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ARIMA MODELS, THE STEADY STATE OF ECONOMIC VARIABLES AND THEIR ESTIMATION

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Abstract

This paper presents a procedure to breakdown the forecast function for a base period t of an ARIMA model in terms of its permanent and transitory components. The former is an estimate of the equilibrium level or steady state path of the corresponding economic variable and the latter describes the approach towards the permanent component. Within the permanent component a distinction is made between the factors which depend on the initial conditions of the system, and those which are deterministic.

Key words
Forecasting function; Long Term Growth; Seasonal Components; Trends.

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1. INTRODUCTION

A disadvantage frequently attributed to ARIMA models is the difficulty involved in interpreting them in terms of the classical trend, seasonal and irregular components (Chatfield, 1977; Harvey and Todd, 1983). Though it is well known (Box and Jenkins, 1976) that the forecasting function of a seasonal multiplicative ARIMA model can be represented as a combination of an adaptative trend and a seasonal component, until the work of Box, Pierce and Newbold (1987), no simple, direct procedures had been developed for determining these components. These authors use the eventual forecasting function together with signal extraction theory to perform a breakdown of the series into its components and to detail its application for the IMA model (1.1) x (1.1), commonly known as the airline model.

In this work these ideas are generalised to obtain a breakdown of the forecasting function into a permanent term, which is the one produced by the model's non-stationary structure, and a transitory term, which is the one produced by the stationary operators. In seasonal series the permanent term can be easily broken down into a trend component and a seasonal component.

Calculating these components has three important advantages: it makes interpretation of the model easier; it is a useful diagnostic tool for identifying interventions which may affect trend or seasonal nature; and it offers a means of comparison between ARIMA models and models in state representation space in their Bayesian version (Harrison and Stevens, 1977) or the structural one (Harvey and Todd, 1983).

The work is structured as follows: in Section 2 the permanent and transitory components of a prediction function of an ARIMA model are defined, and the breakdown of the permanent component into trend and seasonal factors is described. In section three an economic interpretation of the components of the forecasting function is given. In section four these components are determined on the basis of the ARIMA model's predictions, by solving a system of linear equations, and section five presents some applications.

2. SEASONAL MODELS, THEIR FORECASTING FUNCTIONS AND THEIR COMPONENTS

2.1. Seasonal models and their forecasting functions

As it is well known, according to Wold's theorem every linear stationary stochastic process without deterministic components can be represented by:

$$X_t = \Psi(L)\ a_t\ , \qquad (2.1)$$

where $\Psi(L)=1+\Psi_1 L+\Psi_2 L^2+\ldots$ is an infinite convergent polynomial in the lag operator L, and a_t is generated by a white noise stochastic process. Approximating this polynomial by means of a ratio of two finite order polinomials the result is the ARIMA representation,

$$\Phi(L)\ X_t = \Theta(L)\ a_t\ , \qquad (2.2)$$

where $\Psi(L)=[\Phi(L)]^{-1}\Theta(L)$, and the operator $\Phi(L)$ has all the roots outside the unit circle so that the process is stationary. The previous formulation is extended to non-stationary processes, by allowing one or more roots of the operator $\Phi(L)$ to lie on the unit circle. For seasonal processes, Box-Jenkins (1970) simplify (2.2) by factorising the polynomials in two operators one on L and another on L^s, where s is the seasonal period. These two contributions are backed by the factorisation properties of polynomial operators which as we shall see, are crucial in determining the structure of the model.

In general, a seasonal multiplicative ARIMA model is represented by :

$$\phi_p(L)\Phi_P(L^s)(\Delta^d\Delta_s X_t-\mu)=\Theta_q(L)\Theta_Q(L^s)a_t \ , \qquad (2.3)$$

where $\Delta=1-L$ is the regular difference operator, $\Delta_s=1-L^s$ is the seasonal difference operator, μ is the mean of the stationary series, $\phi_p(L)$ and $\Theta_q(L)$ are finite operators (with roots outside the unit circle) in the lag operator L and $\Phi_P(L^s),\Theta_Q(L^s)$ are the seasonal operators on L^s , also with stationary roots. Calling

$$\tilde{\phi}_r^*(L) = \phi_p(L)\Phi_P(L^s)\ \Delta^d\Delta_s \ ; \qquad r=p+d+s(1+P)$$

$$\Theta_m^*(L) = \Theta_q(L)\Theta_Q(L^s) \ ; \qquad m=q+sQ$$

$$c = \phi_p(L)\ \Phi_P(L^s)\ \mu = \phi^*(L)\ \mu$$

and $\hat{X}_t(\ell)$ the prediction of $X_{t+\ell}$ from the origin t, we have that this prediction is given by:

$$\hat{X}_t(\ell) = \sum_{i=1}^{r} \tilde{\phi}_i^* \hat{X}_t(\ell-i) + \sum_{j=1}^{m} \Theta_j\ a_{t+\ell-j} + c \ , \qquad (2.4)$$

where the predictions $\hat{X}_t(\ell-i)$ coincide with the values observed when the horizon is negative and the disturbances $a_{t+\ell-j}$ are zero if $\ell>j$ and they coincide with the estimated values if $j>\ell$.

For $\ell>m$ the MA part of the model will have no effect on prediction. Consequently, for a relatively far-off time horizon the so-called eventual forecasting function is obtained in which:

$$\hat{X}_t(\ell) = \tilde{\Phi}_1^* \ \hat{X}_{t(\ell-1)} + \ldots + \tilde{\Phi}_r \ \hat{X}_{t(\ell-r)} + c \ . \qquad (2.5)$$

The solution of this difference equation provides the structure of the forecasting function. To obtain this solution we are going to use the following theorem.

