



A CHARACTERIZATION OF GROMOV HYPERBOLICITY OF SURFACES WITH VARIABLE NEGATIVE CURVATURE

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ABSTRACT. In this paper we show that, in order to check Gromov hyperbolicity of any surface with curvature $K \leq -k^2 < 0$, we just need to verify the Rips condition on a very small class of triangles, namely, those contained in simple closed geodesics. This result is, in fact, a new characterization of Gromov hyperbolicity for this kind of surfaces.

Key words and phrases: Gromov hyperbolicity, Riemannian surface, negatively curved Riemannian surface.

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1. INTRODUCTION.

The theory of Gromov hyperbolic spaces is a useful tool in order to understand the connections between graphs and Potential Theory (see e.g. [6], [14], [19], [25], [26], [27], [28], [36], [37], [41]). Besides, the concept of Gromov hyperbolicity grasps the essence of negatively curved spaces, and has been successfully used in the theory of groups (see e.g. [20], [21], [22] and the references therein).

A geodesic metric space is called hyperbolic (in the Gromov sense) if there exists an upper bound of the distance of every point in a side of any geodesic triangle to the union of the two other sides (see Definition 2.3). This condition is known as Rips condition.

In general it is not easy to determine whether a given space is Gromov hyperbolic or not. In recent years several investigators have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For example, the Gehring-Osgood metric ([24]) is always Gromov hyperbolic, and the Klein-Hilbert metric ([10], [29]) is also Gromov hyperbolic for certain domains. However, the Vuorinen metric ([24]) is not Gromov hyperbolic except in a particular case. Recently, Balogh and Buckley [7] have made significant progress on the hyperbolicity problem of Euclidean domains with respect to the quasihyperbolic metric (see also [12], [42] and the references therein). Another interesting instance is that of a Riemann surface endowed with the Poincaré metric. With such metric structure a Riemann surface is always negatively curved, with $K = -1$, but not every Riemann surface is Gromov hyperbolic, since topological obstacles may impede it: for instance, the two-dimensional jungle-gym (a \mathbb{Z}^2 -covering of a surface of genus two) is not hyperbolic. In [4], [30], [31], [32], [38], [40] there are results about the hyperbolicity of Riemannian surfaces.

We are interested in studying when Riemannian surfaces with negative curvature are Gromov hyperbolic. The following theorem is the main result of this paper, which is a new characterization of Gromov hyperbolicity for negatively curved surfaces (see Theorem 4.1):

A Riemannian surface S with $K \leq -k^2 < 0$ is hyperbolic if and only if the triangles contained in simple closed geodesics of S satisfy the Rips condition.

In general, one has to check the Rips condition for all triangles. Our result states that, for negatively curved surfaces, you only have to check it for quite a smaller class of triangles.

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Furthermore, this theorem provides a bound for the hyperbolicity constant: in Riemannian surfaces with $K \leq -k^2 < 0$, if the triangles contained in simple closed geodesics satisfy the Rips condition with constant δ_0 , then every geodesic triangle also satisfies it, with constant $\delta < 62/k + \delta_0$.

This result is known if the curvature K is constantly equal to -1 (see [39]). The argument in the proof of Theorem 4.1 is similar to the one in the proof of the constant curvature case. Unfortunately, *every* standard fact used in the proof in [39] is false when the curvature is not constant (in fact, the proof in [39] uses heavily hyperbolic trigonometry; however, if $K \leq -k^2$ we just have some trigonometric inequalities). Hence, it is necessary to prove alternative results that are valid for variable curvature. Here we develop new techniques that also improve several results in [39]. We also prove in Section 5 several consequences of Theorem 4.1, which are new in the context of variable negative curvature.

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2. BACKGROUND IN GROMOV SPACES AND RIEMANNIAN SURFACES.

In our study of hyperbolic Gromov spaces we use the notation of [20]. We give now the basic facts about these spaces. We refer to [20] for more background and further results.

Definition 2.1. *Let us fix a point w in a metric space (X, d) . We define the Gromov product of $x, y \in X$ with respect to the point w as*

$$(x|y)_w := \frac{1}{2} (d(x, w) + d(y, w) - d(x, y)) \geq 0.$$

We say that the metric space (X, d) is δ -hyperbolic ($\delta \geq 0$) if

$$(x|z)_w \geq \min \{ (x|y)_w, (y|z)_w \} - \delta,$$

for every $x, y, z, w \in X$. We say that X is hyperbolic (in the Gromov sense) if the value of δ is not important.

It is convenient to remark that this definition of hyperbolicity is not universally accepted, since sometimes the word *hyperbolic* refers to negative curvature or to the existence of Green's function. However, in this paper we only use the word *hyperbolic* in the sense of Definition 2.1.

Examples:

- (1) Every bounded metric space X is $(\text{diam} X)$ -hyperbolic (see e.g. [20, p. 29]).
- (2) Every complete simply connected Riemannian manifold with sectional curvature bounded from above by $-k$, with $k > 0$, is hyperbolic (see e.g. [20, p. 52]).
- (3) Every tree with edges of arbitrary length is 0-hyperbolic (see e.g. [20, p. 29]).

Definition 2.2. *If $\gamma : [a, b] \rightarrow X$ is a continuous curve in a metric space (X, d) , the length of γ is*

$$L(\gamma) := \sup \left\{ \sum_{i=1}^n d(\gamma(t_{i-1}), \gamma(t_i)) : a = t_0 < t_1 < \dots < t_n = b \right\}.$$

We say that γ is a geodesic if it is an isometry, i.e. $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$. We say that X is a geodesic metric space if for every $x, y \in X$ there exists a geodesic joining x and y ; we denote by $[x, y]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but convenient as well).

Definition 2.3. *If X is a geodesic metric space and J is a polygon whose sides are J_1, J_2, \dots, J_n , we say that J is δ -thin if for every $x \in J_i$ we have that $d(x, \cup_{j \neq i} J_j) \leq \delta$. If $x_1, x_2, x_3 \in X$, a geodesic triangle $T = \{x_1, x_2, x_3\}$ is the union of three geodesics $[x_1, x_2]$, $[x_2, x_3]$ and $[x_3, x_1]$. The space X is δ -thin (or satisfies the Rips condition with constant δ) if every geodesic triangle in X is δ -thin.*

If we have a triangle with two identical vertices, we call it a “bigon”. Obviously, every bigon in a δ -thin space is δ -thin. It is also clear that every geodesic polygon with n sides in a δ -thin space is $(n - 2)\delta$ -thin.

Definition 2.4. *Given a geodesic triangle $T = \{x, y, z\}$ in a geodesic metric space X , let T_E be a Euclidean triangle whose sides have the same lengths as those of T . Since there is no possible confusion, we will use the same notation for the corresponding points in T and T_E . The maximum inscribed circle in T_E meets the side $[x, y]$ (respectively $[y, z]$, $[z, x]$) in a point z' (respectively x' , y') such that $d(x, z') = d(x, y')$, $d(y, x') = d(y, z')$ and $d(z, x') = d(z, y')$. We call the points x', y', z' , the internal points of $\{x, y, z\}$. There is a unique isometry f of the triangle $\{x, y, z\}$ onto a tripod (a tree with one vertex w of degree 3, and three vertices x'', y'', z'' of degree one, such that $d(x'', w) = d(x, z') = d(x, y')$, $d(y'', w) = d(y, x') = d(y, z')$ and $d(z'', w) = d(z, x') = d(z, y')$). The triangle $\{x, y, z\}$ is δ -fine if $f(p) = f(q)$ implies that $d(p, q) \leq \delta$. The space X is δ -fine if every geodesic triangle in X is δ -fine.*

A basic result is that hyperbolicity is equivalent to Rips condition and to the property of being fine:

Theorem 2.5. ([20, p.41]) *Let us consider a geodesic metric space X .*

- (1) *If X is δ -hyperbolic, then it is 4δ -thin and 4δ -fine.*
- (2) *If X is δ -thin, then it is 4δ -hyperbolic and 4δ -fine.*
- (3) *If X is δ -fine, then it is 2δ -hyperbolic and δ -thin.*

We present now the class of maps which play the main role in the theory.

Definition 2.6. *A function between two metric spaces $f : X \longrightarrow Y$ is a quasi-isometry if there are constants $a \geq 1$, $b \geq 0$ such that*

- (1) $\frac{1}{a} d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq a d_X(x_1, x_2) + b$, for every $x_1, x_2 \in X$,
- (2) for any $y \in Y$, there is some $x \in X$ with $d_Y(y, f(x)) \leq b$.

Such a function is called an (a, b) -quasi-isometry. An (a, b) -quasigeodesic in X is an (a, b) -quasi-isometry between an interval of \mathbb{R} and X . An (a, b) -quasigeodesic segment in X is an (a, b) -quasi-isometry between a compact interval of \mathbb{R} and X .

