluntarily giving a part of their own profits to their rivals. Thus, in a first stage, firms decide simultaneously what part of their profits to give away to their rival and then, in a second stage, the equilibrium price and quantities are determined. Notice that the action of giving away profits is decided unilaterally and unconditionally in a non-cooperative framework. After the decision to give profits is done, the giving firm is committed to it.

We use a general model to show the direct (negative) and indirect (may be positive) effects of this strategy, and, then, concentrate on the effects of its inclusion in some oligopolistic models (Cournot, Bertrand, Hotelling). We show that giving away profits is a rewarding strategy for firms in some (but not all) of the above oligopolistic competition models. Our analysis of the standard models shows that firms may have incentives to share profits in equilibrium in the Bertrand and Hotelling models (at least for some values of the parameters that define the model) and not so much in the Cournot ones (requiring an unrealistic value for the relevant parameter).

For the linear demand cases of study, and for the most standard assumptions on both Cournot and Bertrand models, a necessary (but not sufficient) condition to find a profit-sharing equilibrium is that strategies are complementary for substitutes goods. This, in turn, implies that profit sharing is never an equilibrium in the standard Cournot models. The complementarity of the strategies is necessary for the equilibrium to internalize the effect on the other firm's profits.

The sufficiency is achieved if this effect, whenever is positive, is enough to overcome the negative effect of the profit-giving strategy. This is more likely to be the case the more fierce is the competition, which, in general, occurs in the Bertrand models (see Amir and Jin, 2001, and references within.)

The rationale behind this is that in the cases of greater competition there is more to gain from collusion. The intuitive explanations of why Bertrand competition is higher than Cournot typically refer to the fact that

“firms have less capacity to raise prices above marginal cost in Bertrand competition because the perceived elasticity of demand of a firm when taking the price of the rival as given is larger than that which the firm perceives when taking the quantity of the rival as given” (Singh and Vives, 1984).

One important consequence of the model is that it shows the possibility of attaining partial collusion in one-shot oligopolistic interactions, in contrast with models of partial cross-ownership (see Gilo *et al.*, 2006), where the collusion is facilitated only in the repeated game.

In our model, the profit-giving strategy is performed directly. However, in real life, the strategy that makes the rival firm internalize, at least partially, one’s well-being may be more indirect, and could be hidden behind a more complicated relation. After the presentation of the basic model, we suggest one such possibility, namely, the use of a joint venture.

The effects of cross ownership on competition has been explored at least since the work by Reynolds and Snapp (1986), where it is shown that cross ownership serves to internalize free rider problems associated with policing collusion. Since then, other authors have worked out many related issues. For example, Farrell and Shapiro (1990) study a one-way cross ownership model where a big firm wants to acquire assets from another firm. Throughout a single-period Cournot oligopoly model, they show that, as the degree of cross ownership among rivals increases, the equilibrium in the market becomes less competitive.

In a dynamic setting, Malueg (1992) shows that, if firms interact repeatedly, increasing cross ownership may reduce the likelihood of collusion. A high level of cross ownership may even entail a lower likelihood of collusion than no-cross-ownership would. Gilo *et al.* (2006) explore this issue with more detail to show that, in general, the incentives to tacitly collude depend in a complex way on the entire partial cross ownership.

More related to our work, Reitman (1994) considers an oligopolistic model with conjectural variations in which firms buy claims to profits of other firms, and find that in the more rivalrous competition (i.e., more than Cournot), firms are willing to form partial ownership agreements to take advantage of a reduced competition. Later on, Alley (1997) develops a conjectural variation model that allows for partial ownership arrangements to the Japanese and US automobile industries to study the degrees of competitiveness and collusion in both countries.

Other recent works include Jackson and Wilkie (2005), that characterize the outcomes of games when players can make binding offers of strategy-contingent side payments before the game is played. In a game-theoretical setting, Jackson and Wilkie (2005) have characterized the outcomes of games when players can make binding offers of strategy-contingent side payments before the game is played. Contrary to their paper, our firms, far from making binding agreements, play non-cooperatively through all the game.

All these works, except Reitman (1994), have in common the exogeneity of the degree of cross-ownership. Our thesis real life problems are the same as those of this literature. In our thesis, by contrast, this is endogenously determined in the model. Our thesis contribution is to explore issues on collusion if the cross-ownership is the product of an endogenous, unilateral choice by firms through the strategy of giving away profits. This way we may learn more about the likelihood of cross-ownership. In fact, the choice of how much of the profits to give away is not done contingent on the other firm's choice and that, in this sense, there is no agreement or commitment. Once the decision of sharing a proportion of its profits is done, the firm is bound by the decision and, only in this sense there is a commitment.

In Reitman (1994), the cross ownership is decided in a mechanism in which firms buy claims to other firms' profits. In practice, this price needs not be a cash payment, but any sort of contribution to production, marketing, etc., that does not affect the variable costs of production. There are two main differences with our model. First, we show that there is no need for this cash payment, and, second, that the choice of how much of the profits to share does not need to be part of an agreement between two firms, as we only need that the level of profit sharing be decided by the giving firm. Remarkably, we observe the same kind of results (the willingness to share profits is higher if the competition is stronger) even with these differences.

In Section 2 we present the basic model to show the direct (negative) and indirect (perhaps positive) effects of the profit-giving strategy. Section 3 presents three models of either price or quantity competition with product differentiation. Section 4 relates our results with known differences between price and quantity competition in oligopolies by exhibiting the necessary and sufficient conditions for the profit sharing equilibrium. Sections 5 and 6 relax the profit sharing key assumptions of our model to understand the extent to which they are necessary conditions. For example, in Section 5 we explore the possibility that only one firm has the possibility to share profits. In other terms, we had before a two-side profit-sharing while we focus here on just one-side profit-sharing. We study this new version only for the cases (Hotelling and Bertrand models with heterogeneous goods) where we observed the existence of profit sharing equilibria when both firms were al-

lowed to share profits, and find that profit sharing between firms is a winning strategy only if both firms are sharing when competing in prices. In Section 6 we explore how the profit sharing strategy may be concealed behind other arrangements. In particular, we see that the implication of the firms in a joint venture may be such a strategy. This is important, as the standard profit sharing strategy may be easy to detect. Finally, Section 7 concludes.

2 The basic model

Consider a duopolistic market in which two firms have s_1 and s_2 as their respective strategic variables. These variables can be price, quantity, location, or any other standard strategic variable in oligopoly theory. Assume now that, previous to the strategic choice of s_1 and s_2 , Firm i can give part of its profits to Firm j , and that Firm j accepts. Let $\alpha_i \in [0, 1]$ denote the part of the profit that Firm i gives to Firm j . The profit functions of Firm i can be written as $\Pi_i(s_1(\alpha_1, \alpha_2), s_2(\alpha_1, \alpha_2))$. To simplify this exposition, we will assume interior solutions in all maximization problems. In the next section we provide examples where these assumptions are met, as well as examples of corner solutions.

In the second stage of the game Firm i maximizes $P_i = (1 - \alpha_i)\Pi_i(s_i, s_j) + \alpha_j\Pi_j(s_i, s_j)$, whose first-order conditions for an interior solution give:

$$\alpha_i \frac{\partial \Pi_i(s_i, s_j)}{\partial s_i} - \alpha_j \frac{\partial \Pi_j(s_i, s_j)}{\partial s_i} = \frac{\partial \Pi_i(s_i, s_j)}{\partial s_i}. \quad (1)$$

An analogous condition is found for Firm j . The solution of the system thus defined, if it exists, defines the Nash equilibrium $s_1^*(\alpha_1, \alpha_2)$ and $s_2^*(\alpha_1, \alpha_2)$ of the second stage. From there, we can write the profits function $P_i(\alpha_1, \alpha_2)$ that we use to solve the first stage of the game:

$$\max_{\alpha_i} P_i = (1 - \alpha_i)\Pi_i(s_i^*(\alpha_i, \alpha_j), s_j^*(\alpha_i, \alpha_j)) + \alpha_j\Pi_j(s_i^*(\alpha_i, \alpha_j), s_j^*(\alpha_i, \alpha_j)).$$

Using (1), the first-order conditions for an interior solution are given by

$$\frac{\partial P_i}{\partial \alpha_i} = -\Pi_i + \frac{\partial s_j^*}{\partial \alpha_i} \frac{\partial (\Pi_i + \Pi_j)}{\partial s_j} = 0, \quad (2)$$

which, together with a similar condition for Firm j , defines the system of equations that yields the equilibrium choice of α_1^* and α_2^* in the first stage.

This means that, if the equilibrium is found solving the first order conditions for an interior solution, we are in a situation where α_1^* and α_2^* are

greater than zero. I.e., firms are willing to give away part of their profits to their rival. The last expression shows that sharing profits has two opposite effects. First, a direct and negative effect given by the first term $-\Pi_i$ and then, a strategic effect (positive in an interior solution) given by the second term, $\frac{\partial s_j^*}{\partial \alpha_i} \frac{\partial (\Pi_i + \Pi_j)}{\partial s_j}$. Hence, if the strategic effect is positive and large enough, firms may find profitable to give away profits to their rival. In section 4, we will come back to those two effects for a deeper analysis. But, let us first study how our profit sharing model reacts to the oligopoly models (Bertrand, Cournot, Hotelling).

3 Profit sharing in oligopolies

3.1 The Bertrand model with heterogenous goods

Let us apply the basic model to the Bertrand model with heterogeneous goods, where $p_i = 1 - q_i - \gamma q_j$ is the inverse demand function of Firm i . Then, profits are $\Pi_i(q_1, q_2) = p_i q_i(p_1, p_2)$, with $q_i = \frac{1}{1-\gamma^2}(1 - \gamma - p_i + \gamma p_j)$. We assume that $\gamma^2 < 1$, that is, the own-price effect dominates the cross-price effect. In the second stage the first-order conditions with respect to p_i in the problem of maximizing $P_i = (1 - \alpha_i)p_i q_i + \alpha_j p_j q_j$ with respect to q_i give:

$$p_i(p_j) = \frac{1 - \gamma}{2} + \gamma \frac{1 - \alpha_i + \alpha_j}{2(1 - \alpha_i)} p_j.$$

By solving the system formed by $p_1 = p_1(p_2)$ and $p_2 = p_2(p_1)$, we have

$$p_i^*(\alpha_i, \alpha_j) = \frac{(1 - \alpha_j)(1 - \gamma) [2(1 - \alpha_i) + \gamma(1 - \alpha_i + \alpha_j)]}{4(1 - \alpha_i)(1 - \alpha_j) - \gamma^2(1 - \alpha_i + \alpha_j)(1 - \alpha_j + \alpha_i)}. \quad (3)$$

Substituting, in q_i :

$$q_i^*(\alpha_i, \alpha_j) = \frac{(2 + \gamma)(1 - \alpha_j)(1 - \alpha_i) - \gamma(1 + \gamma)\alpha_j(1 - \alpha_j) - \gamma^2\alpha_i\alpha_j}{(1 + \gamma)[4(1 - \alpha_i)(1 - \alpha_j) - \gamma^2(1 - \alpha_i + \alpha_j)(1 - \alpha_j + \alpha_i)]}.$$

In the first stage, profits have the form $P_i(\alpha_i, \alpha_j) = (1 - \alpha_i)p_i^* q_i^* + \alpha_j p_j^* q_j^*$, $i, j = 1, 2$, $i \neq j$. The equilibrium choice of (α_1^*, α_2^*) (assuming interior solutions) is given by maximizing $P_i(\alpha_i, \alpha_j)$ with respect to α_i , and solving the system formed by the reaction functions $\alpha_1 = \alpha_1(\alpha_2)$, $\alpha_2 = \alpha_2(\alpha_1)$ to get (α_1^*, α_2^*) . Once this is done, it is of interest to know whether there exist values of γ for which $(\alpha_1^*, \alpha_2^*) > (0, 0)$. Our objective is to find out, first how α in equilibrium given by interior solutions varies with γ , second how

profits change with γ and third whether such profits are greater or less than profits where $\alpha = 0$. However, the analytical expression for $\alpha_i^*(\gamma)$, as shown in Appendix is very complicated and it is difficult to find a explicit value for α^* . The model is thus tractable only numerically and for simplicity reasons, whenever we have interior solutions, we concentrate on symmetric equilibria and consider only $\alpha_1^* = \alpha_2^* = \alpha^*$. However, we do not assume that the only equilibria are the interior solutions or, within the interior, the symmetrical case.

For negative values of γ there is no profit-sharing equilibrium. Thus, we illustrate the equilibria for positive γ . Figure 1 shows the values for which there exists an equilibrium with $(\alpha_1, \alpha_2) > (0, 0)$. For all other values of γ there is only a corner solution $((\alpha_1, \alpha_2) = (0, 0))$ for the equilibrium. The discontinuity around 0.8 is due to the fact the equilibria for $\alpha > 0$ must satisfy the equations for interior solution and the inequality that makes profits greater than those if $\alpha = 0$. The discontinuity takes place when this inequality changes, as the solution changes from $\alpha = 0$ to the positive α that satisfies the conditions for interior solution. In other words, when looking for equilibria, the interior solutions will give the pair of best replies, which, for every individual firm implies that it is located in a local maximum. I.e., the firm is maximizing provided it decides to share its profits. For this to be an equilibrium, it must be the case that the non-sharing unilateral decision does not give higher payoffs. The situation is not different than the one in entry models where the equilibrium conditions provided a firm enter give this firm lower profits than the “do not enter” decision.

Figure 1: Equilibrium alpha as a function of gamma

Recall that the function the $P_i(\alpha_1, \alpha_2)$ gives the equilibrium profits for every

pair of sharing decisions. This function is parametrized by γ , the degree of heterogeneity. Next we present the profits associated with three different choices of sharing pairs (α_1, α_2) for the different positive values of γ , thus defining a function $\mathbf{P}_i(\alpha_1, \alpha_2, \gamma)$, where now (α_1, α_2) are parameters and γ is a variable. As there is no profit-sharing equilibrium for negative values of γ , we illustrate the equilibrium profits only for positive γ . In Figure 2 profits $\mathbf{P}_i(\alpha_1, \alpha_2, \gamma)$ are shown for three cases, the equilibrium case with $(\alpha_1, \alpha_2) = (\alpha_1^*, \alpha_2^*)$, the case of not-profit sharing with $(\alpha_1, \alpha_2) = (0, 0)$, and the case of collusion to share monopolistic profits with $(\alpha_1, \alpha_2) = (0.5, 0.5)$. The equilibrium values of (α_1^*, α_2^*) are given by the solution of the reaction functions $\alpha_1 = \alpha_1(\alpha_2)$, $\alpha_2 = \alpha_2(\alpha_1)$ whenever $\mathbf{P}_i(\alpha_1 = \alpha_2 = \alpha^*, \gamma) \geq \mathbf{P}_i(\alpha_1 = \alpha_2 = 0, \gamma)$.

The differences between $\mathbf{P}_i(\alpha_1 = \alpha_2 = 0.5, \gamma)$ and $\mathbf{P}_i(\alpha_1 = \alpha_2 = \alpha^*, \gamma)$ provide an idea of how much it is to be gained if some sort of collusion is reached between the firms. These comparisons are interesting and will be discussed later in Section 4.

Figure 2: Profits in
equilibrium: $\mathbf{P}_i(\alpha_1 = \alpha_2 = \alpha^*, \gamma)$ (boxed points),
non-sharing: $\mathbf{P}_i(\alpha_1 = \alpha_2 = 0, \gamma)$ (solid line), and
collusion (half monopoly): $\mathbf{P}_i(\alpha_1 = \alpha_2 = 0.5, \gamma)$ (dotted line)

We can summarize the results in this section in the next claim.

Claim 1 *The equilibrium of the model of Bertrand competition with heterogeneous goods and a profit sharing stage as defined above satisfies:*

(i) *The equilibrium implies that firms share profits for high values of γ , namely $\gamma \in [0.8, 1)$.*

(ii) *The threshold value of γ for which the equilibrium implies profit sharing occurs at a point where the difference between the non-sharing profits and the collusion profits is high enough.*

(iii) *The equilibrium approaches to the monopolistic outcome as γ approaches to one.*

3.2 The Cournot model with heterogenous goods

Consider now the case of Cournot with heterogenous goods, with demand functions given by $p_i = 1 - q_i - \gamma q_j$ ($i, j = 1, 2, i \neq j$). Again assume $\gamma^2 < 1$.