Theorem

Let the homogenous difference equation be

$$A(L)X_t = 0 \qquad (2.6)$$

where $A(L)=1+a_1L+\ldots+a_kL^k$ is a finite polynomial in the lag operator which can be factorised as:

$$A(L) = P(L) \ \ Q(L), \qquad (2.7)$$

where the polynomials P(L) and Q(L) are prime (they do not have common roots. Then, the general solution to this equation can always be written as:

$$Z_t = p_t + q_t, \qquad (2.8)$$

where the sequences p_t and q_t are the solutions to each prime polyomial, that is to say:

$$P(L) \ p_t = Q(L) \ q_t = 0, \qquad (2.9)$$

The proof of this theorem is given in the appendix.
To apply this theorem let us note that the eventual forecasting function can be written, for $\ell > m$:

$$\Phi^{*}(L)\ (\Delta^{d}\Delta_{s}\ \hat{X}_{t}(\ell) - \mu) = 0\ , \tag{2.10}$$

where $\Phi^{*}(L)=\Phi_{p}(L)\ \Phi_{P}(L^{s})$ and the L operator acts on the index ℓ and t, the origin of the prediction, is fixed. The stationary operator $\Phi^{*}(L)$ has all the roots outside the unit circle, the operator $(1-L)^{d}$ has a unit root repeated d times, while the operator $(1-L^{s})$ can be written:

$$(1-L^{s}) = (1-L)\ S(L) \tag{2.11}$$

where:

$$S(L) = (1 + L +...+ L^{s-1})\quad .$$

This operator has s-1 roots, all of them in the unit circle. If s is even, these s-1 roots include L=-1 and other s-2 complex conjugated roots with a unit module and distributed symmetrically in the unit circle. Consequently, the stationary operators $\Phi^{*}(L)$ and the non-stationary ones $\Delta^{d}\Delta_{s}$ have no root in common and the eventual forecasting function can always be broken down into two components:

$$X_{t}(\ell) = P_{t}(\ell) + t_{t}\ (\ell)\ ,$$

where

(1) $P_{t}(\ell)$ is the permanent component of the long-term forecast, which is determined only by the non-stationary part of the model and is the solution to:

$$\Delta_{d}\Delta_{s}\ P_{t}(\ell) = \mu \tag{2.12}$$

(2) $t_t(\ell)$ is the transitory component, which is determined by the stationary autoregressive operator. This component defines how the approach towards the permanent component tends to be produced. The transitory component is defined by means of the equation:

$$\Phi(L)\ \Phi(L^s)\ t_t(\ell) = \Phi^*(L) t_t(\ell) = 0\ . \qquad (2.13)$$

Now we will study the form of these components on the basis of the ARIMA model and in section four we analyse how to calculate them.

2.2 The transitory component

The transitory component of the eventual forecasting function is the solution to (2.13). The general solution to this homogenous difference equation, supposing that the n=p+P.s roots of the polynomial $\Phi^*(L)$ are different, is

$$t_t(\ell) = b_1^{(t)}\ G_1^{\ell} + \ldots + b_n^{(t)}\ G_n^{\ell}\ , \qquad (2.14)$$

where $G_1^{-1}, \ldots, G_n^{-1}$ are the roots of the autoregressive polynomial and $b_j^{(t)}$ are coefficients depending upon the origin of the prediction. Since, by hypothesis, the operator AR is stationary, its roots will be outside the unit circle or,which is equivalent, the terms G_j are all in module less than the unit. Consequently,:

$$\lim_{\ell \to \infty} t_t(\ell) = \Sigma\ b_j^{(t)} \lim_{\ell \to \infty} G_j^{\ell} = 0 \qquad (2.15)$$

and the transitory component will be zero in the long term. This same reasoning is valid when h identical roots exist, since in that case the term associated with those h equal roots, G , will be:

$$[b_1^{(t)}+b_2^{(t)}\ell +\ldots+ b_h^{(t)}\ell^{h-1}]G_h^{\ell} ,$$

which will tend once more to zero when $\ell\to\infty$ if $|G_h|<1$.

Consequently, the transitory component specifies how the transition towards the permanent component is produced and disappears for high prediction horizons.

2.3. The permanent component

By using the factorisation (2.11) the permanent component of the long-term forecasting function can be written (2.12) as follows:

$$\Delta^{d+1}\ S(L)\ P_t(\ell) = \mu\ . \qquad (2.17)$$

According to the theorem of the previous section the solution to this equation can in turn be broken down into two terms associated with the prime polynomials Δ^{d+1} and S(L), the first of which we will call trend component, T_t, and will be the solution of:

$$\Delta^{d+1}\ T_t(\ell) = c\ , \qquad (2.18)$$

where $c = \mu/s$ and the second term we will call seasonal component, E_t, and is the solution of:

$$S(L)\ E_t(\ell) = 0\ . \qquad (2.19)$$

It can be immediately checked that these components satisfy the equation (2.17). In the following section its properties are analysed.

2.4. The trend component

The trend component of the model is the solution of (2.18) which can be written:

$$T_t(\ell)=c_0^{(t)}+c_1^{(t)}\ell+\ldots+c_d^{(t)}\ell^d+c^*\ell^{d+1} \qquad (2.20)$$

where $c^* = \dfrac{\mu}{s(d+1)!}$,

and is a polynomial of degree d+1 with coefficients varying with the origin of the prediction, except for the latter which is constant and equal to c^*. The trend, therefore, of an ARIMA model is always polynomial: if there are no seasonal differences and $\mu=0$, the order of the polynomial is d-1, whilst in the same case if $\mu\neq0$ the order is d. When there is a seasonal difference the trend polynomial is of degree d, if $\mu=0$, and d+1 if $\mu\neq0$

2.5. The seasonal component

The seasonal component of the model is the solution of (2.19) which is any function of period s with values summing zero each s lags.
We will call

$$S_\ell^{(t)} = E_t(\ell) \;, \qquad \ell=1,\ldots s \;,$$

the s solutions of the equation (2.19), which are the seasonal coefficients of the forecasting function. It should be noted that the seasonal coefficients observe the restriction

$$\sum_{\ell=1}^{s} S_\ell = 0 \;,$$

so that when they are unknown we only have (s-1) unknown factors.