Notice that a quasi-isometry can be discontinuous.

Quasi-isometries are important since they are maps which preserve hyperbolicity:

Theorem 2.7. ([20, p.88]) *Let us consider an (a, b) -quasi-isometry between two geodesic metric spaces $f : X \longrightarrow Y$. If Y is δ -hyperbolic, then X is δ' -hyperbolic, where δ' is a constant which only depends on δ , a and b .*

Definition 2.8. *Let us consider $H > 0$, a metric space X , and subsets $Y, Z \subseteq X$. The set $V_H(Y) := \{x \in X : d(x, Y) \leq H\}$ is called the H -neighborhood of Y in X . The Hausdorff distance between Y and Z is defined by $\mathcal{H}(Y, Z) := \inf\{H > 0 : Y \subseteq V_H(Z), Z \subseteq V_H(Y)\}$.*

The following is a beautiful and useful result:

Theorem 2.9. ([20, p.87]) *For each $\delta \geq 0$, $a \geq 1$ and $b \geq 0$, there exists a constant $H = H(\delta, a, b)$ with the following property:*

Let us consider a δ -hyperbolic geodesic metric space X and an (a, b) -quasigeodesic g joining x and y . If γ is a geodesic joining x and y , then $\mathcal{H}(g, \gamma) \leq H$.

This property is known as *geodesic stability*. Mario Bonk has proved that, in fact, geodesic stability is equivalent to hyperbolicity [11].

In the next sections of this paper we will work with submanifolds of Riemannian surfaces. There is a natural way to define a distance in subspaces contained in geodesic metric spaces.

Definition 2.10. Let X be a path-connected subset of a Riemannian surface. The intrinsic metric on X is defined as follows:

$$d_X(x, y) := \inf \{L(\gamma) : \gamma \subset X \text{ is a continuous curve joining } x \text{ and } y\}.$$

The following result in [38] shows that we can just deal with triangles whose sides are simple curves.

Lemma 2.11. ([38, Lemma 2.1]) Consider a geodesic metric space X . If every geodesic triangle in X which is a simple closed curve is δ -thin, then X is δ -thin.

Remark 2.12. This theorem allows to avoid self-intersecting triangles.

Definition 2.13. Let (X, d) be a metric space, and let $\{X_n\}_n \subseteq X$ be a family of geodesic metric spaces such that $\eta_{nm} := \eta_{mn} := X_n \cap X_m$ are compact sets. Further, assume that for any n and m the set $X \setminus \eta_{nm}$ is not connected, and that a and b are in different connected components of $X \setminus \eta_{nm}$ for any $a \in X_n \setminus \eta_{nm}$, $b \in X_m \setminus \eta_{nm}$, with $m \neq n$. If there exist positive constants c_1 and c_2 such that $\text{diam}_{X_n}(\eta_{nm}) \leq c_1$ for every n, m , and $d_{X_n}(\eta_{nm}, \eta_{nk}) \geq c_2$ for every n and $m \neq k$, we say that $\{X_n\}_n$ is a (c_1, c_2) -tree decomposition of X .

Theorem 2.14. ([38, Theorem 2.4] and [30, Theorem 2.9]) Let us consider a metric space X and a family of geodesic metric spaces $\{X_n\}_n \subseteq X$ which is a (c_1, c_2) -tree decomposition of X . Then X is δ -hyperbolic if and only if there exists a constant c_3 such that X_n is c_3 -hyperbolic for every n . Furthermore, δ (respectively c_3) is a universal constant which only depends on c_1, c_2 and c_3 (respectively c_1, c_2 and δ).

Definition 2.15. Any divergent curve $\sigma : [0, \infty) \rightarrow Y$, where Y is a noncompact Hausdorff space, determines an end E of Y . Given a compact set F of Y , one defines $E(F)$ to be the arc component of $Y \setminus F$ that contains a terminal segment $\sigma([a, \infty))$ of σ . A set $U \subset Y$ is a neighborhood of an end E if U contains $E(F)$ for some compact set F of Y . An end E in a surface S is collared if E has a neighborhood homeomorphic to $(0, \infty) \times \mathbb{S}^1$. A neighborhood U of E will be called Riemannian collared if there exists a C^1 diffeomorphism $X : (0, \infty) \times \mathbb{S}^1 \rightarrow U$ such that the metric in U relative to the coordinate system X is written $ds^2 = dr^2 + G(r, \theta)^2 d\theta^2$, where G is a positive continuous function. We say that a closed curve γ bounds a collared end E in S if some arc component of $S \setminus \gamma$ is a neighborhood of E .

Definition 2.16. Given a Riemannian surface S , a geodesic γ in S , and a continuous unit vector field ξ along γ , orthogonal to γ , we define the Fermi coordinates based on γ as the map $Y(\theta, r) := \exp_{\gamma(\theta)} r\xi(\theta)$.

It is well known that the Riemannian metric can be expressed in Fermi coordinates as $ds^2 = dr^2 + G(\theta, r)^2 d\theta^2$, where $G(\theta, r)$ is the solution of the scalar equation

$$(2.1) \quad \frac{\partial^2 G}{\partial r^2}(\theta, r) + K(\theta, r)G(\theta, r) = 0, \quad G(\theta, 0) = 1, \quad \frac{\partial G}{\partial r}(\theta, 0) = 0,$$

(see e.g. [15, p. 247]).

Definition 2.17. A bordered surface is a surface with boundary.

A halfplane is a bordered Riemannian surface which is simply connected and whose border is a single nonclosed simple geodesic.

A funnel is a bordered Riemannian surface which is a neighborhood of a collared end and whose boundary is a simple closed geodesic γ . If the curvature satisfies $K \leq -k^2 < 0$ then there is no other simple closed geodesic freely homotopic to the boundary of the funnel, and γ minimizes the length in its free homotopy class.



A puncture is a collared end whose fundamental group is generated by a simple closed curve σ and there is no closed geodesic $\gamma \in [\sigma]$. If the curvature satisfies $K \leq -k^2 < 0$ then $L([\sigma]) = \inf_{\gamma \in [\sigma]} L(\gamma) = 0$.

A bordered or nonbordered surface is doubly connected if its fundamental group is isomorphic to \mathbb{Z} . Every funnel and every neighborhood of a puncture are doubly connected surfaces.

A Y -piece is a bordered Riemannian surface which is homeomorphic to a sphere minus three open disks and whose boundary curves are simple closed geodesics. If $K = -k^2 < 0$, given three positive numbers a, b, c , there is a unique (up to isometry) Y -piece such that their boundary curves have lengths a, b, c (see e.g. [35, p.410]). Y -piece is a standard tool for constructing Riemannian surfaces with negative curvature. A clear description of these Y -pieces and their use are given in [13, chapter 1] and [15, chapter X.3].

A generalized Y -piece is a Riemannian surface (with or without boundary) which is homeomorphic to a sphere without n open disks and m points, with integers $n, m \geq 0$ such that $n + m = 3$, the n boundary curves are simple closed geodesics and the m deleted points are punctures. Observe that a generalized Y -piece is topologically the union of a Y -piece and m cylinders.

The following result assures that if $K \leq -k^2 < 0$, there always exists a closed geodesic in every free homotopy class, except for punctures, in which case it is impossible to have one.

Theorem 2.18. ([34, Theorem 3.7]) *Let S be a complete Riemannian surface with $K \leq -k^2 < 0$. Assume the boundary of S either is empty or a (finite or infinite) union of pairwise disjoint simple closed geodesics. Let α be a homotopically nontrivial closed curve in S . Then there exists a minimizing closed geodesic $\gamma \in [\alpha]$ if and only if $L_S([\alpha]) > 0$.*

Remark 2.19. *The conclusion of this Theorem does not hold if we replace the hypothesis $K \leq -k^2 < 0$ by the weaker one $K < 0$, as shown by the revolution surface of the graph of $f(x) = 1 + e^x$ around the horizontal axis (with the standard metric induced by the Euclidean metric in \mathbb{R}^3).*

3. TECHNICAL LEMMAS.

For the proof of Lemma 3.3 we will need to compare two functions G_1 and G_2 as in (2.1) corresponding to two different metrics. In order to achieve our objective, we prove now some technical lemmas:

Lemma 3.1. *Let us consider the positive function $G(\theta, r)$ which is the solution of the equation (2.1). If $K \leq -k^2 < 0$, then $G(\theta, r) \geq \cosh kr$ for every $\theta, r \in \mathbb{R}$.*

Proof. According to the equation (2.1), and bearing in mind that $G(\theta, r) > 0$, we have that $\partial^2 G / \partial r^2 = -KG \geq k^2 G > 0$ for every $\theta, r \in \mathbb{R}$. This fact implies that $\partial G / \partial r$ is an increasing function in $r \in \mathbb{R}$ for each fixed θ .