In the second stage, maximization of P_1 and P_2 with respect to q_1 and q_2 , respectively, yields reaction functions

$$q_i = \frac{(1 - \alpha_i) - \gamma(1 - \alpha_i + \alpha_j)q_j}{2(1 - \alpha_i)}, \quad i, j = 1, 2; \quad i \neq j,$$

and equilibrium quantities

$$q_i^*(\alpha_i, \alpha_j) = \frac{(1 - \alpha_j)(2(1 - \alpha_i) - \gamma(1 - \alpha_i + \alpha_j))}{4(1 - \alpha_i)(1 - \alpha_j) - \gamma^2(1 - \alpha_i + \alpha_j)(1 - \alpha_j + \alpha_i)}. \quad (4)$$

Substituting in $p_i = 1 - q_i - \gamma q_j$ we find the equilibrium prices

$$p_i^*(\alpha_i, \alpha_j) = \frac{2(1-\gamma)(1-\alpha_i)(1-\alpha_j) - \gamma^2 \alpha_j(1-\alpha_j + \alpha_i) + \gamma(1-\alpha_j)(1-\alpha_i + \alpha_j)}{4(1-\alpha_i)(1-\alpha_j) - \gamma^2(1-\alpha_i + \alpha_j)(1-\alpha_j + \alpha_i)}.$$

In the first stage, Firm i maximizes $P_i(\alpha_i, \alpha_j) = (1 - \alpha_i)p_i^*q_i^* + \alpha_j p_j^*q_j^*$ with respect to α_i .

The equilibrium choice of (α_1^*, α_2^*) (assuming interior solutions) is given by the solution of the reaction functions $\alpha_1(\alpha_2)$ and $\alpha_2(\alpha_1)$. These expressions, as shown in Appendix are quite complicated, and an explicit solution for α^* is not available. The model is thus tractable only numerically and for simplicity reasons, whenever we have interior solutions, we concentrate on symmetric equilibria and consider only $\alpha_1^* = \alpha_2^* = \alpha^*$. However, we do not assume that the only equilibria are the interior solutions or, within the interior, the symmetrical case. For instance, we find non-interior solutions in Section 5.

When proceeding as in the Bertrand case, our computations (not shown here) show that it never pays to give profits away. (However, by investigating values beyond but close to the corner values, we see that if we allow for values of γ with $\gamma^2 > 1$, there are cases of profit sharing in equilibrium. For instance, if $\gamma = -1.09$ then $(\alpha_1^*, \alpha_2^*) = (0.47, 0.47)$ with profits $P_i(\alpha = 0.47, \gamma = -1.09) = 9.5$, greater than $P_i(\alpha = 0, \gamma = -1.09) = 1.21$)

Figure 3 is the counterpart of Figure 2. However, contrary to the Bertrand case, it shows that equilibrium profits decrease with the value of γ , and that there is a small difference in profits between the equilibrium case (no-sharing, $\alpha_1 = \alpha_2 = 0$) and the sharing of monopolistic profits ($\alpha_1 = \alpha_2 = 0.5$). Thus, in the Cournot case, there is less to gain from collusion.

Figure 3: Profits in
equilibrium: $\mathbf{P}_i(\alpha_1 = \alpha_2 = \alpha^* = 0, \gamma)$ (solid line), and
collusion (half monopoly): $\mathbf{P}_i(\alpha_1 = \alpha_2 = 0.5, \gamma)$ (dotted line)

The next claim summarizes our findings.

Claim 2 *The model of Cournot competition with heterogeneous goods defined in above has no equilibrium in which firms share profits for $\gamma \geq 0$.*

With homogeneous goods, it is easy to see that there is no sharing in equilibrium under Cournot competition¹. Consider two firms, 1 and 2, that compete in a homogeneous market, and that have no production costs. Let the demand curve be given by $p = 1 - q_1 - q_2$ where q_i is Firm i 's output. It is straightforward to see that the equilibrium in the second stage of the profit-giving game is $q_i^*(\alpha_i, \alpha_j) = \frac{1-\alpha_j}{3-\alpha_i-\alpha_j}$, with prices given by $p^*(\alpha_1, \alpha_2) = \frac{1}{3-\alpha_1-\alpha_2}$. Hence $P_i(\alpha_i, \alpha_j) = \frac{1-\alpha_i}{(3-\alpha_i-\alpha_j)^2}$, and $\frac{\partial P_i}{\partial \alpha_i} = \frac{-1-\alpha_i+\alpha_j}{(3-\alpha_i-\alpha_j)^3} < 0$ for $\alpha_i, \alpha_j \in [0, 1]$, and, thus, the maximization problem of the first stage yields $\alpha_1^* = \alpha_2^* = 0$.

3.3 The Hotelling model

Consider a linear city that lies on the interval $[0, 1]$, and where consumers are uniformly distributed with density 1 along this interval. There are two firms which sell the same physical good, and that are located at the extremes of the city. Firm 1 is at $x = 0$ and Firm 2 at $x = 1$. Consumers incur a transportation cost $t > 0$ per unit of length. Thus, a consumer living at x incurs a cost of tx to go to Firm 1 and a cost of $t(1-x)$ to go to Firm 2. Consumers have unit demands, and will choose from which firm to buy minimizing the sum of the price and transportation cost. Firms compete in price, which they choose simultaneously.

A consumer who is indifferent between the two firms is located at $x = D_1(p_1, p_2)$, where x is given by equating generalized costs; i.e., $p_1 + tx = p_2 + t(1-x)$. The firms' respective demands are $D_1(p_1, p_2) = \frac{1}{2} + \frac{p_2-p_1}{2t}$, and $D_2(p_1, p_2) = 1 - D_1$.

To find the equilibrium in the second stage of the game, Firm i maximizes $P_i = (1 - \alpha_i) p_i D_i(p_i, p_j) + \alpha_j p_j D_j(p_i, p_j)$. The first-order conditions with respect to p_i give:

$$p_i(p_j) = \frac{t}{2} + \frac{1 - \alpha_i + \alpha_j}{2(1 - \alpha_i)} p_j, i, j = 1, 2, i \neq j.$$

The solution of the system yields

$$p_i^*(\alpha_i, \alpha_j) = t(1 - \alpha_j) \frac{3 - 3\alpha_i + \alpha_j}{(1 - \alpha_i - \alpha_j)(3 - \alpha_i - \alpha_j)}, i, j = 1, 2, i \neq j.$$

Substituting equilibrium prices in D_i , we have:

$$D_i(p_i^*, p_j^*) = \frac{1}{2} \frac{3 - 2\alpha_j}{3 - \alpha_i - \alpha_j}.$$

¹However, under Bertrand competition Waddle (2005) shows that any price between perfect competition and monopoly can be achieved, yielding positive profits to the industry. The ambiguity of the result and the difference in methodology suggest a separate treatment of the case.

Now, profits in the first stage are given by $P_i(\alpha_i, \alpha_j) = (1 - \alpha_i) p_i^* D_i(p_1^*, p_2^*) + \alpha_j p_j^* D_j(p_1^*, p_2^*)$, or

$$P_i = \frac{t(1-\alpha_i)}{2(3-\alpha_i-\alpha_j)^2(1-\alpha_j-\alpha_i)} (2\alpha_j^3 - 8\alpha_j^2 - 3\alpha_j + 12\alpha_i\alpha_j - 2\alpha_i^2\alpha_j + 9 - 9\alpha_i).$$

Maximization with respect to α_i gives $\alpha_1 = \alpha_2 = \alpha^* = 0.15$, with profits $P_1^* = P_2^* = 0.6t$ and prices $p_1^* = p_2^* = 1.2t$. However, if firms decided not to share their profits, that is, if $\alpha^* = 0$, we find $P_1^* = 0.5t$.

3.4 The Cournot case with homogeneous goods

Consider two firms, 1 and 2, that compete in a homogeneous market, and that have no production costs. Let the demand curve be given by $p = 1 - q_1 - q_2$ where q_i is Firm i 's output. It is straightforward to see that the equilibrium in the second stage of the profit-giving game is $q_i^*(\alpha_i, \alpha_j) = \frac{1-\alpha_j}{3-\alpha_i-\alpha_j}$, with prices given by $p^*(\alpha_1, \alpha_2) = \frac{1}{3-\alpha_1-\alpha_2}$. Hence $P_i(\alpha_i, \alpha_j) = \frac{1-\alpha_i}{(3-\alpha_i-\alpha_j)^2}$, and $\frac{\partial P_i}{\partial \alpha_i} = \frac{-1-\alpha_i+\alpha_j}{(3-\alpha_i-\alpha_j)^3} < 0$ for $\alpha_i, \alpha_j \in [0, 1]$, and, thus, the maximization problem of the first stage yields $\alpha_1^* = \alpha_2^* = 0$.

4 The two effects of profit-sharing

In this section, we will relate the results with known differences between price and quantity competition in oligopolies by exhibiting the necessary and sufficient conditions for the profit sharing equilibrium. Recall condition (2) in Section 2:

$$\frac{\partial P_i}{\partial \alpha_i} = -\Pi_i + \frac{\partial s_j^*}{\partial \alpha_i} \frac{\partial (\Pi_i + \Pi_j)}{\partial s_j} = 0, \quad (5)$$

This condition shows two effects in the profit-sharing strategy. One is the direct and negative effect of giving away profits, and the other is the indirect effect via the changes in the rival's motivations when it has to take into consideration the part of the profits coming from the other firm. As the direct effect is always negative, a necessary condition for the profit sharing equilibrium is that the indirect is positive, which is the case, for substitutes goods when quantities and prices are strategic complements as we will show in Section 4.1. With a positive indirect effect, a sufficiency condition for sharing profits in equilibrium is achieved if the indirect effect is big enough to overcome the negative effect of the profit-giving strategy. This is more

likely to occur when the competition is more aggressive, which, in general, is the case in the Bertrand models as we will show in Section 4.2.

4.1 The indirect effect

Here, we will provide necessary conditions for the indirect effect to be positive in our linear models of Bertrand and Cournot competition with heterogeneous goods.

First, recall the concepts of substitute and complementary goods in linear demands and its relation with the parameter γ .

For the Cournot case (competition in quantity), let the demand functions be given by $p_i = 1 - q_i - \gamma q_j$, with $i, j = 1, 2$, and $i \neq j$. Solving for q_i , we have: $q_i = \frac{1}{1-\gamma^2} (1 - \gamma - p_i - \gamma p_j)$. Now we can state that (i) goods are substitutes if $\frac{\partial q_i}{\partial p_j} = \frac{1}{1-\gamma^2} (-\gamma) > 0$, which is satisfied if $\gamma < 0$ whenever $\gamma^2 < 1$ (usually the case), and that (ii) goods are complementary if $\gamma > 0$ whenever $\gamma^2 < 1$.

Second, recall the concepts of strategic substitutes and complements in the Cournot and Bertrand models.

For the Cournot case, write the formula for profits as: $\Pi_i = (1 - q_i - \gamma q_j) q_i$. Now solve for reaction functions to get: $q_i = \frac{1-\gamma q_j}{2}$. We now have that (iii) quantities are strategic substitutes $\left(\frac{\partial q_i}{\partial q_j} < 0\right)$ if $\frac{\partial q_i}{\partial q_j} = -\frac{\gamma}{2} < 0$ or $\gamma > 0$ and that (iv) quantities are strategic complements if $\gamma < 0$.

For the Bertrand case, write the formula for profits as:

$\Pi_i = p_i \frac{1}{1-\gamma^2} (1 - \gamma - p_i - \gamma p_j)$, and solve for reaction functions. Then we have: $p_i = \frac{1-\gamma+\gamma p_j}{2}$, and now we can see that (v) prices are strategic complements $\left(\frac{\partial p_i}{\partial p_j} > 0\right)$ if $\frac{\partial p_i}{\partial p_j} = \frac{\gamma}{2} > 0$ or $\gamma > 0$ and that (vi) prices are strategic substitutes if $\gamma < 0$.

Now, take the indirect effect in condition (5), which implies that

$$\frac{\partial s_j^*}{\partial \alpha_i} \frac{\partial (\Pi_i + \Pi_j)}{\partial s_j} > 0, \quad (6)$$

where s is the strategy variable, is a necessary (but not sufficient) condition for a profit-sharing equilibrium.

In the Bertrand case with heterogenous goods, using expression (3) in Section 3, the sign of the expression (6) takes the following form:

$$\text{sign} \left(\frac{\partial p_j^*}{\partial \alpha_i} \frac{\partial (\Pi_i + \Pi_j)}{\partial p_j} \right) = \text{sign} \left(2 \frac{1-\gamma}{1+\gamma} \gamma (1-\alpha)(2+\gamma-2\alpha) \right), \quad (7)$$

where, given the symmetry of the problem, we have set $\alpha_i = \alpha_2 = \alpha$ after differentiation. It can be seen that expression (7) is satisfied if $\gamma > 0$ (recall that $\alpha \in [0, 1]$) so that the goods are substitutes, and the prices are strategic complements, which is the standard case in Bertrand models.

In the Cournot case, use expression (4) in Section 3 to get

$$\text{sign} \left(\frac{\partial q_j^*}{\partial \alpha_i} \right) = \text{sign} \left(\gamma [4 - \alpha(\alpha^2 - 3\alpha - 3)] + [\alpha\gamma(\gamma - 2)] \right), \quad (8)$$

where, again, we set $\alpha_i = \alpha_2 = \alpha$ after differentiation. Since $\alpha \in [0, 1]$, the first term in brackets is greater than 4, and the second is never smaller than -1 for any positive value of γ , and, thus, expression (8) is positive if $\gamma > 0$. Now,

$$\frac{\partial (\Pi_i + \Pi_j)}{\partial q_j} < 0$$

in the equilibrium point in Cournot competition, as maximum total profits come after the monopolistic quantity, and decrease beyond it. Therefore, the necessary condition (6) can only be satisfied if $\gamma < 0$, and the goods are substitutes and the quantities are strategic complements.

Thus, with substitutes goods we see that a necessary condition to find a profit-sharing equilibrium is that strategies are complementary with the standard assumptions. This makes good sense as strategic complements reinforce each other. The profit sharing strategy may exploit this fact until this reinforcement overcomes the prisoners' dilemma nature of the duopoly.

4.2 The direct effect

Here, we will provide and characterize sufficient conditions for sharing profits in equilibrium with the models of Bertrand and Cournot with heterogeneous goods and Hotelling. In fact, the sufficiency of conditions for sharing profits in equilibrium is achieved if the indirect effect, whenever is positive, is big enough to overcome the negative effect of the profit-giving strategy. This is more likely to occur when the competition is more aggressive, which, in general, is the case in the Bertrand models. See the works of Amir and Jin (2001), Singh and Vives (1984), and references within. The reason why more fierce competition is better for the profit-sharing equilibrium is that there is

more to gain from collusion. To further understand what makes a sufficient condition, we consider some numerical cases.

In Section 3 we already saw that the differences in profits between the non-profit sharing case and the monopoly case were much higher for the Bertrand competition than for the Cournot competition. We can see this in more detail by fixing the value of γ and observing the changes in profits in the symmetric case as the shares go from $\alpha_1 = \alpha_2 = 0$ to a positive value $\alpha_1 = \alpha_2 = \alpha$.

Consider, then, first the Bertrand case with different values of γ , make both firms choose the same sharing rules ($\alpha_i = \alpha_j = \alpha$), and show the changes in prices and profits as α changes.

Figure 4

Figure 5

In figures 4 and 5 we observe that, as α increases, both prices and profits increase until they reach a maximum and, then, decline rapidly. For the values of γ for which there is a profit sharing equilibrium ($\gamma = 0.85$, and $\gamma = 0.99$), the pattern shows a low start and a rapid increase towards the maximum. For the value of γ for which there is no profit sharing equilibrium ($\gamma = 0.2$), on the other hand the pattern shows a much slower increase in profits. This, in turn shows that if the differences in profits between a situation with $\alpha = 0$ and another with $\alpha > 0$ is not big enough to begin with, the incentives to share profits will be small.