The superindex t in the seasonal coefficients indicates that these coefficients vary with the origin of the prediction and are updated as new data are received. The seasonal coefficients will be determined from the initial conditions, as we shall see in section four.

2.6 The long-term forecasting function

As we have seen, in the long term the transitory component of the eventual forecasting function is made zero and only the permanent component remains, that is for a very large ℓ

$$\hat{X}_t(\ell) = T_t(\ell) + E_t(\ell) ,$$

where $T_t(\ell)$ is a polynomial trend and $E_t(\ell)$ is the seasonal component which is repeated every s periods.

3. ECONOMIC INTERPRETATION OF THE COMPONENTS OF THE UNIVARIATE FORECASTING FUNCTION

In this section we attempt to analyse what type of information is provided by the forecasting function of the previous section corresponding to an economic variable. The prediction is the future value which the variable X would have if no type of innovation occurred from the moment in which the prediction is made. Consequently, the predictions described in the previous section for different values of ℓ are the expectations held in the moment t on the values of the variable in t+1, t+2, ..., . It must be noted that these expectations are constructed by using exclusively information on the history of the phenomenon in question.

Supposing that the parameters of the ARIMA model are known, the value $X_{t+\ell}$ can be broken down in the following way:

$$X_{t+\ell} = \hat{X}_{t+\ell} + e_{t+\ell} \quad , \qquad (3.1)$$

where with $e_{t+\ell}$ we denote the prediction error which is equal to

$$e_{t+\ell} = a_{t+\ell} + \Psi_1 a_{t+\ell-1} + \ldots + \Psi_{\ell-1} a_{t+1} \quad . \qquad (3.2)$$

The breakdown (3.1) divides the observed value $X_{t+\ell}$ into two parts which are mutually independent:

- $\hat{X}_{t+\ell}$: <u>expectation</u> for $X_{t+\ell}$ which we have in the moment t;

- $e_{t+\ell}$: <u>effect of the surprises</u> which occurred between t+1 and t+ℓ, which is obtained as a weighted sum of the corresponding innovations.

When ℓ tends to infinite, the forecasting function indicates the value to which the long-term variable tends, if in the future the stochastic innovations or disturbances which affect the system were zero. Therefore, giving to ℓ high and ever higher values the forecasting function describes the <u>long-term equilibrium path</u> of the economic variable in question. Thus, if the limit of the forecasting function is a constant we will conclude that the variable tends to a stable equilibrium, while on the contrary, if this function has no limit we will say that the variable tends to a situation of steady state.

In conclusion we have that the economic importance of the forecasting function obeys two fundamental causes: on the one hand it enables us to quantify the different term univariate expectations for a particular phenomenon; on the other, it describes the long-term equilibrium path towards which this phenomenon is moving, and this is given by the trend of the forecasting function.

In adding comment to the concept of integrated variables defined in Engle and Granger (1987) and Escribano (1987) we will say that a variable generated by an ARIMA model is integrated of order (h, 1)if it needs to be differentiated h times to become stationary and the stationary transformation has a mathematical expectation different to zero. If in the previous case the mathematical expectation of the stationary transformation is zero we will say that the variable is integrated of order (h, 0). In general the order of integration is represented by (h, m), where m takes the value zero or one according to whether the mathematical expectation of the stationary transformation is nil or not. From what has been seen in the previous section, we have that the order of integration fully describes the polynomial structure of the trend of the forecasting function, which will be of the order max (o,h+m-1). The trend is purely stochastic, in the sense that all its coefficients are determined by the initial conditions of the system, if m is zero, and it is mainly deterministic if m is different to zero.

This definition of integration makes explicit the presence or otherwise of a constant in the stationary series due to the importance which, as we shall see, this parameter has.

If h+m adds up to zero or one the variable tends to a stable equilibrium, the value of which will be purely deterministic if h is zero, or it will be determined by the initial conditions if h is one.

If h+m adds up to more than one, the variable does not tend to a stable value, but evolves according to a polynomial structure which accords it a steady state. In this polynomial structure the most important thing in the long-term is the coefficient corresponding to the greatest power, since compared to it all other powers have a negligible contibution. Now, this coefficient will be deterministic if m is one, in which case the long-term path will also be so. This means that the factor which contributes most to this path is not altered by changes in the conditions of the system, and therefore the development of an Economic Theory to explain in this case the long term path of a variable, i.e. consumption, in terms of another one, i.e. income, is not of much help. On the contrary, if m is zero all the parameters of the trend of the forecasting function depend on the initial conditions of the system. In such cases the long-term law is determined by a time polynomial of the order (h-1), but the parameters of this polynomial change as new disturbances reach the system.

To complete the description of the long term of an economic variable we must specify the magnitude of the uncertainty we have about it. This uncertainty is expressed by the term $e_{t+\ell}$ in (3.1) when ℓ tends to infinite. If the process is stationary h=0, the polynomial $\Psi(\ell)$ which enters in the definition (3.2) of $e_{t+\ell}$ is convergent and the variance of $e_{t+\ell}$ when ℓ tends to infinite is finite. This result is certain even when bearing in mind the uncertainty associated with the estimation of the parameters (see Box and Jenkins (1970) appendix A7.3). In such a case we say that the uncertainty regarding the future, however far off it may be, is limited. If h is not zero, $\Psi(\ell)$ does not converge, and the variance of $e_{t+\ell}$ tends to infinity with ℓ, so that we say that uncertainty regarding the future is not bounded. It is worth pointing out that the fact that ARIMA models generate, for the case of non-stationary series, predictions regarding the future whose uncertainty is not bounded as the horizon of prediction (ℓ) increases is not a disadvantage of these models, since the nature of uncertainty regarding the future is not a characteristic which indicates to us whether the model is good or bad, but an aspect which defines the real world which we are attempting to make a model of.