If $r \geq 0$ then $\partial G / \partial r \geq 0$ and we can deduce

$$\begin{aligned}
\frac{\partial^2 G}{\partial r^2}(\theta, r) &\geq k^2 G(\theta, r), \\
\frac{\partial^2 G}{\partial r^2}(\theta, r) \frac{\partial G}{\partial r}(\theta, r) &\geq k^2 G(\theta, r) \frac{\partial G}{\partial r}(\theta, r), \\
\frac{\partial G}{\partial r}(\theta, r)^2 - \frac{\partial G}{\partial r}(\theta, 0)^2 &\geq k^2 (G(\theta, r)^2 - G(\theta, 0)^2), \\
\frac{\partial G}{\partial r}(\theta, r)^2 &\geq k^2 (G(\theta, r)^2 - 1), \\
\frac{\partial G / \partial r(\theta, r)}{\sqrt{G(\theta, r)^2 - 1}} &\geq k, \\
\text{Arcosh } G(\theta, r) &\geq kr.
\end{aligned}$$

If $r < 0$ then $\partial G / \partial r < 0$, and an argument similar to the above gives the same inequality.

In both cases we have obtained the desired result: $G(\theta, r) \geq \cosh kr$ for every $\theta, r \in \mathbb{R}$. □

Lemma 3.2. *Let us consider $\mathbb{R}^2 = \{(\theta, r) : \theta, r \in \mathbb{R}\}$ with two different metrics given in Fermi coordinates as $ds_1^2 = dr^2 + G_1(\theta, r)^2 d\theta^2$ and $ds_2^2 = dr^2 + G_2(\theta, r)^2 d\theta^2$, such that their respective curvatures, K_1 and K_2 , satisfy $K_1 \leq K_2 = -k^2 < 0$. Let us consider two curves σ_1 and σ_2 in \mathbb{R}^2 with the same endpoints, such that σ_i is a geodesic for ds_i ($i = 1, 2$). Then, $L_{ds_1}(\sigma_1) \geq L_{ds_2}(\sigma_2)$.*

Proof. Let us notice that $G_2(\theta, r) := \cosh kr$. By Lemma 3.1, we have that $G_1 \geq G_2$ for every $(\theta, r) \in \mathbb{R}^2$. Hence, if $\sigma_1(\theta) = (\theta, r(\theta))$ with $\theta \in [a, b]$,

$$L_{ds_1}(\sigma_1) = \int_{\sigma_1} ds_1 = \int_a^b \sqrt{r'(\theta)^2 + G_1(\theta, r(\theta))^2} d\theta \geq \int_a^b \sqrt{r'(\theta)^2 + G_2(\theta, r(\theta))^2} d\theta = L_{ds_2}(\sigma_1) \geq L_{ds_2}(\sigma_2).$$

□

Lemma 3.3. *Let us consider a simply connected locally geodesic quadrilateral Q in a complete Riemannian surface S with curvature $K \leq -k^2 < 0$, whose sides A, B, C and η have respective lengths a, b, c and l , and such that C meets orthogonally the sides A and B . Then there exists a positive constant $\lambda := \log(5 + 2\sqrt{6})$ such that $a + b + c - \lambda/k \leq l$, for every $c \geq \lambda/k$.*

Remark 3.4. *Let us notice that we have that $l \leq a + b + c$ by the triangle inequality.*

Proof. First of all, let us recall that any trigonometric formula with $K = -1$ involving lengths x_1, \dots, x_n holds for $K = -k^2$ if we replace x_1, \dots, x_n by kx_1, \dots, kx_n , respectively.

We start the proof assuming that C and η are disjoint; then the quadrilateral Q is the boundary of a simply connected open set. Using Fermi coordinates based on C , we represent C as the Euclidean segment joining $(0, 0)$ and $(c, 0)$, A as the Euclidean segment joining $(0, 0)$ and $(0, a)$, and B as the Euclidean segment joining $(c, 0)$ and (c, b) ; the side η is the geodesic joining $(0, a)$ and (c, b) . Now, let us consider a simply connected locally geodesic quadrilateral Q' in a complete Riemannian surface S with curvature $K = -k^2 < 0$, whose sides A', B', C' and η' have respective lengths a, b, c and l' , and such that C' meets orthogonally the sides A' and B' . Applying the same before construction, it is, using Fermi coordinates, based now on C' , we can represent C' as the Euclidean segment joining $(0, 0)$ and $(c, 0)$, A' as the Euclidean segment joining $(0, 0)$ and $(0, a)$, and B' as the Euclidean segment joining $(c, 0)$ and (c, b) ; the side η' is the geodesic joining $(0, a)$ and (c, b) , with length l_0 . By Lemma 3.2, we know that $L_S(\eta) = l \geq l_0$.

We are going to use the next hyperbolic trigonometry formula with constant curvature $K = -k^2 < 0$ (see [18, p. 88]) and the comparison of lengths $l \geq l_0$:

$$\cosh kl \geq \cosh kl_0 = \cosh ka \cosh kb \cosh kc - \sinh ka \sinh kb.$$

Using now the facts that, on the one hand, $2(\cosh t - 1)e^{-t} = (1 - e^{-t})^2$ and, on the other hand, for $\lambda := \log(5 + 2\sqrt{6})$ it is satisfied the equality $\frac{1}{8}(1 - e^{-\lambda})^2 = e^{-\lambda}$, we have:

$$\begin{aligned} e^{kl} &\geq \cosh kl \geq \cosh ka \cosh kb \cosh kc - \sinh ka \sinh kb \geq (\cosh kc - 1) \cosh ka \cosh kb \\ &= \frac{1}{2}(1 - e^{-kc})^2 e^{kc} \cosh ka \cosh kb \geq \frac{1}{8}(1 - e^{-kc})^2 e^{k(a+b+c)} \geq \frac{1}{8}(1 - e^{-\lambda})^2 e^{k(a+b+c)} \\ &= e^{-\lambda} e^{k(a+b+c)}, \end{aligned}$$

for every $c \geq \lambda/k$. Therefore $l \geq a + b + c - \lambda/k$.

If C and η are not disjoint we use the hyperbolic trigonometry formula with constant curvature $K = -k^2 < 0$ (see [18, p. 88]) and, once again, Fermi coordinates and Lemma 3.2 give:

$$\cosh kl \geq \cosh kl_0 = \cosh ka \cosh kb \cosh kc + \sinh ka \sinh kb.$$

By a similar argument to the previous one:

$$e^{kl} \geq \cosh kl \geq \cosh ka \cosh kb \cosh kc + \sinh ka \sinh kb \geq \cosh kc \cosh ka \cosh kb \geq \frac{1}{8} e^{k(a+b+c)},$$

with no restriction about the length c .

Notice that in this case we obtain a sharper inequality $a + b + c - (3 \log 2)/k \leq l$. \square

Lemma 3.5. *Let us consider a simply connected locally geodesic quadrilateral Q in a complete Riemannian surface S with curvature $K \leq -k^2 < 0$, with sides A , B , C and η , such that C meets orthogonally the sides A and B , and $L_S(C) \geq \lambda/k$ with $\lambda := \log(5 + 2\sqrt{6})$. We also assume that η is a geodesic in S . If we define $\sigma := A \cup C \cup B$, then the curve σ is a $(1, \lambda/k)$ -quasigeodesic with its arc-length parametrization.*

Proof. Denoting by a , b , c and l the lengths of the sides A , B , C and η respectively, since $\sigma : [0, a+b+c] \rightarrow S$ is a continuous curve, it is clear that $d(\sigma(t), \sigma(s)) \leq L_S(\sigma([t, s])) = |t - s|$ always. Without loss of generality, from now on we are assuming that $t < s$.

We are going to prove that $|t - s| \leq d(\sigma(t), \sigma(s)) + \lambda/k$. In order to prove it, let us assume that it is false, and let us seek a contradiction. Suppose then that $|t - s| > d(\sigma(t), \sigma(s)) + \lambda/k$ and define a curve σ_0 as the union of the next three curves: $\sigma([0, t])$, a geodesic connecting $\sigma(t)$ with $\sigma(s)$, and $\sigma([s, a+b+c])$. Let us denote by u, v the endpoints of σ (notice that they are also the endpoints of the geodesic η); since σ_0 is a continuous curve connecting u and v , we have that, applying Lemma 3.3,

$$\begin{aligned} d(u, v) &= L_S(\eta) = l \leq L_S(\sigma_0) = L_S(\sigma) - L_S(\sigma([t, s])) + d(\sigma(t), \sigma(s)) \\ &= L_S(\sigma) - |t - s| + d(\sigma(t), \sigma(s)) \\ &< L_S(\sigma) - \lambda/k = a + b + c - \lambda/k \leq l = d(u, v), \end{aligned}$$

which is a contradiction. Therefore we can assert that

$$|t - s| - \lambda/k \leq d(\sigma(t), \sigma(s)) \leq |t - s|, \quad \text{for every } 0 \leq t, s \leq a + b + c.$$

\square

Lemma 3.6. *Given a complete Riemannian surface S , let us consider a continuous curve g joining $u, v \in S$. If $L_S(g) \leq \alpha$, then g is a $(1, \alpha)$ -quasigeodesic with its arc-length parametrization.*

Proof. Let $g : [0, l] \rightarrow S$ with its arc-length parametrization. Since g is continuous, it is clear that $d(g(t), g(s)) \leq L_S(g([t, s])) = |t - s|$. Let us notice now $|t - s| \leq L_S(g) \leq \alpha \leq d(g(t), g(s)) + \alpha$.