Now we do the same with the Cournot model for heterogenous goods, where profit sharing was not an equilibrium strategy. Figures 6 and 7 show the change in both prices and profits as α changes for $\gamma = 0.2$ and $\gamma = 0.85$. We observe the same patterns that we did in figures 1 and 2 for the Bertrand case when $\gamma = 0.2$, the case in which there is no profit-sharing equilibrium.

Recall that in these graphs we study the changes in profits after changes in a common value of α for both firms. These profits show the potential for gaining, that achieves the greatest value at $\alpha_1 = \alpha_2 = 0.5$ when each firm behaves like a monopolist in half of the market. This situation is not an equilibrium, as we saw in Section 3.

Figure 6

Figure 7

Finally, in figures 8 and 9 we show the same graphs for the Hotelling model studied in Section 3, with transportation cost $t = 1$. In this case, the patterns are similar to the Bertrand cases of profit sharing equilibrium.

Figure 8

Figure 9

As the most interesting our findings, we observe that the main features that characterize the situation in which an equilibrium exists in which firm share profits are (i) a rapid increase in profits as we move from $\alpha_1 = \alpha_2 = \alpha = 0$ to $\alpha_1 = \alpha_2 = \alpha = 0.5$, and (ii) a big difference between the oligopoly and monopoly profit. *This is more likely to occur when decision variables are strategic complements, and when competition is more aggressive.*

These conditions explains also why, in the case when only one firm is allowed to share profits, the only oligopoly in which profit sharing is still an equilibrium is the one in which conditions (i) and (ii) are satisfied at a higher degree, as we will see in Section 9 of Chapter 2.

5 The need of reciprocity

We consider here a strategic framework similar to the one used in Section 3 except that we allow only one firm to share its profit whereas the other firm keeps its entire profit and still receives a fraction of its rival's profit. We show that for the models of Hotelling and Bertrand with heterogeneous goods, there is no profit-sharing equilibrium in this situation. For the asymmetric Bertrand with homogenous goods (examined in Chapter 2) there is, however, the possibility of profit sharing.

Let α_1 denote the part of the profit that Firm 1 (the loyal firm, whose original profit is Π_1) wants to share with Firm 2 (the deviating firm, whose original profit is Π_2) We suppose that $\alpha_1 \in [0, 1]$. Consequently, we can write

the *new profit functions* $P_1(p_1(\alpha_1), p_2(\alpha_1))$ and $P_2(p_1(\alpha_1), p_2(\alpha_1))$ (hereafter P_1 and P_2) of each firm as:

$$P_1 = (1 - \alpha_1)\Pi_1(p_1(\alpha_1), p_2(\alpha_1))$$

$$P_2 = \Pi_2(p_1(\alpha_1), p_2(\alpha_1)) + \alpha_1\Pi_1(p_1(\alpha_1), p_2(\alpha_1))$$

We consider a two-stage game whose sequences are thus defined. In the first stage of the game, firm 1 chooses α_1 . In the second stage of the game, firms select p_i .

In the first stage of the game Firm 1 solves $\text{Max}_{\alpha_1} P_1 = (1 - \alpha_1)\Pi_1$, and in the second stage, given α_1 , firms simultaneously choose p_1 and p_2 to solve:

$$\text{Max}_{p_1} P_1 = (1 - \alpha_1)\Pi_1$$

$$\text{Max}_{p_2} P_2 = \Pi_2 + \alpha_1\Pi_1.$$

5.1 The Bertrand model with heterogenous goods

We consider here the Bertrand model (with heterogeneous goods) similar to the one presented in Section 3, where $p_i = 1 - q_i - \gamma q_j$ is the inverse demand function of Firm i . Then, profits are $\Pi_i(q_1, q_2) = p_i q_i(p_1, p_2)$, with $q_i = \frac{1}{1-\gamma^2}(1 - \gamma - p_i + \gamma p_j)$. We assume that $\gamma^2 < 1$, that is, the own-price effect dominates the cross-price effect. The only difference is that *only* Firm 1 shares its profit whereas Firm 2 keeps its entire profit and still receives a fraction of Firm 2's profit.

In the second stage the problem of maximizing $P_1 = (1 - \alpha_1)p_1 q_1$ and $P_2 = p_2 q_2 + \alpha_1 p_1 q_1$ with respect to p_1 and p_2 respectively give, after some algebra:

$$p_1^*(\alpha_1) = \frac{(1-\gamma)(2+\gamma)}{4-\gamma^2(1+\alpha_1)}$$

$$p_2^*(\alpha_1) = \frac{(1-\gamma)(2+\gamma(1+\alpha_1))}{4-\gamma^2(1+\alpha_1)}.$$

Substituting, in q_i , we have:

$$q_1^*(\alpha_1) = \frac{2+\gamma}{(1-\gamma)(4-\gamma^2(1+\alpha_1))}$$

$$q_i^*(\alpha_1) = \frac{(2+\gamma)-\gamma(1+\gamma)\alpha_1}{(1-\gamma)(4-\gamma^2(1+\alpha_1))}.$$

And now, substituting, in P_1 and P_2 :

$$P_1(\alpha_1) = \frac{(1-\gamma)(2+\gamma)^2}{(1+\gamma)(4-\gamma^2(1+\alpha_1))} (1 - \alpha_1)$$

$$P_2(\alpha_1) = \frac{(\gamma-1)(\gamma+2)(-\gamma-2+\alpha_1(\gamma^2-2))}{(1+\gamma)(4-\gamma^2(1+\alpha_1))}$$

In the first stage, the equilibrium choice of α_1^* (assuming interior solutions) is given by maximizing $P_1(\alpha_1)$ with respect to α_1 , which gives $\alpha_1^* = \frac{1}{\gamma^2} (3\gamma^2 - 4)$, for $\gamma \neq 0$. We can check that there exist no values of γ such $\gamma^2 < 1$ for which $\alpha_1^* > 0$. This means that there is no profit-sharing equilibrium when only one firm shares profits in the Bertrand model with heterogeneous goods.

5.2 The Hotelling model

We consider here a different model of oligopolistic competition where goods are differentiated by location and competition is via prices, a model similar to the one presented in Section 4 of Chapter 1. Consider a linear city that lies on the interval $[0, 1]$, and where consumers are uniformly distributed with density 1 along this interval. There are two firms which sell the same physical good, and that are located at the extremes of the linear city lying on in the interval $[0, 1]$. Firm 1 is at $x = 0$ and Firm 2 at $x = 1$. Uniformly distributed consumers incur a transportation cost $t > 0$ per unit of length. Thus, a consumer living at x incurs a cost of tx to go to Firm 1 and a cost of $t(1-x)$ to go to Firm 2. A consumer who is indifferent between the two firms is located at $x = D_1(p_1, p_2)$, where x is given by equating costs; i.e., $p_1 + tx = p_2 + t(1-x)$. The firms' respective demands are $D_1(p_1, p_2) = \frac{1}{2} + \frac{p_2 - p_1}{2t}$, and $D_2(p_1, p_2) = 1 - D_1$.

To find the equilibrium in the second stage of the game, after a first moment of profit-sharing decisions, Firms 1 and 2 maximizes respectively $P_1 = (1 - \alpha_1)p_1q_1$ and $P_2 = p_2q_2 + \alpha_1p_1q_1$ with respect to p_1 and p_2 respectively. The first-order conditions with respect to p_1 and p_2 give, after some calculus: $p_1^*(\alpha_1) = t\frac{3}{3-\alpha_1}$ and $p_2^*(\alpha_1) = t\frac{3}{3-\alpha_1}$.

Now, profits in the first stage are given by: $P_1 = \frac{9}{2}t\frac{(1-\alpha_1)}{(3-\alpha_1)^2}$ and $P_2 = \frac{9}{2}t\frac{(1-\alpha_1)}{(3-\alpha_1)^2}$.

Maximization of P_1 with respect to α_1 gives: $\frac{\partial P_1}{\partial \alpha_1} = \frac{9}{2}t\frac{-1-\alpha_1}{(3-\alpha_1)^3} = 0$, which leads to $\alpha_1^* = -1$. This means that there is no profit-sharing equilibrium when *only* one firm shares profits in the Hotelling model.

This suggests that profit sharing between firms is a winning strategy only if both firms are sharing when competing in prices.

6 A joint venture may serve as a profit sharing strategy

So far, the profit-giving strategy has been performed directly. However, in real life, this strategy may be hidden behind a more complicated relation. To see this, consider the following simple case of a joint venture.

Let $\beta_i \in [0, 1]$ denote the part of its own profits that Firm i is willing to invest in a joint venture along with Firm j . The total investment in the joint venture is, then, given by $\beta_1\Pi_1 + \beta_2\Pi_2$. We will assume a simple joint venture activity with net profits given by $F = k(\beta_1\Pi_1 + \beta_2\Pi_2)$ where $k > 0$. Finally, we assume that each firm receives $s_i F$, where $s_i = \frac{\beta_i}{\beta_i + \beta_j}$. Consequently, we can write the new profit function of each firm as $P_i = (1 - \beta_i)\Pi_i + s_i k(\beta_i\Pi_i + \beta_j\Pi_j)$ or $P_i = [1 - (1 - s_i k)\beta_i]\Pi_i + s_i k\beta_j\Pi_j$. It is now straightforward to see that $(1 - s_i k)\beta_i$ plays the role of α_i , and $s_i k\beta_j$ the role of α_j in the previous model, and that conditions on α_i and α_j can be translated as conditions on β_i , β_j and on k for the profit-giving strategy to be profitable in equilibrium². Or, in Sections 3.1 and 3.3 of Chapter 1 and in Chapter 2, we have clearly shown such conditions in the Bertrand models (with heterogenous and homogenous goods) and in the Hotelling models leading to a profit-sharing equilibrium. As far as we know, only Reynolds and Snapp (1986) make a connection between cross ownership and joint ventures, although, in their case, the connection is made to show an incentive to investing in entry deterrence strategies.

7 Conclusion

This Chapter has shown how two firms in a duopolistic market may be able to limit the competition through the profit-sharing strategy, thus increasing their profits. Using different oligopolistic models, we have brought to light some situations where giving away profits could be a rewarding strategy for firms. For the linear demand cases of study, and for the most standard assumptions on both Cournot and Bertrand models, a necessary (but not sufficient) condition to find a profit-sharing equilibrium is that strategies are complementary for substitutes goods. This, in turn, implies that profit sharing is never an equilibrium in the standard Cournot models. The complementarity of the strategies is necessary for the equilibrium to internalize the effect on the other firm's profits. For the sufficient condition to hold, competition has to be aggressive enough.

²For instance, condition $\alpha_1 + \alpha_2$ above is satisfied for $\beta_1 = \beta_2 = \frac{1}{4}$ and any k .

When the profit-sharing equilibrium exists, profits are higher and, thus, a tacit partial collusion occurs in a one shot oligopolistic interaction.

In our model firms give profits directly to the other firm. In more realistic situations, this strategy may be performed indirectly, perhaps disguised behind some other strategy like a joint venture. If this is the case, our model still serves as a departing point to study these other cases, as one would expect the same qualitative results.

Further, the fact that our findings show that the strategy of profit sharing is more likely to occur in models of price competition opens the door to investigate its influence in other scenarios of price competition, like in the literature of price leadership (cf. Dastidar and Furth, 2005, and Yano and Komatsubara, 2006.)

References

- Amir, R., and J. Jin (2001), “Cournot and Bertrand equilibria compared: substitutability, complementarity and concavity,” *International Journal of Industrial Organization* **19**, 303-317.
- Caminal, P., and X. Vives (1996), “Why market shares matter: Information-based theory,” *RAND Journal of Economics* **27**(2), 221-239.
- Dastidar, K. G., and D. Furth (2005), “Endogenous Price Leadership in a Duopoly: Equal Products, Unequal Technology,” *International Journal of Economic Theory* **1**, 189-210.
- Farrell, J., and C. Shapiro (1990), “Asset ownership and market structure in oligopoly,” *RAND Journal of Economics* **21**, 275-292.
- FitzRoy, F. R., and F. Kraft (1986), “Profitability and profit-sharing,” *Journal of Industrial Economics* **35**, 113-130.
- Gilo, D., Y. Moshe, and Y. Spiegel (2006), “Partial Ownership and Tacit Collusion,” *RAND Journal of Economics* **37**(1), 81-99.
- Jackson, M. O., and S. Wilkie (2005), “Endogenous games and mechanisms: Side payments among players,” *Review of Economics Studies* **72**, 543-566.
- Malueg, D. (1992), “Collusive behavior and partial ownership of rivals,” *International Journal of Industrial Organization* **10**, 27-34.
- Reitman, D. (1994), “Partial Ownership Arrangements and Potential for Collusion,” *Journal of Industrial Economics* **42**, 313-322.
- Reynolds, R.J., and Snapp, B.R. (1986), “The competitive effects of partial equity interests and joint ventures,” *International Journal of Industrial Organization* **4**, 141-153.
- Singh, N., and X. Vives (1984), “Price and quantity competition in a differentiated duopoly,” *RAND Journal of Economics* **15**, 546-554.
- Waddle, R. (2005), “Strategic profit sharing between firms: the Bertrand model,” *Working paper 05-09*, Economic Series: Universidad Carlos III de Madrid.
- Yano, M., and T. Komatsubara (2006), “Endogenous Price Leadership and Technological Differences,” *International Journal of Economic Theory* **2**, 365-383.

Appendix

Computations for Strategic Profit Sharing Between Firms

Section 3.1 The Bertrand model

Here, we explain and show the simulation method used to get α^* in the first stage of the game before computing and comparing $P_i(\alpha_i = \alpha_j = \alpha^*, \gamma)$, $P_i(\alpha_i = \alpha_j = 0, \gamma)$, $P_i(\alpha_i = \alpha_j = 0.5, \gamma)$. We first derive the first order conditions for symmetrical solutions (interior) and then give different values to α ($\alpha \in [0, 1]$) to get equations whose solutions are γ . We keep only γ such $\gamma^2 < 1$. That tells us how α^* varies with γ . Then, replacing γ and α^* in (9) allows us to find P_i^* . Note that if for an α , we have different γ , we keep γ that gives the greatest P_i^* . We also compare P_i^* in interior solutions with P_i^* in corner solutions ($\alpha = 0$) which tells us when giving away profits to rival is the best reply. In Section 3, we have solved the second stage of the game that gives p_i^* , p_j^* , q_i^* , q_j^* and we can now solve the first stage.