In economics the hypothesis that uncertainty regarding the future is not limited seems acceptable. Note that in a structural economic model (SEM) where exogenous variables are generated by non-stationary ARIMA models, long-term predictions are also generated on the endogenous variables with non-bounded uncertainty. The difference with respect to the ARIMA predictions can be found simply in the fact that the uncertainty may tend to infinite more slowly and with a greater delay.

The characteristics of the long-term path deriving from the models from the ARIMA models with the most common values for h and m are shown in Table 1.

An ARIMA model with (h+m=2) implies that in the long term the level of the corresponding variable tends to infinite. Such a characteristic may be considered as unacceptable in Economics, but note that simply substituting one of the positive unit roots included in the differentiations by $(0.99)^{-1}$ will be sufficient for the law of the long-term to become a stable equilibrium. But, the way in which this stable equilibrium is achieved depends on the transitory component of the prediction function $t_t(\ell)$, defined in (2.13). As this component will in this case have a term $b_j^{(t)}$ $(0.99)^{\ell}$ it will not be cancelled out in the medium term and this facet, in practice, will not be able to be distinguished from the first mentioned one in which (h+m) equalled two. In fact the long term in Economics cannot be estimated, since it is not possible to discriminate, with the available sample sizes, between a fixed structure and one which is slowly evolving. Therefore, when we say that an economic variable follows a linear growth path we mean simply that in the medium term it tends to follow such a behaviour path.

From Table 1 it follows that <u>the inclusion of constants in the ARIMA model means severe restrictions on the characterisation of the long term of an economic variable</u>.

Having seen that the parameters of the forecasting function of an ARIMA model, and specifically the slope of the trend of the permanent component change with time, it is important to analyse how we can calculate them. The next section is devoted to this topic.

Table 1

4. THE DETERMINATION OF THE COMPONENTS OF THE FORECASTING FUNCTION

4.1 General Approach

The results of the previous sections indicate that the eventual forecasting function of a seasonal ARIMA model can be written:

$$\hat{X}_t(\ell) = \sum_{j=0}^{d} c_j^{(t)} \ell^j + S_\ell^{(t)} + \sum_{i=1}^{n} b_i\, G_i^\ell , \qquad (4.1)$$

to simplify the analysis we are assuming that $\mu=0$ and $D=1$. Then $n=p+P.s$ and the equation is valid for $\ell>q+sQ$. However, $d+1+p+sP$ initial values are required to determine it, therefore, from $K>q+sQ-d-1-p-sP$ the predictions will already be related among each other according to (4.1). The coefficients of this equation can be obtained through two different procedures: the first is to generate as many predictions as parameters and to solve the resulting system of equations. The equation (4.1) has d parameters c_j, $s-1$ seasonal parameters (since a coefficient can be expressed as the sum of the others with a changed sign) and n coefficients b_i. Therefore, we need to generate a number of predictions equal to $R=d+1+s-1+n=d+s+n$. Calling the prediction vector $\underset{\sim}{\hat{X}}_{t+R}$ and the parameter vector $\underset{\sim}{\Theta}$ we can write from a certain moment ℓ the following expression:

$$\begin{bmatrix} \hat{X}_t(\ell) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \hat{X}_t(\ell+R) \end{bmatrix} = \left[\begin{array}{cccc|cccc|cccc} 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & & G_1 & G_2 & \cdots & G_n \\ 1 & 2 & & 2^d & 0 & 1 & \cdots & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & & & & & & & \\ \vdots & \vdots & & \vdots & 1 & & & & & & & \\ \vdots & \vdots & & \vdots & 1 & 0 & & & & & & \\ \vdots & \vdots & & \vdots & & 1 & & & & & & \\ 1 & R & & R^d & & & & & G_1^R & G_2^R & & G_n^R \end{array}\right] \begin{bmatrix} c_0^{(t)} \\ \vdots \\ c_d^{(t)} \\ S_1^{(t)} \\ S_{s-1}^{(t)} \\ b_1^{(t)} \\ b_n^{(t)} \end{bmatrix}$$

and the coefficient $S_s^{(t)}$ will be equal to

$$- \sum_{j=1}^{s-1} S_j^{(t)} .$$

Writing

$$\hat{\underset{\sim}{X}}_{t+R} = \underset{\sim}{M}\,\underset{\sim}{\theta} \quad ,$$

where $\underset{\sim}{M}$ is the data matrix which contains the known coefficients which multiply the parameter vector $\underset{\sim}{\theta}$ we can express $\underset{\sim}{\theta}$ as:

$$\underset{\sim}{\theta} = \underset{\sim}{M}^{-1}\,\hat{\underset{\sim}{X}}_{t+r} \quad , \tag{4.2}$$

which enables all the parameters for the eventual forecasting function to be obtained.

The second procedure is first to obtain a value r high enough for the transitory component to be cancelled out for k>r. This value depends on the roots of the autoregressive polynomial and is determined in such a way that $|G_1^r| \simeq 0$, where G_1 is the G_i with the highest absolute value. A simple way of checking whether the transitory component is practically nil for K>j.s, consists of taking the differences:

$$\hat{X}_t((j+1)\ s+k) - \hat{X}_t(js+k) \ ,$$

which will be free of the seasonal effect, and to observe whether such a difference stays practically constant for positive values of K. In this case we shall say that from a prediction horizon j.s the transitory component is practically nil. For example this implies that with monthly data the annual differences between the monthly predictions will be constant from a certain year onwards. Thus taking the expression of the general predictions and eliminating from it the transitory component we can set up a system of equations to determine the coefficients of the trend equation and the seasonal coefficients, which are in general those of interest. Let us see some specific cases.