Therefore, we can assert that

$$|t - s| - \alpha \leq d(g(t), g(s)) \leq |t - s|, \quad \text{for every } 0 \leq t, s \leq l.$$

\square

Lemma 3.7. *Let S be a complete Riemannian surface with $K \leq -k^2 < 0$, $\delta_1 := \frac{1}{k} \log(1 + \sqrt{2})$ and let Q be a simply connected locally geodesic quadrilateral in S . Assume the sides, A , B , C and η of Q satisfy the following two conditions: (i) $L_S(C) > 4\delta_1$, (ii) C meets orthogonally the sides A and B .*

Then, we have that $d(z, \eta) \leq 4\delta_1$ for every $z \in C$.

Proof. To start, note that the sides C and η might meet or might not (but if they do, it must be in a single point). However we use an argument which covers both cases.

Without loss of generality we can assume that S is simply connected, since otherwise we can lift Q to the universal covering of S (recall that Q is simply connected and that the distances in the universal cover are greater than in the surface). Consequently, Q is a geodesic quadrilateral (in a simply connected Riemannian surface with $K \leq -k^2$, every local geodesic is, in fact, a geodesic).

Since S is a simply connected and complete Riemannian surface, it is δ_1 -thin (see [2, p. 130] and [20, p. 52]).

Let us consider the geodesic C as an oriented curve joining A and B . By hypothesis $L_S(C) > 4\delta_1$; therefore we can assert that there exist two points α and β (with $\alpha < \beta$) in the oriented geodesic C defined as $\alpha := \max\{z \in C : d(z, A) \leq 2\delta_1\}$, and $\beta := \min\{z \in C : d(z, B) \leq 2\delta_1\}$.

Since Q is $2\delta_1$ -thin, $d(z, A \cup B \cup \eta) \leq 2\delta_1$ for every $z \in C$. If $z \in (\alpha, \beta) \subset C$ then $d(z, A \cup B) > 2\delta_1$ and, therefore, $d(z, A \cup B \cup \eta) = d(z, \eta) \leq 2\delta_1$; consequently, $d(z, \eta) \leq 2\delta_1$ for every $z \in [\alpha, \beta]$. If $z \in C \setminus [\alpha, \beta]$ it is verified that $d(z, [\alpha, \beta]) \leq 2\delta_1$ and, therefore, $d(z, \eta) \leq 4\delta_1$. □

Definition 3.8. *Let λ be the constant in Lemma 3.3 and S a complete Riemannian surface with $K \leq -k^2 < 0$. A λ -triangle in S is defined to be a triangle with continuous injective $(1, \lambda/k)$ -quasigeodesic sides with its arc-length parametrization.*

Lemma 3.9. *Let us consider a complete Riemannian surface S with $K \leq -k^2 < 0$, a simple closed geodesic γ with length l and a λ -triangle T contained in γ and homotopic to γ . Then, every side of T has length less than or equal to $\frac{l}{2} + \frac{\lambda}{k}$. Furthermore, at least two of the sides of T have length greater than or equal to $\frac{l}{4} - \frac{\lambda}{2k}$.*

Proof. If $T := \{a, b, c\}$, let us denote by $l_1 := L_S([a, b])$, $l_2 := L_S([b, c])$ and $l_3 := L_S([a, c])$.

Seeking for a contradiction, let us assume that one side of T , for example $[a, c]$, is “long”, i.e. $l_3 > l/2 + \lambda/k$. However $[a, c]$ is, as a side of T , a $(1, \lambda/k)$ -quasigeodesic with its arc-length parametrization $\sigma_3 : [0, l_3] \rightarrow \gamma$. Then, $l_3 - \lambda/k \leq d(a, c) \leq l_3 + \lambda/k$.

But $d(a, c) \leq l/2$ since a and c belong to a closed curve with length l . Therefore, $l_3 - \lambda/k \leq d(a, c) \leq l/2$, which implies that $l_3 \leq l/2 + \lambda/k$ and this fact contradicts the assumption about l_3 being “long”.

Finally, we prove the last part of the Lemma. Seeking again for a contradiction, let us assume that two sides, for example $[a, b]$ and $[b, c]$, are “short”, that is, $l_1, l_2 < l/4 - \lambda/(2k)$. Then $l_3 > l/2 + \lambda/k$, since $l = l_1 + l_2 + l_3$, which is, in fact, a contradiction. □

Lemma 3.10. *Let us consider a doubly connected complete Riemannian surface S with curvature $K \leq -k^2 < 0$, and let l be the length of the simple closed geodesic γ in it. Then S is $(18\delta_1 + 2l)$ -thin, with $\delta_1 := \log(1 + \sqrt{2})/k$.*

Remark 3.11. *According to the above Lemma, every geodesically convex subset of a surface like the one described in it, is $(18\delta_1 + 2l)$ -thin as well. Further, every funnel F in a Riemannian surface with $K \leq -k^2 < 0$ and $L_S(\partial F) = l$ is $(18\delta_1 + 2l)$ -thin.*

Proof. Let us consider a geodesic triangle $T = \{a, b, c\}$ in S . By Lemma 2.11 we can assume that T is a simple closed curve. Notice that T is homotopic to either to a point or the simple closed geodesic γ . If T is homotopic to a point, then it is the boundary of a simply connected closed set E , and consequently E , with its intrinsic distance, is isometric to some subset of the universal covering surface \tilde{S} ; this implies that T is δ_1 -thin (recall that \tilde{S} is δ_1 -thin). Therefore, to finish the proof we may assume that T is homotopic to the simple closed geodesic γ of S .

In this case, there are two possible situations:

- (1) Two of the vertices of the triangle coincide. Then T is a geodesic bigon whose two sides are two geodesics γ_1 and γ_2 with the same length. Now, we have Figure 1 (drawn in the universal covering surface). In it, the vertices a and a_0 must be considered as coincident, but we have maintained both names to be able to identify clearly γ_1 with $[a, b]$ and γ_2 with $[b, a_0]$. Let us call a', b' and a'_0 their respective projections over γ (where $[a', b']$ is the projection of γ_1 over γ and $[b', a'_0]$ is the projection of γ_2 over γ ; although all the projections are obviously orthogonal, in the Figure 1 we have not used right angles to emphasize that the lengths of the segments $[a', b']$ and $[b', a'_0]$ might be very different, since the curvature is not constant).

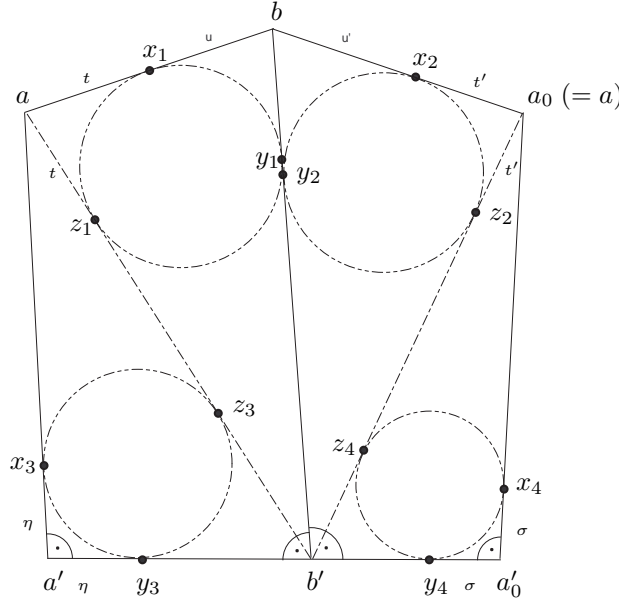


FIGURE 1. T is a geodesic bigon.