First stage of the game: After substitution of p_i^* , p_j^* , q_i^* , q_j^* in the expression (9) the profit function for Firm i is given by

$$P_i = (1 - \alpha_i)p_i^*q_i^* + \alpha_j p_j^*q_j^* \tag{9}$$

$$P_i = (1 - \alpha_i) \frac{(1 - \alpha_j)(1 - \gamma)[2(1 - \alpha_i) + \gamma(1 - \alpha_i + \alpha_j)]}{[4(1 - \alpha_i)(1 - \alpha_j) - \gamma^2(1 - \alpha_i + \alpha_j)(1 - \alpha_j + \alpha_i)]} \\ + \frac{1}{1 + \gamma} \frac{(2 + \gamma)(1 - \alpha_j)(1 - \alpha_i) - \gamma(1 + \gamma)\alpha_j(1 - \alpha_j) - \gamma^2\alpha_i\alpha_j}{[4(1 - \alpha_i)(1 - \alpha_j) - \gamma^2(1 - \alpha_i + \alpha_j)(1 - \alpha_j + \alpha_i)]} \\ + \alpha_j \frac{(1 - \alpha_i)(1 - \gamma)[2(1 - \alpha_j) + \gamma(1 - \alpha_j + \alpha_i)]}{[4(1 - \alpha_i)(1 - \alpha_j) - \gamma^2(1 - \alpha_i + \alpha_j)(1 - \alpha_j + \alpha_i)]} \frac{1}{1 + \gamma} \\ + \frac{(2 + \gamma)(1 - \alpha_j)(1 - \alpha_i) - \gamma(1 + \gamma)\alpha_i(1 - \alpha_i) - \gamma^2\alpha_i\alpha_j}{[4(1 - \alpha_i)(1 - \alpha_j) - \gamma^2(1 - \alpha_i + \alpha_j)(1 - \alpha_j + \alpha_i)]}$$

The first-order conditions with respect to α_i give:

$$\begin{aligned} & [(1 - \alpha_j)(1 - \gamma) \{ [((1 - \alpha_i)(-4 - \gamma) - \gamma(1 - \alpha_i + \alpha_j))] \\ & [((2 + \gamma)(1 - \alpha_j)(1 - \alpha_i) - \gamma(1 + \gamma)\alpha_j(1 - \alpha_j) - \gamma^2\alpha_i\alpha_j)] \\ & - [((2 + \gamma)(1 - \alpha_j) + \gamma^2\alpha_j)](1 - \alpha_i)(2(1 - \alpha_i) + \gamma(1 - \alpha_i + \alpha_j)) \} \\ & + \alpha_j(1 - \gamma) \{ [(-2(1 - \alpha_j) - \gamma(1 - \alpha_j + \alpha_i) + \gamma(1 - \alpha_i))] \\ & [((2 + \gamma)(1 - \alpha_j)(1 - \alpha_i) - \gamma(1 + \gamma)\alpha_i(1 - \alpha_i) - \gamma^2\alpha_i\alpha_j)] \\ & - 2[((2 + \gamma)(1 - \alpha_j) - 2\gamma(1 + \gamma)\alpha_i + \gamma(1 + \gamma) + \gamma^2\alpha_j)] \\ & (1 - \alpha_i)[(2(1 - \alpha_j) + \gamma(1 - \alpha_j + \alpha_i))] \} \\ & [4(1 - \alpha_i)(1 - \alpha_j) - \gamma^2(1 - \alpha_i + \alpha_j)(1 - \alpha_j + \alpha_i)]^2(1 + \gamma) \\ & - 4[4(1 - \alpha_i)(1 - \alpha_j) - \gamma^2(1 - \alpha_i + \alpha_j)(1 - \alpha_j + \alpha_i)] \end{aligned}$$

$$\begin{aligned}
& [(-2(1-\alpha_j) + \gamma^2(a_i - \alpha_j))](1+\gamma)\{(1-\gamma)(1-\alpha_i)(1-\alpha_j) \\
& [(2(1-\alpha_i) + \gamma(1-\alpha_i + \alpha_j))]\} \\
& [((2+\gamma)(1-\alpha_j)(1-\alpha_i) - \gamma(1-\gamma)\alpha_j(1-\alpha_j) - \gamma^2\alpha_i\alpha_j)] \\
& + \alpha_j(1-\gamma)(1-\alpha_i)[(2(1-\alpha_j) + \gamma(1-\alpha_j + \alpha_i))] \\
& [((2+\gamma)(1-\alpha_j)(1-\alpha_i) - \gamma(1-\gamma)\alpha_j(1-\alpha_j) - \gamma^2\alpha_i\alpha_j)]\} = 0
\end{aligned}$$

As explained in Section 3.1 with the model tractable only numerically and for simplicity reasons, whenever we have interior solutions, we concentrate on symmetric equilibria and consider only $\alpha_1 = \alpha_2 = \alpha$

$$\begin{aligned}
& [(1-\alpha)(1-\gamma)\{(1-\alpha)(-4-\gamma) - \gamma] \\
& [(2+\gamma)(1-\alpha)^2 - \gamma(1+\gamma)\alpha(1-\alpha) - \gamma^2\alpha^2] \\
& - [(2+\gamma)(1-\alpha) + \gamma^2\alpha](1-\alpha)[2(1-\alpha) + \gamma]\} \\
& + \alpha(1-\gamma)\{-2(1-\alpha) - \gamma + \gamma(1-\alpha)\} \\
& [(2+\gamma)(1-\alpha)^2 - \gamma(1+\gamma)\alpha(1-\alpha) - \gamma^2\alpha^2] \\
& - 2[(2+\gamma)(1-\alpha) - 2\gamma(1+\gamma)\alpha + \gamma(1+\gamma) + \gamma^2\alpha] \\
& (1-\alpha)[2(1-\alpha) + \gamma]\}[4(1-\alpha)^2 - \gamma^2]^2(1+\gamma) - \\
& 4[4(1-\alpha)^2 - \gamma^2][-2(1-\alpha)](1+\gamma)\{(1-\gamma)(1-\alpha)^2 \\
& [2(1-\alpha) + \gamma][(2+\gamma)(1-\alpha)^2 - \gamma(1-\gamma) \\
& \alpha(1-\alpha) - \gamma^2\alpha^2] + \alpha(1-\alpha)(1-\gamma)[2(1-\alpha) + \gamma] \\
& [(2+\gamma)(1-\alpha)^2 - \gamma(1-\gamma)\alpha(1-\alpha) - \gamma^2\alpha^2]\} = 0
\end{aligned}$$

By solving the above equation, we find $\alpha = -\frac{1}{2}\gamma + 1$, $\alpha = \rho$, $\alpha = 1 + \frac{1}{2}\gamma$, and $\alpha = 1 + \frac{1}{2}\gamma$ where ρ solves

$$\begin{aligned}
& (8 + 8\gamma)\alpha^5 + (-52 - 48\gamma)\alpha^4 + (-2\gamma^3 + 106\gamma + 128)\alpha^3 \\
& + (6\gamma^2 - \gamma^4 + 9\gamma^3 - 110\gamma - 152)\alpha^2 \\
& + (-12\gamma^2 + 88 + 54\gamma - 10\gamma^3)\alpha + 3\gamma^3 - 20 + 6\gamma^2 - 10\gamma = 0
\end{aligned}$$

The corner solutions $\alpha = 1$, $\alpha = 1 + \frac{1}{2}\gamma$, and $\alpha = 1 - \frac{1}{2}\gamma$ give negative values for the variables or a minimum for the objective function, which can be easily checked by replacing such solutions in the expression (3) and the next one after (3). Therefore, we only take into account the solution $\alpha = \rho$ and for different values of $\gamma \in]-1, 1[$ we solve the above expression for ρ numerically and we get the values in the first two columns of Table 1. In this way we get the values for α in Figure 1.

γ	α	$P_i(\alpha_i = \alpha_j = \alpha^*, \gamma)$	$P_i(\alpha_i = \alpha_j = 0, \gamma)$	$P_i(\alpha_i = \alpha_j = 0.5, \gamma)$
0.99	0.500	0.1256	0.0049	0.1256
0.95	0.499	0.1280	0.0232	0.1282
0.90	0.495	0.1307	0.0434	0.1315
0.83	0.480	0.1332	0.0678	0.1366
0.82	0.475	0.1330	0.0710	0.1373
0.80	0.462	0.1321	0.0771	0.1388
0.77	0	0.0858	0.0858	0.1412
0.75	0	0.0914	0.0914	0.1428
0.70	0	0.1044	0.1044	0.1470
0.60	0	0.1275	0.1275	0.1562
0.50	0	0.1481	0.1481	0.1666
0.40	0	0.1674	0.1674	0.1785
0.30	0	0.1863	0.1863	0.1863
0.20	0	0.2057	0.2057	0.2057
0.10	0	0.2266	0.2266	0.2266
0	0	0.2500	0.2500	0.2500

Table 1

Finally, by replacing $\alpha_1 = \alpha_2 = \alpha$ in the expression for $P_i(\alpha, \alpha)$, we find

$$P_i(\alpha, \alpha) = \frac{1-\gamma}{1+\gamma} \frac{1-\alpha}{(2(1-\alpha)-\gamma)^2} \left((2+\gamma)(1-\alpha)^2 - \gamma(1+\gamma)\alpha(1-\alpha) - \gamma^2\alpha^2 \right) + \frac{1}{2(1-\alpha)+\gamma}$$

This is the expression used to compute the values in the last three columns of Table 1 and to draw Figure 2.

To summarize, we use a simple method for the simulation. We first solve the model by finding q_1^* and q_2^* as functions of (α_1, α_2) , then α_1^* , α_2^* , and by finally considering only symmetric equilibria. We then compute profits and compare them to the case $\alpha = 0$ to make sure it is indeed an equilibrium. We present the results of the simulation in Table 1.

Section 3.2 The Cournot model

As for the Bertrand model, we use the same simulation method to get α^* in the first stage of the game before computing and comparing $P_i(\alpha_i = \alpha_j = \alpha^*, \gamma)$, $P_i(\alpha_i = \alpha_j = 0, \gamma)$, $P_i(\alpha_i = \alpha_j = 0.5, \gamma)$

In the first stage of the game Firm i maximizes $P_1(\alpha_1, \alpha_2) = (1-\alpha_1)p^*q_1^* + \alpha_2p^*q_2^*$, which, after substituting p^* , q_1^* , and q_2^* can be written as

$$\begin{aligned}
P_1 = & (1 - \alpha_1) \frac{2(1-\gamma)(1-\alpha_1)(1-\alpha_2) - \gamma^2 \alpha_2 (1-\alpha_2 + \alpha_1) + \gamma(1-\alpha_2)(1-\alpha_1 + \alpha_2)}{4(1-\alpha_1)(1-\alpha_2) - \gamma^2(1-\alpha_1 + \alpha_2)(1-\alpha_2 + \alpha_1)} \\
& \cdot \frac{(1-\alpha_2)(2(1-\alpha_1) - \gamma(1-\alpha_1 + \alpha_2))}{4(1-\alpha_1)(1-\alpha_2) - \gamma^2(1-\alpha_1 + \alpha_2)(1-\alpha_2 + \alpha_1)} \\
& + \alpha_2 \frac{2(1-\gamma)(1-\alpha_1)(1-\alpha_2) - \gamma^2 \alpha_1 (1-\alpha_1 + \alpha_2) + \gamma(1-\alpha_1)(1-\alpha_2 + \alpha_1)}{4(1-\alpha_1)(1-\alpha_2) - \gamma^2(1-\alpha_1 + \alpha_2)(1-\alpha_2 + \alpha_1)} \\
& \cdot \frac{(1-\alpha_1)(2(1-\alpha_2) - \gamma(1-\alpha_2 + \alpha_1))}{4(1-\alpha_1)(1-\alpha_2) - \gamma^2(1-\alpha_1 + \alpha_2)(1-\alpha_2 + \alpha_1)}
\end{aligned}$$

The first-order conditions with respect to α_1 give:

$$\begin{aligned}
\frac{dP_1}{d\alpha_1} = & 0: \\
& ((\gamma^2 \alpha_2 (2\alpha_1 - \alpha_2) + (1 - \alpha_2) (-\gamma \alpha_2 - 2\alpha_1 - 2\alpha_1 \gamma)) \\
& (1 - \alpha_2) (2(1 - \alpha_1) - \gamma(1 - \alpha_1 + \alpha_2)) + (1 - \alpha_2) (-2 + \gamma) \\
& (1 - \alpha_1) (2(1 - \gamma) (1 - \alpha_1) (1 - \alpha_2) - \gamma^2 \alpha_2 (1 - \alpha_2 + \alpha_1) + \gamma(1 - \alpha_2) (1 - \alpha_1 + \alpha_2)) \\
& + (\alpha_2 (-2(1 - \gamma) (1 - \alpha_2) - (\gamma^2 + \gamma) (1 - \alpha_1 + \alpha_2) - \gamma^2 \alpha_1 - \gamma(1 - \alpha_1))) \\
& (1 - \alpha_1) (2(1 - \alpha_2) - \gamma(1 - \alpha_2 + \alpha_1)) + (-2(1 - \alpha_2) + \gamma(1 - \alpha_1 + \alpha_2) - \alpha_1 \gamma (1 - \alpha_1)) \\
& (\alpha_2 (2(1 - \gamma) (1 - \alpha_1) (1 - \alpha_2) - \gamma^2 \alpha_1 (1 - \alpha_1 + \alpha_2) + \gamma(1 - \alpha_1) (1 - \alpha_2 + \alpha_1))) \\
& (4(1 - \alpha_1) (1 - \alpha_2) - \gamma^2 (1 - \alpha_1 + \alpha_2) (1 - \alpha_2 + \alpha_1))^2 \\
& - 4(4(1 - \alpha_1) (1 - \alpha_2) - \gamma^2 (1 - \alpha_1 + \alpha_2) (1 - \alpha_2 + \alpha_1)) (-2(1 - \alpha_2) + \gamma^2 (\alpha_1 - \alpha_2)) \\
& ((1 - \alpha_1) (2(1 - \gamma) (1 - \alpha_1) (1 - \alpha_2) - \gamma^2 \alpha_2 (1 - \alpha_2 + \alpha_1) + \gamma(1 - \alpha_2) (1 - \alpha_1 + \alpha_2)) \\
& + (1 - \alpha_2) (2(1 - \alpha_1) - \gamma(1 - \alpha_1 + \alpha_2)) \\
& + \alpha_2 (-2(1 - \gamma) (1 - \alpha_2) - (\gamma^2 + \gamma) (1 - \alpha_1 + \alpha_2) - \gamma^2 \alpha_1 - \gamma(1 - \alpha_1)) \\
& (1 - \alpha_1) (2(1 - \alpha_2) - \gamma(1 - \alpha_2 + \alpha_1))) = 0
\end{aligned}$$

As explained in Section 3.2 with the model tractable only numerically and for simplicity reasons, whenever we have interior solutions, we concentrate on symmetric equilibria and consider only $\alpha_1 = \alpha_2 = \alpha$

$$\begin{aligned}
\frac{\partial P_1}{\partial \alpha_1} \Big|_{\alpha_1 = \alpha_2 = \alpha} = & 0 : \\
& (\gamma^2 \alpha^2 + (1 - \alpha) (-3\alpha \gamma - 2\alpha)) (1 - \alpha) (2 - 2\alpha - \gamma) + (1 - \alpha)^2 (-2 + \gamma) \\
& (2(1 - \gamma) (1 - \alpha)^2 - \gamma^2 \alpha + \gamma(1 - \alpha)) + \\
& \alpha^2 (-2(1 - \gamma) (1 - \alpha) - \gamma^2 - \gamma - \gamma^2 \alpha - \gamma(1 - \alpha)) \\
& (1 - \alpha) (2 - 2\alpha - \gamma) (-2 + 2\alpha + \gamma - \alpha \gamma (1 - \alpha)) \\
& (2(1 - \gamma) (1 - \alpha)^2 - \gamma^2 \alpha + \gamma(1 - \alpha)) (4(1 - \alpha)^2 - \gamma^2)^2 \\
& - 4(4(1 - \alpha)^2 - \gamma^2) (-2 + 2\alpha) \\
& ((1 - \alpha) (2(1 - \gamma) (1 - \alpha)^2 - \gamma^2 \alpha + \gamma(1 - \alpha)) + (1 - \alpha) (2 - 2\alpha - \gamma) \\
& + \alpha (-2(1 - \gamma) (1 - \alpha) - \gamma^2 - \gamma - \gamma^2 \alpha - \gamma(1 - \alpha)) \\
& (1 - \alpha) (2 - 2\alpha - \gamma)) = 0
\end{aligned}$$

By solving the above equation, we find as solutions $\alpha = 1$, $\alpha = \rho$, $\alpha = 1 + \frac{1}{2}\gamma$, and $\alpha = 1 - \frac{1}{2}\gamma$ where ρ solves

$$(4 - 4\gamma) \alpha^4 + (-32 + 28\gamma + 4\gamma^2) \alpha^3$$

$$\begin{aligned}
& + (72 - 44\gamma - 11\gamma^2 - \gamma^3) \alpha^2 \\
& + (-64 + 20\gamma + 10\gamma^2) \alpha - 3\gamma^2 + 20 = 0
\end{aligned}$$

The corner solutions $\alpha = 1$, $\alpha = 1 + \frac{1}{2}\gamma$, and $\alpha = 1 - \frac{1}{2}\gamma$ yield negative values for the variables or a minimum for the objective function, which can be easily checked by replacing such solutions in the expression (4) and the next one after (4). Therefore, we only take into account the solution $\alpha = \rho$ and for different values of $\gamma \in]-1, 1[$ we solve the above expression for ρ numerically and we thus get the values in tables 2 and 3. The tables show that the strategy of profit-giving is not worth.