4.2 The airline model

A seasonal ARIMA model much used for representing the evolution of monthly economic series is the one called the airline model:

$$\Delta\Delta_{12} X_t = (1-\Theta_1 L)(1-\Theta_{12} L^{12}) a_t \quad . \qquad (4.3)$$

According to what has been previously discussed, the forecasting equation of this model for k>0 can be written;

$$X_t(k) = b_o^{(t)} + b_1^{(t)} k + S_k^{(t)}$$

and contains 13 parameters. (Remember that $\Sigma S_k^{(t)}=0$).
By equalling the predictions for k=1, 2,...,13 obtained with the model (4.3) with the structural form, we shall have:

$$\begin{bmatrix} \hat{X}_{t+1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \hat{X}_{t+12} \\ \hat{X}_{t+13} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & \dots & 0 \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ 1 & 12 & 0 & 0 & \dots & 1 \\ 1 & 13 & 1 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} b_o^{(t)} \\ b_1^{(t)} \\ S_1^{(t)} \\ \cdot \\ \\ S_{12}^{(t)} \end{bmatrix} ,$$

a system of 13 equations and 14 unknowns which with the restriction $\Sigma S_j^{(t)}=0$ enables the parameters $b_o^{(t)}$, $b_1^{(t)}$ and the seasonal coefficients $S_j^{(t)}$ to be obtained. By subtracting the first equation from the last and dividing by twelve, we obtain directly

$$b_1^{(t)} = \frac{\hat{X}_t(13)-\hat{X}_t\ (1)}{12} \quad . \tag{4.4}$$

By adding up the first 12 equations the seasonal coefficients are cancelled out and we obtain:

$$\hat{X}_t = \frac{1}{12} \sum_{1}^{12} X_t(K) = b_o^{(t)}+b_1^{(t)}\left(\frac{1+...+12}{12}\right) ,$$

which gives as a result

$$\hat{b}_o^{(t)} = \hat{\bar{X}}_t - \frac{13}{2}\hat{b}_1 \quad . \tag{4.5}$$

Fynally the seasonal coefficients are obtained by:

$$S_j^{(t)} = \hat{X}_t(j) - \hat{b}_o^{(t)} - \hat{b}_1^{(t)} j \quad . \tag{4.6}$$

It must be noted that if the ARIMA model is specified on the logarithmic transformation of X, then the coefficients $b_1^{(t)}$ can be interpreted as growth rates and the coefficients S_j measure the seasonal nature as a percentage of one on the level of the series.

4.3. General models with a difference of each type

Any ARIMA model which has as non-stationary operators $\Delta\Delta_s$ and $\mu=0$ has a permanent component of the forecasting function which is the sum of a linear trend and a stable seasonal component. To determine the parameters b, which measure the linear trend and the seasonal coefficients S_j, we will use the fact that, taking K=si+j as high enough for the stationary terms to be negligible, and equalling the predictions to the permanent component:

$$\hat{b}_1^{(t)} = \frac{\hat{X}_t(K+s+1)-\hat{X}_t(K+1)}{s} \qquad (4.7)$$

$$\hat{b}_o^{(t)} = \hat{\bar{X}}_t - \hat{b}_1\left(K + \frac{s+1}{2}\right) \qquad (4.8)$$

$$S_j^{(t)} = \hat{X}_t\ (K+j) - \hat{b}_o - \hat{b}_1\ (K+j) \qquad (4.9)$$

equations analogous to those of (4.4) and (4.6), where now $\hat{\bar{X}}$ is the average of the s observations in the interval (K+1, K+s).

5. APPLICATION OF THE CALCULATION OF THE TREND OF THE UNIVARIATE FORECASTING FUNCTION TO SERIES ANALYSIS OF THE SPANISH ECONOMY

In this section an estimate is made, for a certain sequence of months, of growth rates in the trend of the forecasting function of the following series of the Spanish economy: imports, exports and the consumer price index for services. The use of the above-mentioned rate in a relatively complete short-term analysis of an económic phenomenon is put forward and described in Espasa (1990). Following the terminology used in the above-mentioned work, we will call it inertia to the rate of growth of the trend of the forecasting function of a univariant ARIMA model which will be given by the parameter b_1, defined in (4.7), when the model is specified on the logarithmic transformation of the variable.

Regarding Spanish foreign trade on non-energy goods the following univariate monthly models can be used to explain imports (M) and exports (X)

$$\Delta\Delta_{12}\ln M_t = \Delta\Delta_{12}AIM_t + (1-0'77L)(1-0'72L^{12})a_t \ , \qquad (5.4)$$

$$\sigma = 0'092 \ ,$$

$$\Delta\Delta_{12}\ln X_t = \Delta\Delta_{12}AIX_t + (1-0'83L)(1-0'72L^{12})a_t \ , \qquad (5.5)$$

$$\sigma = 0'117 \ ,$$

in which AIM and AIX are particular intervention analyses requiring both series and which have no effect upon the slope of the forecasting function, therefore, henceforth we will ignore them.

The original series with their corresponding trends are given in Figure 1 and the inertias are show in Figure 2.

Figure 1

The last figure shows these inertias from January 1986, the month in which Spain joined the EEC, to December 1989. Thus, this chart can be used to illustrate what happened to Spanish overseas trade, in nominal terms from that date. Obviously, on the basis of this description no causal analysis can be made, since we are not using models incorporating the determining variables of M and X with which an analysis can be made of which explanatory variables are responsible, and to what extent, for the trend changes. Nevertheless, the mere description of these changes is in fact of interest in itself. However, it must be pointed out that the trend evolutions shown in Figure 2 refer to the sale and purchase of goods in nominal terms and, therefore, prices are also influencing these very same trend movements.

We can deduce from Figure 2 that trend growth expectation in nominal imports was increasing systematically throughout 1986 and first three quarters of 1987, that then this expectation has stabilised, with minor oscillations, at around 23% until the second half an 1989, when it started the decrease very slowly. As a result, a worsening of perspectives for Spanish imports of around four percentual points has occurred during this period.