Let us draw now the geodesics $[a, b']$ and $[a_0, b']$ and let l_1 and l_2 be their respective lengths. Next, we are going to prove that $|l_1 - l_2| \leq 3l$. In order to do it, we need to consider the following four geodesic triangles: $T_1 = \{a, b, b'\}$ (with internal points x_1, y_1, z_1), $T_2 = \{b, a_0, b'\}$ (with internal points x_2, y_2, z_2), $T_3 = \{a, a', b'\}$ (with internal points x_3, y_3, z_3) and $T_4 = \{a_0, a'_0, b'\}$ (with internal points x_4, y_4, z_4). Notice that T_1, T_2, T_3 and T_4 are, all of them, $4\delta_1$ -fine, because they are homotopically trivial and then their lifts to the universal covering surface are $4\delta_1$ -fine.

If we denote $\eta := L_S([a', x_3]) = L_S([a', y_3])$ and $\sigma := L_S([a'_0, x_4]) = L_S([a'_0, y_4])$ then

$$l_1 = L_S([a, z_3]) + L_S([z_3, b']) = L_S([a, x_3]) + L_S([b', y_3]) = L_S([a, a']) + L_S([a', b']) - 2\eta, \text{ and}$$

$$l_2 = L_S([a_0, z_4]) + L_S([z_4, b']) = L_S([a_0, x_4]) + L_S([b', y_4]) = L_S([a_0, a'_0]) + L_S([a'_0, b']) - 2\sigma.$$

Hence:

$$\begin{aligned} |l_2 - l_1| &= |L_S([a_0, a'_0]) + L_S([a'_0, b']) - 2\sigma - L_S([a, a']) - L_S([a', b']) + 2\eta| \\ &= |L_S([a'_0, b']) - L_S([a', b']) - 2(\sigma - \eta)| \leq 3l. \end{aligned}$$

Now we claim that both $|d(a_0, x_2) - d(a, x_1)| \leq 2l$ and $|d(b, x_1) - d(b, x_2)| \leq 2l$. For simplicity of notation we are going to give names to the lengths involved in the proof. So, we define $l_3 := L_S([a, b]) = L_S([b, a_0])$, $l_4 := L_S([b, b'])$, $u := L_S([b, x_1])$, $u' := L_S([b, x_2])$, $t := L_S([a, x_1]) = L_S([a, z_1])$, $t' := L_S([a_0, x_2]) = L_S([a_0, z_2])$, $s := L_S([b', z_1])$ and $s' := L_S([b', z_2])$.

Since $l_1 + l_2 + l_3 = 2s + 2t + 2u$ and $l_1 = s + t$, subtracting both equations we get $u = l_3/2 + l_4/2 - l_1/2$. Applying a similar argument, we can also obtain $u' = l_3/2 + l_4/2 - l_2/2$. Hence,

$$(3.2) \quad \begin{aligned} |d(b, x_1) - d(b, x_2)| &= |u - u'| = |l_2 - l_1|/2 \leq 3l/2 < 2l, \text{ and} \\ |d(a_0, x_2) - d(a, x_1)| &= |t' - t| = |l_3 - u' - l_3 + u| = |u - u'| \leq 3l/2 < 2l \end{aligned}$$

as we claimed.

Now we proceed to show that our bigon T is $(16\delta_1 + 2l)$ -thin. To do it, let us take an arbitrary point $z \in T$. Obviously there is no loss of generality in assuming that $z \in \gamma_1$. More precisely, z may belong either to the segment $[b, x_1]$ or to the segment $[a, x_1]$. In the first case, there are two possible situations:

- (a) z is a point such that $d(b, z) \leq \min\{u, u'\}$. Then, since T_1 is $4\delta_1$ -fine, there exists a point $z' \in [b, y_1]$ such that $d(z, z') \leq 4\delta_1$. Applying the same argument to the $4\delta_1$ -fine triangle T_2 , there exists a point $z'' \in [b, x_2]$ such that $d(z', z'') \leq 4\delta_1$. It means that we have been able to find a point $z'' \in \gamma_2$ verifying that $d(z, z'') \leq d(z, z') + d(z', z'') \leq 8\delta_1$.
- (b) z is a point such that $d(b, z) > \min\{u, u'\}$. If this is so, the first thing to do is getting another point $z_0 \in [b, x_1]$ such that $d(b, z_0) \leq \min\{u, u'\}$ and then apply the previous case to that z_0 . Notice that, as $|u - u'| \leq 2l$ by (3.2), z_0 can be found easily by moving to the right at most a distance $2l$ along γ_1 starting at x_1 . Then, by the arguments used before, we are able to find a point $z'' \in \gamma_2$ such that $d(z, z'') \leq d(z, z_0) + d(z_0, z') + d(z', z'') \leq 2l + 8\delta_1$.

Now, if $z \in [a, x_1]$ we repeat the same arguments but taking into account that besides T_1 and T_2 , the triangles T_3 and T_4 are $4\delta_1$ -fine as well (recall that by (3.2) we also have $|t' - t| \leq 2l$). Hence, we know that there exists a point $z'' \in \gamma_2$ such that $d(z, z'') \leq 16\delta_1 + 2l$, which implies our conclusion.

- (2) Now, let T be a general geodesic triangle, with its three vertices different. See Figure 2, in which we have unfolded our triangle in the universal covering surface to make things more understandable. There we have used two different names a and a_0 for the same vertex in order to simplify notation in the following argument (they are identified when the triangle is “glued” again).

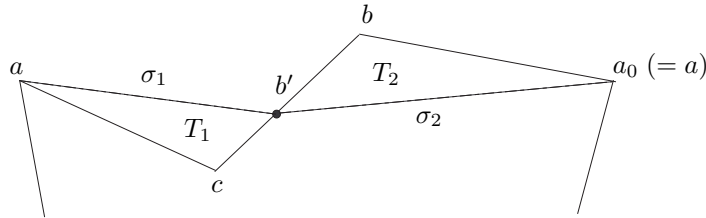


FIGURE 2. T is a general triangle.

Let us define Γ as the set of curves joining c and a which are homotopic to $[c, b] \cup [b, a]$, and Γ' as the set of curves joining a and b which are homotopic to $[a, c] \cup [c, b]$. It is obvious that the distance from c to a (that is to say, the length of the geodesic side $[a, c]$) is less than or equal to the distance between the same two points when we just consider the curves in Γ . That is to say, $d(c, a) \leq d_\Gamma(c, a)$. For the sake of simplicity in the notation of the following argument, we will denote $d_\Gamma(c, a)$ by $d(c, a_0)$ (see Figure 2 for a better understanding).

A similar argument yields $d_{\Gamma'}(a, b) \geq d(b, a)$. Again, in order to simplify the notation (and to maintain the coherence with Figure 2), we will denote $d_{\Gamma'}(a, b)$ simply by $d(a, b)$, and we will refer to the length of the geodesic side $[b, a]$ as $d(b, a_0)$.

Let us define now a function $f(x) := d(x, a) - d(x, a_0)$. Since $f(c) = d(c, a) - d(c, a_0) \leq 0$ and $f(b) = d(b, a) - d(b, a_0) \geq 0$, it means that there must exist a point $b' \in [b, c]$ such that $f(b') = d(b', a) - d(b', a_0) = 0$.

Notice that there are two nonhomotopic geodesics joining the vertex a with the point b' . Let σ_1 be the geodesic homotopic to $[a, c] \cup [c, b']$; σ_2 the geodesic homotopic to $[b', b] \cup [b, a_0]$. Then $\sigma_1 \cup \sigma_2$ is a geodesic bigon whose sides have the same length. If we apply the previous case to this bigon, we can state that it is $(16\delta_1 + 2l)$ -thin.

Next, let us consider the two simply connected triangles $T_1 = [a, c] \cup [b', c] \cup \sigma_1$ and $T_2 = [a_0, b] \cup [b, b'] \cup \sigma_2$, both of them δ_1 -thin. So, given any point $z \in [a, c]$, there exists a point $z' \in [b', c] \cup \sigma_1$ with $d(z, z') \leq \delta_1$. If $z' \in [b', c] \subset [b, c]$ we have finished the proof, and we conclude that $d(z, A) \leq \delta_1$. If $z' \in \sigma_1$, we know that there exists a point $z'' \in \sigma_2$ such that $d(z', z'') \leq 16\delta_1 + 2l$ since the bigon $\sigma_1 \cup \sigma_2$ is $(16\delta_1 + 2l)$ -thin. Hence, $d(z, \sigma_2) \leq 17\delta_1 + 2l$; since T_2 is also δ_1 -thin, it means that $d(z, A) \leq 18\delta_1 + 2l$, and therefore our general triangle T is $(18\delta_1 + 2l)$ -thin, which finishes the proof. \square

Corollary 3.12. *Let us consider a doubly connected complete Riemannian surface S with no simple closed geodesic, and with curvature $K \leq -k^2 < 0$. Then S is $18\delta_1$ -thin, with $\delta_1 := \log(1 + \sqrt{2})/k$.*

Remark 3.13. *According to the above Corollary, every geodesically convex subset of a surface like the one described in it, is $18\delta_1$ -thin as well.*

Proof. As the proof is almost identical to that of Lemma 3.10, we will just explain the only difference. Let us consider a geodesic bigon $T = \{a, b\}$ as in the proof of Lemma 3.10. Since there is no simple closed geodesic in S homotopic to T , and the curvature K is strictly negative, there exist non-trivial closed curves with arbitrarily small length which are homotopic to T . For any fixed $\varepsilon > 0$ let us choose one of these closed curves with length smaller than ε and let us call it σ_0 . Then, we can repeat the same construction used in the proof of Lemma 3.10 for geodesic bigons, but projecting the vertices a and b of T over σ_0 rather than over a simple closed geodesic (which does not exist in our situation). Let us call a' and b' the respective projections of a and b over σ_0 . It is clear that there exist two non-homotopic geodesic segments σ_1 and σ_2 joining a' and b' . We choose now $\sigma = \sigma_1 \cup \sigma_2$. Then we can repeat the same arguments used in the proof of Lemma 3.10 replacing $[a', b']$ and $[b', a'_0]$ by σ_1 and σ_2 , and conclude that the bigon T is $(16\delta_1 + 2\varepsilon)$ -thin.