$\gamma < 0$	α	$\mathbf{P}_i(\alpha, \gamma)$	$\mathbf{P}_i(0, \gamma)$
-0.97	0.51	-24.12	0.94
-0.71	0.6	-1.63	0.6
-0.61	0.64	-1.01	0.51
-0.27	0.8	-0.27	0.33
0	1	0	0.25

Table 2

$\gamma > 0$	α	$\mathbf{P}_i(\alpha, \gamma)$	$\mathbf{P}_i(0, \gamma)$
0	1	0	0.25
0.27	0.94	0.121	0.19
0.47	0.9	0.115	0.16
0.79	0.84	0.106	0.12
1	0.80	0.101	0.111

Table 3

By replacing $\alpha_1 = \alpha_2 = \alpha$ in the expression (9) for profits we find

$$\begin{aligned}
P_i^*(\alpha, \alpha) &= (1 - \alpha)^2 \frac{2(1-\gamma)(1-\alpha)^2 - \gamma^2\alpha + \gamma(1-\alpha)}{4(1-\alpha)^2 - \gamma^2} \frac{2-2\alpha-\gamma}{4(1-\alpha)^2 - \gamma^2} \\
&+ \alpha \frac{2(1-\gamma)(1-\alpha)^2 - \gamma^2\alpha + \gamma(1-\alpha)}{4(1-\alpha)^2 - \gamma^2} (1 - \alpha) \frac{2-2\alpha-\gamma}{4(1-\alpha)^2 - \gamma^2} \\
&= (1 - \alpha) \frac{2(1-\gamma)(1-\alpha)^2 - \alpha\gamma + \gamma(1-\alpha)}{(2(1-\alpha) + \gamma)^2(2(1-\alpha) - \gamma)}
\end{aligned}$$

This is the expression we use to compute the values in Tables 2 and 3 and in Figure 3.

Chapter 2

Strategic Profit Sharing Leads to Collusion in Bertrand Oligopolies¹

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1 Introduction

The present Chapter studies the consequences of introducing a new strategy in a Bertrand oligopoly with homogenous goods that consists on unilaterally and voluntarily giving a part of own profits to rivals. Thus, in a first stage, firms decide simultaneously what part of their profits to give away to their rival and then, in a second stage, they choose a price in a standard Bertrand competition. Notice that the action of giving away profits is decided unilaterally and unconditionally in a non-cooperative framework. After the decision to give profits is done, the giving firm is committed to it.

Chapter 1 explores this strategy in a general setting, and apply it to both Cournot and Bertrand competition with heterogenous goods. We show that the strategy is more likely to allow for tacit collusion when choice variables are strategic complements and when monopolistic profits are big compared to the non-cooperative equilibrium in the oligopoly. In fact, we find that for Bertrand competition with goods with low heterogeneity some degree of tacit collusion is possible. However, our methodology relies on the differentiability of the profits functions, and cannot be applied to Bertrand competition with homogenous goods. The special characteristics of the Bertrand model, with a discontinuous profits function, and the qualitatively different results we find, call for the separate analysis presented in this work.

By adding a stage of profit sharing to the Bertrand model, we show that firms may be able to support prices between the marginal cost and the monopoly price, thus obtaining positive profits in almost all of the equilibria. This remarkable result, that resembles a Folk theorem, is robust to the number of firms and to cost asymmetries. Furthermore, for a given equilibrium price, any share of the market can also be supported in a subgame perfect equilibrium. For the duopoly case we completely characterize the set of pure strategy equilibria. However, in the extension to more than two firms, and due to the increasing complication in the multiplicity of equilibria, we only show the existence of the equilibria in the subgames that is sufficient to support the desired equilibrium price.

One may be tempted to argue that the result is not surprising. To put it in Reitman's words (Reitman, 94):

“To take the simplest example, suppose symmetric Cournot duopolists each own 50% of the profits from its competitor's product. In choosing its own strategy, each firm's objective will be to maximize the sum of the two firms' profits, and will choose the collusive output level in equilibrium.”

However, this intuition is misleading, as one has to check that the in-

dividual incentives make this situation, the choice of sharing a 50% of the rival's profits, an equilibrium. In Chapter 1 we already showed that this is the case only in some scenarios. We refer to the Introduction of Chapter 1 for a discussion that relates our approach with the literature on cross-ownership.

The extension of our model to price competition also contributes to the literature that views the Bertrand model as paradoxical, as it predicts perfect competition when there are only two firms in the market. Some authors have strongly criticized the Bertrand model pointing out its lack of realism. For instance, they think that it could be improved by relaxing some of its crucial assumptions like the timing of the game or the perfect substitutability of products. Others have attempted to find out a solution that fits to the real world. For example, Edgeworth (1897) solved it by introducing the elegant idea of capacity constraints, by which firms cannot sell more than they are able to produce. Later, Kreps and Scheinkman (1983) treated capacities as endogenous decisions previous to price competition, and showed that the new model leads to Cournot outcomes.

One of the problems in Kreps and Scheinkman is that, for some capacity choices, the only equilibria are in mixed strategies, which are not uniformly accepted as a satisfactory explanation of pricing behavior by oligopoly firms¹. After all, in a mixed strategy equilibrium, firms can regret ex post their decisions, and, since prices can easily be changed, the stability of the equilibrium may be called into question². In our model, every price in the range is obtained in a pure strategy equilibria and, thus, is immune to this criticism.

The infinite repetition of the Bertrand competition offers another way out of the paradox, as the Folk Theorem states that, for sufficiently high discount rates, any price between the cost and the monopoly price can be attained in a subgame perfect equilibrium and also that any market share among the different firms can be obtained. We obtain the same result without the need of repetition.

We proceed as follows. In Section 2 we present the analysis for the standard Bertrand model with equal marginal costs. Section 3 modifies the model to allow firms to have different marginal costs. In Section 5, the need for a tie-breaking rule different from the equal division of the market is examined for the case of asymmetric firms in Bertrand competition with homogenous goods³. Section 5 generalizes the previous models to n firms with equal mar-

¹See Shapiro (1989) and Maggi (1996).

²In a more recent work Moreno and Ubeda (2006) are able to provide a more elegant model in which the equilibria exist in pure strategies. Their model uses the capacity and price choices of each firm to construct a supply function game, in which, again, the Cournot outcome is obtained in equilibrium.

³Some references on the role of tie-breaking rules in Bertrand models include Deneckere

ginal costs and to n firms with asymmetric costs. In Section 6 we explore the possibility that only one firm has the possibility to share profits. In other terms, we had before a two-side profit-sharing while we focus here on just one-side profit-sharing. We study this new version only for the Bertrand duopoly and find that only in the case of asymmetric Bertrand competition with homogenous goods, there is indeed the possibility of profit sharing by the firm with the lowest cost. Section 7 concludes.

2 Profit sharing in a Bertrand duopoly

Consider two firms, 1 and 2, that compete *a la* Bertrand in a homogeneous market, and that each firm incurs a cost $c \in [0, 1)$ per unit of production. Let the market demand function be $q = D(p) = 1 - p$, and assume that firms do not have capacity constraints, and always supply the demand they face. The (before sharing) profit function for Firm i is:

$$\Pi_i = \begin{cases} (p_i - c)(1 - p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c)(1 - p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases} \quad i, j = 1, 2 \ (i \neq j)$$

Consider now the following two-stage game of profit-sharing. In the first stage Firm i ($i = 1, 2$) chooses α_i , the proportion of profits Π_i that Firm i gives to Firm j ($\neq i$). In the second stage, Firm i selects price p_i after observing the choices in the previous stage. Profits after sharing are given by $P_i = (1 - \alpha_i)\Pi_i + \alpha_j\Pi_j$. We will be interested in finding the set of prices that can be supported in a subgame perfect Nash equilibrium in this market.

So far, the profit-giving strategy has been performed directly. However, in real life, this strategy may be hidden behind a more complicated relation as described in Chapter 1.

2.1 The second stage

To find a subgame perfect Nash equilibrium (SPNE), we begin by finding the set of pure strategies Nash equilibria in the second-stage. Although for the purposes of finding equilibria that supports the different prices it is not necessary to find all the equilibria in these subgames, we do it for its own interest and for the sake of completeness. In a series of lemmata, we find all equilibria in pure strategies for all subgames in the second stage. Then we summarize our findings in Proposition 5. In what follows, denote the monopoly price by p^m .

and Kovenock, 1996; Hoerning, 2007; and Simone and Zame, 1990.

Lemma 1 *The price configuration (p_1, p_2) s.t. $p_1 = p_2 = c$, is an equilibrium for any (α_1, α_2) .*

Proof. The proof is straightforward, as unilateral deviations result in zero or negative profits. ■

Lemma 2 *The price configuration (p_1, p_2) s.t. $c < p_1 = p_2 = p \leq p^m$ is never an equilibrium for any (α_1, α_2) such that $\alpha_1 + \alpha_2 \neq 1$. In the case $p > p^m$ there is no equilibrium with $p_1 = p_2$ if $\alpha_1 + \alpha_2 < 1$.*

Proof. The expressions for profits before and after sharing take the forms $\Pi_i = \frac{1}{2}(p - c)(1 - p)$, and $P_i = \frac{1}{2}(1 - \alpha_i + \alpha_j)(p - c)(1 - p)$.

First, if $\alpha_1 + \alpha_2 > 1$, Player i has an incentive to deviate from $p_i = p$ to $p'_i > p$, leaving all the market to Firm j . Profits after this deviation are, then, $P'_i = \alpha_j(p - c)(1 - p)$. Clearly, $P'_i > P_i$ as long as $\alpha_j > \frac{1}{2}(1 - \alpha_i + \alpha_j)$, or, equivalently, $\alpha_1 + \alpha_2 > 1$.

Second, if $\alpha_1 + \alpha_2 < 1$, Player i has an incentive to deviate from $p_i = p$ to $p'_i < p - \varepsilon$ if $p \leq p^m$, and to $p'_i = p^m$ if $p > p^m$. In both cases, it takes the whole market to itself.

If $p \leq p^m$, profits after the deviation are given by $P'_i = (1 - \alpha_i)(p'_i - c)(1 - p'_i)$, which can be made arbitrarily close to

$$\sup_{p'_i < p} P'_i = (1 - \alpha_i)(p - c)(1 - p) = P_i^{\sup}.$$

One can see that $P_i^{\sup} > P_i$ as long as $1 - \alpha_i > \frac{1}{2}(1 - \alpha_i + \alpha_j)$, or $\alpha_1 + \alpha_2 < 1$.

If $p > p^m$ firm i can deviate to $p_i = p^m$ and obtain $P_i^m = (1 - \alpha_i)(p^m - c)(1 - p^m)$. Clearly, $P_i^m > P_i^{\sup} > P_i$ if $\alpha_1 + \alpha_2 < 1$. ■

The next lemma is the key to our results. It shows that, for some configuration of profit shares, any price between the cost and the monopoly price may be sustained in an equilibrium where both firms set the same price, and thus share the market equally. Since they also share profits, the incentives are conflicting. The perspectives after deviating to a lower price must balance the increase in $(1 - \alpha_i)\Pi_i$ with the decrease in $\alpha_j\Pi_j$. Similarly, a deviation to a higher price must balance the decrease in $(1 - \alpha_i)\Pi_i$ with the increase in $\alpha_j\Pi_j$. These balances make the existence of an equilibrium with $p > c$ possible if $\alpha_1 + \alpha_2 = 1$.

Lemma 3 *Prices (p_1, p_2) s.t. $p_1 = p_2 = p \in [c, p^m]$ are equilibria for (α_1, α_2) such that $\alpha_1 + \alpha_2 = 1$.*

Proof. Lemma 1 proves the case for $p_1 = p_2 = c$. Suppose, then, that $p \in (c, p^m]$, and that Firm i sets a price p'_i below p and above c , then profits after the deviation are $\Pi'_i = (1 - p'_i)(p'_i - c) \geq 0$, $\Pi'_j = 0$, and

$$P'_i = (1 - \alpha_i) \Pi'_i = (1 - \alpha_i)(p'_i - c).$$

The $\sup_{p'_i} P'_i = P_i^{\sup} = (1 - \alpha_i)(p'_i - c)$, the best deviation, is achieved at $p'_i = p$ as long as $p \leq p^m$. To avoid a profitable deviation, we need $P_i^{\sup} \leq P_i$, or $(1 - \alpha_i) \leq \frac{1}{2}(1 - \alpha_i + \alpha_j)$, which gives

$$\alpha_1 + \alpha_2 \geq 1. \quad (1)$$

Similarly, for any price $p''_i > p$, we have $\Pi''_i = 0$ and $\Pi''_j = (1 - p_j)(p_j - c) > 0$

$$P''_i = \alpha_j \Pi''_j = \alpha_j(1 - p_j)(p_j - c) = \alpha_j(1 - p)(p - c).$$

To avoid a profitable deviation, we need $P''_i \leq P_i$, or $\alpha_j \leq \frac{1}{2}(1 - \alpha_i + \alpha_j)$, which implies

$$\alpha_1 + \alpha_2 \leq 1. \quad (2)$$

Inequalities (1) and (2) represent the non-deviation conditions, and both are satisfied when $\alpha_1 + \alpha_2 = 1$. ■

The next lemma completes the set of equilibria that one can find using pure strategies. It shows the possibility of equilibria with different prices. The following notation will be useful for the next statements: $\Pi(p) = (1 - p)(p - c)$. I.e., $\Pi(p)$ denotes the total profits in the market if firms set price p .

Lemma 4 *The only cases in which (p_1, p_2) s.t. $p_i \neq p_j$ constitute an equilibrium are given by the conditions $\alpha_1 + \alpha_2 \geq 1$ and $p_i = p^m < \bar{p} \leq p_j$ for $\bar{p}_j$ satisfying $\frac{\Pi(p^m)}{\Pi(\bar{p})} > \frac{\alpha_j}{1 - \alpha_i}$.*

The strategy of the proof is similar to that in Lemma 3. However the conditions are more complicated. It should be clear that, if $p_i < p_j \leq p^m$, Firm i could set a price closer to p_j and increase $(1 - \alpha_i)\Pi_i$ without changing $\alpha_j\Pi_j = 0$, thus increasing P_i . If $p_i = p^m < p_j$, things are more complicated. Certainly, the deviation to set p_i closer to p_j , but still under it, will not work. However, it could be the case that Firm i wants Firm j to have all the market (or part of it) to take advantage of the increase in $\alpha_j\Pi_j$. We need to find the conditions to ensure that this deviation does not work. The details of the proof are left to the Appendix.

The next proposition summarizes the Nash equilibria in pure strategies that can be found in the subgames after the choice of (α_1, α_2) .

Proposition 5 *The Bertrand game with a profit-sharing previous stage has the following pure strategy Nash equilibria in the subgames:*

(a) *If $\alpha_1 + \alpha_2 = 1$, then the equilibria are (p_1, p_2) s.t. $p_1 = p_2 = p \in [c, p^m]$ or $p_i = p^m < \bar{p} \leq p_j$ for $\bar{p}$ satisfying $\frac{\Pi(p^m)}{\Pi(\bar{p})} = \frac{\alpha_j}{1-\alpha_i}$.*

(b) *If $\alpha_1 + \alpha_2 > 1$, then the equilibria are (p_1, p_2) s.t. $p_1 = p_2 = c$ or $p_i = p^m < \bar{p}_j \leq p_j$ for $\bar{p}_j$ satisfying $\frac{\Pi(p^m)}{\Pi(\bar{p})} > \frac{\alpha_j}{1-\alpha_i}$.*

(c) *If $\alpha_1 + \alpha_2 < 1$, then the equilibrium is (p_1, p_2) s.t. $p_1 = p_2 = c$.*

2.2 The whole game

The next proposition shows that any price between perfect competition and monopoly can be achieved, yielding positive profits to the industry.