Figure 2

With exports there was a movement from a growth expectation of 18% at the beginning of 1986 to an expectation of around 14% at the end of that year and during 1987. Since then the expectations have remained fairly stable.

In conclusion we can say that the perspectives for imports worsened (increased) progressively in 1986 and 1987 taking on a relatively stable evolution from that time until the second half of 1989 when a certain improvement occurred. As for expectations for exports,although they worsened (declined) during 1986, they have maintained a fairly high level during the last three years of the sample considered. If the evolutions of imports and exports are compared in order to have a better understanding of of the possible evolution of the Spanish trade deficit, a conclusion can be drawn to the effect that, it is necessary, given that the level of imports is higher than that of exports, at least for the growth rates expected in both series to equal each other fairly quickly, and, insofar as export growth can be considered as optimistic, given the level of world commercial activity and the relative level of Spanish prices compared to the rest of the world, to bring these rates together must perforce require a significant reduction in import growth.

In the consumer price index for the Spanish economy the component referring to the prices of services, which we shall call IPCS, has been showing fairly uneven behaviour with regard to the component referring to the prices for non-energy manufactured goods. Both components make up the IPSEBENE, the consumer price index for services and non-energy manufactured goods, which represents 77.54% of the IPC, and is an appropriate index on which it is worthwhile analysing underlying inflation or the inflationary trend.

By using the sample comprising from May 1984 to January 1989 the following model has been estimated

$$\Delta\Delta_{12}\ln IPCS_t = AIS_t + \frac{1 - 0'85L^{12}}{1 - 0'32L^{3}} a_t \tag{7.6}$$

$\sigma = 0'0014$,

where AIS_t are interventions required by this index. These interventions include a step effect which begins in January 1986 and is due to the introduction of Value Added Tax. The moving average coefficient is fixed at 0.85.

From this model a calculation has been made of the inertia of the IPCS,(corrected of interventions) during the period comprising from January 1986 to December 1989. These calculations are shown in Graph 3. There it can be seen that during these years the medium-term growth expectations of this index have always remained above 7%. It can also be detected that during 1986 expectations on this index increased. That is to say, unlike what occurred with the prices of non-energy manufactured goods, Spain's entry in the EEC meant no improvement in expectations for the prices of services. This result is not surprising if it is borne in mind that entry scarcely brought with it greater competitiveness in the Spanish service sector. Figure 3 also shows that throughout 1987 there was a slight improvement (fall) in the IPCS inertia, which disappeared completely in 1988, and in 1989 this deterioration in the prices of services continued. All this represents a grave threat to the IPC since the services component accounts for 34.24% of this index.

Figure 3

ACKOWLEDGEMENT

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APPENDIX 1

Demonstration of the theorem in Section 2

It can be immediately proved that the condition is sufficient and that (2.3) is a solution of (2.1). Because of the commutability of the operators

$$P(B)Q(B)Z_t=P(B)Q(B)(p_t+q_t)=Q(B)P(B)p_t+P(B)Q(B)q_t=0$$

Let us now prove that the condition is necessary, that is, that any solution of (2.1) can be written as in (2.3). From Bezout's theorem, if $P(B)$ and $Q(B)$ are prime two polynomials exist $T_1(B)$, $T_2(B)$ such that:

$$1 = T_1(B)P(B) + T_2(B)Q(B)$$

therefore:

$$Z_t = T_1(B)P(B)Z_t + T_2(B)Q(B)Z_t$$

calling

$$T_1(B)P(B)Z_t = q_t \tag{A.1}$$

$$T_2(B)Q(B)Z_t = p_t \tag{A.2}$$

is verified

$$Z_t = q_t + p_t$$

multiplying both members of (A.1) and (A.2) by Q(B) and P(B) respectively:

$$Q(B)q_t = T_1(B)P(B)Q(B)Z_t = T_1(B)A(B)Z_t=0$$

$$P(B)p_t = T_2(B)P(B)Q(B)Z_t = T_2(B)A(B)Z_t=0$$

and therefore any solution can be written in the form (2.3) and (2.4) indicated in the theorem. Let us prove that the breakdown is unique. Let us assume another breakdown:

$$Z_t = q'_t + p'_t$$

where q'_t and p'_t verify (2.4). Then:

$$q'_t = T_1(B)P(B)q'_t+T_2(B)Q(B)q'_t=T_1(B)P(B)q'_t=$$

$$= T_1(B)P(B)(q'_t+p'_t) = T_1(B)P(B)(q_t+p_t)=$$

$$= T_1(B)P(B)q_t = (1-T_2(B)Q(B))q_t = q_t$$

analogously it is proved that p_t must be identical to p'_t.

REFERENCES

Box, G.E.P., y G.M. Jenkins (1976), Time series Analysis, Forecasting and Control, Holden Day.

Box, G.E.P., Pierce, D. and Newbold, P. 1987, "Estimating trend and growth rate in alasonal time series", J.A.S.A., 82, pp. 276-282.

Chatfield, C. 1977, "Some recent developments in Time series Analysis", J.R.S.S., A, 140, pp. 492-510.

Engle, R.F., y C.W.J. Granger (1987), "Co-integration and Error Correction: Representation, Estimation and Testing" Econometrica, 55, pp. 251-76.

Escribano, A. (1987), "Co-Integration, Time Co-Trends and Error-Correction Systems: an alernative Approach" CORE Discussion Paper no. 8715, Universite Catholique de Louvain.

Espasa, A. (1990), "Univariate Methodology for short-term economic analysis", Banco de España, Working paper 9003, Madrid, Spain.

Harrison, P.J. and Stevens, C.F., 1976, "Bayesian forecasting" J.R.S.S., B., 38, pp. 205-247.

Harvey, A.C. and Todd, P.H.J., 1983 "Forecasting economic Time series with structural and Box Jenkins models", J. of Bus. and Eco. Stat, 1, 4, pp. 299-315.