Applying this result to a general geodesic triangle T as in the proof of Lemma 3.10, we can conclude that T is $(18\delta_1 + 2\varepsilon)$ -thin.

Since ε can be arbitrarily small, a passage to the limit (making $\varepsilon \rightarrow 0$) implies that our general triangle is also $18\delta_1$ -thin. \square

4. THE MAIN RESULT.

Theorem 4.1. *Let S be a complete Riemannian surface with $K \leq -k^2 < 0$ (with or without boundary); if S has boundary, we also require that ∂S is the union of local geodesics (closed or non-closed). Let $\lambda := \log(5 + 2\sqrt{6})$; then S is δ -hyperbolic if and only if every λ -triangle contained in a simple closed geodesic of S is δ_0 -thin.*

More precisaly, if S is δ -thin, then every λ -triangle contained in a simple closed geodesic in S is δ_0 -thin with $\delta_0 = \delta + 2H$, where $H = H(4\delta, 1, \lambda/k)$ is the constant in Theorem 2.9; and if every λ -triangle contained in a simple closed geodesic in S is δ_0 -thin then S is δ -thin, with $\delta := \max\{18\delta_1 + 20\lambda/k, 12\delta_1 + \delta_0\}$ and $\delta_1 := \frac{1}{k} \log(1 + \sqrt{2})$.

Remark 4.2. *Let us notice that, as $\log(1 + \sqrt{2}) \approx 0.8814$ and $\lambda := \log(5 + 2\sqrt{6}) \approx 2.2924$, the hyperbolicity constant δ can be bounded in the following way: $\delta \leq \delta_0 + 18\delta_1 + 20\lambda/k < \delta_0 + 62/k$*

Proof. Let us assume first that S is δ -hyperbolic and let us check that every λ -triangle contained in a simple closed geodesic of S is δ_0 -thin, with $\delta_0 = \delta + 2H$. Let T be a λ -triangle with sides g_1, g_2, g_3 , contained in a simple closed geodesic γ . Theorem 2.9 implies that for every quasigeodesic side g_i there exists a geodesic γ_i with the same endpoints as g_i , such that $\mathcal{H}(g_i, \gamma_i) \leq H = H(4\delta, 1, \lambda/k)$. We have to prove that the distance from any point in a side of T to the union of the other two sides is bounded from above by δ_0 . Without loss of generality let us assume that $z \in g_1$. Hence, for every $z \in g_1$, there is a point $z_0 \in \gamma_1$ with $d(z, z_0) \leq H$. Since S is δ -thin, we can find $z'_0 \in \gamma_2 \cup \gamma_3$ with $d(z_0, z'_0) \leq \delta$. Finally, we also have a point $z' \in g_2 \cup g_3$ with $d(z', z'_0) \leq H$. Consequently $d(z, g_2 \cup g_3) \leq d(z, z') \leq \delta + 2H$.

Now, let us assume that every λ -triangle contained in a simple closed geodesic of S is δ_0 -thin and let us prove that S is δ -hyperbolic.

First, notice that if S has boundary, the hypothesis implies that ∂S is the union of pairwise disjoint simple local geodesics (closed or non-closed). In this case, we can construct a complete Riemannian surface R without boundary and with $K \leq -k^2$ by pasting to S a cylinder along each simple closed geodesic, and a halfplane in each non-closed simple geodesic. If $\gamma_0 \subseteq \partial S$ is a closed geodesic with length l , we can consider the Fermi coordinates based on γ_0 . The Riemannian metric can be expressed in Fermi coordinates as $ds^2 = dr^2 + G(\theta, r)^2 d\theta^2$, with $G(\theta, r)$ satisfying (2.1), where the function $K(\theta, r)$ is C^∞ in $\mathbb{R} \times (-\infty, 0]$ and l -periodic in θ ; actually, $K(\theta, r) \in C^\infty(\mathbb{R} \times (-\infty, 3\varepsilon))$, for some $\varepsilon > 0$. Let $\psi(r) \in C^\infty(\mathbb{R})$ be a function satisfying

$$\psi(r) = \begin{cases} 1, & r \leq \varepsilon, \\ 0, & r \geq 2\varepsilon, \end{cases}$$

and $0 \leq \psi(r) \leq 1$. We define $\tilde{K}(\theta, r) := K(\theta, r)\psi(r) - (1 - \psi(r))k^2$; then $\tilde{K}(\theta, r) = K(\theta, r)$ in $\mathbb{R} \times (-\infty, \varepsilon]$ and $\tilde{K}(\theta, r) = -k^2$ in $\mathbb{R} \times [2\varepsilon, \infty)$. Therefore, $\tilde{K}(\theta, r)$ is C^∞ in $\mathbb{R} \times \mathbb{R}$, l -periodic in θ and $\tilde{K} \leq -k^2$. Now we consider the function $\tilde{G}(\theta, r)$ associated to $\tilde{K}(\theta, r)$ and satisfying (2.1). This allows to attach a cylinder to S along each simple closed geodesic $\gamma_0 \subseteq \partial S$. If $\gamma_0 \subseteq \partial S$ is a non-closed geodesic we can apply a similar argument to the previous one. We must consider another function $\psi(\theta, r)$ depending on θ and r since, in this case, G need not be periodic. This allows to attach a halfplane to S along each simple non-closed geodesic $\gamma_0 \subseteq \partial S$.

In any case we get a complete Riemannian surface R containing S and with curvature less than or equal to $-k^2$. Since S is geodesically convex in R (every geodesic connecting two points of S is contained in S), then $d_R(z, w) = d_S(z, w)$ for every $z, w \in S$, and any simple closed geodesic in R is contained in S . Therefore, it is sufficient to prove the Theorem for surfaces without boundary.

Since the universal covering map, $\pi : \tilde{S} \rightarrow S$ is a local isometry, the universal covering \tilde{S} (which is a simply connected and complete surface satisfying that $K \leq -k^2$) is δ_1 -thin (see [2, p.130] and [20, p.52]).

Now, let us consider a geodesic triangle $T = \{a, b, c\}$ in S . By Lemma 2.11, we can assume that T is a simple closed curve. There are three possibilities: T is homotopic to a point, T is homotopic to a puncture, or T is freely homotopic to a simple closed geodesic in S .

If T is homotopic to a point, then it is the boundary of a simply connected closed set E , and consequently E , with its intrinsic distance, is isometric to some geodesically convex subset of \tilde{S} ; this implies that T is δ_1 -thin (recall that \tilde{S} is δ_1 -thin).

If T is homotopic to a puncture, then it is the boundary of a closed doubly connected set, which is, with its intrinsic distance, isometric to some geodesically convex subset of a doubly connected surface verifying the hypothesis in Corollary 3.12. This implies that T is $18\delta_1$ -thin.

Otherwise, $T = \{a, b, c\}$ is freely homotopic to a simple closed geodesic γ in S .

We are going to deal with two cases.

If $L_S(\gamma) \leq 10\lambda/k$, we consider a doubly connected complete Riemannian surface, S_0 , with $K \leq -k^2 < 0$ containing a simple closed geodesic $\tilde{\gamma}$ of length l , such that the closed set in S bounded by T and γ is, with its intrinsic distance, isometric to a subset in S_0 , bounded by $\tilde{\gamma}$ and a triangle T_0 . We know that S_0 is $18\delta_1 + 2l$ -thin by Lemma 3.10. These facts imply that T is $(18\delta_1 + 20\lambda/k)$ -thin.

We consider now the case $L_S(\gamma) > 10\lambda/k$.

The main idea of the proof is to project the three vertices of T onto the simple closed geodesic γ to obtain a new triangle $T' \subseteq \gamma$. We will check that T' is a λ -triangle and hence it is δ_0 -thin by hypothesis.