Proposition 6 *Any price pair (p_1^*, p_2^*) such that $c \leq p_1^* = p_2^* = p^* \leq p_m$ can be sustained in a SPNE. Further, for the cases $c < p_1^* = p_2^* = p^* \leq p_m$ the SPNE implies that, in the first stage the shares (α_1^*, α_2^*) satisfy $\alpha_1^* + \alpha_2^* = 1$.*

Proof. Consider the following strategy. In the first step players play (α_1^*, α_2^*) such that $\alpha_1^* + \alpha_2^* = 1$. In the second stage they play (p_1^*, p_2^*) such that $c \leq p_1^* = p_2^* = p^* \leq p_m$ if $\alpha_1 + \alpha_2 = 1$, and $p_1 = p_2 = c$ otherwise.

Since the prices constitute Nash equilibria in the respective subgames, it remains to check that there are no profitable deviations for the firms in the first stage. This is straightforward, as any deviation in the first period has the consequence that $\alpha_1 + \alpha_2 \neq 1$ and, therefore, in the second period, the equilibrium implies $p_1 = p_2 = c$ and zero profits. Thus, no deviation is profitable. ■

Proposition 6 has the flavor of a Folk Theorem, even if it deals with a one shot game. If firms follow the tacit agreement to share profits in a certain way in the first stage, they get a high price in the second. If they do not, they get zero.

Intuitively, it does not seem surprising that $\alpha_1 = \alpha_2 = \frac{1}{2}$ results in monopolistic profits, as both firms control half of the profits of the rival, and that each one of them has the same objective function as a monopolist. However, this intuition does not take into account the entire story. On the one hand, having half the profits of the rival, and giving away half of the own, provides the incentive not to undercut the rival, but this also occurs at any other price. On the other hand, a price below p^m does not provide incentives to unilaterally increase the price, as it would if the firm behaved as a monopolist. This is because this action will not affect the total market profits, as the other firm gets the whole market.

The other interesting aspect of Proposition 6 comes from the fact that this same argument applies whenever $\alpha_1 + \alpha_2 = 1$. For instance, if Firm 1 gives as little as a 10% of its profits to Firm 2, then, it needs to receive 90% of Firm 2's profits in the equilibrium. This results in Firm 1 having 90% of total market profits. Thus, Proposition 6 not only says that any price may be sustained in a SPNE, but also that any final market share can be sustained.

3 Asymmetric costs

We consider the same model as in the previous section except that we allow Firm 1 and Firm 2 to have different marginal costs (assume that $c_1 < c_2 < p_1^m$, where p_1^m is the monopolist price of Firm 1), and that we let Firm 1 supply the entire market whenever its price is less or equal to Firm 2's price. This last condition is the natural one to impose due to the effects of the discontinuity of the profits function on the equilibrium if we do otherwise. If the firm with the lowest cost does not get all the market when both set price $p_1 = p_2 = c_2$, Firm 1 will set a price slightly below c_2 , but since the best response is not well defined, the situation $(p_1 = c_2 - \varepsilon, p_2 = c_2)$ could not be an equilibrium. However, those prices constitute a perfectly good equilibrium if all quantities must be multiples of ε . Imposing the rule that the firm with the lowest cost gets all the market if prices are the same saves the equilibrium in the continuous case. Then, the profit of each firm is:

$$\Pi_1 = \begin{cases} (p_1 - c_1)(1 - p_1) & \text{if } p_1 \leq p_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Pi_2 = \begin{cases} (p_2 - c_2)(1 - p_2) & \text{if } p_2 < p_1 \\ 0 & \text{otherwise} \end{cases}$$

Lemma 7 *The price configuration (p_1, p_2) s.t. $p_1 = p_2 = c_2$, is an equilibrium for any (α_1, α_2) .*

Proof. The proof is straightforward, as a deviation to set a smaller price by Firm 1 (resp., Firm 2) clearly implies smaller profits for Firm 1 (resp., losses for Firm 2). Likewise, if Firm 1 sets a higher price, Firm 2 takes all the market, in which it makes zero profits. Firm 2 does not change anything by increasing its price. ■

Lemma 8 next is the counterpart of Lemma 3, and it shows the possibility of multiple equilibria. As it was the case in Lemma 3, the key is to find conditions in (α_1, α_2) to balance the changes in both $(1 - \alpha_i)\Pi_i$ and $\alpha_j\Pi_j$

after a deviation in order for that deviation not to be profitable. The most interesting part of the lemma is the fact that the conditions in (α_1, α_2) are less restrictive than in the case of symmetric firms.

In the sequel, p_i^m will denote the monopoly price when costs are c_i .

Lemma 8 *If $\frac{\alpha_2}{1-\alpha_1} \leq \frac{p_1^m - c_1}{p_1^m - c_2}$ and $\frac{1-\alpha_2}{\alpha_1} \leq \frac{p_1^m - c_1}{p_1^m - c_2}$, then $p_1 = p_2 = p \in (c_2, p_1^m]$ are equilibrium prices.*

If $\frac{\alpha_2}{1-\alpha_1} \geq \frac{p_1^m - c_1}{p_1^m - c_2}$ or $\frac{1-\alpha_2}{\alpha_1} \geq \frac{p_1^m - c_1}{p_1^m - c_2}$, then the following prices constitute an equilibrium in the subgame, $p_1 = p_2 = p \in (c_2, p_1^m]$ such that $\frac{\alpha_2}{1-\alpha_1} \leq \frac{p-c_1}{p-c_2}$ and $\frac{1-\alpha_2}{\alpha_1} \leq \frac{p-c_1}{p-c_2}$.

Proof. Consider a situation in which $p_1 = p_2 = p$, where profits are given by $P_1 = (1 - \alpha_1)(p - c_1)(1 - p)$ and $P_2 = \alpha_1(p - c_1)(1 - p)$

i) Suppose that Firm 1 sets a price p'_1 above p_2 (by setting $p'_1 < p_2$ we have that Π_1 decreases while Π_2 remains unchanged, so that no improvement in payoffs can be expected), then $\Pi'_1 = 0$ and $\Pi_2 = (p - c_2)(1 - p) \geq 0$, and

$$P'_1 = \alpha_2 \Pi_2 = \alpha_2 (p - c_2)(1 - p)$$

To avoid a profitable deviation, we need $P'_1 \leq P_1$, or $\alpha_2(p - c_2) \leq (1 - \alpha_1)(p - c_1)$, which gives

$$\frac{\alpha_2}{1 - \alpha_1} \leq \frac{p - c_1}{p - c_2} \quad (3)$$

Let us study the deviation for Firm 2

ii) Suppose that Firm 2 sets a price p'_2 below p_1 (setting $p'_2 > p_1$ changes nothing), then $\Pi_1 = 0$ and $\Pi'_2 = (p'_2 - c_2)(1 - p'_2) \geq 0$, and

$$P'_2 = (1 - \alpha_2) \Pi'_2 = (1 - \alpha_2)(p'_2 - c_2)(1 - p'_2)$$

The $\sup_{p'_2} P'_2 = P_2^{\sup} = (1 - \alpha_2)(p'_2 - c_2)(1 - p'_2)$ (the best deviation) is achieved at $p'_2 = p$ as long as $p \leq p_2^m$ (p_2^m is the Firm 2 monopoly price). To avoid a profitable deviation, we need $P_2^{\sup} \leq P_2$, or $(1 - \alpha_2)(p - c_2) \leq \alpha_1(p - c_1)$, which gives

$$\frac{1 - \alpha_2}{\alpha_1} \leq \frac{p - c_1}{p - c_2} \quad (4)$$

Inequalities (3) and (4) represent the non-deviation conditions, and both are satisfied when

$$\frac{\alpha_2}{1 - \alpha_1} \leq \frac{p - c_1}{p - c_2} \text{ and } \frac{1 - \alpha_2}{\alpha_1} \leq \frac{p - c_1}{p - c_2}.$$

The statements of the proposition follow once one realizes that $\frac{p_1^m - c_1}{p_1^m - c_2} \leq \frac{p - c_1}{p - c_2} < \infty$ for all $p \in (c_2, p_1^m]$. ■

Lemma 9 *There is no equilibrium with $p_1 = p_2 > p_1^m$.*

Proof. Consider (p_1, p_2) s.t. $p_1 = p_2 = p > p_1^m$. In this case, $P_1 = (1 - \alpha_1)(p - c_1)(1 - p)$, which can be increased with deviation $p'_1 = p^m$. ■

Finally, Lemma 10 completes the search for equilibria in pure strategies showing the equilibria in which firms set different prices. The proof is left to the Appendix.

Lemma 10 *The necessary and sufficient condition for an equilibrium with different prices is $\frac{1 - \alpha_2}{\alpha_1} \leq \frac{p_1^m - c_1}{p_1^m - c_2}$, and the possible equilibria are given by the pairs (p_1, p_2) s.t. $c_2 < p_1 = p_1^m < \bar{p}_2 \leq p_2$ for $\bar{p}_2$ satisfying $\frac{\Pi(p_1^m)}{\Pi(\bar{p}_2)} > \frac{\alpha_2}{1 - \alpha_1}$.*

The next proposition summarizes our findings about equilibria in the second stage of the two asymmetric firms case.

Proposition 11 *The Bertrand game with a profit-sharing previous stage and costs $c_1 < c_2$ has the following pure strategy Nash equilibria in the subgames:*

- (a) *If $\frac{\alpha_2}{1 - \alpha_1} \geq \frac{p_1^m - c_1}{p_1^m - c_2}$ or $\frac{1 - \alpha_2}{\alpha_1} \geq \frac{p_1^m - c_1}{p_1^m - c_2}$, then the equilibria with equal prices are (p_1, p_2) s.t. $p_1 = p_2 = p \in [c_2, p_1^m]$ and $\frac{\alpha_2}{1 - \alpha_1} \leq \frac{p - c_1}{p - c_2}$ and $\frac{1 - \alpha_2}{\alpha_1} \leq \frac{p - c_1}{p - c_2}$.*
- (b) *If $\frac{\alpha_2}{1 - \alpha_1} \leq \frac{p_1^m - c_1}{p_1^m - c_2}$ and $\frac{1 - \alpha_2}{\alpha_1} \leq \frac{p_1^m - c_1}{p_1^m - c_2}$, then the equilibria with equal prices are (p_1, p_2) s.t. $p_1 = p_2 = p \in [c_2, p_1^m]$.*
- (c) *If $\frac{1 - \alpha_2}{\alpha_1} \leq \frac{p_1^m - c_1}{p_1^m - c_2}$, then the equilibria with different prices are (p_1, p_2) s.t. $c_2 < p_1 = p_1^m < \bar{p}_2 \leq p_2$ for $\bar{p}_2$ satisfying $\frac{\Pi(p_1^m)}{\Pi(\bar{p}_2)} > \frac{\alpha_2}{1 - \alpha_1}$. If $\frac{1 - \alpha_2}{\alpha_1} > \frac{p_1^m - c_1}{p_1^m - c_2}$ there are no equilibria with different prices.*

The next proposition is the counterpart of Proposition 6 and shows that, with different costs, it is also possible to support multiple market prices in a subgame perfect Nash equilibrium. In this case, any price between the cost of the least competitive firm and the monopoly price of the most competitive one can be found in a SPNE.

Proposition 12 *Any price pair (p_1^*, p_2^*) such that $c_2 \leq p_1^* = p_2^* = p^* \leq p_1^m$ can be sustained in a SPNE.*

Proof. The proposition is trivial for $p^* = c_2$, so we only provide the proof for the other cases.

Consider the following strategy. In the first step players play (α_1^*, α_2^*) such that $\frac{\alpha_1^*}{1-\alpha_2^*} \leq \frac{p^*-c_1}{p^*-c_2}$, $\frac{1-\alpha_2^*}{\alpha_1^*} \leq \frac{p^*-c_1}{p^*-c_2}$ and $\alpha_1^* < 1 - \frac{(c_2-c_1)(1-c_2)}{(p^*-c_1)(1-p^*)}$. In the second stage they play $(p_1, p_2) = (p_1^*, p_2^*)$ if in the first stage $(\alpha_1, \alpha_2) = (\alpha_1^*, \alpha_2^*)$, and $p_1 = p_2 = c_2$ otherwise. Notice that the conditions are satisfied for $\alpha_1^* + \alpha_2^* = 1$ and any $p^* \in (c_2, p_1^m]$.

Since the prices constitute Nash equilibria in the respective subgames, we need to check that there are no profitable deviations for the firms in the first stage. Profits in equilibrium are $P_1 = (1 - \alpha_1^*)(p^* - c_1)(1 - p^*)$ and $P_2 = \alpha_1^*(p^* - c_1)(1 - p^*)$.

If Firm 1 deviates to $\alpha_1' \neq \alpha_1^*$ it triggers a subgame in which it gets $P_1' = (1 - \alpha_1')(c_2 - c_1)(1 - c_2)$. Given this continuation, the best deviation is $\alpha_1' = 0$, with $P_1'(\alpha_1' = 0) = (c_2 - c_1)(1 - c_2)$. The deviation is not profitable if $P_1'(\alpha_1' = 0) \leq P_1$, something that must be the case for $\alpha_1^* < 1 - \frac{(c_2-c_1)(1-c_2)}{(p^*-c_1)(1-p^*)}$ as $(p^* - c_1)(1 - p^*) > (c_2 - c_1)(1 - c_2)$ whenever $c_2 < p^* \leq p_1^m$.

Finally, deviations by Firm 2 can only give this firm smaller profits, as price in out of equilibrium subgames is $p = c_2$, which implies $P_2' = \alpha_1^*(c_2 - c_1)(1 - c_2) \leq P_2$. ■

Let $\frac{p^*-c_1}{p^*-c_2} = t$. Clearly $t > 1$, thus conditions $\frac{\alpha_1^*}{1-\alpha_2^*} \leq \frac{p^*-c_1}{p^*-c_2}$ and $\frac{\alpha_2^*}{1-\alpha_1^*} \leq \frac{p^*-c_1}{p^*-c_2}$ are satisfied if $\frac{1}{t}\alpha_1^* + \alpha_2^* \leq 1$ and $\alpha_1^* + \frac{1}{t}\alpha_2^* \leq 1$, which are satisfied for all $\alpha_1^* + \alpha_2^* \leq 1$ and for some values such that $\alpha_1^* + \alpha_2^* > 1$. Also, notice that the restriction that α_1^* be small enough is more restrictive the lower the equilibrium price one wants to support. In particular it is enough that $1 - \alpha_1^* \geq \frac{(c_2-c_1)(1-c_2)}{(p^*-c_1)(1-p^*)}$. Thus, the margins for a profit sharing equilibrium are quite broad.

The fact that we do not have the equivalent to the symmetric case condition $\alpha_1^* + \alpha_2^* \geq 1$ is due to the specific tie breaking rule, that favors the firm with the lowest cost with the whole market, and thus deviations by the firm with the highest cost are not particularly important. In fact, one can prove that the rule that divides the market equally if prices are the same cannot support this kind of equilibria. However, tie breaking rules that assign a proportion of the market higher than $\frac{1}{2}$ but less than 1 to the low cost firm can still support prices between c_2 and p_1^m .

4 The tie-breaking rule

In the previous section, when both firms set the same price, the tie-breaking rule favored the firm with the lowest cost. Here we show that a tie-breaking

rule different from the equal division of the market is necessary to obtain the results.