TABLE 1

Characteristics of the long term path derived from the ARIMA model corresponding to an economic variable

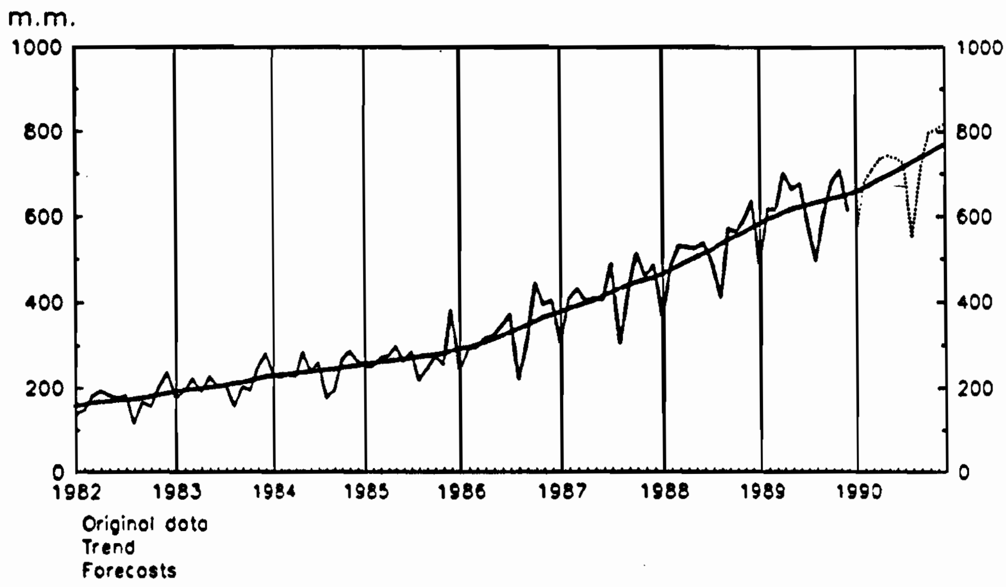
h+m, nature of the long-term path (a)	(h, m) (b)	Influence of initial conditions on the long-term path	Uncertainty regarding long-term	
			On the level	On the growth rates
0. NIL LONG-TERM VALUE	(0,0)	none	finite	nil (growth is zero)
1 ESTABLE EQUILIBRIUM	(0,1)	none	finite	nil (growth is zero)
	(1,0)	they determine the equilibrium value	infinite (it growth lineary with)	nil (growth is zero)
2 LINEAR GROWTH	(1,1)	they determine the ordenate in the origin of the straight line, but have no influence on its slope	infinite (it growht lineary)	finite
	(2,0)	they determine the two parameters which define the line	infinite (it growth quadratically)	infinite (it growth lineary)

(a) h is the total number of differentiations required by the variable to become stationary.

(b) m=0 implies that the mathematical expectancy or the stationary series is not nil.
m=1 implies that this mathematical expectancy is not nil.

FIGURE 1

SPANISH IMPORTS AND EXPORTS OF NON-ENERGY GOODS
(Original data and trend)
Imports



Exports

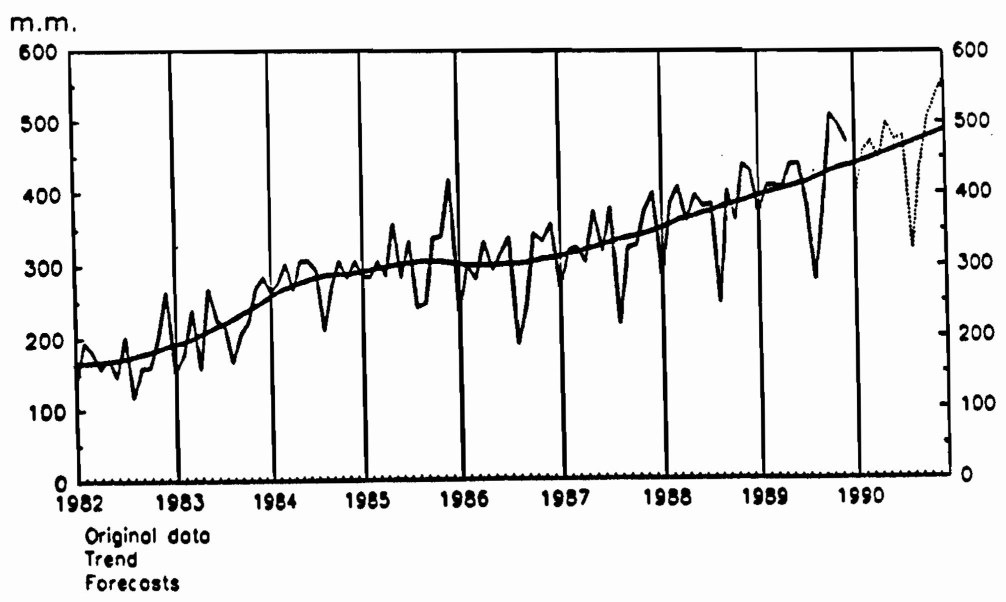
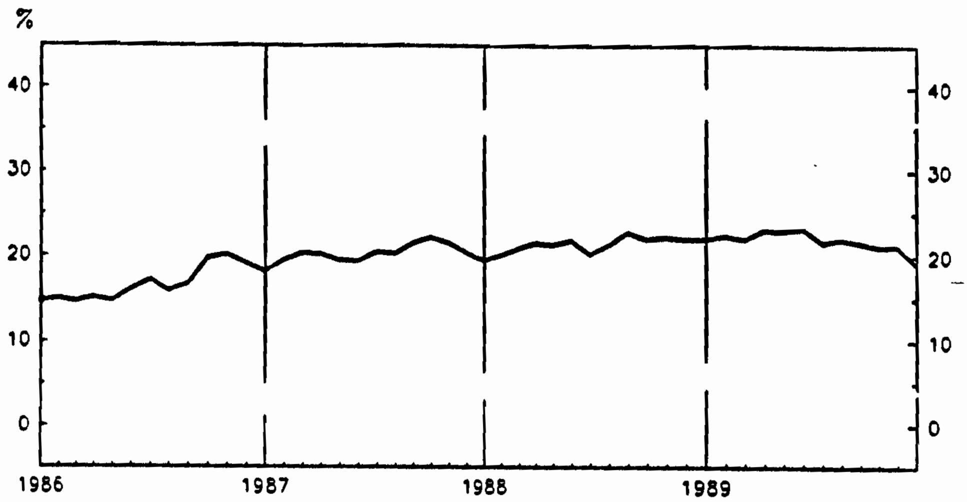