Let a', b' and c' be the projections over the geodesic γ of a, b and c respectively (so $[a', b']$, $[b', c']$ and $[c', a']$ are the projections of the sides $[a, b]$, $[b, c]$ and $[c, a]$ over γ). Thus, we have constructed a new triangle $T' = \{a', b', c'\}$ contained in the simple closed geodesic γ .

Next, we are going to construct some quadrilaterals from the sides of both triangles T and T' , in order to apply the previous lemmas.

As Figure 3 shows, every side of T there is a simply connected locally geodesic quadrilateral:

- (i) for $[a, b]$ we have $Q_1 := [a, b] \cup [a, a'] \cup [b, b'] \cup [a', b']$,
- (ii) for $[b, c]$ we have $Q_2 := [b, c] \cup [b, b'] \cup [c, c'] \cup [b', c']$,
- (iii) for $[c, a]$ we have $Q_3 := [c, a_0] \cup [c, c'] \cup [a_0, a'_0] \cup [c', a'_0]$.

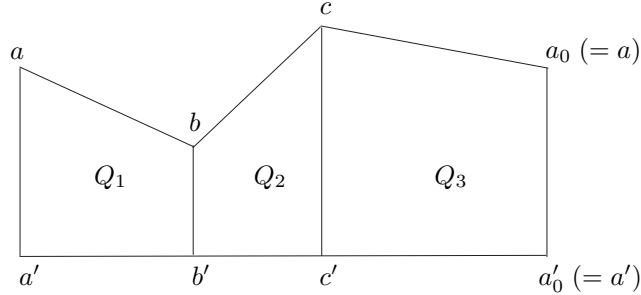


FIGURE 3

In the figure we have unfolded again our triangle in the universal covering surface to make things more understandable, and we have used two different names, a and a_0 , for the same vertex for the sake of simplicity in notation in the following argument. They are identified when the triangle is "glued" again (similarly for a' and a'_0 of γ). Every Q_i , $1 \leq i \leq 3$ is $2\delta_1$ -thin (its lift in the universal cover is a geodesic quadrilateral).

We claim that $T' = \{a', b', c'\}$ is a λ -triangle. Let us check it just for one side, for example $[a', b'] \in Q_1$:

- (1) If $L_S([a', b']) > \lambda/k$ then, by Lemma 3.5, $[a, a'] \cup [b, b'] \cup [a', b']$ is a $(1, \lambda/k)$ -quasigeodesic, so $[a', b'] \subset \gamma$ is also a $(1, \lambda/k)$ -quasigeodesic.
- (2) If $L_S([a', b']) \leq \lambda/k$ then, by Lemma 3.6, the side $[a', b'] \subset \gamma$ is a $(1, \lambda/k)$ -quasigeodesic.

It follows that, on the one hand, T' is δ_0 -thin (by hypothesis) and, on the other hand, the lengths of at least two sides of T' are greater than or equal to $l/4 - \lambda/(2k)$ (by Lemma 3.9). In our current case, as $l > 10\lambda/k$, it is easy to check that, actually, these lengths are greater than $2\lambda/k$. Besides, note that:

$$(4.3) \quad \frac{l}{4} - \frac{\lambda}{2k} > \frac{2\lambda}{k} = \frac{2\log(5+2\sqrt{6})}{k} > \frac{2\log 9}{k} = \frac{4\log 3}{k} > \frac{4\log(1+\sqrt{2})}{k} = 4\delta_1.$$

This means that, at least two sides have length strictly greater than $4\delta_1$.

Let z be a point of one of the sides of T and σ the union of the other two sides; we are going to prove that $d(z, \sigma) \leq 12\delta_1 + \delta_0$. Without loss of generality we can assume that $z \in [a, b]$, and then there exists $z_0 \in [a, a'] \cup [b, b'] \cup [a', b']$ with $d(z, z_0) \leq 2\delta_1$, since Q_1 is $2\delta_1$ -thin.

If $z_0 \in [a', b']$, since the triangle T' is δ_0 -thin, there exists $z'_0 \in [b', c'] \cup [c', a'_0]$ with $d(z_0, z'_0) \leq \delta_0$. There is no loss of generality in assuming that $z'_0 \in [b', c']$. There are two possibilities:

- (1) If $L_S([b', c']) > 4\delta_1$, applying Lemma 3.7 we have that $d(z'_0, [b, c]) \leq 4\delta_1$. Therefore $d(z, \sigma) \leq 6\delta_1 + \delta_0$.
- (2) If $L_S([b', c']) \leq 4\delta_1$, then $d(z'_0, c') \leq 4\delta_1$. Note that $L_S([c', a'_0]) > 4\delta_1$ by (4.3) and by Lemma 3.7, $d(c', [c, a_0]) \leq 4\delta_1$. Therefore $d(z, \sigma) \leq 10\delta_1 + \delta_0$.

If $z_0 \in [a, a'] \cup [b, b']$, without loss of generality we can assume that $z_0 \in [b, b'] \in Q_2$. Since Q_2 is $2\delta_1$ -thin, once again, there exists $z'_0 \in [b, c] \cup [c, c'] \cup [b', c']$ with $d(z_0, z'_0) \leq 2\delta_1$. If $z'_0 \in [b, c]$ then $d(z, \sigma) \leq 4\delta_1$ and we are done. If $z'_0 \in [b', c'] \cup [c, c']$, then there are two possibilities:

- (1) If $L_S([b', c']) > 4\delta_1$, then we can choose z'_0 with $z'_0 \in [b', c']$: this is because we can lift Q_2 to the universal covering \tilde{S} , obtaining in it a new geodesically convex quadrilateral (let us use the same notation for the points in \tilde{S}), hence $d_{\tilde{S}}([b, b'], [c, c']) = L_{\tilde{S}}([b', c']) > 4\delta_1$. The new geodesic quadrilateral in \tilde{S} is $2\delta_1$ -thin as well, so we can take a point $z'_0 \in [b', c'] \subset \tilde{S}$ such that $d(z_0, z'_0) \leq d_{\tilde{S}}(z_0, z'_0) \leq 2\delta_1$. Applying now Lemma 3.7 we have that $d(z'_0, [b, c]) \leq 4\delta_1$, and therefore $d(z, \sigma) \leq 8\delta_1$.
- (2) If $L_S([b', c']) \leq 4\delta_1$, there are two possible situations: either $z'_0 \in [b', c']$ or $z'_0 \in [c, c']$.
In the first case, $d_S(z'_0, c') \leq 4\delta_1$ and, since $L_S([c', a'_0]) > 4\delta_1$ (by (4.3)), applying Lemma 3.7 we have that $d(c', [c, a_0]) \leq 4\delta_1$; thus $d(z, \sigma) \leq 12\delta_1$.
In the second case, when $z'_0 \in [c, c']$, notice that now we are in the quadrilateral Q_3 where $L([c', a'_0]) > 4\delta_1$, and repeating the same arguments we obtain $d(z, \sigma) \leq 12\delta_1$.

Hence, S is δ -thin, with $\delta := \max\{18\delta_1 + 20\lambda/k, 12\delta_1 + \delta_0\}$.

□

One might think that the λ -triangle T' contained in the simple closed geodesic is geodesic; however, the following example shows that T' in the proof of Theorem 4.1 does not need to be geodesic, even in the constant curvature case.

Example. *There is a geodesic triangle T in a complete Riemannian surface S with constant curvature $K = -k^2 < 0$, such that the corresponding T' is not geodesic. That is, at least one of the sides of T' is not geodesic (See Figure 4).*

Given $x_0 > 0$ satisfying $kx_0 < \text{Arcsinh } 1$, let us fix $y > 0$ with $\sinh k(x_0 + y) > \cosh ky$. Then $\sinh k(x + y) > \cosh ky$ for any x satisfying $kx_0 \leq kx < \text{Arcsinh } 1$, and consequently we can choose $x > 0$ such that $kx < \text{Arcsinh } 1$ and $\sinh kx \sinh k(x + \varepsilon) > \cosh ky$.

Let $\varepsilon := \frac{1}{k} \text{Arcsinh}(1/\sinh kx) - x > 0$, then $\sinh kx \sinh k(x + \varepsilon) = 1$. Let us consider a geodesic quadrilateral V in the Cartan-Hadamard surface H_k with constant curvature $K = -k^2$, with three right angles and an angle equal to zero, such that the two finite sides have length x and $x + \varepsilon$ (see e.g. [11, p.157] and [18, p.89]). If we glue together four isometric copies of V , we obtain a generalized Y -piece Y with two punctures and a simple closed geodesic γ with $L_S(\gamma) = 4(x + \varepsilon)$. We obtain S by gluing Y with a funnel F with $K = -k^2$ whose simple closed geodesic has length $4(x + \varepsilon)$.