Consider, then, the same model in Section 3, except that the profits functions are given by

$$\Pi_i = \begin{cases} (p_i - c_i)(1 - p_i) & \text{if } p_i < p_j \\ \frac{1}{2}(p_i - c_i)(1 - p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases} \quad i, j = 1, 2 \ (i \neq j).$$

Suppose that we want to have $p_1 = p_2 \in (c_2, p_1^m]$ in an equilibrium. The expressions for profits take the form $\Pi_i = \frac{1}{2}(p - c_i)(1 - p)$, and $P_i = \frac{1}{2}(1 - \alpha_i)(p - c_i)(1 - p) + \frac{1}{2}\alpha_j(p - c_j)(1 - p)$. Consider now a deviation by Firm 1 to a lower price, $p'_1 < p$. This will provide profits $\Pi'_1 = (p'_1 - c_1)(1 - p'_1)$, $\Pi'_2 = 0$, and $P'_1 = (1 - \alpha_1)(p'_1 - c_1)(1 - p'_1)$. In order for this deviation not to be profitable, we need $\sup_{p'_1 < p} P'_1 = P_1^{\sup} = (1 - \alpha_1)(p - c_1)(1 - p) \leq P_1$, which implies $(1 - \alpha_1)(p - c_1) \leq \frac{1}{2}(1 - \alpha_1)(p - c_1) + \frac{1}{2}\alpha_2(p - c_2)$, or

$$\frac{\alpha_2}{1 - \alpha_1} \geq \frac{p - c_1}{p - c_2}.$$

A deviation to a higher price, on the other hand we result in $\Pi'_1 = 0$, $\Pi'_2 = (p - c_2)(1 - p)$, and $P'_1 = \alpha_2(p - c_2)(1 - p)$. Now the condition for the deviation not to be profitable is $\alpha_2(p - c_2)(1 - p) \leq \frac{1}{2}(1 - \alpha_1)(p - c_1)(1 - p) + \frac{1}{2}\alpha_2(p - c_2)(1 - p)$, or

$$\frac{\alpha_2}{1 - \alpha_1} \leq \frac{p - c_1}{p - c_2}.$$

Thus we need

$$\frac{\alpha_2}{1 - \alpha_1} = \frac{p - c_1}{p - c_2}. \quad (5)$$

Similarly, the conditions for Firm 2 not to deviate imply

$$\frac{\alpha_1}{1 - \alpha_2} = \frac{p - c_2}{p - c_1}. \quad (6)$$

These conditions cannot be satisfied at the same time. Notice that they imply $\frac{\alpha_2}{1 - \alpha_1} = \frac{1 - \alpha_2}{\alpha_1}$, or $\alpha_1 + \alpha_2 = 1$, which in turns means $\frac{\alpha_2}{1 - \alpha_1} = 1 \neq \frac{p - c_1}{p - c_2}$ as $c_1 \neq c_2$.

5 Extension to n firms

5.1 Equal costs

In this section, we consider n firms indexed by $i \in N = \{1, 2, \dots, n\}$ in a homogeneous market. We suppose that each firm incurs a cost c per unit of production. The market demand function is $q = D(p) = 1 - p$. We assume that firms do not have capacity constraints and always supply the demand they face. For a given vector of price choices $(p_1, \dots, p_n)$, consider $p_{\min} = \min\{p_1, \dots, p_n\}$, and define N' as $N' = \{i \in N : p_i = p_{\min}\}$, and let $n' = \text{card}N'$. Therefore, the profit function of Firm i can be written as:

$$\Pi_i = \begin{cases} (p_i - c)(1 - p_i) & \text{if } p_i < p_j \ \forall j \neq i \\ \frac{1}{n'}(p_i - c)(1 - p_i) & \text{if } p_i = \min\{p_1, \dots, p_n\} \\ 0 & \text{otherwise} \end{cases}$$

Let β_{ij} denote the part of the profit that Firm i shares with firm j . We suppose that $\beta_{ij} \in (0, 1)$ and $\sum_{j=1}^n \beta_{ij} = 1$. Consequently, we can write the new profit function P_i of each firm as:

$$P_i = \beta_{ii}\Pi_i + \sum_{j=1(j \neq i)}^n \beta_{ji}\Pi_j.$$

As before, we consider a two-stage game whose sequences are thus defined. In the first stage of the game, Firm i chooses $(\beta_{i1}, \dots, \beta_{in})$, while in the second stage of the game, it selects p_i .

To find the complete set of equilibria in all subgames becomes a complicated, tedious exercise as the number of firms increase. However, we can still prove the existence of a SPNE to support any price between c and the monopoly price.

Proposition 13 *Any price vector $(p_1^*, \dots, p_n^*)$ such that $c \leq p_i^* = p^* \leq p_1^m$ for all $i \in N$ can be sustained in a SPNE. Further, the equilibrium requires that firms share profits to satisfy*

$$\sum_{j=1(j \neq i)}^n \beta_{ij} + \frac{1}{n-1} \sum_{j=1(j \neq i)}^n \beta_{ji} = 1 \text{ for all } i \in N. \quad (7)$$

The proof is similar to that in propositions 5 and 6 and can be found in the Appendix. Condition (5) is the generalization of the condition $\alpha_1^* + \alpha_2^* = 1$ for the case of two symmetric firms. Notice that it is satisfied for $\beta_{ij} = \frac{1}{n}$ for all $i, j \in N$.

5.2 Different costs

Order firms according to costs, so that $c_1 \leq c_2 \leq \dots \leq c_n$, let N_1 be the set of the firms with the lowest cost, i.e., $N_1 = \{i \in N : c_i = c_1\}$, and denote by n_1 the cardinal of N_1 . Now the profit function is:

$$\Pi_i = \begin{cases} (p_i - c_i)(1 - p_i) & \text{if } p_i < p_j \ \forall j \neq i \\ \frac{1}{n'}(p_i - c_i)(1 - p_i) & \text{if } p_i = \min\{p_1, \dots, p_n\} \\ 0 & \text{otherwise} \end{cases}$$

Now we can state our last proposition for the general case of n firms and asymmetric costs.

Proposition 14 *Any price p s.t. $\bar{c} \leq p \leq p_1^m$, where $\bar{c} = c_1$ if $n_1 > 1$, and $\bar{c} = c_2$ if $n_1 = 1$, and where p_1^m is the monopoly price of Firm 1, can be supported in a subgame perfect equilibrium in pure strategies.*

The formal proof is given in the Appendix, but a sketch can be presented as follows. If $n_1 > 1$, the idea is to share the market among the firms with the lowest cost (firms with $c_i = c_1$) in a similar fashion as in Proposition 13. The other firms choose not to share profits and set price equal their marginal costs. If $n_1 = 1$, however, the market will be shared among Firm 1 and the firms with the next-to-lowest costs (firms with $c_i = c_2$) in a way similar to that in Proposition 12, with the added feature that now we can have more than one firm with a higher cost. Thus, conditions in Proposition 12 must be adapted to this possibility. The general arguments, however, still hold.

6 Profit sharing (one-sided) in a Bertrand duopoly

We consider here the Bertrand model similar to the one presented in Sections 2 and 3 above except, as described in Section 5 of Chapter 1, that only Firm 1 shares its profit whereas Firm 2 keeps its entire profit and still receives a fraction of Firm 2's profit. Thus, we must find equilibria in subgames after (α_1, α_2) with $\alpha_1 \in [0, 1]$ $\alpha_2 = 0$. Consider first the case $\alpha_1 \in [0, 1)$, $\alpha_2 = 0$. According to Proposition 5 (c), the only equilibrium in these subgames are $p_1 = p_2 = c$. As profits are zero in this context, any $\alpha_1 \in [0, 1)$ is an equilibrium choice in the first stage. No matter what Firm 1 does, it will get zero profits. Similarly, $\alpha_1 = 1$, $\alpha_2 = 0$ imply, by Proposition 5 (a), that any price may be supported in equilibrium in the second stage. But $\alpha_1 = 1$, $\alpha_2 = 0$ also imply $P_1 = 0$ along the equilibrium path, as well as after any

deviation. Thus, we have multiplicity of equilibrium price in an uninteresting way, and with no multiplicity of market shares.

The introduction of asymmetries, however, allows for the existence of an equilibrium in which only Firm 1 can use the profit sharing strategy. Notice that the case $\alpha_1 > 0$ $\alpha_2 = 0$ lies within the case in Proposition 11 (a), where it is shown the existence of an equilibrium with $p_1 = p_2 \in [c_2, p_1^m]$ as long as $\frac{\alpha_2}{1-\alpha_1} \leq \frac{p-c_1}{p-c_2}$ and $\frac{1-\alpha_2}{\alpha_1} \leq \frac{p-c_1}{p-c_2}$, which is satisfied for all $\alpha_1 > 0$ $\alpha_2 = 0$. That is not the case if $\alpha_1 = 0$ $\alpha_2 > 0$, which implies that there is no equilibrium in which only the second firm shares profits if it is the only one allowed to do so.

When only one firm is allowed to share profits, the conditions for this strategy to be of any success are more difficult, thus should not come at a surprise that the only oligopoly in which it works is the Bertrand model with homogenous goods, where conditions (i) and (ii) discussed in Section 4 of Chapter 1 are satisfied in a higher degree. Further, they are satisfied even more for the firm with the lowest cost in the asymmetric Bertrand scenario.

7 Conclusion

We considered Bertrand oligopolies with homogeneous goods, linear demand and constant marginal costs, and found that it is possible to support equilibrium prices above marginal costs by introducing a previous stage of profit sharing. In this stage, firms voluntarily and independently of each other decide how much of their profits they will give away to rivals.

The range of prices that can be supported in a subgame perfect equilibrium varies between the second lowest marginal cost and the monopoly price of the lowest cost firm. Further, there is also a great range of the final market shares (after counting the effects of profit sharing) that can be supported in equilibrium, thus making the range of possible payoffs similar to those provided by the Folk theorem in a repeated game.

Chapter 1 analyzed the strategic-profit-sharing strategy in oligopolistic scenarios with differentiable payoff functions. In that work it was shown that the strategy was more likely to facilitate some degree of tacit collusion the higher the strategic complementarity of second stage game, and the higher the differences between monopoly and oligopoly equilibria. Both conditions are satisfied in the present model. However, as nothing could be shown in general about the extend of the degree of collusion, if any, a separate analysis was necessary.

References

- Alley, W. (1997), "Partial Ownership Arrangements and Collusion in the Automobile Industry," *Journal of Industrial Economics* 45, 191-205.
- Edgeworth, F. (1897) "The Pure Theory of Monopoly," *Papers Relating to Political Economy* 1, 111-142. London: MacMillan & Co., Ltd., 1925.
- Farrell, J., and C. Shapiro (1990), "Asset ownership and market structure in oligopoly," *RAND Journal of Economics* 21, 275-292.
- Ferreira, J.L, Waddle, R. (2010), "Strategic profit sharing between firms," *International Journal of Economic Theory*, forthcoming.
- Gilo, D., Moshe, Y., and Spiegel Y. (2006), "Partial Ownership and Tacit Collusion," *RAND Journal of Economics* 37(1), 81-99.
- Jackson, M. O., and S. Wilkie (2005), "Endogenous games and mechanisms: Side payments among players," *Review of Economics Studies* 72, 543-566.
- Kreps, D., Scheinkman, J. (1983), "Quantity precommitment and Bertrand competition yield to Cournot outcomes," *Bell Journal of Economics* 14, 326-337.
- Maggi, G. (1996), "Strategic trade policies with endogenous mode of competition," *American Economic Review* 86, 237-258.
- Malueg, D. (1992), "Collusive behavior and partial ownership of rivals," *International Journal of Industrial Organization* 10, 27-34.
- Moreno, D., Ubeda, L. (2006), "Capacity precommitment and price competition yield to the Cournot outcome," *Games and Economic Behavior* 56, 323-332.
- Reitman, D. (1994), "Partial Ownership Arrangements and Potential for Collusion," *Journal of Industrial Economics* 42, 313-322.
- Reynolds. R.J., and Snapp, B.R. (1986), "The competitive effects of partial equity interests and joint ventures," *International Journal of Industrial Organization* 4, 141-153.
- Shapiro, C. (1989), "Theories of oligopoly behavior," Chapter 6 in: Schmalensee, R., Willig, R.D. (Eds.), *Handbook of Industrial Organization* 1, 329-410 North-Holland, Amsterdam.

Appendix

Proofs.

Lemma 4.

Proof. Without loss of generality, let (p_i, p_j) be such that $c \leq p_i < p_j$. Then

$$P_i = (1 - \alpha_i) \Pi_i = (1 - \alpha_i) (p_i - c) (1 - p_i), \text{ and}$$

$$P_j = \alpha_i \Pi_i = \alpha_i (p_i - c) (1 - p_i).$$

Since prices p_i and p_j are different, we have to study separately the deviations for each firm. Let us check first Firm j . We will consider $p_i \leq p_m$ since, otherwise, Firm i will deviate to a lower price. If $c = p_i$ profits by Firm i can increase by increasing p_i . Consider, then that $c < p_i$ and first check for deviations by Firm j .

i) If Firm j sets a price p'_j below p_i , its new profits are given by $P'_j = (1 - \alpha_j) \Pi'_j = (1 - \alpha_j) (p'_j - c) (1 - p'_j)$. The sup P'_j is achieved at $p'_j = p_i$, with $P'_j(p'_j = p_i) = (1 - \alpha_j) \Pi_i$. Thus, to avoid a profitable deviation, we need $P_j = \alpha_i \Pi_i \geq P'_j = (1 - \alpha_j) \Pi_i$, which implies $\alpha_1 + \alpha_2 \geq 1$.

ii) If Firm j sets a price p''_j equal to p_i , then $\Pi''_i = \Pi''_j = \frac{1}{2}(p''_j - c)(1 - p''_j) = \frac{1}{2}\Pi_i > 0$, with $P''_j = \frac{1}{2}(1 - \alpha_j + \alpha_i) \Pi_i$. To avoid a profitable deviation, we need $P_j = \alpha_i \Pi_i \geq P''_j = \frac{1}{2}(1 - \alpha_j + \alpha_i) \Pi_i$, which implies $\alpha_1 + \alpha_2 \geq 1$, as before.

Now, let us check deviations for Firm i .

i') Suppose that Firm i deviates to set a price p'_i above p_i and below p_j , then $P'_i = (1 - \alpha_i) (p'_i - c) (1 - p'_i) = (1 - \alpha_i) \Pi'_i$. This kind of deviation is profitable as long as $p_i < p^m$. Thus, a necessary condition for Firm i not to be willing to deviate is $p_i \geq p^m$. If $p_i > p^m$, a deviation with p'_i slightly below p_i will be profitable. Hence, the equilibrium requires $p_i = p^m$.

ii') Suppose, then, that $p_i = p^m$, and that Firm i considers a deviation to set its price $p''_i > p_j$ (any other deviation below p_j is clearly unprofitable), then $\Pi''_i = 0$ and $\Pi''_j = (p_j - c) (1 - p_j)$, with $P''_i = \alpha_j \Pi''_j$. For the deviation not to be profitable we need $P_i = (1 - \alpha_i) \Pi_m \geq P''_i = \alpha_j \Pi''_j$, which implies

$$\frac{\Pi(p^m)}{\Pi(p_j)} \geq \frac{\alpha_j}{1 - \alpha_i}. \quad (8)$$

iii') Suppose that $p_i = p^m$, and that Firm i considers a deviation to set its price $p'''_i = p_j$. Now $\Pi'''_i = \Pi'''_j = \frac{1}{2}(p_j - c)(1 - p_j)$, and $P'''_i =$

$\frac{1}{2}(1 - \alpha_j + \alpha_i) \Pi(p_j)$. To avoid a profitable deviation, we need $P_i = (1 - \alpha_i) \Pi_m \geq P_i''' = \frac{1}{2}(1 - \alpha_i + \alpha_j) \Pi(p_j)$, which implies

$$\frac{\Pi_m}{\Pi(p_j)} \geq \frac{1 - \alpha_i + \alpha_j}{2(1 - \alpha_i)}. \quad (9)$$

A deviation to a price $p_i^{iv} < p_j$ is clearly unproductive. Inequalities (6) and (7) are satisfied for all $p_j \geq \bar{p}$ if they are satisfied for $\bar{p}$. Of them, (6) is more restrictive and, therefore, both are satisfied for all $p_j \geq \bar{p}$ such that

$$\frac{\Pi(p^m)}{\Pi(\bar{p})} \geq \frac{\alpha_j}{1 - \alpha_i}$$

is satisfied, which provides the condition for the equilibrium. ■

Lemma 10.