FIGURE 2

SPANISH IMPORTS AND EXPORTS OF NON-ENERGY GOODS
(Original data and trend)
Imports



Exports

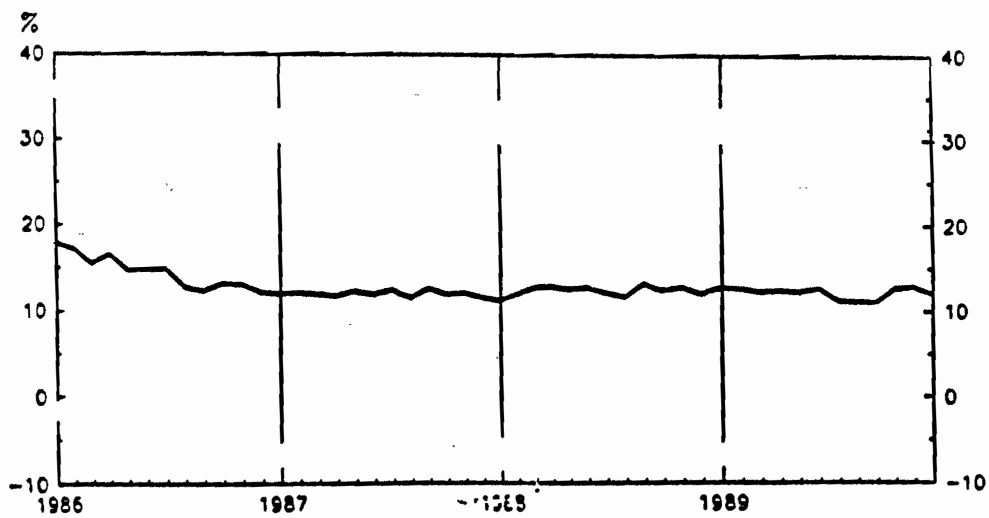
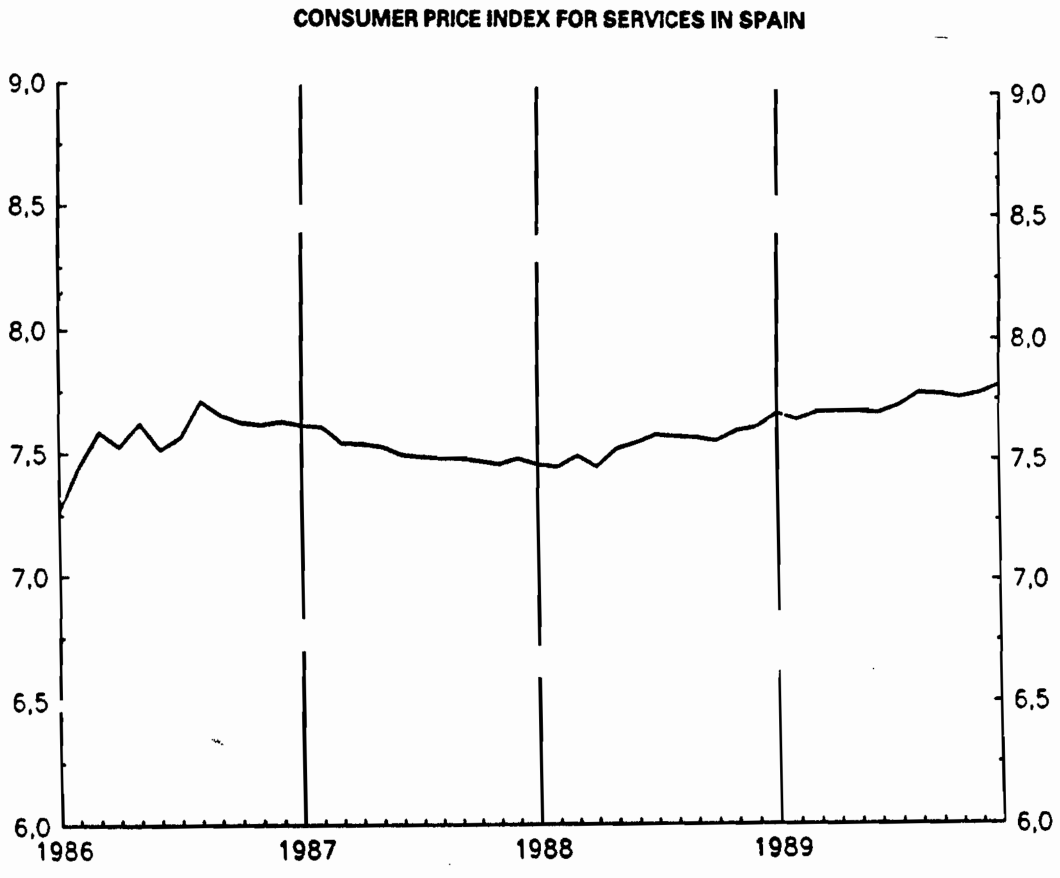


FIGURE 3


CONSUMER PRICE INDEX FOR SERVICES IN SPAIN
9,0
8,5
8,0
7,5
7,0
6,5
6,0
1986
1987
1988
1989