Let us denote by μ_0 the geodesic in Y with $L_S(\mu_0) = 2x$, joining γ with itself which is not homotopic to any curve contained in γ . We denote by p', q' the endpoints of μ_0 . Let us consider the unbounded geodesic μ in S which contains μ_0 , and the two points $p, q \in \mu \cap F$ at distance y of γ .

Let us define the triangle T as the union of the two geodesics α, β in F joining p and q (in fact, T is a geodesic “bigon”). The length of the segment of μ between p and q is $2x + 2y$; by [18, p.89] we have $\sinh(kL_S(\alpha)/2) = \sinh k(x + \varepsilon) \cosh ky = \cosh ky / \sinh kx < \sinh k(x + y)$. So we obtain $L_S(\alpha) < 2x + 2y = L_S([p, q])$, and consequently α, β are in fact geodesics in S . However, $T' = \{p', q'\}$ is contained in γ and $L_S(\alpha') = L_S(\beta') = 2x + 2\varepsilon > 2x = L_S(\mu_0) = d(p', q')$; hence α', β' are not geodesics in S .

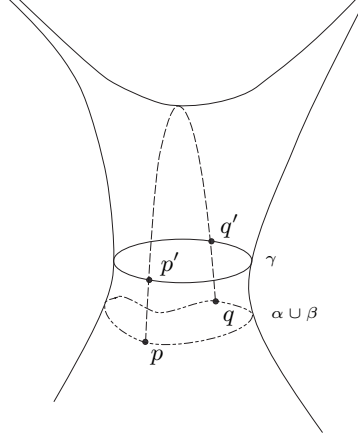


FIGURE 4

5. APPLICATIONS.

In this section we will give several applications of Theorem 4.1.

Definition 5.1. *We say that a complete Riemannian surface S (with or without boundary) is of finite type if its fundamental group is finitely generated.*

Corollary 5.2. *Let us consider a complete Riemannian surface S (with or without boundary) with $K \leq -k^2 < 0$; if S has boundary, we also require that ∂S is the union of local geodesics (closed or non-closed). If S is of finite type, then it is hyperbolic. More precisely, if S is of finite type and every simple closed geodesic γ in S verifies $L_S(\gamma) \leq l$, then S is δ -thin, with $\delta = \max \{18\delta_1 + 20\lambda/k, 12\delta_1 + l/4 + \lambda/(2k)\}$, where $\delta_1 := \frac{1}{k} \log(1 + \sqrt{2})$ and $\lambda := \log(5 + 2\sqrt{6})$.*

Proof. The number of simple closed geodesics in S which are homotopic to a geodesic triangle is finite: $\{\gamma_1, \dots, \gamma_k\}$, and hence we have $L_S(\gamma_j) \leq l$. Every continuous injective $(1, \lambda/k)$ -quasigeodesic with its arc-length parametrization $g \subset \gamma_j$ verifies $L_S(g) \leq l/2 + \lambda/k$ by Lemma 3.9; hence $d(z, \partial g) \leq l/4 + \lambda/(2k)$ for every $z \in g$. Then the hypothesis of Theorem 4.1 is verified with $\delta_0 := l/4 + \lambda/(2k)$. Hence S is δ -thin with $\delta = \max \{18\delta_1 + 20\lambda/k, 12\delta_1 + l/4 + \lambda/(2k)\} \leq 18\delta_1 + 20\lambda/k + l/4$. \square

A consequence of this corollary is the following result.

Corollary 5.3. *Every generalized Y -piece S with $L_S(\gamma_i) \leq l$, where γ_i ($i = 1, 2, 3$) are the simple closed geodesics in ∂S , is δ -thin, with $\delta = \max \{18\delta_1 + 20\lambda/k, 12\delta_1 + l/4 + \lambda/(2k)\}$, where $\delta_1 := \frac{1}{k} \log(1 + \sqrt{2})$ and $\lambda := \log(5 + 2\sqrt{6})$.*

Remark 5.4. *As usual, we view a puncture as a simple closed geodesic with length equal to zero.*

It is clear that a funnel contains infinitely many halfplanes.

Two additional results can be deduced from Theorem 4.1. The first one (see Theorem 5.5 below) allows us to simplify the topology of a surface in order to study its hyperbolicity: it assures that deleting funnels and halfplanes does not change the hyperbolicity of a Riemannian surface.

If one thinks of possible conjectures about hyperbolicity, the following one might sound reasonable: "If a Riemannian surface has a sequence of funnels $\{F_n\}_n$ with $\lim_{n \rightarrow \infty} L_S(\partial F_n) = \infty$, then it is not hyperbolic". However, in [39] we prove that this reasonable result is false indeed, and besides it turns out to be an important tool in the proof is Theorem 5.5.

Theorem 5.5. *Let us consider a complete Riemannian surface S (with or without boundary); if S has boundary, we also require that ∂S is the union of local geodesics (closed or non-closed). Let us denote by F the union of some pairwise disjoint funnels and halfplanes of S . Let S_0 be the bordered complete Riemannian surface obtained by deleting from S the interior of F . Then S is hyperbolic if and only if S_0 is hyperbolic.*

More precisely, if S is δ -thin (respectively, δ -hyperbolic) then S_0 is δ -thin (respectively, δ -hyperbolic); and if S_0 is δ' -hyperbolic, then S is δ -thin, with $\delta = \max\{18\delta_1 + 20\lambda/k, 12\delta_1 + 4\delta' + 2H(\delta', 1, \lambda/k)\}$ where $\delta_1 := \frac{1}{k} \log(1 + \sqrt{2})$, $\lambda := \log(5 + 2\sqrt{6})$ and H is the constant in Theorem 2.9.

Remark 5.6. *We want to emphasize that in Theorem 5.5 there is no hypothesis about the length of the boundary curves of the funnels (this is not the case in corollaries 5.2 and 5.3). This is an important fact since there are complete hyperbolic Riemannian surfaces containing funnels F_n with $L_S(\partial F_n) \rightarrow \infty$ as $n \rightarrow \infty$.*

Proof. Let us assume that S is δ -thin (respectively, δ -hyperbolic). As S_0 is geodesically convex in S (every geodesic connecting two points of S_0 is contained in S_0), we have $d(z, w) = d_{S_0}(z, w)$ for every $z, w \in S_0$. Therefore S_0 is also δ -thin (respectively, δ -hyperbolic).

Let us assume now that S_0 is δ' -hyperbolic. By [39, Lemma 3.3] (or the first part of the proof of Theorem 4.1), every λ -triangle T in S_0 is $(\delta' + 2H(\delta', 1, \lambda/k))$ -thin, where H is the constant in Theorem 2.9. Let us observe that any simple closed geodesic in S is contained in S_0 . Since $d(z, w) = d_{S_0}(z, w)$ for every $z, w \in S_0$, every λ -triangle in S (contained in a simple closed geodesic in S) is also a λ -triangle in S_0 . Let us observe also that $H \geq 1 > \log(1 + \sqrt{2})$. Then Theorem 4.1 implies that S is δ -thin, with $\delta = \max\{18\delta_1 + 20\lambda/k, 12\delta_1 + 4\delta' + 2H(\delta', 1, \lambda/k)\}$. \square

Many Riemannian surfaces can be decomposed as a union of funnels and generalized Y -pieces (see [5] and [34, Theorem 4.1]). The following result uses this decomposition in order to obtain hyperbolicity.

Theorem 5.7. *Let us consider a complete Riemannian surface S with $-k_1^2 \leq K \leq -k_2^2 < 0$ (with or without boundary), with genus equal to zero. If there is a decomposition of S as a union of funnels $\{F_m\}_{m \in M}$ and generalized Y -pieces $\{Y_n\}_{n \in N}$ with $L_S(\gamma) \leq l$ for every simple closed geodesic $\gamma \subset (\cup_n \partial Y_n) \cup (\cup_m \partial F_m)$, then S is δ -hyperbolic, where δ is a constant depending only on k_1, k_2 and l .*

Proof. The Collar Lemma in variable negative curvature (see [16]) states that there exists a constant c_0 , which only depends on k_1, k_2 and l , such that $d(\gamma_1, \gamma_2) \geq c_0$ for every $\gamma_1, \gamma_2 \subset (\cup_n \partial Y_n) \cup (\cup_m \partial F_m)$ with $\gamma_1 \neq \gamma_2$.

Hence, $\{F_m, Y_n\}_{m,n}$ is a $(l/2, c_0)$ -tree decomposition of S .

By Lemma 3.10, F_m is $\tilde{\delta}$ -hyperbolic for every $m \in M$, where $\tilde{\delta}$ is a constant which just depends on k_2 and l . By Corollary 5.3, Y_n is δ^* -hyperbolic for every $n \in N$, where δ^* is a constant which just depends on k_2 and l . Now the result follows from Theorem 2.14. \square

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