Proof. Let (p_1, p_2) such that $c_2 \leq p_1 < p_2$. Then $\Pi_1 = (p_1 - c_1)(1 - p_1) > 0$ and $\Pi_2 = 0$, with

$$P_1 = (1 - \alpha_1) \Pi_1 = (1 - \alpha_1)(p_1 - c_1)(1 - p_1), \text{ and}$$

$$P_2 = \alpha_1 \Pi_1 = \alpha_1(p_1 - c_1)(1 - p_1)$$

Since prices p_1 and p_2 are different, we have to study separately the deviation for each firm. Let us check first Firm 2. We will consider $p_1 \leq p_1^m$ since, otherwise, Firm 1 will deviate to a lower price.

i) If Firm 2 sets a price p'_2 below p_1 (other deviations are trivially shown not to provide higher profits), its new profits are given by $P'_2 = (1 - \alpha_2) \Pi'_2 = (1 - \alpha_2)(p'_2 - c_2)(1 - p'_2)$. The sup P'_2 is achieved at $p'_2 = p_1$, with $P'_2(p'_2 = p_1) = (1 - \alpha_2)(p_1 - c_2)(1 - p_1)$. Thus, to avoid a profitable deviation, we need $P_2 = \alpha_1(p_1 - c_1)(1 - p_1) \geq P'_2 = (1 - \alpha_2)(p_1 - c_2)(1 - p_1)$, which implies

$$\frac{1 - \alpha_2}{\alpha_1} \leq \frac{p_1 - c_1}{p_1 - c_2}. \quad (10)$$

Now, let us check for Firm 1.

i) Suppose that Firm 1 deviates to set a price p'_1 above p_1 and below p_2 , then $P'_1 = (1 - \alpha_1)(p'_1 - c)(1 - p'_1) = (1 - \alpha_1) \Pi'_1$. This kind of deviation is profitable as long as $p_1 < p_1^m$. Thus, a necessary condition for Firm 1 not to be willing to deviate is $p_1 \geq p_1^m$. If $p_1 > p_1^m$, a deviation with p'_1 slightly below p_1 will be profitable. Hence, the equilibrium requires $p_1 = p_1^m$, and condition (8) becomes

$$\frac{1 - \alpha_2}{\alpha_1} \leq \frac{p_1^m - c_1}{p_1^m - c_2}.$$

ii) Suppose, then, that $p_1 = p_1^m$, and that Firm i considers a deviation to set its price p_1'' above p_2 (any other deviation below p_2 is clearly unprofitable), then $\Pi_1'' = 0$ and $\Pi_2 = (p_2 - c_2)(1 - p_2)$, with $P_1'' = \alpha_2 \Pi_2$. For the deviation not to be profitable we need $P_1 = (1 - \alpha_1) \Pi(p_1^m) \geq P_1'' = \alpha_1 \Pi_2$, which implies

$$\frac{\Pi(p_1^m)}{\Pi(\bar{p}_2)} \geq \frac{\alpha_2}{1 - \alpha_1} \quad (11)$$

The above inequality provides the condition for the equilibrium. It will be satisfied if p_2 is set so high that $D(p_2) = 0$.

Now, it remains to show that there are no other equilibrium prices if $\frac{1 - \alpha_2}{\alpha_1} \leq \frac{p_1^m - c_1}{p_1^m - c_2}$. To that effect, let us consider (p_1, p_2) such that $p_1 < p_2$ and $p_1 < p_1^m$. In this case, Firm 1 would set a price p_1''' greater than p_1 , but lower than p_2 to get a profit $P_1''' = (1 - \alpha_1) \Pi_1'''(p_1''', p_2)$, which is greater than $(1 - \alpha_1) \Pi_1(p_1, p_2)$ as long as $p_1''' < p_1^m$. If $p_1 > p_1^m$, the profitable deviation occurs with a price $p_1''' < p_1$. ■

Proposition 13.

Proof. First we start by specifying the equilibria in the subgames. It is straightforward to see that, in every subgame, the price vector $(p_1, \dots, p_n)$ s.t. $p_1 = \dots = p_n = c$ constitutes an equilibrium. Next we find equilibrium prices above the cost. The profits of Firm i ($i = 1, \dots, n$) in a situation in which $c < p_1 = \dots = p_n = p \leq p^m$ are $\Pi_i = \frac{1}{n} (p - c)(1 - p)$ and $P_i = \left(1 - \sum_{j=1(j \neq i)}^n \beta_{ij}\right) \Pi_i + \sum_{j=1(j \neq i)}^n \beta_{ji} \Pi_j$ or

$$P_i = \frac{1}{n} \left(1 - \sum_{j=1(j \neq i)}^n \beta_{ij} + \sum_{j=1(j \neq i)}^n \beta_{ji}\right) (p - c)(1 - p)$$

Let us study possible deviations by Firm i .

i) Suppose that Firm i deviates to a price $p_i' < p$, then $\Pi_i' = (1 - p_i')(p_i' - c) > 0$ and $\Pi_j = 0$, and

$$P_i' = (1 - \sum_{j=1(j \neq i)}^n \beta_{ij} \Pi_i') = \left(1 - \sum_{j=1(j \neq i)}^n \beta_{ij}\right) (1 - p_i')(p_i' - c)$$

Since $p \leq p^m$, the best deviation provides at most

$$\sup_{p_i' < p} P_i' = P_i^{\sup} = \left(1 - \sum_{j=1(j \neq i)}^n \beta_{ij}\right) (1 - p)(p - c).$$

Therefore, to avoid a possible deviation we need $P_i^{\text{sup}} \leq P_i$, which implies $\left(1 - \sum_{j=1(j \neq i)}^n \beta_{ij}\right) \leq \frac{1}{n} \left(1 - \sum_{j=1(j \neq i)}^n \beta_{ij} + \sum_{j=1(j \neq i)}^n \beta_{ji}\right)$, and that is satisfied if

$$(n-1) \sum_{j=1(j \neq i)}^n \beta_{ij} + \sum_{j=1(j \neq i)}^n \beta_{ji} \geq n-1 \text{ for all } i \in N \quad (12)$$

ii) Suppose that Firm i sets a price $p_i'' > p$ then $\Pi_j = \frac{1}{n-1} (1-p)(p-c) > 0$ and $\Pi_i'' = 0$, with

$$P_i'' = \sum_{j=1(j \neq i)}^n \beta_{ji} \Pi_j = \frac{1}{n-1} \sum_{j=1(j \neq i)}^n \beta_{ji} (1-p)(p-c).$$

To avoid a possible deviation we need $P_i'' \leq P_i$, implying $\frac{1}{n-1} \sum_{j=1(j \neq i)}^n \beta_{ji} \leq \frac{1}{n} \left(1 - \sum_{j=1(j \neq i)}^n \beta_{ij} + \sum_{j=1(j \neq i)}^n \beta_{ji}\right)$, which is satisfied if

$$(n-1) \sum_{j=1(j \neq i)}^n \beta_{ij} + \sum_{j=1(j \neq i)}^n \beta_{ji} \leq n-1 \quad (13)$$

Inequalities (10) and (11) represent the non-deviation conditions and both are satisfied when

$$\sum_{j=1(j \neq i)}^n \beta_{ij} + \frac{1}{n-1} \sum_{j=1(j \neq i)}^n \beta_{ji} = 1. \quad (14)$$

The proof is complete by making firms choose $\beta_{ij} = \beta_{ij}^*$, where $(\beta_{ij}^*)_{i,j}$ satisfies (12), in the first stage, and $(p_1^*, \dots, p_n^*)$ such that $c \leq p_i^* = p^* \leq p_1^m$ if $(\beta_{ij}^*)_{i,j}$ was indeed chosen in the first stage, and $p_i^* = c$ otherwise. ■

Proposition 14.

Proof. First we start by finding subgames in which the different prices are equilibria. To this end, start by finding the expression for P_i , in a situation in which $c_n < p_i = p \leq p_1^m$ for all $i \in N$. Profits in this situation are given by

$$\Pi_i = \frac{1}{n_1} (p - c_1) (1 - p) \text{ for all } i \in N_1,$$

$$\Pi_i = 0 \text{ for all } i \notin N_1,$$

$$P_i = \left(1 - \sum_{j=1(j \neq i)}^n \beta_{ij}\right) \Pi_i + \sum_{j=1(j \neq i)}^{n_1} \beta_{ji} \Pi_j$$

$= \frac{1}{n_1} (1 - \sum_{j=1(j \neq i)}^n \beta_{ij} + \sum_{j=1(j \neq i)}^{n_1} \beta_{ji}) (p - c_1) (1 - p)$ for all $i \in N_1$,
and

$$P_i = \frac{1}{n_1} (\sum_{j=1(j \neq i)}^{n_1} \beta_{ji}) (p - c_1) (1 - p) \text{ for all } i \notin N_1.$$

Now we can consider deviations from this point.

(i) Case 1: $n_1 > 1$, $i \in n_1$, deviation to $p'_i < p$.

The consequence is $\Pi'_i = (p - c_1) (1 - p)$, while $\Pi'_j = 0$ for all $j \neq i$. Since $p \leq p_1^m = p_i$, its maximum new profits P'_i will be computed as

$$\sup_{p'_i < p} P'_i = P_i^{\sup} = \left(1 - \sum_{j=1(j \neq i)}^n \beta_{ij}\right) (p - c_1) (1 - p).$$

To get $P_i^{\sup} \leq P_i$ we need

$$1 - \sum_{j=1(j \neq i)}^n \beta_{ij} \leq \frac{1}{n_1} \left(\left(1 - \sum_{j=1(j \neq i)}^n \beta_{ij}\right) + \left(\sum_{j=1(j \neq i)}^{n_1} \beta_{ji}\right) \right), \text{ or}$$

$$\sum_{j=1(j \neq i)}^n \beta_{ij} + \frac{1}{n_1 - 1} \sum_{j=1(j \neq i)}^{n_1} \beta_{ji} \geq 1. \quad (15)$$

(ii) Case 2: $n_1 > 1$, $i \in N_1$, deviation to $p''_i > p$.

The new profits are $\Pi''_i = 0$, $\Pi''_j = \frac{1}{n_1} (p - c_1) (1 - p)$ for all $j \in N_1 \setminus \{i\}$, and $\Pi''_j = 0$ for all $j \notin N_1$. This gives

$$P''_i = \frac{1}{n_1 - 1} (\sum_{j=1(j \neq i)}^{n_1} \beta_{ji}) (p - c_1) (1 - p).$$

In order for the deviation not to be profitable we need $P''_i \leq P_i$:

$$\frac{1}{n_1 - 1} \sum_{j=1(j \neq i)}^{n_1} \beta_{ji} \leq \frac{1}{n_1} \left(1 - \sum_{j=1(j \neq i)}^n \beta_{ij} + \sum_{j=1(j \neq i)}^{n_1} \beta_{ji} \right), \text{ or}$$

$$\sum_{j=1(j \neq i)}^n \beta_{ij} + \frac{1}{n_1 - 1} \sum_{j=1(j \neq i)}^{n_1} \beta_{ji} \leq 1. \quad (16)$$

Conditions (13) and (14) are satisfied when

$$\sum_{j=1(j \neq i)}^n \beta_{ij} + \frac{1}{n_1 - 1} \sum_{j=1(j \neq i)}^{n_1} \beta_{ji} = 1 \text{ for all } i \in n_1. \quad (17)$$

(iii) Case 3: $n_1 = 1$, in which case $N_1 = \{1\}$, and the deviation of $i \in N_1$ is the deviation of Firm 1. In this case $P_1 = (1 - \sum_{j=2}^n \beta_{1j}) (p - c_1) (1 - p)$. Consider $p'_1 < p$.

With this deviation $\Pi'_1 \leq \Pi_1$, and $\Pi'_j = \Pi_j = 0$ for all $j \neq i$, with $P'_1 = (1 - \sum_{j=2}^n \beta_{ij})(p'_1 - c_1)(1 - p'_1)$. The consequence is $P'_1 \leq P_1$ as long as $p \leq p_1^m$.

(iv) Case 4: $n_1 = 1$, deviation by Firm 1 to $p'_1 > p$.

After this deviation, the market will be shared among the firms with costs at the level of Firm 2, c_2 . Denote this set of firms by N_2 ; i.e., $N_2 = \{i \in N : c_i = c_2\}$, n_2 will denote the cardinal of N_2 . Profits are $\Pi'_i = \frac{1}{n_2}(p - c_2)(1 - p)$ for all $i \in N_2$, and $\Pi'_i = 0$ for all $i \notin N_2$. Profits of Firm 1 are

$$P'_1 = \frac{1}{n_2}(\sum_{j=2}^{n_2+1} \beta_{j1})(p - c_2)(1 - p).$$

The deviation is not profitable if

$$\frac{1}{n_2}(\sum_{j=2}^{n_2+1} \beta_{j1})(p - c_2)(1 - p) \leq (1 - \sum_{j=2}^n \beta_{1j})(p - c_1)(1 - p), \text{ or}$$

$$\frac{\frac{1}{n_2}(\sum_{j=2}^{n_2+1} \beta_{j1})}{1 - \sum_{j=2}^n \beta_{1j}} \leq \frac{p - c_1}{p - c_2}. \quad (18)$$

(v) Case 5: $n_1 = 1$, deviations by $i \neq 1$.

A deviation to a higher price changes nothing, so consider $p'_i < p$, with the effect that $\Pi'_i = (p'_i - c_i)(1 - p'_i)$, and $\Pi'_j = \Pi_j = 0$ for all $j \neq i$. Then

$$\sup_{p'_i < p} P'_i = P_i^{\sup} = \left(1 - \sum_{j=1(j \neq i)}^n \beta_{ij}\right)(p - c_i)(1 - p).$$

The deviation is not profitable if $P_i^{\sup} \leq P_i = \beta_{1i}(p - c_1)(1 - p)$

$$\left(1 - \sum_{j=1(j \neq i)}^n \beta_{ij}\right)(p - c_i)(1 - p) \leq \beta_{1i}(p - c_1)(1 - p), \text{ or}$$

$$\frac{1 - \sum_{j=1(j \neq i)}^n \beta_{ij}}{\beta_{1i}} \leq \frac{p - c_1}{p - c_i} \text{ for all } i \neq 1. \quad (19)$$

Condition (17) is the counterpart of condition (12) for the case of n firms with equal costs, and is satisfied if $\beta_{ij} = \frac{1}{n}$ for all $i, j \in N$, while conditions (15) and (16) are the counterpart of conditions (3) and (4) for two firms with different costs, and they are satisfied whenever $\frac{1}{n_2}(\sum_{j=2}^{n_2+1} \beta_{j1}) + \sum_{j=2}^n \beta_{1j} \leq 1$, and $\sum_{j=1(j \neq i)}^n \beta_{ij} + \beta_{1i} \leq 1$ for all $i \neq 1$, as $\frac{p - c_1}{p - c_i} \geq 1$, which in turn are satisfied if β_{ij} are small enough.

To support a price $p \in (c_i, c_{i+1}]$ with $c_i < c_{i+1}$, one can set $p_j = p$ for all j such that $c_j \leq c_{i+1}$, and $p_j = c_j$. In this case, firms with a cost higher than c_{i+1} do not receive or give any profits. This way, the new game is as the one before with firms with the highest costs are out of it, and we can the argument above.

The price $p = c_1$ if $n_1 > 1$ or $p = c_2$ if $n_1 = 1$ is supported straightforward.

Finally, to show the SPNE in the whole game, proceed as follows. Choose a price p s.t. $\bar{c} \leq p \leq p_1^m$, find the conditions on $(\beta_{ij}(p))_{ij}$ for which the price p can sustained in equilibrium. We just made sure that such conditions exist. Now the equilibrium is as follows

- (a) in stage 1, choose $(\beta_{ij}(p))_{ij}$,
- (b) if $n_1 > 1$, in stage 2 choose $p_i = p$ if $(\beta_{ij})_{ij} = (\beta_{ij}(p))_{ij}$, and $p_i = c_i$ otherwise for all $i \in N$ or
- (c) if $n_1 = 1$, in stage 2 choose $p_i = p$ if $(\beta_{ij})_{ij} = (\beta_{ij}(p))_{ij}$, and $p_i = \max\{c_2, c_i\}$ for all $i \in N$

If $n_1 > 1$ Firms in N_1 do not deviate to a different β_{ij} as that will imply zero profits in the second stage.

If $n_1 = 1$, Firm 1 will not be willing to deviate if β_{1j} is small enough. The best deviation implies not to share profits with any rival, thus getting $P'_1 = (c_2 - c_1)(1 - c_2)$ in the second stage, which is lower than $P_1 = (1 - \sum_{j=2}^n \beta_{1j})(p - c_1)(1 - p)$ if β_{1j} are small enough as $p > c_2$. ■